Secondary Preservice Teachers’ Mathematical Discourses
On Geometric Transformations

BY

WENJUAN LI
B. S., Guangzhou University, Guangdong, China, 2004
M.S., East China Normal University, Shanghai, China, 2007

THESIS

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Defense Committee:

Mara Martinez, Chair and Advisor
Alison Castro Superfine
Joshua Radinsky
Anatoly Libgober, Mathematics
Sasha Wang, Boise State University
This dissertation is dedicated to

Professor Emeritus Philip Wagreich (1941-2013)

Dr. Philip Wagreich was my advisor, mentor, role model, and dear friend. The first time I met Dr. Phil in his office, he was planning a course using Geometry Sketchpad. At that time, I would never have known that that would be the topic I would study for my dissertation. It was Dr. Phil’s encouragement and support that made this journey possible.

Dr. Phil was a constant source of ideas, insightful feedback, and general optimism. During the first three years of my study in the program, Dr. Phil supervised my teaching apprentices, and we co-taught several courses for preservice teachers. He was a passionate educator. He always sat down with me for hours to discuss the lesson plan. One time, he saw his son, Alexander, doing fractions division by dividing the numerators and the denominators. He then proposed, “How about we ask our preservice teachers to help figure out this new method?” This was Dr. Phil, someone who would never stop seeking meaningful learning material for his students.

Dr. Phil is my role model. He made great contributions to the field of mathematics education. He established important organizations and projects that have had a significant influence on teacher development and student learning. He created the Office of Mathematics
and Computer Education within UIC's math department, and the Institute for Mathematics and Science Education (IMSE) within the Learning Science Research Institute. The most influential contribution was the Teaching Integrated Math and Science Project (TIMS), and the development of the Math Trailblazers elementary mathematics curriculum created by Dr. Phil and Dr. Howard Goldberg.

Finally, Dr. Phil was a kind and wonderful human being. Not only was he knowledgeable in mathematics and mathematics education, but he would always share with me interesting aspects of American culture, historical anecdotes and stories about his journeys to French and Spain.

This dissertation is a direct result of Dr. Phil’s inspiration. The dissertation ideas came from my years of co-teaching with Dr. Phil in a geometry course. Dr. Phil’s guidance was invaluable throughout the proposal, data collection, and early analysis. I regret that he passed away before he could see the final product.
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SUMMARY

Given the fundamental role of geometric transformations in mathematics and the high expectations on this topic in the Common Core State Standards for Mathematics, it is imperative to support secondary preservice teachers’ learning of geometric transformations. To uncover their knowledge and evidence of learning on geometric transformations, this study examined the nature and the development of four preservice teachers’ mathematical discourses about geometric transformations from a commognitive perspective.

The design of the study was guided by the following research question: What is the nature of secondary preservice teachers’ mathematical discourses on geometric transformations, and how does their discourses change over time? This study employed a descriptive case study approach with interviews and participant-observation to understand what preservice teachers have learned during a five-week instructional unit. The commognitive framework developed by Sfard (2008) was utilized to analyze preservice teachers’ mathematical discourses.

Data analysis revealed that various types of keyword use (e.g., operational word use, structural word use), routines (e.g., identification routines, specification routines), and narratives (e.g., geometric transformations as reflection and rotation; geometric transformations as one-to-one and onto functions) emerged from preservice teachers’ discourses on geometric transformations, which illustrated the complex nature of their discourses. In addition, preservice teachers’ discourses on geometric transformations shifted toward an objectified and mathematically sophisticated direction at the end of the instructional unit. This analysis suggested that preservice teachers developed deeper knowledge about geometric transformations, and that studying preservice teachers’ learning from a discursive perspective can provide explicit indicators for observing learning.
1. INTRODUCTION

From a mathematical perspective, transformation is a fundamental topic, which connects geometry and algebra, provides new insights into geometric concepts, and offers powerful and elegant tools for analyzing mathematical and real-life situations (Coxford, 1973). From the perspectives of pedagogy and curriculum, the topic of transformations provides valuable opportunities for the learning of geometry. Specifically, studying geometric transformations can develop spatial visualization ability and reasoning ability with dynamic geometric objects, and provide opportunities to make connections between different mathematical concepts and representations (Clements and Battista, 1992). In addition, the topic of geometric transformations is an important part of K-12 mathematics curricula. National Council of Teachers of Mathematics (NCTM) Standards (2000) has included transformations as one of the four pivotal content areas in K-12 geometry, and declares, “Instructional programs from pre-kindergarten through grade 12 should enable students to apply transformations and use symmetry to analyze mathematical situations (p. 41).” According to the Common Core State Standards for Mathematics (CCSSM, 2010), the topic of transformations plays an even more important role because it was considered as a fundamental component facilitating the coherent transition from middle school to high school geometry. In the Common Core State Standards for Mathematics, the concepts of geometric transformations learned in middle school are used to explain concepts of congruence and similarity in high school geometry, rather than merely used for art appreciation (Wu, 2012). Specifically, high school students should be able to explain concepts of congruence, similarity, and symmetry based on geometric transformations, examine congruence and similarity figures using geometric transformations, and represent geometric transformations using different representations (CCSSM, 2010).
It is imperative to support teachers and enhance their knowledge about the topic of geometric transformations, given the advocacy of developing students’ thorough and integrated knowledge about geometric transformations in the Common Core State Standards for Mathematics. In fact, the Conference Board of the Mathematical Sciences (2012) proposed that high school teachers should be exposed to the detailed explanation of how geometry developed from axioms based on a transformational perspective. This would allow them to have the knowledge to accurately assess students’ learning and maximize learning opportunities once they enter the classroom. This new emphasis on geometric transformations requires teachers have comprehensive knowledge and skills about geometric transformations. However, the topic of geometric transformations is a neglected area in which teachers have little experience in their coursework (Wang & Smith III, 2011). Specifically, for preservice teachers, the learning of geometric transformations in the K-12 level mainly draws on the intuitive aspect of this concept, while during their collegiate education, the study of transformations focuses on manipulating symbols and formulas. It is a challenge for preservice teachers to connect these ideas and apply them to explore the geometric figures (Ada & Kurtulus, 2010), not to mention uniting other geometric concepts, such as congruence and similarity, through a transformational perspective. Thus, as recommended in current curricular documents, it is urgent to help preservice teachers develop thorough and integrated knowledge about transformation geometry. Indeed, some mathematicians (Madden, 2011; Wu, 2012) have argued that such deep knowledge about geometric transformations is mostly unknown to teachers and mathematics educators, and a massive effort is required for successful implementation of the Common Core State Standards.

Research seldom reported secondary preservice teachers’ knowledge about geometric transformations, while a number of studies have been devoted to investigating the conceptions of
geometric transformations in K-12 students (Edward, 1991; Hollebrands, 2003, 2004) as well as in elementary and middle grades preservice teachers (Edward & Zazkis, 1993; Ada & Kurtulus, 2010; Yanik, 2011). Results from these studies have shown that learners, despite differences in grade level, can perceive geometric transformations as a daily life phenomenon and use the words “flip, turn, slide” to describe these movements. However, they seldom consider geometric transformations as mathematics objects, which is one of the essential elements of the topic. It is important to investigate secondary preservice teachers’ knowledge about geometric transformations because little is known about their thinking on geometric transformations, and secondary preservice teachers’ knowledge about advanced mathematics might provide them with unique experience in learning geometric transformations.

Furthermore, existing research on the knowledge of geometric transformations from the cognitive perspective has focused on mental representations in learners’ minds. Some researchers provided evidence for the existence of various primitive conceptions of geometric transformations that differ from the standard concept (e.g., Edward, 1991; Hollebrands, 2004; Yanik, 2011), and some others documented the growth of learners’ knowledge by identifying the schemas that learners held at different stages (e.g., Flanagan, 2001; Yanik, 2006). The underlying assumption for these studies is that conceptions are mental representations located in individuals’ minds, and learning is to develop mental representations. Exploring learners’ conceptions based on mental representations provides information about how to organize the content that learners need to learn, but seldom attend to their experiences of interacting with the physical and social world (Greeno, 1996; Cobb & Yackel, 1996). There is an increasing effort to reconsider the concept of conception, which is influenced by the reconceptualization of knowledge and cognition. For example, knowledge and cognition can considered as distributed
in the individuals and their interactions with tools and between people (Hutchins, 1995). More specifically, researchers have proposed that mathematical conceptions should be reformulated as multiplicities. That is, gestures, manipulatives, figures, and utterances should be considered as constitutive parts of a conception (Roth, 2009; Nemirovsky, 2012). Moreover, conceptual growth can be viewed as change in discourse, or change in the ways an individual participates in discourse (Greeno, 1998; Sfard, 2008).

It is in this context that this study aims to investigate secondary preservice teachers’ discourses about geometric transformations. Anna Sfard’s commognitive framework (2008) is used to examine the nature and the development of preservice teachers’ discourses. The commognitive framework regards thinking and communication as the same phenomenon, and considers learning as change in discourse. Moreover, the framework is to examine mathematical discourse in terms of four key elements: keywords and their use, visual mediators, routines, and narratives. Therefore, the overarching research question guiding this study is: What is the nature of secondary preservice teachers' mathematical discourses on geometrics transformations, and how do their discourses change over time?

Specifically, the overarching question is addressed in terms of three elements of mathematics discourse:

1. What are the keywords used in preservice teachers’ mathematical discourses on geometric transformations, how are the keywords being used, and how does the use of keywords in preservice teachers’ discourses change over time?

2. What are the routines that emerged in preservice teachers’ mathematical discourses on geometric transformations, and how do the routines in preservice teachers’ discourses change over time?
3. What are the narratives that emerged in preservice teachers’ mathematical discourses on geometric transformations, and how do the narratives in preservice teachers’ discourses change over time?

By examining the nature and the development of preservice teachers’ mathematical discourses about geometric transformations, we can better understand preservice teachers’ knowledge of and evidence of learning on geometric transformations. Thus, we can better support their learning of geometric transformations.

This dissertation has seven chapters. In this chapter, I have briefly introduced the context and research questions for this study. Chapter 2 provides an in-depth background information for this study, which includes the fundamental role of transformations in mathematics, the importance of transformations in K-12 mathematics curricula, and previous studies on students’ understanding of geometric transformations. Chapter 3 provides the rationales for studying secondary preservice teachers’ learning from a discursive perspective, and presents the commognitive framework for examining secondary preservice teachers’ mathematical discourses. Chapter 4 details the research design and methods utilized in this study. Chapters 5, 6, and 7 present the results from this study to address each of the research questions. Chapter 5 addresses question 1 focusing on the element of keywords; Chapter 6 answers question 2 on the element of routines; and Chapter 7 is devoted to question 3 on the element of narratives. Each chapter includes detailed descriptions of the focused element and the patterns of discourse changing in the element. Chapter 8 discusses the results by comparing them with previous findings. It also includes implications of this study on the larger field of education research, implications for teaching geometric transformations, and future research questions. Together,
these chapters create a comprehensive picture of secondary preservice teachers’ discourses on geometric transformations.
2. THE BACKGROUND OF THE STUDY

This chapter provides in-depth background information for this study, which is divided into three sections. The first section provides the mathematical background about the topic of transformations. The second section highlights the importance of transformations in K-12 mathematics curricula. Finally, the last section reviews findings of existing studies and highlights the limitations of these studies.

2.1 Transformations as a Fundamental Approach to Geometry

A transformation is an important mathematical object in geometry and algebra. What is a transformation? A transformation of a plane can be informally defined as the procedure that moves all the points in the plane. However, a formal definition of a transformation relies on the concept of function or mapping. That is, a geometric transformation of a plane is a one-to-one and onto mapping or function that maps the plane to itself (Coxford, 1973). This definition highlights some important ideas that are missing in an informal definition. A transformation has input (i.e., preimage) and output (i.e., image) as a function or a mapping; the inputs and outputs could be points or sets of points. All the points on the plane form the domain and range. Moreover, a transformation is one-to-one and onto; that is, each point of the plane is paired with exactly one point of the plane. There are different types of geometric transformations of the plane; this study primary focuses on reflection, rotation, translation, dilation and shear. For each type of geometric transformations, the important features include the parameters, relationship between preimage and image, fixed points, and properties preserved by the transformations. In addition, geometric transformations can be represented by sketches, coordinates, vectors, function notations, and matrices. In the following sections, the geometric transformations presented include reflection, rotation, translation, and dilation. These sections have referenced to
documents or geometry textbooks written for teachers or college students (e.g., Coxford, 1973; Usiskin et al. 2003; Barker & Howe R, 2007; Reynolds & Fenton 2006)

2.1.1 Reflection

Reflection plays a fundamental role in the study of isometries (i.e., rigid motions or congruence transformations). A reflection $r_l$ in line $l$ assigns each point $P$ in the plane its image point $P'$ over line $l$, which can be written as $r_l(P) = P'$. The reflection line $l$ is the parameter, and is the perpendicular bisector of the line segment from preimage $P$ to image $P'$ (Shown in Figure 1A).

A. Reflection $r_l$ over line $l$

B. Reflection $r_m$ over line $m$

Figure 1. Reflections
Figure 1B illustrates several properties of reflection: 1) If point D is on line m, the image of D is itself. Point D is called the fixed point; 2) Reflection over a line preserves lines; that is, the image of any straight line is a straight line; 3) Reflection over a line preserves distance between points (e.g. \( BC = B'C' \)) and the measure of angles (e.g., \( \angle BCA = \angle B'C'A' \)); and 4) Reflection reverses the orientation of three noncollinear points; that is, the order of preimages A-B-C is counterclockwise, while the order of images A’-B’-C’ is clockwise.

If two reflections \( r_1 \) and \( r_2 \) are equal, then the reflection lines are the same, and vice versa. In the coordinate plane, reflection images over some lines are very easy to find. Table I displays a point and its reflection images over the x-axis, the y-axis, and the line \( y=x \).
### TABLE I
REFLECTION OVER THE X-AXIS, REFLECTION OVER THE Y-AXIS AND REFLECTION OVER LINE Y = X

<table>
<thead>
<tr>
<th>Reflections</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Reflection over the x-axis:</strong></td>
<td>![Graph of Reflection over the x-axis]</td>
</tr>
<tr>
<td><strong>Function form</strong></td>
<td>$P(x, y)$</td>
</tr>
<tr>
<td>$(x', y') = r_{x-axis}(x, y) = (x, -y)$</td>
<td>![Graph of Reflection over the x-axis]</td>
</tr>
<tr>
<td><strong>Matrix form</strong></td>
<td>$P'(x', y') = r_{x-axis}(x, y) = (x, -y)$</td>
</tr>
<tr>
<td>$\begin{bmatrix} x' \ y' \end{bmatrix} = \begin{bmatrix} 1 &amp; 0 \ 0 &amp; -1 \end{bmatrix} \begin{bmatrix} x \ y \end{bmatrix}$</td>
<td>$\begin{bmatrix} x \ y \end{bmatrix}$</td>
</tr>
</tbody>
</table>

| **Reflection over the y-axis:** | ![Graph of Reflection over the y-axis] |
| **Function form** | $P(x, y)$ |
| $(x', y') = r_{y-axis}(x, y) = (-x, y)$ | ![Graph of Reflection over the y-axis] |
| **Matrix form** | $P'(x', y') = r_{y-axis}(x, y) = (-x, y)$ |
| $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ | $\begin{bmatrix} x \\ y \end{bmatrix}$ |

| **Reflection over the line y=x:** | ![Graph of Reflection over the line y=x] |
| **Function form** | $P(x, y)$ |
| $(x', y') = r_{x=y}(x, y) = (y, x)$ | ![Graph of Reflection over the line y=x] |
| **Matrix form** | $P'(x', y') = r_{x=y}(x, y) = (y, x)$ |
| $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ | $\begin{bmatrix} x \\ y \end{bmatrix}$ |
These reflections can also be represented in the form of function or matrix. In fact, all reflections can be represented in the form of function or matrix. Due to their complexity, the general forms are not introduced here.

2.1.2 Rotation

A rotation about point O of $\theta$ ($-\pi < \theta < \pi$) takes each point P in the plane to its image point $P'$ such that $P$ and $P'$ lie on the same circle center at O and the measure of angle $POP'$ is $\theta$, which can be written as $R_{O, \theta}(P) = P'$. The center O and angle measure $\theta$ are the parameters of the rotation (Shown in Figure 2A).

A. Rotation $r_{O, \theta}$ about center O by $\theta$

B. Rotation $r_{O, \alpha}$ about center $e$ by $\alpha$

Figure 2. Rotations
Figure 2B illustrates several properties of rotation: 1) The center of rotation is the only fixed point; 2) Rotation preserves lines; 3) Rotation preserves distance between points and the measure of angles; 4) Rotation preserves the orientation of three noncollinear points; that is, the order of images A’-B’-C’ is the same as the order of the preimages A-B-C.

If two rotations $r_1$ and $r_2$ are equal, then the rotation center and angle measure $\theta$ are the same, and vice versa. In the coordinate plane, rotation image about center $O$ and angle measure $\theta$ can be represented in the form of function or matrix as the following:

Function form: $r_{O, \theta}(x, y) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta)$

Matrix form: \[
\begin{bmatrix}
  x' \\
  y'
\end{bmatrix}
= \begin{bmatrix}
  \cos \theta & -\sin \theta \\
  \sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
\]

2.1.3 Translation

A translation by vector $\nu$ takes each point $P$ in the plane to its image point $P'$ by a certain distance and a certain direction defined by $\nu$, which can be written as $T_{\nu}(P) = P'$. The translation vector $\nu$ is the parameter of the translation (shown in Figure 3A).

Figure 3B illustrates several properties of translation: 1) Translation has no fixed points; 2) Translation preserves lines; 3) Translation preserves distance between points, angles, and the measure of angles; and 4) Translation preserves the orientation of three noncollinear points.
If two reflections $T_1$ and $T_2$ are equal, then translation vectors are the same, and vice versa. In the coordinate plane, translation image by vector $v = (m, n)$ can be represented in the form of function or matrix as the following:

**Function form:** $T_v (x, y) = (x + m, y + n)$

**Matrix form:**

\[
\begin{bmatrix}
  x' \\
  y'
\end{bmatrix} = \begin{bmatrix}
  x + m \\
  y + n
\end{bmatrix}
\]

### 2.1.4 Dilation

A dilation by center $O$ and scale factor $k$ ($k \neq 0$) maps each point $P$ in the plane to its image point $P'$ such that $OP' = k \cdot OP$. This dilation can be written as $D_{O,k}(P) = P'$. The center $O$ and scale factor $k$ are the parameters of the dilation (Shown in Figure 4A).

Figure 4B illustrates several properties of dilation: 1) The only fixed point of dilation is its center point $S$; 2) Dilation preserves lines; 3) Dilation preserves angles and the measure of angles; 4) Dilation preserves the orientation of three noncollinear points; and 5) Dilation does not preserve distance.
In the coordinate plane, dilation can be represented in the form of function or matrix as the following:

Function form: \( D_{O,k} (x, y) = (kx, ky) \)

Matrix form: \[
\begin{bmatrix}
  x' \\
  y'
\end{bmatrix} = \begin{bmatrix}
  k & 0 \\
  0 & k
\end{bmatrix} \begin{bmatrix}
  x \\
  y
\end{bmatrix}
\]

2.1.5 Shear

There are two major kinds of shear transformations: horizontal shear (Shown in Figure 5A) and vertical shear (Shown in Figure 5B). A horizontal shear takes each point of the plane in a fixed direction parallel to line AB and by a distance proportional to the perpendicular distance from line AB, which can be written as \( S (P) = P' \). For example, a horizontal shear moves point C to point C’ by the direction parallel to line AB and by the distance \( k \cdot CC' \), where \( k = CC' / CA \).

A vertical shear moves each point of the plane in a fixed direction parallel to line HE and by a
distance proportional to the horizontal distance from line HE, which can be written as $S(P) = P'$. For example, a vertical shear moved point G to point G’ by the direction parallel to line HE and by the distance $m \cdot GG'$, where $m = GG'/HG$.

![Shear Diagram]

A. Horizontal shear by segment AB  
B. Vertical shear by segment HE

Figure 5. Shears

Figure 5 illustrates several properties of shear: 1) Shear does not preserve distance; 2) Shear does not preserve angles and the measure of angles; 3) Shear preserves lines; and 4) Shear preserves area (e.g., the area of square ABCD is equal to the area of the parallelogram ABC’D’).

In the coordinate plane, shear can be represented in the form of function or matrix as the following:

Function form: $S_h(x, y) = (x + ky, y)$; $S_v(x, y) = (y, y + kx)$
Matrix form:

\[
\begin{bmatrix}
    x' \\
    y'
\end{bmatrix} = \begin{bmatrix} x + ky \\
    y \end{bmatrix} = \begin{bmatrix} 1 & k \\
    0 & 1 \end{bmatrix} \begin{bmatrix} x \\
    y \end{bmatrix} ;
\begin{bmatrix}
    x' \\
    y'
\end{bmatrix} = \begin{bmatrix} kx + y \\
    y \end{bmatrix} = \begin{bmatrix} k & 0 \\
    1 & 1 \end{bmatrix} \begin{bmatrix} x \\
    y \end{bmatrix}
\]

2.1.6 The Composition of Transformations

Transformations can be composited as functions. When function \( g \) composed with function \( f \), it can be written as \( g (f(x)) = (f \circ g)(x) \), and two functions become one function \( (f \circ g)(x) \). When transformation \( R(P) \) is composed with \( S(P) \), the two transformations become one transformation \( (R \circ S)(P) \). For example, the composition of two translations \( T_m \circ T_n(P) \) or \( T_n(T_m(P)) \) is a translation \( T_{m+n}(P) \) (Shown in Figure 6).

Figure 6. The composition of two translations \( T_m \circ T_n(p) \) is a translation \( T_{m+n}(p) \)

Transformations are important tools to study geometry. The development of transformation geometry was one of the pivotal evolutions in the history of mathematics (Chern, 1990). Indeed, in the Erlanger Program launched by the German mathematician Felix Klein in...
1872, symmetry and transformations were used to classify and characterize different types of geometries. That is, geometry is considered as the study of the properties in a space that are invariant under a group of transformations. Each geometry corresponds to a group of transformations acting on a space, such as isometries, similarity transformations, affine transformations, and projective transformations. Isometries—which are also called congruence transformations or rigid motions—preserve distance, measure of angles, and so on. Isometries include reflection, rotation, translation, and glide reflection. Similarity transformations—which consist of isometries and dilation—do not preserve distance, but do preserve measure of angles, betweenness, and so on. Euclidean geometric can be studied in terms of isometries and similarity transformations. Affine geometry is a study of properties of geometric objects that remain invariant under affine transformations, which include similarity transformations and shear transformations. Affine transformations do not preserve distance and the measure of angles, but preserve betweenness, collinearity, area, and so on. Table II below provides detailed information about different group of geometric transformations, and is based on Kidder’s (1976) summary about transformations and their properties.

Moreover, transformations are useful tools in the modern world. For example, in the field of computer graphing, game designers create motions by applying geometric transformations.
TABLE II
PROPERTIES OF TRANSFORMATIONS GROUPS

<table>
<thead>
<tr>
<th>Transformations group</th>
<th>Invariant</th>
<th>Geometric context</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isometries (reflection, rotation, translation and glide reflection)</td>
<td>Distance, measure of angles, betweenness of points, ratio of distance, parallelism of lines, cross-ratio, colinearity and concurrency</td>
<td>Metric</td>
</tr>
<tr>
<td>Similarity transformations (isometries and dilation)</td>
<td>Measure of angles, betweenness of points, ratio of distance, parallelism of lines, cross-ratio, colinearity and concurrency</td>
<td>Similarity</td>
</tr>
<tr>
<td>Affine transformations (similarity transformations and shear)</td>
<td>Betweenness of points, ratio of distance, parallelism of lines, cross-ratio, colinearity and concurrency</td>
<td>Affine Geometry</td>
</tr>
<tr>
<td>Projective transformations (affine transformations and others)</td>
<td>Cross-ratio, colinearity and concurrency</td>
<td>Projective Geometry</td>
</tr>
</tbody>
</table>

2.2 Transformations as an Important Topic in the K-12 Curriculum

The topic of transformations has been an important part of K-12 mathematics curricula (National Council of Teachers of Mathematics, 1989, 2000). First, studying transformations can provide opportunities to explore the dynamic aspects of geometry, and develop spatial visualization of and reasoning abilities regarding dynamic geometric objects. Second, it provides opportunities for students to make connections between mathematics topics, such as algebra and geometry. Last, students are provided opportunities to develop an accurate and broad concept of congruence and similarity. Defining congruence using transformations improved Euclid’s definition using superposition, which is not a mathematical construct that undermines the rigor of
geometry. The definitions of congruence and similarity based on geometric transformations can be extended to all the figures, not just triangles.

