Mathematics as Meaning-Making Activity:
Describing Preservice Teachers’ Discourse during Meaning Making

BY

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THESIS

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This dissertation is dedicated to my mother who celebrated every small achievement and who always had confidence that one day I would finish my dissertation.
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LIST OF ABBREVIATIONS

CCSS-M  Common Core State Standards—Mathematics
IMAP    Integrating Mathematics and Pedagogy
LMT     Learning Mathematics for Teaching
NCTM    National Council of Teachers of Mathematics
PST     Preservice Teachers
Summary

This is a qualitative study exploring the characteristics of preservice teachers’ discourse while they engage in meaning making in the context of mathematical activity. This study employs an interpretive case study approach and includes such ethnographic tools as videotape, audiotape, and participant-observation. The data analysis incorporates processes derived from grounded theory (Glaser & Strauss, 1967) and draws on research related to various approaches to discourse analysis in the field of mathematics education (e.g., Cobb & Bauersfeld, 1995; Cobb, Boufi, McClain, & Whitenack, 1997; Hufferd-Ackles, Fuson, & Sherin, 2004; Sfard, 2001a, 2007). The need for this study is based on the fact that mathematics reformers have been calling for learning mathematics with understanding, which requires individuals to negotiate the meaning of the mathematics. Although research on elementary school students’ meaning making exists, there is little research on preservice teachers’ meaning making.

Results from this study indicate that preservice teachers engage in meaning making using conceptual talk—talk relying on conceptual coherence that connects mathematical ideas to explanations, explorations, and strategies. Throughout this study, observable meaning making never exists in the absence of gestures, including deictic gestures (pointing) and iconic gestures (related to the semantic concept they support). Preservice teachers build on each other’s ideas when they pick up language from classmates or when working together to come to consensus on language use for some presentation or written record of their work. These various results have possible implications for the design of preservice teacher courses that might enhance opportunities for meaning making.
1 Introduction

1.1 Background

Expectations for how mathematics is taught and even the general goals of mathematics education in the U.S. have evolved and changed over the last 20 years (Common Core State Standards Initiative, 2010; Fraivillig, Murphy, & Fuson, 1999; NCTM, 1989b, 1991, 2000). Mathematics educators have called for mathematics classrooms that are “communities of inquiry, a problem-posing and problem-solving environment in which developing an approach to thinking about mathematical issues would be valued more highly than memorizing algorithms and using them to get right answers” (Schifter, 1996, p. 495). A mathematics classroom should foster the development of ways of communicating and thinking about mathematics as well as supporting students in their development of understandings of mathematical concepts (cf. Common Core State Standards Initiative, 2010; Conference Board of the Mathematical Sciences, 2012; Hiebert et al., 1997; Kilpatrick, Swafford, & Findell, 2001). To establish classrooms with these characteristics, teachers need a range of skills and practices for teaching mathematics that they may not be familiar with and for which they may not have a pre-existing model (Ball, Lubienski, & Mewborn, 2001). Teachers’ development of these skills and practices may be further complicated because “by the time they begin professional education, teachers have already clocked more than 2,000 hours in a specialized ‘apprenticeship of observation’ (Lortie, 1975, p. 61), which not only has instilled traditional images of teaching and learning, but also has shaped their understanding of mathematics (Ball, 1988)” (Ball et al., 2001, p. 437).

Generally, in their own K-12 education, U.S. elementary school teachers have not had the opportunity to participate in mathematics classrooms like those they are expected to establish
One context for providing preservice teachers (PSTs) with models and images of teaching and learning in these types of mathematics classrooms is during their teacher preparation in undergraduate mathematics content courses. Although research from various perspectives and with diverse foci has explored the features and development of desired elementary school mathematics classrooms, much more needs to be known about teacher preparation for teaching in those classrooms. In particular, “[w]e need to know more about the nature and quality of subject matter preparation, including the impact on teacher learning of various instructional methods in high quality, undergraduate and graduate discipline-based education” (Wilson, Floden, & Ferrini-Mundy, 2001, p. 11). To address this need for research into teacher preparation for teaching subject matter, this study proposes to examine one facet of an elementary school preservice teacher mathematics content course, that of PSTs’ discourse during the process of meaning making in mathematical activity.

1.2 **Context of Evolving Classroom Expectations**

Although elementary school teachers cannot be experts in all of the subjects they teach, they must have sufficient knowledge of each subject to know what students are expected to understand and apply, and to teach the content and practices that are components of the subject. There are several ways teachers might come by this knowledge. For example, the school or district may produce a scope and sequence document that outlines content. Written curriculum materials adopted by a school or district may be the teachers’ guide to discerning what students should know in terms of content and practices. Standardized tests for each grade level may provide some information—although it is likely that standardized tests will be more narrowly focused than the expectations outlined in school or state documents. For a more complete
description, teachers may look to standards documents—district, state, or national. Finally, teachers might rely, at least to some degree, on their own experiences in school.

In a departure from how most teachers had engaged in school mathematics, the late 80’s witnessed the beginning of a new set of widely accepted standards that redefined the expectations for educating students to be mathematically proficient (Kilpatrick et al., 2001; NCTM, 1989b, 2000). Many state standards began to reflect the content of these national standards documents. One of the features of the new standards that had not existed previously in various state standards documents was the inclusion of standards not only for content, but also for the processes in which students should engage as they work on mathematical tasks. The process standards described how students should: communicate about mathematics; represent their thinking in a variety of ways; problem solve; reason about mathematical tasks and situations and prove or justify their reasoning; and make connections in their investigations between mathematical ideas and/or representations. Teachers who were required to teach content and process standards that had evolved considerably since the teachers themselves were in school generally had few personal resources to draw on—especially for knowing how to support students in developing competency with the process standards (D. K. Cohen, 1991; Spillane & Zeuli, 1999).

As the national focus on mathematics teaching has intensified, the U.S. government supported the development of a set of national standards The Common Core State Standards or CCSS-M (Common Core State Standards Initiative, 2010) that, following those documents developed earlier, continue to define basic mathematical proficiency as including both content

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1 Massachusetts standards from 2004, Michigan standards from 2004, and Minnesota standards from 2003 provide examples of states standards that incorporate some of the content of the national standards documents.
and processes (recast as practices). The expectations described in the CCSS-M, which have been adopted by most of the fifty states, require that elementary school teachers provide opportunities for students to develop proficiency in a broad range of mathematics content including: operations and algebraic thinking, geometry and measurement, data and statistics, number theory, ratios and proportional reasoning, expressions and equations, and functions.

In addition to this broad list of content domains, the Common Core State Standards call for teachers to explicitly provide students with opportunities to develop mathematical practices including making sense of problems, constructing and critiquing arguments, reasoning quantitatively, modeling problem situations with mathematics, using tools strategically, being precise, and identifying and using structure and regularity to solve problems (Common Core State Standards Initiative, 2010). These comprehensive standards describe what teachers are responsible for having their students know and be able to do. The next section of this chapter describes recommendations from research about what elementary school teachers need to know and be able to do in order to implement the CCSS-M for mathematical content and practices.

1.3 Mathematical Knowledge for Teaching

As a result of the widespread adoption of the Common Core State Standards, there has been a renewed call for identifying what teachers must know in order to teach mathematics so that students will meet the expectations laid out in the CCSS-M. One way this call has been answered is through revision of the Mathematical Education of Teachers, a national document produced by the Conference Board of the Mathematical Sciences (Conference Board of the Mathematical Sciences, 2001, 2012), outlining what teachers need to know to teach mathematics in alignment with the Common Core State Standards for Mathematics. In addition to the Mathematical Education of Teachers, various researchers have conducted research that provides
further and varied insights into what teachers need to know, how they need to know it, and what skills are required in order to teach mathematics and to implement the new national Standards.²

One way to investigate the knowledge necessary for teaching mathematics is to consider what knowledge mathematics teachers employ in their work. Ma (1999) took this approach to exploring the mathematics content knowledge teachers know and use by examining differences in how Chinese and American teachers apply their content knowledge to various teaching situations. She found, for example, that Chinese teachers drew on a deep understanding of operations in crafting questions to probe children’s errors in a multiplication algorithm. The Chinese teachers asked questions that emphasized underlying mathematical ideas, whereas their American counterparts, when presented with the same situation, tended to ask questions focused on the multiplication procedure itself. In addition, American teachers sometimes avoided questions altogether and simply demonstrated the desired procedural steps repeatedly to correct children’s errors. Ma described several mathematical situations that highlighted the more conceptual approach of Chinese teachers and the more procedural approach of American teachers. In order to meet the expectations of the Standards, however, American teachers will need to adopt a more conceptual approach to teaching, which may require knowledge and experiences that are not a part of their current repertoire.

Differences between how teachers approach applying their knowledge may be related to their differences in content knowledge, but these differences may also be related to teachers’ prior experiences while learning mathematics—that is the ways in which they engaged in mathematical activity during their own schooling. Stigler and Hiebert (1999) compared teachers

² From this point forward, Standards will be used to designate the Common Core State Standards.
from several countries, including Japan and the United States. In each country, particular patterns of teaching mathematics dominated most classrooms in which they observed—patterns that differed considerably from one country to the next. In the United States, for example, teachers tended to demonstrate procedures, have children practice the procedures, assign related homework problems, and then go over the assigned problems together with the class the following day. The Japanese teachers had a notably different pattern for teaching mathematics though. Generally, in Japanese classrooms, the teacher posed a problem for children to work on together and solve, and then the children shared the various strategies they used to solve the problem, explaining and comparing their methods.

In sharp contrast to what Japanese teachers may have experienced, it is likely that many American teachers learned in environments such as those described above for the U.S. classrooms. While U.S. teachers likely focused on the development of procedural knowledge with problems that involve recall, definition, and applying procedures, their Japanese counterparts may have had much more experience engaging in problem solving around problems involving making connections, and constructing and applying relationships among mathematical concepts and procedures. Though U.S. standards documents of the last twenty years have more broadly defined the mathematics content and practices children should have opportunities to learn, changes in mathematics classrooms in the United States have been slow in coming (D. K. Cohen, 1991; Jacobs et al., 2006; Tarr et al., 2008). One reason may be that cultural patterns in classrooms and in teaching tend to be perpetuated from generation to generation. Based on their research, Stigler and Hiebert (1998) propose that teaching is a cultural activity and hypothesize that how teachers teach is at least somewhat dictated by the culture established during their own past learning experiences.
By providing the context in which individuals develop their beliefs about what mathematics teaching and learning is and should look like, the classroom culture teachers participated in as learners might influence both what mathematical knowledge they identify as relevant and how they apply that knowledge to teaching. At San Diego State University the Integrating Mathematics and Pedagogy (IMAP) project has been investigating PSTs’ beliefs about mathematics teaching and learning (Philipp, 2007). Some of the reasons Philipp describes for viewing beliefs as critical to the preparation of PSTs are that beliefs influence perception (Pajares, 1992), shape how we interpret events (Grant, Hiebert, & Wearne, 1998), and influence the actions we take (Cooney, Shealy, & Arvold, 1998). The IMAP project has identified a set of productive beliefs that are well aligned with the opportunities teachers must provide for children in support of meeting expectations described in the Standards. These beliefs include: seeing mathematics as a set of connected concepts and procedures; recognizing that the ability to apply procedures does not necessarily imply understanding; taking the view that understanding concepts is more mathematically powerful than only knowing procedures; appreciating the importance of learning concepts before the related procedures; respecting that children can solve problems for which they have not been taught procedures; perceiving that children do not always think about mathematics in the same ways as adults and; realizing that when children explore mathematical ideas, children should do as much of the thinking as possible (Philipp et al., 2007). The research from the IMAP project has indicated that few PSTs arrive in their program with many, if any, of the productive beliefs the project has identified, which likely influences how

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3 More thorough discussion about mathematics teacher’s beliefs is beyond the scope of this manuscript. See Philipp’s chapter in the Second handbook of research on mathematics teaching and learning (Philipp, 2007) for more information.
the PSTs apply their knowledge as they engage in mathematical activity and how they initially
expect mathematical knowledge to develop for children.

Finally, to elaborate a more complete picture of what the relevant body of knowledge for
mathematics teaching consists of, research begun at the University of Michigan over twenty
years ago (Ball, 1988; Ball et al., 2001; Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling,
2008) has focused on more fully and specifically describing this body of knowledge. Researchers
in the Learning Mathematics for Teaching (LMT) project have developed and refined various
instruments that aim to identify the knowledge that teachers use in their work with students.
Their explorations of teachers’ applied mathematical knowledge has resulted in conceptualizing
knowledge for teaching mathematics as more than just *everyday content knowledge*—that is, the
mathematical knowledge that anyone might require, for example, to estimate if they have enough
money at the grocery store or to calculate how much money they have left in their bank accounts.
They propose that there are many facets to the knowledge teachers require and use. These
researchers are still broadening and refining their descriptions of the knowledge teachers must be
able to access and use in their teaching, but a second and fairly well-described type of
mathematical knowledge is something they call *specialized content knowledge*—for example,
being able to analyze how students figure out the solution to the problem of “if they have enough
money,” or to evaluate which student strategies might offer rich learning opportunities if shared
with the class and how to have students share their strategies to maximize those opportunities.
The LMT project continues to develop ideas about content knowledge and specialized content
knowledge, and as their model of the mathematical knowledge for teaching evolves, they are adding to and enhancing their descriptions and categories of knowledge.\(^4\)

Research such as that described above demonstrates how complex it can be to prepare teachers to teach mathematics. It is not just about them being able to do the mathematics that they themselves learned in elementary school, as many PSTs expect (Philipp et al., 2007). Rather, PSTs must be prepared, for example, to think conceptually about mathematics, to identify, recognize, and build on their students’ mathematical ideas, and to facilitate the development of mathematical tools and language. They must hold certain beliefs that will motivate them to act in productive ways so that their students will not only learn mathematical concepts with understanding but also demonstrate mathematical proficiency by engaging in the mathematical practices described in the Standards.

For many PSTs, making sense out of the mathematics and engaging in the mathematical practices described in the Standards do not resonate with their own experiences in mathematics. As students, they have often participated in mathematics lessons involving memorizing and applying rules and procedures rather than lessons involving making meaning of the mathematics (Philipp et al., 2007). Explaining their own thinking and critiquing other’s ideas, as well as the development of mathematical arguments based on evidence and understanding, for example, were not part of the expectations and norms in their elementary school math classes. So mathematics teacher educators must prepare PSTs to engage children in exploring mathematical ideas in ways that the PSTs themselves are not likely to have experienced when they were learning mathematics (Lortie, 1975). University content courses for PSTs offer a possible context

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\(^4\) For a more complete description of the current knowledge categories explored in the University of Michigan’s Mathematical Knowledge for Teaching research see (Ball et al., 2008).
for providing PSTs with exposure to meaning making in mathematical activity that is more consistent with the ways in which they will be expected to teach mathematics to their future students.

1.4 Courses for Preparing Teachers to Teach Mathematics

At many institutions of higher education, the mathematical education of teachers is divided into one or more courses that focus on developing mathematics content, and an additional course or courses on teaching methods. The Conference Board of the Mathematical Sciences (CBMS) currently recommends that teacher preparation at the elementary grades (K-4) should include at least twelve semester-hours in mathematics courses that focus on the fundamental ideas of elementary grades mathematics as well as on connections to early childhood and middle-grades mathematics (Conference Board of the Mathematical Sciences, 2012). For the middle grades (5-8) teacher preparation, there should be a program of at least twenty-four semester-hours of which at least twelve should focus on the fundamental ideas appropriate for middle-grades teachers.

At the University of Illinois at Chicago, where this study is situated, all preservice elementary school teachers are required to take at least one mathematics methods course in the College of Education (3 semester hours) and two mathematics content courses in the Mathematics Department (6 semester hours). Although this is fewer hours than is currently recommended, this did meet the previous recommendations of the CBMS (2001). The scope and sequence of the methods courses may vary somewhat, depending on the instructor, but generally the mathematics methods course in the College of Education provides a context for PSTs to explore instructional practices and children’s mathematical thinking in a variety of ways. For example, PSTs may analyze classroom discourse through case studies or video clips with the
goals of identifying and describing the instructional practices of the teachers and the ways in which teachers engage children in mathematical activity. Preservice teachers may read about, view, or conduct interviews with children in order to analyze how children think about and solve problems. Preservice teachers may analyze student strategies and student work to better understand the mathematics students use and the misconceptions students have. Preservice teachers also consider how to identify the mathematics in student strategies and how to respond to student thinking. Finally in their methods courses, PSTs may explore resources for and how to plan and prepare mathematics lessons.5

In the Mathematics Department6, the first content course focuses on developing skills and concepts related to number sense (including number theory, number systems and place-value, and rational numbers), algebraic reasoning emphasizing work with patterns, and symbolic manipulation and representation, operations focused on comparing and contrasting computational procedures for rational numbers; and proportional reasoning and ratios. The second content course focuses on developing skills and concepts related to data and statistics, and geometry and measurement. In both courses, the content is taught with an emphasis on engaging PSTs in the mathematical practices as described in the Standards, such as making sense of problems, constructing and critiquing arguments, reasoning quantitatively, modeling problem situations with mathematics in meaningful ways, using tools strategically, and being precise in their work. The first content course in the sequence typically includes problem-solving sessions as well as sessions focused on interpreting children’s thinking and on analyzing videotaped

5 This information comes from observations of two full semesters of methods courses in the College of Education during the 2006-7 and 2007-8 school years as well as from discussions with instructors of these courses.
6 From this point forward, Mathematics Department will be used to designate the University of Illinois at Chicago Department of Mathematics Statistics and Computer Science.
elementary school classroom episodes. The sessions include discussions of what is occurring in terms of children’s mathematical thinking and strategies and also instructional strategies the teacher uses to facilitate student engagement in mathematical activity.

The content course that is the focus of this study, has been designed with attention to the following: 1) engaging PSTs in mathematical activity that emphasizes meaning making, which might contrast with their prior experiences as mathematics learners; 2) stressing learning content with understanding and making sense of the mathematics in ways that might support later explorations of children’s thinking and teachers’ instructional practice; and 3) having PSTs reflect on and analyze their meaning-making experiences in the content course and connect these experiences to what they will be expected to do as teachers. Although the content foci and highlighted mathematical practices for the course described in this study remain basically the same as other sections of this course, the tasks and activities for the course in this study are specifically chosen because they have the potential to foster meaning making.

To better understand the process of meaning making, this study investigates the ways in which PSTs communicate during mathematics lessons—that is, how they engage in discourse during small-group and whole-group discussions. The central question of this study is: What are characteristics of PSTs’ discourse during mathematical activity as they engage in meaning making? Based on the literature about meaning making and discourse, there are several discourse components that I explore in pursuing answers to this question. First, if PSTs engage in meaning making around the mathematics, what differences might exist in how and when their discussions involve conceptual talk versus calculational talk. Second, as PSTs explain, justify, and build arguments for their thinking and strategies, as they engage in meaning making around the mathematics, in what ways will they build on and incorporate each other’s ideas into their
narratives and arguments? Finally, how does their engagement in meaning making affect their use of the language of authority from either the text or the teacher versus their use of language connected to their own experiences with mathematics?

In the next chapter, I provide an overview of the literature that lays the foundation for this study—how meaning making has been described, how learning for understanding and meaning making are connected, what is involved in the meaning making process, and why investigating how the PSTs are communicating while engaged in meaning making might be a reasonable and productive approach to studying meaning making.
2 Conceptual Framework

This study proposes to identify and describe characteristics of PSTs’ discourse during meaning making in mathematical activity. Many PSTs may not have had opportunities to engage with mathematics in this way in their own schooling (Hiebert et al., 1997; Lortie, 1975; Stigler & Hiebert, 1998). Without experiences to draw on, they may have difficulty creating opportunities for children to learn mathematics with understanding, which has been called for in the mathematics reforms (Common Core State Standards Initiative, 2010; Hiebert et al., 1997; Kilpatrick et al., 2001; National Council of Supervisors of Mathematics, 1988; NCTM, 1989b, 2000; Porter, McMaken, Hwang, & Yang, 2011).

To provide PSTs with relevant experiences in learning with understanding, the course that is central to this study incorporates mathematical tasks and activities that are specifically chosen to offer opportunities for PSTs to engage in meaning making. Learning with understanding requires making sense out of the mathematics—that is, “solving problems” refers to making sense of the problem situation, and the investigation and solution processes rather than correctly applying procedures that may or may not be connected to mathematical understanding. Making sense requires a negotiation of meaning.

A negotiation of meaning involves dialogue and therefore, discourse\(^7\) (Bakhtin, 1981; van Oers, 1996). In this study, I explore the PSTs’ process of meaning making by analyzing patterns in their discourse. Thus, this study primarily draws on three areas of research:

\(^7\) Note that use of the term *discourse* is consistent with Gee’s (Gee, 1995) description of “little d” discourse, that is, language-in-use. The term *language* is broadly construed to include communication in its various forms, for example, words, symbols, gestures, pictures, and so on.
1) learning theory that helps define learning with understanding; 2) theories of meaning making; and 3) theories about the negotiation of meaning through discourse in a community of learners.

2.1 Developing a Perspective of Learning with Understanding

2.1.1 Learning isolated skills versus learning through experience and reflection

The first of two different and conflicting theories about how students learn that came to prominence early in the last century is represented by Dewey (1933/1960, 1938), with his utilitarian view of education. Dewey proposed that, to learn, students need to experience and reflect on new situations, problems, and ideas. Dewey described experiences as opportunities for students to problematize situations and figure out what they need to do to solve the problems generated therein. Dewey described reflection as a mental activity students engage in because of their intellectual curiosity about their experiences. According to Dewey, experience and reflection are generative activities that result in learning. Dewey did not describe knowledge that emerged through experience and reflection as something static that existed out there in the world waiting to be discovered; but rather, he described it as something useful in the context of student’s lives and the problems they were solving.

The second and competing theory proposed by Thorndike (1922) and others around the same time took a behaviorist perspective. This perspective claimed that the process of learning mathematics could be viewed as a series of stimulus-response type relationships. Students would learn and practice particular responses to arithmetic and algebraic stimuli that once learned, they would be able to apply appropriately to solving problems (NCTM, 1970). Thorndike’s approach to learning resulted in experiencing mathematics as a set of fragmented, isolated, and individual rules, skills, and procedures. Learning and practicing isolated skills and procedures made sense
from this perspective because mathematical knowledge was viewed as based on objective and unchanging truths that could be learned piecemeal.

Dewey’s (Dewey, 1933/1960) criticism of the behaviorist approach pointed out some of the inherent problems with the stimulus-response view of learning. Besides the curiosity-killing world view that “nothing remains to be found out (p.41),” Dewey described the general process and outcomes for students as follows:

Sheer imitation, dictation of steps to be taken, mechanical drill, may give results most quickly and yet strengthen traits likely to be fatal to reflective power. The pupil is enjoined to do this and that specific thing, with no knowledge of any reason except that by so doing he gets his result most speedily; his mistakes are pointed out and corrected for him; he is kept at pure repetition of certain acts till they become automatic. Later teachers wonder why the pupil reads with so little expression, and figures with so little intelligent consideration of the terms of his problem (p. 63).

Many students over the past century have experienced Thorndike’s fragmented, drill-focused approach to learning, and for these students, perhaps Dewey’s admonitory prediction may have come to fruition (Jacobs et al., 2006; National Commission on Excellence in Education, 1983; U.S. Department of Education, 2008).

These two contrasting theories of how students learn still compete today in the context of the “math wars.” Grounded heavily in behaviorist theories, the back-to-basics movement tends to focus on introducing and practicing skills and procedures before developing understanding. Hiebert et al. (1997) have suggested that this approach to teaching and learning is more easily assessed than a more experiential approach. With the current, intense attention to evaluating children, teachers, and schools with standardized tests, the back-to-basics movement may be enjoying renewed support. The mathematics reformers rely more on Dewey’s theories. They have been dubbed by the other side as proponents of “fuzzy math.” The reformers aim to first
support students in learning with understanding and then to provide opportunities for students to explore and apply procedures based on those understandings (Hiebert, 2003). Dewey’s ideas about experience and education are apparent in the design of the reformers’ curricula, from the “new math” of the 1960’s to the various standards documents created by the National Council of Teachers of Mathematics through the 1990’s and 2000’s. In 2010, some of Dewey’s ideas have been elevated to new prominence and validated in the first set of national Standards, which recommend that learning with understanding should generally come before procedures.8

Even as “math wars” continue, when PSTs begin their teaching careers, they will need to be prepared to engage children in learning mathematics with understanding. As discussed above, this is unlikely to be consistent with their learning experiences in mathematics. The content course in this study is designed so that it aligns with Dewey’s perspective that for learning to occur, individuals need to experience and reflect on new situations, problems, and ideas. They need to engage in meaning making and make sense out of the mathematics. The emphasis is on learning with understanding.

2.1.2 Learning “in the head” versus learning through social interactions

After World War II, the United States refocused on what mathematics students should learn and on how they might best learn it (National Commission on Excellence in Education, 1983; NCTM, 1970). Cognitive science with its emphasis on mental representations (Varela, Thompson, & Rosch, 1991) offered opportunities to further explore the process of learning, and again, two main schools of thought evolved. Those with a constructivist perspective placed the

8 One example supporting this claim is that operations with whole numbers are explored conceptually for several years before standardized procedures are expected to be mastered.
process of learning as primarily occurring in the individual’s head, and those with a sociocultural perspective placed the process primarily in social interactions (Sfard, 2000). Both perspectives viewed humans as goal-directed agents who come with prior knowledge, skills, and beliefs with which they organize what they see and their development in the world (Bransford, Brown, & Cocking, 2000). Like Dewey, both perspectives include the individual and the social world of the individual in their views of learning (Cobb, 1994), but these perspectives go a step farther in describing where the learning process occurs—in the individual or in social interactions—and here they differ.

The constructivist school of thought describes learning as happening within the head—that is, learning is a process of active cognitive reorganization (Cobb, 1994). The research of Piaget provides the constructivist foundation for understanding learning as a progression through a set of seemingly universal cognitive categories of thinking and development (Confrey, 1991). In their heads, individuals hold particular concepts they have developed as what has been called conceptual structures or schemes (Bransford et al., 2000). As individuals come into contact with “other” in the world (human and object being treated equally in this regard), they process new information. In processing, they have to either assimilate or accommodate this new information connecting it to their existing schemes. *Assimilate* means they incorporate the information into their existing schemes. *Accommodate* means they change their existing schemes forming new ones as a result of the new information. The individual does all the work involved for both mechanisms—assimilation and accommodation. Although contact with the world outside the individual leads to learning, constructivists tend to minimize the role of the social world and place the burden for learning on the individual’s personal creative and problem-solving abilities
Sfard (1998) has used an acquisition metaphor to describe how the individual possesses new knowledge as a result of the process of learning.

In contrast, sociocultural theorists emphasize interactions and the social world as the location for the process of learning (Cobb, 1994; Cobb et al., 1997; Confrey, 1991; Sfard, 1998). A key premise of a sociocultural theory of learning is that interaction is a primary mediating factor for development (Vygotsky, 1978; Wertsch & Stone, 1985). It is through interactions between people that understanding is developed and moved forward. Since dialogue or communication is at the heart of interaction, discourse and communication become key factors for understanding development (Bakhtin, 1981). In addition, learning is viewed as a social activity involving cultural tools (including language as well as, for example, books and calculators) (Lampert, 1990; Lerman, 2001; Radford, 2006; Rogoff & Lave, 1984/1999; Vygotsky, 1978).

Both perspectives suggest that learning happens in a context, the context of social interaction in a community. Taking a sociocultural perspective involves perceiving students’ understandings as relevant within and products of the community and culture in which they interact and participate (e.g., Rogoff, 1990; Vygotsky, 1978; Wenger, 1998). Sfard (1998) has used a participation metaphor to describe one group of sociocultural theorists who view knowledge as residing in the community (not the individual), and learning as changes in the ways members of the community participate in community activities—in this case, in mathematical activity. Echoing Dewey, concepts are tools understood through their use in the community (Barab & Duffy, 2000).

Consistent with a sociocultural perspective, this study is situated in the context of a classroom community that privileges interactions with others and with the world as a primary
vehicle for learning. Although it might be possible to interpret the work in this study through the lenses of both constructivist and sociocultural perspectives, the design of the content course draws heavily on a sociocultural perspective in that the PSTs work together to explore mathematical situations where they negotiate first, the meaning of the problems and second, solutions to the problems. Through their negotiations, the PSTs make sense of the mathematical situations, which results in opportunities for learning of the embedded concepts. Learning with understanding, in the context of this study, can be viewed as PSTs’ ability to make sense of and leverage mathematical concepts as tools for solving problems.

2.2 Learning with Understanding as a Process of Meaning Making

In the community context, coherence and connectedness become considerations so that “…large amounts of mathematics can be learned as sensible answers to sensible questions—that is, as part of mathematical sense making, rather than by ‘mastery’ of bits and pieces of knowledge” (Schoenfeld, 1994, p. 59). Hearkening back to Thorndike and Dewey, researchers continue to emphasize learning mathematics with understanding versus learning isolated skills and procedures in a disconnected and piecemeal fashion. There seems to be a wide consensus both in the United States and worldwide that mathematics education is a process of enculturation—that is, students should pursue making sense of mathematics based on their experiences in the world (van Oers, 1996). Pursuit of learning new mathematical concepts becomes a meaning-making activity.

2.2.1 Two kinds of meaning making

Van Oers (1996) suggests that there are two kinds of meaning important to learning mathematics—cultural meaning and personal meaning. He describes cultural meaning as
knowledge, skills, ways of talking, and ways of acting that are needed to be successful in the community. The components of cultural meaning have been negotiated over time by the community members who participate in the culture. These components are not static, but rather, they are constantly renegotiated as members of the community engage in cultural meaning making (van Oers, 2001). The mathematics in textbooks provides one example of where cultural meaning making might occur. Textbooks reify the knowledge of the community that produces the textbooks. Because of this, the textbook offers students a basis for making sense of what the community has negotiated historically around mathematical ideas, skills, language, and ways of acting. Teachers provide another site for students’ cultural meaning making as they offer views and insights into the culturally established and negotiated ideas. Van Oers (1996) proposes that personal meaning must be attached to cultural meaning in order for real sense-making to occur. He describes personal meaning making as the process of attaching personal value to actions and goals in the mathematical activity. Therefore, making sense means that what has been developed as cultural meaning has personal significance for the student. Dewey’s (1933/1960) description of meaning clarifies what personal significance might include:

To grasp the meaning of a thing, an event, or a situation is to see it in its relations to other things: to note how it operates or functions, what consequences flow from it, what causes it, what uses it can be put to (p. 138).

For van Oers as with Dewey, meaning derives its relevance from the community context. Similar to Dewey, Gee (1995) explains that meaning is situated—that is, meaning is “grounded in actual practices and experiences” (p. 53). Similarly, knowledge does not exist outside of and disconnected from the community. It is not something developed and possessed by individuals. In fact, it serves no purpose without the existence of a community where it is relevant. As
described earlier, knowledge resides within the community, and learning is about changes in participation in the community—that is, concepts become meaningful because they serve as tools used to do work in the community. One striking example of relationships between mathematical meaning making and the community where the meaning making occurs comes from the research of Carraher, Carraher, and Schliemann (1985). These researchers studied a group of children who worked in Recife Brazil evaluating the children’s competence with and understanding for computational strategies. The children used a different set of strategies for and were far more successful with solving computation problems related to their real-life context than they were with school word problems or naked-number computation problems, even though the computational demands were the same. From van Oer’s perspective, personal meaning could be connected to cultural meaning when children engaged in computation for the real-life purpose of commerce.

### 2.2.2 The meaning-making process

Reliance on a community context implies that community members must be able to communicate to facilitate the process of meaning making (e.g., Bakhtin, 1981; Bruner, 1990; Dörfler, 2000; Greenleaf & Katz, 2004; Knoeller, 2004; Sfard, 2008; van Oers, 1996). Since meaning is culturally mediated, meaning making relies on shared language, symbols, and representations within the community (Bruner, 1990).³

As van Oers (1996) describes the process of meaning making, through negotiations, actions become associated with symbols (which includes, for example, language, representations, representations as per Bruner (1990).³

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³ Note that from this point on, the term symbols will be used to indicate the combination of language, symbols, and representations as per Bruner (1990).
and gestures). Symbols do not present themselves with meaning attached (Wertsch & Stone, 1985). According to van Oers, when actions become associated with symbols, the actions become part of the shared meaning of that symbol. Because of this association, the action can now be stored in conscious memory, recalled, and communicated about through the use of the symbol. When individuals come together to negotiate the meaning of symbols—that is, how symbols are connected to actions—meaning making and therefore, learning is happening in the community.

