Meshfree Methods for Geotechnical Disaster Simulation and Prediction

by

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THESIS

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This thesis is dedicated to,

my wonderful parents,

Amnuayporn and Kannika Siriaksorn,

and my lovely sister,

Kornkanok Siriaksorn
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<td>Drucker-Prager yield surface in (a) principal stress space and (b) $\pi$-plane</td>
</tr>
</tbody>
</table>
## LIST OF ABBREVIATIONS

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>MLS</td>
<td>Moving Least Squares</td>
</tr>
<tr>
<td>RK</td>
<td>Reproducing Kernel</td>
</tr>
<tr>
<td>EFG</td>
<td>Element-Free Galerkin</td>
</tr>
<tr>
<td>SPH</td>
<td>Smoothed Particle Hydrodynamics</td>
</tr>
<tr>
<td>DEM</td>
<td>Discrete Element Method</td>
</tr>
<tr>
<td>FEMs</td>
<td>Finite Element Methods</td>
</tr>
<tr>
<td>LEM</td>
<td>Limit Equilibrium Method</td>
</tr>
<tr>
<td>BBM</td>
<td>Barcelona Basic Model</td>
</tr>
<tr>
<td>RKPM</td>
<td>Reproducing Kernel Particle Method</td>
</tr>
<tr>
<td>RKCM</td>
<td>Reproducing Kernel Collocation Method</td>
</tr>
<tr>
<td>DNI</td>
<td>Direct Nodal Integration</td>
</tr>
<tr>
<td>SCNI</td>
<td>Stabilized Conforming Nodal Integration</td>
</tr>
<tr>
<td>SNNI</td>
<td>Stabilized Non-Conforming Nodal Integration</td>
</tr>
<tr>
<td>MSCNI</td>
<td>Modified Stabilized Conforming Nodal Integration</td>
</tr>
<tr>
<td>MSNNI</td>
<td>Modified Stabilized Non-Conforming Nodal Integration</td>
</tr>
<tr>
<td>VCI</td>
<td>Variationally Consistent Integration</td>
</tr>
</tbody>
</table>
SUMMARY

An effective model for both slope stability analysis and run-out simulation helps to predict and mitigate the environmental and social impacts caused by landslide activities. Due to distinctive characteristics in each landslide process, different numerical techniques are usually employed to model landslide processes at different stages. Continuum-based methods with the Lagrangian formulation, such as finite element method (FEM), are preferable in slope stability analysis before material separation. On the other hand, discontinuum-based methods, such as the discrete element method (DEM) and smoothed particle hydrodynamics (SPH), are common for addressing fluid-like behaviors in the run-out simulation. It is extremely challenging, nevertheless, to apply one of the aforementioned numerical methods to accurately and effectively analyze entire landslide processes.

This thesis provides meshfree frameworks based on the semi-Lagrangian Reproducing Kernel (RK) approximations to handle extreme geotechnical events. The semi-Lagrangian RK approximation combines advantages of the Eulerian and Lagrangian formulations, that is, state variables follow material points while the approximation function is updated in the current configuration to allow extreme deformation and material separation. The approximation is extended to a two-field (displacement-pressure) formulation based on Biot's theory for considering the poromechanics of geomaterials. Considering the complexity of geomaterials, a Drucker-Prager plasticity model with a single-parameter damage model is employed to properly represent the behavior of the solid phase of geomaterials. In addition, variationally consistent stabilized nodal integration schemes and kernel contact algorithms are extended to the two-field
semi-Lagrangian RK framework to enhance accuracy and stability of the domain integration and to naturally model arbitrary contacts between different material regions. Detailed studies of the temporal stability of the frameworks are performed by using the von Neumann method to provide a guideline of time step selection when explicit time integration schemes are adopted. The robustness and effectiveness of the proposed two-field semi-Lagrangian RK formulation is verified with FEM solutions in several slope stability analyses. The post-failure or run-out simulation capability of the presented meshfree method is validated with experimental data and actual data from a landslide site. The proposed framework can be applied to study other geotechnical events under extreme conditions, which is demonstrated in simulating the penetration process of a projectile penetrating into the soil.

Additionally, the strong form collocation method with the RK approximation is introduced to study the poromechanics of geomaterials, as an alternative approach. The governing equations in the method are directly solved in the strong formulation with a point collocation method, in which the domain integration is not required in contrast to the Galerkin weak formulation. The effectiveness of the method is studied and demonstrated first in hyperelasticity problems. The method is then reformulated for elastodynamic problems. Ultimately, the method is extended to the two-field formulation and its performance is examined by a soil consolidation problem.
CHAPTER 1

INTRODUCTION

1.1 Motivation

A natural disaster is a catastrophic event that results in tremendous fatalities, loss of properties, and impacts on economy. One of the most critical natural disasters is landslide. As an example of the severity of landslide events, according to Close and McCormick [1] and Feng and Guo [2], the deadliest landslide in the 20th century happened in Haiyuan, China, 1920; the incident is estimated to have fatalities of more than 100,000. In addition, the impacts of landslides on properties or economies can also be crucial. For instance, the Thistle landslide in Utah, 1983, costs more than US$600 million (based on U.S. dollars in 1983) [3], [4].

<table>
<thead>
<tr>
<th>Type of movement</th>
<th>Type of material</th>
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<tbody>
<tr>
<td></td>
<td>Bedrock</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Falls</td>
<td>Rock fall</td>
</tr>
<tr>
<td>Topples</td>
<td>Rock topple</td>
</tr>
<tr>
<td>Slides</td>
<td>Rotational</td>
</tr>
<tr>
<td></td>
<td>Translational</td>
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<tr>
<td>Lateral spreads</td>
<td>Rock spread</td>
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<tr>
<td>Flows</td>
<td>Rock flow (deep creep)</td>
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</tbody>
</table>

Complex: Combination of two or more principal types of movement

TABLE I. VARNES’ LANDSLIDE CLASSIFICATION [5]

Landslides can be sometimes triggered by other natural disasters, e.g., earthquakes, tsunami, etc. Additionally, human activities, snowmelt, liquefaction, and erosion can also be the
causes of landslides. Nevertheless, the causes of landslides are commonly attributable to rainfall since it changes groundwater level, water content of geomaterials, and weight of soil. These changes result in erosion, liquefaction, increasing load, or decreasing of matric suction which can lead to slope instability.

One of the most commonly used means to categorize the types of landslide is by its movement and geomaterial compositions [5], as shown in Table I. For example, the difference of failure modes between debris flow (Figure F\textsuperscript{1} and Figure G\textsuperscript{1}) and earth flow (Figure H\textsuperscript{1}) is due to the primary constituent of the soil. Debris flow primarily consists of coarse-grained soils, while fundamental elements of earth flow are fine-grained soils. Additionally, topology and causes of landslide can also play an important role on the types of landslide. Falls and topples usually occur on very steep slopes (Figure D\textsuperscript{1} and Figure E\textsuperscript{1}), whereas slides commonly take place on moderate slopes (Figure A\textsuperscript{1} - Figure C\textsuperscript{1}). Lateral spreads are often triggered by earthquake, and hence can occur in less moderate slopes or even plane landscapes (Figure J\textsuperscript{1}). Generally, precipitation is the main cause of debris flow and slides but it can also cause falls, topples, and creep (Figure I\textsuperscript{1}). Creep is a very slow-moving landslide which can be resulted from the changing of temperature or moisture content of geomaterials. All of the previously described background information portrays that landslides are intricate and detrimental.

To determine whether a landslide event will occur, extensive studies have been devoted to develop approaches that analyze or monitor slope stability. Due to its simplicity, limit equilibrium method (LEM) is a popularly used class of techniques to analyze the slope stability. However, LEM considers only the equilibrium condition of a pre-defined slip surface, which limits its applicability to conduct more realistic slope stability analyses. The other widely used type of

\footnote{Referred to the figures from http://pubs.usgs.gov/fs/2004/3072/images/Fig3grouping-2LG.jpg}
approaches is the real-time landslide monitoring technique. Most of the techniques in this class utilize remote sensing, satellite, aerial photographs, optical fiber sensors, or GPS to keep monitoring deformation of the landscape. This type of methods, however, is suitable only for slow-moving landslides and is not sufficient for slope stability analyses. A great example to illustrate the advantages and disadvantages of the monitoring technique is the landslide in Utah mine (Bingham Canyon Mine) in 2013, which is the largest landslide in the US history (not including landslides from volcanic eruptions). One aspect of the example, which demonstrates the capability of the method, is that it can detect the slow movement of the slope very early resulting in 0 casualties. The other aspect, however, shows a substantial drawback of the technique because this man-made landslide can be prevented from happening in the first place if the slope stability can be predicted before the excavation is performed. This landslide results in the economic loss of more than US$100 million. Therefore, an effective means that can predict slope instability is required.

In addition to the capability to analyze slope stability, the method should be able to simulate post-failure stage because potential areas that would be affected by the run-out of a landslide is also a vital concern as run-out distance can be ranged in the scale of meters (e.g., landslide in La Conchita, California, 1995) to kilometers (e.g., debris flow in Venezuela, 1999). These areas are called elements at risk which, combined with statistics and probability, is significant information for geologists to determine the risk assessment for each risk scenario. Subsequently, the strategies to mitigate the risk can be proposed by taking into account cost-benefits, environmental impacts, ecological impacts, and impacts on people’s lives or communities. Therefore, the necessity of sophisticated yet effective methods that can both analyze slope stability and simulate landslide after the failure is proliferated.
Over the past few decades, computer performance has been gradually improved; it leads to considerable development in numerical methods such as finite element methods (FEMs), smoothed particle hydrodynamics (SPH), discrete element method (DEM), and meshfree methods. Therefore, there have been attempts to apply these methods with landslides for many years.

FEM is an exceptional and efficient numerical method. However, it is extremely challenging for FEM to model run-out simulation or post-failure stage of landslides because they involve considerably large deformation, material defragmentation, or liquefaction. Conventional FEM suffers from mesh distortion issues when modeling landslide processes due to excessive deformation and material separation.

Smoothed particle hydrodynamics is excellent in simulating the run-out process of a landslide and representing fluid-like behavior because the method is particle based. Therefore, it is also a great method when the slope has complex geometry and topology. Nevertheless, the challenge of SPH arises when using the method to represent solid-like behavior. It also suffers from boundary deficiency, which causes inaccurate results around the boundary of the domain. The other issue of SPH is the tensile instability, which can occur under tension resulting in a bunch of particles moving toward each other instead of separating.

Discrete element method demonstrates outstanding performance for modeling granular materials. It can effectively simulate the propagation stage of landslides because the discrete bodies allow large deformations, separation, rotations, and contact. However, it is difficult to incorporate poromechanics constitutive models into DEM to represent material behaviors.
Meshfree methods based on the Reproducing Kernel (RK) approximation are particle based methods; therefore, they do not require mesh to construct approximation functions, which eliminates the aforementioned difficulties of FEM for run-out simulations, yet they are as effective as in FEM for strength analyses (small to moderate deformation). They can also represent fluid-like behavior similar to SPH without suffering from tensile instability and boundary deficiency. Lastly, material constitutive models can be incorporated in meshfree methods with no difficulties unlike DEM. Therefore, meshfree methods based on the (RK) approximation will be introduced for landslide simulation.

1.2 **Objectives**

The objective of the research is to develop a meshfree method that can effectively simulate and predict landslide activities, in order to mitigate the impact of such disasters. The meshfree approximation based on the RK approximation is introduced in the weak formulation and strong formulation to handle extreme geotechnical problems. Numerical algorithms for handling large deformation and material separation are studied and developed. Multi-field formulations are implemented to include poromechanics constitutive models for geomaterials. The stability issues arising from the low-order integration in the weak formulation and temporal discretization in the multi-field meshfree formulations are investigated and addressed. Specific research developments are summarized as follows:
(1) *u-p semi-Lagrangian Reproducing Kernel (RK) formulation for saturated porous media*

The semi-Lagrangian Reproducing Kernel approximation is introduced in the two-field, displacement-pressure (*u*-*p*), formulation for saturated porous media. Variationally consistent stabilized nodal integration schemes and kernel contact algorithms have been extended into the *u*-*p* semi-Lagrangian RK framework. Drucker-Prager with a single-parameter damage model is employed to represent geomaterial behaviors from elastic to separation. The framework is tested with small-deformation benchmark problems, then verified the slip surface results with FEM for slope stability analysis, and validated the landslide simulations with experimental and field data. In addition, the *u*-*p* semi-Lagrangian RK is also introduced for the simulation of the munitions penetrating into the soil.