The role of transformations in the geometry curricular documents has been revamped from the NCTM Principles and Standards for School Mathematics (2000) to the Common Core State Standards for Mathematics (2010), from an isolated topic in geometry to an idea that permeates into geometry and serves as its foundation. The following section is devoted to comparing the differences between the two standards, which provide a rationale for deepen teachers’ mathematics knowledge to teach geometry in a way that required by the Common Core State Standards for Mathematics.

In the NCTM Principles and Standards for School Mathematics, the topic of transformations is listed as one of the four components in geometry (i.e., apply transformations and use symmetry to analyze mathematical situations). In Grades K-5, students investigate the effect of transformations through experiments. Later, in Grade 6-8, students explore the properties of transformations, such as preserving distance and preserving angle. In high school, multiple ways of expressing transformations are introduced, along with the effects of compositions of transformations. The expectations for this component for different grade levels are listed in Table III.
### TABLE III
TRANSFORMATIONS IN THE NCTM PRINCIPLES AND STANDARDS FOR SCHOOL MATHEMATICS

<table>
<thead>
<tr>
<th>Kindergarten through Grade 2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1-1 Recognize and apply slides, flips, and turns.</td>
<td></td>
</tr>
<tr>
<td>1-2 Recognize and create shapes that have symmetry.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Grades 3-5</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1-3 Predict and describe the results of sliding, flipping, and turning two-dimensional shapes.</td>
<td></td>
</tr>
<tr>
<td>1-4 Describe a motion or a series of motions that will show that two shapes are congruent.</td>
<td></td>
</tr>
<tr>
<td>1-5 Identify and describe line and rotational symmetry in two- and three-dimensional shapes and designs.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Grades 6-8</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1-6 Describe sizes, positions, and orientations of shapes under informal transformations such as flips, turns, slides, and scaling.</td>
<td></td>
</tr>
<tr>
<td>1-7 Examine the congruence, similarity, and line or rotational symmetry of objects using transformations.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Grades 9-12</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1-8 Understand and represent translations, reflections, rotations, and dilations of objects in the plane by using sketches, coordinates, vectors, function notation, and matrices.</td>
<td></td>
</tr>
<tr>
<td>1-9 Use various representations to help understand the effects of simple transformations and their compositions.</td>
<td></td>
</tr>
</tbody>
</table>

In contrast, in the Common Core State Standards for Mathematics, transformation is at the core foundation of geometry. Indeed, in Grade 4, line symmetry is introduced, given that students have real-life experience in line symmetry. In Grades 7-8, students explore rotations, reflections, and translations through experiments. They also develop the meaning of congruence in terms of rigid motions. In high school, students develop these ideas more rigorously, learning
standard terminology and fundamental facts about rigid motions and dilations of the plane as a foundation for the study of Euclidean geometry. The expectations for this component for different grade levels are listed in Table IV.
### TABLE IV
TRANSFORMATIONS IN THE COMMON CORE STATE STANDARDS FOR MATHEMATICS

| Grade 4 | 2-1 Understand that a line of symmetry for a geometric figure is a line across the figure, such that the figure can be folded along the line into matching parts. |
| Grade 4 | 2-2 Identify line-symmetric figures; given a horizontal or vertical line and a drawing that is not a closed figure, complete the drawing to create a figure that is symmetric with respect to the given line. |

| Grade 7 | 2-3 Verify experimentally the fact that a rigid motion (a sequence of rotations, reflections, and translations) preserves distance and angle, e.g., by using physical models, transparencies, or dynamic geometry software. Lines are taken to lines, and line segments to line segments, of the same length. Angles are taken to angles of the same measure, and parallel lines are taken to parallel lines. |
| Grade 7 | 2-4 Understand the meaning of similarity: a plane figure is similar to another if the second can be obtained from the first by a similarity transformation (a rigid motion followed by a dilation). |
| Grade 7 | 2-5 Use informal arguments involving approximation by lines, squares, and cubes to see that a similarity transformation with a scale factor of $k$ leaves angle measures unchanged, changes lengths by a factor of $k$, changes areas by a factor of $k^2$, and changes volumes by a factor of $k^3$. |

| Grades 9-12 | 2-6 Understand that two geometric figures are congruent if there is a sequence of rigid motions (rotations, reflections, translations) that carries one onto the other. This is the principle of superposition. |
| Grades 9-12 | 2-7 Use two-dimensional representations to transform figures and to predict the effect of translations, rotations, and reflections. |
| Grades 9-12 | 2-8 Use two-dimensional representations to transform figures and to predict the effect of dilations. |
| Grades 9-12 | 2-9 Understand that dilations can be used to show that all circles are similar. |
| Grades 9-12 | 2-10 Understand that transforming the graph of an equation corresponds to substitutions in the equation by reflecting in the axes, translating parallel to the axes, or applying a dilation in one of the coordinate directions. |
Besides considering transformations as the foundation of geometry, there are several notable features about transformations in the Common Core State Standards for Mathematics. First, informal words for geometric transformations—which are usually used in an everyday-life sense—are not encouraged. Words, such as “flips,” “turns,” “slides,” and “scaling” did not appear on the Common Core State Standards, but they were used in grades K-8 in the NCTM Principles and Standards for School Mathematics. As Usiskin (2010) indicated, despite the fact that informal words can describe transformations intuitively, they do not carry sufficient information compared to formal mathematics words. For example, the informal word “turn” for rotation is usually used to talk about making a right turn or a U-turn, which do not have centers.

Second, the NCTM Principles and Standards for School Mathematics focus on the properties of shapes (See Item 1-6 in Table III), while the Common Core State Standards highlight the properties of transformations (See Items 2-3, 2-5 in Table IV). Third, the relationships between congruence and transformations are different in the NCTM standards and the CCSSM standards. In the former, congruence is a property of the shape under an isometry (See Item 1-7 in Table III); in the latter, congruence is defined by isometries (See Item 2-6 in Table IV). Last, transformations are used to explore the graph of an equation in the Common Core State Standards for Mathematics (See Item 2-10 in Table IV). Giving such prominence to transformations and the fact that most of the traditional textbooks take an axiomatic approach to Euclidean geometry, secondary preservice teachers need opportunities to experience a transformation approach to Euclidean geometry.

2.3 Studies on Knowledge of Geometric Transformations

There has been some research devoted to the understanding of geometric transformations. Much of it has been devoted to understanding K-12 students’ and elementary preservice teachers’
conceptions of geometric transformations. Of the existing studies, some are focused on the thinking about geometric transformations (e.g., Edward & Zazkis, 1993; Hollebrands, 2003; Yanik, 2009, 2011; Portnoy et al., 2006), while some are focused on the representations of geometric transformations (e.g., Jung, 2002; Ada & Kurtulus 2010). The first part of this section reviews the studies on the thinking about geometric transformations. The second part reviews the studies on using various representations for geometric transformations.

2.3.1 Studies on Thinking about Geometric Transformations

Edward & Zazkis (1993) conducted a study to explore elementary preservice teachers’ primitive understanding of transformations (reflections and rotations). Fourteen elementary preservice teachers learned transformations by playing a game in a computer environment. Interviews were conducted before these teachers were exposed to this learning environment, and a written assessment was administrated after the learning unit. Elementary preservice teachers’ strategies to solve transformations tasks produced in the course of the study were examined. The results of the study showed that the most common strategy employed to identify reflections or rotations was the decomposition strategy, which involved breaking down a reflection or a rotation into more than one move. Also, more than half of the elementary preservice teachers recognized one single-move solution only in the case of reflection in a horizontal or vertical line. When asked to predict the result of rotations, more than one quarter of teachers showed two primitive interpretations of reflections and rotations. These interpretations are named “reflection bug” and “rotation bug.” In the reflection bug interpretation, elementary preservice teachers slid the shape to the mirror and reflected it. In the rotation bug interpretation, these teachers slid the shape to the center of rotation and rotated it. The research also found that these teachers who show naïve concepts of transformations did not consider transformations as functions, as they are
defined in mathematics. These naïve concepts were strongly influenced by learners’ experiences with physical motion in real life.

Hollebrands (2003) studied high school students’ understandings of geometric transformations (translations, rotations, reflections, and dilations) in a technological environment. Four high school students from an honors geometry class were interviewed three times during the time they were studying a transformations unit with the researcher. The task-based interviews were analyzed using the Action Process Operation Schema (APOS) theory, which was developed by Dubinsky and his colleague (1992) based on Piaget’s concept of reflective abstraction (Piaget, 1997). The analysis showed that all four students had an action conception of transformations at the beginning of the unit; two of them developed a process conception of transformations after finishing the unit. According to Flanagan (2001), an action conception of transformation sees transformation as a physical or mental manipulation that transforms objects. More specifically, an action conception means transformations are considered as motions that are applied to a single object, and that the parameter(s) of transformations are viewed as an indicator of an action rather than as a mathematical object. A process conception of transformations sees transformations as processes beginning with objects of some kind, doing something to these objects, and obtaining new objects as a result of what was done. An object conception of transformations sees transformations as functions that map points in the plane; transformations can also be composited as functions in this conception.

Based on Flanagan’s (2001) study of high school students, Yanik (2009) conducted individual teaching experiments to study four elementary preservice teachers’ understanding of and development in understanding rigid transformations. Pre-interview, post-interview and teaching episodes were analyzed from the lens of the AOPS theory to examine preservice
teachers’ learning of transformations. The results of the study indicated that all elementary preservice teachers held an action conception of transformations before the individual teaching experiment. After the teaching experiment, two elementary preservice teachers developed some process conception of transformations; one teacher developed firm process conception of transformations, and the other developed some initial object conception of transformations. Further analysis on the latter teachers’ trajectory of developing action conception to object conception showed that the teachers followed an order: (1) understanding transformations as undefined motions of a single object; (2) understanding transformations as defined motions of a single object; and (3) understanding transformations as defined motions of all points on the plane.

Portnoy et al. (2006) investigated four secondary preservice teachers’ concepts of geometric transformations and the influence of their concepts on understanding and constructing proof in transformation geometry. Interview data, concept maps, and journal entries were collected. Secondary preservice teachers’ concepts of transformations were analyzed through the theoretical perspective from Tall et al. (2000), who highlight the duality of mathematics objects. This includes the process and object duality, as well as the perceived and conceived duality. In the process and object duality, the process view refers to transformations being viewed as a procedure, while the object view refers to transformations being viewed as an object. In the perceived and conceived duality, the perceived view focuses on the perceptual aspect of transformations, while the conceived view focuses on the abstract aspect of transformations that has no equivalent in real life. The results of the research showed that secondary preservice teachers viewed transformations as processes and had difficulties conceiving of transformations as objects; secondary preservice teachers viewed transformations perceivably and had not been
able to develop a complete conceived view after the instruction units. Because they viewed transformations as processes and as perceived objects, secondary preservice teachers who were aware of the need for a general proof were not able to construct a proof for general cases.

Yanik (2011) studied forty-four middle grades preservice teachers’ prior knowledge of rigid geometric transformations. Semi-structured clinical interviews were conducted to collect these teachers’ responses. The result of the study showed that middle grades preservice struggled in recognizing, describing, executing, and representing geometric translations. Also, these teachers held various views of rigid geometric transformations. These included viewing translations as rotational or translational motion, and interpreting the vector of a translation as either a force, line of symmetry, direction indicator, or displacement.

In summary, existing studies have described the primitive concepts and the developmental stage in geometric transformations in terms of the facts that learners knew. These studies contribute a general and broad understanding of learners’ thinking of geometric transformations.

2.3.2 Studies on Using Representations for Geometric Transformations

Jung (2002) studied the effect of representations (pictorial, oral, and written) on secondary preservice teachers' understanding of geometric transformations and the growth of the understanding in a technology-based collegiate mathematics classroom. The researcher observed two teachers’ activity in the geometry course, interviewed the teachers nine times during the course, and administrated pre- and post-course diagnostic tests. He found that teachers relied on pictorial representations to analyze geometric transformations and that they had difficulties in using mathematical symbols to express geometric transformations. He also pointed out that using technology to study transformations can help students develop solid conjectures.
Ada and Kurtulus (2010) studied elementary preservice teachers’ performance in transformation geometry after these teachers took an analytical geometry course. They examined teachers’ responses to questions on the final exam with emphasis on the geometric meaning of a transformation, representing the image and parameter of a transformation using equations and combinations of transformations. The results of the study showed that performing rotations was a challenge to elementary preservice teachers. Also, these teachers were able to represent geometric translations and rotations using the algebraic representation, but they struggled with providing explanation for the meaning of geometric transformations. For instance, they were able to represent a translation as $T(x, y)=(x+2, y+3)$, but could not determine the image of the preimage under the translation.

These studies provided two contrasting pictures of preservice teachers’ ways of understanding geometric transformations. They showed that preservice teachers’ learning of geometric transformations does not necessarily follow the sequence from concrete to abstract, as the development stages from APOS theory (i.e., action-process-object-schema) suggested. In fact, how preservice teachers communicate geometric transformations heavily depends on their learning experiences.

2.3.3 Limitation of Existing Studies

From the literature review, we can see that current research on the students’ knowledge of transformations is explored from cognitive perspectives that focus on mental representations. According to Sfard (1998), this approach is based on the acquisition metaphor of learning, which “[makes] us think about the human mind as a container to be filled with certain materials and about the learner as becoming an owner of these materials” (p. 5). The acquisition metaphor of learning considers learning as constructing or refining mental representations. Although this
perspective significantly enhanced our knowledge of students' thinking on geometric transformations, an exploration of this topic from another viewpoint is needed. First, while knowing the mental representations or the structure of a concept does provide information about how to organize the content the learners need to learn, this perspective is not enough for describing learner’s knowledge and evidence of learning. Because the interactions with the physical and social world during the learning process has been teased out from the scope of the acquisitionist (Sfard, 1998, 2007).

Second, mental representations and misconceptions are abstract constructs that cannot be directly observed. Thus, the more nuanced aspects of learning might be captured. Moreover, research from the acquisition paradigm does not take into account learners’ use of gestures, manipulatives, figures and utterances as integrated parts of their conceptions.

In fact, research (Wang, 2011; Sinclair & Moss, 2012) from a commognitive perspective presented high-solution analysis of geometric discourse, which illustrated a nonlinear development in student geometric thinking. For these reasons, this study explored the preservice teachers’ knowledge of geometric transformations from the commognitive framework.
3. THEORETICAL FRAMEWORK

The aim of this study is to explore the nature of preservice teachers’ mathematical discourses on geometric transformations, and describe the change in their mathematical discourses. This chapter provides the rationales for studying preservice teachers’ learning from a discursive perspective, and presents the commognitive framework for examining secondary preservice teachers’ mathematical discourses.

This chapter is divided into three sections. The first section describes the nature of mathematics learning from the cognitive perspectives and the sociocultural/situative perspectives. The second section describes the role of discourse in mathematics learning. Finally, the third section describes the commognitive framework developed by Sfard (2008).

3.1 The Nature of Mathematics Learning

This section begins with the review of central theoretical perspectives for thinking about learning: the cognitive perspectives and the sociocultural/situative perspectives. The nature of learning viewed through various lenses displays diversity. From the cognitive perspectives, learning is to acquire cognitive structures, such as mental representations. The sociocultural or situative theory viewed learning as participation in the practice or discourse of a group or community. The review of the two perspectives aims to provide foundations for understanding the phenomena of and the research on learning or learning mathematics, as well as the research on the relationship between mathematics learning and mathematical discourse.

3.1.1 Cognitive Perspectives on the Nature of Learning

Inspired by Jean Piaget (e.g., Piaget, 1970; Piaget, 1985), cognitive theorists—who might be labeled differently as radical constructivism (von Glasersfeld, 1995), social constructivism (Ernest, 1998), and so on—are interested in cognitive processes and structures, such as metal
representations. From the cognitive perspectives, learning is considered as acquiring or constructing metal representations in an individual’s head.

According to Hatano (1996), there are key ideas central to the cognitive approach to learning mathematics:

- learning requires active construction to acquire mathematical knowledge;
- learning involves reorganizations of mathematical knowledge through reflection;
- learning is constrained by an individual’s prior knowledge and misconceptions; and
- learning as acquisition of mathematical knowledge should be facilitated by social context.

The cognitive approach to study learning is to focus on understanding how information from the external world can be acquired or constructed, how the information is represented in the mind, and how the mental representation can be transferred to a new situation when people need to use it. The cognitive process and structure are expected to be cross-contextually invariant. The universal rules or mental representations remain basically the same across different social, cultural, historical, and situational settings. The unit of analysis in the cognitive approach is usually the individual.

3.1.2 Sociocultural/ Situative Perspective on The Nature of Learning

The sociocultural/ situative perspective has its origin in the work of Lev Vygotsky and his colleagues (e.g., Vygotsky & Kozulin, 1986; Vygotsky & Cole 1978). Under the sociocultural/situative perspectives, there are a variety of different theories or models on learning, which include guided participation (Rogoff, 1990), situative learning (Lave & Wenger, 1991), distributive cognition (Hutchins, 1995), and so on. Despite the variety in above-mentioned theories, the notions are strikingly similar to Vygosky’s argument in that learning is an inherently social activity. Learning is socially constructed through individuals’ interactions.
with one another. I group these theories under the sociocultural/situative perspectives following the same rationale as Gee (2008), and attempt to understand the common themes underlying these perspectives.

Sociocultural/situative theorists are interested in the social and situational aspects of learning. A sociocultural/situative theorist does not conceptualize learning in terms of representations in the head, but consider learning in terms of a relationship between an individual with both a mind and a body and the environment in which the individual thinks, feels, acts, and interacts. They often use metaphors such as enculturation or legitimate peripheral participation to characterize learning. For example, Lave & Wenger (1991) describe learning as individuals moving from the periphery of a community to the center. They become more active participants within that community, adopting new discourse and participating in the routine activities of a particular group.

According to van Oers (1996) and Forman (1996), understanding mathematics learning from a sociocultural/situative perspectives means:

- learning needs to be viewed as a form of participation;
- learning mathematics is a discursive activity;
- learning involves the negotiation of meaning within the context of situated activity; and
- Social, cultural, and institutional context are inherent characteristics of learning, but not merely factors that facilitate or impede learning.

Taking a sociocultural/situated perspective to study learning, researchers seek to understand the transformations of participation in sociocultural activity, including participants’ transformations in role, identity, practice, discourse, and so on. Under the sociocultural/situated framework, some researchers advocate that analysis of transformations of participation in
sociocultural activity over time is necessary to unpack what a particular social act means in context for a specific person (Rogoff, 1998; Lerman 1996; Forman, 1996). The focus on distributed aspects of cognition in sociocultural treatments is often at the expense of a coordinated analysis of the cognitive activity of individuals.

3.1.3 Coordinating Between Perspectives

The contrasting features of the two perspectives seem to problematize the relationship between individual and context. However, neither perspective supersedes the other in its power to uncover the nature of learning. Indeed, researchers have pointed out the necessity of coexistence between the two approaches. For example, Sfard (1998) argued that the two perspectives could be “incommensurable rather than incompatible.” Cobb and Yackel (1996) proposed an emergent approach to study classroom, which situates analyses of individual students' constructive activities in social context.

As to coordinating multiple perspectives, Simon (2009) pointed out that it is important to distinguish between what one is looking at (e.g., individual, classroom lesson, school district) and what one is looking with (e.g., social perspective, cognitive perspective). This distinction implies that a cognitive perspective is not always associated with the analysis of an individual interview. Similarly, a social construct from the sociocultural/situative perspective might not be used to document learning on a group or community level. Moreover, Simon argued that researcher should choose the perspectives based on what might be useful to look with rather than based on what is looked at. For example, Simon & Blume (1994) designed a teacher experiment to understand classroom interactions in terms of the various students’ mathematics ideas. Instead of resorting to a sociocultural/situative perspective on classroom interactions, the authors used a cognitive lens to look at a group situation, with the focus on the mathematical conceptualizations
underlying particular mathematical discussions.

Indeed, I agree that learning should be viewed as both a process of active individual construction and a process of participation in the practices of community or society. Thus, learning mathematics is a process of thinking and communicating mathematically. In fact, the transformations of participation can be observed through the changes in role, identity, practice, and goals of the activity (Lave and Wenger, 1991; Rogoff 1998; Saxe 2002; van Es, 2009), which are the constructs that capture the joint or collective development in a community. These constructs, however, do not provide a means to unpack participants’ thinking, if thinking is conceptualized as originating within the individual. The construct, discourse, does provide a means to connect the individual thinking and the activities of the community, especially when discourse is defined as it is in Sfard’s communicational framework.

3.2 Discourse and Mathematics Learning

In the last two decades, as NCTM Principles and Standards for School Mathematics (2000) have included communication as a central aspect of the standards, discourse has attracted much attention in mathematics education research. Discourse, originally referring to a unit above the level of sentence in linguistics (van Dijk, 1997), has been developed into a multifaceted construct. Professional standards for teaching mathematics (1991) prepared by National Council of Teachers of Mathematics provides a definition of discourse:

The discourse of a classroom—the ways of representing, thinking, talking, agreeing and disagreeing—is central to what students learn about mathematics as a domain of human inquiry with characteristic ways of knowing. Discourse is both the way ideas are exchanged and what the ideas entail: Who talks? About what? In what ways? What do people write, what do they record and why? What questions are important? How do ideas change? Whose ideas and ways of thinking are valued? Who determines when to end a discussion?
This definition indicated that discourse could be considered as an interactive process and a product of social, historical, and institutional formation. For example, Sherin (2002) examined classroom discourse in terms of its process and content. The process of mathematical discourse is defined as the way in which the teacher and the students interact with each other, while the content of mathematical discourse refers to the mathematical ideas embedded in the communication.

Discourse as a social interaction process has been manifested in research on classroom discourse and discourse communities in the field of mathematics education. For example, White (2003) argued that it is important to include all students in classroom discussions, and highlighted the some effective strategies for promoting productive mathematical classroom discourse, such as valuing students’ ideas, exploring students’ answers, incorporating students’ background knowledge, and encouraging student-to-student communication. In addition, Krussel and colleagues (2004) focused on teachers’ discursive moves (e.g., challenge, probe, and request for clarification) and developed a framework for analyzing discourse in mathematics classroom. Moreover, Hufferd-Ackles and colleagues (2004) describe components and levels of mathematical discourse communities as a learning environment, which guides teachers to listen to their students, draw out students’ ideas, and encourage students to listen to each other. Researchers also attended to the norms that regulate classroom discussion (Yackel & Cobb, 1996; Cobb et al., 1997). The studies on the process of mathematical discourse contributed to the effort of building a learning environment that fosters mathematics communication.

On the other hand, discourse as the product of the interaction is an important aspect that is critical to mathematics leaning. Mathematics learning involves far more than students being able to talk and write in mathematics class. It also involves ways of talking and forms of
reasoning that characterize the discipline. Sfard (2000) indicated that although research has been extensively investigated rules and norms in classroom discourse, little attention has been directly given to the mathematical contents in the communication and almost none to mathematical objects being communicated. In fact, mathematics can be conceptualized as discourse about mathematical objects (e.g., O’Halloran, 2005; Moschkovich, 2002; Sfard, 2007), which affords the investigation of discourse with the focus on mathematics content or mathematics objects during communications. Research from this perspective strives to understand how students’ discourse can be turned into mathematical one.

Indeed, discourse is a complex and multifaceted construct (Ryve, 2011). In this study, I followed Moschkovich (2002), O’ Halloran (2005), and Sfard (2008) by conceptualizing mathematics as discourse, rather than focusing on the patterns of interaction in the classroom. More specifically, I was concerned with to what extent preservice teachers communicate geometric transformations mathematically. Communicating mathematically is more than using mathematics vocabulary and certain mathematical register, but involves using social, linguistic, and material resources to participate in a certain type of discourse called mathematics (Moschkovich, 2002; Sfard, 2008).

In this study, I examined preservice teachers’ mathematics discourses on geometric transformations, and aimed to depict preservice teachers’ participation in the mathematical discourse from a commognitive standpoint, which allowed the focus on mathematics content and mathematical objects in preservice teachers’ discourses.

3.3 Commognitive Framework

By combining the words ‘‘communication’’ and ‘‘cognition,’’ Sfard (2008) coined the term “commognition” to highlight the idea that cognitive processes and interpersonal
communication are the same phenomenon. More specifically, thinking is considered as a form of
communication with oneself, an “individualized version of interpersonal communication” (Sfard
2008, p.81). In the commognitive framework, communication is not interpreted as an action of
exchanging messages and feelings between actors, but “a collectively performed rule-driven
activity that mediates and coordinates other activities of actors” (p.86). The reconceptualization
of communication stems from the idea that the information exchange between individuals is not
directly observable, unlike the patterns of actions and reactions. Discourses are defined as
different kind of communications, which include talking, writing, drawing, gesturing and so on.
Discourse can bring people together to form a community of discourse. There are various
overlapping communities of discourse. To be members of a discourse, individuals participate in
the communication activities of certain discursive practices.