Negotiation of meaning involves a dialogue—among individuals as well as with the historical and cultural associations of actions and symbols (Khisty, 1998; van Oers, 1996). Individuals negotiate meaning by discussing, gesturing, drawing, showing, and so on. They endeavor to figure out what connections they can agree to make between actions and symbols with the goal of developing shared meanings. Personal significance comes to be attached to meanings when individuals have participated in the higher levels of cognitive demand associated with active dialogue (Khisty, 1995). In addition to dialogue among individuals, there is a dialogue between what the individuals in the community are negotiating and what connections between action and symbols have been historically accepted in the community.

Bakhtin (1981) described this dialogic negotiation as an interaction between two discourses—an authoritative discourse and an internally persuasive discourse. As individuals struggle with meaning making, they have their own discourse to which they are listening and responding. This is their internally persuasive discourse. At the same time, they are considering, listening to, and responding to the discourse in the community that has cultural and/or historical weight. According to Bakhtin, this is the authoritative discourse. In the context of a mathematics activity, this dialogue may take the form of students discussing their ideas about a problem and
trying to align their thinking and strategies with a presentation in a textbook or from a teacher.

As individuals consider existing cultural meanings in their negotiations, they are also possibly reconstituting the cultural meaning. As described by Greenleaf and Katz (2004), “…individuals are always in the act of responding to their social world, and in making meaning through their responses to that world, they are also reshaping and authoring it” (p. 173).

The dialogic process of negotiating meaning is facilitated by language tools. (See above for language as one category of symbols.) One example of a language tool that supports the process of meaning making is Bruner’s (1990) conception of narrative. According to Bruner, narrative is a type of human communication that is culturally determined. Narrative involves four features essential to the negotiation of meaning: 1) Narrative is a way to present human actions that are directed towards particular goals; 2) narrative presents events in a sequential order that is accepted as standard by the community; 3) narrative possesses a sensitivity to what is and is not canonical in interactions; and 4) narrative provides a particular perspective as dictated by the narrator (p. 77). In narrative, individuals present their ideas as coherent stories that describe the ways in which they (and we) interact with the world. A narrative has a trajectory from beginning, to middle, to end, and acceptable trajectories are culturally determined—that is, they are historically negotiated and therefore valid. Constructing a narrative that conforms to expectations not only indicates membership in the community, but also provides opportunities for negotiation. Bruner states that narrative provides the basis for meaning making because it is “…our capacity to render experience in terms of narrative…[that is] an instrument for making meaning” (p. 97).

The language tool of narrative is one way that individuals may engage in a negotiation of meaning because the structure of narrative is at least to some degree, already established for and by the community. Members of a community can participate within the narrative structure to
share their various ways of interpreting and understanding. Sometimes in the meaning-making process of constructing the narrative, individuals may engage in personal negotiations between authoritative discourses and internally persuasive discourses (Knoeller, 2004).

Argumentation provides a second language tool for negotiating meaning. If individuals work cooperatively and solve problems in different ways, sharing narratives of their ideas serves as a first step. The negotiation process also requires a mechanism for allowing individuals to compare their ideas, productively question each other, and to justify their ideas or solutions with evidence. Argument can serve as this mechanism (Andriessen, 2006). Argument also provides a structure for negotiating what and when ideas and solutions have been sufficiently compared, explained, and justified.

Van Eemeren and Grootendorst (2004) conceptualize argumentation as “a verbal, social, and rational activity aimed at convincing a reasonable critic of the acceptability of a standpoint by putting forward a constellation of propositions justifying or refuting the proposition expressed in the standpoint” (p. 1). Van Eemeren and Grootendorst explain that argumentation is *verbal* because it always involves some language use. Argumentation is *social* because it is directed at other people. Argumentation is *rational* because it stems from intellectual considerations. Argumentation involves having *standpoints*—similar to a narrator’s perspective but more specific. As van Eemeren and Grootendorst explain, a standpoint is a point of view that the arguer will defend to another individual who doubts or does not hold that point of view. Argument takes place where individuals adopt different standpoints and their purpose is to negotiate between the standpoints. In the context of a classroom community, students engage in what Andriessen (2006) calls arguing to learn where argumentation is a collaborative process that promotes the meaning making and the development of taken-as-shared meanings.
Mathematical activity provides a context in which members of a community use the language tools of both narrative and argumentation—narrative to share their ideas about the meaning of problems and possible solution paths, and argumentation to compare, explain, and justify their ideas. The design of the content course that is the focus of this study incorporates mathematical situations that require community members to participate in mathematical activity by sharing their ideas through narrative and testing and defending standpoints through argumentation with an emphasis on meaning making.

2.2.3 Mathematical activity as meaning-making activity

Brownell (1947) proposes a variety of reasons to emphasize and expect meaning making in mathematical activity. He explains that “[computational] skills learned mechanically [as supported by Thorndike’s theories] quickly deteriorate” (p. 261). Rather, students must have a meaningful grasp of even the most basic computation to ensure that they will remember and be able to flexibly use the procedures. In addition, Brownell suggests that if students grasp how and why simpler arithmetic examples like dividing a three-digit number by a two-digit number work, they can extrapolate to more complex examples. According to Brownell, being able to make sense of mathematics also provides students with confidence in and respect for what they are learning. Finally, like Dewey (1933/1960), Brownell points out that meaning making in mathematical activity is far more motivating than simply remembering or practicing facts and procedures.

More recently, Hiebert et al. (1997) describe why it is essential for students to learn mathematics with understanding, which requires meaning making during mathematical activity. They argue that the ways in which students must be mathematically proficient to be successful underlie the emphasis on learning with understanding. For example, because of the new
technologies permeating society today (and evolving by the minute), students need flexibility in defining problems; they need to be able to adapt what they know to new situations; and they need to be able to approach, model, and develop new strategies for solving new kinds of problems. In addition, new technologies have made the skill-and-drill approach that Thorndike proposed obsolete. Computers and calculators can take care of more mundane tasks that involve computation and statistics, for example, but they must be programmed by individuals who understand the mathematical ideas of the problems they are trying to solve.

In mathematical activity, meaning making—as described above—requires a social context, interaction with others (and with the world), and opportunities for students to make connections among, identify relationships between, and construct personal significance for mathematical concepts and ideas (c.f., Bruner, 1990; Dewey, 1933/1960; Hiebert et al., 1997; van Oers, 1996). In the context of this study, mathematical meaning making occurs in the setting of a classroom community. In the process of negotiating mathematical meaning for the community as well as for the individual community members, PSTs discuss, explain, argue, justify, question, in short, engage in discourse with each other about their mathematical ideas and strategies. As Khisty (1995) explains, this kind of “talk” is “the critical vehicle by which an individual internalizes meanings, becomes enculturated, and develops a sense of personal power in mathematics” (p. 290). Because discourse is central to mathematical meaning making, this study proposes to investigate meaning making by identifying and analyzing the discourse patterns that emerge when PSTs in one specific classroom community communicate during the process of arguing and explaining—that is meaning making.
2.3 **Negotiating Meaning in a Community through Discourse Practices**

As explained by Rogoff (1995), both Vygotsky’s sociocultural view and Dewey’s experiential focus foreground individuals as participating in a community with others. One way individuals become enculturated into a community is through opportunities to discuss and debate their ideas (Khisty, 1995) as they participate in the activities of the community. Furthermore, in the tradition of sociocultural theorists, learning and development can be explained as changes in the practices and participation occurring in a community (Sfard, 2002; Wenger, 1998). As discussed above, participation changes as new concepts are understood through their use in the community. For new concepts to be understood, members of a community negotiate meaning for the concepts by talking through, sharing, comparing, and contrasting their ideas.

2.3.1 **A community context for meaning making**

Wenger (1998) describes three defining features that provide coherence for what he calls a community of practice—features which apply to the classroom community in this study. Wenger uses the term *practice* to indicate that members of the community are doing something that is in a historical and social context. That context provides structure and meaning for the activity in which community members are engaged. According to Wenger, the concept of practice is far-reaching and includes, but is not limited to, artifacts, symbols, roles, procedures, norms, conventions, perceptions, assumptions, and so on. The first defining feature of a community of practice is that community members are mutually “engaged in actions whose meanings they negotiate with one another” (p. 73). This mutual engagement includes such components as doing things together, working towards maintaining the community, and establishing relationships. Second, community members are engaged in joint enterprise, which might involve negotiated goals as well as mutual accountability. Third, community members
have a shared repertoire that is comprised of elements such as stories, ways of acting, ways of
talking, historical events, and artifacts.

Since the context of this study is a community of PSTs in a mathematics content course,
the community practices of interest are those related to a mathematics classroom. Consistent with
Wenger (1998), Cobb, Stephan, McClain, and Gravemeijer (2001) describe classroom
mathematical practices as normative ways of participating in mathematical activity where the
norms are co-constructed by the community members. Again, similar to Wenger, they explain
that classroom mathematical practices “focus on the taken-as-shared ways of reasoning, arguing,
and symbolizing established while discussing particular mathematical ideas” (p. 126). Cobb et al.
develop the construct of classroom mathematical practices as sites for investigating and
discussing the learning trajectories of a classroom community—that is, changes in the practices
of the community, how members participate in the community, provide information about the
learning trajectory of the community. In their discussion of learning trajectories, Cobb et al.
mainly explore changes in how students reason, argue, and discuss their mathematical ideas or,
more generally, how they engage in the practice of mathematical discourse.

2.3.2 Discourse practices related to meaning making

Researchers have explored discourse practices in mathematics classrooms from a variety
of perspectives. Yackel and Cobb (1996), for example, consider features of the classroom
environment in the form of sociomathematical norms, including norms for discussion\(^{10}\). They
find that particular norms are associated with the development of intellectual autonomy for

\(^{10}\) Fully describing characteristics that define sociomathematical norms is beyond the scope of this manuscript. For
more information about the distinctions between social norms and sociomathematical norms see Yackel and Cobb
(1996).
students. Their study also investigates the ways in which the teacher represents the views and introduces the practices of a larger mathematics community—in Bakhtin’s terms, how the teacher provides the authoritative discourse for the dialogic process of negotiating meaning.

In addition to considering the relationship between the classroom environment and the practices of the community, Hufferd-Ackles, Fuson, and Sherin (2004) explore a trajectory for the development of discourse practices in a mathematics classroom community. They describe what different levels of discourse look like in a mathematics classroom—that is, they describe a learning trajectory for development of what they call “Math Talk.” The descriptions of various points along the trajectory capture how the teacher’s and students’ participation in discussion evolves over the course of the year from lower levels of engagement in discourse where the students’ participation is minimal to higher levels where students become full participants in the discourse. As full participants in the mathematical discourse, students take on the role of questioner, explain and articulate their math ideas more fully, become a resource that influences the course of the lesson, and take responsibility for their learning and for making sense of the mathematics. At the lowest level of Math Talk, the classroom community views the teacher as the mathematical authority. At the highest level, the community of practice values both teacher and student contributions to the discourse. Students become active and essential participants in the discussion responding to and building on each other’s contributions instead of only responding to the teacher’s comments, prompts, and questions. At the highest level of math talk, students’ internally persuasive discourses hold equal footing with the teacher’s authoritative discourse in terms of negotiating meaning in mathematical activity.

The importance and role of student contributions as they become full participants in the community discourse practices provide the focus for Cobb, Boufi, McClain, and
Whitenack (1997) in their study of reflective discourse and collective reflection. According to Cobb et al., *reflective discourse* refers to some mathematical activity that is objectified and during discussion, this objectified activity itself becomes the explicit topic for investigation requiring students to respond to, question, analyze, evaluate, and build on contributions made by their classmates. To illustrate this concept, Cobb et al. describe how students explore the ways that ten monkeys might be sitting in two trees—for example, there could be three in one tree and seven in another, or five in each. The teacher records student contributions in a table. After students make their suggestions for the possibilities, the teacher then asks them if they have found all possible combinations and how they would know. Cobb et al. go on to describe how, in order to respond to the teacher’s questions, students have an opportunity to “reflect on and objectify their prior activity” (p. 264). They describe the process of collective reflection as a process parallel to what Piaget called *reflective abstraction*, which describes how individuals reorganize their mathematical activity. Cobb et al. claim that students potentially engage in collective reflection as they participate in the practice of reflective discourse—that is, they engage in a communal activity of reflecting on their prior activities possibly reorganizing they ways in which they think about the prior work. Individual and collective student contributions become the objects of mathematical discussion for the community. Full student participation in the discourse is essential for developing mathematical ideas and for providing opportunities to think more deeply about the mathematical content.

Strengthening the case for investigating discourse patterns as one way to better understand meaning making, the research of Wood, Williams, and McNeal (2006) identifies correlations between interaction patterns and the kinds of thinking students are doing as they participate in mathematical discussions. They first describe the interaction patterns they observe.
These range from basic teacher-directed interactions like collecting answers and IRE (teacher Initiates, students Respond, teacher Evaluates the student’s response) (Hoetker & Ahlbrand, 1969) involving limited student participation to patterns like inquiry (about making sense of other’s ideas) and argument (about students disagreement and resolution through argument) where students participate more fully in the discourse. Wood et al. find that interaction patterns related to students’ full participation in the discourse practices tend to correlate with student statements representing higher levels of verbalized thinking such as comprehending (understanding concepts), applying (knowing when to use an idea), or building with analyzing (applying known procedures in new contexts).

Going a step beyond just correlating discourse patterns with learning and thinking, Sfard’s (1998, 2002, 2007, 2008) work over the last decade has taken what she calls a communicational approach to cognition that presents a vision of learning involving a conceptualization of thinking as communication (Sfard, 2002). Consistent with theories of both Vygotsky and Bakhtin, Sfard outlines how communication with oneself and communication with others are almost the same, since both are dialogic processes. An individual can enter into dialogue—arguing, questioning, and negotiating—with his or her own internally persuasive discourses as well as with the discourse of others or with authoritative discourses. Sfard (2002) is specific about her use of the word discourse. Discourse for Sfard includes any instance of communication, not restricted to verbal or textual language, including symbolic systems. She claims that there is no way to conceive of thought as separate from the symbolic embodiment of thought; thus, thinking is communication. Sfard proposes that changes in discourse practices represent changes in thinking; and therefore, changes in discourse patterns can provide evidence of learning and sense making. In one instance that exemplifies the process, Sfard (2002)
demonstrates how a failure to learn can be directly related to specific failures in communication when students work together on a joint task. She explains that students must be engaged in a mutual endeavor of making sense out of each other’s communication in order for the discourse to be productive. Ari, one student in Sfard’s study, does not fully participate in the discourse with his classmate, although he seems to be able to utilize concepts to solve the problem on which they are working. Gur, his partner, attempts to but does not meaningfully participate in the discourse and seemingly cannot make sense out of Ari’s comments and gestures. Further analysis of their interactions indicate that, although Gur expresses a desire to make sense out of Ari’s contributions to the discourse, Ari has no investment in Gur’s participation. Sfard posits that Gur’s failure to learn and therefore to participate in any meaningful way is because of the breakdown in communication between the partners. In her discussion of the situation, Sfard suggests that had Gur’s partner been invested in the discourse between them, Gur’s participation in the task and in the discourse may have been more successful.

According to Sfard (2002), the dialogic process of communicating (i.e., thinking) may result in community members changing the ways in which they participate in the discourse of the community (i.e., learning). These changes may occur based on interactions among individuals or based on cues supplied by both the learning environment and authoritative discourse from other sources in the community (artifacts, text, or discussion, for instance). This echoes van Oers (1996) description of the dialogic process of negotiating meaning both with individuals in the community and with the cultural and historical context provided by the community. Sfard (2002) provides an example of what this negotiation with the authoritative discourse of the community context might look like. She describes how a student who initially responds that there is a largest number is coaxed, through discursive moves, to answer that, in fact, there can never be a largest
number because you can always add one. Sfard discusses, as follows, how the change in discourse might represent a first step to learning that there is no largest number. In the process of making sense out of the discourse, the student recognizes the answer her interlocutor (who represents the culture and history of the community) expects and will now compare the questioner’s (authoritative) discourse with her own (internally persuasive) discourse and adjust accordingly. Sfard (2007) later comes to call this dialogic process commognitive conflict and to describe thinking as the individualization of interpersonal communication. The result of commognitive conflict, according to Sfard, is not “transformations in individuals, but rather in what and how people are doing—in patterned human processes, both individual and collective (p. 568), ” in what Wenger would describe as practices of the community (1998).

Research such as that described above illustrates the necessary connection between discourse practices in a community and the meaning making process. The research cited here took place in elementary school classroom contexts. In this study, I intend to extend the study of meaning-making through discourse to a content course for PSTs. Discourse analysis, because of the intimate relationship between discourse practices and meaning making, offers one method for unpacking and understanding the meaning-making process.

2.4 Framework Summary

The conceptual framework outlined above provides warrants for designing a preservice teachers’ mathematics content course that presents opportunities for meaning making in mathematical activity and for analyzing discourse patterns as a way of understanding the meaning making process in mathematical activity. The framework describes research from three distinct areas: 1) learning theory that helps define learning with understanding; 2) theories of meaning making; 3) theories about the negotiation of meaning through discourse in a community
of learners. The learning theories included in the first section lay the foundation for the design of the content course and its focus on meaning making. The second section on theories about meaning making provides the basis for recognizing the process that is the focus of this study. The third section includes research that illuminates the close connection between the process of meaning making and the discourse where meaning making occurs, which implies that discourse analysis is likely to yield valuable insights into the meaning-making process. The focus of this study is to see what characteristics of preservice teachers’ discourse during mathematical activity emerge as the preservice teachers engage in meaning making. To investigate these characteristics, I specifically focus on 1) how and when PSTs engage in conceptual talk or calculational talk; 2) to what extent and in what ways PSTs incorporate each other’s ideas into the discourse of their small groups; and 3) how and when PSTs’ incorporate authoritative language (language derived from the text or the teacher) versus language that is more connected to their own experiences during their discussions. I describe how the conceptual framework is applied to the course design and the research design in the following chapters.
3 Course Design\textsuperscript{11}

The context for this study is the first in a series of two mathematics content courses designed to prepare elementary school teachers to be generalists teaching kindergarten through ninth grade. Because this study, situated in a PST content course, examines the characteristics of PST discourse during mathematical activity as they engage in meaning making, I have specifically designed this course—choosing tasks and fostering a classroom environment—to provide opportunities for PSTs to engage in discourse while working together to solve problems. Understanding the outcomes of this study relies on a familiarity with the structure and composition of the course. For these reasons, in this chapter, I describe the curriculum and classroom environment of the course.

3.1 The Curriculum

Historically, the content of this course develops mathematical ideas about 1) number sense including number theory, number systems and place-value, and rational numbers; 2) algebraic reasoning emphasizing work with patterns and symbolic representation and manipulation; 3) operations focused on comparing and contrasting computational procedures for rational numbers; and 4) proportional reasoning and ratios. When PSTs have finished the course sequence, they should be prepared to take the basic skills test required for K-8 teacher certification.\textsuperscript{12}

\textsuperscript{11} I have been one of the instructors for this course for the last two years and will be the instructor for the course in the fall that is the subject of this study. Therefore, I have knowledge of the design of the course.

\textsuperscript{12} A brief course description for MATH 140 can be found at http://www.uic.edu/ucat/courses/MATH.html. The university’s course description is revised every two years, so there may be some differences between the course description and the course design described here. Information about the historical content of the course has also been
The mathematical content goals for the course are derived from a combination of past course syllabi and recommendations from authoritative resources containing rich descriptions of what children and teachers need to know about mathematics. One of the resources, the publications of the Conference Board of the Mathematical Sciences (2001, 2012), for example, combine the ideas of sixteen national professional societies concerned with the mathematical education of teachers and children. Another, *Adding It Up* (Kilpatrick et al., 2001), synthesizes research on mathematics education and how children learn. Standards documents from the National Council of Teachers of Mathematics (NCTM, 1989a, 1989b, 1991, 2000) support teachers in understanding what content and processes children should have opportunities to learn, as well as how teachers need to be prepared to adequately meet those expectations for children. Resources such as these provide guidance for determining what content must be central to the course that is the focus of this study and also what widely-accepted mathematical language should be used to describe that content.

Beyond goals addressing common mathematics content knowledge, there are other kinds of knowledge about mathematics to be considered when preparing teachers to teach mathematics. The Learning Mathematics for Teaching (LMT) project at the University of Michigan has been developing models to represent the various components of mathematical knowledge for teaching for decades (Ball, 1990; Ball et al., 2001; Ball et al., 2008; Hill, Blunk, et al., 2008; Hill, Schilling, & Ball, 2004; Hill, Sleep, Lewis, & Ball, 2007). They began by decomposing knowledge for teaching into Shulman’s (1987) broad categories of subject matter knowledge and pedagogical content knowledge. They have further subdivided each of these
broad categories (Ball et al., 2008). The LMT project continues to refine and extend their framework of knowledge needed for teaching, but one area that is fairly well developed in their work is the area of specialized content knowledge—the knowledge that teachers require to perform the tasks of teaching. They identify tasks specific to teaching, including tasks such as actively listening to, interpreting, and analyzing children’s strategies and explanations; identifying which children’s contributions might be mathematically productive to pursue and knowing how to maximize the potential of these contributions; and planning questions that will promote rich mathematical discussion and increase opportunities for children to make sense of the mathematics.

In addition to the common content goals, the course goals include several specialized content goals derived from the LMT research on specialized content knowledge. Specific specialized content goals emphasized in the course design consist of listening to and interpreting others’ strategies, representations, and explanations; explaining strategies and thinking to others; unpacking the mathematical ideas in specific tasks, strategies, and representations; and analyzing affordances and constraints (Greeno, 1994) of tasks, models, strategies, and representations. In order to meet specialized content goals, the first level of engagement with the designed course tasks and activities provides PSTs with opportunities to do mathematics while developing the practices that they will be expected to promote and support in their classrooms—practices specified in the Standards such as making sense of problems, constructing and critiquing arguments, reasoning quantitatively, modeling problem situations with mathematics, using tools strategically, being precise, and identifying and using structure and regularity to solve problems (Common Core State Standards Initiative, 2010). The second level of engagement invites PSTs
to make connections among 1) their experiences in doing mathematics in the course, 2) the practices they are developing as learners, and 3) their future work as teachers.

I have selected and refined course tasks and activities for meeting the content and specialized content goals based on a combination of past iterations of the course and on other resources soundly based in research about learning mathematics, as described above. Textbooks designed to support this kind of course, such as those by Beckmann (2011), Bassarear (2001), and Sowder, Sowder, and Nickerson (2009), and Bennett, Burton, and Nelson (2007) are significant resources. In addition to textbooks, the design of course tasks and activities draws on in-depth research into particular topics, such as Lamon (1999) for fractions and ratios; Carpenter et al. (Carpenter, Fennema, Franke, Levi, & Empson, 1999) for computation strategies; and a variety of books on algebraic reasoning (e.g., Carpenter, Franke, & Levi, 2003; Driscoll, 1999; Moses, 1999).

In considering how to choose tasks that specifically promote meaning making, Hiebert, et al. (1997) offer a description of several necessary features of tasks that promote learning for understanding, and therefore, meaning making in mathematical activity. First, if we expect students to discuss the mathematics and try to make sense out of it, we need tasks that are really a problem for the students—something that they will be intrigued by and therefore want to figure out. Hiebert et al. caution that a task may be intriguing but not mathematical or mathematical but not really problematic. In addition, a task needs to be at an appropriate level of difficulty. All of these are important considerations when choosing tasks for the course. Second, Hiebert et al. explain that if we want to encourage students to reflect on the mathematics in a task or activity, they must possess tools that will allow them to engage with that task or activity. They define tools as including strategies with which students are familiar and are able to apply in new
situations. Third, Hiebert et al. elucidate what the focus of tasks should be in terms of developing important mathematical concepts and ideas:

Many of us have been brought up to think that the best way to teach mathematics is to teach important concepts, like place value or common denominators, by explaining them clearly and demonstrating how to use them and then having students practice them. Our recommendation is that we change our way of thinking and teaching so that students are allowed to develop concepts, such as place value and common denominators, in the context of solving problems. This means that when selecting tasks or problems, we need to think ahead about the kinds of relationships that students might take with them from the experience (p. 22).

Their rationale for choosing and implementing tasks as described above is to counter the traditional separation of concepts and procedures. They explain that by “teaching” the procedures and concepts separately, we have limited the connections students are able to make and the mathematical relationships students are able to identify and apply. Instead of conceptualizing the teaching of mathematics as teaching a series of concepts and procedures, Hiebert et al. suggest that we should focus on providing opportunities for students to develop an understanding of meaningful mathematical relationships.

I have selected a series of tasks for the course based on the characteristics described above as well as on prior experience with some of the tasks during previous iterations of the course. Each task selected for its meaning-making potential focuses on content identified as essential to the course, includes opportunities for students to engage in the development of mathematical content knowledge and specialized content knowledge as intended for the course, and is derived from a textbook with some authoritative standing in the community of mathematics teacher educators. In addition, the tasks are chosen because they are potentially real problems for PSTs to solve, and hopefully, PSTs will find them engaging. The tasks provide an appropriate level of challenge for PSTs, and PSTs should have the tools, background, and
experiences that allow them to successfully engage in the tasks. The tasks invite discussion of the
problem parameters, negotiation of the meaning of the problem—that is, what the problem is
asking for—explanations and comparisons of strategies and solutions, and the tasks present
multiple points of entry and access for PSTs. Finally, the tasks emphasize the development of
relationships among, for example, mathematical concepts, tools, strategies, representations, and
language. Figure 1 on the next page shows a task that fits the above criteria.

I piloted the base-5 addition/subtraction exploration in Figure 1 during Fall 2011. One of
the content foci for the course specifies that PSTs will have opportunities to explore a variety of
computation algorithms—not only the U.S. traditional algorithms. The Standards specify that
students should have several years to explore computation strategies that make sense to them.
The practice standards emphasize using representations and models, and making connections
among strategies, representations, and models. The Base-5 Addition and Subtraction exploration
provides an opportunity for PSTs to explore these connections together. Before engaging in this
task, the PSTs have explored various number systems—including different place-value systems.
They have also informally explored working in different bases, including building
representations for quantities. This task is presented as an early exploration of operations. When
this task was piloted in an earlier iteration of the course, PSTs discussed and worked on the task
for about 30 minutes, at the end of which, most of them understood what the problem was about
and were starting to make some connections between the problem types and the representations.
The next task was to connect their work on Base-5 Addition and Subtraction with what
elementary school students do in solving these problems.
Base-5 Addition and Subtraction

Instructions for: Adding and Subtracting with Base-5 “Blocks”

1. Below are three base-5 blocks strategies that people have used to solve the problem $23_5 - 14_5$.
2. Work through each strategy with base-5 blocks.
3. Decide how you would describe each strategy.
4. Compare notes with your group and come up with a group description of each strategy.

5. Listed below are three different subtraction situations that all result in the number sentence: $23_5 - 14_5 = 4_5$
   For each situation, use base-5 blocks to model and draw a solution strategy that you think fits the situation and describe how it fits.
   a) I have $14_5$ apples, and I need $23_5$ apples. How many more do I need?
   b) I have $23_5$ apples, and my friend has $14_5$. How many more apples do I have than my friend?
   c) I have $23_5$ apples, and I give away $14_5$. How many apples do I have now?

6. The three types of addition/subtraction situations have been described as **Comparison**, **Change (to more or to less)**, and **Parts-Total situations**. Discuss what you would call each of the situations described in Problem 5 and why.

Figure 1. Base-5 Addition and Subtraction exploration.
The course is built around tasks such as the one presented above. The lengths of the implemented tasks during this study ranged from part of one class period to four class periods. Sometimes a task consists of a single problem followed by sharing and comparing strategies. For example, \textit{Find the sum. (Use patterns—no calculators allowed.)} \[13 + 15 + 17 + 13 + 15 + 17 + 13 + 15 + 17 + 13 + 15 + 17 + 13 + 15 + 17 + 13 + 15 + 17.\] At other times, the task is a problem situation that involves multiple stages interspersed with discussions and presentations. For example, PSTs create a number system based on a set of constraints. During the course of this task, PSTs work in small groups, model their number system with base-5 blocks, and design presentations of their number system to share with the class. (See Figure 3 in Section 4.5.1 of this paper for the full set of task instructions.) After they have presented the various number systems, PSTs compare and contrast the advantages and disadvantages and overall features of the systems. Generally, all of the tasks require discussion either during or after their implementation.

3.2 \textbf{Community and Classroom Environment}

In order to study meaning making in a mathematics classroom, the design of the classroom environment, and the norms and expectations for participation that I, as the instructor, establish play a key role. As educational philosophers have claimed and researchers have discovered, meaning derives its relevance from the community context—that is, meaning is situated in the practices and experiences of a community (cf., Dewey, 1938; Gee, 1995; van Oers, 1996). In the community that I and my students co-constructed as the context for this study, meaning provided a key component of what made our mathematics classroom a community of practice (Wenger, 1998). I as the instructor selected problems and problem situations for the PSTs to explore that I believed presented opportunities for mathematical meaning making around the required content specified in our course goals. In addition, I
facilitated discussions about the mathematics embedded in those problems; PSTs’ ideas about the meaning of the problems; their strategies for solving the problems; as well as how they thought our work in the course might support their future work with students. The PSTs engaged in the joint enterprise of learning mathematics for teaching.

Most PSTs have not participated in a classroom community where the norm is to work together to make sense of intriguing, and sometimes, complex mathematical ideas (Jacobs et al., 2006; Lortie, 1975). Because of this, they do not necessarily come equipped with the belief that this is a desirable way to engage in the study of mathematics (Ambrose, Clement, Philipp, & Chauvot, 2004; Philipp et al., 2007). Therefore, the norms and expectations of the learning environment must facilitate and set expectations for collaborative work and therefore, communication and negotiation among community members. Our classroom, for example, includes the norms of PSTs explaining, questioning, and justifying their ideas and strategies as they work in small groups. All PSTs are expected to contribute, to be respectful of each other’s contributions, and to listen as classmates share their ideas.

Supporting productive interactions aligned with the norms and expectations described above suggests that the community must value the mathematical ideas PSTs generate and discuss in their joint problem solving activities. Following Cobb et al. (1997), PSTs’ contributions to small group and whole group discussion often become the focus of our mathematical explorations in what Cobb et al. call collective reflection. PSTs publicly share and compare their ideas about and strategies for solving problems, which provides opportunities for focusing our mathematical discussions on their ideas (Cobb et al., 1997) and for assigning competencies based on what individuals offer (E. G. Cohen, 1994; E. G. Cohen, Lotan, Scarloss, & Arellano, 1999)
The dynamics of student interactions, which are influenced both by the types of tasks and by community norms and expectations, may dramatically influence what students learn when working together in joint problem solving (Anderson et al., 2001; Barron, 2003; E. G. Cohen et al., 1997; E. G. Cohen, Lotan, Abram, Scarloss, & Schultz, 2002). In order to facilitate productive interactions, the tasks must be what Lotan (2003) calls *groupworthy*—that is, the tasks have to be open-ended and have multiple access points so all students have a place at which they can enter into the problem-solving process. When group members work productively together on *groupworthy* tasks, they are acting in concert with norms for listening to and building on each other’s ideas (Anderson et al., 2001; Barron, 2003). They take advantage of each other’s strengths and value each other’s competencies (E. G. Cohen, 1994; E. G. Cohen et al., 1997; E. G. Cohen et al., 2002). This implies that community members must come to value these norms and practices. With this in mind, the content course begins with an activity designed to promote familiarity and team work. In the course overview, I present the course expectations that PSTs will work together on mathematical problems; that the course activities will provide opportunities for them to develop the practices of listening to, interpreting, and building on each other’s ideas. Finally, I emphasize that everyone has strengths and weaknesses, and in our community, we want to promote identifying other’s strengths and determining how individual strengths can contribute to the overall processes of problem solving and meaning making. I reinforce these expectations throughout the semester and attempt to create opportunities for highlighting key contributions individuals make and therefore highlighting individual strengths.