(2) *Von Neumann stability analysis for the u-p Reproducing Kernel (RK) formulation*

The von Neumann stability analysis is employed to study the temporal stability of the *u*-*p* formulation based on the Reproducing Kernel approximation and nodal integration schemes. Both quasi-static and dynamic *u*-*p* formulations are considered. Effects of material parameters, RK shape function influence domain, discretization methods, and nodal integration schemes on temporal stability are studied, while using stability of FEM with full integration as a reference. Numerical examples are provided to validate the critical time step size estimations from the von Neumann analysis with the observed critical time step size in full-scale analysis.
(3) Reproducing Kernel Collocation Method (RKCM) for hyperelasticity and poroelasticity

The RK approximation is introduced in the strong-form collocation method for hyperelasticity to account for finite elastic deformation problems, which circumvents the complexity due to domain integration in the weak formulation. Several numerical examples are considered to verify the results from the method. The strong-form collocation method based on the RK approximation is then extended to the two-field formulation for dynamic poroelasticity problems. Lastly, the accuracy and effectiveness of the $u$-$p$ RKCM is verified with a consolidation problem.

The remainder of the thesis is as follows: Chapter 2 reviews the critical literature on soil constitutive models, meshfree methods, and methods for slope stability analysis and run-out simulation. The single-field Galerkin weak form framework for excessive deformation and material separation problems is described in Chapter 3. In Chapter 4, the framework in Chapter 3 is extended to the two-field formulation to properly account for geomaterial behaviors. Temporal stability of the $u$-$p$ RK approximation is investigated by the von Neumann method in Chapter 5. Strong-form collocation method based on the RK approximation is introduced for hyperelasticity and $u$-$p$ elastodynamic problems in Chapter 6. Chapter 7 presents the performance of the Galerkin weak form frameworks for the simulations of landslides and munitions penetration into the ground. Lastly, conclusions and future work are discussed in Chapter 8.
CHAPTER 2

LITERATURE REVIEW

2.1 Constitutive Models for Geomaterials

Over the past century, theories and mathematical models to describe mechanics of porous media have been proposed and developed. Terzaghi [6] employed the definition of effective stress to explain the mechanics of soil consolidation in a one-dimensional system. The theory was then generalized into a three-dimensional case and became the theory of linear poroelasticity by Biot [7] and extended to anisotropic cases by Biot [8]. The general solutions to the isotropic case of the Biot theory was developed in [9]. The extension of the Biot theory to a non-linear regime for the soil skeleton was proposed by Prevost [10]. The formulations of two-phase saturated and partially saturated soil following the Biot theory were solved in the framework of the finite element method for both static and dynamic cases by Zienkiewicz et al. [11], [12]. Meroi et al. [13] developed the finite strain formulation with non-linear (plasticity) material behavior in solid phase for both static and dynamic cases in partially saturated soil. Chao Li et al. [14] developed a mathematical framework for dynamics analysis of fully saturated soil under finite strain by employing the Neo-Hookean hyperelasticity model. The formulation was then extended to an unsaturated case by Uzuoka and Borja [15].

Alternatively, there are other models that use different stress variables besides the aforementioned effective stress. Complete formulation of an elastoplastic model for unsaturated soil was proposed by Alonso et al. [16]. The model is based on the Modified Cam-Clay model which can be recovered when the soil is fully saturated. It employs the definition of net stress and
also includes suction as an additional variable. It has become one of the most widely used constitutive models for unsaturated soil and has been referred to as the Barcelona Basic Model (BBM). The model was extended by Gens and Alonso [17] to describe the behavior of expansive clays, which exhibit irreversible swelling when the soil undergoes an imbibition process, with a complete mathematical model given by Alonso et al. [18]. A fully couple hydro-mechanical constitutive model for unsaturated soil to take into account hydromechanics was proposed by Wheeler et al. [19]. A detailed review of constitutive models for unsaturated soil was given by Gens et al. [20].

2.2 **Meshfree Methods**

The fundamental rationale of the idea of meshfree methods is to alleviate the difficulties of mesh distortion in mesh-based methods such as the finite element methods (FEMs) and finite difference methods, while achieving high accuracy primarily by satisfying the partition of unity [21].

The early development of meshfree methods can be traced back to 1977 when the smoothed particle hydrodynamics (SPH) was proposed by Lucy [22] and Gingold and Monaghan [23]. The method was introduced for the solutions of astrophysics problems by using kernel functions as the approximation functions. SPH is extended to account for frictional contact by Wang et al. [24]. Nayroles et al. [25] proposed the diffuse element method by employing the moving least-squares method [26] for the approximation of field variables in the Galerkin weak formulation. Later, Belytschko et al. [27] proposed the Element-Free Galerkin (EFG) method to extensively improve the accuracy of the diffuse element method by using higher-order quadrature
for numerical integration, including the certain terms omitted in the diffuse element method, and imposing Dirichlet boundary conditions using Lagrange multipliers. Liu et al. [28] proposed the Reproducing Kernel Particle Method (RKPM) to improve the consistency and accuracy of the meshfree approximation by introducing the correction function in the kernel estimate to satisfy the consistency conditions and partition of unity. The RKPM is then extended for large deformation problems by Chen et al. [29] and Chen et al. [30]. Wu [31] proposed the semi-Lagrangian RKPM for extremely large deformation problems. The method constructs the RK shape functions in the current configuration, which readily allows extreme deformation and material separation; however, convective term and correction of nodal mass are required. Stability analyses of the semi-Lagrangian RKPM using nodal integration methods with and without the convective term were performed by Chen and Wu [32] and Wu [31]. The semi-Lagrangian RKPM was introduced for modeling penetration problems by Guan et al. [33] and for the earth-moving simulation by Guan et al. [34]. An algorithm to detect contact, based on the moment matrix of meshfree shape functions, was proposed by Li et al. [35]. Natural kernel contact algorithm using concept of partition of unity as the contact detector was introduced under the semi-Lagrangian RKPM by Wu [31]. The frictional kernel contact algorithm to account for friction between contacting bodies was proposed by Guan [36]. Chi [37] and Chi et al. [38] proposed the use of the level-set algorithm to properly represent the contact surfaces of the kernel contact algorithm.

Unlike shape functions of FEM, meshfree shape functions do not possess the Kronecker delta property, and hence imposition of the Dirichlet boundary conditions is not straightforward. Nevertheless, there are many proposed methods to overcome such a difficulty. The EFG of Belytschko et al. [27] used the Lagrange multipliers to impose the Dirichlet boundary conditions,
while Chen et al. [29] introduced the transformation method to recover the Kronecker delta property of the shape functions. Later, Chen and Wang [39] proposed the mixed transformation method to improve the calculation efficiency of the transformation method by transforming only the nodes associated with the Dirichlet boundaries. To enforce the Dirichlet boundary conditions under the EFG method framework, the penalty method was used by Zhu and Atluri [40], while Nitsche’s method [41] was employed by Fernández-Méndez and Huerta [42].

Numerical integration for meshfree methods in the Galerkin weak formulation is not as trivial as in FEM since the use of quadrature rules requires a background mesh that is not preferable due to the desire to be completely free from the mesh. In addition, quadrature rules are computationally expensive and can yield considerable errors if the integration cells do not align with the influence domains of the shape functions, as reported by Dolbow and Belytschko [43]. As a result, researchers in the meshfree methods community have shown much interest in nodal integration methods. Bonet and Kulasegaram [44] and Beissel and Belytschko [45] proposed the least-square stabilization to stabilize the rank deficiency due to the under-integration of the Direct Nodal Integration (DNI). Chen et al. [46] proposed the Stabilized Conforming Nodal Integration (SCNI) by using the idea of assumed strain to satisfy the integration constraints for linear exactness. The method was then extended for large deformation problems by Chen et al. [47]. Duan et al. [48] proposed the Consistent Element-Free Galerkin method, which can ensure higher-order exactness by satisfying the orthogonality condition [49] of stress and strain from the Hu-Washizu variational principle. The Stabilized Non-Conforming Nodal Integration (SNNI) was proposed by Wu [31] for the integration of extremely large deformation problems by relaxing the conforming condition in SCNI, which helps SCNI to satisfy the integration constraints. The Modified Stabilized Conforming Nodal Integration (MSCNI) was proposed by Chen et al. [50] to
enhance the stability of the SCNI by adding a higher-order term in strain energy to help suppress the spurious low energy modes. Puso et al. [51] used a similar idea as in [50] to stabilize the SNNI and proposed the Modified Stabilized Non-Conforming Nodal Integration (MSNNI). The Variationally Consistent Integration (VCI) was proposed by Chen et al. [52] to correct the inconsistency occurred by the numerical integration in the Galerkin weak formulation. The method can satisfy the specified order of the integration constraint, thus achieving the desired order of exactness.

2.3 Landslides Prediction and Simulation

Landslide modeling has been a challenge for many decades because of the difficulties of analyzing mechanics and characteristics of landslide processes. A commonly used class of techniques to analyze slope stability is the limit equilibrium method (LEM). These techniques compute the factor of safety, which is a ratio of the available shear strength to the required shear force for equilibrium conditions, to determine slope stability. The concept of the safety factor was first used by Fellenius with his proposed Swedish Circle Method [53]–[55], in which the slip surface is assumed to be an arc of a circle. Taylor [56] introduced the Infinite Slope Method by assuming that the slip surface is parallel to the slope, which is suitable only when the depth-to-length ratio of the slope is small. Frohlich [57] assumed the slip surface to be an arc of a logarithmic spiral and proposed the Logarithmic Spiral Method. With different assumptions of the forces between slices, there are many proposed methods that use the concept of the method of slices, in which the earth mass is partitioned into slices. Some of the most popularly used methods under such concept are the Ordinary Method of Slices [58], [59], Simplified Bishop’s Method
[60], Modified Bishop’s Method [60], Janbu’s method [61], [62], Spencer’s method [63], Morgenstern-Price’s method [64], and Sarma’s method [65]. Nevertheless, the main issue of LEM is that it requires the pre-defined potential slip surface. In addition, many uncertainties associated with soil can lead to the slope failure even if the factor of safety indicates that the slope is stable. Reliability analysis, which considers the probability of failure from quantified uncertainties of various parameters, has been proposed to mitigate this issue [66].

With the advancement of computation capabilities in the past two decades, numerical methods have been extensively employed and developed for landslide analyses. However, due to distinctive numerical challenges, numerical analysis of landslide processes is commonly separated into slope stability analysis and run-out simulation. To analyze slope stability, whose characteristics consist of complex poromechanics and solid-like behaviors, finite element methods (FEMs) with Lagrangian formulations are usually employed [67]–[70]. Although FEMs can accurately and effectively account for complex soil behaviors while the slope undergoes small deformation, they suffer significant difficulties due to mesh distortion and separation when slope experiences extreme deformation and propagates. As a result, run-out simulation, which involves extremely large deformation and material separation, is usually handled by smoothed particle hydrodynamics (SPH) [71]–[77] and discrete element method (DEM) [78]–[81]. This is due to the discontinuum-based nature of SPH [22], [23] and DEM [82], which is suitable with simulating flow-like phenomena. Within the framework of SPH and DEM, a simplified model called the depth-integrated model was introduced [71], [83]–[85] to reduce the 3-D system into a 2-D system for the simulations of the granular avalanches and mudflows. Nevertheless, SPH suffers from tensile instability and boundary deficiency since the kernel approximations are inconsistent [26], while parameters of element-to-element contact models used in DEM are
difficult to calibrate and can significantly impair the accuracy [86]. Since it is extremely challenging or even impractical to apply one of these numerical methods to effectively and accurately analyze whole landslide processes, some researchers have suggested or employed coupled methods such as FEM-SPH [87], [88] or FEM-DEM [89], [90], to handle different stages of landslide processes with suitable numerical methods by defining certain strain criteria to switch between methods. However, mathematical verification of the transition between two distinctive numerical methods is still not robust.
CHAPTER 3

MESHFREE METHODS FOR GALERKIN WEAK FORMULATION

In this chapter, the overall frameworks of the Galerkin meshfree methods based on the Reproducing Kernel approximation are described by considering equation of motion as the governing equation. These frameworks focus on two important components for a numerical method to effectively handle extremely large deformation and material separation. The first component is an effective numerical discretization scheme to capture both continuous and discontinuous material deformations, which are the essential characteristics in landslide simulations. The second constituent is an effective domain integration method in the Galerkin formulation, which can significantly affect the stability, efficiency, and accuracy of the analysis.