From the commognitive perspective, mathematics is considered as a form of discourse.
Learning mathematics is viewed as change in mathematical discourse. Becoming fluent in
mathematical discourse that can be recognized as mathematical by expert interlocutors is the
expected change. As learners engage in the process of learning, they change their communication
with themselves and with others by interacting with the learning environment. What is learned
during this process of interactions can be revealed by examining to what extent the changes in
discourse aligned with the desirable discourse. In the case of mathematics, if the learner had
learned, the learner is more able to engage in an appropriate mathematical communication with
him/herself and others.

According to Sfard (2008), mathematical discourse can be characterized in terms of four
aspects: word use (the words used to describe the mathematical objects or the products of
mathematical discourse), visual mediators (pictures, graphs, symbols, and gestures), routines
(repetitive patterns characteristic of a given discourse) and narratives (any text that is framed as description of objects, or relations between objects that is subject to being endorsed or rejected). These four aspects will be dramatically different in the colloquial discourse and in the literate discourse. For example, regarding word use, we say “flip” to name a reflection in the colloquial discourse, but call it “reflection around reflection line L” in literate mathematics discourse.

The four elements of discourse—word use, visual mediators, narratives, and routines—are described as follows:

*Word use* refers to the way of mathematics words are used in discourse. Mathematics words refer to words used to describe the mathematical objects and mathematical actions, such as “triangle” and “rotation.” In a colloquial discourse, words are used in the everyday sense while, in a literate mathematical discourse, words are mathematical as they are used by the members of the mathematics community. Word use is important because the meaning of a word in a discourse is the same as how the word is used. An important feature of word use in mathematical discourse is objectification. Objectification means that words are used as though they are self-sustained entities in the world. Words functioning either as determiners or as nouns are one of the indicators of objectified discourse. Literate discourse is objectified discourse (Sfard & Lavie, 2005).

*Visual mediators* are visible objects that are operated upon as part of the process of mathematical communication. There are three categories: concrete (e.g., transparencies, paper), iconic (e.g., pictures, figures), and symbolic (e.g., algebraic expressions). Concrete and iconic visual mediators are the main mediators in colloquial discourse, whereas symbolic visual mediators are key in literate mathematics discourse. According to Sfard (2008), there are two goals for school mathematics learning in terms of visual mediators. The first is to develop
students’ ability to use symbolic representations in communicating mathematics. The second is
to develop their ability to flexibly use a variety of visual mediators.

*Routines* are repetitive patterns when interlocutors are engaging in mathematical
activities, or a set of rules regulating the repeated discursive activities. These activities include
using words, generating visual mediators, and endorsing narratives, as well as other activities,
such as categorizing and constructing definitions (Sfard, 2007). There are two kinds of routines:
the how routine and the when routine. The how routine determines how an interlocutor carries
out actions, while the when routine determines when an interlocutor carries out certain repetitive
actions. Routines in a full-fledged mathematical discourse show mathematical ways of attending
to the situations and solving the problems. Routines are interpretive, implicit, unjustifiable,
nondeterministic, normative and dynamic (Sfard, 2000). The interpretive nature of routines
means that routines are constructs defined by researchers in order to explain the phenomena.
Thus, routines are implicit and not directly observable. Given their implicit nature, the existence
of routines cannot be justified rationally or by logical inference, but can be supported by reasons.
Routines can function as norms, which are not deterministic but enabling. This means that the
routines do not direct our communications, but make communication possible.

*A narrative* is defined as any spoken or written text that is framed as a description of
objects, of relations between objects and of activities with or by objects. A narrative is also
subject to endorsement or rejection, meaning it can be labeled true or false (Sfard, 2008). There
are three ways to endorse a narrative in mathematics: recalling, constructing, and substantiating.
Recalling means summoning a previously endorsed narrative; constructing is to create a
narrative; and substantiating means proving a narrative. In terms of endorsed narrative, the goal
of school mathematics learning is to move students from the everyday use of empirical evidence to more formal mathematical methods of narrative endorsement.

These four elements of discourse are helpful in describing the nature of preservice teachers’ discourse and examining the changes in their discourses. Using these tools to explore what preservice teachers had learned provides new ways to present their learning.
4. METHOD

The purpose of this study is to describe preservice teachers’ learning through a discursive lens. In particular, this study examines the nature of preservice teachers’ mathematical discourses on geometric transformations, and the change in their mathematical discourses. This study is a descriptive case study using interviews and participant-observation to understand what preservice teachers learned during a five-week instructional unit.

By using these qualitative research methods, this study aims to understand the nature of and the change in preservice teachers’ mathematical discourses. In particular, pre-interview and post-interview were conducted before and after preservice teachers were exposed to the instructional unit. The main focus of this study was preservice teachers’ mathematical discourses at the pre-interview and the post-interview. Class observations provided a fuller picture about preservice teachers’ learning.

This chapter is divided into eight sections. The first section provides a rationale for employing a case study design to explore preservice teachers’ discourses. The second and third sections describe the context of the research site and the instructional unit. The fourth section provides the sampling methods used for selecting the focus participants for this study. The fifth and sixth sections describe the process of data collection and data analysis. Finally, the last two sections of this chapter discuss the reliability, validity, and limitations of the study design. Together, these sections describe the research design employed in this study to examine preservice teachers’ mathematical discourse on geometric transformations.

4.1 Case Study

To understand the nature of preservice teachers’ discourse, this study employs a case study design to thoroughly investigate this phenomenon. Case study, as defined by Yin (2008,
p.13-14), refers to “an empirical inquiry that investigates a contemporary phenomenon within its real-life context, especially when boundaries between the phenomenon and its context are not clearly evident.” Case study involves a very small geographical area or number of subjects of interest, which can be examined in detail. Case study relies on multiple sources of evidence, and can be based on any mix of quantitative and qualitative evidence.

The phenomenon of interest in the proposed case study is how preservice teachers communicate about geometric transformations, and how the ways in which preservice teachers communicate about geometric transformations change before and after the intervention of a five-week instructional unit. The concern of this study is what these teachers have learned during the five-week instruction unit. Because this study is designed to explore preservice teachers’ learning by examining their discourse, rather than their solutions to mathematics tasks, case study is an appropriate method for examining preservice teachers’ learning in a natural context. In addition, case study is pertinent because the questions to be addressed are descriptive questions (i.e., “What is the nature of secondary preservice teachers' mathematical discourses?” and “How do their discourses change over time?”). This case study is a descriptive case study (Yin, 2008), which sets out to depict preservice teachers' discourse as it occurs.

4.2 Research Setting

The study was conducted in a geometry-content course for secondary preservice teachers at a university in the Midwest. The university enrolls approximately 25,000 undergraduates, graduates, and professionals. About 65% of the students are undergraduates.

The Mathematics Teaching Program is designed for students who want to teach at the secondary level. The geometry course is a requirement in the teaching program offered by the mathematics department. Before the geometry course, preservice teachers have already taken
Calculus I, Calculus II, Calculus III, and either Applied Linear Algebra or Linear Algebra I. The content covered by each course, as detailed in the program handbook, is described in Table V.

<table>
<thead>
<tr>
<th>COURSE</th>
<th>DESCRIPTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculus I</td>
<td>Differentiation, curve sketching, maximum-minimum problems, related rates,</td>
</tr>
<tr>
<td></td>
<td>mean-value theorem, antiderivative, Riemann integral, logarithm, and</td>
</tr>
<tr>
<td></td>
<td>exponential functions.</td>
</tr>
<tr>
<td>Calculus II</td>
<td>Techniques of integration, arc length, solids of revolution, applications,</td>
</tr>
<tr>
<td></td>
<td>polar coordinates, parametric equations, infinite sequences and series,</td>
</tr>
<tr>
<td></td>
<td>and power series.</td>
</tr>
<tr>
<td>Calculus II</td>
<td>Vectors in the plane and space, vector-valued functions, functions of several</td>
</tr>
<tr>
<td></td>
<td>variables, partial differentiation, maximum-minimum problems, double and</td>
</tr>
<tr>
<td></td>
<td>triple integrals, applications, and Green's theorem.</td>
</tr>
<tr>
<td>Introduction to Advanced</td>
<td>Introduction to methods of proofs used in different fields in mathematics.</td>
</tr>
<tr>
<td>Mathematics</td>
<td></td>
</tr>
<tr>
<td>Applied Linear Algebra</td>
<td>Matrices, Gaussian elimination, vector spaces, LU-decomposition, orthogonality,</td>
</tr>
<tr>
<td></td>
<td>Gram-Schmidt process, determinants, inner products, eigenvalue problems,</td>
</tr>
<tr>
<td></td>
<td>applications to differential equations, and Markov processes.</td>
</tr>
<tr>
<td>Linear Algebra I</td>
<td>Linear equations, Gaussian elimination, matrices, vector spaces, linear</td>
</tr>
<tr>
<td></td>
<td>transformations, determinants, eigenvalues, and eigenvectors.</td>
</tr>
</tbody>
</table>

The title of the geometry course is Advanced Euclidean Geometry. The course goals are to help preservice teachers 1) to understand the foundations of the three major approaches to
Euclidean geometry: synthetic, coordinate, and transformational; 2) to know the history of geometry from antiquity until now, and understand how history and other factors have influenced the current teaching of geometry; 3) to understand the importance of geometry as an academic subject, in the development of modern thought in all fields; 4) to have sufficient facility with the techniques and methods of Euclidean geometry; and 5) to be comfortable teaching any of the current approaches to geometry in high school.

The content covered in this course includes ruler-and-compass constructions, congruence, Euclid’s and Hilbert’s axiom systems, transformations, similarity, measurement, coordinate geometry, trigonometry, history of geometry, history of the teaching of geometry, and modern approaches to the teaching of geometry. Readings for this course are edited from various texts.

The course was co-taught by the author and an experienced mathematics professor. The main instructor, Dr. P., was a professional mathematician and educator in the mathematics department. His major research interests are algebraic geometry and K-12 mathematics curricula. Dr. P was teaching this course for the second time. He used PowerPoint presentations to present the class outline, theorems, and tasks, and used blackboard and chalk when orchestrating discussions about problem-solving processes. He also integrated dynamic geometry software in his instruction. For example, he used the software to visually demonstrate concepts, have preservice teachers construct geometry objects and verify geometric relationships, and design activities that support preservice teachers to make conjectures. Instruction was a mixture of lecture and group work. He encouraged preservice teachers to share ideas and listen to each other’s ideas. The author served as a second instructor, leading whole-group discussion for the last two classes of the intervention unit and participating in small-group discussion during the five classes.
4.3 Instructional Unit

As mentioned before, the purpose of this study is to explore and describe the nature and change of preservice teachers’ mathematical discourse in transformation geometry. A five-week instruction unit was designed collaboratively by the instructor and the researcher to support preservice teachers’ learning of transformation geometry. The goal of the unit was to develop preservice teachers’ understanding of geometric transformations and their ability to solve problems involving geometric transformations. Ultimately, preservice teachers should be able to conceive geometric transformations as a mathematical object or mathematical entity, such as function or mapping; represent transformations using different representations; and use the notion of function to analyze transformations, composition of transformations, and symmetry. There were five classes, which focused on transformations, composition of transformations, representing transformations with matrices, symmetry, and similarity. The instructional plan timetable is described in Table VI:
<table>
<thead>
<tr>
<th>Date</th>
<th>Content</th>
</tr>
</thead>
</table>
| 3/15/12    | Transformations: reflections, translations, rotations, glide reflection, similarities, and shears  
Activity: Find transformations  
Activity: Definition of a transformation of a plane and identify transformations  
Activity: Consider a transformation of the plane as a function from the plane to itself. Come up with definitions for each type of transformations of the plane.  
Activity: Definition of an isometry  
Activity: GSP activity from College Geometry activity 1-5 |
| 3/29/12    | Isometries, fixed points, orientation, composite of transformations, and inverse of transformations  
Activity: Identify fixed points and orientations for different types of isometries  
Activity: Find inverse of a transformation  
Activity: GSP activity from College Geometry activity 6-10 |
| 4/5/12     | Isometries, fixed points, orientation, composite of transformations, and inverse of transformations  
Activity: If f is an isometry, prove $f^{-1}$ is a isometry  
Activity: Prove that the set of isometries in the plane is a group  
Activity: Prove ASA Criterion for congruent triangles |
| 4/12/12    | Representing transformations with matrix and vector  
Activity: GSP activity from College Geometry activity 11 |
| 4/19/12    | Symmetry and similarity  
Activity: Symmetry group of a equilateral triangle  
Activity: GSP activity from College Geometry activity 12 |

Each session lasted three hours. During the class, students were organized in groups of four per table. At least one computer, with Geometer’s Sketchpad installed, was provided to each group.
4.4 Participants

All students enrolled in the course received an invitation to participate in the research project. Twenty-two preservice teachers were enrolled in the geometry content course during the spring 2012 semester. Eight were female, and fourteen were male. Two of them were graduate students pursuing master's degrees in mathematics teaching, while the other twenty preservice teachers were undergraduate students studying to be high school teachers. Participants in the study were eight preservice teachers selected from thirteen teachers who volunteered to participate in the research project. The eight participants were selected based on the following guidelines: participants are willing to share and articulate their thoughts and problem-solving strategies; and participants preferably have different levels of mathematics performance, based on their performance on the content survey, quizzes, and homework assignments from the first half of the course. In class, the eight participants were assigned to two groups. Each group sat at a round table and worked together during small-group discussion.

For this study, four preservice teachers—Nancy, Brian, Owen, and Nathan\(^1\)—were selected as focus participants. The four preservice teachers sat at the same table during class. The four teachers were chosen because they actively participated in class discussion and collaborated with their group members solving mathematics problems in class, and they varied with respect to their mathematics performance in the course. By the time of implementing the instructional unit, grades on content survey, quizzes, and homework assignments showed that Nancy’s mathematics performances was at high level with fluctuations; and Brian’s mathematics performance was at a consistent high level; Owen’s mathematics performance was at

\(^1\) All names are pseudonyms.
intermediate level with the tendency of raising; and Nathan’s mathematics performance was at a relatively low level.

4.5  **Data Collection**

In this case study, a variety of instruments were used to collect data. First, a student background survey and a content survey were conducted at the beginning of the course, which provided information about the courses taken by the preservice teachers (Appendix 1) and preservice teachers’ prior knowledge about geometric transformations (Appendix 2). The surveys provided information for selecting focus participants for interview, and for improving interview protocol design and instructional design. Second, individual semi-structure and task-based interviews were conducted one week before the instructional unit began and one week after the instruction ended. The interviews were videotaped and audiotaped. During the instructions, whole-group discussion was videotaped. Also, preservice teachers’ artifacts were collected, including written notes, homework assignments, and worksheets used in class meetings. The background survey, content survey, classroom observations, and artifacts informed instructional design and provided additional information about preservice teachers’ learning. The data collected from the task-based pre-interview and post-interview was the main resource drawn on to answer research questions.

4.5.1  **Interview**

To examine preservice teachers’ learning of transformation geometry, task-based pre-interviews and post-interviews were conducted with each participant. Generally, the pre-interviews were conducted during the week before the transformations units, and the post-interviews were conducted within one week after the instruction. The only exception was that Brian’s interview was conducted after the first class due to the participant’s time conflict. All
interviews were audiotaped and videotaped. However, due to a technical issue, Brian’s work on the Reflection Task at the pre-interview was not videotaped. Field note and audio were used to fill this gap during data analysis.

Identical interview tasks were used in pre-interviews and post-interviews, because identical interview tasks allowed a straightforward comparison between preservice teachers’ discourse at the pre-interview and post-interview. The interview included three tasks: identify transformations, transformations and isometry, and composition of transformations and isometry. Only the response to the first interview task was analyzed because preservice teachers spent nearly two-thirds of the time on the first task.

The first task was designed to explore how preservice teachers identify different type of transformations and how they describe different types of transformations. Preservice teachers were presented with six diagrams (each were printed on A5 paper) and asked to identify the geometric transformations that map one polygon to another polygon in the diagrams. For the purpose of convenient discussion, the identification task that involved Diagram 1 is called the Reflection Task; the one involving Diagram 2 is called the Translation Task; the ones involving Diagram 3 and 4 are called the Rotation A Task and the Rotation B Task; Diagram 5 is used for the Dilation Task; and Diagram 6 is used for the Shear Task (Shown in Figure 7).
Task 1 (Identify Transformations):
Identify the transformations that map one polygon to another polygon in the following diagrams:

Diagram 1                          Diagram 2                          Diagram 3

Diagram 4                          Diagram 5                          Diagram 6

Figure 7. Interview Task I: Identify transformations

The transformations in each diagram could be: reflection over line $l$, translation along vector, rotation about center $O$ by $90^\circ$, rotation about center $O$ by $60^\circ$, dilation with center $O$ and scale factor $1/2$, and shear (Shown in Figure 8).
Task 1 (Identify Transformations): Identify the transformations that map one polygon to another polygon in the following diagrams:

Answers:

- Reflection over line $l$
- Rotation about center $O$ by $90^\circ$
- Rotation about center $O$ by $60^\circ$
- Translation by vector $v$
- Dilation with center $O$ and scale factor $1/2$
- Shear $T(x, y) = (x+2y, y)$

Figure 8. Transformations in Interview Task I

After the preservice teachers identified the geometric transformation in the diagram, I asked them how they figured it out, to elicit their thinking about what information they used to identify the transformations. If they only mentioned the types of transformations, I asked them to
provide more specifics about the type of transformations and how they figured it out. I asked them to describe how the transformation(s) they identified works. Preservice teachers’ responses to these questions could reveal the words used to communicate about the geometric transformations, the routine of identifying geometric transformations, and the definition they used for the geometric transformations.

4.5.2 **Classroom Observations**

Given the role of the author as an instructor, classroom observations that aimed to provide a full picture of preservice teachers’ learning were documented mainly by audio and video records. As illustrated in Figure 9 below, a camera operated by a professional photographer focused on the group of four participants. When there was a whole-group activity, the camera records the whole-group discussion; when there was small-group activity, the camera recorded small-group discussion.

![Figure 9. Classroom layout](image-url)
4.6 Data Analysis

Again, the data sources for this study included videotapes and transcripts of the pre-interviews and post-interviews, as well as written records. There were eight videotapes in total (4 participants x 2 trials). The length of the videotapes varied from 30 to 60 minutes.

This section describes the process of data analysis (see Table VII). I started with describing my general approach to the data, and then I focused on the specific analysis of the three elements in preservice teachers’ discourse.

First, an undergraduate assistant transcribed the videotapes of interviews using Inscribe™ software. The speech of both the interviewer and the preservice teachers was transcribed. Talking turns between the preservice teacher and the interviewer were identified, and were assigned time codes. I verified all the transcripts before all the transcripts were imported into Nvivo 10™. Given that preservice teachers’ discourse includes utterances, text, gestures, and drawings, Inscribe and Nvivo10 were used together to complement each other during the process of analysis. I read the transcripts and watched the video in Inscribe, and then coded the same transcripts in Nvivo 10™ qualitative data analysis software.

Second, I watched the interview videos with the transcripts. For each preservice teacher, I first divided the transcripts based on the mathematical tasks, and then examined preservice teachers’ performance on the tasks with the support of relevant written work produced during the interview. Meanwhile, I wrote memos on the salient instances relevant to the elements of mathematics discourse in word documents.
### TABLE VII
**DATA ANALYSIS PROCESS**

<table>
<thead>
<tr>
<th>Stage 1</th>
<th>Verify and organize transcripts</th>
</tr>
</thead>
</table>
| Stage 2 | View interview videotapes and transcripts for each individual  
Identify three elements of discourse as the focus of the analysis: geometric transformation words and their use, identification routine, and working definitions as endorsed narratives |
| Stage 3 | Analyze interview videotapes and transcripts by the elements of discourse  
- Analyze geometric transformation words and their use  
  - **Phase 1:**  
    - Identify geometric transformation words used by preservice teacher  
    - Examine the nature of geometric transformations by category  
    - Examine the percentages of geometric transformation words by category  
  - **Phase 2:**  
    - Identify types of word use  
    - Examine the nature of different types of word use  
  - **Phase 3:**  
    - Identify the patterns of change in word use  
    - Examine the significance of the change  
- Analyze identification routine  
  - **Phase 1:**  
    - Identify types of determination routines  
    - Examine the nature of different types of determination routines  
    - Identify the patterns of change in determination routines  
  - **Phase 2:**  
    - Identify types of specification routines  
    - Examine the nature of different types of specification routines  
    - Identify the patterns of change in specification routines  
- Analyze working definition as endorsed narratives  
  - Identify working definitions  
  - Examine the nature of the working definitions  
  - Identify the patterns of change in working definitions |
The goals of the first two stages of data analysis were to become familiar with the data and narrow the focus of analysis. The following questions were used to guide my observations:

- What are the keywords that stood out in preservice teachers’ discourses?
- What are the visual mediators that stood out in preservice teachers’ discourses?
- What are the routines that happened regularly in preservice teachers’ discourses?
- What are the narratives that stood out in preservice teachers’ discourses?

Third, after the two passes through the data, these three elements—geometric transformation words, identification routines, and the working definitions of geometric transformations—were selected as the focus of the analysis. The rationale for choosing these elements as the focus of analysis is provided in the coming subsections. For visual mediators, representational gestures and mathematics symbols in preservice teachers’ discourses were attended during the observations. However, due to the low numbers of occurrence of representational gestures and mathematical symbols, visual mediator was not analyzed by itself, but was included in the analysis of other elements.

In what follows, I describe the detailed analysis on geometric transformation words, identifying routines, and working definitions in preservice teachers’ discourses.

4.6.1 Analysis of Geometric Transformation Words and Their Use in Preservice Teachers’ Discourses

As described previously, keywords are words that signify quantities, shape, and operations. Keywords in a mathematical discourse on geometric transformations included, but were not limited to: words that refer to geometric transformations (e.g., reflection, flipping); words that denote the elements of geometric transformations (e.g., preimage, image, inverse); and words that refer to the properties of geometric transformations (e.g., orientation-preserving).
The goal of analyzing keywords and their use is to understand the objectification of geometric transformations in preservice teachers’ discourses. Focusing on the keywords that signified geometric transformations provided straightforward access to understand to what extent, preservice teachers talk about geometric transformations as mathematics entities by themselves.

The first phase in analyzing keywords involved identifying a list of geometric transformation words used by preservice teachers. I began by highlighting the words that are used in formal mathematics discourse from preservice teachers’ discourses. These include “reflect,” “reflection,” “rotate,” “rotation,” “translate,” “translation,” “dilated,” “dilation,” “scale,” and “enlargement.” Next, I highlighted the informal words that refer to geometric transformations based on preservice teachers’ speech, gestures, and drawings. Given that geometric transformations are prevalent in daily life, preservice teachers might use informal words, such as “flip” and “turn,” to describe geometric transformations. With the identified geometric transformation words, I further examined the nature of these keywords under both categories. I checked and made sure that the words used by preservice teachers denoted geometric transformations, rather than signifying other different meanings. For example, the word “translation” that refers to “the process of translating text from one language into another” was not included for further analysis. Furthermore, I examined the percentages of geometric transformation words by categories to give a sense of what kinds of geometric transformation words were used by preservice teachers during pre-interviews and post-interviews.

In the second phases of the analysis, the goal was to identify types of the use of geometric transformation words, or types of word use, in preservice teachers’ discourses. Given that the dual nature of the notion of geometric transformations, geometric transformations can be considered as an operation or an action (e.g., translating a point horizontally five units), and as a
structure or an object (e.g., a composition of two reflections could be a rotation). Moreover, as mentioned before, transition from operational talk to structural talk is a significant process in objectifying a mathematics entity. Thus, I focused on the two types of use of geometric transformation words:

- **Operational word use**: If the geometric transformation word is used to describe a geometric transformation as a process or action, operational word used is identified.

- **Structural word use**: If the geometric transformation word is used to describe a geometric transformation as a structure or an object, structural word use is identified.

The two types of word use were used to initially identify potential ways of using geometric transformation words. With a further examination under these two types of word use, subcategories emerged in preservice teachers’ discourses for using geometric transformation words emerged in preservice teachers’ discourses: the different kinds of operations and the different facets of geometric transformations. The different kinds of operations under operational word use include operational word use with operation on polygon(s) and operational word use with operation on point(s). The former indicates that a geometric transformation word is used to describe a geometric transformation as operation on polygon(s), and the latter refers to a geometric transformations word used to describe a geometric transformation as operation on point(s). In addition, the different facets of geometric transformations as object refer to different aspects of geometric transformations presented in preservice teachers’ discourses, which includes: a) geometric transformation is what is signified by the diagram; b) geometric transformation has parameter(s); c) geometric transformation can be acted on; and d) geometric
transformation has agency. For each type of word use, I created detailed characterization and analyzed its nature in terms of the mathematical goal of the instructional units.