4 Study Design and Research Methods

4.1 Overview

This study examines the characteristics of PSTs’ discourse during mathematical activity as they engage in meaning making. The goal of this study is to gain insights into and new understandings about the meaning-making process that occurs in this community of PSTs. In this chapter, I describe the overall design of this study investigating three themes embedded in the discourse of the meaning-making process: 1) how and when PSTs engage in conceptual talk or calculational talk; 2) to what extent and in what ways they incorporate each other’s ideas into the discourse of their small groups; and 3) how and when PSTs’ discussions incorporate authoritative language (language derived from the text or the teacher) versus language that is more connected to their own experiences.

The intimate connection between discourse and the meaning-making process suggests that an analysis of discourse patterns in interactions might produce opportunities for developing insights into and understandings about the meaning-making process. Discourse analysis (cf. Gee, 1995; Sawyer, 2006) for meaning making requires an in-depth examination of the interactions among PSTs as they engage in meaning making in the context of the classroom community. In addition, the discourse being analyzed cannot be separated from the mathematical and community contexts—it is situation specific. Because of these qualities of this investigation, I employ an interpretive case study design (Merriam, 1998; Yin, 2006).

I studied PSTs’ interactions from what Lampert (1998) calls the “perspective of practice.” I served as the instructor for the course, which was the context of this study, and therefore, I was a direct participant in designing the course to foster certain behaviors among the
study’s participants described in the next section, co-creating the classroom environment and
discourse with the PSTs, and in some cases directing the discourse—by design and in-the-
moment. As such, this is a special case of case studies (Ball, 2000) with the researcher taking a
researcher-participant role (Merriam, 1998). Ball (2000) describes the special features of this
kind of case study as follows:

In some ways, these researcher-teacher, first-person projects might appear to be
highly similar to other case study research. However, two features characterize
important distinctions. First, design—not only of the methodology but also of the
phenomenon and its context as well—plays a critical role. Instead of merely
studying what they find, they begin with an issue and design a context in which to
pursue it. The issue with which they begin is at once theoretical and practical,
rooted in everyday challenges of practice but also situated in a larger scholarly
discourse, and they create a way to examine and develop that issue further. What
they ultimately focus on may emerge out of the situation and its unfolding, but
they have an important hand in constructing fundamental features of the arena of
study (p. 386).

In the role of researcher-participant for this study, I bring a set of perspectives, beliefs, and
expectations to the design and implementation of the course that guide the decisions I make in
my data analysis—for example, what I attend to, what I ignore, and how I analyze, categorize,
and ultimately synthesize the data.

The following sections of this chapter more completely describe the four main
components of the methodology and study design:

- Rationale for the methodology
- Participants and course context
- Data collection methods
- Data analysis
4.2 Rationale for the Methodology

The research question that guides the design of this study is: What are characteristics of PSTs’ discourse during mathematical activity as they engage in meaning making? This research aims to illuminate the characteristics of interactions in a mathematics content course designed to foster and support the meaning-making process. The course design centers on tasks selected because they provide opportunities for participants to discuss, make sense of, and develop ways of mathematical thinking that they can pass on to their own students. The course design also sets norms for this course’s classroom community including the expectation that all PSTs will engage in making sense of problems; explaining, justifying, and describing their ideas and strategies; listening to, responding to, and building on the ideas of others; and reflecting on what they are learning and how they are learning it. The norms and tasks work together to set the stage for PSTs to engage in discourse in the context of mathematical activity as meaning-making activity.

The qualitative nature of the research question and the location of the research in a real-life context make this research ideal for an interpretive case study design. Using an interpretive approach means delving into the complex set of characteristics of which the socially-constructed phenomenon—in this case PSTs’ discourse—is composed. This study’s exploration of discourse patterns includes an examination of such questions as what happens, how does it happen, when does it happen, and what does it look like when it happens—all questions that imply descriptive answers. Answers to questions like these can provide multi-dimensional images of events or interactions that allow for an in-depth analysis of how the PSTs engage in the meaning-making process.

This study includes multiple sources of data which provide a way to triangulate the data and to check for patterns across the data sources. For example, how PSTs engage in meaning
making during in-class, small group work is compared with how PSTs engage in meaning making during a special problem-solving session with only two students. PSTs’ verbal explanations when describing their strategies during small group work is compared to both their verbal and written explanations when they are asked to work together to solve a problem in a special-problem solving session.

In order to maximize the identification of characteristics of interactions during the process of meaning making, I began by looking for general discourse patterns in the various videotaped sources. Next, I began to formulate categories related to meaning making in the context of interactions and to refine and fully describe the properties of these categories. Finally, I further investigated the categories by exploring other variables that potentially affect the characteristics of the interactions.

4.3 Participants and Course Context

The participants in this study are preservice teachers (PSTs) in the first of two mathematics content courses specifically designed to prepare elementary school teachers. The PSTs are primarily freshman enrolled in a large, urban university in the Midwest. The class has 26 PSTs with significant ethnic diversity—21 of the PSTs are students of color. Generally, about one fourth to one third of the group first must take a remedial mathematics content course in preparation for this preservice teacher content course. The PSTs enrolled in this course have chosen one of two possible sections, but there is no special preselection for being in the section that is the focus of this study.

Initially, I proposed to select six participants as the focus participants for this study guided by their responses on the IMAP (Integrating Mathematics and Pedagogy) Web-Based
Beliefs Survey developed at San Diego State University as part of the Integrating Mathematics and Pedagogy project (Ambrose et al., 2004; Ambrose, Philipp, Chauvot, & Clement, 2003) (See Appendix D), and the LMT (Learning Mathematics for Teaching) rational numbers assessment developed at the University of Michigan (Ball et al., 2001; Ball et al., 2008) (See Appendix E). The Beliefs Survey was designed specifically to be used in mathematics methods courses to collect data on PSTs’ beliefs about children and mathematics, and more generally about teaching and learning in the area of mathematics. The LMT was designed to collect data on what mathematics content inservice teachers know, and how they can apply that knowledge in their teaching. Unfortunately, these early course assessments did not adequately distinguish PSTs from each other.

Although these assessments did not distinguish among PSTs, as the semester progressed, videotape of the small group interactions suggested that there were differences worth exploring. Based on differences in how PSTs’ participated in discussions and how they engaged classmates, nine PSTs were selected for more intensive study. I chose the nine PSTs to represent the range of participation, including PSTs who often dominated their small group discussions to PSTs who participated only occasionally.

4.4 Data Collection

To investigate the process of meaning-making in the course context, I collected data that provides opportunities to explore 1) how and when PSTs engage in conceptual talk versus calculational talk, 2) how they build on and incorporate each other’s ideas into their narratives and arguments, and 3) how their engagement in meaning making affects their use of the language of authority versus language connected to their own experiences of mathematics. The two main data sources are 1) videotaped and audiotaped small-group and whole-group discussions and
2) videotaped special problem-solving sessions. In addition, I collected and scanned all student written work and homework, as well as data from the previously described pre- and post-tests for the IMAP Beliefs Survey and LMT. Although these instruments did not aid in selecting focus PST, comparing the pre-and posttest scores on these assessments did become useful after I noticed differences in how PSTs typically engaged in interactions. I discuss the findings related to these assessments later in Chapter 6.

I also collected several other data sources related to the design of the course. To evaluate the success of the mathematical tasks and activities for providing opportunities for meaning making, I draw on course planning documents and my written reflections from after I implemented lessons. The planning documents reference course reading assignments that provide useful comparative information for thinking about the language PSTs’ use.

### 4.4.1 Videotaped lessons

Every class session, a videographer who was not the instructor recorded the class using one video camera. Although, the videographer occasionally taped whole-group presentations and discussions to capture the general flow of the discussion, the videographer mainly focused on taping small-group problem solving sessions and discussions. The videographer moved the camera from small group to small group with the aim of capturing mathematical discussion. Part of the design of the course is that, after about six class sessions, groups change so that PSTs work with new partners within new small groups. Because of the distribution of the nine selected PSTs at any one time and because the composition of the groups varied throughout the semester, the videographer moved among the five table groups and did not follow only the nine selected PSTs. The videographer attempted to capture full discussions, but when a discussion or
negotiation about the mathematics seemed to come to a natural stopping point in one group, the videographer moved the camera to a group that was engaged in discussion.

Each table group was audiotaped every day as a way to back up the video. At times, it was difficult to sort out the interaction on the videotape, and I used the audio tape to clarify the utterances. PST work from the class sessions also provided a secondary source to support the interpretation of utterances and gestures on the video tape. As PSTs described their strategies or indicated places in their work to accompany their utterances, I looked at the scanned student work to more fully make sense of their interactions. In addition, I took notes about each class session after the class ended. I briefly describe these data sources in later sections.

4.4.2 Videotaped special problem-solving sessions

Initially, the data collection plan included a set of individual PST interviews where PSTs would be asked to think aloud as they made sense of and solved a problem on their own. Because the focus of the study is on discourse, the plan to do individual interviews evolved into a plan to have PSTs engage in meaning making in a context that might broaden and enhance the data already collected during small-group problem solving in the classroom setting. In order to provide points of contrast and comparison in coding, a non-routine problem was developed for use in a partner problem-solving situation. All PSTs participated in one problem-solving session with a partner. These sessions took place during the last two weeks of class. A stationary camera was set up, and all sessions were videotaped. Either myself or a volunteer administered the task.

The task administrator engaged minimally with the PSTs, basically just framing and introducing the task. (See Appendix A for a copy of the task implementation instructions.) PSTs were told that the results of the interview could contribute positively to their final grade, but that
it could not negatively affect their grade. The problem-solving session emphasized collaborative problem solving around a non-routine problem. Because the semester strongly emphasized interpreting, creating, and making connections among representations; justifications; and properties of arithmetic, the task was grounded in these three areas. The task was presented to PSTs on three separate pages (one page for each model) to provide ample room for diagrams and explanations. (See Appendix A for the task parameters.)

4.4.3 Transcripts of videotaped small group interactions

After class sessions and special problem-solving sessions, video episodes were viewed to look for patterns related to how PSTs engaged in discourse while problem solving. Several focus episodes were selected to be transcribed for a more thorough analysis of discourse during the meaning-making process. I describe the selection details and the selected focus episodes in the Data Analysis and Results sections of this paper. The transcripts were used to better discern discourse patterns in the interactions and to unpack precisely how the PSTs were using mathematical language.

4.4.4 Course planning documents

Before the course began, I created a detailed plan of the course. (See Appendix B for an example.) As the course progressed, I modified the plan to reflect in-the-moment instructional decisions and de facto modifications that occurred as I implemented lessons. Modifications resulted in the production of a final enacted course plan (see Appendix C for an example), which included my notes about what occurred during the lesson that might have influenced or necessitated changes in the lesson plans. These planning documents served as a reference as to
where in the scope and sequence the focus episodes fell and as to how the content in which the
PST discourse was embedded had progressed.

4.4.5 **Course reading assignments**

The course reading assignments derive mainly from one textbook designed to be used in
PST mathematics content courses (Bassarear, 2001). The language used in the textbook is the
mathematical language accepted by many mathematicians as composing the mathematical
discourse PSTs should be able to understand, participate in, and/or use in their own teaching. The
course textbook serves as the main resources for determining what terms and vocabulary to
include in my analysis of PSTs’ use of authoritative language.

4.4.6 **Instructor’s reflective notes after lessons**

During both an earlier iteration of the course and during the current study, I recorded
written reflections after each class session about how the lesson plan was enacted—that is, from
my perspective as the instructor, what actually occurred during the lesson. These reflections
included descriptions of how mathematical tasks from the lesson seemed to (or failed to) provide
opportunities for PSTs to engage in meaning making around mathematical ideas; questions about
how the PSTs engaged in the task; speculations about how the task presentation (or tasks) might
be modified, enhanced, or perhaps redesigned completely to better foster meaning making; and
finally observations of and reflections about what PSTs describe, discuss, question, present,
represent, and so on. These notes, along with the original and modified lesson plans provided
opportunities to connect lesson context and tasks with PST interactions.
4.4.7 **Written work**

I collected digital images of PSTs’ written work from their class logs, reflections (written during class), written assignments from class and homework, and for various assessments. PSTs completed a class log daily where they described their strategies and thinking during problem-solving and discussions. Once each week, PSTs responded to a reflection prompt about some content they had been studying. Because the focus PSTs were not chosen until late in the semester, everyone’s work was digitized and archived. The written work served as a reference when it was difficult to see what PSTs indicate in their discussions on the videotape. In addition, the written work often provides a record of the steps PSTs explore and explain as they are engaged in solving problems together.

4.4.8 **Assessments**

In addition to the written assessments completed as part of the course (which includes a mid-term and a final exam requiring explanation on at least 50% of each exam), at both the beginning and at the end of the course, PSTs completed the IMAP Survey (Ambrose et al., 2004; Ambrose et al., 2003) (Appendix D) and the LMT (Ball et al., 2001; Ball et al., 2008) (Appendix E). Over one hundred PSTs at the University of Illinois Chicago have completed these assessments as pre-and posttests over the last three years. At some point, the course design could be evaluated by comparing the outcomes for this group of PSTs with the norms established in the other classes. This kind of evaluative work is beyond the scope of this study, but conceivably, this class’s results on the assessments might inform future revisions to the course design.
Initially, I had planned to use the PSTs’ performance on these instruments to select focus PSTs. However, their scores on the pretest did not differentiate among PSTs in any discernible way. I returned to these instruments after selecting the nine PSTs based on their participation in interactions to look for patterns in their performance. In Chapter 6, I discuss how changes in scores on these assessments seem to correlate to individual PSTs’ discourse patterns.

4.5 Data Analysis

4.5.1 Getting to know the data

The total number of hours of video came out to just about 50 hours. I began by watching all the video from days that involved PSTs solving problems or working through problem situations in small groups—a total of about 40 hours. By solving problems, I mean the task presented PSTs with a particular non-routine mathematical problem to solve, and engaged PSTs in working together or individually to solve the problem. They generally used strategies that were either student-generated or based on their prior knowledge. By problem, I mean a question or set of questions related to a single task that can generally be solved in one class session or less. (See Figure 2 on the next page for a sample problem from the course. This sample problem is taken from a middle school mathematics textbook.\textsuperscript{13} By problem situation, I mean the task involves more than solving one particular problem. Generally, a problem situation involves analyzing or evaluating a situation and engaging in mathematical activity in the context of that situation or perhaps generating problems or creating a presentation about the mathematics embedded in a specified situation. A problem situation that PSTs grappled with, for example, is

\textsuperscript{13} Excerpted from Mathematics in Context: Comparing Quantities copyright 1998 by Encyclopedia Brittanica Educational Corporation.
the one pictured in Figure 3 derived from a problem found in an elementary school preservice teacher textbook (Bassarear, 2001). (See Figure 3 for the complete instructions for this problem situation.)

Figure 2. Example of a problem in the content course.

I watched the classroom video through the first time with an eye towards accomplishing three goals: to identify general patterns, to figure out how to segregate the video, and to highlight video that would be potentially useful in my investigation. First, I made notes about the general
Numbers in Alphabatia

Imagine that you are a member of a small tribe that lived thousands of years ago, when people were making the transition from being hunter-gatherers to becoming farmers. You have a numeration system that is alphabetically based, so you are called Alphabitians. As is true of many other ancient peoples, your numeration system is finite. For any amount greater than Z, you have no symbol; you just call that amount “many.”

<table>
<thead>
<tr>
<th>Amount</th>
<th>Alphabitian numeral</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
</tr>
</tbody>
</table>

Now that your tribe has settled down, you have “many” sheep and “many” ears of corn. Without an adequate numeration system, figuring out how many more sheep you have this year than last year and determining each family’s share of the corn harvest are very tedious. Recently a young woman in your tribe had excitedly announce that she had invented a new counting system with which she can represent any amount using only the symbols A, B, C, D, and a new symbol she calls zero and writes as 0. Unfortunately for you tribe, this young woman died on a hunting trip. However, she left behind some artifacts that she was going to use to help you learn the new system. These artifacts are called flats, longs, and units.

Because the visionary member of your tribe is no longer with you, it is up to you to invent the new numeration system that your tribe desperately needs. That is, you need to develop a system that lets you represent any amount using only the symbols A, B, C, D, and 0.

1. It is up to you and your classmates to create a numeration system using only the symbols A, B, C, D, and 0.
2. Once you have agreed on a system, consider the following questions:
   a. Does your system make sense to every member of your group?
   b. How would you explain your system to the council of elders?
   c. What made you choose this system? What are its advantages?
3. Count up to double Z in your system (Z + Z in the Alphabitian system).
4. Make up and solve some simple number stories in your system. (e.g., selling sheep or planning rows of trees or dividing ears of corn between multiple families.)

Figure 3. Complete instructions for the number-system problem situation.
patterns I was seeing and questions I had as I watched the video. For example, I noted that PSTs often explained strategies to each other, and I wondered both about their purposes for explaining, as well as how they knew whether their group members understood their explanations.

Second, during the course of the first viewing, I devised a method for segregating the video into sections in some meaningful way. That is, after watching several hours of tape, I went back and segregated the tape into episodes. I defined an episode as a collection of interactions, related by a problem or problem situation. An episode begins with an opening comment by one of the PSTs or an instructor and ends when the individuals involved in the episode change, the topic changes, or when the camera moves to a new group or setting. I made notes about the episodes as I went. (For an example, see Figure 4 on the next page. For record-keeping, the blue areas and associated Xs in the V and T columns indicate that a clipped video segment (V) was made and that the episode was transcribed (T).) During the first viewing, I gave each episode a segment number (far left); identified its location on the video (Disk column), briefly described the episode; noted whether I thought it would be useful for further analysis (highlighted in yellow—a less positive identification is green); and recorded general notes or questions.

When one of the PSTs or an instructor presents to the whole class, I also count this as an episode. An interaction consists of two or more individuals engaged in communicating around some problem or problem situation. The individuals were usually PSTs, but occasionally, the instructor was also one of the participants in the interaction. An interaction begins when one individual makes a comment to or asks a question of someone else. An interaction ends when individual contributions are no longer related to the initial comment or question. For the first
Figure 4. Record keeping sample from first video viewing.

video viewing, I focused on sectioning the video into episodes. I identified 209 episodes in the 
40 hours of tape. In the transcript excerpt below, Lines 1-7 are one episode. The camera first 
focusses on the group in Line 1. Since there was not much mathematical discussion occurring, a 
few seconds later at Line 8 the camera moves and focuses on a different group.

1 S1: I got 1282.
2 S2: I gotta wait till, which one, this one?
3 S1: Yeah
4 S2: I got 1122.
5 S1: How ‘bout this one.
6 S2: That one? Oh. I got 2002. This one is right [pointing at the problem]
7 S1: 1 times 2 is 2
8 [CAMARA MOVES]

The above example is one of the shortest episodes. In contrast, the longest episode is twenty-two 
minutes during one of the Special Problem-Solving Sessions when PSTs worked with a partner
to solve the problem with models for properties of arithmetic. Finally, as I watched the video through the first time, I identified 83 episodes that might have potential for further analysis in terms of understanding meaning making.

4.5.2 A Process for developing a coding system

Consistent with grounded theory (Glaser & Strauss, 1967), I sifted through the 83 episodes I identified in the first video viewing to select several episodes for more intensive study that would maximize the diversity of contexts and participants. The purpose for attending to diversity was to provide a data set, with the broadest possible range of characteristics, which would support the development of robust preliminary interaction categories. For example, in the first viewing, my notes include references to a wide range of interactions such as, questions asked for clarification of directions, questions asked for clarification of meaning, statements made to describe representations, negotiations around verbiage used to describe an action or idea, reporting out on strategy steps, negotiations around what a problem actually required in order to set it up and then solve it, and so on. I attempted to include as many types of exchanges as possible in the initial episodes for identifying preliminary categories.

I watched the episodes repeatedly, comparing the interactions in these episodes, in order to formulate a working description of what I was looking for as meaning-making activity. The description I used to identify episodes for a more in-depth analysis was as follows: The episodes include interactions between two or more PSTs as they attempt to negotiate how to make sense of a problem or problem situation and/or how to accomplish an assigned task. This then became the hallmark of episodes that potentially involved meaning-making activity—that is, PSTs communicating while, on some level, negotiating around a problem or problem situation.
I chose four episodes to examine more closely for the purpose of developing categories. These episodes took place in different contexts. The first episode involved solving several multiplication problems in base-5 using arrays. The second and third episodes were related to each other and involved a problem situation. The PSTs were given a non-routine subtraction algorithm and their task was twofold—to unpack the mathematics behind how the algorithm worked and then to plan how to teach this algorithm to their classmates. The “teaching” included not only the procedure, but also how the algorithm worked mathematically. The second episode shows one PST explaining to her group how their subtraction algorithm works. She uses base-10 blocks to represent the mathematics embedded in the procedure. In the third episode, the camera follows one of her team members to a new group where he presents the subtraction algorithm to classmates who have not seen it before. The final episode is one of the Special Problem-Solving Sessions where two students work together to figure out which models map to which properties. The context differs across these episodes. The first captures a group working together to solve a multiplication problem. The second involves one group member supporting her group in understanding a new algorithm. The third follows up on the second as one PST attempts to apply what he learned in a prior negotiation of meaning. The last presents the negotiations that resulted when two PSTs were asked to work in a partnership to solve the special interview problem requiring an interpretation and comparison of models to properties of arithmetic. Once I had selected the four episodes that would support my initial efforts at category development, I divided each episode into interactions. Table 1 shows the variation among episodes for number and length of interactions.
<table>
<thead>
<tr>
<th>Episode Number</th>
<th>Number of Interactions</th>
<th>Median Length of Interaction</th>
<th>Total Length of Episode</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>11</td>
<td>1.5 minutes</td>
<td>20 minutes</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>2.5 minutes</td>
<td>14.5 minutes</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>2.75 minutes</td>
<td>16.5 minutes</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>1.5 minutes</td>
<td>13 minutes</td>
</tr>
</tbody>
</table>

The development of categories and corresponding codes occurred simultaneously with segmenting the episodes into interactions. This was an iterative process where, as I identified the breaks for interactions, I developed categories for coding the interactions. After coding several interactions for a given category, I attempted to define and describe the properties of the category. Using the constant comparative method, I repeatedly compared the interactions with the same code to be sure the properties made sense. I attended closely to properties I was describing to check my categories. For example, with the category of reporting, I examined the interaction to see if PSTs were explaining their strategy steps or ideas in the absence of any justifications or persuasive arguments. In addition, I compared across codes to minimize the overlap in the properties for different categories. For example, the category of fully engaged might appear similar to the category of reporting except that PSTs accompany their explanations with justifications or persuasive arguments inviting their peers into a conversation about what they are explaining. Once the categories seemed stable for some subset of the interactions, I continued coding previously uncoded interactions, and repeated the process.

When I could consistently apply the codes to the first four episodes, I revisited the larger data set. I discarded video that involved PSTs working silently and independently and incidents
where the instructor was simply explaining without inviting PST participation. Then I took a subset of what was left, sampling from early in the semester, the middle of the semester, and the end of the semester—including the Special Problem-Solving Sessions. Although I was applying the codes I had developed, I repeatedly refined the properties and their descriptions whenever an interaction was not an exact match for the existing properties of a category. (See Table II in Chapter 5 for a complete list of the codes and their detailed descriptions.) After each refinement, I went back to similar interactions in the original four episodes to check the revised descriptions with the initial coding to make sure the revisions were consistent with the earlier coding. I continued in this way until the categories were theoretically saturated—that is, until I did not identify any new properties or properties requiring revision. At this point, I could consistently apply the existing codes to all subsequent interactions. In total, I coded about thirteen and a half hours of video. I discuss the results of this process beginning in Chapter 5 of this paper.

4.6 **Reliability and Validity**

A case study of this nature, of a qualitative approach to research, involves different characteristics of reliability and validity than what would be found in a quantitative study. Consistent with a case study design, validity can be construed in qualitative terms such as dependability and consistency. First, I transcribed the focus episodes, and therefore, I was able to rely on consistent records of what PSTs actually said in the episodes in developing codes and categories. Second, using a grounded-theory approach resulted in a set of codes and categories that were theoretically well-saturated—that is, no matter what videotaped episode I analyzed, no new qualities or properties could be added to the codes and categories based on that episode—and that could be consistently applied to the data. Third, I engaged in researcher reflexivity by carefully and thoughtfully making explicit the lens I used (based on my experiences and beliefs)
to make important decisions about the design of the research. My decisions ultimately affected what data was available for analysis, how I analyzed it, and what inferences I drew from that analysis. Fourth, after developing the codes and coding categories, I inquired how reliably the inferences from the data analysis accurately portrayed the experiences of the case study participants by doing some member checking with them (Creswell & Miller, 2000). The member checking occurred in informal conversations and through the review of their class journals as related to specific tasks. For example, in one section of the video, one PST coaches another PST through how to use blocks to model a base-5 multiplication problem. It appears that the second PST has made sense of the representation, but the video ends without conclusive evidence that the second PST has grasped the base-5 block model. Upon examination, the second PST’s journal entry reveals that he did in fact make sense of both the problem and the representation. (See Figure 5.) The member-checking conversations focused on reminiscing about particular tasks and the takeaway from those tasks. Fifth, because of the multiple data sources, I had opportunities to triangulate the data across the different sources. For example, I could compare

![PST journal reflection describing how he came to understand a problem.](image)

Figure 5. PST journal reflection describing how he came to understand a problem.
how a PST discusses a problem during the small group work and then look to the copy of their written work compare the language used.
5 Results: Describing Meaning Making

5.1 Original Expanded Code Categories

The initial analysis of four focus episodes resulted in a total of twelve codes that I used to characterize the discourse across episodes: 1) Explaining, 2) Fully Engaged, 3) None, 4) Note Taking, 5) Off Task, 6) Making Sense of Instructions, 7) Making Sense with the Teacher, 8) Partially Engaged, 9) Question and Answer, 10) Reporting, 11) Teacher Questioning, and 12) Teacher Whole Group. As the coding progressed beyond the focus episodes, two codes that had seemed distinct from the others initially began to overlap with other codes. Making Sense of Instructions overlapped with Partially Engaged or Question and Answer. Making Sense with the Teacher overlapped with Teacher Questioning or Reporting. About a quarter of the way through the remaining episodes in the full data set, I went back to attempt to revise the description for these two categories to make them more distinct from other codes. The end result was, I slightly revised the overlapping code descriptions so that they would include every instance of the two codes in question. I therefore eliminated Making Sense of Instructions and Making Sense with the Teacher from the list.

The process of revisiting codes that overlapped resulted in a final list of ten codes. I reviewed all of the interactions previously coded with one of the two deleted codes, of which there were eight, and brought them into alignment with one of the ten final codes. (See Table II for a description of when each of these ten codes is applied.) This set of ten codes set the foundation for me to explore subsets of related interactions in my quest for understanding more about the meaning making process of PSTs engaged in mathematical activity.
### TABLE II
DESCRIPTIONS FOR EXPANDED SET OF CODES

<table>
<thead>
<tr>
<th>Code Title</th>
<th>Code Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Explaining*</td>
<td>PSTs present their thinking in a whole-class format. They may be explaining the way they are thinking about something, how they solved a problem, or questions that they have. Only one PST is speaking.</td>
</tr>
<tr>
<td>Fully Engaged</td>
<td>Most of the table group is engaged in the interaction. PSTs’ comments, questions, and explanations relate to each other and are part of a discussion about a problem, a problem situation, or a solution strategy. PSTs not only directly comment on, add to, and question each other’s statements and explanations, but they also address and comment on prior related contributions.</td>
</tr>
<tr>
<td>None*</td>
<td>There are no relevant interactions or contributions to consider.</td>
</tr>
<tr>
<td>Note Taking*</td>
<td>PSTs working individually at the end of a “Fully Engaged” interaction. PSTs record the results of their group discussion or work. Sometimes, there is direct and explicit evidence that they are recording notes based on what they did as a group, such as asking a clarifying question of the table group or confirming that others in the group are recording the same results.</td>
</tr>
<tr>
<td>Off Task*</td>
<td>PSTs finish a task and wait for the next steps; or they should be engaged in a task, but instead, they appear to be discussing something unrelated.</td>
</tr>
<tr>
<td>Partially Engaged</td>
<td>Only a subset of a table group engages in a discussion that involves features of “Fully Engaged” activity and interactions. Often this may include just two members of a four or five member table group. The rest of the group either appears to be working independently or to be off task.</td>
</tr>
<tr>
<td>Question and Answer</td>
<td>PSTs engage in a series of questions and answers—sometimes without justifications attached to the answers.</td>
</tr>
<tr>
<td>Reporting</td>
<td>Students share what they have written down or how they have thought about something. This may involve explanations as well as just reporting their answers or the steps of their strategies. Their explanations are reports of what they have been doing and generally do not include justifications or persuasive arguments.</td>
</tr>
</tbody>
</table>
Since I was investigating meaning making, the next step in the coding process brought me to consider how the codes may or may not represent instances of PSTs making meaning together—PSTs negotiating meaning around a problem, problem situation, or solution strategy. The Fully Engaged and Partially Engaged codes, by definition, consistently applied to interactions showing PSTs in negotiations around making sense of a problem, a problem situation, or a strategy. Question and Answer and to a lesser extent Teacher Questioning sometimes applied to interactions that possessed features similar to those of the Fully Engaged and Partially Engaged Codes—that is, interactions that seemed to have some negotiation of meaning at their core. The remaining codes never included interactions that involved negotiations or observable sense making. In fact, five of the six codes that never involve negotiations or sense making in the interactions—Explaining, None, Note Taking, Off Task, and Teacher Whole Group—were only important in enhancing the flow of coding episodes. During an episode, for example, PSTs will discuss the problem (Fully Engaged) and then record what
they discussed (Note Taking) and finally return to the discussion (Fully Engaged). Rather than work with disjointed transcripts and coding, it made sense to include codes for the interleaved interactions even though they did not include relevant discourse among PSTs.

5.2 Aggregated Code Categories and Meaning Making

An overarching analysis of the content of the ten codes led to three broad categories for the codes—Type 1) those consistently involving the negotiation of meaning or making sense of a problem, a problem situation, or a strategy; Type 2) those that sometimes involved this type of negotiation; and Type 3) those that never involved this type of negotiation. As seems often to be the case, a code that was inconsistent, one that crossed the boundaries between including negotiations of meaning and not including negotiations (Type 2 above), seemed to offer the best opportunity for initially teasing out how to determine when meaning making through discourse among PSTs might be occurring.

The Question and Answer code is one of the two boundary-crossing codes (Type 2 above) where PSTs may or may not be negotiating meaning in the interactions. This code describes episodes in which PSTs are asking questions of each other about a problem, a problem situation, or a solution strategy, and then answering those questions. Sometimes, the questions elicit thoughtful answers that are direct responses to the questions, possibly including a justification or a relevant explanation. For example:

S1: So that equals 42, right?
S2: Oh and C would be 6 because you have C on both sides, and you have 6 on both sides [she indicates this on her paper as S1 looks over], in both blocks.
S1: I just, kept it the same but changed numbers, uh letters. [S2 looks at S1’s paper] Does it matter?
S2: No, because if you distribute it, you have both As and 6 is on both sides. [she points to where this is on his paper] That’s what I mean by both sides.
This type of interaction coded as Question and Answer would fit into the larger category of negotiating meaning. At other times, the Question and Answer code refers to interactions where the answers do not address the nuances or intent of the questions. For example, a PST may simply repeat a report of his or her thinking or the steps he or she followed without any consideration for the actual question asked.

S₃: Equals BA ...and the other one's like B plus A. Wait. That's the same one, isn't it?
S₄: I know there's one that switches. But... Ach…

This type of interaction coded as Question and Answer would fit into the larger category of no negotiation of meaning.

In comparing and contrasting the features of the boundary-crossing coded interactions, one feature that distinguishes the interactions is the verbal cues that signal a negotiation of meaning is occurring. First, the PSTs’ word choice and word use vary between when they are negotiating meaning and when they are not. For example, PSTs say “we” far more often when discussing and negotiating the meaning of and how to solve a problem. One of the best contexts to explore this in my data set is in the Special Problem setting because the partnerships are addressing the same problem in the same setting, with the same time constraints. In comparing two of the more sharply contrasting Special Problem episodes, Partnership A, who is more often engaged in meaning-making activity (approximately 13 out of 20 minutes or about 65% of the time), uses the pronoun “we” 32 times versus only 5 instances of using “we” in Partnership B, who is not engaged in meaning making as often (approximately 6 out of 20 minutes or about 30% of the time). In addition, there is a qualitative difference in the usage. Partnership A consistently uses the pronoun “we” to discuss what they are doing, whereas Partnership B
consistently uses “we” in the context of asking the teacher how “we” are to proceed. See the illustrative excerpts below:

Sample of Partnership A’s use of we (meaning making)

S₁: Okay. So then, if we want to do A plus B, we have to switch this. [S₂ moves his paper even closer to his partner’s, as he watches the changes S₁ is making.] …and make 6, C, 3, A, 4, B. That’s A [S₁ labels the model] …B… C times A plus B equals CA, CB, which is what we got.