For single-field formulation, the governing equation is expressed as

\[ \sigma_{ij,j} + b_i = \rho \ddot{u}_i \quad \text{in } \Omega \]  \hspace{1cm} (3.1)

with boundary conditions described as

\[
\begin{align*}
\sigma_{y} n_j &= h_i \quad \text{on } \Gamma_b \\
u_i &= u_i^s \quad \text{on } \Gamma_g 
\end{align*}
\]  \hspace{1cm} (3.2)

and initial conditions defined as

\[
\begin{align*}
u_i(x,t)|_{t=0} &= u_i^0(x) \quad \text{in } \Omega \\
\dot{u}_i(x,t)|_{t=0} &= \dot{u}_i^0(x) \quad \text{in } \Omega
\end{align*}
\]  \hspace{1cm} (3.3)
where $\Gamma_h$ and $\Gamma_g$ are Neumann and Dirichlet boundaries in current configuration, which obey the following rules: (1) $\Gamma_h \cup \Gamma_g = \Gamma$, (2) $\Gamma_h \cap \Gamma_g = \emptyset$, with traction $h_i$ and prescribed displacement $u_i^g$. The body force $b_i = \rho g_i$ is considered in the deformed configuration, where $\rho$ is the density in the deformed configuration and $g_i$ is the gravitational acceleration. The initial values of displacement and velocity are denoted by $u_i^0(x)$ and $u_i^0(x)$, respectively.

Applying the boundary conditions (3.2) into the governing equation (3.1), the Galerkin weak formulation are obtained as

$$\int_{\Omega} \delta u_i^h \sigma_{ij}^h d\Omega + \int_{\Omega} \delta u_i^h \rho u_t^h d\Omega = \int_{\Gamma_h} \delta u_i^h h_i d\Gamma + \int_{\Omega} \delta u_i^h b_i d\Omega \tag{3.4}$$

Equation (3.4) is to be discretized in space and time. The Lagrangian RK and semi-Lagrangian RK are adopted for spatial discretization. To take into account large deformation, the Cauchy stress is updated by following the approach from [91], see Appendix B. Explicit time integration method (e.g., central difference) is employed for temporal discretization.

The outline of this chapter is as follows: Reproducing Kernel approximation [28], [29] is reviewed in Section 3.1. The semi-Lagrangian Reproducing Kernel approximation is described in Section 3.2. Then, the spatial discretization of the single-field formulation is demonstrated in Section 3.3, while its temporal discretization is presented in Section 3.4. Domain integration methods are discussed in Section 3.5. Contact algorithms are presented in Section 3.6. Note that throughout this chapter, the Einstein notation is used when lower-case indices are repeated in a term unless otherwise noted.
3.1 **Reproducing Kernel Approximation**

Moving Least Squares (MLS), adopted in Element-Free Galerkin (EFG) method [27], and Reproducing Kernel (RK), adopted in RKPM [28], [29], are two commonly used meshfree approximations. It has been shown that MLS and RK are equivalent in the discrete form and when polynomials are used as basis functions. A consistent way of deriving the RK approximation in the discrete form is to enforce the approximation function to be able to reproduce polynomial functions.

An approximated function of a function \( R_i \) can be expressed as the linear combination of RK shape functions as

\[
R_i^h(\mathbf{X}, t) = \sum_{I=1}^{NP} \psi_I(\mathbf{X}) r_{it}(t)
\]

where \( \mathbf{X} \) is the Lagrangian coordinates, \( NP \) is the total number of nodes, \( \psi_I(\mathbf{X}) \) is the RK shape function of node \( I \) constructed in the reference configuration, and \( r_{it}(t) \) is the corresponding nodal coefficient at time \( t \). The shape function \( \psi_I(\mathbf{X}) \) can be expressed by

\[
\psi_I(\mathbf{X}) = C(\mathbf{X}; \mathbf{X}_I - \mathbf{X}_I) \Phi_a(\mathbf{X}_I - \mathbf{X}_I)
\]

where \( C(\mathbf{X}; \mathbf{X}_I - \mathbf{X}_I) \) is the correction function to ensure the reproducibility of the RK approximation, and \( \mathbf{X}_I \) is the nodal coordinates in the undeformed configuration. Subscript \( a \) in the kernel function \( \Phi_a(\mathbf{X}_I - \mathbf{X}_I) \) is a number defining the influence domain or so-called support size of the function as demonstrated in Figure 1. The support size is usually regarded after normalized by nodal distance and called normalized support size \( \bar{a} = a/\Delta x \) (see Figure 1 for the
case when $\Delta x = 1$). The kernel function controls the smoothness and locality of the approximation function, and hence it should be selected depending on the characteristics of the problem, e.g., the order of partial differential equations. In general, a required order of spline function is employed as a kernel function, for instance, a cubic spline function (Figure 1), which is of class $C^2$.

![Figure 1](image1.png)

(a)  

(b)  

Figure 1. RK shape functions in 2-D using cubic spline functions as kernel functions with normalized support sizes equal to (a) 1.5 and (b) 2.5

For $m$-dimensional analysis, the $m$-D kernel function is constructed by the product of 1-D kernel function, that is

$$
\Phi_{a}^{m-D}(\mathbf{X} - \mathbf{X}_i) = \prod_{i=1}^{m} \Phi_{a}^{1-D}(Z_i) \quad (3.7)
$$

where

$$
Z_i = \frac{|X_i - X_{i|d}|}{a_i} \quad (3.8)
$$

The correction function $C(\mathbf{X}; \mathbf{X} - \mathbf{X}_i)$ is introduced to enforce the reproducing conditions to achieve reproducibility for monomials up to the specified $n^{th}$ order consistency, that is
\[
C(X; X - X_i) = \sum_{|\alpha| = 0}^{\alpha} (X - X_i)^\alpha b_\alpha (X) 
\]  
(3.9)

where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \) with its length defined as \( |\alpha| = \sum_{i=1}^{m} \alpha_i \) and \( b_\alpha \equiv b_{\alpha_1, \alpha_2, \ldots, \alpha_m} \) is the coefficient of the monomials \((X - X_i)^\alpha \equiv (X_1 - X_{1i})^{\alpha_1} (X_2 - X_{2i})^{\alpha_2} \ldots (X_m - X_{mi})^{\alpha_m}\). The coefficients \( b(X) \) are obtained by satisfying the \( n \)th order reproducing condition stated as

\[
\sum_{i=1}^{NP} \Psi_j(X) X_i^\alpha = X^\alpha \quad ; \quad |\alpha| \leq n 
\]  
(3.10)

By using Equation (3.6) and Equation (3.9) and imposing the reproducing condition (3.10), the corresponding coefficients \( b(X) \) can be determined as

\[
b(X) = M^{-1}(X)H(0) 
\]  
(3.11)

where \( M(X) \) is the moment matrix which can be described in discrete form as

\[
M(X) = \sum_{i=1}^{NP} H(X - X_i) H^T(X - X_i) \Phi_\alpha (X - X_i) 
\]  
(3.12)

and \( H(X - X_i) \) is the vector of monomial basis functions.

Substituting Equation (3.11) into Equation (3.9) and Equation (3.6), the RK shape function \( \Psi_j(X) \) reads

\[
\Psi_j(X) = H^T(0)M^{-1}(X)H(X - X_j) \Phi_\alpha (X - X_j) 
\]  
(3.13)
3.2 Semi-Lagrangian Reproducing Kernel Approximation

In the Lagrangian RK approximation, mapping between current configuration and initial configuration is required. The mapping between the Lagrangian coordinates $\mathbf{X}$ and Eulerian coordinates $\mathbf{x}$ are described as

$$\mathbf{x} = \varphi(\mathbf{X}, t)$$  \hspace{1cm} (3.14)

where $\varphi$ is the mapping function. This mapping between the two configurations may break down when the Jacobian becomes zero or negative due to excessive deformation and a poorly formed constitutive law. To circumvent such issue, the shape functions of semi-Lagrangian RK is constructed in the current configuration, thus eliminating the necessity of mapping. The discretization (i.e., nodal points) of semi-Lagrangian RK is, however, still under Lagrangian description to track internal variables of the same material points at each time step. These properties of semi-Lagrangian RK, illustrated in Figure 2, are advantageous for problems involving extremely large deformation and material separation [39], [42].

![Figure 2](image_url)

Figure 2. Comparison between 2-D (a) RK shape functions in initial configuration, (b) Lagrangian RK shape functions in current configuration, and (c) semi-Lagrangian RK shape functions in current configuration.
The semi-Lagrangian RK shape function is defined as

\[ \Psi'_I(x) = C(x; x - x_I) \Phi_a(x - x_I) \] (3.15)

where \( \Psi'_I(x) \) is the shape function of node \( I \) constructed in the current configuration and \( x_I = x(X_I, t) \) is the nodal position of node \( I \) in the current configuration. The correction function \( C(x; x - x_I) \) and the kernel function \( \Phi_a(x - x_I) \) are also determined in the current configuration. These are in contrast to the Equation (3.6), where all of the terms are considered in the reference configuration. The multi-dimensional kernel function in the current configuration can be constructed similar to that in Equation (3.7), that is

\[ \Phi^{m-D}_a(x - x_I) = \prod_{i=1}^m \Phi^{1-D}_a(z_i) \] (3.16)

where

\[ z_i = \frac{|x_i - x_{i,j}|}{a_i} \] (3.17)

By following similar derivation procedures as in the previous section, the correction function \( C(x; x - x_I) \) in deformed configuration can be obtained as

\[ C(x; x - x_I) = H^T(0)M^{-1}(x)H(x - x_I) \] (3.18)

where \( H(x - x_I) \) is the basis function in the current configuration, which defines the order of completeness of the reproduced function and \( M(x) \) is the moment matrix obtained by

\[ M(x) = \sum_{I=1}^{NP} H(x - x_I) H^T(x - x_I) \Phi_a(x - x_I) \] (3.19)
Substituting Equation (3.18) into Equation (3.15), the semi-Lagrangian RK shape functions read

$$
\Psi_i(x) = H^T(0)M^{-1}(x)H(x-x_i)\Phi_a(x-x_i)
$$

(3.20)

By using semi-Lagrangian RK approximation $\Psi_i(x)$, a function $Y_i$ can be approximated by a set of distinct points $\{x_{j_i}\}_{i=1}^{NP}$ and its corresponding nodal coefficient $y_{a}(t)$, that is

$$
Y_i^h(x, t) = \sum_{i=1}^{NP} \Psi_i(x) y_{a}(t)
$$

(3.21)

Subsequently, the approximation of the temporal derivative of function $Y_i$ is expressed as

$$
\dot{Y}_i^h(x, t) = \sum_{i=1}^{NP} (\Psi_i(x) \dot{y}_a(t) + \Phi_a(x-x_i))
$$

(3.22)

where $\dot{y}_a(t)$ is the nodal coefficient corresponding to $Y_i^h(x, t)$. $\Phi_a(x-x_i)$ is the change of the RK shape function with respect to time, due to the reconstruction of the shape function in semi-Lagrangian RK, which can be interpreted as a convection term to carry the information history during the transition between the old shape function and the new one. It is defined as

$$
\Phi_a(x-x_i) = C(x; x-x_i)\Phi_a(x-x_i)
$$

(3.23)

where temporal derivative of the correction function is omitted since the function is constructed by solving the corresponding coefficient $b(x)$ under the current configuration. The temporal derivative of the kernel function $\Phi_a(x-x_i)$ is constructed by using Equation (3.16) and performing chain rule

$$
\Phi_a(x-x_i) = \prod_{i=1}^{m} \left( \frac{1}{a_i} \frac{\partial \Phi_{a}^{m,D}(z_i)}{\partial (z_i)} \frac{x_i - x_{a}}{|x_i - x_{a}|} \frac{\partial (x_i - x_{a})}{\partial t} \right)
$$

(3.24)
3.3 **Spatial Discretization**

3.3.1 **Lagrangian Reproducing Kernel**

For the Lagrangian RK approximation, the displacement is discretized as

\[
u_i^h (X, t) = \sum_{l=1}^{NP} \Psi_{l,i} (X) d_{il} (t)
\]

(3.25)

where \( d_{il} (t) \) is the nodal coefficient of displacement or so-called generalized displacement.

Since \( d_{il} (t) \) is independent from space, the spatial derivative with respect to initial configuration of Equation (3.25) can be written as

\[
u_{i,j}^h (X, t) = \sum_{l=1}^{NP} \Psi_{l,j} (X) d_{il} (t)
\]

(3.26)

where \( \cdot^j \) denotes spatial derivative with respect to reference coordinates.

By taking second-order temporal derivative of Equation (3.25), the acceleration is determined as

\[
\ddot{u}_i^h (X, t) = a_i^b (X, t) = \sum_{l=1}^{NP} \Psi_{l,i} (X) a_{il} (t)
\]

(3.27)

where \( a_{il} (t) \) is the nodal coefficient of acceleration or so-called generalized acceleration.