In the third phase of the analysis, the goal was to examine the change in the use of geometric transformation words in preservice teachers’ discourses, with the attempt to understand preservice teachers’ learning. First, transcripts were blind coded with the types of word use. I did not know whose interview I was coding, or whether the interview was from the pre-interview or post-interview. The unit of analysis was the verbal utterances with geometric transformation words, which were defined as any individual units of speech that contained geometric transformation words, such as sentences or words like “rotation.” Gestures and actions accompanying the utterances, if any, were examined to determine the type of word use. Second, the frequency and percentage of the occurrence of each kind of word use in the pre-interview and post-interview were calculated. The patterns of change emerging at the group level and at the individual level were identified by examining these percentages. Third, statistical methods were used to assess the significance of the changes in preservice teachers’ word use from pre-interview to post-interview. Given that this study involves a sample that was not randomly chosen from a particular population, significance levels were reported here to indicate the strength of patterns identified, as opposed to indicating that such patterns would hold in a general population. Specifically, a Chi-square test was conducted to determine whether there was a significant relationship between pre-interview and post-interview in terms of word use. In particular, I examined differences in the percentages of preservice teachers’ overall word use, structural word use, and operational word use, both in the group and in individually. However, if the number of utterances of a certain type of word use was less than five, then a Chi-square was
not conducted, because the expected cell value for a Chi-square is not less than five (Gravetter & Wallnau, 2005).

4.6.2 Analysis of Routine in Preservice Teachers’ Discourses

To explore routine, I first identified the notable routines emerging from preservice teachers discourses, which was the routine of identifying geometric transformations. In fact, the tasks provided great opportunities for preservice teachers to engage in the activity of identifying geometric transformations, which elicited a set of identifications routines. However, compared to the activity of identifying a shape, the activity of identifying a geometric transformation was more complex, given the different natures of shapes and geometric transformations as mathematical objects (this difference is discussed in the mathematics section). Thus, the procedures and modalities for identifying shapes and geometric transformations are different. Identifying a shape involves recalling past experiences associated with the present experience and attaching a word to the recognized shape, which is mediated by and documented in speech and written language (Sfard, 2007). Identifying geometric transformations might not end with naming the geometric transformation by talking or writing, but could involve specifying the parameters of the geometric transformation by drawing. Furthermore, naming geometric transformations might not rely on the mode of speech. It could also be mediated by gestures, which “name” the geometric transformations by indicating the path of movement. Thus, I broke the activity of identifying a geometric transformation into two processes: determining the type of the geometric transformation and specifying the parameters of the geometric transformation. Determining the type of a geometric transformation was similar to identifying a shape, which includes the steps of recognition and naming. Again, this “naming” process could be done by indicating movements through gestures. Specifying the parameters of a geometric transformation
was the process of drawing out the parameters based on the diagram provided in the task, which
cannot be accomplished solely through the mode of speech. Note that these two processes were
not always separate; in fact, they were intertwined.

In the second phase of the analysis of routine, the goal was to identify different
determination routines and specification routines in preservice teachers’ discourses. For each
routine, I created detailed characterization and analyzed its nature in terms of the mathematical
goal of the instructional units.

In the third phase of the analysis, the goal was to examine the change in routines of
identification in preservice teachers’ discourses, with an attempt to understand preservice
teachers’ learning. First, transcripts were blind coded with the different routines. The unit of
analysis was the episodes in which identification activity was involved. The activity of
identifying geometric transformations began as preservice teachers started solving the problem
and ended with questions asked by the interviewer to initiate another activity, such as finding the
image of a given point. Forty episodes with activity of identification routine were identified from
the four preservice teachers working on five tasks at the pre-and post-interview (4 participants x
5 tasks x 2 trials). Second, the frequency and percentage of the occurrence of each identification
routine in the pre-interview and post-interview were calculated. The patterns of change emerging
at the group level and individual level were also identified by examining the frequency and
percentage. Given the small number of occurrences in certain routines, no statistical test was run;
Instead, descriptive statistics were presented to characterize the patterns identified.

4.6.3 Analysis of Narratives in Preservice Teachers’ Discourses

When analyzing narratives, I focused on the working definition of geometric
transformation in preservice teachers’ discourses. Working definition refers to a temporary
definition that preservice teachers used for working with geometric transformations. The working definition from individual preservice teachers might be under development and not fully aligned with the definitions endorsed by the mathematics community (O’Connor, 1998). Thus, preservice teachers’ working definitions provided a window to explore their discourses on geometric transformations. In addition, the conversation about working definitions was initiated when preservice teachers encountered difficulty in identifying a geometric transformation, at which point they were explicitly asked for working definitions. For example, after Nathan had solved the Reflection Task, Rotation Tasks and Translation Task, he started hesitating and looking at the Shear Task for more than thirty seconds. I then posed the question, “What is the definition of geometric transforms you are using?” This asked Nathan to provide a working definition. Eliciting preservice teachers’ working definitions as they encountered difficulty created a situation in need of definition. This provided an opportunity for preservice teachers to generate the definition from the work they had done during the interview, which avoided preservice teachers simply recalling definitions of geometric transformations. This also indicated that preservice teachers’ working definitions were endorsed narratives that were “tested” in their previous, implicit use of the definition. Therefore, it was worth taking a deep look at preservice teachers’ working definitions.

After identifying the eight episodes (4 participants x 2 trials) containing the communication of working definition, I fully examined the eight working definitions in terms of their characteristics, the extent to which they aligned with definitions endorsed by the mathematics community, and the effect on solving the Shear Task. The change in working definition in preservice teachers’ discourses was also examined to understand individual trajectory.
4.7 **Reliability and Validity**

In designing and conducting this study, there were limitations inherent to interpreting preservice teachers’ discourses. Regarding the issues of reliability and validity, this study used multiple strategies to ensure the consistency, credibility, and applicability of the results. Reliability refers to the extent to which the researcher’s approach can be repeated with the same results across different researchers and different projects, while validity refers to the extent to which the researcher provided an accurate representation of the phenomena being studied. The degree to which the research provides a true picture of the group being studied is referred to as internal validity, and external validity refers to the extent to which the data collected from the group studied can be generalized to a wider population.

Two strategies were used to support the reliability of the research findings. The first strategy, transcripts checking, was used before data analysis to detect obvious mistakes made during transcription and avoid major changes in transcripts during data analysis. This enhanced the consistency throughout the study. The second strategy, cross-checking codes, was used to measure reliability. Cross-checking codes were preformed for the analysis of word use. Two researchers independently coded a subset of data from both the pre-interview and post-interview. Overall inter-rater reliability was initially 88%. Any differences between the two coders were discussed and resolved through consensus.

Three strategies were used enhance validity. First, triangulation was used to confirm the consistency in factual data, which involved checking data collected from multiple sources. Specifically, transcription data was triangulated with the video data, researcher’s memo and preservice teachers’ written records to determine whether the transcription data carried the same
meaning as in the interview video and written documents. In this way, multiple data sources were used to interpret preservice teachers’ mathematical discourses.

Moreover, peer debriefing was used to enhance the accuracy of the account. This process involved seeking verification of the data, result, and interpretation from other peer researchers. According to Gribbs (2007), involving an interpretation beyond the researcher and invested in another person adds validity to an account. A peer researcher in the PhD program reviewed and asked questions about methods, results of the study, and the interpretations, so that the account resonates with people other than the author.

Finally, as the first two strategies were targeted for internal validity, the third strategy of rich and thick description was used throughout the analysis to enhance the study’s internal and external validity. Thick description was used in order to provide sufficient details so as to not make the reader question the legitimacy of the findings. For example, detailed descriptions were provided when presenting different types of word use, routines, and narratives. The author did not, however, merely provide continuous description of each preservice teacher’s discourses. In addition, thick description allows readers to make decisions regarding enhance the external validity, because the author describes in detail the participants or setting under study. With such detailed description, the researcher enables the reader to transfer information to other settings.

4.8 Limitations

There are several limitations of the methodology that should be brought to the reader’s attention. First, the role of author as instructor and interviewer might have impacted preservice teachers making their thinking explicit. Preservice teachers might take into account that the interviewer was the instructor in the same class; thus, the interviewer knew what they were talking about. Shared knowledge from the class might be a factor that leads to implicit
communication between the preservice teacher and the interviewer. However, being an active participant observer allows the interviewer gain close familiarity with the nature of participants’ discourse in various contexts, which adds reliability to current research.

A second limitation of the study is that the results from the case study cannot be generalized to a large population. However, the goal of the case study was to provide detailed descriptions of preservice teachers’ discourses. Readers can make decisions regarding whether the results can be used in their study.
5. GEOMETRIC TRANSFORMATION WORDS AND THEIR USE

According to Sfard’s communicational framework (2007), if preservice teachers communicate geometric transformations mathematically, their discourses should feature mathematical words that are used in a mathematical way. The mathematical words in this analysis are those that can be used to signify types of geometric transformations (reflection, rotation, translation, dilation), which I refer to as geometric transformation words. As is mentioned in the data analysis process, this dissertation focused on geometric transformation words because they provided straightforward access to uncover the extent to which preservice teachers communicated geometric transformations as mathematics object or mathematics entities.

This chapter presents the results of the analysis of geometric transformation words and their use to address three research questions: What are the geometric transformation words used in preservice teachers’ discourses on geometric transformation? What are the different ways of using geometric transformation words that emerge from preservice teachers' discourses? And how does the use of geometric transformation words in preservice teachers’ discourses change over time?

To answer the first question, the first section provides a list of words that can be used by preservice teachers to signify geometric transformations. The second section presents a closer examination of these geometric transformation words in terms of the different types of geometric transformation words: formal geometric transformation words and informal geometric transformation words. This examination of geometric transformation words served two purposes. One is to bond the geometric transformation words for the next step in analysis, because not every occurrence of the word was used to denote a geometric transformation. Specifically, for
formal geometric transformation words, two special occasions are presented to illustrate the various meanings of geometric transformation words in preservice teachers’ discourses: formal geometric transformation words signified everyday-life objects and formal geometric transformation words signified other mathematical objects. Another purpose is to reveal the complex nature of preservice teachers’ discourses through words that appeared in their discourses. Besides the complexity shown in formal geometric words, the informal geometric transformation words in preservice teachers’ discourses have two subcategories: specific informal geometric transformations and general informal geometric transformation words.

To address the second question, the second section presents the ways in which geometric transformation words were used in preservice teachers’ discourses, which I called “word use” for short. Specifically, it presents operational word use, structural word use, and their subcategories. Detailed descriptions and transcripts from interviews are presented to illustrate each type of word use. In addition, this section presents the analysis of word use in terms of the goal of the instructional unit, in order to understand the nature of each type of word use.

The third section provides patterns of change in word use in preservice teachers’ discourses to address the third research question. The patterns of change in word use are presented by category: the pattern of change in overall word use, the patterns of change in structural word use, and the patterns of change in operational word use. For each category, group-level patterns were presented and followed with patterns at the individual level.

5.1 Geometric Transformation Words in Preservice Teachers’ Discourses

Preservice teachers used various words to denote each type of geometric transformations. Table VIII provides a list of words related to four types of geometric transformations. There were no geometric transformation words identified for the shear, because the shear
transformations are not familiar to preservice teachers. To solve the Shear Task, preservice teachers typically applied other geometric transformations in a certain way to transform one polygon into another. More details about the preservice teachers' discourses on shear are presented in Chapter 7.

<table>
<thead>
<tr>
<th>Geometric Transformations</th>
<th>Geometric Transformation Words Used by Preservice Teachers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Formal</td>
</tr>
<tr>
<td>Reflection</td>
<td>reflection, reflect</td>
</tr>
<tr>
<td>Rotation</td>
<td>rotation, rotate</td>
</tr>
<tr>
<td>Translation</td>
<td>translation, translate,</td>
</tr>
<tr>
<td></td>
<td>dilation, dilate, shrinkage,</td>
</tr>
<tr>
<td></td>
<td>shrink scaling, enlarging</td>
</tr>
</tbody>
</table>

As shown in the above table, some of the geometric transformation words are the words used in formal mathematical discourse, and some are used in informal discourse. In this study, the words “reflection,” “reflected,” “rotation,” “rotate,” “translation,” “translate,” “dilation,” “dilate,” “scale,” and “enlargement” are considered formal geometric transformation words, given that these words are widely used in the mathematics community. Words such as “flip,” “shift,” “shrink,” “flip,” “move,” “turn,” and “resize” are considered to be informal geometric
transformation words, which might be used in K-12 textbooks, but their occurrences in formal mathematics communication is limited.

5.1.1 **Formal Geometric Transformation Words**

For the formal geometric transformation words, there was one instance in which a preservice teacher used a formal word to refer to an everyday-life object, and three instances in which a preservice teacher used a formal word to refer to other mathematical notions.

Owen used a formal geometric transformations word to signify an everyday-life object when he explained how he knew that the geometric transformation was a reflection in the Reflection Task at the pre-interview. Owen pointed out that reflection works like a mirror. “If I was looking at a mirror, it (polygon A’B’C’D’) would be like its (polygon ABCD’s) reflection,” he said. Meanwhile, he pointed from polygon A’B’C’D’ to polygon ABCD (Shown in Figure 10).

![Figure 10](image)

**Figure 10.** Owen’s written work for the Reflection Task at the pre-interview
When using the word “reflection” here, Owen did not regard “reflection” as a type of the geometric transformation, but rather referred to it as the image in a mirror. By pointing back and forth between the polygons, Owen indicated that one polygon could be a reflection or a image of another polygon. The use of “reflection” to denote the image in a mirror is reasonable in this instance, because the reflection phenomena in a mirror or water provide real-life connections to reflection as a type of geometric transformation.

Brian's discourse demonstrated the use of formal geometric transformation words to signify other mathematics notions. When explaining how he determined a rotation in the Rotation A Task at the post-interview, Brian stated, “By the fact that its (polygon ABCD’s) orientation is preserved, but the position of the figure (polygon ABCD) has been changed, as well as its rotation has been changed.”

![Diagram of rotation with labeled points](image)

Figure 11. Brian’s written work for the Rotation A Task at the post-interview
The word “rotation” in his statement was used to describe not whether the polygon ABCD was rotated, but rather a type of geometric transformation. In other words, the word “rotation” is regarded as a property of a figure, such as its orientation. The use of formal geometric transformation words to signify other mathematics notions suggested that preservice teachers used geometric transformation words in an unconventional way to express their mathematical thinking.

A closer look at these four instances sheds light on the complex nature of the geometric transformation words in preservice teachers’ discourses. Geometric transformation words are not always used to signify geometric transformations. During the second phase analysis (i.e. analysis of word use), the four instances are not included because they were not used to signify geometric transformations.

5.1.2 Informal Geometric Transformation Words

There are two categories of informal geometric transformation words identified in the list. One category is what I refer to as “specialized informal words,” the geometric transformation words that describe only one type of geometric transformation. For example, “flip” is for reflection, while “turn” denotes rotation. The other category is what I call “general informal words,” the geometric transformation words not specifically associated with a certain type of geometric transformation. For instance, preservice teachers could use the general informal word “move” to signify a rotation or a translation. General informal words were usually used to communicate the geometric transformations that preservice teachers were not familiar with, and the occurrences of general informal words were usually accompanied by gesturing, with diagrams to indicate the path of movement. Therefore, despite the vagueness of general words when used by themselves, they were included for next-step analysis as long as the gestures
and diagrams provided sufficient information about the meaning of the words. For example, when Owen talked about translating point A to point A’ (Shown in Figure 12) during the post-interview, he said, “move A to A’ this way.” At the same time, he traced a path from A to A’ with his index finger.

**Interviewer:** How do you decide that's the vector of the translation?

**Owen:** Move A to A’ [Tracing a path from A to A’ with his index finger], B to B’ move the same amount; C to C’, D to D’ in the same direction.

![Figure 12. Owen’s written work for the Translation Task at the post-interview](image)

In this situation, his index figure tracing on the diagram illustrated the moving path from A to A’, which demonstrated that the word “move” signified a translation. The occurrence of general informal geometric transformation words with gesturing on diagrams indicated that preservice teachers relied on various modes of communication (e.g., speech, diagram, gesture) to convey mathematics notions.

Although the purpose of the analysis of geometric transformation words is not to analyze the percentages of the change in each category of words changes over time, it is useful to have a sense of the changes in the type of geometric transformation words from pre-interviews to post-interviews. As illustrated in Table IX, the occurrence of informal geometric transformation words in preservice teachers’ discourses decreased from about 40% in the pre-interviews to
about 10% in the post-interviews. This change aligned with the goal of the instructional unit: preservice teachers should be able to communicate geometric transformations mathematically.

<table>
<thead>
<tr>
<th>Interview</th>
<th>Formal geometric transformation words</th>
<th>Informal geometric transformation words</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-interview</td>
<td>122</td>
<td>75</td>
</tr>
<tr>
<td></td>
<td>61.93%</td>
<td>38.07%</td>
</tr>
<tr>
<td>Post-interview</td>
<td>226</td>
<td>19</td>
</tr>
<tr>
<td></td>
<td>92.24%</td>
<td>7.76%</td>
</tr>
</tbody>
</table>

The changes in the different word categories could be superficial without further examining the changes in the use of geometric transformation words, given that “word use” is equivalent to “word meaning” (Sfard, 2007). In the next section, I focus on the use of geometric transformation words in preservice teachers’ discourses. The analysis and the results are described below.

5.2 **Word Use in Preservice Teachers’ Discourses**

After geometric transformation words were identified, the goal of the following analysis was to characterize different ways in which geometric transformation words were used in preservice teachers’ discourses, which I simply call it as “word use.” According to Sfard (2008),
objectification is an important feature of word use in mathematical discourse, and transition from operational word use to structural word use is an important indicator of objectification.

5.2.1 Structural Word Use in Preservice Teachers’ Discourses

In this section, structural word use and its subcategories are introduced. If the geometric transformations word is used to describe a geometric transformation as an object that has mathematical structure or properties (e.g., “Geometric transformations are mappings between points”), it is called structural word use. Generally, structural word use can be identified when the geometric transformations word is a noun, such as “reflection,” “rotation,” “translation,” or “dilation.” There is an exception; the noun form of a geometric transformations word following the verb “do” is considered as operational word use. For example, the phrase “I can do a rotation” indicated that rotation is a set of procedures that one can carry out. In this narrative, “rotation” is not referring to a structure or object. Describing a geometric transformation as a structure or an object is a significant hallmark that preservice teachers can objectify a geometric transformations as a mathematical object, that is, they can talk about a geometric transformation as a self-sustained object (Sfard, 2005).

When preservice teachers talked about geometric transformations as a structure or an object, different facets of geometric transformations as an object emerged from their discourses. A facet could be a property or a feature of geometric transformations. The goal of the analysis of different facets is to understand how geometric transformations as objects were presented in preservice teachers’ discourses. As preservice teachers participating in a course, they should be able to focus on facets that provide essential information about geometric transformations, or communicate multiple facets of geometric transformations.
In what follows, I present the facets of geometric transformations emergent in preservice teachers' discourses, and illustrate each facet with example utterances. These facets include: a) geometric transformations are what is signified by the diagram; b) geometric transformations have parameter(s); c) geometric transformations can be acted on; and d) geometric transformations have agency. Again, the natures of these facets are different. One facet might provide more information than the other; one facet might occur more often than the other. However, they are not hierarchically related to each other.

5.2.1.1 Geometric Transformations are What is Signified by the Diagram

For the geometric transformation words that were identified as structural word use, if the words were used to describe the diagram on the worksheet, that meant the words were used to describe the facet of geometric transformations as what was signified by the diagram.

When preservice teachers communicated geometric transformations as what was signified by the diagram, they either looked at the diagram, pointed to the diagram as a whole, or circled the diagram while uttering the geometric transformations word. For a specific example, let’s review the excerpts from Brian as he solved the Rotation A Task at the pre-interview. For the transcript in the gray box below, verbal utterances, gesture utterances, and written work are presented. Gesture utterances are in the bracket, and the underlines indicate that the underlined verbal utterances and gesture utterances occurred simultaneously.

Brian: It is a rotation… I oriented them [Points to polygon 1 and polygon 2] based on their longest side. This [Points to polygon 1] would be able to lie perfectly on top of this [Points to polygon 3]. So it (polygon 1) hasn't been reflected or its shape or size hasn’t been changed.
Figure 13. Brian’s written work for the Rotation A Task at the pre-interview

Brian looked at the diagram on the worksheet, and reported, “It is a rotation.” The word “rotation” here referred to what was signified by the diagram in Figure 13. Within this verbal utterance, no further information about the rotation was provided. Even though Brian’s latter utterances indicated that two polygons were congruent, only the utterance containing the geometric transformation words was considered. This was because the goal of the analysis is to reveal the facets of geometric transformations from the use of geometric transformation words, and not attempt to uncover the facets by making inferences from ideas conveyed in preservice teachers’ discourses. Below are some examples from preservice teachers’ discourses, in which geometric transformation words in italics are associated with what is signified by the diagram, and the underline indicated that the verbal utterances and gesture utterances or gaze in the bracket happened at the same time.
It could be a reflection, too. [Looks at the diagram on the worksheet.]

I decide that it is a rotation. [Circles the two polygons on the worksheet.]

So I am going to assume that it is a translation. [Points to the worksheet.]

This would be a dilation. [Looks at the diagram on the work sheet.]

When geometric transformation words are used to denote what is signified by the diagram, the verbal utterances containing the word include declarative sentences, such as “This would be…” or “It is …” The pronouns refer to the diagram.

5.2.1.2 Geometric Transformations Have Parameter(s)

If the geometric transformation words identified as structural word use appeared with other words to denote the parameters of geometric transformations, this was defined as the facet that geometric transformations have parameter(s). The use of geometric transformation words indicated that preservice teachers could see a geometric transformation as an object that has parameters. Below are the examples of utterances in preservice teachers’ discourses, in which geometric transformation words are italicized for emphasis, and the underline indicates the parameters of the transformations.

So, Line L would be the line of reflection.

We could find the angle of rotation.

So then this is going to be the center of rotation, point P.

So it's a translation of vector AA'.

It would be our center of dilation.

Our scale factor of dilation is 1/2.

Compared to the facet of geometric transformations as what is signified by the diagram, the facet of geometric transformations having parameter(s) is not merely what is signified in the
diagram. In addition, it reveals that parameter(s) as an integral part of a geometric transformation.

### 5.2.1.3 Geometric Transformations Can be Acted On

In preservice teachers’ discourses, geometric transformations were presented as objects that can be acted on, which include: geometric transformations can be composited; geometric transformation can be compared; and geometric transformations can be categorized. Next, I present more details about these actions on geometric transformations.

When solving the Rotation B Task at the post-interview, Nathan used translation and rotation. He then concluded that, “I can composite the translation and the rotation.” In this case, Nathan realized that two types of geometric transformations were used to move one polygon to the other, and one geometric transformation could be sufficient. Thus, he proposed to composite the two geometric transformations. This suggests that Nathan considered “composite” as an action that applies to geometric transformations.

When finding the angle of rotation at the post-interview, Nancy first labeled the angle of rotation. She pointed out that the angle of rotation could be 70 degrees if going counterclockwise, or the angle of rotation could be 290 degrees if going clockwise. Thus, she reported the target rotation could be a rotation about 70 degrees or a rotation about 290 degrees. She then claimed, “It’d be the same rotation,” which referred to the two rotations she mentioned as the same. The utterance, “It’d be the same rotation” indicated that in Nancy’s discourses, rotation was an object that can be compared.

When describing the relationship between preimage and image under a reflection, Brian stated, “Well, since reflection is an isometry, our image will be congruent to our source (preimage).” Brian’s statement “reflection is an isometry” indicated that he considered reflection
as an object that has a generic relationship to isometry, which reveals that geometric
transformation is an object that can be categorized.

Communicating geometric transformations as objects that can be acted on is a different
facet of geometric transformations from the first two facets mentioned. While the first two facets
present “what geometric transformations are” and “what geometric transformations have,” this
facet presents “what one can do to geometric transformations.”

5.2.1.4 Geometric Transformations Have Agency

Under structural word use, geometric transformations can be described as an object that
has agency. That is, a geometric transformation is talked about as an agent that can act on other
objects. Below are some examples in which a geometric transformation is described as an object
that has agency, in which geometric transformation words are italicized for emphasis, and the
underline indicates the actions that produced by the geometric transformations.

[12] Well, I know it is not a reflection because it changes orientation.
[14] A 90-degree rotation would produce this image.

In utterances [11]-[14], geometric transformations were the agents that acted on points,
orientations, distance, and image. This facet of geometric transformations indicated that
geometric transformations are not merely static objects illustrated by the three aforementioned
facets, but also objects that can act on other objects.