Sample of Partnership B’s use of we (clarifying instructions)

S₃: Do we have to write more than one sentence or is that okay? [looking at the teacher]

Second, use of the pronoun “you” differs significantly between meaning-making versus non-meaning making interactions. For example, in the Special Problem setting, Partnership A (65% meaning making) uses the pronoun “you” 51 times of which 48 of those usages (about 94%) are in service of describing the steps “you” could take as part of their negotiation of how they will proceed. Three times (about 6%) one partner of Partnership A says “you” when directly questioning the other partner. In contrast, the partners in Partnership B use the pronoun “you” only 18 times of which six of these uses (about 33%) occur when one partner directly questions the other partner. See the illustrative excerpts below:

Sample use of you in service of negotiating around the problem

S₁: Oh!... Would Model B be associative? Because you split it up… into… into, fifths? And then you can split it up… because then, it doesn’t matter how you associate it [S₁ gestures with her hands to indicate different groups] as long as you have the same answer, right? [S₁ is still bouncing hands in a gesture that looks like she is “holding” separate groups as S₂ looks on] ‘cause you don’t…

S₂: Yeah. ‘Cause either way it’s going to be the same amount of boxes in each, um, model. It’s just split differently. So… but wouldn’t that be?

Sample use of you in addressing a partner

S₃: Okay. How would this be associative? That’s the one you said was associative, right? [looking at the problem on one of her pages]

S₄: This one? No, I said this one. [looking at the problem on one of her pages]
S3: Oh, this one? [shifting the papers around and putting another on top]
S4: Then, I'm confused now, so... [looking at her paper]

A third verbal indicator that PSTs are negotiating meaning involves fact checking. By fact checking, I mean that PSTs check in with other group members to see if what they are thinking or doing aligns with the direction the group discussion is taking or with what the group has discussed and/or decided. See the transcript excerpt below for an example:

[PSTs write]
S1: I just want... why's this one associative again? Because there’s the same amount of blocks no matter the ways that it’s split, right? ...er, the different ways that it’s split does not change the fact that there are... that the sum...
S2: Cause you have the same amount, the same amount, like the same total.
S1: Okay... same total is constant.

[PSTs write]
S1: Distributive. The original block has... it like it’s all group together.
S2: And then you split it up...
S1: And then you split it up... to... to... add them up individually. Right?
S2: Yeah.
S1: Like the first block is all grouped together, er, the numbers are all grouped together... er the blocks or the squares are all grouped together?
S2: Um hmmm.
S1: So, let’s write that.
S2: So, what did we say?
S1: The squares are grouped together.
S2: They’re all grouped together
S1: ...and when it’s split into two groups, you can multiply and add it together
S2: Yeah, but you can, multiply each side separately...
S1: You can multiply each side separately.

Indicators of meaning making include physical as well as verbal components. Although a detailed analysis of the physical indicators that negotiation is occurring is beyond the scope of
this study, with further study, physical actions ultimately may prove to be more reliable and consistent indicators of an interaction where meaning is being negotiated. The three verbal indicators described in the paragraphs above generally appear in my data set in the context of negotiations of meaning, but these indicators also sometimes appear when no negotiation is taking place. For example, when the teacher asks a PST, “…and what will you do next,” a PST responds “You could just add them,” which is similar to the example above where partners are exploring how they might proceed on a task. In this example though, the PST is simply reporting the next step in a process in response to the teacher’s question. Additionally, in the data collected for this study, the verbal indicators did not ever exist in the absence of physical indicators.

The physical actions associated with the meaning-making interactions in this study fall into the category of gesture as Roth (2001) describes it:

Gestures can be distinguished from other movements…by four characteristics (Kendon, 1980, 1996). First, gestures begin from a position of rest, move away from this position, and then return to rest. Second gestures have a peak structure, also referred to as the stroke, which is generally recognized as a moment of accented movement to denote the function of meaning of a movement. Third, the stroke phase is preceded by a preparation phase and succeeded by a recovery phase in which the hand and arm move back to their rest position. Consequently, gestures have a clear beginning and ending. Fourth, gestures are often symmetrical (and, for instance, would therefore be difficult to use as a basis for deciding whether videotape is run forward or backward) (p. 369).

Several gestures are prevalent during interactions that involve the negotiation of meaning. First, PSTs sometimes change their posture and lean into a common space. Figure 6 on the next page shows what this shift might look like when PSTs are working with a partner.
Figure 6. A shift in position from individual focus to joint focus.

Second, PSTs may move their attention and gaze toward another group member’s paper or some common representation or set of manipulatives involved in a negotiation of meaning. Third PSTs sometimes move their own papers or manipulatives they are using to ameliorate another group member’s access to the work. Fourth, PSTs may use what McNeill (1992) and others call deictic gestures—that is, pointing to indicate or clarify what they are referring to, describing, or explaining. (See Figure 7 below for an example of deictic gesture. Note that the focus for both PSTs is in the same location on one paper in each frame of the figure.)

Figure 7. Deictic gestures or pointing to direct attention and clarify referent.
Finally, PSTs sometimes use iconic gestures, gestures that are closely related to the semantic content of what they are verbally describing (McNeill, 1992), to enhance their communication of an idea or a process that might be difficult to verbally communicate (Singer, Radinsky, & Goldman, 2008). In the example below, one PST emphasizes how the blocks in the model are grouped to show the associative property. As she explains, she uses her hand gestures to show how the three groups of blocks are associated and says, “It doesn’t matter how you associate it, as long as you have the same answer.” (See Figure 8 below.)

![Figure 8. Example of iconic gesture for a grouping situation.](image)

5.3 **Focus Episodes**

Once I established some guidelines for identifying and describing possible interactions where observable meaning making is occurring, I selected focus episodes to examine in greater depth. I selected the episodes to maximize the variety in contexts and to focus on particular
aspects of my research subquestions. Episodes 1 and 2 occur at the end of the semester. PSTs are instructed to work together to match the properties with the models. These episodes last about 20 minutes. Episode 3 takes place in mid-October (about half of the way through the course), table groups have been asked to figure out how an unfamiliar subtraction algorithm works mathematically and then to plan how to teach the algorithm to other groups of PSTs in the class. (See Figure 9 below for the algorithm they are using.) Episode 4 follows one member of Episode 3 as he moves to a new group unfamiliar with the subtraction algorithm. He is charged with teaching the algorithm to his new group. This episode takes place on the same day as Episode 3. Episode 5 takes place in the second week of November (about three fourths of the way through the course), PSTs are solving a multidigit multiplication problem in base 5.

\[
\begin{align*}
\text{Example:} \\
826 &- 438 \\
8 &\quad 12 \\
-4 &\quad 8 \\
3 &\quad 8 \quad 8
\end{align*}
\]

Figure 9. Subtraction algorithm PSTs are using in Episodes 3 and 4.
6 Results: Calculational and Conceptual Talk

In my analysis of PST discourse, I aimed to first make distinctions between conceptual and calculational talk, and then to investigate any differences between conceptual and calculational talk in PSTs’ meaning-making versus non-meaning-making interactions. Following the work of Cobb, Stephan, McClain, and Gravemeijer (2001) and Thompson, Philipp, Thompson, and Boyd (1994), I define conceptual talk as discourse more focused on sharing and exploring mathematical ideas. PSTs involved in conceptual talk focus on making connections between ideas or between ideas and the steps of the strategy or procedure they are explaining or questioning. Their explanations rely on conceptual coherence—that is, they attach mathematical meaning to the statements they make or the steps they describe. Conceptual talk describes PST interactions that are more focused on the aspects or characteristics of problems and problem situations or on understanding a mathematical process. Also consistent with Cobb et al. (2001) and Thompson et al. (1994), I view calculational talk as more focused on stating or providing mathematical information or steps of a strategy or procedural process. PSTs involved in calculational talk might be reviewing answers or strategies for correctness or reporting on strategies or computations being used. Calculational talk describes PST interactions that are more focused on providing or sharing mathematical information.

6.1 Codes Associated with Calculational Talk and Conceptual Talk

When PSTs engage in conceptual talk versus calculational talk, the goals in the interaction seem to differ. When PSTs are engaged in conceptual talk, they are often sharing with each other the mathematical ideas and opinions that they have. Other group members respond to those ideas in any number of ways—for example, with questions about details or justifications,
with contrary statements of ideas or opinions, with examples and counterexamples. Significantly, the initiator of the interaction seems to expect a response. The interactions involving conceptual talk can regularly be found coded as Fully Engaged and Partially Engaged. In addition, some conceptual talk interactions are assigned the Question and Answer and Teacher Questioning codes.

When PSTs engage in calculational talk, the goals for the interactions tend to focus on delivering information about a process, a strategy, or an answer. When PSTs are engaged in calculational talk, they are often reporting to their classmates about what they thought or did, or about what they are thinking. Their group members might ask clarifying questions about details or order of steps, for example, but seldom go beyond this kind of questioning. Rarely does the initiator of the interaction appear to expect a response. These interactions generally appear in sections coded as Explaining or Reporting. Additionally, calculational talk interactions are sometimes assigned to interactions coded as Question and Answer and Teacher Questioning.

Although both conceptual talk and calculational talk may involve meaning making for individuals, conceptual talk figures much more prominently in the interactions included in the larger category of those consistently involving the negotiation of meaning or making sense of a problem with others (in the way I have construed meaning making for this study). Conceptual talk relies on an underlying mathematical coherence and establishing connections among mathematical ideas. PSTs engaged in conceptual talk present their ideas for evaluation and argument. They provide mathematical justifications and are receptive to critique and questioning. These features of conceptual talk lend themselves well to a negotiation of meaning around problems or problem situations.
Although it does not invite or require negotiation, presenting information in the context of calculational talk may establish a common starting point for exploring mathematical content. When calculational talk is assigned to boundary-crossing codes (Question and Answer or Teacher Questioning) and is juxtaposed with conceptual talk, the calculational talk may provide a basis for further discussion and negotiation. In the brief excerpt below, S₁ is explaining the steps of a subtraction algorithm. Other PSTs in his group are trying to figure out, based on his presentation, how the algorithm works. This following calculational interaction is coded as Question and Answer (a boundary-crossing code).

**Calculational Talk Lead In**

S₁: This would be a 16. But this would be, I think, 4. Is that a 4? [showing what he wrote] Because you're taking away. You're adding a ten from this column. You're taking ten units. So, and you're adding, and you're taking another... you're adding another ten to this column. So that will still be a plus ten. So we added ten to both columns

S₂: Wait, on the second column, where did you add a ten?

S₁: Right here. [indicating the tens column in the subtrahend]

**Conceptual Talk Follow up**

S₂: Okay. That turns into fourteen or four... or thirt...? wait, I don't get the four. [pause while S₁ looks at the work] Where are you getting three to four? You're adding?

S₁: Wait, now how do you explain... [Pause while S₁ looks at the work] I should do it with the blocks. I think that would...I think it's better with the blocks.

S₂: Okay. Use the blocks. I don't know.

S₃: Use the blocks.

From this point in the transcript, the PSTs continue to negotiate how the algorithm works using the blocks to try and model what happens in the algorithm.
6.2 **Comparing Calculational Talk and Conceptual Talk in Episodes**

To further explore how conceptual talk and calculational talk differ, I performed a close examination of and compared two focus episodes. I selected two episodes that shared as many features as possible so that I could focus on the calculational and conceptual talk. Episodes 1 and 2 both occur on the same day of class at the end of the semester. The setting is the Special Problem-Solving session where partnerships are instructed to work together to solve a problem. The session begins with the instructor reviewing the introductory script (see Appendix A). The instructor informs the PST partnerships that the emphasis in these sessions is on working together, not on necessarily arriving at the “correct” answer. Although there is no penalty for failing to solve the problem with the one “best” answer, the instructor explains that their level of engagement in discussion could potentially contribute positively to their grades.

In comparing the episodes, several similarities beyond the context and set up became apparent. In both episodes, the PSTs successfully arrive at the best answer after about eighteen minutes. In both cases, the PSTs check in with each other and shared their answers or what they are thinking as they attempted to match the models to the properties of real numbers. The partnerships begin with the Commutative Property, and in both cases, they explain that this property is easiest because you can see that the product is the same and the model depicts flipping or turning the same rectangle so that the dimensions will be in different places. As they work, the PSTs in both partnerships periodically record and work on his or her own paper. Finally, throughout the episodes, PSTs appear to talk to each other as they solve the problem.
There are also significant differences in the episodes. Episode 1 with Mark\textsuperscript{14} and Linda lasts about 20 minutes. About 13 minutes of this episode fall into the larger category of meaning making and about 7 minutes fall into the larger category of no meaning making. Episode 2 with Samantha and Justina also lasts about 20 minutes. About 6 minutes of this episode fall into the larger category of meaning making and about 14 minutes fall into the larger category of no meaning making.

Just over half of the interactions in Episode 1 (Mark and Linda), are coded as Fully Engaged (about 11 of 19 minutes). Mark and Linda’s discussion stays connected to and focused on solving the problem. In Episode 2 (Samantha and Justina), the profile of the codes shows a much different distribution. Only about a minute and a half is coded as Fully Engaged. Almost half of the time (about 11 of 19 minutes) is divided between the Question and Answer, and Teacher Questions codes. As might be expected based on the explanation of how conceptual talk and calculational talk are coded, Episode 1 includes a higher concentration of conceptual talk.

In order to better compare conceptual talk and calculational talk, I identify brief segments of each episode that best illustrate the two kinds of talk. For Episode 1, I analyze 2 minutes and 41 seconds of the problem-solving session chosen as an opportunity to further investigate features of conceptual talk. This segment occurs about 4 minutes into the session. For Episode 2, I select 2 minutes and 43 seconds chosen as an opportunity to further investigate features of calculational talk. This segment occurs about 6 minutes into the session. Because the initial instructions for Episode 2 take over $4\frac{1}{2}$ minutes (2 minutes longer than with the PSTs in Episode 1), the segments occur at approximately the same point in the problem-solving process.

\textsuperscript{14} All student names are pseudonyms.
In the excerpt from Episode 1 (Table III) that begins on the next page, \( S_1 \) is Mark and \( S_2 \) is Linda. I have included the transcript and notes about any gestures that occurred during each transcript section. The transcript has been segmented so that each new speaker’s turn is represented as a row. Occasionally, the same turn has multiple rows to facilitate recording the gesture with the appropriate accompanying speech. The diagram in the “Statement Information” column can be read as follows: Line 1 shows Mark making a statement with Linda as the intended audience. The black circle and black arrow, in this case, show Mark to be the speaker. Line 2 follows up on the statement in Line 1. For this reason, the speaker Linda (who was the intended audience in Line 1) is circled in brown in Line 1 and the brown arrow indicates that Linda is speaking to Mark in Line 2. The continuous flow of brown circles and brown arrows indicates that each row of text is connected to and follows up on the previous row. In Line 16, where this pattern ends temporarily, Linda’s question assumes Mark as the audience, but the question does not follow from what they were just discussing. The brown line segment between Lines 16 and 17 shows that there is a connection between those two sections of text, but Line 17 does not follow exactly from Line 16. Once again, in Line 17, Linda is talking to Mark, but this does not follow up from the previous row. In Line 19, a new pattern emerges. Linda is talking
### TABLE III
EXCERPT OF TRANSCRIPT FROM EPISODE 1

<table>
<thead>
<tr>
<th>Statement Information</th>
<th>Transcript</th>
<th>Accompanying Gesture</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>S₁: CA, CB... right? /[with emphasis]/ plus CB</td>
<td>S₁ and S₂ look at their own papers.</td>
</tr>
<tr>
<td>2</td>
<td>S₁: So then... And then what would... This is distributive... And then associative would be... A... That would just be the parenthesis, like it doesn't matter where you do the parenthesis.</td>
<td>S₁ and S₂ look at their own papers. As S₂ says the last sentence, she indicates parenthesis by curving her hands and she names one group—in the air in front of her—then moves her hands to show a second grouping, then moves her hands back to the original position—bouncing back and forth. S₁ looks over.</td>
</tr>
<tr>
<td>3</td>
<td>S₁: Yeah, if, well, yeah if, A plus B... plus C or times C₁₁, plus C?</td>
<td>S₁ glances at her and wrinkles her brow.</td>
</tr>
<tr>
<td>4</td>
<td>S₁: But that's the distributive.</td>
<td>S₁ and S₂ record on their own papers. S₁ has placed his paper between them and they glance back and forth as they write.</td>
</tr>
<tr>
<td>5</td>
<td>S₁: Hold on... and then A, B, C. Look. So it would be...</td>
<td>S₁ and S₂ record on their own papers. S₁ looks at S₂'s paper in response to his &quot;look.&quot;</td>
</tr>
<tr>
<td>6</td>
<td>S₁: Oh, yeah, yeah, yeah /quoting/ A plus B, B plus C, plus C...</td>
<td>S₂ records on her own paper.</td>
</tr>
<tr>
<td>7</td>
<td>S₁: So then Model A is a 4 by, 1, 2, 3, 6, and then a 3 by 6.</td>
<td>S₁ and S₂ look at their own papers.</td>
</tr>
<tr>
<td>8</td>
<td>S₁: B or A?</td>
<td>S₁ looks at S₂'s paper.</td>
</tr>
<tr>
<td>9</td>
<td>S₁: A?</td>
<td>S₁ moves his hand so S₂ can see.</td>
</tr>
<tr>
<td>10</td>
<td>S₁: A?</td>
<td>S₁ and S₂ record on their own papers.</td>
</tr>
<tr>
<td>11</td>
<td>S₁: Yeah, 'cause it's 4. So then, they split it into...</td>
<td>S₁ and S₂ record on their own papers.</td>
</tr>
<tr>
<td>12</td>
<td>S₁: So it's a 4 by... it's a 6 by 4 and a 6 by 3.</td>
<td>S₂ points to the representations on her paper.</td>
</tr>
<tr>
<td>13</td>
<td>S₁: Right</td>
<td>S₁ glances over.</td>
</tr>
<tr>
<td>14</td>
<td>S₁: And then...</td>
<td>S₁ and S₂ look at their own papers.</td>
</tr>
<tr>
<td>15</td>
<td>S₁: And then they split it together... but</td>
<td>S₁ and S₂ look at their own papers.</td>
</tr>
<tr>
<td>16</td>
<td>S₁: Oh!... Would Model B be associative?</td>
<td>S₁ sits up straight and S₁ looks over and then checks one of his other papers and looks over at S₂ again.</td>
</tr>
<tr>
<td>17</td>
<td>S₁: Because you split it up... into... into, fifths? And then you can split it up... because then, it doesn't matter how you associate it as long as you have the same answer, right? /cause you don't...</td>
<td>S₁ cups her hands indicating different groups. As she goes through the description, she bounces her hands in the form of &quot;holding&quot; groups different groups. S₁ looks back at his own paper.</td>
</tr>
<tr>
<td>18</td>
<td>S₁: Yeah, 'cause either way it's going to be the same amount of boxes in each, um, model. It's just split differently. So... but wouldn't that be?</td>
<td>S₂ relaxes a bit in her chair and looks at S₁'s paper as he points to the relevant work.</td>
</tr>
</tbody>
</table>
TABLE III (continued)
EXCERPT OF TRANSCRIPT FROM EPISODE 1

<table>
<thead>
<tr>
<th>Statement Information</th>
<th>Transcript</th>
<th>Accompanying Gesture</th>
</tr>
</thead>
<tbody>
<tr>
<td>19</td>
<td>S: …And, this one would be distributive because...</td>
<td>S: sits up straight again, alert and with energy. She taps repeatedly on S's arm and points to the representation on her paper. S: looks at her paper.</td>
</tr>
<tr>
<td>20</td>
<td>S: [quietly] Oh... right...</td>
<td>S: goes from sitting straight up to relaxing (almost collapsing) in her seat as S: turns back towards his paper, which is still between them.</td>
</tr>
<tr>
<td>21</td>
<td>S: because... [pause]</td>
<td>S: and S: look at their own papers.</td>
</tr>
<tr>
<td>22</td>
<td>S: Wait, and when you added it, it becomes... Wait, would the Model A be the associative? [grow progressively quieter with this sentence]?</td>
<td>S: angles her head towards S: as she asks the question.</td>
</tr>
<tr>
<td>23</td>
<td>S: I mean cause either way, if you like, split it up, if you cause, they're together, wait...</td>
<td>S: shifts his gaze from his paper (placed between them) to her paper.</td>
</tr>
<tr>
<td>24</td>
<td>S: Yeah, because this would be your, your A and your...</td>
<td>S: points to her paper as S: looks over. He appears to be holding his finger on the same representation on his own paper.</td>
</tr>
<tr>
<td>25</td>
<td>S: Would this be your A, and your B, and then your C?</td>
<td>S: taps here paper as S: holds his place and watches As she finishes her statement. S: shakes her head no.</td>
</tr>
<tr>
<td>26</td>
<td>S: Hold on. Which one would look like the distributive?</td>
<td>S: and S: look at their own papers.</td>
</tr>
<tr>
<td>27</td>
<td>S: It's 50, 50. So...</td>
<td>S: smiles as he makes this statement, and then S: joins him in a smile.</td>
</tr>
</tbody>
</table>

aloud, but she does not seem to have Mark as her intended audience. She appears to be thinking through some ideas and verbalizing as she considers the problem. Notice that she makes a series of related comments. In Lines 21 and 22, Linda is talking to Mark. In Line 23, Mark picks up from Line 22 and responds to Linda’s thinking. From there, the initial pattern repeats. Note that, when there is no link to the previous row and Mark is speaking, the line segment (or ray) is black. Similarly, for Linda, the line segment (or ray) is blue.

Notably, many of the rows of transcript in Episode 1 are connected to previous rows. For example, in Lines 1–16, each time a PST comments, the comment connects to a remark or
statement made earlier. Mark and Linda provide each other with ideas, for example, in Line 11, Linda says, “So then, they split it into...” Mark adds, “So it’s a six by four and a six by three.” Linda then confirms that this is correct with, “Right.” The conversation comes to a momentary conclusion as Mark and Linda check their papers, and then it continues with Linda using her cupped hands to demonstrate that the diagrams show groupings. Throughout the episode, Mark and Linda also question each other about next steps, check in with each other to verify that they have a taken-as-shared meaning for their statements about the problem, and confirm that they agree. Most of the discussion that occurs in Episode 1 is conceptual talk, where the PSTs are relating the mathematics in the models to what they know about the properties of real numbers or they are comparing their ideas about how to proceed based on their understanding of the underlying mathematics.

In the excerpt from Episode 2 (Table IV) that begins on the next page, S₁ is Samantha and S₂ is Justina. Similar to Episode 1, I include the transcript and notes about any gestures that occur during each transcript section; generally, each new speaker’s turn has a row; and occasionally, the same turn has multiple rows to facilitate recording the gesture with the appropriate accompanying speech. The diagram in the “Statement Information” column can be read as follows: In Line 1, Samantha says quietly, “So then, this would be the Associative?” She continues what seems to be a conversation with herself in Line 2 with, “I would just be confused about how this would be distributed then.” She closes in Line 4 with “That’s okay.” Samantha does not appear to be directing her statement to Justina. In fact, she concentrates on her own paper as she talks. Similarly in Lines 3 and 5, Justina makes statements, but she is focused on her own paper and seems to be verbalizing thoughts to herself. Beginning with Line 6, there are a string of transcript segments that are connected in content, but they do not follow up
<table>
<thead>
<tr>
<th>Statement Information</th>
<th>Transcript</th>
<th>Accompanying Gesture</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>S₁: So then, this would be the Associative? S₂: I would just be confused about how this would be distributed then.</td>
<td>S₁ and S₂ look at their own papers</td>
</tr>
<tr>
<td>2</td>
<td>S₁: That's okay.</td>
<td>S₁ and S₂ look at their own papers</td>
</tr>
<tr>
<td>3</td>
<td>S₁: I thought you said B. Um. [13 second pause]</td>
<td>S₁ and S₂ look at their own papers</td>
</tr>
<tr>
<td>4</td>
<td>S₁: Ooo! Oh yeah, because I think this one would be distributive. Because if you [continued below where both students talk over each other]</td>
<td>S₁ is pointing to the model on her paper and looking her own paper. S₂ is pointing to her own paper and quickly glances at S₁'s paper.</td>
</tr>
<tr>
<td>5</td>
<td>S₁: combine these two S₂: [quietly] oh, if you combine like</td>
<td>S₁ and S₂ are pointing to areas on their own papers and looking at their own papers.</td>
</tr>
<tr>
<td>6</td>
<td>S₁: There two, it would be one, wouldn’t it?</td>
<td>S₁ points to her paper and glances at S₂'s paper. S₂ stops and looks at S₁'s paper.</td>
</tr>
<tr>
<td>7</td>
<td>S₁: Or am I seeing it a completely different way? [For 4 seconds, S₁ waits for a reply. For an additional 8 seconds, both students work on their own]</td>
<td>Students briefly look at each other.</td>
</tr>
<tr>
<td>8</td>
<td>S₁: Like</td>
<td>S₁ and S₂ look at their own papers</td>
</tr>
<tr>
<td>9</td>
<td>S₁: Like... I combine these two and it would give me one of these; and then I combine these two and it would give me this and these two would be that and this would be that</td>
<td>S₁ records on her paper as she describes the steps. S₂ watches as she works.</td>
</tr>
<tr>
<td>10</td>
<td>S₁: I don’t know S₂: Then wouldn’t that be [continued below]</td>
<td>S₁ glances at S₂'s paper.</td>
</tr>
<tr>
<td>11</td>
<td>S₁: ... too big to make that?</td>
<td>S₁ and S₂ look at their own papers</td>
</tr>
<tr>
<td>12</td>
<td>Teacher: Maybe it would help to label the numbers in the drawings to see how the number sentences look. So, like, if you were going to label this one that you’ve already done? How would you label the model with numbers?</td>
<td>Students briefly look up at the teacher then work on their own papers.</td>
</tr>
</tbody>
</table>
on each other. When Samantha does talk to Justina (in Lines 8 and 9, for example where Samantha says, “These two, it would be one, wouldn’t it? Or am I seeing it a completely different way?” while directing her gaze towards Justina), Justina does not respond. Only towards the end of the excerpt do the rows begin to connect to each other with Samantha asking Justina to clarify what she is thinking and Justina responding.

Episode 2 contains both conceptual and calculational talk. There are a few places where Samantha and Justina use mathematical language to comment on how they are thinking about the problem, but they do not communicate with each other in this process. For example, in Lines 6-8, Samantha and Justina both comment on one of the models. Samantha says, “Oo! Oh yeah.

<table>
<thead>
<tr>
<th>Statement Information</th>
<th>Transcript</th>
<th>Accompanying Gesture</th>
</tr>
</thead>
<tbody>
<tr>
<td>S3: [quietly]...so this would be um... [continued below]</td>
<td>S1 and S2 work on their own papers</td>
<td></td>
</tr>
<tr>
<td>[speaking at the same time] S3: and then this one S1: [quietly] so then this is seven times five...</td>
<td>S1 and S2 work on their own papers</td>
<td></td>
</tr>
<tr>
<td>[speaking at the same time] S1: and this would be five times seven. [14 second pause]</td>
<td>Students each reorganize their own set of papers and looking at new model on their respective papers.</td>
<td></td>
</tr>
<tr>
<td>S1: Okay. How would this be associative?</td>
<td>S1 looking at her own paper.</td>
<td></td>
</tr>
<tr>
<td>S1: That’s the one you said was associative, right?</td>
<td>S1 points to her own paper and looks at her partner.</td>
<td></td>
</tr>
<tr>
<td>S1: This one? No, I said this one.</td>
<td>S1 looking at her two of her own papers. Points to one and then the other.</td>
<td></td>
</tr>
<tr>
<td>S1: Oh, this one?</td>
<td>S1 moves another of her own papers to the top of her pile.</td>
<td></td>
</tr>
<tr>
<td>S1: Then, I’m confused now, so... [5 second pause] Didn’t you say this one was distributive?</td>
<td>S1 and S2 look at their own papers. S1 points to one of her pages.</td>
<td></td>
</tr>
<tr>
<td>S1: No. I was thinking this one was because I don’t see how this can be because there’s nothing that it really...</td>
<td>S1 quickly glances at S1’s paper then back at her own.</td>
<td></td>
</tr>
<tr>
<td>S1: that it’s relating... yeah.</td>
<td>S1 and S2 work on their own papers</td>
<td></td>
</tr>
</tbody>
</table>
Because I think this one would be distributive. Because if you combine these two... These two would be one, wouldn’t it?” Overlapping with Samantha’s underlined words, Justina says, “oh, if you combine like…” They are both referring to what they see in the model, but they do not seem to be talking with each other. Rather, they seem to be describing their own steps or thinking—that is, they are sharing information about how they are thinking about the problem, but irrespective of whether they have an audience. Notice that the connections between rows often do not flow from one PST to the other. Rather the flow appears to be more like a set of parallel roads that all arrive at dead ends. I do not mean to suggest that this episode is devoid of meaning making or sense making, but meaning making may be happening only at the level of individuals with little observable negotiation of meaning. The negotiation of meaning between Samantha and Justina seems to be much less prevalent than the negotiations Mark and Linda engage in throughout Episode 1.

Based on a closer inspection of Episodes 1 and 2, I propose that calculational talk alone may be less likely to engender a negotiation of meaning than conceptual talk alone. Calculational talk often consists of stating familiar information or steps—familiar to an individual or known by some mathematical authority (teacher/textbook). Calculational talk might provide a springboard into conceptual talk once the known information is shared in a group, but, for example, without connecting explanations to the underlying mathematical ideas, making connections between PSTs’ turns, or pursuing a negotiation where ideas are compared to each other and/or to a set of known information, generally, no negotiation of meaning takes place.

6.3 Comparing Calculational and Conceptual Talk for Individual PSTs

Once I could articulate characteristics of calculational and conceptual talk, I noticed that, when looking across episodes, some individual PSTs tend to engage in interactions involving
only one or the other kind of talk. That is, some PSTs participate almost exclusively in
calculational talk in the video-taped episodes I analyzed while other PSTs regularly interact with
classmates engaging in conceptual talk. For example, in Episode 2 discussed above, Samantha
tends to report out her own thinking rather than to engage her partner in discussion. Her
comments, explanations, and questions do not connect to comments and questions from Justina.
Rather, she almost seems to be thinking aloud. Whereas in Episode 1 discussed above, Mark and
Linda both use conceptual talk quite consistently as they negotiate how they will solve the
problem of matching models to properties. Samantha and Mark followed their respective patterns
of using calculational or conceptual talk in the majority of the coded episodes in which they
appear. This finding led to a consideration of whether there might be tangible differences among
PSTs that could be somehow noted or observed.