Rewriting Equation (3.4) in initial configuration, the new equation reads

\[
\int_{\Omega^0} \delta u_{i,j}^h P_{i}^0 d\Omega + \int_{\Gamma^0} \delta u_{i}^h \rho^0 \dot{u}_{i}^h d\Gamma = \int_{\Gamma^0} \delta u_{i}^h h^0 d\Gamma + \int_{\Omega^0} \delta u_{i}^h \rho^0 g_i d\Omega
\]

(3.28)
where \( \Omega^0 \) and \( \Gamma^0_h \) are domain and Neumann boundary in initial configuration, respectively. \( \rho^0 \) is initial material density and \( h^0_i \) is traction in reference configuration. \( P_{ij} \) is the first Piola-Kirchhoff (PK) stress which can be related to Cauchy stress \( \sigma_{ij} \) as

\[
P_{ij} = F^{-1}_{ij} \sigma_{ij} J
\]

(3.29)

where \( F_{ij} \) is the deformation gradient expressing the mapping between the initial configuration and current configuration, i.e.,

\[
F_{ij} = \frac{\partial x_j}{\partial X_i}
\]

(3.30)

\( J \) is the Jacobian denoted by

\[
J = \det\left( F_{ij} \right)
\]

(3.31)

The Cauchy stress is calculated by the approach in Appendix B. The Drucker-Prager plasticity with damage model as shown in Appendix A is also incorporated for the calculation of the stress.

Substituting Equation (3.25) - Equation (3.27) into Equation (3.28), the semi-discrete equation reads

\[
\delta d_{\tilde{\alpha}} \int_{\Omega^0} \Psi_{1,ij} P^{h}_{ij} d\Omega + \delta d_{\tilde{\mu}} \int_{\Omega^0} \Psi_{1} \rho^{0} \Psi_{j} d\Omega a_{ij} = \delta d_{\tilde{\mu}} \int_{\Gamma^0} \Psi_{j} h^{0}_{i} d\Gamma + \delta d_{\tilde{\mu}} \int_{\Omega^0} \Psi_{1} \rho^{0} g_{i} d\Omega
\]

(3.32)

or

\[
Ma = F^{ext} - F^{int}
\]

(3.33)
where mass matrix $M$, external force matrix $F^{\text{ext}}$, and internal force matrix $F^{\text{int}}$ are defined in Table II. Note that Einstein summation convention is used in Equation (3.32) for subscripts $I$ and $J$; hence, the summation symbols are vanished.

<table>
<thead>
<tr>
<th></th>
<th>RK</th>
<th>Semi-Lagrangian RK</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{IJ}$</td>
<td>$\int_{\Omega^0} \Psi_i \rho^0 \Psi_j d\Omega$</td>
<td>$\int_{\Omega} \Psi_i \rho \Psi_j d\Omega$</td>
</tr>
<tr>
<td>$F^{\text{ext}}_{IJ}$</td>
<td>$\int_{\Gamma_a^0} \Psi_i h_j^0 d\Gamma$</td>
<td>$\int_{\Gamma} \Psi_i h_j d\Gamma$</td>
</tr>
<tr>
<td>$F^{\text{int}}_{IJ}$</td>
<td>$\int_{\Omega^0} \Psi_{i,j} P_{ij}^b d\Omega - \int_{\Omega^0} \Psi_i \rho^0 g_i d\Omega$</td>
<td>$\int_{\Omega} \Psi_{i,j} \sigma_{ij}^b d\Omega - \int_{\Omega} \Psi_i h_i d\Omega$</td>
</tr>
<tr>
<td>$G_{IJ}$</td>
<td>$-$</td>
<td>$\int_{\Omega} 2\Psi_i \rho \tilde{\Psi}_j d\Omega$</td>
</tr>
<tr>
<td>$\overline{G}_{IJ}$</td>
<td>$-$</td>
<td>$\int_{\Omega} \Psi_i \rho \tilde{\Psi}_j d\Omega$</td>
</tr>
</tbody>
</table>

TABLE II. DEFINITIONS OF EACH MATRIX IN EQUATION (3.33) AND EQUATION (3.38)

3.3.2 Semi-Lagrangian Reproducing Kernel

In the semi-Lagrangian framework, all of the formulations are considered in the current configuration. Hence, the displacement and its spatial derivative with respect to current coordinates are, respectively, discretized as

$$u_i^b(\mathbf{x}, t) = \sum_{I=1}^{NP} \Psi_I(\mathbf{x}) d_i(t)$$ (3.34)

$$u_{i,j}^b(\mathbf{x}, t) = \sum_{I=1}^{NP} \Psi_{i,j}(\mathbf{x}) d_{ij}(t)$$ (3.35)

where $\mathbf{\cdot}_j$ denotes spatial derivative with respect to deformed configuration.
From Equation (3.22) and Equation (3.34), the acceleration can be determined as

\[ \ddot{u}^b_i(x,t) = a^b_i(x,t) = \sum_{i=1}^{NP} \left( \Psi_i(x)a_{ij}(t) + 2\bar{\Psi}_i(x)v_i(t) + \bar{\Psi}_i(x)d_i(t) \right) \quad (3.36) \]

where \( v_i(t) \) denotes the nodal coefficient of velocity and \( \bar{\Psi}_i(x) \) is the change of \( \Psi_i(x) \) with respect to time.

The Cauchy stress is calculated by the similar approach as in the previous section (see Appendix A and Appendix B). This is advantageous since the approach is suitable for large deformation [91], which is a common characteristic of problems where semi-Lagrangian RK is required.

Finally, Equation (3.4) is spatially discretized by substitutions of Equation (3.34) - Equation (3.36)

\[
\delta d \int_{\Omega} \Psi_{i,j} \sigma_{ij}^h d\Omega + \delta d \int_{\Omega} \Psi_{i,j} \rho \Psi_{j} \Psi d\Omega a_{ij} + \delta d \int_{\Omega} 2\Psi_{i,j} \rho \bar{\Psi}_{j} d\Omega v_{ij}
\]

\[
+ \delta d \int_{\Omega} \Psi_{i,j} \rho \bar{\Psi}_{j} d\Omega d_{ij} = \delta d \int_{\Omega} \Psi_{i,j} h_{ij} d\Gamma + \delta d \int_{\Omega} \Psi_{i,j} b_{ij} d\Omega
\]

or

\[ \mathbf{M}a + \mathbf{G}v + \bar{\mathbf{G}}d = \mathbf{F}^{\text{ext}} - \mathbf{F}^{\text{int}} \quad (3.38) \]

where mass matrix \( \mathbf{M} \), external force matrix \( \mathbf{F}^{\text{ext}} \), internal force matrix \( \mathbf{F}^{\text{int}} \), and convection matrices \( \mathbf{G} \) and \( \bar{\mathbf{G}} \) are defined in Table II. Note that similar to Equation (3.32), the Einstein summation convention is used for subscripts \( I \) and \( J \).
3.4 Temporal Discretization

Considering the central difference scheme, displacement and velocity can be discretized at time step \( n+1 \) as

\[
d_{ij}^{n+1} = d_{ij}^n + \Delta t v_{ij}^n + 0.5 \Delta t^2 a_{ij}^n
\]

(3.39)

\[
v_{ij}^{n+1} = v_{ij}^n + 0.5 \Delta t \left( a_{ij}^n + a_{ij}^{n+1} \right)
\]

(3.40)

where \( \Delta t \) is the time step size. From Equation (3.39), it is easily noticeable that \( d_{ij}^{n+1} \) can be obtained at the beginning of each time step as all information is from previous time step; hence, the predicted displacement \( \hat{d}_{ij}^{n+1} \) is equal to \( d_{ij}^{n+1} \). On the other hand, the predicted velocity

\[
\hat{v}_{ij}^{n+1} = v_{ij}^n + 0.5 \Delta t a_{ij}^n
\]

(3.41)

needs to be corrected at the end of each time step after solving for \( a_{ij}^{n+1} \).

Applying Equation (3.39) - Equation (3.41) into Equation (3.33) and (3.38), then including damping matrix \( \mathbf{C} \), the fully discrete equations read

**RK:**

\[
(\mathbf{M} + 0.5 \Delta t \mathbf{C})a_{ij}^{n+1} = \mathbf{F}^{\text{ext}} - \mathbf{F}^{\text{int}} - \mathbf{C} \hat{v}_{ij}^{n+1}
\]

(3.42)

**Semi-Lagrangian RK:**

\[
(\mathbf{M} + 0.5 \Delta t (\mathbf{C} + \mathbf{G}))a_{ij}^{n+1} = \mathbf{F}^{\text{ext}} - \mathbf{F}^{\text{int}} - (\mathbf{C} + \mathbf{G}) \hat{v}_{ij}^{n+1} - \tilde{\mathbf{G}} d_{ij}^{n+1}
\]

(3.43)

For simplicity, the lumped mass scheme by the method of row summation is employed for the mass matrix \( \mathbf{M} \) and 5% mass proportional damping is used for the damping matrix \( \mathbf{C} \). This allows the system of equations in (3.42) to be solved separately, and hence significantly improve computational efficiency. Further, the convection matrices \( \mathbf{G} \) and \( \tilde{\mathbf{G}} \) in Equation (3.43) can be
negligible since the effect of this term on the accuracy of the solution is trivial [31]. Therefore, the linear equations in (3.43) can also be solved independently. At each time step, the acceleration $a_{n+1}^i$ is solved from Equation (3.42) or Equation (3.43), then the predicted velocity $\hat{v}_{n+1}^i$ is updated as

$$v_{n+1}^i = \hat{v}_{n+1}^i + 0.5\Delta t a_{n+1}^i$$

(3.44)

### 3.5 Domain Integration Methods

Solving problems by Galerkin meshfree methods can become challenging since domain integration is required. Although domain integration methods that rely on quadrature rules on structured grids can be used (e.g., the Gauss integration), nodal integration methods are preferable in the Galerkin meshfree method for two main reasons. The first reason is that the idea of "mesh-free" is well preserved in nodal integration methods, which is in contrast to finite element methods (FEMs) where mesh is needed, as shown in Figure 3.

![Figure 3](image-url)

Figure 3. Discretizations, in 2-D, of (a) meshfree methods and (b) finite element methods.
The other reason is that nodal integration methods are computationally more efficient than Gauss integration and the internal variables are stored at nodes, which is beneficial for large deformation problems since the integration point attaches to the node.

The followings are some of the nodal integration methods to be discussed here:

1. Direct Nodal Integration (DNI)
2. Stabilized Conforming Nodal Integration (SCNI) [46], [47]
3. Stabilized Non-Conforming Nodal Integration (SNNI) [50]
4. Modified Stabilized Conforming Nodal Integration (MSCNI) [50]
5. Modified Stabilized Non-Conforming Nodal Integration (MSNNI) [51]
6. Variationally Consistent Integration (VCI) [52]

For Direct Nodal Integration (DNI), the locations of the integration points are taken at the same positions as nodal points, for instance each term of Equation (3.42) can be numerically integrated as

\[
F_{\text{ext}}^\text{int} = \sum_{K=1}^{NP} \Psi_i(\mathbf{x}_K)h_i^0(\mathbf{x}_K)A_K^0, \quad F_{\text{int}} = \sum_{L=1}^{NP} \Psi_{I,j}(\mathbf{x}_L)P_{ij}(\mathbf{x}_L)V_L^0 - \sum_{L=1}^{NP} \Psi_i(\mathbf{x}_L)\rho_i^0(\mathbf{x}_L)g_iV_L^0 \quad (3.45)
\]

and each term of Equation (3.43) can be integrated by DNI as

\[
F_{\text{ext}} = \sum_{K=1}^{NP} \Psi_i(\mathbf{x}_K)h_i(\mathbf{x}_K)A_K, \quad F_{\text{int}} = \sum_{L=1}^{NP} \Psi_{I,j}(\mathbf{x}_L)\sigma_{ij}(\mathbf{x}_L)V_L - \sum_{L=1}^{NP} \Psi_i(\mathbf{x}_L)b_i(\mathbf{x}_L)V_L \quad (3.46)
\]

where \(A_K^0\) and \(A_K\) are the boundary integration weights associated with node \(K\) in reference and current configurations, respectively. \(V_L^0\) and \(V_L\) are the nodal volumes associated with node \(L\) in initial and deformed configurations, respectively.
However, DNI causes rank deficiency and instability due to under-integration [44], [45]. It also does not satisfy linear exactness and has suboptimal convergence. Hence, the Stabilized Conforming Nodal Integration (SCNI) scheme [46], [47] was proposed to remedy such issues. SCNI introduced the assumed strain $\bar{e}^h_{ij}$ to avoid taking direct derivatives of the shape function at the node, which causes the rank deficiency in the stiffness matrix. The method assumes constant strain within the domain of each node that is conforming to each other and the problem boundary. This particular domain of each node is called nodal representative domain, which is usually constructed by using the Voronoi diagram (Figure 4a). As such, the assumed strain $\bar{e}^h_{ij}$ can be expressed in 3-D as

$$\bar{e}^h(x_L) = \sum_{l=1}^{NP} \overline{B}_l(x_L)d_l$$

(3.47)

where

$$\overline{B}_l(x_L) = \begin{bmatrix} \overline{b}_{11}(x_L) & 0 & 0 & 0 & \overline{b}_{13}(x_L) & \overline{b}_{12}(x_L) \\ 0 & \overline{b}_{12}(x_L) & 0 & \overline{b}_{13}(x_L) & 0 & \overline{b}_{11}(x_L) \\ 0 & 0 & \overline{b}_{13}(x_L) & \overline{b}_{12}(x_L) & \overline{b}_{11}(x_L) & 0 \end{bmatrix}$$

(3.48)

with

$$\overline{b}_h(x_L) = \frac{1}{V_L} \int_{\Gamma_L} \Psi_I(x)n_i(x)d\Gamma$$

(3.49)

and $\Gamma_L$ is the boundary of the representative domain of node $L$.