The four facets that emerged in preservice teachers’ discourses as examining structural
word use represent different aspects of geometric transformation: what geometric transformation
are; what geometric transformations have; what can be done to geometric transformations; and
what geometric transformations can do to other objects. The natures of these facets are different, and they are not hierarchically related to each other.

5.2.2 Operational Word Use in Preservice Teachers’ Discourses

In this section, I mainly focus on describing operational word use and its subcategories. Operational word use refers to the use of the geometric transformation words to describe a geometric transformation as a procedure or action. Operational word use can be identified if the geometric transformations word is a verb, as in, “This polygon would be reflected over that line,” in which reflect is a verb in passive-voice form. In addition, operational word use can be found if the geometric transformations word appears in noun form following the word “do.” For example, in the utterance, “So you'd have to do the rotation here.” “do the rotation” indicated that rotation is a set of procedures that one can carry out.

A closer examination of operational word use revealed two different kinds of operations or actions in preservice teachers’ discourses: operational word use with operation on polygon(s) and operational word use with operation on point(s). In the following paragraphs, I provide descriptions and examples for these two subcategories, followed with two excerpts from the interviews to illustrate discourses featuring operational word use with operation on polygon(s) and discourses featuring operational word use with operation on point(s).

5.2.2.1 Geometric Transformations as Operation on Polygon(s)

If preservice teachers used geometric transformation words to describe a geometric transformation as an operation on polygons or their sides, I refer to the use of geometric transformation as operational word use with polygon(s). When preservice teachers talked about a geometric transformation as an operation or an action on polygon(s), they referred to the
polygons or their sides shown in the diagram. Some examples of operational word use with polygon(s) are presented in the following:

[15] This polygon would be reflected over that line.

[16] What I'm thinking here is we're going to reflect this polygon along our Y-axis.

[17] The polygon from this point is rotated 90 degrees to put it into this position.

[18] It (the polygon) has been rotated in some way.

[19] Since the whole thing is shifting.

[20] Segment AB is translated 10 units to the right.

[21] Even if it (the polygon) is dilated, it is still the same shape.

[22] This side would be scaled by 2.

In utterance [15], the word “reflect” is a verb, which indicates that the geometric transformation is described as an operation. Moreover, “this polygon” is the object that is being operated on. Therefore, the geometric transformation word “reflect” is used to denote an operation on a polygon. In utterances [15]-[21], the objects being operated on are polygons. A side on a polygon could be the operation receiver, as illustrated in utterance [22].

5.2.2.2 Geometric Transformations as Operation on Point(s)

If preservice teachers used geometric transformation words to describe a geometric transformation as an operation on point(s), I refer this as operational use with operation on point(s). Given that a polygon could be considered as a set of points, the reference to a set of points has to be made explicit to fall into this category. Preservice teachers had to say aloud “point” or “points,” or indicate the point or points on the diagram explicitly. Some examples of operational word use with point(s) are presented:

[23] It's the same point. If you reflect it [point] on the line, it [point] stays on the line.
I could have *rotated* point A clockwise, and end up with this point here be A'.

We'll *move* Q (point Q) down 13 then to the right 9.

I have to *shift* O (point O) to E.

So I *scaled it* (point C) in 1/2 in the direction of this.

So if I *rotate* them (the vertices) all around the circle to those points.

In utterance [24], the word “rotate” is a verb, which indicates that the geometric transformation is an operation. Moreover, “point A” is the object that is being operated on. Therefore, the geometric transformations word “rotate” denotes an operation on point A. In utterances [23]-[27], the objects operated on are all one point. As illustrated in utterance [28], the object operated on could be more than one point.

Two excerpts from the interviews were presented to illustrate discourse featuring operational word use with polygon(s) and discourse featuring operational word use with point(s). One excerpt was from Brian as he was working on the Rotation A Task at the pre-interview. The other was from Nancy as she was solving the Translation Task at the pre-interview.

**Excerpt 1: Discourse Featuring Operational Word Use With Operation on Polygon(s)**

At the pre-interview, Brian first determined the type of geometric transformation as rotation for the Rotation Task A. He then showed that the one polygon could lie on top the other one if they were repositioned. He explained that the polygon had not been reflected; the shape and the size of the polygon had not been changed. He claimed that it was a rotation of 90 degrees. He further pointed out that the figure had also been translated down by three units and to the right by one unit.

Brian: It is a rotation… I oriented them [Points to polygon 1 and polygon 2] based on their longest side. This [Points to polygon 1] would be able to lie perfectly on top of this
[Points to polygon 3]. So it (polygon 1) hasn't been reflected or its shape or size hasn’t been changed… And so in some way, it (polygon 1) must have been rotated 90 degrees in this direction… Simply rotating (polygon 1) would have produced [Draws down polygon 2.]

…

Brian: So that’s a 90-degree rotation alone along that point [Labels point A]. But the actual figure (polygon 2) has also been moved…Down five and over one.

Figure 14. Brian’s written work for the Rotation A Task at the pre-interview

Brian’s discourses in this excerpt featured operational word use on polygon(s). Despite the occurrences of structural word uses, the majority is the operational word use. Among these operational word uses, all of them refer to operation on the polygon(s). The annotated utterances from the excerpt below illustrate operational word use with operation on polygon(s):

[29] So it (polygon 1) hasn't been reflected or its shape or size hasn’t been changed. …
And so in some way, it (polygon 1) must have been rotated 90 degrees in this direction.

... Simply rotating (polygon 1) would have produced [Draws down polygon 2].

But the actual figure (polygon 2) has also been moved.

When solving the Rotation Task A at the pre-interview, as indicated in his discourse, Brian attended to the polygons and how polygon 1 was relocated to polygon 3, rather than correspondent points on polygon 1 and polygon 3. As was shown in his written work in Figure 14, Brian was able to successfully relocate polygon 1 to polygon 3 by acting on the polygon twice, rotating 90 degrees by point A and translating 5 units down and 1 unit to the right. However, the essential information of the geometric transformations—such as the direct relationship between polygon 1 and polygon 3, and relationship between correspondent points—were not made explicit in this “polygon talk.”

Excerpt 2: Discourse Featuring Operational Word Use with Operation on Point(s)

When working on the Translation Task at the pre-interview, Nancy did not immediately report what type of geometric transformations was involved. She observed the polygons and pointed out that the corresponding sides were all in the same direction; meanwhile, she used a pen to highlight the corresponding sides that were parallel. She then focused on how the translation works. She specifically stated that she focused on a pair of correspondent points, A and B, because all the points shifted in the same way.

Nancy: We could pick any point to find out how the transformation works. Since the whole thing (Polygon 1) is shifting, if we take A and B [Labels point A and point B], all the points along the polygon [Traces the four sides of polygon 1] would be transformed to this [Traces the four sides of polygon 2]. So if we focus on A being transformed to B, we can see that
it is shifted down six [Counts six units down] and over to the right eight [Counts eight units to the right].

Figure 15 Nancy’s written work for the Translation A Task at the pre-interview

Nancy’s discourse in this excerpt featured operational word use on point(s). Despite the occurrences of operational word use on polygon(s), the emphasis is operational word use on a point or a set of points. The annotated utterances from the excerpt below highlighted the operational word use on point(s):

[33] All the points along the polygon [Traces the four sides of polygon 1] would be transformed to this [Traces the four sides of polygon 2].

[34] So if we focus on A being transformed to B, we can see that it (point A) is shifted down six [Counts six units down] and over to the right eight [Counts eight units to the right].

Nancy’s discourse featuring operational word use on points revealed 1) the relationship between correspondent points A and B and 2) the relationship between one pair of correspondent points and the rest of the points on the polygons. Even though geometric transformations were described as an operation transforming one point to the other, the polygons in this discourse were
broken down into a set of points. The formation of the image polygon was the result of translating infinite preimage points on polygon 1. In fact, the attention to “points” allowed Nancy to make a connection to a coordinate system and to represent the translation as x+8, y-6.

The purpose of presenting these two contrasting excerpts is twofold. On one hand, the two excerpts present two types of operational word use embedded in participants’ discourses. On the other hand, by comparing the two participants’ discourses, the significance of distinguishing operational word use on polygon(s) and operational word use on point(s) was unpacked. As illustrated in Nancy’s example, discourse featuring operational word use on point(s) has a connection to other mathematical discourse, such as coordinate system. However, this is not to say that operational word use on polygon(s) should be replaced by operational word use on point(s), or that operational word use on points definitely leads to a correct answer. The distinction between operational word use with operation on polygon(s) and with operation on point(s) is that one is closer than the other to mathematical discourse. Indeed, according to Edward (2009), one salient difference between learner and mathematician in geometrical thinking is whether or not they consider points as the basic element in geometry. Mathematician can see lines, circles, triangles, and planes as a set of points, and can study lines, circles, triangles, and planes by examining the points on these elements.

In summary, I have presented the two different types of word use and their subcategories, and discussed their nature in terms of the goals of the instructional unit. In the next section, I present the changes in word use in preservice teachers’ discourses.

5.3 **Changes of Word Use in Preservice Teachers’ Discourses**

In the third phase of the analysis, the goal is to examine the change in the use of geometric transformation words in preservice teachers’ discourses and to attempt to understand
their learning. The patterns emerging at the group level were presented, and patterns emerging at
the individual level were examined to reveal the complex nature of change in word use.

Analysis revealed that there were three changes of word use in preservice teachers’
discourses from the pre-interview to the post-interview: 1) Word use shifted dramatically from
operational word use to structural word use; 2) Structural word use revealed more facets of
geometric transformations; and 3) Operational word use with operation on polygon(s) decreased
while operational word use with operation on point(s) increased.

5.3.1 Word Use Shifted Dramatically from Operational Word Use to Structural Word Use

The analysis of word use by comparing operational word use versus structural word use
revealed that there was a shift in word use from operational word use at the pre-interview to
structural word use at the post-interview. As shown in Table X, operational word use was
prevalent in preservice teachers’ discourses from the pre-interview (70.56%), and the structural
word use predominated in preservice teachers’ discourses from the post-interview (64.08%). The
change in word use from pre-interview to post-interview was statistically significant, $\chi^2(1, N =
442) = 52.45$, $p<0.05$. 
This shift from operational word use to structural word use after the intervention of the instructional unit is an expectation of the instructors. Before the instructional unit, preservice teachers’ discourses mainly presented geometric transformations as operations, and communicating geometric transformations as mathematical objects is an important goal of the instructional unit. Thus, the expectation of a decrease in operational word use by no means suggests that operational word use is not important or should not appear in preservice teachers’ discourses.

In fact, the trend of moving from operational word use to structural word use was found in all four preservice teachers, as illustrated in Table XI.
Chi-square test analysis revealed that the shifts of the word use in Nancy’s discourse ($\chi^2 (1, N = 80) = 40.896, p<0.05$), Owen ($\chi^2 (1, N = 106) = 9.723, p<0.05$), and Nathan ($\chi^2 (1, N = 155) = 33.29, p<0.05$) were statistically significant. A closer examination showed that the shifts in the three preservice teachers’ word use varied. For Nancy, the percentage of structural word use in her discourse at the pre-interview was relatively low (13.33%), while the percentage leap to a relatively high level (86.00%) at the post-interview. For Nathan, the percentage of structural word use in his discourse at the pre-interview was very low (5%), but the percentage reached only a medium level (49.47%) at the post-interview. For Owen, the percentage of structural word use in his discourse at the pre-interview was below the medium (37.04%). However, his
structural word use did not move to a relatively high level as Nancy’s did, instead moving to slightly above the medium level (66.67%).

In addition, even though Brian’s word use followed the same trend, the change was not significant. The percentage of structural word use in Brian’s discourse was above the medium at the pre-interview (58.49%) and post-interview (66.67%). This slight change might be due to the fact that the pre-interview with Brian was conducted after the first class, in which all kinds of geometric transformations were introduced. This indicated that the first class contributed substantially to the shift from operational word use to structural word use.

5.3.2 Structural Word Use Reveals More Facets of Geometric Transformations

The next step in this analysis is to focus on the occurrences of each subcategory of structural word use. Statistics analysis revealed that structural word use in preservice teachers discourses at the pre-interview was significantly different from that at the post-interview, \( \chi^2 (1, N = 215) = 14.35, p<0.05, \) which indicated that the presentation of the facets of geometric transformations in preservice teachers’ discourses differed from the pre-interview to post-interview. As table XII shows, there were two major patterns in regard to the appearance of the subcategories or facets. First, the changes in the frequency of each facet from pre-interview to post-interview—especially the facet of geometric transformations being acted on or having agency—suggested that preservice teachers were able to talk about more and more facets of geometric transformations. This is an important indicator of objectification, that is, treating geometric transformations as an object. In the pre-interview, the facet of geometric transformations being acted on and the facet of geometric transformations having agency only appear once, which suggested that only one or two preservice teachers talked about more than two facets of geometric transformations at the pre-interviews. The occurrences of the facets of
geometric transformations being acted on or having agency had increased in the post-interview, and all of the preservice teachers talked about more than two facets of geometric transformations. However, it is worth noting that, compared to other facets, the occurrences of the facets of geometric transformations being acted on or having agency were relatively low in both the pre-interview and post-interview. This indicated that the facets of geometric transformations being acted on or having agency were more abstract than the other two facets. Indeed, the action on geometric transformations (e.g., composite) or geometric transformations as an agent were not as visible or transparent as diagrams or parameters on the diagrams.

Second, the facet of geometric transformations as what is signified by the diagram was prevalent in preservice teachers’ discourses at the pre-interviews, while the facet of geometric transformations as objects that have parameter(s) was predominant at the post-interviews. This shift indicated that preservice teachers considered parameter(s) as an integral part of geometric transformations.
### TABLE XII

STRUCTURAL WORD USE IN ALL PRESERVICE TEACHERS’ DISCOURSES IN THE PRE-INTERVIEW AND POST-INTERVIEW

<table>
<thead>
<tr>
<th>Interview</th>
<th>Geometric transformations are what is signified by the diagram</th>
<th>Geometric transformations have parameter(s)</th>
<th>Geometric transformations can be acted on</th>
<th>Geometric transformations has agency</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Frequency</td>
<td>Percentage</td>
<td>Frequency</td>
<td>Percentage</td>
</tr>
<tr>
<td>Pre-interview</td>
<td>38</td>
<td>65.52%</td>
<td>18</td>
<td>31.03%</td>
</tr>
<tr>
<td>Post-interview</td>
<td>59</td>
<td>37.58%</td>
<td>77</td>
<td>49.04%</td>
</tr>
</tbody>
</table>
Next, I will unpack the patterns at the individual level. Due to the notably low occurrence in the facets of being acted on and having agency in some preservice teachers’ discourses (See table XII), Chi-square test analysis is not applicable to reveal the difference in preservice teachers’ discourses based on the subcategories in structural word use. Thus, I present descriptive analysis focusing on patterns shown in Table XIII.
TABLE XIII
STRUCTURAL WORD USE IN EACH PRESERVICE TEACHER’S DISCOURSES
IN THE PRE-INTERVIEW AND POST-INTERVIEW

<table>
<thead>
<tr>
<th>Pre-service teacher</th>
<th>Interview</th>
<th>Geometric transformations is what is signified by the diagram</th>
<th>Geometric transformations has parameter(s)</th>
<th>Geometric transformations can be acted on</th>
<th>Geometric transformations has agency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nancy</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pre-interview</td>
<td>2</td>
<td>50.00%</td>
<td>2</td>
<td>50.00%</td>
<td>0</td>
</tr>
<tr>
<td>Post-interview</td>
<td>17</td>
<td>39.53%</td>
<td>22</td>
<td>51.16%</td>
<td>1</td>
</tr>
<tr>
<td>Brian</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pre-interview</td>
<td>17</td>
<td>54.84%</td>
<td>12</td>
<td>38.71%</td>
<td>1</td>
</tr>
<tr>
<td>Post-interview</td>
<td>14</td>
<td>43.75%</td>
<td>15</td>
<td>46.88%</td>
<td>2</td>
</tr>
<tr>
<td>Owen</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pre-interview</td>
<td>16</td>
<td>80.00%</td>
<td>4</td>
<td>20.00%</td>
<td>0</td>
</tr>
<tr>
<td>Post-interview</td>
<td>14</td>
<td>40.00%</td>
<td>15</td>
<td>42.86%</td>
<td>0</td>
</tr>
<tr>
<td>Nathan</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pre-interview</td>
<td>3</td>
<td>100.00%</td>
<td>0</td>
<td>0.00%</td>
<td>0</td>
</tr>
<tr>
<td>Post-interview</td>
<td>14</td>
<td>29.79%</td>
<td>25</td>
<td>53.19%</td>
<td>6</td>
</tr>
</tbody>
</table>
The first pattern, of shifting from focusing on the facet of geometric transformations as what is signified by the diagram to focusing on the facet of geometric transformations as objects that have parameter(s), remained for each preservice teacher. As mentioned above, this shift might indicate that preservice teachers moved beyond the diagrammatic representation of geometric transformation and considered parameter(s) as an integral part of geometric transformations.

The second pattern, of communicating more facets of geometric transformations at the post-interview, held in all preservice teachers’ discourses except for Brian. Nancy, Owen, and Nathan only talked about two facets of geometric transformations at the pre-interview. At the post-interview, Nancy communicated the facets of geometric transformation as an object, which was evidenced in the following utterances:

• Then the reflection maps it (point A) to itself.

• This would be a transformation, a translation (that) have T which would be (-6, 8), that takes P to P’.

• So the transformation would be a dilation because it takes the polygon to a similar polygon but not a congruent one.

In these utterances, the reflection, translation, and dilation are considered as agents that can produce operation on points. Despite the small number of the occurrences, the four narratives illustrated the multiple facets of geometric transformations in Nancy’s discourses. Similar narratives were found in Owen’s and Nathan’s discourses.

However, this shift to presenting more facets was not found in Brian’s discourse. Brian was able to present all four facets of geometric transformations at the pre-interview. The utterances, “It's the same rotation basically” and “That is a 90-degree rotation that would produce
this image,” indicated that he talked about geometric transformation as an object that can be compared and an object that has agency. At the post-interview, one more instance of geometric transformations as objects was presented in Brian’s discourse, which was geometric transformations as objects that can be acted on. Brian articulated that reflection is an isometry, which indicated he could treat reflection as an object that has a generic relationship to isometry. As discussed before, the slight change might be due to the fact that the pre-interview with Brian was conducted after the first class, in which all kinds of geometric transformations were introduced.

5.3.3 Operational Word Use with Operation on Polygon(s) Decreased while Operational Word Use with Operation on Point(s) Increased.

Despite the shift from operational word use to structural word use, it is worth exploring the changes in operational word use because the two subcategories emerging from preservice teachers’ discourses—operational word use with operation on polygon(s) and operational word use with operation on point(s)—shed light on whether preservice teachers’ discourses became more mathematical. A deep analysis of operational word use in four preservice teachers’ discourses showed that the occurrence of operational word use with operation on polygon(s) decreased, and the occurrence of operational word use with operation on point(s) increased. From the pre-interview to the post-interview, operational word use with operation on polygon(s) decreased from 82.01% to 67.05%, and operational word use with operation on point(s) increased from 17.99% to 32.95% (See table XIV). The change from pre-interview to post-interview in operational word use is statistically significant, \( \chi^2 (1, N = 227) = 6.66, p<0.05 \).
Despite of the significant change in operational word use from the pre-interview to the post-interview, the shift from operational word use with operation on polygon(s) to operational word use with operation on point(s) was not observed. As shown in Table XIV, the operational word use with operation on polygon(s) predominated in preservice teachers’ discourses in both the pre-interview (82.01%) and the post-interview (67.05%). A closer examination of each preservice teacher’s discourse is presented below to unpack the prevalent appearance of operational word use with operation on polygon(s).

At the individual level, Chi-square test analysis revealed that the Owen’s and Brian’s discourses had significant changes in operational word from pre-interview to post-interview (Owen \(\chi^2 (1, N = 51) = 12.35, p<0.05\), and Brian \(\chi^2 (1, N = 38) = 4.34, p<0.05\)). Owen’s operational word use showed the shift from operation on polygon(s) to operation on point(s) (See table XV). His operational word use focused on polygon(s) at the pre-interview (70.59%) and moved to focus on point(s) at the post-interview (82.35%). For Brian, operational word use in his discourse was dominated by operation on polygon at the pre-interview (81.82%), and had equal occurrences of operation on polygon(s) and operation on point(s) at the post-interview. It is
worth noting that Brian’s discourse did not show any significant changes, except for this shift in operational word use. As discussed before, the non-significant changes might be the result of conducting Brian’s pre-interview after the first class of the unit. However, the change in operational word use might indicate that later classes had contributed to his shift from operational word use with operation on polygon(s) to operational word use with operation on point(s).

<table>
<thead>
<tr>
<th>Preservice teacher</th>
<th>Interview</th>
<th>Operational word use with operation on polygon(s)</th>
<th>Operational word use with operation on point(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nancy</td>
<td>Pre-interview</td>
<td>25</td>
<td>96.15%</td>
</tr>
<tr>
<td></td>
<td>Post-interview</td>
<td>6</td>
<td>85.71%</td>
</tr>
<tr>
<td>Brian*</td>
<td>Pre-interview</td>
<td>18</td>
<td>81.82%</td>
</tr>
<tr>
<td></td>
<td>Post-interview</td>
<td>8</td>
<td>50.00%</td>
</tr>
<tr>
<td>Owen*</td>
<td>Pre-interview</td>
<td>24</td>
<td>70.59%</td>
</tr>
<tr>
<td></td>
<td>Post-interview</td>
<td>3</td>
<td>17.65%</td>
</tr>
<tr>
<td>Nathan</td>
<td>Pre-interview</td>
<td>47</td>
<td>82.46%</td>
</tr>
<tr>
<td></td>
<td>Post-interview</td>
<td>42</td>
<td>87.50%</td>
</tr>
</tbody>
</table>
In addition, Nathan’s operational word use did not show significant change from the pre-interview to the post-interview. In fact, Nathan’s discourse showed the opposite trend in operational word use. This might indicate that Nathan’s discourse is not full-fledged mathematical discourse.

Given only one occurrence of operational word use with operation on point(s) in Nancy’s discourse, Chi-square test is not applicable to test the change in Nancy’s operational word use. In fact, Nancy’s operational word use showed the tendency of decrease in operational word use with operation on polygon(s) from the pre-interview (n=25) to the post-interview (n=6). And there was no change in the number of occurrences of operational word use with operation on points from the pre-interview to post-interview (n=1). This might indicate that operational word use with operation on polygon(s) transitions to structural word use, without needing a transition to operational word use with operation on point(s). Indeed, as previous analysis showed, Nancy’s structural word use was at a relatively high level at the post-interview.

5.4 Summary

The three research questions addressed in this chapter are: “What are the geometric transformation words used in preservice teachers’ discourses on geometric transformation?” “What are the different ways of using geometric transformation words that emerge from preservice teachers’ discourses?” and “How does the use of geometric transformation words in preservice teachers’ discourses change over time?” Geometric transformation words are the words that can be used to signify geometric transformations (reflection, rotation, translation, dilation).

Preservice teachers used various words to denote geometric transformations, which include formal words and informal words. Among the formal words, some were used to denote
an everyday-life object or other mathematics notion. In addition, there are two types of informal words: specific informal words (e.g., turn, flip) and general informal words (e.g., move, shift). General informal words could be used to denote multiple geometric transformations, and were usually accompanied by gesturing, with diagrams to indicate the path of movement involved in the geometric transformations.

Structural word use and operational word use were examined in preservice teachers’ discourses. Structural word use means that geometric transformations word were used to describe a geometric transformation as an object that has mathematical structure or properties (e.g., “Geometric transformations are mappings between points”). With structural word use, preservice teachers were able to communicate different facets of geometric transformations: a) geometric transformations are what is signified by the diagram; b) geometric transformations have parameter(s); c) geometric transformations can be acted on; and d) geometric transformations have agency. Operational word use indicates that the geometric transformation words are used to describe a geometric transformation as a procedure or action (e.g., “This polygon would be reflected over that line”). There are two kinds of operations: operations on polygon(s) and operations on point(s).

From pre-interview to post-interview, word use in preservice teachers’ discourses shifted dramatically from operational word use to structural word use. More specifically, the change in structural word use revealed that preservice teachers were able to communicate more facets of geometric transformations, while the change in operational word use indicated that operations on polygon(s) decreased for three of the preservice teachers whose discourses had 50% or more structural word use.
6. ROUTINES OF IDENTIFYING GEOMETRIC TRANSFORMATIONS

This chapter presents the results to address the two research questions on routines: What are the routines emerging from preservice teachers’ discourses on geometric transformation? And how do routines in preservice teachers’ discourses change over time?