To pursue the investigation of individual differences among PSTs, I looked to their LMT
and Beliefs Survey results. Although the preliminary implementations of these tools had not
helped me differentiate PSTs early in the semester, I compared the pre- and posttests for Focus
PSTs who could be described as engaging in distinctly different patterns of talk during
discussions—more conceptual or more calculational. The comparison yielded some interesting
results. Six of my nine focus PSTs had both pre- and posttest results for both instruments. (Due
to computer issues and absences, all PSTs did not complete both implementations of both
assessment tools.) I found that, for the two PSTs who engage in primarily calculational talk
throughout episodes, their scores on both the LMT and the Beliefs Survey do not change much
from the beginning of the semester to the end of the semester. Samantha (in Episode 2 discussed
earlier) scored the same on the scored Beliefs Survey prompts and she had a score of 21% both
times on the LMT. Similarly, Diana, (in Episode 3 discussed in the next section), scored lower
on one scored item of the Beliefs Survey and the same on the others. Diana’s LMT score was
43% both times. Interestingly, Samantha and Diana answered different LMT problems
incorrectly from pre- to posttest but their overall scores remained static. Two PSTs who
primarily entered into discussions with conceptual talk showed positive gains on both
instruments. Both Mark (in Episode 1 discussed earlier) and David (in Episode 4 discussed in the
next section) scored higher on all of the scored Beliefs Survey prompts. On the LMT, Mark’s
scores were 14% on the pre-test and 43% on the posttest. David’s LMT increase was not as
dramatic but still positive, from 21% to 29%. Additionally, the last two PSTs (Linda discussed
above from Episode 1 was one of these) were less consistent in how they worked with others
during the videotaped episodes. For these two PSTs, sometimes they participated in calculational
talk, sometimes they sat quietly and listened during small group work, and sometimes they
engaged in conceptual talk. The scores for these two PSTs varied across the two instruments.
Linda received the same pre- and posttest scores on the scored Beliefs Survey prompts, but her
score on the LMT rose from 29% to 36 correct. Verneccia’s score on the Beliefs Survey score
was higher for every scored item. Her score on the LMT decreased from 36% to 14%. Clearly,
there are many factors affecting test scores, and how PSTs engage with classmates during
discussions may not be one of the factors; however, the consistency of this small set of results
suggest the possibility that there may be some connection worth studying further. In the
Discussion section of this paper, I revisit some of the possible implications of these results.
7 Results: Building on Ideas

7.1 Building on Ideas during Meaning Making Negotiations

To determine where PSTs may be building on each other’s ideas during their small group discourse, I analyze interactions and the larger episodes within which the interactions reside. Because the videotaped sessions do not follow particular PSTs or maintain a focus on a single group over the course of a class period or over several class periods, my access to observing this phenomenon is limited. Even with this limitation, the data present sets of related interactions demonstrating that PSTs do take up or incorporate each other’s ideas into their discourse, similar to Anderson, et al.’s (2001) snowball effect. The snowball effect, as described by Anderson, et al., occurs during a discussion where as new ideas or strategies (narratives) offered by an individual become accepted (endorsed) by a partnership or small group, the language and actions associated with those ideas or strategies is incorporated into the group discussion. In addition, the endorsed narratives tend to build and evolve during the course of discussions.

Barron’s (2003) research exploring the level of connectedness among group members during small group discussions provides a starting point for choosing episodes for my analysis of how PSTs build on each other’s ideas. Barron describes successful groups as having the highest frequency of group members connecting back to prior turns and agreeing with, responding to, or elaborating on earlier statements, ideas, strategies, questions, and so on. Because the goal of this theme in my research is to identify how and when PSTs built on each other’s ideas, it seems that the kinds of connections Barron describes would be essential in episodes that include discourse resembling Anderson, et al.’s (2001) snowball effect. Barron concludes that when group members invest in each other’s contributions, the discussions last longer and tend to involve
higher levels of reasoning. Consistent with Barron’s research, the focus episodes I chose to investigate tended to be longer and to have a higher concentration of conceptual talk.

Using connectedness as a tool for determining episodes for closer examination, I chose two episodes that were related to each other in their content and context and that provided evidence of connectedness based on my analysis of conceptual talk and calculational talk. Episodes 3 and 4 take place in mid-October (about half of the way through the course) and demonstrate connections not only within each episode, but also across episodes. An investigation of these episodes provides an opportunity to observe the evolution of discourse around unpacking the mathematics of a nonstandard subtraction algorithm. (See Figure 10.)

Example:

\[
\begin{array}{c}
826 - 438 \\
\hline
8 \quad 1 \quad 6 \\
-5 \quad 0 \quad 8 \\
\hline
3 \quad 8 \quad 8
\end{array}
\]

Figure 10. Non-standard subtraction algorithm

In Episode 3, a small group is trying to understand the mathematics of the algorithm and how the algorithm works—what the steps are for solving a problem. One PST, Diana, seems to have figured out the underlying mathematical ideas and is explaining those ideas to her group and re-explaining in response to their questions. She uses base-10 blocks, when required, to demonstrate how the numbers being subtracted are revised using a compensation strategy. Episode 3 lasts about 14.5 minutes. About 8.5 minutes of this episode fall into the larger
category of meaning making and about 6 minutes fall into the larger category of no meaning making.

In Episode 4, a PST who was in the small group in Episode 3, David, has been charged with “teaching” the algorithm to a group of PSTs unfamiliar with the mathematics of the algorithm. This episode takes place on the same day as Episode 3 and lasts about 16.5 minutes. All of the 16.5 minutes of this episode fall into the larger category of meaning making and there are no interactions that fall into the larger category of No Meaning Making.

7.2 Language Usage in Developing a Taken-as-Shared Meaning

In Episode 3, Diana explains to the four classmates in her group how she believes the algorithm works—that is, what the steps are for using this algorithm. Many of the interactions in the transcript excerpt from Episode 3 (Table V beginning on the next page) were coded as partially or fully engaged and therefore fell into the larger category of negotiating meaning. Because the gestures associated with the transcript are integral to the explanations in both episodes I have again included the gestures column with the transcript. There is one new symbol
### TABLE V
EXCERPT OF TRANSCRIPT FROM EPISODE 3

<table>
<thead>
<tr>
<th>Statement Information</th>
<th>Transcript</th>
<th>Accompanying Gesture*</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>S1: She [teacher who wrote the algorithm] added a 100 to the bottom.</td>
<td>S1 picks up ten longs (base-10 blocks that represent tens) and stacks them on the hundreds place of the blocks representing the subtrahend while holding up whiteboard.</td>
</tr>
<tr>
<td>2</td>
<td>S1: I'll just do one more of these.</td>
<td>S1 removes the longs and replaces them with a flat (base-10 block that represents 100).</td>
</tr>
<tr>
<td>3</td>
<td>S1: She added a one hundred to the bottom, apart from what was already there. So she added...</td>
<td>S1 points to the subtrahend on her whiteboard.</td>
</tr>
<tr>
<td>4</td>
<td>S1: She also added 10 tens to the top part, to what was already there. So she added...</td>
<td>S1 picks up ten longs and places them on the tens place in the blocks representing the minuend.</td>
</tr>
<tr>
<td>5</td>
<td>S1: Isn't that a flat then? Shouldn't it be a flat right there?</td>
<td>S1 indicating the ten longs S1 just placed on the minuend.</td>
</tr>
<tr>
<td>6</td>
<td>[speaking at the same time] S1: It's either way. S1: But I guess it helps to see it. It helps you to see the ten 10s she added to the top.</td>
<td>S1 points to the crossed out tens place on the white board. S1 looking at the blocks.</td>
</tr>
<tr>
<td>7</td>
<td>S1: But here she added ten tens rather than [writes off]=</td>
<td>S1 points to the crossed out tens place on the white board.</td>
</tr>
<tr>
<td>8</td>
<td>S1: Yeah... Yeah... It helps you to see the ten 10s that she added.</td>
<td>S1 looking at and pointing generally towards the blocks.</td>
</tr>
<tr>
<td>9</td>
<td>S1: So she added it to the top and to the bottom. Now when you subtract it, you subtract a positive number it's basically, you know. A negative number. So this cancels out what was here, and the difference is still the same. Do you see that? Do you see that? =</td>
<td>S1 pointing to whiteboard and looking back and forth between the board and the blocks =</td>
</tr>
<tr>
<td>10</td>
<td>S1: Like if we were just to subtract it, like how it is now, the difference is the same.</td>
<td>S1 moves hand back and forth over block representation of problem.</td>
</tr>
<tr>
<td>11</td>
<td>S1: Aren't they... I thought they stay the same because we did whatever, we did to the top.</td>
<td>S1 looks over at S1.</td>
</tr>
<tr>
<td>12</td>
<td>[speaking at the same time] S1: we did, we did to the bottom. S1: Like whatever we did to the top, we did to the bottom.</td>
<td>S1 looks over at S1 and indicates blocks representing the problem by flattening her hand over the blocks.</td>
</tr>
</tbody>
</table>
### TABLE V (continued)
EXCEPNT OF TRANSCRIPT FROM EPISODE 3

<table>
<thead>
<tr>
<th>Statement Information</th>
<th>Transcript</th>
<th>Accompanying Gesture*</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>S₁: So, both numbers are increasing sort of... so... it would be the same</td>
<td>S₂ is off camera. S₃ looks at blocks.</td>
</tr>
<tr>
<td></td>
<td>value, just a bigger number.</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>S₂: Well, it was different values, but...the same difference.</td>
<td>S₁ looks at blocks.</td>
</tr>
<tr>
<td>15</td>
<td>S₁: It was different values but...same... difference.</td>
<td>S₂ looks at S₁. S₃ looks at blocks.</td>
</tr>
<tr>
<td>16</td>
<td>S₁: The number of the difference is the same.</td>
<td>S₁ looks at blocks.</td>
</tr>
<tr>
<td>17</td>
<td>S₂: Is it better with blocks?</td>
<td>S₁ is off camera. S₃ looks at blocks.</td>
</tr>
<tr>
<td>18</td>
<td>S₀: Um. It depends. Blocks? I don't know. I understood it more when we did it like way</td>
<td>S₂ indicates the written algorithm on the whiteboard.</td>
</tr>
<tr>
<td>19</td>
<td>S₁: But if that helps the individual... You can see me adding the ten</td>
<td>S₃ picks up the previously added handful of tens longs</td>
</tr>
<tr>
<td></td>
<td>tens</td>
<td></td>
</tr>
<tr>
<td></td>
<td><img src="image" alt="Image of person adding blocks" /></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>S₁: You can see me adding these ones</td>
<td>S₃ picks up a two longs (one from the ones place of the minuend and one from the tens place of the subtrahend). S₁ looks at S₂.</td>
</tr>
<tr>
<td></td>
<td><img src="image" alt="Image of person adding blocks" /></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>S₁: And you can see me adding this one</td>
<td>S₁ picks up a flat.</td>
</tr>
<tr>
<td>22</td>
<td>S₁: So this, these two would be the same?</td>
<td>S₀ indicates the subtrahend and then the minuend that S₁ made with blocks.</td>
</tr>
<tr>
<td>23</td>
<td>S₁: What do you mean?</td>
<td>S₁ looks at S₀.</td>
</tr>
</tbody>
</table>
**TABLE V (continued)**
EXCERPT OF TRANSCRIPT FROM EPISODE 3

<table>
<thead>
<tr>
<th>Statement Information</th>
<th>Transcript</th>
<th>Accompanying Gesture*</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>S1: These, you said, the difference would be the same? S1 points to blocks.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>S2: Yeah. The difference... the outcome would be the same.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>S3: It's like doing... It's like doing 30 minus 10, and then you add ten to the top and the bottom. Forty minus 20. S3: takes S1's whiteboard and records a new problem of 30-10 and solves it with a compensation algorithm adding 10 to both the minuend and subtrahend. S3 watches.</td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>S3: The answer's still 20. The original problem is 30 minus 10, and then I added 10 to both the top and the bottom. S3 holds up the whiteboard with the algorithm. S3 looks at the problem. S3 looks at S1.</td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>S2: Um hm. S2 and S3 look at the problem.</td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>S3: The difference is still 20. S3 and S2 look at the problem.</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>S3: Right. S2 and S3 look at the problem.</td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>S1: Is that like adding it to get the same? ummm. What strategy is this? Well, I don't know if it's one of the three, but it kind of reminds me of the third one. Where you add and make it look like the same. But I don't think it's it. S1 looks at the problem and then switches back to her own whiteboard. Then looks up at S1 who seems to be pointing to a paper he is holding up with all of the algorithms. S3 is off camera.</td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>S3: So, a ten was added here, a ten was added here, a one hundred was added here, and one hundred was added here. S3 shows the problem on her paper to S1.</td>
<td></td>
</tr>
<tr>
<td>33</td>
<td>S2: Well, one hundred was added here because she borrowed a whole one hundred from this, well, not borrowed. She added 100. Cause this is... when you borrow a one here, you're adding a ten. S3 takes S3's paper and points to the problem as she describes the algorithm again.</td>
<td></td>
</tr>
</tbody>
</table>

* S1: gaze alternates between whiteboard and blocks throughout this episode unless otherwise noted.
Diana repeatedly refers to adding tens to the “top” [minuend] and adding a hundred to the “bottom” [subtrahend]. See Lines, 1, 3, 4, 6–8, 19–21, and 26–27. When Diana explains using the base-10 blocks to represent the numbers in the problem, she emphasizes that when you add ten 10s to the top number [longs (10s) that are added to the tens place in the minuend], you also add one hundred to the bottom number [a flat (100) is added to the hundreds place in the subtrahend]. As she explains, she picks up blocks and acts out adding them to the numbers she has built with base-10 blocks (the original numbers of the problem 826 [minuend] and 438 [subtrahend]). In Line 32, one of Diana’s group members, Colleen, picks up on Diana’s language and repeats that they are adding ten tens to the top number and one hundred to the bottom number. In Lines 12 and 13, David clarifies that Diana is doing the same thing to both numbers [minuend and subtrahend] in that “whatever we did to the top, we do to the bottom.” By repeating his statement in Line 13, Diana confirms that David’s statement accurately describes the procedure she is using. Throughout this episode, PSTs are negotiating a taken-as-shared meaning\textsuperscript{15} for the subtraction algorithm that they have been assigned.

7.3 **Transferred or Evolving Ideas Present in Language Usage**

In Episode 4, David travels to a new group where his assignment is to “teach” the group the algorithm that Diana explained in Episode 3. The excerpt of Episode 4 (Table VI beginning on the next page) includes David using both phrases and actions with base-10 blocks reminiscent of the language and actions Diana used in her presentation of the procedure for reproducing

\textsuperscript{15} See Cobb, Stephan, McClain, and Gravemeijer (2001)
<table>
<thead>
<tr>
<th>Statement Information</th>
<th>Transcript</th>
<th>Accompanying Gesture</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td><strong>S:</strong> This would be a sixteen. But this would be, I think, four. Is that a four? Because you're taking away. You're adding a ten from this column. You're taking ten units. So, and you're adding, and you're taking another... you're adding another ten to this column. So that will still be a plus ten. So we added ten to both columns.</td>
<td><strong>S:</strong> records on his small whiteboard and then shows what he wrote</td>
</tr>
<tr>
<td>2</td>
<td><strong>S:</strong> Wait, on the second column, where did you add a ten?</td>
<td><strong>S:</strong> looking at <strong>S</strong>'s whiteboard.</td>
</tr>
<tr>
<td>3</td>
<td><strong>S:</strong> Right here.</td>
<td><strong>S:</strong> points to the tens column in the subtrahend on his whiteboard while maintaining eye contact with <strong>S</strong>.</td>
</tr>
<tr>
<td>4</td>
<td><strong>S:</strong> Okay. That turns into fourteen or for... or thir...? wait, I don't get the four. [7 second pause] Where are you getting three to four? You're adding?</td>
<td><strong>S:</strong> turns the whiteboard to examine his work.</td>
</tr>
<tr>
<td>5</td>
<td><strong>S:</strong> Wait, now how do you explain... [3 second pause]. I should do it with the blocks. I that would... I think it's better with the blocks.</td>
<td><strong>S:</strong> continues looking at the work on his whiteboard.</td>
</tr>
<tr>
<td>6</td>
<td><strong>S:</strong> Okay. Use the blocks. I don't know.</td>
<td><strong>S:</strong> continues looking at the work on his whiteboard.</td>
</tr>
<tr>
<td>7</td>
<td><strong>S:</strong> Use the blocks.</td>
<td><strong>S:</strong> continues looking at the work on his whiteboard.</td>
</tr>
<tr>
<td>8</td>
<td><strong>S:</strong> But then, do we have enough? We don't have enough. Okay, I'm going to try to explain it the best way by using the [unintelligible]</td>
<td><strong>S:</strong> turns the whiteboard for <strong>S</strong> to see.</td>
</tr>
<tr>
<td>9</td>
<td><strong>S:</strong> Okay so, we're adding ten, we're adding ten to this column</td>
<td><strong>S:</strong> points to the tens column of the minuend.</td>
</tr>
<tr>
<td>10</td>
<td><strong>S:</strong> and we added ten, added ten</td>
<td><strong>S:</strong> points to the tens column in the subtrahend but then stops and looks at the whiteboard as though unsure of his steps. He puts his whiteboard down.</td>
</tr>
<tr>
<td>11</td>
<td><strong>S:</strong> I'm going to use the blocks</td>
<td><strong>S:</strong> reaches for the base-10 blocks.</td>
</tr>
<tr>
<td>12</td>
<td><strong>S:</strong> [unintelligible] He was explaining that he got three. Then he said he added a ten, but there is only a 4 there [in the tens place of the subtrahend].</td>
<td><strong>S:</strong> looking at the teacher as she talks about <strong>S</strong>'s explanation.</td>
</tr>
</tbody>
</table>

Some discussion between **S** and Teacher ensuing.
## TABLE VI (continued)
EXCERPT OF TRANSCRIPT FROM EPISODE 4

<table>
<thead>
<tr>
<th>Statement Information</th>
<th>Transcript</th>
<th>Accompanying Gesture</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>$S_1$: That was a good question. I was confused, that was a good question though. So, I added a ten to this column and that turns it into a 40. And it's basically the same with this one, but this time, I'm going to add a hundred. Oh god, I'm gonna use the blocks, because if I use the blocks, it will be easier.</td>
<td>$S_1$ displays his whiteboard to the group and points generally to the problem. At the end of this statement, he gets up to get blocks.</td>
</tr>
<tr>
<td>14</td>
<td>$S_1$: I just counted was that 873, 4, 7, 8</td>
<td>$S_1$ returns with base-10 blocks and counts out and places the flats on the table.</td>
</tr>
<tr>
<td>15</td>
<td>$S_1$: four thirty-eight. Okay, So. Add a ten, so basically, whatever we do to the top, we have to do to the bottom to keep the numbers even, like, equivalent apart. So, like I said, when I added like tens to the ones column, I would have to add ten units to the ones column. I would have to add another ten, a long, to the tens column.</td>
<td>$S_1$ gets out longs and unit cubes and makes both numbers (438 and 835) with base-10 blocks.</td>
</tr>
<tr>
<td>16</td>
<td>$S_1$: So, instead of using the longs, I'm gonna use ten. So, what I added to the ones column, this is what I'm gonna add to the tens column.</td>
<td>$S_1$ counts out ten unit cubes and adds them to the ones place of the minuend. He then adds one long to the tens place of the subtrahend. (See the blocks in the red circles.)</td>
</tr>
<tr>
<td>17</td>
<td>$S_1$: So now, for this one, when I'm moving over, I'm gonna, I have to add a hundred or ten longs. So, I'm gonna add ten longs or like a</td>
<td>$S_1$ points to the problem on the whiteboard and moves pen from one's column in the minuend to the tens column. He grabs a handful of longs and shows the group the longs.</td>
</tr>
<tr>
<td>18</td>
<td>$S_1$: flat. I'm gonna use ten longs, cause that will probably be more easier. I'm gonna add a flat ... so I add ten longs to this, to the top number</td>
<td>$S_1$ points to the subtrahend represented in blocks and then moves his hand to cover the hundreds place of the minuend.</td>
</tr>
<tr>
<td>19</td>
<td>$S_1$: and a flat to the hundreds place</td>
<td>$S_1$ again moves back and forth covering first the tens place of the subtrahend and then the hundreds place of the minuend.</td>
</tr>
<tr>
<td>20</td>
<td>$S_1$: because what I have to add to the top, I have to do the same thing to the bottom.</td>
<td>$S_1$ counts out the longs in his hand and finds that he does not have ten of them. $S_1$ goes for more blocks.</td>
</tr>
</tbody>
</table>
the subtraction algorithm. David focuses on describing the addition of the ten ones and one ten as well as the ten tens and one hundred to preserve the difference between the numbers. The
numbers are revised but their difference remains the same, as Diana and then David explained. David transferred (i.e., carried with him) Diana’s language to the new situation.

Unlike Diana, David seems to struggle with using only whiteboard notation to explain the subtraction algorithm. See Lines 1–10 above. Although he uses the language that Diana used earlier, it is not apparent that all of his statements are coherent to him. The evidence of this is that he seems to get confused and hesitates as though he might doubt that he has explained the procedure clearly (e.g., Lines 5 and 13). The fact that David does not interact much with Diana—asking questions or contributing statements—during her presentation of the algorithm, may indicate that he is not successfully negotiating meaning as the group in Episode 3 is working. As David pauses to reconsider his explanation of the algorithm, his classmates encourage him to use the blocks to demonstrate how the algorithm works.

Once David assembles base-10 block representations of the original numbers from the problem, the blocks become integral to his explanation. Note that, in Line 15, he repeats Diana’s language and emphasizes that it is necessary to do the same thing to both the top and the bottom numbers. Parallel to what Diana did with the blocks, in Lines 17–19, David demonstrates how to add ten tens to the tens place of the minuend and one hundred to the hundreds place of the subtrahend. He appears confident and clear on how to manipulate the blocks. Perhaps he had not had an opportunity to fully process the meaning-making negotiations in Episode 3, and this resulted in a less than firm grasp of the meaning behind the language that Diana used earlier and that he was attempting to use.

In Lines 25–33, Tamika negotiates an agreement with David on how to describe the procedure for the algorithm that he has demonstrated. She uses language similar to language that Diana used originally—e.g., “because you added ten, you gotta add ten to this one (Line 31).”
David confirms that Tamika is correct. Tamika builds on the previous description of ten tens in the tens place (in the minuend) balanced with one hundred in the hundreds place (in the subtrahend) noting that, in both cases, you add a hundred to the original number (Line 33).

### 7.4 Negotiating Meaning through Building on Each Other’s Ideas

After analyzing the focus episodes, a closer inspection of where PSTs incorporate each other’s language reveals that interactions coded as Fully or Partially Engaged, Reporting, Question and Answer, and Teacher Questions all include examples of PSTs using and/or building on each other’s language. As reported earlier, PSTs seem to successfully use each other’s language once they have a chance to negotiate the meaning of the language in relationship to a mathematical situation. PSTs use each other’s language to report what they are doing, to question each other, and more generally, to negotiate meaning in a mathematical situation. As in the examples above, the discussion often seems to involve some clarification and alignment of language in the process of negotiating meaning. The brief excerpt below from Episode 1, which is one of the episodes described earlier in Chapter 6, provides one example of this negotiation:

Linda: Distributive. The original block has… it like it’s all group together.
Mark: And then you split it up…
Linda: And then you split it up… to… to… add them up individually. Right?
Mark: Yeah.
Linda: Like the first block is all grouped together, er, the numbers are all grouped together… er the blocks or the squares are all grouped together?
Mark: Um hmmm.
Linda: So, let’s write that.
Mark: So, what did we say?
Linda: The squares are grouped together.
Mark: They’re all grouped together
Linda: …and when it’s split into two groups, you can multiply and add it together
Mark: Yeah, but you can, multiply each side separately…

Notice that, in the transcript, Linda and Mark repeat, clarify, and enhance each other’s words and phrases as they negotiate how they are going to describe why one of the models in the special problem-solving session represents the Distributive Property. Linda and Mark did not copy each other’s papers, but their individual written records of the work they did in the special problem-solving session align closely, reflecting their language negotiations. (See Figure 11 and Figure 12 for their written records.)
Mark’s and Linda’s papers are consistent with each other in presenting their final conclusions about how the models depict particular properties. They use language that is similar, and in a similar sequence. They provide clear and thorough explanations.

Some PST partnerships during the special problem-solving session engaged heavily in calculational talk rather than the more connected conceptual talk. In these cases, although there are often similarities in the partners’ written records, their records are not as well aligned as Mark’s and Linda’s, reflecting more of an individual approach to the problem-solving session. Papers from Samantha and Justina from Episode 2, whose interactions were described earlier, illustrate the kinds of differences that might appear. (See Figures 13 and 14 for Samantha’s and Justina’s written record of the problem-solving session.) Samantha writes about separating something and then uses numbers to illustrate her reasoning while Justina focuses more on
the diagram in her written description of this model of the Distributive Property. Note that the papers shown above tend to use rather informal language—for example, “split into two groups” or “sections divided by the line” rather than writing about resulting arrays or areas. The next section provides a more focused investigation of using informal versus authoritative language.
8 Results: Authoritative Language

8.1 What Counts as Authoritative Language

For interpreting how and when PSTs use authoritative language, I look to Bakhtin (1981) and Greenleaf and Katz’s (2004) research that incorporates the Bakhtinian construct of *dialogism*, which implies that an individual is “always in the act of responding to the social world, and in making meaning through their responses to that world” (Greenleaf & Katz, 2004, p. 173). The social world in this case is the mathematics course that provides the context for this study. Since this course focuses heavily on algebraic thinking, number, and operations, a majority of class periods develop concepts related to these topics. Because of this, my analysis of authoritative language focuses on the language associated with these topics. In addition, rather than examine all possible categories of authoritative language (in the Bakhtinian sense, language used by peers might also be considered authoritative language because it is employed in the community), this study focuses specifically on authoritative language that is introduced by the instructor or the text.

To determine when and how PSTs might use authoritative language in their response to the mathematical world that was our course of study, I looked to course readings, course planning documents, and instructor utterances during small group work as sources for relevant words and phrases. I found that the course textbook included all of the language also used in the planning documents and in instructor utterances. I therefore generated a list of terms related to our work with addition, subtraction, and multiplication, and algebraic properties based on the course text. (See Table VIII.) I organized the list of terms according to the main lesson content where the terms were introduced.
### TABLE VII
LIST OF AUTHORITATIVE TERMS FROM COURSE TEXT

<table>
<thead>
<tr>
<th>Addition &amp; Subtraction</th>
<th>Multiplication</th>
<th>Properties &amp; Place Value</th>
<th>Other</th>
</tr>
</thead>
<tbody>
<tr>
<td>algorithms</td>
<td>area model</td>
<td>* associate/ Associative</td>
<td>complex</td>
</tr>
<tr>
<td>add up</td>
<td></td>
<td>* Commutative</td>
<td>diagram</td>
</tr>
<tr>
<td>combine</td>
<td></td>
<td>* Distributive</td>
<td>double</td>
</tr>
<tr>
<td>compensation</td>
<td></td>
<td>* base 10 (or base 5)</td>
<td>efficient</td>
</tr>
<tr>
<td>decomposition</td>
<td></td>
<td>* exchange</td>
<td>equal</td>
</tr>
<tr>
<td>* difference</td>
<td></td>
<td>* flat, long, unit</td>
<td>estimate/estimation</td>
</tr>
<tr>
<td>join</td>
<td></td>
<td>(base-10 block names)</td>
<td>exact/accurate</td>
</tr>
<tr>
<td>* minuend</td>
<td></td>
<td>* part/whole</td>
<td>number model/</td>
</tr>
<tr>
<td>negative number</td>
<td></td>
<td>place value terms</td>
<td>sentence</td>
</tr>
<tr>
<td>strategy</td>
<td></td>
<td>(e.g., tens, tens place,</td>
<td>reasonable</td>
</tr>
<tr>
<td>regroup (instead of</td>
<td></td>
<td>or tens column)</td>
<td>represent/</td>
</tr>
<tr>
<td>borrow or carry)</td>
<td></td>
<td>* property</td>
<td>representation</td>
</tr>
<tr>
<td>* subtrahend</td>
<td></td>
<td>* variable</td>
<td>rounding</td>
</tr>
<tr>
<td>sum</td>
<td></td>
<td></td>
<td>* section</td>
</tr>
<tr>
<td>take apart/ take away</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>trade</td>
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</table>

If the term in the Table VII has an asterisk, the instructor uses the term during small group or partner work in focus episodes. If the term is underlined, the PSTs use the term during small group or partner work in focus episodes. In the five focus episodes, PSTs use fourteen of the listed terms—ten of which are also used by the instructor during the episodes. In my data set for this study, there is no documentation that the instructor uses all of the vocabulary during lesson presentations. However, in the descriptions of course activities, the various planning documents include all of the terms listed in the table, indicating that the instructor is at least aware of these terms in the context of the content being developed.
8.2 **Contrasting the Use of Authoritative Language across Episodes**

In searching for patterns or regularities in PSTs’ language usage within and across the focus episodes during interactions broadly categorized as negotiating meaning, I found in Episodes 3 and 4 involving work with the subtraction algorithm that the frequency of using terms from the list was fairly consistent. Episode 3 included 9 uses (out of 521 words) 1.7% in 8.5 minutes categorized broadly as negotiating meaning, and Episode 4 included 19 uses (out of 718 words) 2.6% in 16.5 minutes categorized as negotiating meaning. The two groups of five PSTs had only one PST in common, but both of these episodes involved one PST explaining to his or her group how the subtraction algorithm worked. (See Figure 15 for an image of the explanation.) Since how the algorithm works is not transparent to the group, PSTs question each other’s explanations and thinking as they try to figure out the underlying mathematics of the algorithm. Most of the terms the PSTs incorporate into their discussion relate to place value—for example, describing when

![Figure 15. The subtraction algorithm explanation from Episodes 3 and 4.](image-url)
they are working in the *tens column* or the *hundreds column* or how they are compensating for *adding ten ones* to the “top” number (minuend) in the ones place by *adding one ten* to the “bottom” number (subtrahend) in the tens place.

In Episode 5, from mid-November, PSTs are working together to figure out how to model and solve a base-5 multiplication problem. In this episode, the frequency of using terms from the list is slightly lower with 19 uses (out of 772 words) 2.5% in 10.5 minutes categorized as negotiating meaning. In this episode, the PSTs had built an array for the problem $23_5 \times 43_5$ using base-5 blocks. (See Figure 16.)

![Figure 16. Base-5 array for $43_5 \times 23_5$](image)

About half of the time (9 instances) when PSTs use terms from the list, they are clarifying what base they are multiplying in and comparing the features of similar arrays built for base-10 versus base-5 multiplication problems. In the remaining instances, PSTs use place-value terms (e.g., *multiplying tens by tens*) or reference the different *sections* of the array they are describing.
Deictic gestures play a key role in the communication in Episodes 3, 4, and 5, and this seems to be correlated with a lower frequency of using terms associated with authoritative language. In fact, PSTs only use the identified authoritative language terms in Episodes 3, 4, and 5 between 1.5% and 2.7% of the time. PSTs may not have relied solely on language, and perhaps authoritative language from the instructor or the text is less relevant in these contexts. For example, rather than describe exactly what they are doing as they perform the algorithm in Episodes 3 and 4, PSTs record on whiteboards and then point to the numbers on the whiteboards to identify where they are “crossing out,” or “making these [numbers] the same” to support their explanations (See Figure 15). In Episodes 3, 4, and 5, PSTs work with base-10 blocks or base-5 blocks, and they are able to point to sections as they describe the array of numbers and how they find totals. The use of deictic gestures may have reduced the need for language in general in these episodes. (See Figure 17.) In Episode 4, when David begins with only a verbal explanation of how the subtraction algorithm worked, two of his four classmates in the group request that he use

Figure 17. Base-5 array for $43_5 \times 23_5$ with deictic gestures.
the base-10 blocks to demonstrate what he was describing. This request may indicate that language alone is insufficient for the negotiation of meaning in this situation.

Episodes 1 and 2, which do not rely heavily on deictic gestures, provide additional evidence that PSTs’ use of authoritative language from the instructor or the text may be minimal in discussions when they are negotiating meaning. Episodes 1 and 2 involve no manipulatives, and the only deictic gestures PSTs employ in these episodes is to point to locations on their papers in order to clarify what they are referring to in their explanations or questions. Both episodes occur during the last week of the semester with PSTs working in partnerships in special problem-solving sessions. Since it is the end of the semester, PSTs may reasonably be expected to employ language they had been learning all semester as they engage in the problem at hand. In addition, by the end of the semester, PSTs likely have gained some familiarity both with their classmates and with the course expectations around problem solving and discussing problems together. The episodes involve different PSTs, but the overall class performance of the PSTs when looking at their partnerships is similar. (See Table VIII.) All four PSTs in these episodes turn in all of their homework, do a thorough and thoughtful job on their class notebooks, and have excellent attendance and participation records. Within each partnership, one PST received an A on the final exam and the other PST scored significantly below 90% (the lowest score for an A). Though the involved PSTs have similar performances and the context is the same for Episodes 1 and 2, the episodes differ from each other, as well as from the other focus episodes, in the frequency of terms used. Episode 1 exhibits the highest frequency of authoritative language use of the five episodes with a frequency of 59 uses in about 13 minutes categorized as negotiating meaning. Episode 2 has the lowest frequency of authoritative language use of the five episodes with a frequency of 11 uses in about 6 minutes categorized as negotiating meaning.
Episodes 1 and 2 differ from each other in several ways, some of which may have contributed to contrasting results for the use of the identified terms of authoritative language in the episodes. One way the episodes differ, as described earlier, is that Episode 1 involves many more connections between partners’ utterances with Mark and Linda explaining, addressing, commenting on, questioning, clarifying, and confirming each other’s statements and questions. Mark and Linda spend much of the time in conceptual talk and in negotiating meaning. Mark and Linda interact more and exchange related ideas and comments throughout the episode while Samantha and Justina spend significant amounts of time reviewing their own papers and thinking silently or seemingly talking to themselves. Samantha and Justina engage more in calculational talk and tend to report their thinking and answers rather than to connect to each other’s ideas.