The method satisfies integration constraints resulting in achievement of the first order accuracy of solutions. The integration constraints are described as

$$\sum_{L} \overline{b}_h(x_L)V_L = 0 \quad \text{for} \quad \{ I : \text{supp}(\Psi_I) \cap \Gamma = \emptyset \}$$

(3.50)

$$\sum_{L} \overline{b}_h(x_L)V_L = \int_{\Gamma_h} \Psi_I(x)n_i(x)d\Gamma \quad \text{for} \quad \{ I : \text{supp}(\Psi_I) \cap \Gamma_h \neq \emptyset \}$$

(3.51)
Nevertheless, the requirement of conforming nodal representative domains in SCNI is computationally inefficient when encountering large deformation, fracture, separation, or penetration problems. The Stabilized Non-Conforming Nodal Integration (SNNI) was proposed by [31] to handle such problems by allowing the nodal representative domain to be pre-defined and independent of nodal distribution, as can be seen in Figure 4b.

![Figure 4](image)

Figure 4. Nodal representative domains of (a) SCNI and (b) SNNI in 2-D

By using SCNI or SNNI, Equation (3.42) can be numerically integrated as

\[
F_{il}^{\text{ext}} = \sum_{K=1}^{NP} \Psi_I(X_K) h^0_l(X_K) A_K^0, \quad F_{il}^{\text{int}} = \sum_{L=1}^{NP} \mathbf{B}_l(X_L) P^{\delta}(X_L) V^0_L - \sum_{L=1}^{NP} \Psi_I(X_L) \rho^0(X_L) g_j V^0_L
\]  

(3.52)

and Equation (3.43) can be integrated as

\[
F_{il}^{\text{ext}} = \sum_{K=1}^{NP} \Psi_I(X_K) h_l(X_K) A_K, \quad F_{il}^{\text{int}} = \sum_{L=1}^{NP} \mathbf{B}_l(X_L) \sigma^*_{ij}(X_L) V_L - \sum_{L=1}^{NP} \Psi_I(X_L) b_i(X_L) V_L
\]  

(3.53)
where the superscript * indicates that the corresponding stress term is calculated by using assumed strain. $\mathbf{B}_i^*(X_L)$ is the smoothed gradient of shape function in initial configuration, which can be defined by rewriting Equation (3.48) and Equation (3.49) in initial configuration.

Although zero energy modes can be suppressed in SCNI and SNNI, low energy modes may still be triggered and cause instability; this calls for an additional stabilization [50], [51]. The Modified Stabilized Conforming Nodal Integration (MSCNI) and Modified Stabilized Non-Conforming Nodal Integration (MSNNI) offer a penalty-type stabilization term, which is constructed based on subdomains of each nodal representative domain (e.g., see Figure 4b). The stabilization term is added in the internal energy $F^\text{int}_{\text{ud}}$ to improve stability, it is defined as

$$
\tilde{\alpha} \sum_{L=1}^{NP} \sum_{c=1}^{N_{Lc}} \left( \mathbf{B}_i^T(X_L) - \mathbf{B}_i^T(X_c) \right) C^e \left( \mathbf{B}_j^*(X_L) - \mathbf{B}_j^*(X_c) \right) V_c^0 \mathbf{d}_j^{n+1}
$$

and

$$
\tilde{\alpha} \sum_{L=1}^{NP} \sum_{c=1}^{N_{Lc}} \left( \mathbf{B}_i^T(x_L) - \mathbf{B}_i^T(x_c) \right) C^e \left( \mathbf{B}_j^*(x_L) - \mathbf{B}_j^*(x_c) \right) V_c \mathbf{d}_j^{n+1}
$$

for Equation (3.42) and Equation (3.43), respectively, where $\tilde{\alpha}$ is the stabilization parameter ranging between 0 and 1, $C^e$ is the elastic material tangent tensor, $V_c^0$ and $V_c$ are the nodal volumes associated with subdomain $c$ (Figure 4b) in initial and current configurations, respectively, with $\sum_{c=1}^{N_{Lc}} V_c^0 = V_L^0$ and $\sum_{c=1}^{N_{Lc}} V_c = V_L$, and $N_{Lc}$ is the number of subdomains in the nodal representative domain of node $L$.

When the numerical integration is performed in the Galerkin weak formulation, it causes numerical inconsistency in the discrete equations, which impairs the accuracy of solutions [52]. In order to recover the consistency, Variationally Consistent Integration (VCI) modifies the
approximations of the test functions [52] to satisfy the specified order of integration constraint, thus achieving the desired order of exactness.

The gradient of the test function is modified to satisfy the following equation

$$\int_{\Omega} \delta \tilde{u}^{h}_{i,j} \sigma^{h}_{g} d\Omega = -\int_{\Omega} \delta u^{h}_{i,j} \sigma^{h}_{g} d\Omega + \int_{\Gamma} \delta u^{h}_{j} \sigma^{h}_{g} n_{j} d\Gamma$$  \hspace{1cm} (3.56)$$

where $\delta \tilde{u}^{h}_{i,j}$ is the modified version of $\delta u^{h}_{i,j}$. $\Gamma$ is the boundary in current configuration. $n_{j}$ is the unit normal vector in the outward direction of its corresponding boundary. $\int_{\Omega} \cdot d\Omega$ and $\int_{\Gamma} \cdot d\Gamma$ indicate numerical integrations of the domain and boundary, respectively.

By enforcing Equation (3.56) to achieve linear exactness (i.e., $\sigma^{h}_{g} = \text{constant}$ and $\sigma^{h}_{g,j} = 0$), the integration constraint can be derived as

$$\int_{\Omega} \hat{\Psi}_{i,j} d\Omega = \int_{\Gamma} \hat{\Psi}_{i} n_{j} d\Gamma$$ \hspace{1cm} (3.57)$$

where $\hat{\Psi}_{i,j}$ is the approximation function associated with $\delta \tilde{u}^{h}_{i,j}$, which can be expressed by

$$\hat{\Psi}_{i,j} = \Psi_{i,j} + R_{i} \xi_{j\ell}$$  \hspace{1cm} (3.58)$$

where $R_{i}$ can be any functions that are linearly independent of $\Psi_{i,j}$; however, local functions are preferred to preserve locality of the approximation functions. The corresponding coefficient $\xi_{j\ell}$ can be solved by substituting Equation (3.58) into Equation (3.57).
Note that the VCI for Equation (3.33) can be derived by considering Equation (3.56) - Equation (3.59) in reference configuration.

### 3.6 Kernel Contact Algorithms

One of the advantages of the semi-Lagrangian RK shape function is that it can naturally detect the contacting bodies without the necessity of pre-defined contacting surfaces as in conventional contact algorithms for FEM [92]. This is considerably beneficial for modeling contact problems involving arbitrary new free surfaces such as penetration and landslide problems. The natural kernel contact algorithm [33] uses the partition of unity of semi-Lagrangian RK shape functions, i.e. \( \sum_{i=1}^{NP} \Psi_i(x) = 1 \), as a contact detection algorithm to determine contact between two bodies (Figure 5). One may recall that RK shape functions satisfy the partition of unity everywhere in the continuum object domain; therefore, when the partition of unity is formed between contacting surfaces, the contacting bodies are naturally considered as a single continuum body. Hence, the contact forces are directly determined from pair-wise compressive stresses induced by the overlapping of the semi-Lagrangian RK shape functions, which naturally prevents interpenetration between contacting bodies. The contact force acting on point \( I \) (\( F_{I}^{\text{cont}} \)) is computed from the pair-wise compressive effective stress, that is

\[
F_{I}^{\text{cont}} = \sum_{L \in N_I} \overline{B}_I^{T}(x_L)\sigma(x_L)V_L ; \quad n_{IL} \cdot \sigma(x_L) \cdot n_{IL} < 0
\]  

(3.60)
where \( \mathbf{n}_{IL} \) is a unit vector from node \( L \) to node \( I \), \( N_I^* \) is the set of nodal points in the body that does not contain node \( I \). The total force acting on point \( I \) is computed by the summation of internal force at point \( I \) and contact force at point \( I \).

![Figure 5. Contact between two bodies using kernel contact algorithms](image)

Although the natural kernel contact algorithm is efficient and can naturally satisfy the non-penetration condition, the algorithm cannot represent stick and slip conditions as it only considers pair-wise compressive contact forces.

The frictional kernel contact algorithm [38] employs the elasto-perfectly-plastic model to represent the friction forces in contact region (Figure 5) by following the Coulomb’s law of friction. The yield surface of the plasticity model is designed such that elastic region represents stick condition and plastic region represents slip condition [38]. By accounting for friction forces, Equation (3.60) is corrected as

\[
\mathbf{F}_{I}^{\text{cont}} = \sum_{L \in N_I^*} \mathbf{B}_I^T(x_I) \left( \mathbf{\sigma}(x_I) + \lambda \mathbf{\xi}(x_I) \right) \mathbf{V}_L + \mathbf{n}_{IL} \cdot \mathbf{\sigma}(x_I) \cdot \mathbf{n}_{IL} < 0
\]  

(3.61)
with
\[
\lambda = \begin{cases} 
0 & ; \text{elastic} \\
\mu f_n - \|t_T\| & ; \text{plastic}
\end{cases}
\] (3.62)

and
\[
\xi(x_L) = (n_{ll} \otimes \sigma(x_L) \cdot n_{ll} + n_{ll} \cdot \sigma(x_L) \otimes n_{ll} - 2t_N n_{ll} \otimes n_{ll})
\] (3.63)

where \( t_N = n_{ll} \cdot \sigma(x_L) \cdot n_{ll} \) is the amplitude of normal traction, \( t_T = \sigma(x_L) \cdot n_{ll} - t_N n_{ll} \) is the tangential traction, and \( \mu \) is the friction coefficient.

Nevertheless, the pair-wise normal/tangential vector used in the aforementioned kernel contact algorithms is just a simplification, which may not properly represent actual normal/tangential vector of contacting surfaces and can impair the accuracy. The simplified vector is used due to the difficulty of determining physical boundaries in mesh-free discretization. To overcome the issue, a level set algorithm is introduced in [38] to adequately represent contacting surfaces.

The normal contact surface \( n \) can be estimated by the zero level set \( \phi(x) = 0 \) as
\[
n = -\frac{\nabla \phi}{\|\nabla \phi\|} \quad \text{on} \quad \phi(x) = 0
\] (3.64)

with
\[
\phi(x) = \sum_{I \in N^A \cup N^B} \Psi_I(x)c_I
\] (3.65)

and
\[
c_I = \begin{cases} 
1 & ; \ I \in N^A \\
-1 & ; \ I \in N^B
\end{cases}
\] (3.66)

where \( N^A \) and \( N^B \) are the sets of nodal points in bodies \( A \) and \( B \), respectively. However, it is computationally impractical to explicitly use the zero level set \( \phi(x) = 0 \) as a criterion to define
normal contact surface; therefore, it is proposed in [38] to search for the zero level set only on the lines that connect a pair of potential contacting nodal points, that is

\[ \phi(x_i + \xi_{IJ} n_{IJ}) = 0 ; \quad \forall J \in N^*_I \]  

where Newton-Raphson is used for iterations until \( \xi_{IJ} \) that satisfies Equation (3.67) is determined. Ultimately, the contact region or so-called contact processing zone (Figure 5), which is used for calculating friction forces as described above, is defined by the closest layers of nodal points to the contacting surfaces determined from Equation (3.67).
CHAPTER 4

SEMI-LAGRANGIAN REPRODUCING KERNEL FOR GALERKIN WEAK u-p FORMULATION

In this chapter, the semi-Lagrangian Reproducing Kernel is introduced to the u-p formulation in the Galerkin weak formulation. Two frameworks of the u-p semi-Lagrangian RK are presented by considering two forms of the u-p formulation with explicit and implicit temporal discretizations, while the temporal stability of the frameworks are analyzed in the next chapter. The frameworks are verified with benchmark problems and will be applied to landslide simulations in subsequent chapter. The summation convention is used when lower-case indices are repeated in a term, otherwise noted.