Recall that routines are repetitive patterns when interlocutors are engaging in mathematical activities, or a set of rules regulating the repeated discursive activities. Routines can be observed in any act of communication, and routines found in a full-fledged mathematical discourse show mathematical ways of attending to the situations and solving the problems. As mentioned in the data analysis process, I focused on the routine of identification emerging from preservice teachers’ discourses, because the tasks provided great opportunities for preservice teachers to engage in the activity of identifying geometric transformations.

Specifically, the two research questions on routines are answered separately through the examination of determination activity and specification activity:

For determination activity, what are determination routines emerging from preservice teachers’ discourses on geometric transformation? And how do determination routines in preservice teachers’ discourses change over time?

For specification activity, what are specification routines emerging from preservice teachers’ discourses on geometric transformation? And how do specification routines in preservice teachers’ discourses change over time?

The first section presents results from the analysis of determination routines. First, different determination routines emerged from preservice teachers’ discourses. Detailed descriptions and transcripts from interviews are presented to illustrate each determination routines. In addition, this section presents the analysis of the determination routines in terms of
the mathematical ways of attending to the situation and solving problems, to understand the nature of each determination routines. Lastly, patterns of change in determination routines in preservice teachers’ discourses are examined at the group level and at the individual level to address the third research question. The second section follows a similar structure to present the results about specification routines.

6.1 Determination Routines

6.1.1 Types of Determination Routines in Preservice Teachers’ Discourses

Determination routines are the rules that regulate the process of determining the type of a geometric transformation. Forty episodes of determining the type of a geometric transformation were identified as the four preservice teachers worked on five tasks at the pre-interview and post-interview (4 participants x 5 tasks x 2 trials). There were four types of routines prominent in preservice teachers’ discourses for determining the type of a geometric transformation: perception-based, partially property-based, property-based, and theorem-based determination routines.

6.1.1.1 Perception-Based Determination Routine

Preservice teachers who used a perception-based determination routine relied solely on perception to determine the type of a geometric transformation. Consistent with the perception routine for identifying shapes in Sinclair and Yurita’s (2008) and Wang’s (2011) studies, the perception-based routine involves only a visual reading of the diagram without the support of other acts, such as counting or defining.

An example of a perception-based routine occurred during the pre-interview of Nathan. While working on the Reflection Task, Nathan reported that the target transformation is a
reflective transformation along a line. He said, “I just picture these,” and drew a line in between the polygons. I asked how he knew.

Interviewer: How did you know about that?"

Nathan: I took early geometry. Right away when I saw it, I said, “Ok, it is reflective upon a point like these.” I just knew right away.

Figure 16. Nathan’s written work for the Reflection Task at the pre-interview

The perception-based routine was identified in Nathan’s utterances. First, the utterance “Right away when I saw it, I said, ‘Ok, it is reflective upon a point like these’” indicated that the determination of the reflection was based on what Nathan visually read from the diagram. Second, the utterance “I just knew right away” suggested the act of naming the reflection was self-evident. In this case, there was no mention of specific information about the diagram.

However, with the perception-based routine, preservice teachers could provide specific information about the diagram, such as relationships between the polygons from the diagram, but what is read is self-evident. An example of this situation was found in Brian’s solution of the Reflection Task at the pre-interview. After reading the question, Brian looked at the diagram for
about thirty seconds. He then named the geometric transformation he recognized as a reflection. Next, he proceeded to find the reflection line. He drew a line segment freehand, which connected points B and B’. He then visually estimated the distance between B and B’ to find the midpoint of the line segment BB’. Brian used a rule to draw a line (l) through the midpoint perpendicular to the line segment BB’ (Shown in Figure 17). I asked him how he knew it was a reflection. He explained:

Brian: It is actually how I look at it. I could think of it as rotation, but then, the... I guess what I look at is the angle here [Marks and labels \(\angle A\)], and here [Marks and labels \(\angle A'\)]... Since these lines (AB and AB’) are oriented opposite to each other, basically, then, it has to be a reflection, instead of to be a rotation...

Brian: I think I saw it as reflection by the fact that it is inverted, and they are not the same image of each other.

Figure 17. Brian’s written work for the Reflection Task at the pre-interview

The routine that guided Brian in determining that the transformation was a reflection is classified as a perception-based routine for three reasons. First, the utterance, “It is actually how
I look at it” indicated he drew on the approach of visual observation. Second, the utterances “…these lines (AB and AB’) are oriented opposite to each other …” and “…it is inverted, and they are not the same image of each other” described the relationship between line segments AB and AB’ and the relationship of the shapes. Nevertheless, the observation was not supported by other acts, such as defining opposite position between line segments or defining inverted shape. Third, the utterance “basically, then, it has to be a reflection, instead of to be a rotation” revealed that he named the geometric transformation based on comparing his visual observations with visual prototypes of reflection and rotation.

As shown in both Nathan’s and Brian’s cases, the perception-based determination routine was effective in solving the tasks, and perception is important to the learning of geometry. However, perception-based determination routine does not involve mathematical ways of solving the problem (e.g, using mathematical properties, using mathematical theorems) and is not a legitimate routine in full-flag mathematics discourse. Thus it is not the ultimate goal of the instructional unit.

6.1.1.2 Partially Property-Based Determination Routine

A partially property-based routine refers to a case where a preservice teacher determined the type of a geometric transformation based in part on properties of the geometric transformations that were verified by an act of counting, measuring, defining, and so forth. Here, the term “properties” includes shape-preserving, angle-preserving, orientation-preserving, orientation-reversing, and the number of fixed points. In fact, a partially property-based routine involves a visual reading of general information (e.g., it looks like a rotation) or of properties from the diagram, followed by a verification of some of the properties from the visual reading.
Owen provided one example of partially property-based routine. When solving the Rotation Task A at the post-interview, Owen looked at the diagram for fifteen seconds, and then started to label the polygons ABCD and A’B’C’D’. Next, he softly spoke the labels for the vertices in the order A-B-C-D while pointing to the vertices with the pencil. After this, he drew an arrow inside the polygon ABCD (Shown in Figure 18A). He repeated the procedure for polygon A’B’C’D’. I asked him what the arrow meant:

Interviewer: What do the two arrows represent?
Owen: I was just checking the orientation.

[Look at the diagram]
Interviewer: What are you thinking?
Owen: Well, I know it is not a reflection because it (reflections) changes orientation. Rotation. I just don’t see how I could rotate to get there [Traces an imaginary curve (Shown in Figure # [b]) back and forth from one polygon to the other.] I feel like if I rotated it [Traces an imaginary curve (Shown in Figure 18B) from polygon ABCD to polygon A’B’C’D’], this would be like upside down. Translation [Traces an imaginary straight line from polygon ABCD to polygon A’B’C’D’ Shown in Figure 18C].

Interviewer: You see a translation?
Owen: Wouldn't work.

Interviewer: Why?
Owen: I just don’t know how it (translation) would turn the shape [Grabs a imaginary polygon ABCD and turns it]. Maybe it is a rotation around some point and then a translation.
Owen first claimed that the target transformation was a rotation, and then he corrected it as a rotation around some point, and then a translation. The partially property-based routine was evident in his process of geometric transformation determination. First, Owen indicated in his drawing that the orientations of the polygons were the same. Orientation-preserving is one of the properties of rotation. In addition, the orientation-preserving property was verified by the act of reading out the labels of the vertices along with the gesture of pointing to the vertices as their labels were read. However, no other properties were indicated, and orientation-preserving is not a unique property to rotations. This suggested that Owen also drew on visual reading to determine that the geometric transformation included a rotation. Moreover, the determination of the geometric transformation included a translation using a visual reading, because no properties of translation were identified and verified.
The key characteristic of the partially property-based determination routine is that mathematical properties of geometric transformations are used to verify perceptual observations. This routine requires preservice teachers to go beyond perceptual observations and attend to the mathematical properties. This is true for the property-based determination routine that will be discussed in the coming section. Compared to the partially property-based determination routine, property-based determination routine requires the presence of sufficient mathematical properties.

6.1.1.3 Property-Based Determination Routine

A property-based routine refers to a case where a preservice teacher determined the type of a geometric transformation based on sufficient set of verified properties of the geometric transformations. Again, the properties of geometric transformations included shape-preserving, angle-preserving, orientation-preserving, orientation-reversing, and the number of fixed points.

An example of the property-based routine occurred during Brian’s post-interview on the Rotation Task A. Brian started by counting squares to measure the side lengths of the polygons and checking the shapes of polygons. Then, he labeled the vertices to check orientations.

Brian: I am going to start with counting to make sure that they are the same side and the same shape. We see this segment length is 4 [Labels the length of segment AB]; this segment length is 4 [Labels the length of segment A’B’]; Segment length 8 [Labels the length of segment DC], segment length 8 [Labels the length of segment D’C’]. Label these vertices as well, ABCD [Labels vertices A, B, C, D following a clockwise direction].

[TURNS the worksheet 90 degrees so that the parallel sides of polygon A’B’C’D’ were horizontal. Follows the same clockwise direction to label vertices A’, B’, C’, D’.]

Brian: By the fact that its orientation is preserved but the position of the figure has
been changed, as well as its rotation has been changed. That is, this side length DC, which is 8, which is horizontal in the original figure [Traces line segment DC] is now vertical in the other figure [Traces line segment D’C’]. So, I would conclude that this is a rotation of 90 degrees.

Figure 19. Brian’s written work for the Rotation A Task at the post-interview

Brian indicated that the corresponding sides were the same, the sizes of the polygons were the same, the orientation was preserved, and the rotation was preserved. Recall that Brian used the word “rotation” in two ways: 1) whether the shape was rotated or not to indicate parallelism; and 2) rotation as a geometric transformation. The word “rotation” was used here to describe whether the shape was rotated. This is the property-based routine of determination. Brian found the lengths of two pairs of corresponding sides to determine the sizes of the shapes. Note that counting the lengths of two pairs of correspondent sides is not actually sufficient for comparing sizes. Nevertheless, given the fact that the polygon was drawn on grid paper and the focus of the unit was not on congruent shapes, I considered that Brian intended to verify the sizes of shapes. Furthermore, Brian verified the orientation-preserving property by labeling vertices of
the two polygons in the clockwise direction and verified that corresponding sides were non-parallel: “That is, this side length DC, which is 8, which is horizontal in the original figure [Traces line segment DC] is now vertical in the other figure [Traces line segment D’C’].” In sum, the determination routine that Brian employed in the Rotation Task A was a property-based determination routine.

6.1.1.4 **Theorem-Based Determination Routine**

Preservice teachers who used the theorem-based determination routine determined the type of a geometric transformation by reasoning deductively from an endorsed narrative, that is, a theorem. The theorem-based determination routine is similar to the discursively-mediated routine for identifying shapes (Sinclair & Yurita, 2008). However, there is a slight distinction between the theorem-based determination routine and the discursively-mediated routine of identification. The theorem-based determination routine requires the appearance of endorsed narrative (e.g., theorem), but does not necessarily involve explicit deductive reasoning statements or structures as the discursively-mediated routine does. For example, a preservice teacher might indicate the theorem (s)he used, but (s)he might not provide specific deductive reasoning statements or structures such as “because…” or “given that…”

Brian provided the following example of the theorem-based determination routine at the post-interview. Brian began by identifying the reflection as soon as he looked at the diagram. He then explained that it was a reflection because the orientation had been reversed.

In response to my request for further explanation, Brian labeled the vertices on the polygons, taking into account corresponding points, and examined the slopes of the line segments by tracing them. Then, he gestured above the polygons to indicate that the orders of the vertices on the polygons were counterclockwise and clockwise. If he had stopped at this point
and identified the target transformation, it would have been categorized as the property-based routine. However, his utterance, “From class, that's two different kinds of transformations,” indicated he used a narrative that was endorsed in class, namely that isometrics can be categorized into two kinds of transformations in terms of orientations. Therefore, Brian verified orientation-reversing by gesturing counterclockwise and clockwise to denote the orders of the vertices of the polygons, and he determined the transformation was a reflection by reasoning deductively using the theorem learned in class. I refer to this routine as the theorem-based determination routine.

With theorem-based determination routine, mathematics theorems are included to justify or warrant the determination process. This routine requires preservice teachers to not only attend to the mathematical properties of the geometric transformations, but also to make clear the theorems used in their reasoning.

6.1.2 Changes of Determination Routines in Preservice Teachers’ Discourses

6.1.2.1 Changes of Determination Routines at the Group Level

The examination of the change in routines of determination in preservice teachers’ discourses aimed to understand what they had learned. The patterns emerging at the group level were presented, and patterns emerging at the individual level were examined to reveal the complex nature of change in routines of determination.

In analysis of the change in routines of determination, frequency and percentage are used to discover the trends. Given the small numbers of occurrences of some routines, statistical analysis—such as the Chi-square test—is not applicable to confirm the significance of the change in determination routine.
As was shown in Figure 20, the most notable change in determination routine was the shift from perception-based routine to property-based routine. At the pre-interview, perception-based routine occurred 17 times, which was 85% of the total routines. At the post-interview, the dominant routine was the property-based routine, which appeared 10 times and was 50% of the total. This shift was a positive change and suggested that preservice teachers’ discourses became more mathematical discourses. In addition, all but the perception-based routine had increased number of occurrences at the post-interview, which indicated that there were multiple trajectories of changes in determination routines.

Figure 20. Number of occurrences of each determination routine in all preservice teachers’ discourses
Despite the notable positive shift, there were two aspects that did not show desirable changes. First, the theorem-based determination routine appeared only once, in Brian’s discourse at the post-interview. Theorem-based determination routine is an appropriate goal for secondary preservice teachers. However, it did not occurred as frequently as expected. This might be due to the nature of the task, which asked preservice teachers to identify geometric transformations, but rather focused on proving or deductively reasoning. In addition, perception-based determination routine was still the second-most-prevalent routine. It which appeared six times and was 33.33% of the total.

### 6.1.2.2 Changes of Determination Routines at the Individual Level

In fact, the same pattern—that perception-based determination routine dominated at the pre-interview, and property-based determination routine prevailed at the post-interview—was found in all four preservice teachers, except Nathan. In what follows, I examined the determination routines in each preservice teacher’s discourse.

![Figure 21. Number of occurrences of each determination routine in Nancy’s discourse](image-url)
As indicted in Figure 21, the overall determination routines used in Nancy’s discourse shifted toward more mathematically sophisticated routines. Besides the pattern found above, only perception-based routine (n=4) and partially property-based routine (n=2) were found in Nancy’s discourse at the pre-interview. As discussed before, these routines involve limited or some mathematics knowledge. At the post-interview, there was no occurrence of perception-based routine. Partially property-based routine (n=2) and the property-based routine (n=4) were found during the post-interview. In fact, all the perceptions-based routine at the pre-interview shifted to partially property-based routine or property-based routine. And the only partially property-based routine shifted to property-based routine. In addition, there is an important observation about Nancy’s performance and discourse on the Reflection Task; Nancy’s final answers had no change from the pre-interview to the post-interview. However, the analysis of Nancy’s discourse revealed that the routine of determination in her discourse moved from a perception-based determination routine to a property-based determination routine, which is a substantial change in terms of mathematics learning.

![Number of occurrences of each determination routine in Brian’s discourse](image)

Figure 22. Number of occurrences of each determination routine in Brian’s discourse
Brian’s routine use was diverse with a focus on the perception-based routine (n=3) at the pre-interview. He used perception-based routine for the Reflection Task, Rotation A Task, and Rotation B Task. Partially property-based routine was used for the Translation Task and the property-based routine was used for the Dilation Task. During the post-interview, Brian’s routine use was still diverse with a focus on the property-based routine (n=3). He used property-based routine for the Reflection Task, Rotation A Task, and Translation Task. The routine used for the Rotation Task was still the perceptual routine he used in the pre-interview, and the routine for the Dilation Task was still the property-based routine. For Brian, three out of five routine uses moved to more mathematical routine. One routine use remained low and one routine use remained high. In addition, the similar observation about Nancy’s performance and determination routine on the Reflection Task was found in Brian’s. The final answers that Brian provided to the Reflection Task were the same, but the determination routine in his discourse moved from a perception-based determination routine to a theorem-based determination routine, which is an important indication of learning.

![Figure 23](image-url)

**Figure 23.** Number of occurrences of each determination routine in Owen’s discourse
Owen used only the perception-based routine at the pre-interview. The use of the routine was diverse, with the focus on property-based routine (n=3) at the post-interview. Besides property-based routine, Owen used partially property-based routine for the Rotation A task. The routine for the Rotation B Task was still perception-based. In sum, four out of five routine uses shifted to more mathematical routine, and one routine used remained the same. In addition, for the Reflection Task, the determination routine in his discourse moved from a perception-based determination routine to a property-based determination routine, while his final answers to the task showed no change.

![Graph showing number of occurrences of each determination routine in Nathan’s discourse](image)

**Figure 24.** Number of occurrences of each determination routine in Nathan’s discourse

Nathan’s discourse did not show the same pattern concerning determination routine that was discerned at the group level. In fact, Nathan’s determination routine use did not show
substantial change. As illustrated in Figure 24, Nathan used the perception-based routine to
determine the type of geometric transformations during both pre-interview and post-interview,
with the exception of using property-based routine for the Dilation Task at the post-interview.

6.2 Specification Routines

6.2.1 Types of Specification Routines in Preservice Teachers’ Discourses

The specification routines are the rules that regulate the process of specifying the
parameters of a geometric transformation. Thirty-eight instances of specifying the parameter of a
geometric transformation were identified as the four preservice teachers worked on five tasks at
the pre-interview and post-interview. The other two instances involved misidentification of a
geometric transformation as a non-geometric transformation. Five types of routines emerged
from preservice teachers’ discourses on geometric transformations: perception-based, path-
based, convenient parameter, property-check, and construction-based specification routine.

6.2.1.1 Perception-Based Specification Routine

Perception-based specification routine refers to the case where a preservice teacher relied
solely on perception to determine the parameters of a geometric transformation. Similar to the
perception-based determination routine, the process of specifying the parameter merely drew on
visual reading from the diagram.

Below is an example of the perception-based specification routine. Working on the
Reflection Task at the pre-interview, Nathan reported that the target transformation is a reflective
transformation along a line. Then, he drew a line / in between the polygons. I asked how he knew
and how he determined the reflection line:
Interviewer: How did you know about that?

Nathan: I took early geometry. Right away when I saw it, I said, “Ok, it is reflective upon a point (line) like this.” I just knew right away.

Interviewer: Can you tell me more about how you determine this line?

Nathan: I have no idea how to term it. I just know that when you have two pictures are like these... I just know they’re reflected by this line right here that cuts in the middle of the X and Y-axis.

Figure 25. Nathan’s written work for the Reflection Task at the pre-interview

In this example, the identifying of geometric transformations can be separated as two processes. The first is determining the reflection by uttering, “It is a reflective transformation along a line.” The second is to specify the line of reflection by drawing line / between the polygons. The perception-specification routine involved in the process of specifying the parameter is evident in Nathan’s drawing action and his utterance. First, the appearance of the line was accomplished in a single drawing step. There was no other action, such as measuring, involved in determining the location of the line. Furthermore, the utterance, “I just know that when you have two pictures are like these. I just know they’re reflected by this line right here
that cuts in the middle of the X and Y-axis,” confirmed that Nathan found the reflection line by
drawing on perception.

It is clear that perception-based specification routine was effective in solving the task. However, the perception-based determination routine shown above mainly involves perceptual observation and freehand drawing, and does not involved mathematical ways of finding the reflection line, such as measuring the distance between correspondent points. Therefore, the perception-based specification routine is not a legitimate routine in a full-flag mathematics discourse, and it is not the ultimate goal of the instructional unit.

**6.2.1.2 Path-Based Specification Routine**

Path-based specification routine can be detected if the process of specifying parameter of a geometric transformation draws on a moving trajectory. Preservice teachers who used the path-based routine did not explicitly talk about the parameter of a geometric transformation, but focused on the path by which a geometric transformation changes the location of the points or the polygons.

Nathan used the path-based specification routine to solve the Translation Task. After the diagram was presented, he observed the polygons for 20 seconds and reported that:

Nathan: I would have no idea how to do this one. First thing I'm thinking, how did you get from here to here? [Points to polygon OPQR with his left hand, and points to polygon EFHG with his right hand.] Without knowing anything about transformations, I would say we moved Q down six units or squares [Moves finger from point Q to point A along QA]. From there, you moved it seven that way. [Moves finger from point A to point G along AG]. That's the only way I would be able to explain how you got from here to there [Points to polygon OPQR first, and then points to polygon EFHG]. You moved the polygon down 6, and then to
the right seven units [Put two hands over polygon OPQR as if holding the polygon, and move it down along QA and to the right along AG ].

Figure 26. Nathan’s written work to the Translation Task at the pre-interview

Nathan’s utterance, “First thing I'm thinking, how did you get from here to here?” indicated that his goal was to find the moving path by which polygon move from one location to the other. Next, his utterances “…we moved Q down six units/squares, …moved it seven that way” and “You moved the polygon down six, and then to the right seven units” revealed that he communicated the movement of the point or polygon. Moreover, his action of tracing the path by which point Q was moved to G, and the action of moving polygon OPRQ to follow the same path, confirmed that moving a shape from one position to another position in a certain trajectory was what he resorted to when specifying the translation. In fact, neither the translation nor its parameter was considered as an object in Nathan’s discourse.

While path-based specification routine works in Nathan’s case, path-based specification routine does not necessarily result in correct parameter. Owen used the path-based specification
routine at the pre-interview to find the parameters of a rotation. He first labeled the polygons ABCD and A’B’C’D’, then described the polygon ABCD as spun a certain way around some certain point. Meanwhile, he drew a curve $c$ to show how polygon ABCD was spun to get to polygon GHIJ. He then drew circle O and point O freehand, and reported that he could rotate point D around circle O. Later, I asked him to draw the circle for one more pair of corresponding points. He drew down circle M, which went through a pair of corresponding points C and I. Furthermore, he indicated that each pair of corresponding points had its own circle.

Figure 27 Owen’s written work for the Rotation A Task at the pre-interview
The path-based specification routine was evident in Owen’s drawings: 1) Curve $c$ indicating how one polygon could move to the other polygon; 2) Circle O as the trajectory for Point D and Point J; and 3) Circle M showing how Point C could move to Point I. Possible paths of moving were considered. However, the focus was not on trajectories particular for one pair of corresponding points, but rather a trajectory for all the points. Therefore, Owen did not find the parameters of the rotation.

Compared to perception-based specification routine, path-based specification routine does not merely draw on self-evident visual reading. It also includes the exploration of the path by which point(s) or polygon(s) move, which is a necessary steps in identifying geometric transformations. However, path-based determination routine does not result in the objectification of the geometric transformations and its parameters. A similar example of counting would be that one can count out “one,” “two,” “three,” but cannot tell how many objects in the set. Therefore, the path-based specification routine is not a legitimate routine in a full-flag mathematics discourse, and it is not the ultimate goal of the instructional unit.

6.2.1.3 Convenient Parameter Specification Routine

Compared to those who used the path-based specification routine, preservice teachers who used the convenient-parameter specification routine could explicitly talk about the parameter of certain geometric transformations. The process of specification involved randomly selecting convenient parameter(s) to define a certain geometric transformation, which was not the target geometric transformation.

Convenient parameter specification routine was seen when Nancy solved the Rotation Task A at the pre-interview. She first looked at the polygons for a while, and then she reported, “So it's rotated 90 degrees counterclockwise. And then shifted down 5 and to the right 1.” She
then clarified that she was not sure how to put the two geometric transformations into one transformation. After that, she started to show how to perform the two translations. She chose vertices A as center of rotation, and rotated polygon 1 by 90 degrees around vertex A to polygon 2. Next, she shifted polygon 2 down by 5 units and to the right 1 unit, so that polygon 2 lay onto polygon 3.

Figure 28 Nancy’s written work for the Rotation A Task at the pre-interview

In this case, vertex A is the convenient parameter that Nancy selected. She indicated vertex A as the center of the rotation and the angle of the rotation as 90 degrees. However,
rotation around vertex A by 90 degrees was not the target geometric transformation. Thus, she conducted a translation moving polygon 2 to polygon 3. In fact, given that the parameter was randomly selected, there was generally more than one transformation involved.

6.2.1.4 Property-Check Specification Routine

Property-check specification routine starts with drawing out the parameter of a geometric transformation based on perception, and usually ends with checking if the parameter drawn beforehand fulfills certain properties. Preservice teachers who used property-check specification routine would need to know the existence of the parameter and be familiar with some of the properties of the parameter.

At the post-interview, Nathan used property-check specification routine as he specified the reflection line. When he saw the Reflection Task, Nathan called the geometric transformation a reflection along a line right away, and then he drew a line with a rule. I asked him how he knew that it was a reflection along a line. His explanation remained the same as what he had provided in the pre-interview, which was perception-based. Then, I asked him how he determined the line:

Interviewer: Can you tell me, how did you determine this line?