There are 1,867 spoken words in the twenty-minute transcript for Mark and Linda, which is more than double the 818 words in the twenty-minute transcript for Samantha and Justina. In their twenty-minute episode, Mark and Linda use just over twice as many words as Samantha and Justina, but over five times as many terms from the identified list of authoritative terms. Mark and Linda use terms from the list about 3.1% of the time and Samantha and Justina about...
1.3% of the time. Although Episodes 1 and 2 represent the extremes of PSTs’ use of authoritative language during discussion in the focus episodes, the frequencies of use remain quite low. These results suggest that PSTs tend to use a combination of their own personally meaningful language with gestures instead of authoritative language from the instructor or the text, at least when they explore and discuss new or unfamiliar content or problems or when they negotiate meaning.

### 8.3 Authoritative Language in PST Writing

The language usage patterns described in the previous section led me to consider if and when PSTs might use authoritative language, since it is not prevalent in their discourse. PSTs kept notebooks during the course that include records of the problems presented in class and their work on those problems. They also recorded responses to instructor prompts as well as reflections on their challenges and successes during class time. As PSTs recorded problems and related work, they sometimes reported on and/or labeled the work following examples provided in class. In these cases, PSTs tended to use terms similar to those the instructor used in presenting content—terms from the list of authoritative language. PSTs’ notes and reflections also tended to use terms that the instructor used when introducing, presenting, or reviewing content. (See Figure 18 on the next page for notebook samples with terms from the list of authoritative language underlined.) The samples illustrate how, in their written work, PSTs incorporate terms from the list with considerable more frequency than during their discussions in the videotaped episodes. For example, in Sample A, every label or reference on the page includes terms from the list. In Sample B, both explanations and labeling include authoritative language. Even in their personal reflections (in Samples C and D), PSTs used language that mirrors the language used in the instructor’s lesson plans for the content. (See Appendices B and C for sample lesson plan documents.)
Figure 18. Notebook samples illustrating PST use of authoritative language.
In addition to PSTs’ regular use of authoritative language as they record in their notebooks, their responses on the IMAP Beliefs Survey (Appendix D) provided an opportunity to compare how PSTs use of authoritative language in their written explanations and reflections might have changed during the semester. Because, in the survey, PSTs responded to identical prompts early in the course and at the end of the course, I examined parallel questions/prompts that related to the content I focused on with my language analysis—in particular, computation questions. In one set of related questions (Problem 3 in the Survey), PSTs explain how some of the computation strategies hypothetical “students” use might be productive to present to a class with the goal of moving the class forward in their understanding of operations. The survey asks PSTs to predict if one of the students using a traditional algorithm would understand another student’s non-traditional algorithm. In a second set of questions (Problem 4 in the Survey), PSTs evaluated two strategies hypothetical “students” present and what PSTs believe the strategies demonstrate about student knowledge. Below are examples of how some PSTs’ answers evolve from the beginning of the semester to the end of the semester. (See Table IX.) Each row contains the responses of the same PST at the beginning of the year and the end of the year. The responses in each of the three rows come from different PSTs. Although not all PST responses change in ways similar to the examples, there seems to be a trend in this direction with PSTs providing clearer, more detailed responses including more language reflective of the list of authoritative terms. The results of this analysis suggest that PSTs have access to the authoritative language from the instructor and the text but that they do not use this language when negotiating meaning. I will explore implications of this in the next section.
### TABLE IX
CHANGES IN PST LANGUAGE FROM PRE- TO POSTTEST

**From Problem 3: PSTs have 5 addition strategies and describe which ones they would share with their students and why.**

<table>
<thead>
<tr>
<th>PST 1 EARLY IN THE YEAR</th>
<th>PST 1 END OF THE YEAR</th>
</tr>
</thead>
<tbody>
<tr>
<td>YES: Carlos’ approach makes the most sense because it is the standard way to solve an addition problem, and can also be seen as the simplest. NO: Henry's approach involves far too much math that a child would need to go through. NO: Elliot's reasoning behind her answer does not reflect the work shown at the beginning of the problem and thus does not seem to be a plausible response to the answer. YES: Sarah's makes more sense than Henry's but still goes through too much work to complicate the problem. YES: Maria uses a sound logic to solve the answer however, again, involves too much work to achieve this answer.</td>
<td>YES--CARLOS: It explains the idea of carrying numbers over and allows for the adding of single columns one by one until the full sum is gained. YES--HENRY: I would explain this method in certain situations. It seem confusing at first but it is good because it shows the idea of estimating and how it helps to find the sum. YES--ELLIOT: It clearly shows the value located in each place value, and shows how to add the different place values together in order to find the sum. YES--SARAH: Sarah also shows the positives to estimating, and helps students to better rationalize the ways by which to find the answer. YES--MARIA: This method greatly illustrates to a child how the numbers work but in a more visual way. It allows the student to see the number in front of them and manipulate the blocks to change the numbers around in finding the sum.</td>
</tr>
</tbody>
</table>

**From Problem 3: PSTs are asked to assess whether a student who uses a traditional algorithm (Carlos) would be able to understand another student’s alternative strategy.**

<table>
<thead>
<tr>
<th>PST 2 EARLY IN THE YEAR</th>
<th>PST 2 END OF THE YEAR</th>
</tr>
</thead>
<tbody>
<tr>
<td>If Elliot’s problem was written out in a clearer manner, perhaps. But because the differences in strategy used, it is more likely that Carlos will be unable to explain Elliot’s strategy clearly.</td>
<td>Perhaps he would be able to because Elliot’s strategy had to do with adding each corresponding place value and then adding all the sums together (partial sums).</td>
</tr>
</tbody>
</table>

**From Problem 4: PSTs have two subtraction strategies (Adriana’s partial differences and Lexi’s traditional algorithm) and explain which shows greater mathematical understanding.**

<table>
<thead>
<tr>
<th>PST 3 EARLY IN THE YEAR</th>
<th>PST 3 END OF THE YEAR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adriana's strategy took a lot longer to solve however, you can see that she understands the meaning of each number and its digit and also she is able to explain what it is she did step by step. However if you was teaching I would use Lexi's strategy.</td>
<td>Adriana way shows that she understands the problem better than Lexi because when she explains it she says the correct place value of the number. For example Lexi instead of saying 8 tens she say 8 this can make people think that she is working with the one's place instead of the tens, while Adriana way shows which place value belongs to what. Although Lexi's way is the way most people learn to subtract, unless you are able to explain it you cannot fully understand the problem.</td>
</tr>
</tbody>
</table>
9 Discussion and Conclusion

9.1 Discussion

The fact that teachers in the U.S. are being asked to teach mathematics in ways they did not learn it motivated the focus of this study. The kinds of mathematics classrooms called for by today’s standards documents and by mathematics educators require teachers to have a set of specialized skills and practices for which they may have no models based in their own education. One aspect of these classrooms is that mathematics should be taught for understanding. That is, students should have opportunities to make sense of the mathematics and to negotiate meaning in the course of discourse around mathematical activity. This study hypothesizes that one way to provide teachers with experiences in negotiating meaning is to design their mathematics content courses to foster meaning making. Although there is a growing body of research on elementary school students’ meaning making and communicating during mathematical activity, there is little research on PSTs in this area. In order to develop a better understanding of the process of mathematical meaning making with PSTs, the central question of this study is: What are characteristics of PSTs’ discourse during mathematical activity as they engage in meaning making? Three more specific questions provided the structure for the investigation of this question:

1) How and when do PSTs engage in conceptual talk or calculational talk?

2) To what extent and in what ways do PSTs incorporate each other’s ideas into the discourse of their small groups?

3) How and when do PSTs’ incorporate authoritative language (language derived from the text or the teacher) versus language that is more connected to their own experiences during their discussions?
In the previous four chapters, I describe the results of this qualitative study as related to my overarching research question and the three sub-questions related to language use and meaning making. In focusing on meaning making during mathematical activity, I investigate the characteristics of PST interactions when they enter into a negotiation about the meaning of a problem or a solution strategy for a problem attaching personal value to the agreements at which they arrive with their group. In the remainder of this chapter, I first summarize and discuss the results of my analysis organized around the three specific sub-questions that guide this research on PSTs’ discourse in meaning making activity. Second, I describe limitations of this study. Third, I explore some of the implications of this work. Finally, I present possible future extensions of this research attending to, in particular, how the research described here might inform planning and implementing preservice teacher mathematics courses focused on facilitating a negotiation of meaning around mathematical ideas.

9.1.1 Discussing conceptual talk, calculational talk, and meaning making

The data analysis around the question of how and when PSTs use conceptual talk versus calculational talk may inform what instructors of preservice teachers attend to as PSTs discuss mathematical problems and problem situations. At least since the 1980’s when the first set of standards from the National Council of Teachers of Mathematics arrived on the scene, teachers (and instructors of teachers) have been encouraged to include mathematical discussion in their classrooms (NCTM, 1989b). Just what that discussion should look like and sound like has often been left to be determined by the individual teacher. Although researchers have investigated what might constitute productive discourse in an elementary school classroom (e.g., Ball & Friel, 1991; Barron, 2003; Chazan & Ball, 1999; Cobb & Bauersfeld, 1995; Cobb et al., 1997; Hufferd-Ackles et al., 2004; Khisty & Chval, 2002; Moschkovich, 2007; Sfard, 2001b, 2007; Wood et al.,
2006; Yackel & Cobb, 1996), there has been far less attention to the discourse in a PST classroom (e.g., Darke, 2010; Li, 2013).

9.1.1.1 Promoting conceptual talk

The results of this study invite a closer examination of how PSTs are connecting questions, answers, and comments with the questions, answers, and comments made by classmates. Consistent with research that has been done in elementary school classrooms (Cobb et al., 2001; Thompson et al., 1994), this study found that PSTs may be talking in small groups, but in fact, the way in which they are asking and answering questions, making statements and comments, or even sharing observations may not constitute a negotiation of meaning. Similar to the elementary school students, when PSTs engage in calculational talk, they are sometimes not connecting to what classmates have said. Calculational talk may take the form of simply reporting on what someone is thinking or the steps taken in solving a problem. Calculational talk may occur when PSTs answer questions without attending to the questioner’s exact inquiry. That is, instead of addressing the exact intent of a question, a PST may answer by repeating a description of an idea or problem-solving step in the same way it was previously stated without adjusting for the information requested or without actually answering the question at all. For example,

S₁: Equals BA ...and the other one's like B plus A. Wait. That's the same one, isn't it?
S₂: I know there's one that switches. But... Ach...

When PSTs engage in calculational talk, to the observer it may appear that PSTs are negotiating around the meaning of some mathematical idea or problem, but instead, they may be talking to, around, or even past each other.
Although sometimes calculational talk is heard at the beginning of a discussion when PSTs negotiate the meaning of an idea or problem, in the data set for this study, the majority of the negotiation of meaning involves conceptual or connected talk. It makes sense that a new discussion might begin with calculational talk because a negotiation could require first getting everyone on the same page, if possible, and then connecting to, reflecting on, and building on each other’s contributions. One elementary school study (Cobb et al., 2001) reported that engaging in conceptual discourse afforded students opportunities to explore the underlying structure of mathematical situations in ways that calculational explanations could not. One way an instructor may promote PSTs’ engagement in conceptual talk, might be to listen carefully for a transition from reporting and stating information without reference to each other’s comments to PSTs engaging in conceptual, thus, connected talk. When PSTs do not appear to be making the transition, an instructor might step in and direct PSTs’ attention to an individual’s ideas, inviting comment and questions specifically related to an individual PST’s contributions. By providing comments and prompts such as, *I hear Lucy saying that she thinks this a is model of the number 25₆ (in base 6). Lucy, is that what you are saying? Would you describe how you see 25₆ in this model? Jordan explain why you agree or disagree with Lucy,* an instructor might not only support the engagement of PSTs in conceptual talk but also furnish a model for participating in conceptual talk.

Instructors of PSTs exploring how to promote discussions involving PSTs in conceptual talk might collaborate to develop questions or prompts along with models of interactions. Lists of “good” questions circulate among teachers and teacher educators such as, *How did you solve the problem? or How do you know that your answer is correct?* Teacher educators sometimes implement questions modeling how future teachers might engage their own students in content
such as, *What does this number six represent in your strategy?* or *Why did you record the six in the tens column?* Although questions like these will possibly elicit thoughtful, mathematical responses from PSTs, they may not provide the impetus for or a model of engaging in conceptual talk. Several elementary school curricula currently provide such models of teacher–student conversations with the intention of supporting mathematical discussion (e.g., Foresman, TERC, Pearson Education, Scott, & Foresman, 2008; Wagreich, Goldberg, & Staff, 1997).

Because textbooks for PSTs, at least those available for the course in this study, do not include models of instructor–PST interactions, instructors’ collaborations and reification of the results of these collaborations may be essential for developing useful models. Researchers who have begun to explore the power of lesson plans for strengthening the implementation of mathematics lessons with PSTs have included instructor prompts and questions and anticipated PST responses in the development of their lessons plans (e.g., Ball, Sleep, Boerst, & Bass, 2009; Castro Superfine & Pitvorec, under review; Hiebert & Morris, 2009). Currently, instructors who teach the course that is the focus of this study collaboratively create lessons plans for the course. These lesson plans include sample prompts and questions for instructors, as well as anticipated PST responses—not quite a complete model of instructor–PST interactions, but moving in that direction. Although such models might be helpful for instructors of PSTs who want to promote conceptual talk in small group and whole group discussions, the tension between a manageable lesson plan and a lesson plan that includes all possible information may dictate the limitations of this artifact for providing models of and promoting conceptual talk.

As instructors of PSTs, we may need to consider more than just engaging PSTs in conceptual talk. In one study in an elementary school context, Thompson, et al. (1994) found that teachers may teach with a more conceptual orientation or a more calculational orientation. The
teacher’s orientation affects the classroom expectations. A teacher with a more conceptual orientation expects students to engage in discourse practices involving explanation and reasoning—discourse practices found to be more frequently associated with meaning making. A teacher with a calculational orientation expects students to focus on sharing correct answers—a process found to be less frequently associated with meaning making. This seems to imply that, as instructors of PSTs, we need to attend to the orientation of our future teachers, hoping to inspire in them a more conceptual orientation. In addition to providing models to support PSTs’ engagement in conceptual talk, Thompson et al.’s (1994) findings seem to imply that, as instructors of PSTs, we should perhaps be explicit about what expectations teachers should have in elementary school classrooms—that is, teachers should value students’ explanations of strategies and reasoning rather than just having students provide correct answers. In this way, we as instructors of PSTs may influence them towards having the desired conceptual orientation.

9.1.1.2 Gesturing in conceptual talk

Throughout the videotape in this data set, gestures accompany every instance of conceptual talk. Often PST gestures consist of deictic gestures—pointing at a place on someone’s paper or at some aspect of a representation to indicate or clarify what is being referred to, described, or explained. Sometimes PST gestures consist of shifting their positions or postures to create a shared space. PSTs might move their gazes to another’s paper or to a set of manipulatives used to support a negotiation of meaning. Finally, PSTs sometimes use iconic gestures to “show” what they were describing—for example, cupped hands to show *grouping* or slicing up and down to show *separating*.

The consistent use of gestures with conceptual talk might indicate that gestures promote a more successful negotiation of meaning, and therefore, that they are a hallmark of meaning.
making. If instructors strive to promote meaning making, they must be able to assess when meaning making is occurring. The data in this study suggest that gestures may provide one assessment tool to help make that determination. If PSTs are leaning into a common space, using their hands to help them describe ideas or to model words, or even pointing to their own or each other’s papers, it is possible, even likely, that some negotiation of meaning is taking place.

Alternatively, even if PSTs are talking, if they are gazing only at their own paper or pointing only to their own paper without inviting another’s gaze to the same spot, they may be engaged in a thought process by themselves. Perhaps in these instances, PSTs are verbalizing either to meet the classroom expectations of “discussion” or because the verbalizing provides a useful strategy for developing their own thinking.

9.1.1.3 Exploring individual PST differences in using conceptual talk

As mentioned earlier in Chapter 6, some PSTs primarily engage in conceptual talk while others primarily engage in calculational talk. The most striking example of this contrast in the data involves Mark (from Episode 1) and Samantha (from Episode 2). Mark and Samantha both frequently make contributions during small group discussions. Because the composition of the small groups changes approximately every three weeks, both PSTs work with a variety of classmates during the semester. In every episode where Mark appears, he primarily participates in the discussion using conceptual talk. In every episode where Samantha appears, she generally employs calculational talk for her contributions.

Since conceptual talk involves PSTs relating back to contributions by classmates and calculational talk tends to be unrelated to the contributions of others, it seems counterintuitive that PSTs could engage in discussion using only calculational talk. In Episode 2, Samantha appears to be talking more to herself than to her partner. She gazes primarily at her own paper
and on several occasions when she poses a question, she often then answers the question herself and keeps working through the problem focused on her own paper. This observation leads to considering that one way a PST might appear to be contributing to a discussion while engaged essentially in only calculational talk would be for the PST to talk regardless of whether or not he or she has an attending audience. The links of conceptual talk from one PST to the next better represent the idea of a negotiation. This connecting back and forth would seem to be, in fact, essential to negotiating meaning. Using conceptual talk requires an attending and responsive audience.

Based on the data in this study, PSTs who generally engage in calculational talk over conceptual talk tend to be in groups that appear to struggle to maintain a discussion. One indicator is that these groups tend to remain quiet for relatively long periods. For example, in a comparison of the special problem solving sessions, Mark and his partner speak more than twice as many words in the twenty-minute problem solving session as Samantha and her partner do in their twenty-minute session. Samantha and Justina have long pauses in between their comments. Mark and Linda occasionally pause, but even these usually occur after one or the other have delivered a prompt to consider some particular aspect of the problem. I will revisit this finding in the Implications section of this chapter.

9.1.2 Discussing how PSTs build on each other’s ideas

In a high-level Math Talk community as described by Hufferd-Ackles, Fuson, and Sherin (2004), students are essential participants who build on each other’s contributions. This study explores the relevance of this characteristic of a Math Talk community for meaning-making with PSTs. In the second grade classroom where Hufferd-Ackles, Fuson, and Sherin (2004) did their research, they described this aspect of the community as, “Students
spontaneously compare and contrast and build on ideas” (p. 90). The course that is the subject of this study expects PSTs to participate at the highest level of a Math Talk community and therefore, building on each other’s ideas and contributions is an essential component of the community. The results of this study provide a thick description of what building on each other’s ideas looks like at the college level with one group of PSTs and connects Hufferd-Ackles, Fuson, and Sherin’s (2004) high-level Math Talk community to the construct of conceptual talk.

Building on each other’s ideas and contributions consistently occurs in the context of conceptual talk where PSTs’ comments and questions relates back to or interrogates what another PST has previously contributed. Episodes 3 and 4, which are the focus episodes for investigating how and when PSTs build on each other’s ideas, illustrate that one form of building on each other’s ideas involves a direct pick up of language and actions as the starting point. David connects back to what Diana presented and interprets her statements and explanations for his new group. In doing so, he records the algorithm in the same way that she did. He uses similar language to describe the procedure he is displaying. Finally, when his classmates request that he use blocks to demonstrate how the algorithm works, David manipulates the blocks similar to how Diana did earlier. At this point, he becomes confused about the demonstration and the building and clarifying begins. As David attempts to make sense of the explanation and accompanying demonstration, his new group members work with him to figure out how the algorithm and the demonstration are coordinated. David moves back and forth between repeating Diana’s language, actions, and descriptions, and incorporating new language and actions that he negotiates with members of his new group. Sfard’s (2001b) communicational approach may inform an understanding of the process as David and his classmates reach a successful conclusion in making sense of how the algorithm functions. Because the group members
communicate effectively—which according to Sfard (2001b) means they have a common
discursive focus, referring to the same things with the same words—they think and learn together
successfully.

A second form of building on each other’s ideas occurs when PSTs negotiate together to
arrive at a single way of expressing, describing, or explaining some idea, problem-solving
strategy, or solution. For example, Linda and Mark from Episode 1 work together to determine
how to describe their solution the problem and to justify that solution. In the process, they repeat
words and phrases to each other, they clarify and enhance each other’s words, and they negotiate
a final result to record on their papers. As shown earlier in Chapter 7, their final papers use
similar language and a similar organization that reflect this negotiation.

Instructors of PSTs should consider multiple forms of building on each other’s ideas as
they plan their lesson implementations. Instructors can set the stage for PSTs so that they are
required to engage in different types of building on ideas. By chance, the course in this study
presents PSTs with specific opportunities to do so and therefore, exposes both of these forms of
building on each other’s ideas. In David and Diana’s case, the first small group (with both David
and Diana) figures out the steps of the algorithm and the underlying mathematics. Because Diana
recognizes the steps of the algorithm quickly, she takes the role of “explainer,” and leads her
group through a description of how the algorithm works. After a jigsaw into new small groups,
David is in a position to build on what Diana said and did in the first group. The two-part
structure of this activity yields the opportunity for PSTs to build on each other’s ideas. In the
second instance with Mark and Linda, the instructor informs PSTs that they are expected to solve
the problem together and that the focus, in terms of expectations, is on how well they work
together, not on whether they are correct. Partnerships in this setting vary considerably in how
well they work together, but the opportunity to negotiate, to build on each other’s contributions, and then to come to consensus exists. It is an opportunity that Mark and Linda successfully seize. By providing situations previously identified as conducive to PSTs building on each other’s ideas, instructors may offer PSTs an opening to participate in a higher level Math Talk community and therefore in a community with opportunities for meaning making.

9.1.3 Discussing PSTs’ use of authoritative language

In general, PSTs do not incorporate much language from the instructor’s presentations or the text during their negotiations of meaning unless there are no informal substitutes for the necessary terms. For example, PSTs are likely to say “the answer” instead of “the sum.” They do, however, use terms like The Distributive Property, for which there is no informal substitute. This coupled with the fact that all interactions categorized as negotiating meaning involve gestures of some kind suggests PSTs do not require authoritative language to make sense of situations and to negotiate meaning. Rather informal language with taken-as-shared meanings and gestures involving pointing or acting out appear to be more useful for meaning making.

To go further, perhaps incorporating authoritative language from the instructor or the text or their authoritative discourse as Bakhtin (1981) describes it may actually interfere with the process of negotiating meaning. This hypothesis resonates with the Whole Language movement of the 1980’s derived in part from Noam Chomsky’s (1975) theories that viewed language as natural, active, and learner-driven rather than conditioned and behavioristic in nature (Shafer, 1998). “Chomsky argued that language is meaning-centered, complex, and forever interwoven in the life and energy of the learner” (Shafer, 1998, p. 18). Invented spelling (to which I am drawing a parallel here to PSTs’ informal and personally meaningful language) was part of the whole-language movement. Clarke (1988) found that first graders who used invented spelling
wrote on their own earlier, could write longer pieces, and performed better in spelling and in word analysis in reading. If using authoritative language from an instructor or the text in a mathematical context can be compared to the phenomena of invented spelling as I suggest, PSTs would more likely use their own natural, personally meaningful language in the process of negotiating meaning, and in fact, may even be required to rely on language that is personally meaningful for successful communication and negotiation with their peers.

9.2 Limitations

Several limitations may have affected the results and interpretations associated with this study. First, because I was a participant observer, I was limited in the amount of observational data I was able to collect. I taught the course that is the subject of this study, and during the class sessions, I was actively engaged in working with PSTs and maintaining the flow of the lesson. I was not in any way thinking about or focused on the research while teaching. Instead, I relied on the video- and audiotape data as the main sources of information about PST discourse. The videographer generally moved around the room according to which groups seemed to be engaged in a negotiation of meaning during their mathematical activity. There was only one camera taping one working group at a time. As mentioned earlier, the camera did not follow a single group for any specified length of time. Although both of the videographers I used have had extensive experience working in classrooms, neither they nor I knew exactly what we were expecting to see as related to meaning making, so the interactions captured on videotape may not always be the most salient interactions available. In addition, the camera frame is not always centered on what I later determined through my analysis to be of the great interest—for example, where someone’s gaze is directed during a question or explanation or on which paper a conversation focuses. Which groups were videotaped had a filtering effect on which episodes I
could adequately analyze. Although the audiotapes captured the conversations at every table, sound quality varied tremendously. Additionally, there are sometimes complications because it is not possible to see who is talking on the videotape and therefore, difficult to match the audiotape.

Second, to some degree, I relied on my reflective notes taken after the lesson implementation for information about the mathematical activities—that is, how did I implement the activity and what did I think the general results of the implementation was. Therefore, my personal biases and past experiences provided a lens for my interpretations of events.

Finally, because this is a case study, it is relevant to a certain group of people in a particular real-life setting. It is context-specific for this class at this time. Case studies are not designed so that they can be generalized to a larger population. Rather, case studies are used when the questions we are asking require in-depth analysis, and detailed and descriptive answers. Although I coded about fourteen and a half hours of class discourse, there was probably close to 250 hours of potential discourse to record and analyze—five groups (only one of which was recorded at a time) over about 50 hours of class time (assuming 10 hours of the 60-hour total may have been spent in assessment situations or during introductory or instructional lessons). Therefore, I coded only about 6% of the total 250 possible hours of discourse. Clearly, the results of this study cannot be generalized to other PST mathematics courses or other situations.

Even with the limitations intrinsic to this study, I believe the goal of providing rich descriptions of PST discourse during the meaning-making process has been attained. Although the results of the case study cannot be generalized, the resulting descriptions are meant to inform the work of mathematics teacher educators by possibly supplying examples and images of the process of meaning making.
9.3 Implications

As argued in Chapter 2, mathematics courses for PSTs should provide opportunities for meaning making in the context of exploring mathematical ideas and problems. This study focuses on describing characteristics of preservice teachers’ discourse during mathematical activity as they engage in meaning making. The characteristics uncovered in this study establish a set of thick descriptions for what meaning making looks like and how PSTs negotiate meaning around mathematical problems. This study has several implications for the design and implementation of PST mathematics content courses. Three of these implications are discussed below. The first implication relates to PSTs’ prevalent use of gestures. The second implication explores connections between the types of talk PSTs engage in, conceptual or calculational, and possible effects PST talk may have on the functioning of small groups. The third implication described below focuses on how and when the use of authoritative language from the instructor or the text might be further highlighted and encouraged.

During this study, observable meaning making related to language use and kinds of talk never exists in the absence of gestures. Gestures always accompany a negotiation of meaning. At the most basic level, deictic gestures (pointing) to indicate what PSTs are referring to during a discussion exist throughout episodes—primarily during conceptual talk (the talk most closely associated with meaning making), but also, on occasion, when the talk is calculational and PSTs are still attempting to communicate with each other. When an episode involves manipulatives (as in Episodes 3-5), these deictic gestures naturally support PSTs’ questions, comments, and explanations. In addition, when PSTs engage in reasoning together, as in Episode 1 where PSTs work together in the special problem-solving session, PSTs use iconic gestures strongly related
to the semantic concept to carry some of the meaning of their statements, questions, and comments.

That gesture plays such a key role in PST communications as they negotiate meaning in small groups implies that perhaps how and when an instructor uses gestures may make a difference in what PSTs experience during their content courses. For example, in this study, PSTs consistently use gestures when they are working together to negotiate meaning. Therefore, perhaps an instructor’s strategic use of gestures could improve PSTs’ opportunities for making sense of the instructor’s statements, explanations, and questions. Additionally, instructors might model gestures chosen for their semantic connections to mathematical ideas. Doing so may enhance PSTs’ communication toolkits by expanding their “vocabulary” of gestures. PSTs might benefit from this toolkit both when negotiating meaning with each other during small group work, but also when working with their future students. Finally, instructors may intentionally observe and then incorporate into their own repertoire PST-generated gestures to both reinforce PSTs’ use of gesture in negotiations of meaning, but also to communicate in ways with which PSTs are familiar.

A second implication of this study potentially informs the composition of small groups in a PST mathematics content course. The data suggest that characteristics of small group discussions may be somewhat dictated by PSTs who fall at the extremes of using conceptual talk or calculational talk. For example, in this study, as the camera moves from group to group over the semester, the episodes with Samantha show her fairly consistently participating in somewhat one-sided conversations—that is, she often explains the way she solved the problem or asserts the way she understands the problem. When her classmates ask her a question, she often repeats what she just said or ignores the question all together. In general, in the episodes where
Samantha appears the discussions tend to involve more calculational talk, less gesturing, and longer pauses. In sharp contrast, in the episodes where Mark appears the discussions generally involve more conceptual talk, more gesturing, and shorter pauses.

This study provides some evidence that an individual PST may in some way enhance, regulate, or limit the ways in which other group members contribute to a discussion. In Mark’s groups, most of the group members usually contribute during the small group discussions. Some PSTs may only interact by asking questions, but, for the most part, everyone tends to participate. In addition, much of the discussion is connected, where PSTs are commenting on, questioning, and responding to what others contributed earlier. In Samantha’s groups, the story differs. Often, at least one group member (and sometimes several) does not contribute to the discussion. The talk tends generally to be one way—that is, PSTs seem to be reporting out or talking out their thought process without engaging other group members.

This implies that an instructor might draw on the knowledge of which PSTs potentially limit versus promote discussion to help determine the composition of small groups. As explained above, PSTs may be more likely to enhance participation opportunities by engaging a small group with conceptual talk. Contrarily, PSTs may diminish opportunities for group discussion with a consistent use of calculational talk. When an instructor circulates to evaluate how PSTs participate in small group work, PSTs like Mark and Samantha may appear to participate equally in that they are both making significant contributions to the group by talking. The current study suggests that, instead of simply gaging “talkers” versus quiet PSTs when assembling groups, instructors might attend to some of the characteristics of interactions associated with a negotiation of meaning, observing which PSTs tend to report their own thinking versus PSTs who listen to, interrogate, and respond to each other’s prior contributions. Identifying the
common interaction patterns for individual PSTs may provide instructors with tools for structuring more balanced small groups thus providing accessible discussion spaces in which all PSTs can participate.

A third implication of this study concerns the use of authoritative language from the instructor or the text. As described in Chapter 8, PSTs generally do not use authoritative language as they negotiate meaning around a problem or problem situation. Even as they negotiate what to record after the problem solving ends, PSTs tend to use more informal language that is personally meaningful and endorsed by the group. This might imply that authoritative language from the instructor or the text does not provide a useful vehicle for PSTs in terms of communicating mathematical ideas. In the previous section of this chapter, I drew comparisons between this phenomena and the Whole Language movement of the 1980’s (Shafer, 1998).

If, as instructors of preservice teachers, we value PSTs’ development and use of authoritative language from the instructor or the text, the results of this study might imply that instructors need to redesign lessons to increase PST use of authoritative language. Following the example of whole language, one way to possibly further the use of authoritative language would be for instructors to require two levels of product from PSTs—one informal and one more formal. The parallel in writing would be a draft version of a written piece versus a final copy that has been edited and polished. In the context of a mathematics classroom, this might take the form of a more formal paper, report, or outline for solving a problem after the initial work has been completed and recorded informally. PSTs might be tasked with taking their informal work and intentionally revising that work to reflect their continuing development of mathematical expertise. PSTs’ use of appropriate authoritative language and symbols would serve as one
hallmark of this development. More formal work would likely also include a higher level of organization and precision in its presentation. Since PSTs incorporate more authoritative language even in their informal written work, it seems reasonable to require them to, on occasion, produce formal written work that more extensively integrates authoritative language. PSTs might be asked to identify appropriate authoritative language from the instructor’s presentations and from the text. If PSTs keep notes in a daily journal, perhaps mandating that the key vocabulary terms are included in a list in that daily journal would increase awareness and perhaps use of those terms. Finding the time for implementing this approach presents a challenge. This approach would likely also require finding mechanisms for building bridges between PSTs’ use of authoritative terminology in formal written products and PSTs’ everyday and informal usage.