The outline of this chapter is as follows: The mechanics of porous media are described in Section 4.1. The u-p semi-Lagrangian RK frameworks are presented in Section 4.2. The verifications of the frameworks are demonstrated with benchmark problems in Section 4.3. Conclusions are given in Section 4.4.

4.1 Saturated Deformable Porous Media

In this section, poromechanics of saturated deformable porous media are described using a saturated two-phase formulation extended from the Biot theory [93]. This is to take into account the coupling effect between solid and fluid phases (Figure 6), which is a unique mechanical behavior in most geomaterials. While the scope of this study is for saturated porous media, one
can easily extend the frameworks described in subsequent sections to more general formulations such as three-phase formulation, where displacement field for solid skeleton and the pressure fields for two fluid phases (water and air) are considered.

In this study, slow to moderate speed phenomena (e.g., consolidation and landslide) is considered, so the fluid acceleration terms can be neglected [93]. By considering no phase change and isothermal condition, the governing equations of the dynamic displacement-pressure (u-p) formulation can be described by

\[
\begin{align*}
\sigma_{ij,j} + b_i &= \rho u_i \quad \text{in } \Omega \tag{4.1} \\
\alpha u_{i,i} + \frac{P^f}{M} + q_{i,i} &= 0 \quad \text{in } \Omega \tag{4.2}
\end{align*}
\]

where \( \sigma_{ij} \) is the total stress, \( b_i = \rho g_i \) is the body force, \( g_i \) is the acceleration due to gravity, \( \rho = (1 - n) \rho^s + n \rho^f \) is the saturated density of porous medium, \( n \) is the porosity, \( \rho^f \) is the

Figure 6. Illustration of soil composition and interface of each phase
density of solid skeleton, \( \rho^f \) is the density of fluid \( f \), \( u_i \) is the displacement of solid skeleton, \( \Omega \) is the problem domain, \( \alpha = 1 - K/K' \) is the Biot coefficient, \( K \) is the bulk modulus of porous medium, \( K' \) is the bulk modulus of solid grain, \( M \) is regarded as Biot compressibility modulus, and \( P^f \) and \( q^f_i \) are the pore fluid pressure and superficial velocity of fluid, respectively.

The Biot compressibility modulus \( M \) can be denoted by

\[
\frac{1}{M} = \frac{\alpha - n}{K^s} + \frac{n}{K^f}
\]  

(4.3)

where \( K^f \) is the bulk modulus of fluid.

Darcy’s law is employed to describe fluid flow by considering isotropic intrinsic permeability \( k \),

\[
q^f_i = -\frac{k}{\mu^f}(P^f_i - \rho^f g_i)
\]  

(4.4)

where \( \mu^f \) is the dynamic viscosity of fluid.

By using the Biot theory, the total stress \( \sigma_y \) can be decomposed to effective stress of solid phase \( \bar{\sigma}_y \) and pore fluid pressure \( P^f \) by

\[
\sigma_y = \bar{\sigma}_y - \alpha P^f \delta_y
\]  

(4.5)

where \( \delta_y \) is the second-order identity tensor.

Since \( \bar{\sigma}_y \) is fully decomposed from the fluid phase, it can be computed using constitutive models of solid. In this study, the Drucker-Prager plasticity model with associated or non-
associated flow rule and a damage model adapted from [94] are used to represent geomaterial behaviors up to the point of separation (see Appendix A and Appendix B).

The corresponding boundary and initial conditions of the governing equations (Equation (4.1) and Equation (4.2)) are defined as

\[
\begin{align*}
\sigma_{ij} n_j &= h_i & \text{on } \Gamma_h \\
u_i &= u_i^g & \text{on } \Gamma_g \\
-q_i^f n_i^f &= v_s^f & \text{on } \Gamma_s \\
P^f &= P_r^f & \text{on } \Gamma_r \\
u_i(x,t)|_{t=0} &= u_i^0(x) & \text{in } \Omega \\
\dot{u}_i(x,t)|_{t=0} &= \dot{u}_i^0(x) & \text{in } \Omega \\
P^f(x,t)|_{t=0} &= P_0^f(x) & \text{in } \Omega
\end{align*}
\] (4.6)

where \( n_j \) and \( n_i^f \) are the unit normal vectors in the outward direction of corresponding boundaries \( \Gamma_h \) and \( \Gamma_s \). \( h_i \) is the prescribed traction on \( \Gamma_h \), \( u_i^g \) is the prescribed displacement on \( \Gamma_g \). \( v_s^f \) is the prescribed fluid inflow on \( \Gamma_s \). \( P_r^f \) is the prescribed pore fluid pressure on \( \Gamma_r \). \( \Gamma \) is the boundary of \( \Omega \) and has the following relationships: \( \Gamma_h \cup \Gamma_g = \Gamma \), \( \Gamma_h \cap \Gamma_g = \emptyset \), \( \Gamma_s \cup \Gamma_r = \Gamma \), and \( \Gamma_s \cap \Gamma_r = \emptyset \). \( u_i^0(x) \) is the initial value of displacement of the porous medium. \( \dot{u}_i^0(x) \) is the initial value of velocity of the porous medium. \( P_0^f(x) \) is the initial value of pore fluid pressure.

The fluid phase considered in this work is only water, and hence the superscript \( f \) will be replaced by \( w \) hereafter. Additionally, the superscript \( w \) on the pore water pressure \( P^w \) and superficial velocity of water \( q_i^w \) will be dropped for simplicity.

Applying the boundary conditions (4.6) into the governing equations (Equation (4.1) and Equation (4.2)), the variational equations of the \( u-p \) formulation are obtained as
\[ \int_{\Omega} \delta u_{i,j} \sigma_{ij} d\Omega - \int_{\Omega} \delta u_{i,j} \alpha P d\Omega + \int_{\Omega} \delta u_{i} \rho \dot{u}_{i} d\Omega = \int_{\Gamma_{h}} \delta u_{i} h d\Gamma + \int_{\Omega} \delta u_{i} b d\Omega \quad (4.7) \]

\[ \int_{\Omega} \delta P \alpha \dot{u}_{i,j} d\Omega + \int_{\Omega} \delta P \frac{\dot{P}}{M} d\Omega + \int_{\Omega} \delta P \frac{k}{\mu^w} P_d d\Omega = \int_{\Gamma_{r}} \delta P \nu^w d\Gamma + \int_{\Omega} \delta P \frac{k}{\mu^w} \rho^w g d\Omega \quad (4.8) \]

with corresponding Galerkin approximation as follows

\[ \int_{\Omega} \delta u^h_{i,j} \sigma^h_{ij} d\Omega - \int_{\Omega} \delta u^h_{i,j} \alpha P^h d\Omega + \int_{\Omega} \delta u^h_{i} \rho \dot{u}^h_{i} d\Omega = \int_{\Gamma_{h}} \delta u^h_{i} h d\Gamma + \int_{\Omega} \delta u^h_{i} b d\Omega \quad (4.9) \]

\[ \int_{\Omega} \delta P^h \alpha \dot{u}^h_{i,j} d\Omega + \int_{\Omega} \delta P^h \frac{\dot{P}^h}{M} d\Omega + \int_{\Omega} \delta P^h \frac{k}{\mu^w} P^h_d d\Omega = \int_{\Gamma_{r}} \delta P^h \nu^w d\Gamma + \int_{\Omega} \delta P^h \frac{k}{\mu^w} \rho^w g d\Omega \quad (4.10) \]

where superscript \( h \) denotes approximated function of the corresponding term.

### 4.2 Semi-Lagrangian Reproducing Kernel For Saturated Porous Media

There are two cases of \( u-p \) formulation considered in this work: dynamic analysis and quasi-static analysis. The formulations are as shown in Equation (4.1) and Equation (4.2) for dynamic case. However, for quasi-static case, the inertial term in Equation (4.1) is omitted, which is only suitable with slow motion phenomena [93]. Both cases are spatially discretized by the semi-Lagrangian RK approximation but temporal discretization for each case is different. Implicit temporal discretization is employed for quasi-static case, whereas explicit time discretization is used for dynamic analysis due to considerably higher efficiency as this numerical framework is used for larger scale and faster motion problems such as landslides. For landslide problems,
similar contact algorithms as described in Section 3.6 are employed, the only difference is that the effective stress $\bar{\sigma}_{ij}$ is used in all equations in Section 3.6 instead of the total stress $\sigma_{ij}$.

4.2.1 Spatial Discretization

The approximated displacement $u^b_i$, pore water pressure $P^b$, and their spatial derivatives with respect to current configuration in Equation (4.9) and Equation (4.10) are spatially discretized by the semi-Lagrangian RK approximation (3.21) as

$$u^b_i(x,t) = \sum_{I=1}^{NP} \Psi_I(x)d^b_I(t), \quad u^b_{i,j}(x,t) = \sum_{I=1}^{NP} \Psi_{I,j}(x)d^b_{I}(t)$$ (4.11)

$$P^b(x,t) = \sum_{I=1}^{NP} \Psi_I(x)p_I(t), \quad P^b_{I,j}(x,t) = \sum_{I=1}^{NP} \Psi_{I,j}(x)p_{I}(t)$$ (4.12)

where $d^b_I(t)$ and $p_I(t)$ are the nodal coefficients of displacement and pore water pressure, respectively. Notice that, for simplicity and efficiency, the same semi-Lagrangian RK approximation $\Psi_I(x)$ is used for displacement and pore pressure, which is a well-known issue that can create numerical oscillations in pressure field and ultimately lead to numerical instability, especially for nearly or fully incompressible problems. This is due to the choice of interpolations of displacement and pore pressure that violate the \textit{inf-sup} or Ladyzhenskaya-Babuška-Brezzi (LBB) condition [95]–[97]. However, the domain integration methods used herein are nodal integration methods (see Section 3.5 and Section 4.2.3) that under-integrates the functions, which makes it similar to the reduced integration technique proposed in FEM [93] to remedy the LBB condition issue. As a result, no instability from the violation of LBB condition has been observed.
in any of the numerical examples shown in this study. It is however reported in [98] that the instability is still observed for problems with low permeability or under undrained conditions, when RK approximation with SCNI is employed.

The temporal derivatives of approximated displacement $u^h_i$ and pore water pressure $P^h_i$ in Equation (4.9) and Equation (4.10) are spatially discretized by the semi-Lagrangian RK approximation (3.21) as

$$\dot{u}^h_i(x,t) = v_i^h(x,t) = \sum_{I=1}^{NP} \left( \Psi_I(x) v_{iI}^h(t) + \bar{\Psi}_I(x) d_{iI}^h(t) \right)$$

$$\ddot{u}^h_i(x,t) = a^h_i(x,t) = \sum_{I=1}^{NP} \left( \Psi_I(x) a_{iI}^h(t) + 2\bar{\Psi}_I(x) v_{iI}^h(t) + \bar{\Psi}_I(x) d_{iI}^h(t) \right)$$

$$\dot{P}^h_i(x,t) = \sum_{I=1}^{NP} \left( \Psi_I(x) \dot{p}_I(t) + \bar{\Psi}_I(x) p_I(t) \right)$$

where $v_i$ denotes the velocity, $v_{iI}^h(t)$ is the nodal coefficient of velocity, $a_i$ is the acceleration, $a_{iI}^h(t)$ is the nodal coefficient of acceleration, and $\bar{\Psi}_I(x)$ is the change of $\bar{\Psi}_I(x)$ with respect to time.

The semi-discrete form of Equation (4.9) and Equation (4.10) are obtained by applying Equation (4.11) - Equation (4.15) as

$$\delta d_{h_i} \int_{\Omega} \Psi_{I,j} \bar{\sigma}_{ij}^h d\Omega - \delta d_{h_i} \int_{\Omega} \Psi_{I,j} \alpha \Psi_{j} d\Omega \rho p_j + \delta d_{h_i} \int_{\Omega} \Psi_{I,j} \rho \Psi_{j} d\Omega a_{iI}$$

$$+ \delta d_{h_i} \int_{\Omega} 2\Psi_{I,j} \rho \bar{\Psi}_{j} d\Omega v_{iI} + \delta d_{h_i} \int_{\Omega} \Psi_{I,j} \bar{\Psi}_{j} d\Omega d_{iI} = \delta d_{h_i} \int_{\Gamma_s} \Psi_{I,j} h_{i} \sigma \Gamma + \delta d_{h_i} \int_{\Omega} \Psi_{I,j} b_{i} d\Omega$$

(4.16)
where Dirichlet boundary conditions are imposed by the boundary singular kernel method [26]. Einstein notation is used in Equation (4.16) and Equation (4.17) for subscripts $I$ and $J$.