Nathan: … the line needs to be equidistant of the preimage and the image, so by seeing it visually, I see that A is 1 unit away from the line of reflection [Draws segment AM]. Let's call this line L. And I know that A’ is reflected 1 unit from the line of reflection [Draws segment MA’], which is L…This one [Marks XN] congruent to this line [Marks NX’]. Assuming that the same would be true for the rest of the points, from the preimage to the image. That's how I determine my line of reflection.
In this example, to determine the reflection, Nathan reported one of the properties of the reflection line and then verified the reflection line drawn beforehand by checking if the line L had equal distance to corresponding points A and A’, as well as to X and X’.

It is clear that property-check specification routine was effective in solving the task. Property-check specification routine involves drawing out the parameters and checking if what is drawn fulfills certain properties, which is similar to guess and check. Property-check specification routine requires the knowledge about the geometric transformation and its parameters and the experiment of testing out the parameters, which indicated a certain degree of mathematics complexity involved. However, the property-check routine emphasizes the experiment of testing out the parameters, rather than the construction of the parameters, which is not the ultimate goal of the instructional unit.

6.2.1.5 Construction-Based Specification Routine

Construction-based specification routine refers to the appearance of a parameter of a geometric transformation resulting from constructing intersection points, or from constructing
lines from relevant, well-defined points. Construction is by no mean defined as compass and straightedge geometric construction in this study. Construction-based specification routine requires working from the corresponding points and making use of the relationship between corresponding points and the parameters, which is a routine in a full-flag mathematics discourse. Thus, it is the ultimate goal of the instructional unit.

Brian provided a construction-based specification routine when solving the Rotation B Task B at the post-interview. He first indicated that the polygons were parallelograms, and then he claimed that the geometric transformation was rotation. He pointed out that it was hard to count because one of the polygons was off kilter, so he decided that it was a rotation because it looked like one. When it came to specifying the center and the angle of the rotation, he explained:

...  
Brian: I'm going to use the fact that in a rotation, there is a central point, and that a source point (preimage point) and its image (image point) both lie on the outside of a circle around that center point. First, I'm going to find the midpoints of these [Points to line segments AA’ and BB’]. I think that I'm going to do it not very rigorously and by measuring them. That would probably give me enough information in this case at least. So this segment is at least 3 inches long, so I place a mark at 1 and 1/2 inches. [Measures line segment AA’ and draws the midpoint of AA’]. This segment is just under 4 inches long by 1/16th of an inch, so I'll place a mark at 2 inches less half of a 1/16th inch mark. [Measures line segment BB’ and draws the midpoint of BB’].

Now I'm going to draw imprecise perpendiculars by lining up these lines with marks on my ruler [Matches a tick mark on the ruler with the midpoint of the line segment AA’].
To specify the center of rotation and angle of rotation, Brian indicated that correspondent points lay on the circle center at the center of the rotation. Based on that, he connected two pairs of correspondent points AA’ and BB’, and then found the midpoints on line segments AA’ and BB’ by measuring their length. Next, he drew perpendicular lines through the midpoints with a ruler, and labeled the intersection points of the perpendicular bisectors as the centers of rotation O. Brian then connected OD and OD’ to measure the angle of rotation \( \angle DOD’ \) with a protractor.

First, Brian’s utterance, “I'm going to use the fact that in a rotation, there is a central point, and that a source point (preimage point) and its image (image point) both lie on the outside of a circle around that center point,” indicated he was aware of the relationship between the corresponding points and the parameters of the rotation. Second, center of the rotation is defined by the intersection of two perpendicular bisectors to line segments connecting corresponding points. Similarly, the rays forming the center of rotation were the perpendicular bisectors that
were defined by the line segments connecting corresponding points. Nevertheless, how Brian made use of the relationship between corresponding points and parameters to construct the parameters was not explicit. Despite the fact that it was worth further investigation, it was not a general-specification process, but rather a deeper-layer phenomenon involved as a step in the specification process.

6.2.2 Changes of Specification Routines in Preservice Teachers’ Discourses

The change in specification routines in preservice teachers’ discourses was examined with an attempt to understand what they learned. The patterns emerging at the group level were presented, and patterns emerging at the individual level were examined to reveal the complex nature of change in routines of specification.

As with determination routine, the examination of the change in routines of specification in preservice teachers’ discourses aimed to understand what they had learned. The patterns emerging at the group level were presented, and patterns emerging at the individual level were examined to reveal the complex nature of change in routines of specification.

In analyzing the change in routines of specification, frequency and percentage are used to discover the trends. Given small numbers of occurrences of some routines, statistical analysis, such as the Chi-square test, is not applicable to confirm the significance of the change in specification routine.

6.2.2.1 Changes of Specification Routines at the Group Level

As was shown in Figure 31, the most notable change in specification routine was that diverse routine use at the pre-interview shifted to construction-based routine. Specifically, the perception-based specification routine occurred three times at the pre-interview. Perception-based specification was not as commonly used as the perception-based determination routine,
which indicated that the process of specifying a parameter could be more complex than the process of determination. Visually reading from the diagrams does not provide sufficient information to find the parameter.

Figure 31. Number of occurrences of each specification routine in all preservice teachers’ discourses

The path-based specification routine (n=6) was found only in the pre-interview. Recall that preservice teachers who used a path-based specification routine might not be aware of the existence of the parameter as an important component of a geometric transformation. The zero occurrence of path-based specification at the post-interview might suggest that preservice
teachers can communicate about the parameters explicitly after the intervention of the instructional unit.

The convenient parameter specification routine was found five times at the pre-interview and two occurrences at the post-interview. All five occurrences took place as construction-based specification routine at the post-interview. Property-check specification appeared three times during the pre-interviews and two times during the post-interview. Among the three, two remained the same at the post-interview. The construction-based specification routine was found seventeen times out of thirty-eight total instances across the eight interviews. All but one occurred during the post-interview. Overall, these changes indicated that preservice teachers could use mathematically sophisticated routines after the instructional unit.

6.2.2.2 Changes of Specification Routines at the Individual Level

Again, the shift of moving to construction-based determination routine prevailing at the post-interview was found in all four preservice teachers except Nathan. In what follows, I examined the determination routines in each preservice teacher’s discourse.

During the pre-interview, Nancy used perception-based specification routine for the Rotation B Task, path-based specification routine for the Translation Task, property-check specification routine for Reflection Task and Dilation Task, as well as convenient parameter specification routine for Rotation A. She could use construction routine for all the tasks at the post-interview. Although the specification routine in Nancy’s discourse moved from the routine of specification in her discourse moved from a property-check specification routine, this substantial change in learning was not reflected in her final answers to the task.
Figure 32. Number of occurrences of each specification routine in Nancy’s discourse

Brian’s specification routine use was similar to Nancy’s specification routine use (Shown in Figure 33). During the pre-interview, Brian used various routines for different tasks. He used path-based specification routine for the Translation Task, convenient parameter specification routine for Rotation Tasks, and Dilation Task, and construction routine for the Reflection Task. He could use construction routine for all the tasks at the post-interview.
Only path-based routine and property-check routine occurred in Owen’s discourse during pre-interview. Owen used path-based routine for the Rotation A Task and the Rotation B Task, and property-check routine for the Reflection Task. Recall that Owen determined only the type of geometric transformation for the Reflection Task, the Rotation A Task, and the Rotation B Task at the pre-interview. Therefore, there was no specification routine found in the Translation Task and the Dilation Task. During the post-interview, he mainly drew on the construction routine with the exception of using property-check routine for the Dilation Task. In addition, a similar observation about Nancy’s performance and specification routine on the Reflection Task was found in Owen’s. The final answers that Owen provided to the Reflection Task were the same, but the specification routine in his discourse moved from a property-check specification routine to a construction-based specification routine, which is an important indication of learning.
Only perception-based routine and path-based routine occurred in Nathan’s discourse during pre-interview. Nathan used perception-based routine for the Reflection Task, the Rotation A Task, and the Rotation B Task, and used path-based routine for the Translation Task and Dilation Task. During the post-interview, Nathan used property-check routine for the Reflection Task; used convenient parameter routine for the Rotation A Task and the Rotation B Task; and used construction-based routine for the Dilation Task and the Translation Task. Despite the diverse routine use at post-interview, all routines used at the pre-interview shifted to more mathematically sophisticated routines. Specifically, for the Reflection Task, the specification routine in his discourse moved from a perception-based determination routine to a property-based determination routine, while his final answers to the task showed no change.
6.3 Summary

The two research questions addressed in this chapter are, “What are the routines emerging from preservice teachers’ discourses on geometric transformation?” and “How do routines in preservice teachers’ discourses change over time?” Identification routines were examined, which include determination routines and specification routines.

Determination routines are the rules that regulate the process of determining the type of geometric transformations. There were four types of routines prominent in preservice teachers’ discourses for determining the type of a geometric transformation: perception-based, partially property-based, property-based and theorem-based determination routines. From pre-interview to post-interview, the most notable change in determination routine was the shift from perception-based routine to property-based routine. However, theorem-based routine, which requires the

![Figure 35. Number of occurrences of each specification routine in Nathan’s discourse](image-url)
most sophisticated mathematical knowledge, did not occur as frequently as expected, and perception-based routine was the second most prevalent routine at the post-interview.

The specification routines are the rules that regulate the process of specifying the parameters of a geometric transformation. Five types of routines emerged from preservice teachers’ discourses on geometric transformations: perception-based, path-based, convenient parameter, property-check, and construction-based specification routine. From pre-interview to post-interview, the specification routines shifted to the construction-based routine.
7. NARRATIVES

In this chapter, I present preservice teachers’ narratives about geometric transformations to address the research questions: “What narratives emerged in preservice teachers’ discourses in geometric transformations?” and “How do the narratives in preservice teachers’ discourses change over time?” Narrative is “any text, spoken or written, that is framed as a description of objects, or of relations between objects or activities with or by objects, and that is subject to endorsement or rejection, that is, to being labeled as true or false” (Sfard, 2007, p.572, Italics in original). Narratives could be definitions, theorems, axioms, and so on. During the interviews, preservice teachers provided a great number of narratives about each type of geometric transformations. These narratives were examined through the lens of operational word use and structural word use in Chapter 5. As the analysis of geometric transformation words and their use has shown, preservice teachers’ discourses on specific geometric transformations were mostly presented as an operation or the process of an operation. In the investigation of narratives, I considered preservice teachers’ working definitions of geometric transformations, including their working definitions of each type of geometric transformations. Working definition refers to a temporary definition that preservice teachers used for a certain situation. The working definition from individual preservice teachers may be under development and not fully aligned with the definitions endorsed by the mathematics community. Thus, preservice teachers’ working definitions of geometric transformations provide a window to explore their discourses on the mathematical object.

The first section presents different working definitions in preservice teachers’ discourses. Detailed descriptions and transcripts from interviews are presented to illustrate each determination routine. In addition, the goal of the analysis of working definitions is to explore to
what extent the working definition in of individual preservice teachers aligned with the definition endorsed by the mathematics community (i.e. A transformation of a plane can be informally defined as the procedure that moves all the points in the plane.). The second section describes the change in working definition in each preservice teacher’s discourse to address the second research question.

7.1 Working Definition as Endorsed Narrative in Preservice Teachers’ Discourses

In preservice teachers’ discourses, geometric transformations were defined as 1) any change to a given figure; 2) reflection and rotation; 3) transformations preserving shape and proportionality; 4) one-to-one and onto functions.

7.1.1 Geometric Transformations as Any Change to a Given Figure

As Brian was solving the problem about the shear transformations in the pre-interview, I asked for a working definition of a geometric transformation. Brian stated, “I think of transformation as any change to a given figure.” Thus, he attempted to find a set of rules for performing the Shear Task. This set of rules was provided in his written work on the following figure:
Brian wrote, “Only points A and B have been translated 10 units to the right, in addition, the segments AC and BD have to be moved according to the translation of their endpoints.” This description of the Shear Task suggests that Brian considered geometric transformations as any change, and thought different points could follow different rules of change. For example, points A and B were translated 10 units to the right, while points on segments AC and BD were determined, respectively, by the location of points A and C, and the location of points B and D.

In the post-interview, Brian reported a working definition of geometric transformations similar to the one he mentioned in the pre-interview. In addition, he stated that a useful geometric transformation preserves size, shape, and angle. As he was solving the Shear Task, he commented, “Yes [there is a geometric transformation], We changed what the figure looks like. We did something to some of it at least. But I don't think it is necessarily a useful transformation.
in most of the cases because it doesn’t preserve any of the things that we usually want to
preserve.” Thus, he claimed this type of geometric transformations is not useful transformations,
and therefore cannot be described by a general rule.

Defining geometric transformations as any change to a figure allowed Brian to claim that
the Shear Task involved a geometric transformation and led him to explore a set of rules for
changing a figure. The lack of the idea that geometric transformations change points in a
consistent way resulted in his failure to solve the Shear Task. The same working definition of
geometric transformations as any change was also found in Nathan’s discourse from pre-
interview and post-interview.

7.1.2 Geometric Transformations as Reflection and Rotation

During the pre-interview, after solving the Reflection Task and the Rotation Tasks, Owen
started working on the Dilation Task. He looked confused and did not do or say anything for
more than thirty seconds. Then, I posed the question, “What is the definition of geometric
transformations you are using?” He reported, “I just know them as a rotation or a reflection.” In
his narrative about geometric transformations, he mentioned two specific, defined types of
transformations. The fact that he considered geometric transformations as rotation and reflection
showed a failure to identify translation, dilation, and shear. For example, when working on the
Translation Task, he stated that, “I'm just trying to think of any way of rotating it or reflecting it.
I'm not really getting anything.”

Owen provided a working definition of a geometric transformation by listing two types of
geometric transformations he was familiar with. Defining geometric transformation as rotation
and reflection might suggest that Owen was confused about the concept of geometric
transformations and the concept of symmetry. Geometric transformations are actions on the
plane, while symmetry is the properties of figure. There are four kinds of geometric transformations introduced in K-12 curricula, while only two kinds of symmetry are discussed in K-12 curricula: reflection symmetry, that is, line of symmetry, and rotation symmetry (i.e., point symmetry).

7.1.3 Geometric Transformations as Transformations Preserving Shape and Proportionality

During the Shear Task portion of the pre-interview, when I requested a working definition of a geometric transformation, Nancy reported that geometric transformations should preserve proportional sides and congruent angles, but could change location and size. She further clarified that if a rectangle were transformed into a parallelogram, then the action or change is not called a geometric transformation. Nancy said, “A transformation is still similar but it doesn't have to be congruent,” by which she meant to communicate that under a geometric transformation, the preimage and image geometric figures could be similar or congruent.

In sum, Nancy provided a working definition of a geometric transformation by describing some of the properties of a certain geometric transformation. Her working definition included isometries and dilation as geometric transformations, but not shear transformations. Thus, she claimed that there was no geometric transformation involved in the Shear Task. Isometries and dilation are the types of geometric transformations introduced in K-12 curriculum. The working definition describing geometric transformation in terms of shape preserving and proportional was also found by Owen at the post-interview.

7.1.4 Geometric Transformations as One-to-One and Onto Functions

Looking at the Shear Task at the post-interview, Nancy hesitated and stated that, “I wonder if this is a transformation or not.” I then asked for the definition of geometric
transformations that she was using. She reported, “A transformation takes a point to another point. Specifically, a transformation is one-to-one and onto. It doesn't have to be an isometry. It doesn't have to preserve the distance.” Though Nancy mentioned the properties of a geometric transformation as she did at the pre-interview, she included essential information that a transformation is one-to-one and onto: that is, a geometric transformation is an onto and one-to-one function.

Next, Nancy checked the distance between correspondent points and recorded her observation with symbol notations. Then, she checked where the different points of the preimage were mapped to the points on the image and recorded the result with the function notations as shown in the following figure. She then stopped.

![Figure 37. Record of distance in Nancy’s written work for the Shear Task at the post-interview](image)
Figure 38. Record of correspondent points in Nancy’s written work for the Shear Task at the post-interview

Based on the fact that she used function notations to record the relationship between the corresponding points, I suggested Nancy further unpack the point using coordinates of the point, but not just using the label to record the points as recording her observations. Nancy defined point D as the origin (0, 0), and found the coordinates for all the vertices (Shown in Figure 39).

Figure 39. Points with coordinates in Nancy’s written work for the Shear Task at the post-interview
After that, Nancy listed the corresponding points with the coordinates as shown in Figure 40. Nancy reported a pattern that the y-coordinates for preimage point and image point did not change. Later, she reported that the x-coordinate of the image point was related to both the x-coordinate and the y-coordinate of the preimage. Then, she wrote down a general rule that revealed the relationship between preimage point and image point, as shown in Figure 41:

Figure 40. Correspondent points with coordinates in Nancy’s written work for the Shear Task at the post-interview

Figure 41. Rule in Nancy’s written work for the Shear Task at the post-interview
Using the working definition of geometric transformations as onto and one-to-one function, Nancy recorded the relationship between corresponding points with function notation. With my suggestion, Nancy found out the coordinate for each vertex and determined the rule of geometric transformation in the Shear Task.

7.2 Changes of Endorsed Narratives in Preservice Teachers’ Discourses

Two trends of change in endorsed narrative in preservice teachers’ discourses were observed. As shown in Table XVI, one trend is that working definitions changed dramatically from the pre-interview to the post-interview, such as in Nancy’s and Owen’s discourses. The other trend is that working definitions changed slightly from the pre-interview to the post-interview, such as in Brian’s and Nathan’s discourses.

<table>
<thead>
<tr>
<th>Working Definitions</th>
<th>Nancy</th>
<th>Brian</th>
<th>Owen</th>
<th>Nathan</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometric Transformations as Any Change to a Given Figure</td>
<td>PRE &amp; POST</td>
<td>PRE &amp; POST</td>
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<tr>
<td>Geometric Transformations as Reflection and Rotation</td>
<td>PRE</td>
<td></td>
<td></td>
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<tr>
<td>Geometric Transformations as Transformations Preserving Shape and Proportionality</td>
<td>PRE</td>
<td></td>
<td>POST</td>
<td></td>
</tr>
<tr>
<td>Geometric Transformations as One-to-One and Onto Functions</td>
<td></td>
<td>POST</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
In Nancy’s and Owen’s discourses, working definitions at the pre-interview and the post-interview were not the same. Nancy defined geometric transformations as transformations preserving shapes and proportionality at the pre-interview, while she was able to define geometric transformations using the concept of function. Owen’s working definitions of transformations included only reflection and rotations at the pre-interview, while his working definitions expanded to include translation and dilation. Even though he was not able to provide a working definitions from a function perspective as Nancy did, his working definitions were qualitatively changed from listing examples to focusing on properties.

In Brian’s and Nathan’s discourses, the overall working definitions were the same at the pre-interview and the post-interview, while some nuanced narratives were added at the post-interview. As presented above, Brain defined geometric transformations as any change, both at the pre-interview and the post-interview. Compared to the pre-interview, he was able to distinguish transformations based on some of the properties, such as preserving size, shapes, and angles.

7.3 **Summary**

The two research questions addressed in this chapter are: “What narratives emerged in preservice teachers’ discourses in geometric transformations?” and “How do the narratives in preservice teachers’ discourses change over time?” One kind of narrative, working definitions, was salient in preservice teachers’ discourses. Therefore, working definitions were examined to address the research questions.

There were four different working definitions of geometric transformations in preservice teachers’ discourses: 1) geometric transformations as reflection and rotation; 2) geometric
transformations preserving shape and proportionality; 3) geometric transformations as any change to a given figure; and 4) geometric transformations as one-to-one and onto functions.

As to the change in preservice teachers’ discourses, Nancy—who defined geometric transformations in terms of some of their properties—was the only preservice teacher who could explicitly talk about geometric transformations as one-to-one and onto functions. Owen’s working definitions changed from listing examples of some geometric transformations to focusing on the properties that some of geometric transformations preserved. Brian and Nathan, who used the working definition of geometric transformations, did not change remarkably in how they defined geometric transformations from pre-interview to post-interview.
8. DISCUSSION AND CONCLUSIONS

In this chapter, I return to the needs that motivated this study and the questions that guided my analysis. First, it is the need to develop comprehensive knowledge about geometric transformations in secondary preservice teachers, which is a requirement of successful implementation of the Common Core State Standards for Mathematics (2010). Second, it is the need for research on secondary preservice teachers’ knowledge about geometric transformations. Because the existing research focused either on high school students, or on elementary or middle school preservice teachers, it might have not confirmed to what extent the results can be generalized to learners at different levels. Moreover, the existing research did not highlight the uniqueness of secondary preservice teachers. Third, there is the need for research from multiple perspectives of learning. Previous research investigated the students’ learning of geometric transformations primarily based on the assumptions of a cognitive framework that highlighted the nature of students’ mental representations of geometric transformations. Research from other perspectives is needed to provide informative knowledge about the preservice teachers' learning of geometric transformations.

The first section of this chapter provides a summary of results to address each research question about discourse on geometric transformations: What is the nature of secondary preservice teachers’ mathematical discourses on geometrics transformations, and how do their discourses change over time? The second section is devoted to situating some of these results in order to respond to the three needs. The third section provides implications of this study for the field of mathematics education and teaching geometric transformations, and also discusses future research questions.
8.1 **Summary of the Results**

Results from this study show that various types of word use (e.g., operational word use, structural word use), routines (e.g., identification routines, specification routines), and narratives (e.g., geometric transformations as reflection and rotation; geometric transformations as one-to-one and onto functions) emerged from preservice teachers’ discourses on geometric transformations, which illustrated the complex nature of their discourses. In addition, preservice teachers’ discourses on geometric transformations shifted toward an objectified and mathematically sophisticated direction at the end the instructional unit. Specifically:

- The occurrences of formal geometric transformation words increased to more than 90% of all the geometric transformations words used in preservice teachers’ discourses.
- Word use in preservice teachers’ discourses shifted dramatically from operational word use to structural word use.
- The change in structural word use suggested that preservice teachers were able to communicate more facets of geometric transformations, and to focus on the parameters of geometric transformations.
- Operational word use on polygon(s) decreased for three of the preservice teachers, whose discourses were at a relatively high level of objectification.

Four determination routines were prominent in preservice teachers’ discourses for determining the type of geometric transformations: perception-based, partially property-based, property-based, and theorem-based determination routines. From pre-interview to post-interview, the most notable change in determination routine was the shift from perception-based routine to property-based routine.
Five specification routines emerged from preservice teachers’ discourses on geometric transformations: perception-based, path-based, convenient parameter, property-check, and construction-based specification routines. From pre-interview to post-interview, preservice teachers' specification routines showed a trend of shifting to the construction-based routine.

The changes in working definitions in preservice teachers' discourse showed two trends. One trend was that working definitions transformed dramatically from the pre-interview to the post-interview. The other trend was that working definitions did not show profound change, but presented nuanced improvement.

Despite of the existence of variances among individuals, the overall trends at the group level showed that: 1) This is a preliminary but important sign of getting more familiar with geometric transformations as a mathematical phenomenon, rather than just a real-life phenomenon, because the occurrence of formal geometric words increased after the instructional units; 2) Preservice teachers were able to communicate geometric transformations toward an objectified direction. This means that preservice teachers enhanced their knowledge of geometric transformations (i.e., Geometric transformations are mathematics objects that have unique elements, features and structures such as parameters, properties, or functions.), rather than merely knowing how to conduct a set of procedures resulting from a certain geometric transformation; 3) Preservice teachers were able to use more sophisticated routines when identifying geometric transformations. This indicates not only that preservice teachers could solve the problems presented during the interview, but also that the ways in which they solved the problems presented mathematical sophistication; and 4) most of the preservice teachers’ working definitions were not yet fully aligned with the endorsed definitions in the mathematics community, but the differences between the working definitions from pre-interview to post-
interviews did indicate encouraging changes. This suggests that preservice teachers were more familiar with the geometric transformations as mathematics objects and knew more properties of geometric transformations.

Observing trends at the group level aimed to provide a general landscape of preservice teachers’ discourses and what they had in common. Besides these overall changes, changes at the individual level can add some detail to this general picture. In the following section, I summarize each preservice teacher's discourse with an attempt to unpack each preservice teacher’s knowledge of and learning about geometric transformations.

8.1.1 Nancy’s Mathematical Discourse about Geometric Transformations

Nancy’s discourse not only showed the most notable changes among those of the four participants, but also became highly objectified, highly sophisticated and fully aligned with discourse within a mathematics community. Nancy’s discourse showed significant changes in all the major categories within the discourse elements. The use of geometric transformation words in Nancy’s discourse shifted dramatically from operational word use to structural word use. The identifying routines in her discourse all moved to ones that involved high levels of sophistication. Most importantly, she was able to communicate geometric transformations as one-to-one and onto functions. Indeed, Nancy’s discourse after the instructional unit provided an example of full-fledged mathematical discourse about geometric transformations.