9.4 Future Research and Conclusion

This study endeavored to unpack and describe characteristics of PSTs’ discourse during mathematical activity as PSTs engage in meaning making. This research produced rich descriptions as well as models of how PSTs engage in discourse during the process of meaning making. The analysis of how PSTs participate in a negotiation of meaning during small group and partner work produced important results regarding the characteristics of these negotiations. First, the results indicate that PST talk may be conceptual, where PSTs’ comments, explanations, and questions connect to previous PST contributions, or calculational and more individual, where PSTs tend to report out on what they are doing or thinking with little or no connection to prior contributions. Although conceptual talk tends to be associated with a negotiation of meaning, on occasion, calculational talk lays the foundation for the conceptual talk to follow. Second, throughout the videotaped episodes, gestures consistently accompany negotiations of meaning.
No instances of meaning making in this study exist devoid of gestures. Third, when PSTs engage in conceptual talk, they may also build on each other’s contributions and ideas. Finally, whether involved in negotiating meaning while solving problems or while determining how to describe a thought process or strategy, PSTs rarely use authoritative language from the instructor or the text. Although these descriptions of small group interactions provide images of what meaning making looks like in the context of mathematical activity, the findings of this study, perhaps offer many more questions than answers.

The importance of gesture, one of the prominent characteristics of meaning making activity uncovered in this study, requires further investigation. The videotape in this study focuses on small group work and generally does not include the instructor. I could not locate research specific to how the instructor of a PST content course uses gesture in presenting mathematics content, but perhaps a body of research could be built on existing research on gesture from other contexts (e.g., Singer et al., 2008). Exploring which gestures more effectively support PSTs in making sense of a mathematical idea or problem and when and how an instructor uses gesture to communicate about mathematical ideas could inform planning for the way course content is introduced and developed. Investigating how PSTs do or do not adopt an instructor’s or each other’s gestures for communicating in mathematically meaningful ways might inform an instructor’s selection of which gestures to incorporate into presentations and explanations with the goal of expanding PSTs’ communication resources.

Because the focus of this study was to describe characteristics of discourse during meaning making in mathematical activity, the results do not provide any insight into how the discourse in the process of meaning making may change or evolve over the course of a semester for small groups or for the individuals participating in small groups. Following several small
groups and possibly several individuals for the duration of a course might yield a hypothetical developmental trajectory for the discourse of meaning making. In addition, analyzing the difference between successful and unsuccessful negotiations of meaning may contribute to understanding some of the transitions in that trajectory. Having a hypothetical trajectory may support instructors in designing course experiences to enhance PSTs’ progress along the trajectory.

One unexpected finding in this study suggests that individual PSTs may adhere to particular interaction patterns and that these patterns may affect classmates’ opportunities for and engagement in meaning making. To further investigate this phenomenon, research exploring the relationship among individual interaction patterns and the interaction patterns in groups would be useful. Following several PSTs from group-to-group and from partner to small group to whole group settings might provide insight into how individuals may affect others as well as the group as a whole.

The findings of this study suggest that PSTs who engage mostly in calculational talk may limit opportunities for classmates to participate in the negotiation of meaning while PSTs who engage mostly in conceptual talk may enhance those opportunities. There are no videotaped episodes with groups composed of the PSTs who, in this study, were identified at both extremes of that interactional spectrum—those incorporating mostly calculational talk and those engaging primarily in conceptual talk; but it would be interesting to see how the individuals in a group including both PSTs would participate. Learning more about the relationship among different interaction patterns during meaning making in small group work may provide instructors with the perspicacity to improve all PSTs’ opportunities to engage in meaning making during mathematical activity.
In conclusion, instructors of preservice teachers are likely always to have more to learn about how PSTs engage in meaning making during mathematical activity. My PSTs have sometimes struggled with the expectation that they will make sense of the mathematics. About two thirds of the way through the course that is the context of this study, one of my PSTs beautifully expressed the argument for why meaning making is so important:

Monica: This class is unfair because you are asking us to make sense out of the math. Even if I can do the skills, that isn’t enough. Why, if it’s so important to understand the math didn’t our teachers teach this way before now?

Instructor: Maybe the teachers didn’t understand the math so well either…so that would make it tough to teach.

Monica: How can anyone learn to understand if the teachers don’t…[pause] Ohhhh.
References


Darke, Kelly Marie. (2010). *An examination of the questioning interactions of prospective teachers during mathematical discussions.* (Doctor of Arts in Mathematics dissertation), University of Illinois at Chicago, Chicago, IL. Proquest database. (3417332)


Sfard, Anna. (2001b). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics, 46*(1/3), 13-57.


APPENDICES

Appendix A

Instructions for Special Problem-Solving Session

Time: 15–20 minutes

Framing the task:

To introduce the task, first explain that the goals are for them to discuss the mathematics and to come to consensus about their ideas before recording their answers. Explain that the interview will be graded according to the following criteria*:

A. Their use of mathematically meaningful language
B. How they engage their partner with questions, suggestions, and/or explanations
C. Whether there is a match between the problem situation and the conversation (that is, they are focused on discussing meaningful aspects of the models such as how they are divided and what the salient differences are between the two models representing a particular property).

* Inform them that their work on this task cannot hurt their grades, but it could bump them up if they are borderline.

Invite PSTs to think privately first, discuss their ideas, and then write. If they have a tough time getting started, the suggested scaffolding for the session might include prompts like the following:

1. Describe what each property is. Give several examples, including one with variables.
2. Describe what you see happening in the models. Write number models that could describe each model.

Suggestion 2 above might require further prompting. Since Model C is the simplest, that seems like a good place to start. Possibly ask PSTs to choose the model they think is simplest and start with that one. This means emphasizing that they don't need to go in order.
Task for Special Problem-Solving Session

For each model below:

1. Tell which property the model illustrates (Commutative, Associative, Distributive). *Hint: Recording each property with variables first might help you identify the properties.*

2. Explain how the model illustrates the property. Use labels to help with your explanation.

3. Write number sentences to describe the model. (These should also illustrate the property.)

**Model A**

![Model A grid]

**Model B**

![Model B grid]

**Model C**

![Model C grid]
# Appendix B

## Initial Course Plan

<table>
<thead>
<tr>
<th>Week #</th>
<th>Topic</th>
<th>Objectives (Students will be able to…)</th>
<th>Activities</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Patterns</td>
<td>• Recognize and extend patterns</td>
<td><strong>Post Agenda on Board</strong> (with Warm up)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Identify and describe patterns</td>
<td><strong>Find your Table:</strong> Take a number and find the table that best describes</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Use patterns to solve problems</td>
<td>your number. [Multiple of 7—7, 14, 21, 28, 35; 2 more than a multiple of</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>• Represent pattern rules symbolically</td>
<td>5—22, 27, 32, 37, 42; a factor of 24—2, 3, 4, 6, 8; a power of 10—0.01,</td>
</tr>
<tr>
<td>Class 1</td>
<td></td>
<td>• Describe characteristics of whole</td>
<td>100, 1,000, 0.1, 10; a prime number greater than 20—23, 29, 31, 39, 41]</td>
</tr>
<tr>
<td>8/27</td>
<td>Patterns</td>
<td>numbers, counting numbers, and integers (also where the sets intersect; how they differ).</td>
<td><strong>Make a nametag:</strong> Take a piece of cardstock from your folder and record your first name in some dark marker (large enough for your old instructor to see it).</td>
</tr>
<tr>
<td>(M)</td>
<td></td>
<td></td>
<td><strong>Warm up</strong> (on board): Find the sum… (Use patterns—no calculators allowed.) (6 triplets of 13, 15, 17): 13 + 15 + 17 + 13 + 15 + 17 +... + 13 + 15 + 17</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td><strong>Getting Started</strong> (need HO [HO stands for “Handout”]—overview, log, bingo, info page): Follow up on “Find Your Table” by briefly talking about types of numbers.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Introductions with Bingo. Course Overview “Who am I as a Math Student?” info page.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td><strong>Warm-up follow up:</strong> Briefly share strategies. Course Overview “Who am I as a Math Student?” info page.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td><strong>Content—Patterns</strong> (need HO of problems): Find sum of the first 100 consecutive counting numbers. (They work on problem &amp; go over strategies… how to break it down into a simpler problem; applying simpler versions to the bigger problem; Gauss’ method) Introduce Gauss story Identify patterns in Pascal’s Triangle (if time permits)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Reflection (on board):</td>
<td></td>
</tr>
</tbody>
</table>
### Appendix B (continued)

<table>
<thead>
<tr>
<th>Week #</th>
<th>Topic</th>
<th>Objectives (Students will be able to…)</th>
<th>Activities</th>
</tr>
</thead>
</table>
| 1 Class 2 8/29 (W) | Patterns | - Identify primes and composites based on number of factors  
- Recognize and describe unique property of prime factors of square numbers | [Substitute Instructor]  
LMT  
*Factor Captor*  
- Focus on developing vocabulary around factor, multiple, prime numbers, square numbers, product, divisible, divisor  
- Seeing that a product would have different factor pairs  
- Using math ideas to development more efficient winning strategies  
**Homework**—no additional homework. They are doing the Belief’s Survey |
| 2 LABOR DAY FOR Class 1 9/5 (W) | Number Theory—prime/composite/odd/even/factor/multiple/divisibility | - Identify special kinds of numbers (prime, composite, even, odd)  
- Identify 0 as an even number  
- Recognize relationship between number of factors and definitions of prime & composite  
- Recognize relationship | **Warm up** (on board): Dartboard problem (See Basserear p. 46) You have a dartboard like the one pictured. (4 rings: out is 1, then 3, then 5, inner is 7.) Your throw 4 darts, all of which land on the dartboard. What kinds of scores are possible and what kinds of scores would be impossible. Be ready to share how would you convince someone your answer is correct?  
**Getting Started**  
Turn in 1st homework  
*Warm-up follow up:* Briefly share ideas—answer must be even & why—encourage images for proof as well as lists. Review that 0 is even (as a sidebar). If someone says they listed all possible, ask them to describe how they know they got all possible combinations—that is how did they systematically approach solving the problem. Might ask, “How would it change with 3 darts?”  
**Content**—(need grids of 201 - 409 (30) and sheet protectors 1 per student with grid inside; vis-à-vis markers): |
<table>
<thead>
<tr>
<th>Week #</th>
<th>Topic</th>
<th>Objectives (Students will be able to…)</th>
<th>Activities</th>
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|       |       | between factors, multiples, & divisibility and use symbolic notation to represent the relationship  
|       |       | • Find the prime factorization of a number  
|       |       | • Use symbolic representations as appropriate | Review important points from Sieve of Eratosthenes  
|       |       | How did you use patterns to help you cross out composite numbers?  
|       |       | How do you know that the circled numbers are prime numbers?  
|       |       | What was the first multiple of 23 that would be crossed out on the grid? Why?  
|       |       | Why was it okay to remove the even numbers from the grid?  
|       |       | Compare the 2 grids (with and without even numbers). How are they the same and/or different? Why can you still count by multiples without worrying about the even numbers?  
|       |       | FIND primes from 201 to 409 “bootstrapping” from sieve (ala Teitelbaum)  
|       |       | • Set up includes asking them how they know what number to circle first  
|       |       | • How do they know how far to go? (must do all primes up to the square root of the number… how do they know?)  
|       |       | • Afterwards, one of the patterns they should be able to use is counting 15 by going down a row and then up or back according to their prime.  
|       |       | • Each student will do one prime on sheet protector and put them together over a page to see primes.  
|       |       | • Finally, they circle the primes on their group page.  
|       |       | Stretching Problem from Fostering Algebraic Thinking p. 54  
|       |       | Introduce finding prime factorization using trees (they have to figure out slide in homework)  
|       |       | Reflection (on board):  
|       |       | Record something about numbers that you thought about today for the first time.  
|       |       | Reading: Bassarear—Prime Factorization  
|       |       | Homework: prime factorization of 357 & 10,780? [The Fundamental Theorem of Arithmetic states that every natural number can be expressed as the product of prime numbers in only one way. Find the prime factorizations of… maybe noting outcomes of odd/even comp] Follow up—There are 3 prime factors of 357, 3, 7, 17; There are 6 prime factors of 10,780, 2², 5, 7², 11. If the number one were considered a prime number, then |
### Week 3

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<th>Topic</th>
<th>Objectives (Students will be able to…)</th>
<th>Activities</th>
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<tr>
<td><strong>Class 1</strong>&lt;br&gt;9/10 (M)</td>
<td>Number Theory—prime/composite/odd/even/factor/multiple/divisibility&lt;br&gt;• Recognize relationship between number of factors and definitions of prime &amp; composite&lt;br&gt;• Recognize relationship between factors, multiples, &amp; divisibility and use symbolic notation to represent the relationship&lt;br&gt;• Find the prime factorization of a number&lt;br&gt;• Find GCF &amp; LCM (for word problems if time permits)&lt;br&gt;• Use symbolic representations as appropriate</td>
<td><strong>Warm up</strong> (on board): <em>Goldbach’s Conjecture</em>? It is believed that every even number greater than 4 can be expressed as the sum of two primes. <em>Examples</em>: $68 = 7 + 61; 12 = 5 + 7$. Can you find sums for all of the two-digit even numbers?&lt;br&gt;<strong>Getting Started</strong>&lt;br&gt;Warm up follow-up: Why is the prime number 2 never used? E.g., only even prime, and even + odd cannot be even. When could 2 be used? E.g., if looking for prime addends for odd number.&lt;br&gt;<strong>Content</strong>— (need HO of problems &amp; Cuisenaire rods):&lt;br&gt;<strong>GCF</strong>: Do “Cutting Squares” problem from Bassarear (p. 239)&lt;br&gt;Review ways to find GCF—list factors; slide; prime factorization&lt;br&gt;• Intro Cuisenaire rods…&lt;br&gt;Organize all of the rods from smallest to largest.&lt;br&gt;How many different rods are there?&lt;br&gt;What are some of the relationships you see between the rods?&lt;br&gt;• Model GCF with Cuisenaire rods?&lt;br&gt;<strong>LCM</strong>: Begin with “What do you think the LCM is?”&lt;br&gt;• Review ways to find LCM—list multiples; prime fact; <strong>Venn</strong> diagrams of factors&lt;br&gt;• Model with Cuisenaire rods?&lt;br&gt;Do word problems if time permits…&lt;br&gt;Challenge question: Why does $a \times b = \text{GCF} (a,b) \times \text{LCM} (a,b)$&lt;br&gt;<strong>Reflection</strong> (on board):&lt;br&gt;Describe some way that a classmate’s contribution influenced your learning.&lt;br&gt;<strong>Reading</strong>: N/A</td>
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<td>Week #</td>
<td>Topic</td>
<td>Objectives (Students will be able to…)</td>
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| 3     | Class 2 9/12 (W) | Number Theory—prime/composite/odd/even/factor/multiple/divisibility | **Additional Resources** GCF & LCM from Bassarear (posted on Blackboard)  
**Homework:** GCF & LCM word problems & naked numbers  

**Warm up** (on board need HO and rods): Cuisenaire Rods with GCF & LCM problems  
**Getting Started:**  
Convert homework scores to decimals & percents  
*Warm up Follow-up:* Share some discoveries they made with Cuisenaire rods (don’t spend a lot of time here)  

**Content:**  
Have groups who did Beads present their methods—(First group for each problem should also explain the problem.) One group each for slide, prime factorization, and lists; For hotdogs, have 3 present—same methods… On chart paper  
*Summary & Reflection* on GCF & LCM—advantages to slide and advantages to factor trees  
*Notes on place value*—have them write a brief statement about what they know about our base-10 number system  
Bassarear activity on developing a number system (2.7)—creating an Alphabatian number system.  
Maybe jigsaw with the different number systems—Egyptian, Roman, Babylonian, and Mayan. They become an “expert” so that they can write numbers in their prescribed system. Then they jigsaw back into a mixed group. Prompts focus on comparing features of their systems with base-10 system.  
Maybe play base-10 exchange game (maybe base 5)—for sure, no time…  

**Reflection** (on board):  
Describe what you learned about number systems.  

**Reading:** N/A  
**Homework:** Find all the factors of 100 and use the information to find all of the factors of 200.
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| 4 Class 1 9/17 (M) | **Finish:** Number Theory  
**Intro:** Place value; Base-10 System | • Recognize and describe underlying structure of Base-10 number system  
• Describe how the Base-10 system is similar to and different from other number systems  
• Describe benefits of the Base-10 system  
• Name numbers using expanded notation  
• Use symbolic representation to show how to write number in expanded notation | **Warm up** (on board & need toothpicks & HO): move toothpicks for Roman Numeral puzzles  
**Getting Started** (need—put out toothpicks and HO):  
1) Convert homework scores to percents.  
2) Stack you homework assignments in the middle of your table.  
3) Do the Warm-up Problem.  
**Content**— (need):  
1) Use prime factorization to… find factors… find GCF… find LCM  
\[ 2^5 \times 3^3 \times 5^3 \quad \& \quad 3^3 \times 7^2 \times 11^2 \quad \& \quad 2^3 \times 3 \times 5^3 \times 7 \]  
2) First they will come up with a number system using only A, B, C, D, and 0 (from Bassarear Investigation). They will work in their table groups to devise a system; make a poster to explain their system; and share their system with the whole group.  
3) Read about different historical number systems and share what they learn about those systems. For this, they will jigsaw into 4 expert groups… That means that some tables will have more than one person from the “expert” group.  
4) Solve the muffin problem from EM. Discuss features of this number system (if time allows?) no way…  
**Reflection** (on board):  
Describe what you learned about number systems and how you learned it.  
**Reading:** Hindu/Arabic # system from Bassarear (following up on class assignments Tu)  
**Homework:** GCF & LCM homework (problems from Bassarear) |
### Appendix C

#### Normal Enacted Course Plan

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<tr>
<th>Week #</th>
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| 1 Class 1 8/27 (M) | Patterns | • Recognize and extend patterns  
• Identify and describe patterns  
• Use patterns to solve problems  
• Represent pattern rules symbolically This has not yet come up…  
• Describe characteristics of whole numbers, counting numbers, and integers (also where they intersect; how they differ). | **Post Agenda on Board** (with Warm up)  
**Find your Table:** Take a number and find the table that best describes your number. [Multiple of 7—7, 14, 21, 28, 35; 2 more than a multiple of 5—22, 27, 32, 37, 42; a factor of 24—2, 3, 4, 6, 8; a power of 10—0.01, 100, 1,000, 0.1, 10; a prime number greater than 20—23, 29, 31, 39, 41]  
**Make a nametag:** Take a piece of cardstock from your folder and record your first name in some dark marker (large enough for your old instructor to see it).  
**Warm up** (on board): Find the sum… (Use patterns—no calculators allowed.) (6 triplets of 13, 15, 17): 13 + 15 + 17 + 13 + 15 + 17 +…+ 13 + 15 + 17  
**Getting Started** (need HO—overview, log, bingo, info page):  
Follow up on “Find Your Table” by briefly talking about types of numbers.  
Introductions with Bingo.  
Course Overview  
“Who am I as a Math Student?” info page.  
*Warm-up follow up:* Briefly share strategies.  
**Content—Patterns** (need HO of problems):  
Find sum of the first 100 consecutive counting numbers. (They work on problem & go over strategies… how to break it down into a simpler problem; applying simpler versions to the bigger problem; Gauss’ method)  
Introduce Gauss story  
Identify patterns in Pascal’s Triangle (if time permits) We started this, but they finished with Wenjuan on Thurs. They used the worksheet from Bassarear’s book (listing pwrs |
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<td>Class 2 8/29 (W)</td>
<td>Patterns</td>
<td>of 2, triangular numbers, natural numbers, and counting numbers). They found these patterns. Wenjuan showed them patterns on her computer after they were done. NO: Use patterns to solve the problem—how many ways to make 25 cents using P, D, and N. We did not do this. The purpose of it was to begin building a case for why identifying and using patterns is important—even in elem school. Reflection (on board): Describe mathematical ideas you worked with in today’s patterning problems. Reading: Prime numbers (Bassarear p. 229-236) Additional Resources: Gauss story (posted in Blackboard) Homework: Sieve of Eratosthenes for homework (both original form and with only odd numbers) &amp; Beliefs Survey Due 8/30</td>
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<td>2</td>
<td>Class 1 LABO R DAY Class 2 9/5 (W)</td>
<td>Number Theory — prime/composite/odd/even/factor/multiple</td>
<td>[ Substitute Instructor] LMT Factor Captor ● Focus on developing vocabulary around factor, multiple, prime numbers, square numbers, product, divisible, divisor ● Seeing that a product would have different factor pairs ● Using math ideas to development more efficient winning strategies</td>
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</table>

Warm up (on board): Dartboard problem (See Basserear p. 46) You have a dartboard like the one pictured. (4 rings: out is 1, then 3, then 5, inner is 7.) Your throw 4 darts, all of which land on the dartboard. What kinds of scores are possible and what kinds of scores would be impossible. Be ready to share how would you convince someone your answer is correct? The problem was okay, but I was not really thoughtful about where it would lead. I brought up the “proof” content from last year—unsuccessfully… We looked at 2x and 2x-1… and then they added these… We also looked at pictures…arrays… which I drew. We spent about 45 minutes on this, and I am not sure what we got out of it. Exposing them to some of the work we did last year… I need to be sure to be focused on my goals. I was
### Appendix C (continued)

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<td>/ divisibility</td>
<td>• Recognize relationship between number of factors and definitions of prime &amp; composite</td>
<td>thinking that they should talk about making a list… but I think I might have shortchanged this by moving quickly to where I wanted to go instead of letting it unfold. Also, this was only meant to be a warm up, so we spent more time than we should have as is… Should have focused more on systematically “proving” going beyond examples… and leaving it where they took it to… not trying to get the “proving” in.</td>
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<td>• Recognize relationship between factors, multiples, &amp; divisibility and use symbolic notation to represent the relationship—Did not bring in symbolic notation here, but they seemed to be doing okay with talking about factors, multiples, and divisibility.</td>
<td>Getting Started:</td>
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<td>• Find the prime factorization of a number—This was briefly introduced at the end of the gum stretching problem. They will follow up with reading and a</td>
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<td>Warm-up follow up: Briefly share ideas—answer must be even &amp; why—encourage images for proof as well as lists. Review that 0 is even (as a sidebar). If someone says they listed all possible, ask them to describe how they know they got all possible combinations—that is how did they systematically approach solving the problem. Might ask, “How would it change with 3 darts?”</td>
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<td>Content— (need grids of 201 - 409 (30) and sheet protectors 1 per student with grid inside; vis-à-vis markers):</td>
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<td>Review important points from Sieve of Eratosthenes—They talked for a few minutes… seemed to go well. We briefly reviewed some of the patterns they used on the OH. And they talked about the first multiple of 23 they would have to look at is 23 x 23.</td>
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<td>How did you use patterns to help you cross out composite numbers?</td>
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<td>How do you know that the circled numbers are prime numbers?</td>
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<td>What was the first multiple of 23 that would be crossed out on the grid? Why?</td>
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<td>Why was it okay to remove the even numbers from the grid?</td>
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<td>Compare the 2 grids (with and without even numbers). How are they the same and/or different? Why can you still count by multiples without worrying about the even numbers?</td>
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<td>FIND primes from 201 to 409 “bootstrapping” from sieve ala Teitelbaum: They did this in about 20 minutes. We spent about 5 more minutes talking about it. From my perspective, this was a useful activity in that it extended what they were doing from their homework; got them using desirable language (e.g., divisible, multiple, factor); got them re-examining the process together; and devising strategies for speeding up the process.</td>
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<td>homework assignment tonight.</td>
<td>Wenjuan said she felt the entire activity was just a waste of time.</td>
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<td>• Use symbolic representations as appropriate—we used this with odd/even numbers, but Wenjuan suggested it was misplaced.</td>
<td>• Set up includes asking them how they know what number to circle first</td>
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<td>• [8/27] Wenjuan &amp; I had a long conversation about what a pattern is (e.g., is Goldbach’s conjecture a pattern?)</td>
<td>• How do they know how far to go? (must do all primes up to the square root of the number… how do they know?)</td>
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<td>• We decided that we would say that Goldbach noticed a pattern—but that it has not yet been determined if it is a pattern that will always hold true. (See Week 2 Class 2 Warm up.) We decided that mathematics begins with a search for</td>
<td>• Afterwards, 1 of the patterns they should be able to use is counting 15 by going down a row and then up or back according to their prime.</td>
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<td>Stretching Problem from <em>Fostering Algebraic Thinking</em> p. 54—We spent about 15-20 minutes on working on the problem. They got through the first 2, but most did not finish the page. I had decided that the purpose of this problem was to get them to think about prime factorization… Next time, I would propose to use the problem in that way—that is, only give them the first 2 questions (maybe) and make the last 2 challenge questions. Then the follow up would—1st, solicit their various combinations of machines for each number; and 2 thinking about the fewest number of machines for the entire set. Then we would extrapolate from that to prime factorization.</td>
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<td>Introduce finding prime factorization using trees (they have to figure out slide in homework)—Anna mentioned this. I drew a tree on the board to remind them of this. I think they will be okay, because I think this is familiar.</td>
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<td>Reflection (on board): Record something about numbers that you thought about today for the first time. Asked them to record something they thought about today for the first time (since we were down to the wire with time)</td>
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<td>Reading: Bassarear—Prime Factorization</td>
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<td>Homework: prime factorization of 357 &amp; 10,780? [The Fundamental Theorem of Arithmetic states that every natural number can be expressed as the product of prime numbers in only one way. Find the prime factorizations of… maybe noting outcomes of... ]</td>
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### Week #  | Topic | Objectives (Students will be able to…) | Activities (possibilities & notes)
---|---|---|---
3  | Number Theory — prime/ composite/ odd/ even/ factor/ multiple / divisibility |  **patterns and that the next step is to verify whether the pattern is one that will always be true.**  
- Recognize relationship between number of factors and definitions of prime & composite
- Recognize relationship between factors, multiples, & divisibility and use symbolic notation to represent the relationship
- Find the prime factorization of a number
- Find GCF & LCM (for word problems if time permits)
- Use symbolic representations as appropriate
- 9/11/11: Wenjuan |  **odd/even comp] Follow up—** There are 3 prime factors of 357, 3, 7, 17; There are 6 prime factors of 10,780, 2, 5, 7, 11. If the number one were considered a prime number, then the fundamental theorem of arithmetic would not be true. Why not? **Challenge:** Something Nu (v) from Fostering Algebraic Thinking p. 55—challenge solutions would have to be posted on Blackboard. Did no include the challenge or a bonus question.

3 Class 1 9/10 (M) | Number Theory — prime/ composite/ odd/ even/ factor/ multiple / divisibility |  **Warm up (on board):** Goldbach’s Conjecture? We did not do Goldbach’s Conjecture for 2 reasons—1 it didn’t seem connected enough anymore… and 2) we had almost finished our work with the gum stretching problem, and I wanted to follow up. So I asked them to go back and work on that problem for a while—finish 3-5. They worked on it for about 30 min and then we briefly (and additional 10 min) went over their answers—that the Fundamental Rule of Arithmetic says any number can be written in exactly one way as the product of primes. They then shared their answers for 4 & 5. I pushed them to think about how they know if they found them all… We talked about finding a boundary beyond which all answers would be too big… and to be systematic so they know they did not miss anything.
It is believed that every even number greater than 4 can be expressed as the sum of two primes. **Examples:** 68 = 7 + 61; 12 = 5 + 7. Can you find sums for all of the two-digit even numbers?  
**Getting Started:**  
**Warm up follow-up:** Why is the prime number 2 never used? E.g., only even prime, and even + odd cannot be even. When could 2 be used? E.g., if looking for prime addends for odd number.
**Content**— (need HO of problems & Cuisenaire rods):  
**GCF:** Do “Cutting Squares” problem from Bassarear (p. 239) This worked well. They spent most of an hour working through this. Several students started by finding the area and then looking for divisors for the area. I have to think more about how to ask questions to steer students away from this approach (so they don’t spend all of their time unproductively pursuing this.) They shared how they thought about the problem as a picture and how they used prime factorization to find the greatest common factor.
and I talked about two possible areas that cause confusion and/or misconception… that is

- First—students may think of factors as part of a pair, which might cause some of their confusion when they are asked to list factors;
- Second—when we talk about prime factorization, there seems to be some confusion about whether we mean the process of finding the prime factorization (which is what the students seem to often think) or if we are talking about the product of prime factors that result in some number. We will try to be more specific in our language and

Although we used this language, we did not emphasize it yet.

STOPPED HERE… after they completed the Bead and Hot Dog problem, but with no sharing out yet.

- Review ways to find GCF—list factors; slide; prime factorization—At the end of the period (the last 10-15 min), they solved the hot dog and bead problems (from last semester). I gave one problem to each side of the room. On Tuesday, I am planning to distribute the other problem to each half of the room and have their classmates share how they solved the problems. Then we will go into finding GCF and LCM with factor trees, by listing factors/ multiples, or using slide. We will also do some naked # problems. They expressed great relief and satisfaction that they were able to do the bead and hot dog problems fairly easily.

- Intro Cuisenaire rods… We did not do this… I decided that this would not have enough impact as part of the lesson. However… I think I will start with this as the warm up. The warm ups have not engaged anyone… Well, the last one wasn’t really a warm up anyway. It was really just to continue from last time. I was starting to lose the purpose of doing a “warm up.” So… I am thinking that whatever it is should be quick, fun, and if someone is late, they don’t get to do it. Cuisenaire rods seem to foot that bill. Now the challenge is to create the assignment!

  - Organize all of the rods from smallest to largest.
  - How many different rods are there?
  - What are some of the relationships you see between the rods?

- Model GCF with Cuisenaire rods?

**LCM:** Begin with “What do you think the LCM is?”

- Review ways to find LCM—list multiples; prime fact; *Venn* diagrams of factors
- Model with Cuisenaire rods?

  - Do word problems if time permits…

  Challenge question: Why does $a \times b = \text{GCF (a,b)} \times \text{LCM (a,b)}$?

**Reflection** (on board):
### Appendix C (continued)

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<td>not use prime factorization to mean both the process and the product.</td>
<td>Describe some way that a classmate’s contribution influenced your learning. This was only given orally. They had about 5 min to write. Most took about 3 min.</td>
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<td>• Recognize relationship between number of factors and definitions of prime &amp; composite</td>
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<td>• Find the prime factorization of a number</td>
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<td>• Find GCF &amp; LCM (for word problems if time permits)</td>
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<td>3 Class 2 9/12 (W)</td>
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<td>Warm up (on board need HO and rods): Cuisenaire Rods for GCF &amp; LCM This took almost ½ hour to complete. In the process, they were not talking about the math, and it was not clear what they understood about the processes they were using. They did mention comparisons to their homework (with the strips).</td>
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<td>Getting Started (need): Convert homework scores to decimals &amp; percents</td>
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<td>Warm up Follow-up: Share some discoveries they made with Cuisenaire rods (don’t spend a lot of time here) –they needed to spend time with this. I asked them to talk about how the rod work was connected to factors and multiples. They did not talk much, but all said they got it. Part of the purpose was for assessment to see what they got and part of the purpose was so that they would have opportunities to use the language around factors and multiples. After a couple of minutes of mostly whispers and silence, I invited them to make sure everyone at their table understood their conclusions because they were going to move around and share them with each other. This got them talking. When all groups reported being ready, they counted off by 5s and rearranged themselves into new groups. First they were invited to share their group’s insights and then they were asked to craft a statement synthesizing everything from the new group. Following are their final statements:</td>
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For homework, they had problems from 106-111 of Mathematics for Elem T. An Activity Approach (METAA) by Bennett, Burton, and Nelson.
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<td></td>
<td>• Use symbolic representations as appropriate</td>
<td>“All the rods can be represented by the smaller rods. To find the LCM with any rods, you would line them up until they have the smallest same length. [to the left] Every time you line up one color, it would be a multiple of the first one.”</td>
<td></td>
</tr>
</tbody>
</table>
"The factors of the train are represented by a group of rods that are all the same color and the same overall length of the train."
### "The least number of rods of different colors to create trains of equivalent length would be the LCM.

For example, using only using only purple rods (4 units) and only dark green rods (6 units) to create trains of equal length, you would use 3 purple rods and 2 green rods to get 12, the LCM."
### Week # | Topic | Objectives (Students will be able to…)
--- | --- | ---

| 2 | | Activities (possibilities & notes)

“You can use the smaller rods to help find the GCF by taking the difference between the two original trains and subtracting it from the smaller of the two trains until the values are equal, and that should be the GCF.”
### Week # | Topic | Objectives (Students will be able to…) | Activities (possibilities & notes)
--- | --- | --- | ---

“Use the smaller rods as factors to see what combination of rods (of the same color) will be equal to the larger train. This shows the factors of the bigger number.”