4.2.2 Temporal Discretization

4.2.2.1 Dynamic $u$-$p$ Formulation

Consider a dynamic $u$-$p$ formulation discretized as shown in Equation (4.16) and Equation (4.17). For this case, explicit temporal discretization is employed for efficiency of the framework, which will be used for landslide problems in a subsequent chapter. The second order temporal derivative of displacement is discretized by central difference scheme and the first order temporal derivative of pore water pressure is discretized by forward Euler method as follows

$$d_{ij}^{n+1} = d_{ij}^n + \Delta t v_{ij}^n + 0.5 \Delta t^2 a_{ij}^n$$  \hspace{1cm} (4.18)

$$v_{ij}^{n+1} = \hat{v}_{ij}^{n+1} + 0.5 \Delta t a_{ij}^{n+1}$$  \hspace{1cm} (4.19)

$$p_i^{n+1} = p_i^n + \Delta t \hat{p}_i^n$$  \hspace{1cm} (4.20)

where $\Delta t$ is the time step size. Due to the use of explicit schemes, $d_{ij}^{n+1}$ and $p_i^{n+1}$ are known since the beginning of time step $n + 1$. The predicted velocity $\hat{v}_{ij}^{n+1}$ is defined as
\( \dot{v}_{ij}^{n+1} = v_{ij}^n + 0.5\Delta t a_{ij}^n \) \hfill (4.21)

Rewriting Equation (4.16) and Equation (4.17) in matrix forms as

\[
\begin{bmatrix}
M a^{n+1} + G v^{n+1} + \bar{G} d^{n+1}
\end{bmatrix} = \begin{bmatrix} F_{ext} - F_{int} \end{bmatrix}
\] \hfill (4.22)

\[
\begin{bmatrix}
S p^{n+1} + \tilde{G} p^{n+1}
\end{bmatrix} = \begin{bmatrix} \tilde{F}_{ext} - \tilde{F}_{int} - \tilde{G} d^{n+1} \end{bmatrix}
\] \hfill (4.23)

\[
M_{ij} = \int_\Omega \Psi_i \rho \Psi_j d\Omega , \quad G_{ij} = \int_\Omega 2 \Psi_i \rho \bar{\Psi}_j d\Omega,
\]

\[
\bar{G}_{ij} = \int_\Omega \Psi_i \rho \bar{\bar{\Psi}}_j d\Omega , \quad F_{ext}^{int} = \int_\Gamma \Psi_i h_i d\Gamma,
\]

\[
F_{int}^{ext} = \int_\Omega \Psi_i \rho \bar{\bar{\Psi}}_j d\Omega - \int_\Omega \Psi_i \alpha \Psi_j d\Omega \rho_j^{n+1} - \int_\Omega \Psi_i b_i d\Omega ,
\]

where

\[
S_{ij} = \int_\Omega \frac{\Psi_i \Psi_j}{M} d\Omega , \quad \tilde{G}_{ij} = \int_\Omega \frac{\Psi_i \bar{\bar{\Psi}}_j}{M} d\Omega,
\]

\[
\tilde{F}_I^{ext} = \int_\Gamma \Psi_j v_j \bar{\bar{\Psi}} d\Gamma , \quad \tilde{G}_{ij} = \int_\Omega \Psi_i \alpha \bar{\bar{\Psi}}_j d\Omega ,
\]

\[
\tilde{F}_I^{int} = \int_\Omega \Psi_i \alpha \Psi_j d\Omega \rho_j^{n+1} + \int_\Omega \Psi_i \frac{k}{\mu_w} \Psi_j d\Omega \rho_j^{n+1} - \int_\Omega \Psi_i \frac{k}{\mu_w} \rho_j^w g_i d\Omega.
\] \hfill (4.24)

Applying temporal discretization from Equation (4.18) - Equation (4.21) into Equation (4.22) and Equation (4.23) and including mass proportional damping \( C \), the fully discrete equations read

\[
(M + 0.5\Delta t (C + G))a^{n+1} = F_{ext} - F_{int} - (C + G)\dot{v}^{n+1} - \dot{G} d^{n+1}
\] \hfill (4.25)

\[
Sp^{n+1} = \tilde{F}_{ext} - \tilde{F}_{int} - \tilde{G} d^{n+1} - \tilde{G} p^{n+1}
\] \hfill (4.26)

According to [31], the effect of convection term of the semi-Lagrangian RK approximation on the accuracy is trivial, and hence matrices \( G, \bar{G}, \tilde{G}, \) and \( \tilde{G} \) can be negligible resulting in

\[
(M + 0.5\Delta t C)a^{n+1} = F_{ext} - F_{int} - Cv^{n+1}
\] \hfill (4.27)
The lumped mass scheme by the method of row summation is employed for $M$ and $S$ to acquire diagonal matrices and 5% mass proportional damping is used for $C$. This can avoid solving system of equations and significantly improve computational efficiency without impairing accuracy as to be shown in the first numerical example. At each time step, $a_i^{n+1}$ is solved from Equation (4.27), then $v_i^{n+1}$ is updated by Equation (4.19), and $\dot{p}_i^{n+1}$ is subsequently determined by using Equation (4.28).

4.2.2.2 Quasi-Static $u$-$p$ Formulation

The quasi-static $u$-$p$ formulation is considered for a sole purpose of verifying semi-Lagrangian RK approximation for the solution of the $u$-$p$ formulation. The formulation is applicable with slow motion phenomena, and hence it will only be used for benchmark problems. The backward Euler is chosen for the implicit temporal discretization due to its superior temporal stability to the explicit scheme although it is less computationally efficient. The temporal stability analysis of the explicit scheme is performed in Chapter 5, which is shown to be more prone to temporal instability than implicit scheme.

For the quasi-static $u$-$p$ formulation, all inertial terms are neglected and the semi-discrete form of the Equation (4.16) and Equation (4.17) can be rewritten in matrix forms as

\[
(M + 0.5\Delta tC)a_i^{n+1} = F_{\text{ext}}^{n+1} - F_{\text{int}}^{n+1} - \dot{p}_i^{n+1}
\]  

(4.29)

\[
Sp_i^{n+1} = \bar{F}_{\text{ext}}^{n+1} - \bar{F}_{\text{int}}^{n+1}
\]  

(4.30)
\[
K_{ijkl} = \int_{\Omega} \Psi_{i,j} \psi_{k,l}^\text{dmg} \Psi_{j,k} d\Omega, \quad Q_{ilj} = \int_{\Omega} \Psi_{i,j} \alpha \Psi_{j,l} d\Omega,
\]

where
\[
F^*_i = \int_{\Gamma_i} \Psi_i \phi_i d\Gamma + \int_{\Omega} \Psi_i \phi_i d\Omega, \quad H_{ij} = \int_{\Omega} \frac{k}{\mu_w} \Psi_{i,j} \Psi_{j,i} d\Omega,
\]
\[
\tilde{F}^*_i = \int_{\Gamma_i} \Psi_i \psi_i^w d\Gamma + \int_{\Omega} \Psi_i \frac{k}{\mu_w} \rho^w \Phi_i d\Omega
\]

As described in dynamic case, convection terms can be negligible resulted in

\[
Kd^{n+1} - Qp^{n+1} = F^* \tag{4.32}
\]

\[
Q^T v^{n+1} + Sp^{n+1} + Hp^{n+1} = \tilde{F}^* \tag{4.33}
\]

There is only the first order temporal derivative of displacement and pore water pressure in Equation (4.32) and Equation (4.33); therefore, backward Euler method is employed for the temporal discretization of both displacement and pore water pressure. At time step \( n+1 \), the displacement and pore water pressure can be temporally discretized as

\[
d_i^{n+1} = d_i^n + \Delta \nu_i^{n+1} = d_i^n + \Delta d_i^{n+1} \tag{4.34}
\]

\[
p_i^{n+1} = p_i^n + \Delta p_i^{n+1} = p_i^n + \Delta p_i^{n+1} \tag{4.35}
\]

Applying temporal discretization from Equation (4.34) and Equation (4.35) into Equation (4.32) and Equation (4.33), then rearranging the equations to recover symmetry as follows

\[
\begin{bmatrix}
K & -Q \\
-Q^T & -S - \Delta tH
\end{bmatrix}
\begin{bmatrix}
\Delta d_i^{n+1} \\
\Delta p_i^{n+1}
\end{bmatrix}
= \begin{bmatrix}
F^* - Kd^n + Qp^n \\
-\Delta t\tilde{F}^* + \Delta tHp^n
\end{bmatrix} \tag{4.36}
\]
At each time step, the incremental displacement $\Delta d^{n+1}_i$ and incremental pore water pressure $\Delta p_i^{n+1}$ are solved simultaneously from Equation (4.36), then $d^{n+1}_i$ and $p_i^{n+1}$ are updated by Equation (4.34) and Equation (4.35), respectively.

### 4.2.3 $u$-$p$ Stabilized Nodal Integration Method

In this section, stabilized nodal integration methods described in Section 3.5 are extended into the mixed-field frameworks. For stability of the domain integration and applicability to the extreme deformation problems, the SNNI or MSNNI is used in these frameworks and are described here. The smoothed gradient of shape functions are used for both displacement and pore water pressure as

$$
\bar{\varepsilon}^h(x_L) = \sum_{l=1}^{NP} \bar{B}_l(x_L) d_l
$$

$$
\bar{\nabla} P^h(x_L) = \sum_{l=1}^{NP} \bar{B}_l^p(x_L) p_l
$$

with

$$
\bar{B}_l^p(x_L) = \begin{bmatrix}
    \bar{b}_{l1}(x_L) & 0 & 0 & 0 & \bar{b}_{l3}(x_L) & \bar{b}_{l2}(x_L)
    \\
    0 & \bar{b}_{l2}(x_L) & 0 & \bar{b}_{l3}(x_L) & 0 & \bar{b}_{l1}(x_L)
    \\
    0 & 0 & \bar{b}_{l3}(x_L) & \bar{b}_{l2}(x_L) & \bar{b}_{l1}(x_L) & 0
\end{bmatrix}
$$

and

$$
\bar{b}_n(x_L) = \frac{1}{V_L} \int_{\Gamma_L} \Psi_i(x) n_i(x) d\Gamma
$$
where $\bar{\nabla}P^b(x_L)$ is the smoothed gradient of pore water pressure at node $L$.

For dynamic $u$-$p$ formulation, Equation (4.27) and Equation (4.28) are numerically integrated by SNNI as

$$F_I^{\text{ext}} = \sum_{K=1}^{NP} \Psi_I(x_K) \bm{h}(x_K) A_K , \quad \tilde{F}_I^{\text{ext}} = \sum_{K=1}^{NP} \Psi_I(x_K)v^w(x_K) A_K ,$$

$$F_I^{\text{int}} = \sum_{L=1}^{NP} \bar{\bm{B}}_I^T(x_L) \sigma(x_L)V_L - \sum_{L=1}^{NP} \bar{\bm{B}}_I^T(x_L) \alpha \Psi_I(x_L)V_L p^{n+1}_L - \sum_{L=1}^{NP} \Psi_I(x_L) \bm{b}(x_L)V_L ,$$

$$\tilde{F}_I^{\text{int}} = \sum_{L=1}^{NP} \Psi_I(x_L) \alpha \bar{\bm{B}}_I(x_L)V_L v^{n+1}_I + \sum_{L=1}^{NP} \bar{\bm{B}}_I^{p,T}(x_L) \frac{k}{\mu^w} \bar{\bm{B}}_I^p(x_L)V_L p^{n+1}_L - \sum_{L=1}^{NP} \bar{\bm{B}}_I^{p,T}(x_L) \frac{k}{\mu^w} \rho^w \bm{g} V_L$$

where $A_K$ is the boundary integration weight associated with node $K$ and $V_L$ is the nodal volume associated with node $L$.