8.1.2 Brian’s Mathematical Discourse about Geometric Transformations

Brian’s discourse changes were not as dramatic as Nancy’s. In fact, the level of objectification in Brian’s discourse was relatively high at the beginning of the unit, and his discourse became more sophisticated. His discourse changes were mainly found in operational use and identification routines. The operational word use shifted from focusing on polygon(s) to
focusing on point(s), while the routines in Brian’s discourse were diverse low-level routines, and moved to diverse high-level routines, or all high-level routines. In addition, the degree of alignment between Brian's working definition and an endorsed definition in the mathematics community was lower than that found in Nancy’s discourse. These results suggest that Brian’s discourses after the instructional unit were relatively objectified and highly sophisticated, but not yet a full-fledged mathematical discourse about geometric transformations. Despite that, the changes documented here might not truly reflect Brian’s change, due to the interview schedule.

8.1.3 **Owen’s Mathematical Discourse about Geometric Transformations**

Owen’s changes in discourse were found in every category within each discourse element. His structural word use moved to a relatively high level, as Brian’s discourse did. Owen's routine use became more sophisticated. However, compared to Nancy’s and Brian’s routine use, Owen had two routines with a lower lever of sophistication. In addition, although Owen’s working definition did not show as high an alignment as Nancy’s discourse, his working definition qualitatively changed from listing examples to focusing on properties of geometric transformations. Overall, Owen’s discourses after the instructional unit were relatively objectified and relatively sophisticated, but not yet full-fledged mathematical discourses about geometric transformations.

8.1.4 **Nathan’s Mathematical Discourse about Geometric Transformations**

Nathan’s discourse was at a less-objectified and less-sophisticated level before the instruction unit, which could have been a hurdle for his discourse to arrive at the same levels as other participants’ discourses. However, positive and encouraging changes were found in Nathan’s discourse. His word use became more objectified and was at a medium level. Moreover, all his specification routines were transformed into more sophisticated ones. It is
interesting that Nathan’s determination routines did not show the encouraging changes that his specifying routines did. It can be posited that Nathan could specify the parameters in a more mathematical way, but still relied on perception to determine geometric transformations. After the instructional unit, Nathan’s discourse did become more objectified and more sophisticated.

8.2 Discussion

Situating some of the results within the context of research and teaching, this section aims to react to the three needs that gave rise to current study. First, I answer the question, “What have we learned about secondary preservice teachers’ knowledge and learning from their discourses?” in order to respond to the need for research on secondary preservice teachers’ knowledge about geometric transformations. Then, I address the question, “What did the discursive framework offer?” to react to the need for research from a discursive perspective.

Meanwhile, the need for developing comprehensive knowledge about geometric transformations is addressed by answering the question “How can the results be used to support teachers developing comprehensive knowledge about geometric transformations?” In this section, the answers to this question were integrated with the responses to the previous two questions. Systematic response to this question is presented in section 8.3.3.

8.2.1 What Have We Learned About Preservice Teachers’ Knowledge and Learning From Their Discourses?

Claim 1: Geometric transformations words in secondary preservice teachers’ discourses indicate the complexity in their discourses or knowledge.

Despite the encouraging increase in occurrences of formal geometric transformations words, I observed instances of formal geometric transformations words suggesting the complexity of preservice teachers’ discourses or knowledge. One was that formal geometric
transformations words were used to denote everyday concepts and geometric transformations, such as in Owen’s case. This finding seems to be consistent with other research, which has shown that everyday discourse and mathematical discourse are not separate, but interwoven (Forman, 1996). The other instance was when Brian borrowed a formal word of geometric transformations (rotation) to denote another mathematical notion (whether a shape was rotated). The complex nature of preservice teachers’ discourse requires teacher educators pay attention to the meaning of the words used by preservice teachers. The meaning of a word can be unpacked by observing accompanying gestures, drawings and symbols, or by posing further questions such as, “can you give another example of reflection?” or “what do you mean when you say that the rotations of the shapes are the same?”

Claim 2: Secondary preservice teachers do not always convey their knowledge of geometric transformations via verbal or written communications.

Preservice teachers may use gestures to convey what they are not yet ready to communicate in verbal or written form. The observations about the general informal words (e.g., move, shift) show that the occurrences of general informal words were usually accompanied by gestures that indicated the path of the general movements. The use of general informal words with gestures suggests that preservice teachers were familiar with the phenomenon of geometric transformations, but were not yet able to make their knowledge about geometric transformations explicit in verbal or written form. This is by no means to argue that the use of gestures was always associated with inadequate knowledge. In fact, the use of gesture, speech, and writing in a complementary way (McNeill, 1992) can be frequently observed in the discourses of preservice teachers who have comprehensive knowledge about geometric transformations, such as in Nancy’s discourse. Therefore, I argue that preservice teachers should be encouraged to use...
gestures in order to make their premature thinking visible, and communicate their knowledge in various forms of communication.

**Claim 3: Secondary preservice teachers can communicate geometric transformations as mathematical objects.**

Secondary preservice teachers can communicate geometric transformations as mathematical objects after the instructional unit, despite their inadequate prior knowledge making the topic of geometric transformations challenging. This finding differs from previous findings that high school students—as well as elementary and middle school preservice teachers—seldom consider geometric transformations as mathematics objects (Edward, 1991; Edward & Zazkis, 1993; Hollebrands, 2003, 2004; Ada & Kurtulus, 2010; Yanik, 2011). The knowledge of geometric transformations as mathematical objects is demonstrated in that: 1) preservice teachers’ discourses generally moved from operational word use to structural word use; 2) more and more facets of geometric transformations as an object were presented in preservice teachers’ discourses; and 3) the facets that dominated the discourses shifted from the facet of geometric transformations as what is signified in the diagram to the facet of geometric transformations having parameter(s).

However, secondary preservice teachers' prior knowledge about geometric transformations were similar to existing findings on high school students, as well as elementary and middle school preservice teachers. This was evidenced by the appearances of the specific informal words (e.g., flip, turn) in secondary preservice teachers’ discourses. And the use of specific informal words was also found in previous studies (Edward, 1991; Edward & Zazkis, 1993; Yanik, 2011). Another example of evidence was that secondary preservice teachers’ discourses were dominated by operational word use. In addition, secondary preservice teachers’
discourses were mainly mediated by visual diagrams before the instructional unit, which was revealed in the analysis of the structural word use (i.e., the facets of geometric transformations as objects).

This finding about learners at all levels presenting similar prior knowledge about geometric transformations as operations is not surprising, because the goals in existing K-12 standards such as the NCTM standards are focused on the operational aspects of geometric transformations. In addition, I posit that the successful transition from geometric transformations as operations to geometric transformations as objects could benefit from secondary preservice teachers' advanced mathematics learning experience, which is a remarkable difference from other learners. The advanced mathematics learning experience might include—but is not limited to—in-depth study of functions or mappings, the study of linear transformations, and being exposed to a large number of advanced mathematical objects (e.g., parameter, group).

Claim 4: Secondary preservice teachers have difficulties communicating geometric transformations as functions or mappings.

Despite the encouraging transition in secondary preservice teachers’ discourses, the results from the analysis of working definitions indicate that secondary preservice teachers have difficulties in communicating geometric transformations as functions or mappings. Preservice teachers held a variety of views of geometric transformations: 1) geometric transformations as reflection and rotation; 2) geometric transformations preserving shape and proportionality; 3) geometric transformations as any change to a given figure; and 4) geometric transformations as one-to-one and onto functions. Only Nancy provided a working definition based on geometric transformations as functions. This finding was similar to what Hollebrands (2003) reported in her
study, that high school students were not yet able to consider geometric transformations as functions or mappings.

Why is it the case that secondary preservice teachers face the same challenges as high school students? I posit that functions or mappings are usually taught in courses such as linear algebra or calculus, and these courses usually focus on functions or mappings of numbers, number pairs, or vectors. The lack of direct connection between functions and geometric objects could be the main source of this challenge.

Claim 5: Secondary preservice teachers show various degrees of familiarity with the parameters of different types of geometric transformations.

The analysis of specification routines showed that preservice teachers employed a variety of specification routines based on their familiarity with the parameter(s) of geometric transformations. Reflection line was the most familiar parameter for preservice teachers. They were aware of the angle of rotation, but generally neglected the center of rotation. This was also true in the case of dilation, as the center of dilation was seldom mentioned. Preservice teachers struggled with the translation vectors before the instructional unit, but they could talk about the translation vectors with no doubt after the instructional unit.

8.2.2 What Did The Discursive Framework Offer?

Claim 6: Preservice teachers’ evidence of learning was made transparent from the discursive perspective.

Influenced by the cognitive perspectives, especially the APOS theory, learners are expected to “transfer their perceptual understanding to perform the transformations mentally.” This description of learning goal was not made clear. It states, “perform the transformations mentally,” but mental performance cannot be observed directly. However, the commognitive
framework is effective in capturing preservice teachers’ learning in terms of their participation in mathematical discourse. Also, the commognitive framework defines learning in terms of changes in observable elements of discourse, rather than implicit mental representations. These can make transparent preservice teachers’ evidence of learning. The results of types of word use, various routines, and working definitions are explicit indicators for observing preservice teachers’ learning of geometric transformations.

**Claim 7: Preservice teachers’ nuanced but substantial evidence of learning can be captured by taking a discursive approach.**

In addition to making preservice teachers’ evidence of learning explicit, the study from the commognitive framework also captured preservice teachers’ nuanced but substantial learning, which can provide a high-solution learning profile of what was learned. For example, Nathan’s performance on the Reflection Task had no change before and after the instructional unit—he was able to name the reflection and find the reflection line. However, Nathan’s discourse change revealed that there was substantial learning happening, as his routine of specification moved from a perception-based specification routine to a property-check specification routine.

### 8.3 Conclusions

This study aimed to advance our understanding of secondary preservice teachers’ knowledge of and evidence of learning on geometric transformations. By focusing on preservice teachers’ discourses, this study contributes to the filed of mathematics education in several ways.
8.3.1 Implications for Research

Besides adding to a picture of secondary preservice teachers’ knowledge and evidence of learning about geometric transformations, results from this study have immediate implications for the research on secondary preservice teachers’ learning of geometric transformations.

First, the results of the current study shed light on the similarities and differences between secondary preservice teachers and learners at other levels. That is, while secondary preservice teachers share similar prior knowledge about geometric transformations, they can develop comprehensive knowledge about geometric transformations and become able to communicate geometric transformations as mathematical objects. As mentioned before, secondary preservice teachers might have benefited from their learning experience in advanced mathematics courses. Therefore, the research agenda should include the study of what kinds of advanced mathematics learning experiences can support secondary preservice teachers’ learning of geometric transformations.

In addition, the results about types of keyword use, various identification routines, and working definitions have their roots in the commognition theory and go beyond what it has offered. For example, the analysis of keyword use did not stay at the level of operational word use and structure word use, and presented a deeper examination of the subcategories of the two types of word use. Also, routines of geometric discourse that have not yet reported (i.e., specification routines) were introduced.

Moreover, learning scientists have been devoted to investigating how people learn by merging the cognitive approach and the sociocultural approach. For example, Cobb and Yackel (1996) have drawn on the psychological constructivist perspective to analyze individuals’ constructive activities in social context and applied sociocultural perspectives to observe the
characteristics of classroom community and communities of practice in which individuals are situated. Some other researchers (Greeno, 2007; Gresalfi, 2009; Gresalfi et al. 2009) have attempted to add social elements into cognitive constructs (i.e., conception growth, competence, and disposition) by observing individual development in situated activities. Current study provides a promising alternative approach to understanding the nature of learning, employing the construct from the sociocultural/ situative perspective and drawing on protocol analysis (interviews) method from the cognitive perspective. Without ignoring the social nature of learning, this approach allows researchers to concentrate on the learning phenomena being studied and to collect adequate usable data by interacting with individuals in socially impoverished environments. The sociocultural construct, discourse, provides an explicit and direct link between individuals’ learning and learning in groups, classrooms, or communities. Therefore, researchers can unveil how learning happens by comparing individuals’ discourse with discourse of groups, classrooms, or communities.

8.3.2 Implications for Teaching Practice

The study also has some practical implications for teaching geometric transformations. The results of the current study can be used to support teachers’ learning and developing comprehensive knowledge about geometric transformations in the following ways:

First, the results of discursive patterns emergent from preservice teachers’ discourses about geometric transformations—such as keyword and its use, routines, and narratives—provide explicit indicators for observing learning. Specifically, teacher educator should attend to the following shifts or trends:

- An increasing use of formal words to denote geometric transformations;
- A shift from operational word use to structural word use;
• An increasing appearance of aspects of geometric transformations as mathematical objects;
• A shift from attending to polygons to both polygons and points as the pre-images and images of a geometric transformation;
• A shift toward using mathematically sophisticated routines; and
• A shift to describing geometric transformations as functions or mappings.

Indeed, these shifts and trends should not merely be used as assessment indicators. They should also be listed as learning goals guiding instructional design, and directions for leading classroom discussions.

Second, the results of determination routines reveals that preservice teachers seldom used theorem-based routines because they did not expect to do proofs within identifying tasks. This lack demonstrates the need for engaging in communications that moves preservice teachers to the theorem-based routine. Besides asking questions (“What transformations is it?” and “How do you know it is a reflection?”) to elicit the name and the properties of the reflection, other questions such as, “How do you know it is always reflection if the shapes preserves length or angles?” can be asked to guide learners to use theorems to back up their observations.

Third, the use of gestures was frequently observed in preservice teachers’ discourses on geometric transformations, and gestures could convey their knowledge of geometric transformations. Instead of ignoring learners’ gestures, teacher educators can attend to what their gestures “say” about the parameter(s) of a transformation, as well as the movement path of a point or a shape under a transformation.

Last but not least, this research shows that discourse is not just a tool for learning mathematics or a byproduct of mathematics learning. Mathematics is communication. Thus,
when encouraging learners to communicate mathematics, teacher educators should have an explicit goal guiding the changes in learners’ mathematical discourses. In the case of geometric transformations, preservice teachers should be encouraged to communicate geometric transformations as mathematics objects, and to employ routines that are legitimate in mathematics discourse, such as theorem-based determination routines and construction-based routines.

8.3.3 Future Study

This study documented the change in preservice teachers’ discourses before and after a five-week instructional unit, and revealed that learning of geometric transformations can be captured in terms of the change in discourses. However, this study leaves us with a list of important questions and ideas for further research.

First, to get a fuller picture of preservice teachers’ discourses on geometric transformations, researchers can explore the discourse by using tasks about representing or proving geometric transformations. This study of preservice teachers' discourses on geometric transformations is based on the identification task. This might be the reason for the low occurrences of geometric transformations as objects that can be acted on and as objects that have agency. Moreover, this study aims to provide a general landscape of preservice teachers’ knowledge across different types of geometric transformations (reflection, rotation, translation, dilation) without highlighting the variations among the discourses on each type of geometric transformations. As shown in the analysis of specifying routines, preservice teachers presented different degrees of familiarity with each type of geometric transformations, and future research is needed to fill this gap.
In addition, it is still unclear why the changes in preservice teachers’ discourse happened in relation to their course experience. This is a complicated phenomenon that needs further exploration. Based on classroom observations, I posit that instructors’ discourses and the interactions between preservice teachers could have great influence on the changes in preservice teachers’ discourses. In what follows, I present classroom episodes that can shed light on this conjecture and provide possible directions for addressing the why question.

Episodes 1 is from small group discussion, in which the four participants in this study worked collaboratively on the Launch Activity. The task in these two episodes involved rotation. Preservice teachers were asked to use transparencies and to identify geometric transformations from the M.C. Escher’s artwork “Symmetry Drawing E45,” in which one angel can be mapped onto other angels through reflection, rotation, and translation.

Episodes 1: Four participants identifying rotations from M.C. Escher’s artwork

Owen: So, what are we moving?

Brian: We are moving the transparency.

[Nancy was turning the transparency around a point fixed by a pencil.]

Owen: Ok, we can turn it.

[Nancy repositioned the transparency and started to turn the transparency so that the original angel mapped to the other angels.]
From Episode 1, it is clear that the discourse of the group was similar to individuals’ discourses described earlier. Preservice teachers mainly used operational words (i.e., turn, move), and there was only one instance of structural word (rotation). In addition, the fact that Nancy turned the transparency around a point fixed by a pencil indicates that preservice teachers could locate the center of rotation. However, they did not attend to the center or consider it as an integral part of rotation. This is evident in their utterances, “And each one of those is a rotation of...” and “90 degree.”

How could interactions between preservice teachers influence the changes in discourse? As Nancy’s and Brian’s discourses arrived at a high or relatively high level after the instructional unit, let us focus on their discursive activities in Episode 1. One observation is Brian’s role as leader, which evident in his utterances. For example, he initiated counting the number of rotations by saying, “This is one”, which was then taken up by Nancy and Nathan. Brian also launched the identifying activity by saying “And each one of those is a rotation of...,” which was completed by Owen and Nathan. Nancy also played an important role in leading the group discussion. She did not explicitly initiate conversations by asking question or proposing new
idea. However, her actions on the transparency gave rise to other members’ questions and comments. I posit that, it was the role as leader that contributed to the development of Brian’s and Nancy’s discourses. Indeed, research has shown that learning can be mediated by developing roles (Radinsky, 2008; van Es, 2009). Thus, future research can examine various roles emerging from group interactions, and explore how the roles interact with development of discourse.

How could instructors’ discourse influence changes in discourse? Episode 2 from the whole-group discussion on the same geometric transformations shed light on this question. In Episodes 2, Dr. P, the instructor, was eliciting preservice teachers’ responses on rotations.

Episodes 2: Instructor launching the discussion of rotations

Dr. P: To describe a rotation, you have to say, “Where is the center of rotation?”

Dr. P: What point do you rotate around? [Turning his palm around a certain point]? And how far do you rotate? [Turning his palm again]

Figure 43. Dr. P pointed to the center of a rotation.

Dr. P: So, what kind of rotations do you find for this picture? [Pointing to the artwork on a projector screen] Where is the center of rotation? [Pointing to several possible points]
Dr. P’s discourse in Episode 2 displays two important features that could support the development of discourse. One is that Dr. P’s transformations word use (i.e., rotation, center of rotation) was formal and structural at the beginning and at the end, which indicates Dr. P’s attempt to introduce formal mathematical discourse. The other is that Dr. P unpacked the formal mathematical discourse by using operational words and gestures, which connected preservice teachers’ discourses to a formal mathematical discourse. I posit that it was the instructor’s demonstrations of formal mathematical discourse and the connecting technique that contributed to the development of preservice teachers’ discourses. Thus, the nature of instructor’s discourse and the connecting techniques are worthy of further investigations.
REFERENCES


Calfee (Eds.), *Handbook of Educational Psychology* (pp. 15-41). New York, NY: MacMillan.


Portnoy, N., Grundmeier, T. A., & Graham, K. J. (2006). Students' understanding of mathematical objects in the context of transformation geometry: Implications for


APPENDIX 1

Student Background Survey

Math 411- Advanced Euclidean Geometry

Spring 2012

Student Information

Name:

Phone:

Email:

Your year in college:

Mathematics course you had taken in college:

Your favorite mathematics courses or topics:
Mathematics Identity: (Briefly describe yourself as a mathematics learner.)
APPENDIX 2

Transformations Tasks in the Content Survey

Give two examples of transformations of the plane. How do you know they are transformations? (That is, what definition of transformation did you use?)

If you compose two transformations, T1 and T2, is T1T2 always equal to T2T1? Please justify your answer.
APPENDIX 2 (continued)

Let \( A = (-4, 7) \). Find the image of \( A \) under the translation \( T_{2,-5} \). Show your work.

Find a translation that transforms the line determined by the equation \( 2y - x = 3 \) to a line through the origin.
APPENDIX 3

Interview Protocol

I’d like to thank you for meeting with me. I am studying how preservice teacher learn geometric transformations. The purpose of this interview is to help me to understand the way you think about geometric transformations.

To do that, I would like you to solve three tasks, and answer some questions related to the tasks. Your answer will not affect your grade in the course and I am not assessing whether you did the problems correctly or not. Please tell me what you are thinking when you are tackling the tasks, which will be very helpful for me to follow your think. After you thinking through the tasks, please record your responses.

I am going to present you three tasks, which have not covered in this class yet. But you might have learned it before. There are compass, straightedge, protractor, and graph paper. Feel free to use them.

Do you have any questions for me?

Let’s get started. I am going to present you Task 1 (Present preservice teachers Task 1). There are six diagrams here (Present six diagrams drawn on A5 paper in front of the PST). Please identify the transformations that map one polygon to another polygon. There is no particular order you need to follow. You can choose anyone you like to start with.
APPENDIX 3 (continued)

Task 1 (Identify Transformations):
Identify the transformations that map one polygon to another polygon in the following diagrams:

Diagram 1

Diagram 2

Diagram 3

Diagram 4

Diagram 5

Diagram 6

Reflection
- If the PST identifies reflection in the diagram, ask the following question:
  - How did you know that it was a reflection?
  - If PST did not specify the line of the reflection, ask: Can you tell me more information about the reflection?
    - If PST mentioned the line of the reflection, ask: How did you figure it out?
    - If PST did not mention the vector of the reflection, ask: What is the line of the reflection? How did you figure it out?
  - Can you show me how the reflection mapped one polygon to another polygon?
- If the PST did not identify translation as a transformation, ask: How did you know that it was not a transformation? What was the definition of a transformation you are using?
Translation

• If the PST identifies translation in the diagram, ask the following question:
  o How did you know that it was a translation?
  o If PST did not specify the vector of the translation, ask: Can you tell me more information about the transformation?
    ▪ If PST mentioned the vector of the translation in previous questions, ask: How did you figure it out?
    ▪ If PST did not mention the vector of the translation in previous questions, ask: What is the vector of the translation? How did you figure it out?
  o Can you show me how the translation mapped one polygon to another polygon?
• If the PST did not identify translation as a transformation, ask: How did you know that it was not a transformation? What was the definition of a transformation you are using?

Rotation

• If the PST identifies rotation in the diagram, ask the following question:
  o How did you know that it was a rotation?
  o If PST did not specify the center and angle of the rotation, ask: Can you tell me more information about the rotation?
    ▪ If PST mentioned the center and angle of the rotation in previous questions, ask: How did you figure it out?
    ▪ If PST did not mention the center and angle of the rotation, ask: What is the vector of the translation? How did you figure it out?
  o Can you show me how the rotation mapped one polygon to another polygon?
• If the PST did not identify rotation as a transformation, ask: How did you know that it was not a transformation? What was the definition of a transformation you are using?

Dilation

• If the PST identifies dilation in the diagram, ask the following question:
  o How did you know that it was a dilation?
  o If PST did not specify the center and angle of the rotation, ask: Can you tell me more information about the rotation?
    ▪ If PST mentioned the center of dilation and the scale factor, ask: How did you figure it out?
    ▪ If PST did not mention center of dilation and the scale factor, ask: What is the vector of the translation? How did you figure it out?
  o Can you show me how the dilation mapped one polygon to another polygon?
• If the PST did not identify dilation as a transformation, ask: “How did you know that it was not a transformation?” “What was the definition of a transformation you are using?”
Shear

- If the PST identifies shear as a transformation, ask them “how did you know that it was a transformation?” “What was the definition of a transformation you are using?” “Can you show me how the transformation mapped one polygon to another polygon?”
- If the PST did not identify rotation as a transformation, ask: “How did you know that it was not a transformation?” “What was the definition of a transformation you are using?”

Present preservice teacher Task 2:

<table>
<thead>
<tr>
<th>Task 2 (Transformation and Isometry)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the transformation given by $T(x, y) = (x+2, -y)$. Is $T$ an isometry (or congruence transformation, or rigid motion)? Prove you conjecture.</td>
</tr>
</tbody>
</table>

- Have the PST read Task 2 aloud and ask: Does the task make sense to you?
  - If the PST does not understand “isometry or congruence transformation or rigid motion”, explain to them: “an isometry is a transformation that map the original to a congruence one.”

Present preservice teacher Task 3:

<table>
<thead>
<tr>
<th>Task 3 (Composition of transformations and Isometry)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Describe the composition of two translations. Is the combination of two translations an isometry? Prove your conjecture.</td>
</tr>
</tbody>
</table>

- Have the PST read Task 3 aloud and ask: Does the task make sense to you?
  - If the interviewee does not understand “combination of two translations”, explain to them: “Given two translations $T_1$ and $T_2$, the combination of $T_1$ and $T_2$, $T_2T_1$, is the product of applying $T_1$ to the preimage and then apply $T_2$ to the image of the $T_1$.”