The groups started with vague, general statements that did not make sense. For example, “You can use the smaller rods to find GCF and the bigger rods to find LCM.” OR “Use the smaller rods to find the GCF of the bigger rods.” Not only did we work to refine their language, but we also tried to push them to uncover and discuss relationships between numbers, their factors, and their multiples.

This took until about 10:20. Then we moved on…
<table>
<thead>
<tr>
<th>Week #</th>
<th>Topic</th>
<th>Objectives (Students will be able to…)</th>
<th>Activities (possibilities &amp; notes)</th>
</tr>
</thead>
</table>

[Note—This gave them an opportunity to represent their table groups in a new group. Getting to know a few more people…]

**Content:**

Have groups who did Beads present their methods—(First group for each problem should also explain the problem.) One group each for slide, prime factorization, and lists; For hotdogs, have 3 present—same methods… On chart paper So… they don’t “get it” as well as I thought because they had trouble figuring out how to do lists to find the answers. Students did demonstrate these methods though.

Summary & Reflection on GCF & LCM—advantages to slide and advantages to factor trees—They actually responded to “what are advantages of the different methods and when might you use them."

This is as far as we got. Next time, after briefly doing a naked number (in prime factorization form) with GCF & LCM, we will move on to place value.

Notes on place value—have them write a brief statement about what they know about our base-10 number system

Bassarear activity on developing a number system (2.7)

Work with Roman Numerals & Mayan number system pages. Describe differences between these number systems… and compare to our own. Maybe jigsaw with the different number systems. They become an “expert” so that they can write numbers in their prescribed system. Then they jigsaw back into a mixed group. Prompts focus on comparing features of their systems with base-10 system?

Maybe play base-10 exchange game (maybe base 5)?

**Reflection** (on board):

Describe what you learned about number systems.

**Reading:** N/A

**Homework:** Find all the factors of 100 and use the information to find all of the factors of 200.
<table>
<thead>
<tr>
<th>Week #</th>
<th>Topic</th>
<th>Objectives (Students will be able to…)</th>
<th>Activities (possibilities &amp; notes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>Class 1</td>
<td>9/17 (M)</td>
<td><strong>Week #</strong></td>
</tr>
<tr>
<td>4</td>
<td>Class 1</td>
<td>9/17 (M)</td>
<td><strong>Objectives (Students will be able to…)</strong></td>
</tr>
<tr>
<td>4</td>
<td>Class 1</td>
<td>9/17 (M)</td>
<td>• Recognize relationship between number of factors and definitions of prime &amp; composite</td>
</tr>
<tr>
<td>4</td>
<td>Class 1</td>
<td>9/17 (M)</td>
<td>• Recognize relationship between factors, multiples, &amp; divisibility and use symbolic notation to represent the relationship</td>
</tr>
<tr>
<td>4</td>
<td>Class 1</td>
<td>9/17 (M)</td>
<td>• Find the prime factorization of a number</td>
</tr>
<tr>
<td>4</td>
<td>Class 1</td>
<td>9/17 (M)</td>
<td>• Find GCF &amp; LCM (for word problems if time permits)</td>
</tr>
<tr>
<td>4</td>
<td>Class 1</td>
<td>9/17 (M)</td>
<td>• Use symbolic representations as appropriate</td>
</tr>
</tbody>
</table>

**Warm up** (on board & need toothpicks & HO): move toothpicks for Roman Numeral puzzles Someone came in to invite them to a “meet and greet” which took about 8 minutes… then they worked on this. They were finished by about 10:20 and they briefly shared their answers. We did this because I was planning to move to number systems… Which we did NOT get to!

**Getting Started** (need—put out toothpicks and HO):
1. Convert homework scores to percents.
2. Stack you homework assignments in the middle of your table.
3. Do the Warm-up Problem.

**Content:**
1. Use prime factorization to… find factors… find GCF… find LCM
   \[ 2^3 \times 3^3 \times 5^3 \quad \& \quad 3^3 \times 7^2 \times 11^2 \quad \& \quad 2^3 \times 3 \times 5^3 \times 7 \]
   We used the first 2 and not the 3rd one. Also, on the second one, I added x 5 so they would have more than 3^3 in common.

One problem with working with these naked prime factorizations is that students might not see them as numbers—that is, that they want to multiply them out to work with them. They don’t immediately see how they manipulate these in the same way they would manipulate the prime factorization of 12.

This is as far as we got. Most of them had no idea how to find the factors of these numbers. They could list 2, 3, 5 on the first one, for example, but did not recognize that they could derive all factors from using the prime factorization. This will go on homework for tonight. They did not have any idea how to find the GCF and LCM. They talked in their table groups, but they really struggled with trying to figure this out. Sara Z. finally suggested using SLIDE to solve the problem. Even then they had a difficult time. She showed how she would factor out 3^3 on the left and then use what was left and 3^3 to show the GCF. She explained that 3^3 would be the only common factor. I pushed them to think about the format of slide and what the numbers on the left and at the bottom mean. The said that the numbers on the left are the common factors and those at the bottom are the uncommon
<table>
<thead>
<tr>
<th>Week #</th>
<th>Topic</th>
<th>Objectives (Students will be able to…)</th>
<th>Activities (possibilities &amp; notes)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>factors between the two numbers. Progress. I asked them to do a new problem without multiplying out or using slide. Then they shared how they thought about it. This was really difficult for them!! More homework. This took us until the end of the period. Then they wrote about what they have learned about GCF and LCM during today’s lesson. The content below moves to next week!!</td>
<td></td>
</tr>
<tr>
<td>2)</td>
<td></td>
<td>First they will come up with a number system using only A, B, C, D, and 0 (from Bassarear Investigation). They will work in their table groups to devise a system; make a poster to explain their system; and share their system with the whole group.</td>
<td></td>
</tr>
<tr>
<td>3)</td>
<td></td>
<td>3) Read about different historical number systems and share what they learn about those systems. For this, they will jigsaw into 4 expert groups… That means that some tables will have more than one person from the “expert” group.</td>
<td></td>
</tr>
<tr>
<td>4)</td>
<td></td>
<td>4) Solve the muffin problem. Discuss features of this system (if time allows?) no way…</td>
<td></td>
</tr>
</tbody>
</table>

**Reflection** (on board):
Describe what you learned about number systems and how you learned it.

**Reading**: Hindu/Arabic # system from Bassarear (following up on class assignments Tu)

**Homework**: GCF & LCM homework (problems from Bassarear)
Appendix D

IMAP Beliefs Survey

Page 1

First Name

Last Name

Phone

E-Mail Address

1. Class for which you are completing this survey

Submit

This is the first page of the survey

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View Next Page

Page 2

2. Read the following word problem:

Leticia has 8 Pokemon cards. She gets some more for her birthday. Now she has 13 Pokemon cards. How many Pokemon cards did Leticia get for her birthday?

2.1 Do you think that a typical first grader could solve this problem? NOTE. The problem could be read to the child.

© Yes

© No

View Previous Page

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View Next Page
Appendix D (continued)

Page 3

You answered that a typical first grader could solve the following problem:

*Leticia has 8 Pokemon cards. She gets some more for her birthday. Now she has 13 Pokemon cards. How many Pokemon cards did Leticia get for her birthday?*

2.2 If a friend of yours disagreed with you, what would you say to support your position?

Page 4

Here is another word problem. Again, read it and then determine whether a typical first grader could solve it.

*Miguel has 3 packs of gum. There are 5 sticks of gum in each pack. How many sticks of gum does Miguel have?*

2.3 Do you think that a typical first grader could solve this problem? NOTE. The problem could be read to the child.

© Yes
© No

Page 5

You answered that a typical first grader could solve the following problem:

*Miguel has 3 packs of gum. There are 5 sticks of gum in each pack. How many sticks of gum does Miguel have?*

2.4 If a friend of yours disagreed with you, what would you say to support your position?
Appendix D (continued)

3.3 Teachers often ask children to share their strategies for solving problems with the class. Read the following student answers and indicate whether each makes sense to you. Then, click on the button at the bottom of the page to continue.

<table>
<thead>
<tr>
<th>Carlos</th>
<th>Sharri</th>
<th>Elliott</th>
<th>Leah</th>
</tr>
</thead>
<tbody>
<tr>
<td>465 + 125 = 590</td>
<td>149 + 286 = 435</td>
<td>120 + 200 = 320</td>
<td>149 + 286 = 435</td>
</tr>
</tbody>
</table>

- Does Carlos’ reasoning make sense to you? ✔ Yes  ❌ No
- Does Sharri’s reasoning make sense to you? ❌ No
- Does Elliott’s reasoning make sense to you? ✔ Yes
- Does Leah’s reasoning make sense to you? ❌ No

Maria uses manipulatives (base ten blocks) to solve the problem. Maria says, “I took one fat for the 149 and 2 fat for the 200 to 239.

- Each 10 tens 1 for the 149 and 2 for the 200 to 239.
- Each 10 singles for the 149 and the 2 for 239.
- The answer is 435.

- Does Maria’s reasoning make sense to you? ✔ Yes  ❌ No
Appendix D (continued)

Page 7

[Chart and text]

Page 8

4. Here are two approaches that children used to solve the problem 635 – 482.

Lexi

\[
\begin{array}{c}
56135 \\
- \quad 385 \\
\hline
155
\end{array}
\]

Lexi says, “First I subtracted 2 from 5 and got 3. Then I couldn’t subtract 8 from 3, so I borrowed. I crossed out the 6, wrote a 5, then put a 1 next to the 3. Now it’s 13 minus 8 is 5. And then 5 minus 4 is 1, so my answer is 153.”

4.1 Does Lexi’s reasoning make sense to you? 

[Options: Yes or No]

Ariana

\[
\begin{array}{c}
635 - 400 = 235 \\
235 - 30 = 205 \\
205 - 40 = 165 \\
155 - 2 = 153 \\
\hline
482
\end{array}
\]

Ariana says, “First I subtracted 400 and got 235. Then I subtracted 30 and got 205, and I subtracted 50 more and got 155. I needed to subtract 2 more and ended up with 153.”

4.2 Does Ariana’s reasoning make sense to you? 

[Options: Yes or No]

4.3 Which child (Lexi or Ariana) shows the greater mathematical understanding?

[Options: Lexi or Ariana]

Why?

4.4 Describe how Lexi would solve this item: 700 – 573.

[Solution]

4.5 Describe how Ariana would solve this item: 700 – 573.
Appendix D (continued)

Page 9

Question 4 (continued)

Here are those two approaches again so that you can refer to them to finish this section.

<table>
<thead>
<tr>
<th>Lee</th>
<th>Armenia</th>
</tr>
</thead>
<tbody>
<tr>
<td>485</td>
<td>400 + 85</td>
</tr>
<tr>
<td>325</td>
<td>10 + 245</td>
</tr>
<tr>
<td>180</td>
<td>30 + 150</td>
</tr>
<tr>
<td>155</td>
<td>2 + 133</td>
</tr>
<tr>
<td>485</td>
<td>400 + 85</td>
</tr>
</tbody>
</table>

Lee says, "First I subtracted 2 from 8 and got 6. Then I subtracted 8 from 4, also subtracted 2. 4 minus 2 is 2, so I add 2 to the 6. Now it's 8, which means 8 is 8. But then 3 means 6 is 6, or any arrow in 135."

Armenia says, "First I subtracted 400 and got 255. Then I subtracted 10 and got 245. Then I subtracted 8 from 245 and got 237. Then I subtracted 8 from 237 and got 229. Lastly, I subtracted 2 from 229 and ended up with 227."

For the remaining questions, assume that students have been exposed to both approaches.

4.6 Of 10 students, how many do you think would choose Lee's approach?

- [ ] of 10 students would choose Lee's approach.

4.7 If 10 students used Lee's approach, how many do you think would be successful in solving the problem 760 - 573?

- [ ] of 10 students would be successful.

Explain your thinking.

4.8 Of 10 students, how many do you think would choose Armenia's approach?

- [ ] of 10 students would choose Armenia's approach.

4.9 If 10 students used Armenia's approach, how many do you think would be successful in solving the problem 760 - 573?

- [ ] of 10 students would be successful.

Explain your thinking.

4.10 If you were the teacher, which approach would you prefer that your students use?

Choose one of the following:  
- [ ]  

Please explain your choice.

Page 10

5. What were your reactions when you were asked to solve a new kind of problem without the teacher's showing you how to solve it?

Page 11

1.1 When you are a teacher, will you ever ask your students to solve a new kind of problem without first showing them how to solve it?

- [ ] Yes

- [ ] No
5.1 You answered that you would ask your students to solve a new kind of problem without first showing them how to solve it. Please elaborate on your reasons:

5.2 How often will you ask your students to do this?

Page 13

6. At this point in time, which of the following best captures your position?

- I am sure that I want to become a teacher.
- I think that I want to become a teacher.
- I am not sure that I want to become a teacher.
- I am leaning against becoming a teacher.
- I am sure that I do not want to become a teacher.

Page 14

In this part of this survey, you will watch an interview with a child.

The following problem is posed to the child:

There are 20 kids going on a field trip. Four children fit in each car. How many cars do we need to take all 20 kids on the field trip?

Click to see the video. View Video (High Speed Connection) View Video (56K Modem Connection)

7.1 Please write your reaction to the video clip. Did anything stand out for you?
Video Questions (continued)

7.2 Identify the strengths of the teaching in this episode.

7.3 Identify the weaknesses of the teaching in this episode.

Page 16

Video Questions (continued)

7.4. Do you think that the child could have solved the problem with less help?

Please select a choice

7.5 Please explain your choice.
8.1 Place the following four problems in rank order of difficulty for children to understand, and explain your ordering (you may rank two or more items as being of equal difficulty). NOTE: Easiest = 1.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Please explain your rank:</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) Understand $\frac{1}{5} + \frac{1}{8}$</td>
<td>Select Rank</td>
</tr>
<tr>
<td>b) Understand $\frac{1}{5} \times \frac{1}{8}$</td>
<td>Select Rank</td>
</tr>
<tr>
<td>c) Which fraction is larger, $\frac{1}{5}$ or $\frac{1}{8}$, or are they same size?</td>
<td>Select Rank</td>
</tr>
<tr>
<td>d) Your friend Jake attends a birthday party at which five guests equally share a very large chocolate bar for dessert. You attend a different birthday party at which eight guests equally share a chocolate bar exactly the same size as the chocolate bar shared at the party Jake attended. Did Jake get more candy bar, did you get more candy bar, or did you and Jake each get the same amount of candy bar?</td>
<td>Select Rank</td>
</tr>
</tbody>
</table>
Appendix D (continued)

Page 18

Consider the last two choices:
___ c) Which fraction is larger, \( \frac{1}{5} \) or \( \frac{1}{8} \), or are they same size?

___ d) Your friend Jake attends a birthday party at which five guests equally share a very large chocolate bar for dessert. You attend a different birthday party at which eight guests equally share a chocolate bar exactly the same size as the chocolate bar shared at the party Jake attended. Did Jake get more candy bar, did you get more candy bar, or did you and Jake each get the same amount of candy bar?

8.2 Which of these two items did you rank as easier for children to understand?

☐ Item c is easier than item d.

☐ Item d is easier than item c.

☐ Items c and d are equally difficult.

Please explain your answer.

Page 19

8.3 In a previous question, you were asked to rank the difficulty of understanding \( \frac{1}{5} \times \frac{1}{8} \). By understand, were you thinking of the ability to get the right answer?

☐ Yes

☐ No

Page 20

8.4 On the last question you indicated that you were thinking about understanding as “getting the right answer.” Were you also thinking of anything else? Please explain.
Click to see the next interview segment. View Video (High Speed Connection) View Video (56K Modem Connection)

9.1 Please write your reaction to this video clip. Did anything stand out for you?

9.2 What do you think the child understands about division of fractions?

9.3 Would you expect this child to be able to solve a similar problem on her own 3 days after this session took place?
  ○ Yes  ○ No

Explain your answer.

Click to watch another video clip: View Video (High Speed Connection) View Video (56K Modem Connection)

9.4 Comment on what happened in this video clip. (NOTE: This interview was conducted 3 days after the previous lesson on division of fractions.)

9.5 How typical is this child? If 100 children had this experience, how many of them would be able to solve a similar problem 3 days later? Explain.

  of 100 children could solve a similar problem later.

9.6 Provide suggestions about what the teacher might do so that more children would be able to solve a similar problem in the future.
10) When you were a child, how did you feel about mathematics?

<table>
<thead>
<tr>
<th>Very comfortable</th>
<th>Somewhat comfortable</th>
<th>Neither comfortable nor anxious</th>
<th>Somewhat anxious</th>
<th>Very anxious</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

11) When you were a child, how well did you like mathematics?

<table>
<thead>
<tr>
<th>I loved it</th>
<th>I liked it</th>
<th>I felt neutral</th>
<th>I disliked it</th>
<th>I hated it</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

12) When you were a child, how successful were you in your mathematics classes?

<table>
<thead>
<tr>
<th>Usually successful</th>
<th>More successful than not</th>
<th>Successful about half the time</th>
<th>More unsuccessful than not</th>
<th>Usually unsuccessful</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

13) Which statement best describes your elementary-school experiences with word problems?

<table>
<thead>
<tr>
<th>They were very difficult.</th>
<th>They were somewhat difficult.</th>
<th>They were neither easy nor difficult.</th>
<th>They were somewhat easy.</th>
<th>They were very easy.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

14) Is there something else about your experiences learning mathematics in elementary school that you would like to share?

[Text box for additional comments]

15) Choose the description that best fits the majority of your mathematics experiences in elementary school.

- a) The teacher explained a way to solve problems. The children independently practiced solving several problems. Hands-on materials were never used.

- b) The teacher sometimes explained things. Sometimes the children figured out their own ways to solve problems. The children occasionally used hands-on materials. Children occasionally worked in groups.

- c) The teacher gave the children problems that they figured out on their own. The children often used hands-on materials and talked to each other about mathematics.

16) Describe the characteristics of someone who is good at doing mathematics.

[Text box for characteristics]

You have reached the end of the survey.
Appendix E

LMT

Elementary Number Concepts and Operations – Content Knowledge

2004 Form A

A VERY important reminder
** Surveys MUST include LMT/SII copyright information **

Steps to creating a survey (see example survey for guidance):

1. Decide on the scale(s) you will use in your study.
2. If you are administering multiple scales, combine the items from the scale forms into one single document.
3. Create a cover page.
4. Include a page with LMT/SII copyright information.
5. Depending on your human subjects requirements, you may decide to use a passive consent letter.
6. Include an “instructions” page.
7. You may choose to include a set of demographic questions in your survey in addition to the LMT knowledge items. If you decide to do this, we recommend creating a separate section for the demographic questions.
8. Renumber items. Currently all items have the same item number we used in our original research. This provides a means for accessing item level technical information from our technical reports. Once you have compiled your items into your survey form you will need to renumber all items sequentially.
9. End the form with a brief thank you note and information about how participants can reach you if they have questions.
10. Make sure all pages are numbered and footers are as you like them.
11. Check your final form multiple times to make sure you did not drop or change any items in the cutting and pasting process.
12. We recommend you convert your final word survey to a PDF document to preserve your final formatting.
INSTRUCTIONS

- Answer questions by circling your choice, e.g.

1. During a unit on functions, Ms. Lopez asks her students to write journal entries on exponential growth. Which of the following journal entries illustrate exponential growth? (For each item below, circle EXPONENTIAL, NOT EXPONENTIAL or I'M NOT SURE.)

<table>
<thead>
<tr>
<th>Option</th>
<th>EXPONENTIAL</th>
<th>Not exponential</th>
<th>I'm not sure</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) An example of exponential growth would be if you got a 1% raise each year.</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>b) An example of exponential growth would be if a car increases in speed by 10 miles per hour every second.</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>c) Exponential growth is when the y-axis increases faster than the x-axis. For example, if each time the x-coordinate goes up by 2, the y-coordinate goes up by 3.</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

- In completing this questionnaire, you should not spend more than 1-2 minutes on any question. Imagine you are responding to real classroom situations, and select the answer that most closely matches what you would do, say, or answer at that moment.

- Your responses are voluntary and confidential. If you come to a question you do not wish to answer, simply skip it. We hope that you will answer as many questions as possible.
1. Ms. Wilson’s class is working in groups to decompose 391 into hundreds, tens, ones, and tenths. As she walks around, she sees groups have arrived at very different answers. Which of the following ways to represent 391 should she accept as correct? (Mark YES, NO, or I’M NOT SURE for each choice.)

<table>
<thead>
<tr>
<th></th>
<th>Yes</th>
<th>No</th>
<th>I’m not sure</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) 3 hundreds + 90 tens + 1 one</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>b) 2 hundreds + 19 tens + 1 one</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>c) 3 hundreds + 9 tens + 10 tenths</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>d) 39 tens + 1 one</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

2. During a discussion about prime numbers in Mr. Gee’s class, Natalie asked whether or not 1 is a prime number. Mr. Gee attended a workshop where he learned about primes. What is the crucial issue for Mr. Gee to remember in order to explain this to her?

a) 1 is a prime number because a prime number is only divisible by 1 and itself.

b) Some definitions of primes include 1 and some do not.

c) 1 is not considered to be prime, because every number can be expressed uniquely as a product of prime numbers, and if 1 were prime, the factorization would not be unique.

d) 1 is not a prime because it is a perfect square and no other perfect squares are prime.

e) I’m not sure.
3. Mr. Lopez was attending a professional development program in mathematics. One day they worked on the following problem:

A program that is four weeks long is split into two equal sessions. The program’s first session has two instructors (Nina and Amy) and the program’s other session has five instructors (Melly, Rosa, Angela, Timo and Jim). What proportion of the overall program does Jim teach if all the teachers in his session share the work equally?

Several groups came up with different answers. There was a lot of discussion and disagreement. Which answer below is correct? (Mark ONE answer.)

a)  
b)  
c)  
d)  
e)  
f) I’m not sure.
4. Mr. Siegel and Mrs. Valencia were scoring their students’ work on the practice state mathematics exam. One open-ended question on the exam asked:

Write the number that is halfway between 1.1 and 1.11.

Mr. Siegel and Mrs. Valencia were interested to see the different answers students wrote. What should the teachers accept as correct? (Mark ONE answer.)

a) 1.05
b) 1.055
c) 1.105
d) 1.115
e) I’m not sure.

5. Mr. Lee asked his students to compare \( \frac{1}{3} \) to \( \frac{2}{5} \). Which of the following should he accept as a correct explanation? (Mark ONE answer.)

a) is greater than because 5 is greater than 3.
b) They are equal because each is missing four pieces from the whole.
c) They are equal because adding two to the numerator in and two to the denominator in produces.
d) is greater because the pieces will be bigger.
e) is greater because it is more than one-half, while is less than one-half.
f) I’m not sure.
6. A group of Ms. Lee’s students was following a set of directions to move a paper frog along a number line.

Their last direction took them to \( \frac{7}{12} \). The next direction says:

\[ \text{Go \ of the way to } \frac{5}{6} \text{. What number will the frog land on?} \]

The students disagreed about where the frog would land. Which answer should Ms. Lee accept as correct? (Mark ONE answer.)

a. \( \frac{1}{12} \)
b. \( \frac{2}{3} \)
c. \( \frac{7}{12} \)
d. \( \frac{5}{6} \)
e. \( \frac{1}{4} \)
f. I'm not sure.
7. Teachers often offer students “rules of thumb” to help them remember particular mathematical ideas or procedures. Sometimes, however, these handy memory devices are not actually true, or they are not true for all numbers. For each of the following, decide whether it is true all of the time or not. (Mark TRUE FOR ALL NUMBERS, NOT ALWAYS TRUE, or I’M NOT SURE.)

<table>
<thead>
<tr>
<th></th>
<th>True for all numbers</th>
<th>Not always true</th>
<th>I’m not sure</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>If the first of two numbers is smaller than a second, and you add the same number to both, then the first sum is smaller than the second.</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>b)</td>
<td>Multiplying a number makes it larger.</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>c)</td>
<td>A negative number plus another negative number equals a negative number.</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>d)</td>
<td>To multiply any number by 10, add a zero to the right of the number.</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>
8. Mr. Stone is looking through some mathematics materials for some problems relating fractions to number lines. He comes across the following problem:

Which point is closest to \( \frac{1}{2} \)?

He has not used number lines for this kind of problem before and he wants to make sure he is using it correctly. What is the intended answer to this problem? (Mark ONE answer.)

a) A  
b) B  
c) C  
d) D  
e) I’m not sure.
Mr. Lewis asked his students to divide \( \frac{9}{4} \) by \( \frac{3}{2} \). Charlie said, “I have an easy method, Mr. Lewis. I just divide numerators and denominators. I get \( \frac{6}{3} \), which is correct.” Mr. Lewis was not surprised by this as he had seen students do this before. What did he know? (Mark ONE answer.)

a) He knew that Charlie’s method was wrong, even though he happened to get the right answer for this problem.

b) He knew that Charlie’s answer was actually wrong.

c) He knew that Charlie’s method was right, but that for many numbers this would produce a messy answer.

d) He knew that Charlie’s method only works for some fractions.

e) I’m not sure.
10. As Mrs. Boyle was teaching subtraction one day, she noticed a few students subtracted in the following way:

\[
\begin{array}{c}
13 \\
63 \\
\underline{-328} \\
35 \\
\end{array}
\]

What were these students **most likely** doing? (Mark ONE answer.)

a) The students “subtracted up,” by taking 3 away from 8, and then tried to compensate for this mistake.

b) The students compensated by subtracting 30 from 63, then dealt with the 8 and 3 in a second step.

c) The students made a mistake with the standard procedure, crossing out the 2 rather than the 6.

d) The students added ten to both 63 and 28, then subtracted.

e) I’m not sure.
11. Ms. Lawrence is making up word problems for her students. She wants to write a word problem for $3 \div \_$. Which word problem(s) can she include? (Mark YES, NO, or I’M NOT SURE for each problem.)

<table>
<thead>
<tr>
<th></th>
<th>Yes</th>
<th>No</th>
<th>I’m not sure</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) Melissa has 3 pizzas and she wants to give half of them to her friend. How much pizza will her friend get?</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>b) Dan has 3 cups of chocolate chips. He wants to bake cookies, and each batch requires (rac{1}{3}) cup of chocolate chips. How many batches of cookies can Dan make if he uses all of the chocolate chips?</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>c) Three friends each have half of a cookie. How many cookies would they have if they put them all together?</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>d) Jacquie has collected three cans of pennies for her fund-raiser. If she is halfway to her goal, how many cans of pennies had she set as the goal?</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>
12. Luanne suggested the following method for multiplying 14 by 12:

   I know that 7 times 12 is 84, so to get 14 times 12, I double 84, which is 168.

   Of the following diagrams, which BEST illustrates Luanne’s method? (Mark ONE answer.)

   a) Diagram A only
   b) Diagram B only
   c) Both diagrams represent Luanne’s method equally well.
   d) Neither diagram represents Luanne’s method well.
   e) I’m not sure.
13. Imagine that you are working with your class on subtracting large numbers. Among your students’ papers, you notice that some have displayed their work in the following ways:

<table>
<thead>
<tr>
<th>Method A</th>
<th>Method B</th>
<th>Method C</th>
</tr>
</thead>
<tbody>
<tr>
<td>932 356 +4</td>
<td>932 356</td>
<td>932 356</td>
</tr>
<tr>
<td>-356 360 +40</td>
<td>-356 360</td>
<td>-356 360</td>
</tr>
<tr>
<td>400 500</td>
<td>632</td>
<td>-400</td>
</tr>
<tr>
<td>900 +32</td>
<td>976</td>
<td>576</td>
</tr>
<tr>
<td>932 576</td>
<td>582</td>
<td>576</td>
</tr>
<tr>
<td>932</td>
<td>50</td>
<td>6</td>
</tr>
<tr>
<td>576</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Which of these students is using a method that could be used to subtract any two whole numbers? (Mark ONE answer.)

a) A only  
b) B only  
c) A and B  
d) B and C  
e) A, B, and C  
f) I’m not sure.
14. Mr. Nessbaum was teaching division with fractions to his class. “If we purchase 8 big chocolate bars from the school candy sale,” he said, “and we want everyone in the class to have at least of a chocolate bar, do we have enough for our 25 students?” Mr. Nessbaum expected his students to write $\frac{8}{\frac{1}{5}} = 20$. Instead the students came up with several different approaches. Which of the students’ approaches is valid? (Mark APPROACH IS VALID, APPROACH IS NOT VALID, or I’M NOT SURE for each.)

<table>
<thead>
<tr>
<th>Approach is valid</th>
<th>Approach is not valid</th>
<th>I’m not sure</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) After I cut each bar into five pieces, there are 40 pieces and everybody gets two. 40 divided by 2 equals 20, so we don’t have enough.</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>b) There are 25 students. 25 times is 10 so we don’t have enough.</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>c) You have to add twenty times to equal 8 whole chocolate bars. But that is not 25 times, so there are not enough pieces for everyone in our class.</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>d) There aren’t enough because if I start with 8 and keep subtracting, I get to zero before I have subtracted it 25 times.</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>
Vita

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Professional Preparation

University of Illinois at Chicago
Learning Sciences—Mathematics Education Focus
Ph.D., (Expected June 2016)
University of Chicago
Elementary Education
M.S.T., 2001
University of Chicago
Anthropology
B.A., 1987

Employment History/Appointments

Senior Curriculum Developer; and Special Projects Consultant
University of Chicago Center for Elementary Mathematics and Science Education
2007–present

Visiting Lecturer:
mathematics content and methods courses for preservice and inservice teachers
University of Illinois at Chicago
2010–2015

Researcher
Center for the Mathematics Education of Latinos/as at the University of Illinois at Chicago
2008–2011

Assessment Project at the Learning Sciences Research Institute at the University of Illinois at Chicago
2008

Curriculum Developer; and Researcher
Teaching Integrated Mathematics and Science Project at the University of Illinois at Chicago
2005–2007

Associate Director
Everyday Mathematics Implementation Center at the University of Chicago
2000–2004

2nd and 3rd Edition Author
Everyday Mathematics at the University of Chicago School Mathematics Project
1997–2006
Vita (continued)

<table>
<thead>
<tr>
<th>Role</th>
<th>Institution</th>
<th>Years</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematics Education Consultant</td>
<td>National and international consultant supporting leadership development, and mathematics teaching and learning in schools and school districts</td>
<td>1996–present</td>
</tr>
<tr>
<td>Teacher</td>
<td>University of Chicago Laboratory Schools</td>
<td>1989–1995</td>
</tr>
<tr>
<td>Assistant Teacher</td>
<td>University of Chicago Laboratory Schools</td>
<td>1987–1988</td>
</tr>
</tbody>
</table>

Refereed Journal Publications


Book Chapter


Refereed Conference Proceedings

Refereed Conference Proceedings (continued)


Refereed Conference Presentations


Pitvorec, K., & Haake, J. (2010). Every student, every day: Developing the knowledge and skills to ensure high quality core instruction. Presentation given at National Council of Supervisors of Mathematics Conference, Apr 18–21, San Diego, CA.

Refereed Conference Presentations (continued)


Selected Invited Talks


Selected Invited Talks (continued)


Other Professional Activities


Reviewer for AMTE conference proposals (2010–2012)

Reviewer for AERA Division K conference proposals (2008–2014)

Reviewer for the *NCSM Journal* (2008–2011)


University of Chicago Laboratory Schools Curriculum Committee member (1995–96)

University of Chicago Laboratory Schools liaison to globally oriented L.I.F.E. school in Guatemala establishing relationships for curriculum development and student exchange (1993–95)

Vice President of the Faculty Association (1994–95)

Member-at Large on the Executive Board of the Faculty Association (1992–94)

Curriculum Consultant for Valentine Associates on “Wings of Courage” education package distributed in conjunction with the Omnimax movie (1995)

Science Committee member at Laura B. Sprague School (1988–89)
Awards & Grants

2010 Nominated for Dean’s Scholarship Award

1994 Golden Apple Award Finalist

1994 Heinemann Family Grant to support classroom Lego-Logo project

1992 Heinemann Family Grant for cross-curricular project comparing the economic systems of socialism and capitalism