For quasi-static analysis, Equation (4.36) is numerically integrated by SNNI as

$$K_{IJ} = \sum_{L=1}^{NP} \bar{\bm{B}}_I^T(x_L) C^{\text{dmg}} \bar{\bm{B}}_J(x_L)V_L , \quad Q_{IJ} = \sum_{L=1}^{NP} \bar{\bm{B}}_I^T(x_L) \alpha \Psi_J(x_L)V_L ,$$

$$H_{IJ} = \sum_{L=1}^{NP} \bar{\bm{B}}_I^{p,T}(x_L) \frac{k}{\mu^w} \bar{\bm{B}}_J^p(x_L)V_L , \quad S_{IJ} = \sum_{L=1}^{NP} \Psi_I(x_L)\Psi_J(x_L)V_L ,$$

$$F_I^s = \sum_{K=1}^{NP} \Psi_I(x_K) \bm{h}(x_K) A_K + \sum_{L=1}^{NP} \Psi_I(x_L) \bm{b}(x_L)V_L ,$$

$$\tilde{F}_I^{s} = \sum_{K=1}^{NP} \Psi_I(x_K)v^w(x_K) A_K + \sum_{L=1}^{NP} \bar{\bm{B}}_I^{p,T}(x_L) \frac{k}{\mu^w} \rho^w \bm{g} V_L$$

Nevertheless, low energy modes may still be triggered in transient problems and cause instability; hence, the MSNNI is extended to the $u$-$p$ semi-Lagrangian RK frameworks by adding the following terms
\[
\tilde{\alpha} \sum_{L=1}^{NP} \sum_{c=1}^{Ns_L} \left( \mathbf{B}_j^T (x_L) - \mathbf{B}_j^T (x_c) \right) \mathbf{C}^e \left( \mathbf{B}_j (x_L) - \mathbf{B}_j (x_c) \right) V_c \mathbf{d}_j^{p+1}
\]
\[
\tilde{\alpha}^P \sum_{L=1}^{NP} \sum_{c=1}^{Ns_L} \left( \mathbf{B}_i^{p,T} (x_L) - \mathbf{B}_i^{p,T} (x_c) \right) \frac{k}{\mu} \left( \mathbf{B}_j^p (x_L) - \mathbf{B}_j^p (x_c) \right) V_c \mathbf{p}_j^{p+1}
\]
\[
\tilde{\alpha} \sum_{L=1}^{NP} \sum_{c=1}^{Ns_L} \left( \mathbf{B}_i^T (x_L) - \mathbf{B}_i^T (x_c) \right) \mathbf{C}^e \left( \mathbf{B}_j (x_L) - \mathbf{B}_j (x_c) \right) V_c
\]
\[
\tilde{\alpha}^P \sum_{L=1}^{NP} \sum_{c=1}^{Ns_L} \left( \mathbf{B}_i^{p,T} (x_L) - \mathbf{B}_i^{p,T} (x_c) \right) \frac{k}{\mu} \left( \mathbf{B}_j^p (x_L) - \mathbf{B}_j^p (x_c) \right) V_c
\]

\(\tilde{\alpha}\) and \(\tilde{\alpha}^P\) are the stabilization parameters ranging between 0 and 1, \(\mathbf{C}^e\) is the elastic material tangent tensor, \(V_c\) is the nodal volume associated with subdomain \(c\) and \(\sum_{c=1}^{Ns_L} V_c = V_L\), and \(Ns_L\) is the number of subdomains in the nodal representative domain \(L\). In this study, \(\tilde{\alpha}^p\) is chosen to be equal to \(\tilde{\alpha}\).

Following the same idea described in Section 3.5, VCI can be extended to be used for pore pressure field by modifying the gradient of test function to satisfy the following equation

\[
\hat{\delta} P_i^h q_i^h d\Omega = -\hat{\delta} P_i^h q_i^h d\Omega + \hat{\delta} P_i^h n_i d\Gamma
\]

where \(\hat{\delta} P_i^h\) is the modified gradient of test function \(\delta P_i^h\).

Enforcing Equation (4.43) to achieve linear exactness (i.e., \(q_i^h\) is constant and \(q_i^h\) is 0), the integration constraint can be expressed as

\[
\hat{\psi}_i d\Omega = \hat{\psi}_i n_i d\Gamma
\]
Notice that this is the same form as in Equation (3.57), this is due to the use of equal-order approximation for displacement and pore pressure fields.

The modified gradient of the shape function $\tilde{\Psi}_{i,j}$ can be determined by

$$\tilde{\Psi}_{i,j} = \Psi_{i,j} + R_i \xi_{i,j}$$

(4.45)

where

$$\xi_{i,j} = \left( \int_{\Gamma} \Psi_{i} n_i d\Gamma - \int_{\Omega} \Psi_{i,j} d\Omega \right) / \int_{\Omega} R_i d\Omega$$

(4.46)

4.3 Numerical Examples

Three benchmark problems are given to verify the performance of the explicit and implicit \(u-p\) semi-Lagrangian RK frameworks described in Section 4.2. Unless otherwise noted, the following setups are used: semi-Lagrangian RK with VC-MSNNI; \(2^{sd_m}\) subdomains per node with stabilization parameter of 1 (\(m_{sd}\) is the number of dimension); cubic spline function for kernel function; a normalized support size of 1.5 with linear basis function for semi-Lagrangian RK shape functions.

4.3.1 Consolidation

A one-dimensional consolidation problem subjected to a step load, \(T_0(t) = 1000 \sin(5\pi t)\) when \(t < 0.1\) s and \(T_0 = 1000\) N when \(t \geq 0.1\) s, on the top of the column is analyzed. Linear elasticity is employed for the geomaterial response of solid phase and material parameters are
given in Table III. The bottom of the column is fixed and impervious, whereas water can flow out freely \( (P = 0) \) on the top of the column. The column is 30 m in height and is discretized by 61 nodes. The pore water pressure at the point of interest, which is 18 m from the top, is analyzed from 0 to 30 seconds and compared with the analytical solution given in [93]. The accuracy, when the row summation method is used for \( \mathbf{M} \) and \( \mathbf{S} \) in explicit temporal discretization, is also verified by comparing with the result when using consistent \( \mathbf{M} \) and \( \mathbf{S} \).

<table>
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<th>2( \times 10^7 )</th>
<th>3( \times 10^7 )</th>
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<td>2.5( \times 10^7 )</td>
</tr>
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<td>0.2</td>
<td>0.3</td>
</tr>
<tr>
<td>Density (kg/m(^3)), ( \rho )</td>
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<td>1670</td>
<td>2670</td>
</tr>
<tr>
<td>Biot Coefficient, ( \alpha )</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Biot Compressibility Modulus (Pa), ( M )</td>
<td>3.33( \times 10^8 )</td>
<td>1( \times 10^{16} )</td>
<td>1.1( \times 10^{16} )</td>
</tr>
<tr>
<td>Permeability ( \frac{m^2}{(\text{Pa} \cdot \text{s})} ), ( \frac{k}{\mu^n} )</td>
<td>1.02( \times 10^{-6} )</td>
<td>1( \times 10^{-6} )</td>
<td>1( \times 10^{-7} )</td>
</tr>
</tbody>
</table>

**TABLE III. MATERIAL PARAMETERS OF THE ONE- AND TWO-DIMENSIONAL PROBLEMS**

From Figure 7, the results from the explicit and implicit \( u-p \) semi-Lagrangian RK agree well with the analytical solution. Oscillations can be observed from explicit temporal discretization at the beginning but after the oscillations taper off, the results are in good agreement with the analytical solution. It should be noted that the oscillations at the beginning of the results are due to the use of explicit temporal discretization as opposed to the implicit time integration [93]. The detailed study and discussions of the oscillations in this problem and the choice of Newmark parameters to achieve numerical damping, can be found in [93]. Further, the results from explicit temporal discretization demonstrate that the difference in term of accuracy between lumped scheme and consistent matrices is trivial. However, for temporal stability, it is
shown in Chapter 5 that the use of lumped scheme for $M$ and $S$ matrices has significantly better temporal stability.

Figure 7. Semi-Lagrangian RK results of the one-dimensional consolidation problem

4.3.2 Soil Column with a Sinusoidal Loading

A one-dimensional soil column subjected to a sinusoidal loading, $T_0(t) = 1000(1-\cos 16\pi t)$ N, on the top of the column is analyzed. Material parameters are given in Table III [99], whereas the Biot compressibility modulus $M$ is set to a large value to represent nearly incompressible porous media. Linear elasticity is employed for the geomaterial response of solid phase. The analytical solution of the problem for the fluid-saturated incompressible porous media can be found in [99], which is given as (see Appendix C)

$$u(x,t) = -\frac{1}{\sqrt{a}}\int_0^t T_0(t-\tau)e^{-\frac{b}{2a}\tau}I_0\left(\frac{b\sqrt{\tau^2-ax^2}}{2a}\right)U(\tau-\sqrt{a}x)d\tau$$  \hspace{1cm} (4.47)$$

The above equation is directly employed as the boundary condition for displacement on the bottom of the column, while the top of the column is pervious and the bottom of the column is
impervious. The soil column is 10 m in height and is discretized by 21 nodes. Displacement histories of the top node are plotted against the analytical solution from [99]. From Figure 8, the results from the explicit and implicit \( u-p \) semi-Lagrangian RK agree well with the analytical solution. No oscillation is observed in the results.

![Figure 8. Displacement histories at the top of soil column](image)

### 4.3.3 Footing Load on Semi-Infinite Soil

A two-dimensional strip load on infinite half-space as shown in Figure 9, where the boundary conditions for displacement and pore pressure are depicted, is analyzed. Linear elasticity is employed for the geomaterial response of solid phase and material parameters are given in Table III [100]. The problem is discretized by 61 nodes on each direction. The stability of the \( u-p \) semi-Lagrangian RK, when equal-order interpolation is used, is demonstrated in this problem by comparing the contour plot of the pore pressure with conventional FEM result [100].
Figure 10 shows that a stable pore pressure distribution can be obtained by using equal-order implicit $u$-$p$ semi-Lagrangian RK, whereas unstable result is obtained from Q1-P1 FEM (cf. [100]). However, temporal instability is observed in explicit $u$-$p$ semi-Lagrangian RK result, which supports the von Neumann prediction given in Chapter 5 that the unconditionally unstable result is expected for the set of material parameters used in this problem, when explicit temporal discretization is employed. As a result, the Biot compressibility modulus $M$ is reduced to $1.1 \times 10^{12}$ (determined from Chapter 5) and a stable result is obtained (Figure 11).
Figure 11. Contour plot of pore pressure with reduced Biot compressibility modulus using explicit semi-Lagrangian RK

4.4 Conclusions

To properly represent geomaterial behaviors, dynamic and quasi-static $u$-$p$ formulations for saturated porous media are formulated in the framework of the semi-Lagrangian RK. The stabilized nodal integration schemes based on VCI are extended into the framework of the $u$-$p$ semi-Lagrangian RK to stabilize the spurious low energy modes and improve solution accuracy. It is noteworthy that even though an equal-order interpolation is used for displacement and pore pressure, no significant pore pressure oscillation has been observed in the study. Explicit temporal discretization is employed in the framework for efficiency and the effect of using row summation for matrices $M$ and $S$ on accuracy is shown to be negligible. Although, implicit temporal discretization offers better temporal stability, it is only suitable with small-scale problem since the computational cost is considerably higher than the explicit time integration. The results obtained from the presented methods are verified with analytical solutions and FEM results. The methods demonstrate the ability to accurately determine displacement and pore water pressure. However, the temporal stability of the explicit $u$-$p$ semi-Lagrangian RK needs further investigation, and hence the von Neumann stability analysis will be adopted in the next chapter for such purpose.
CHAPTER 5

VON NEUMANN STABILITY ANALYSIS

The temporal stability of the frameworks described in Chapter 4 is analyzed using the von Neumann method. The von Neumann method determines the stability by examining the amplification of the Fourier representation of the numerical errors. It is an effective and commonly used technique to assess the temporal stability without expensive computational cost as in eigen-analysis of the discrete system. The von Neumann analysis was adopted to analyze stability of Biot’s poroelastic equations with an explicit staggered finite difference scheme [101], [102] and with sequential schemes [103], [104]. It was also adopted to analyze the effects of temporal integration and material parameters for the 2-D Biot theory using FEM [105].

Von Neumann method is employed to investigate the temporal stability of the $u$-$p$ formulation for both quasi-static and dynamic cases under the Galerkin RK formulation considering an equal-order interpolation for displacement and pressure fields. For temporal integration in the semi-discrete system, explicit central difference and forward Euler schemes are employed for dynamic case, whereas generalized trapezoidal rule is employed for quasi-static case. The effects on temporal stability due to different integration methods in the $u$-$p$ RK formulation such as DNI, SCNI, SNNI, MSCNI, and MSNNI, are analyzed. The von Neumann result of FEM with full integration is also given as a reference. The influence of the support size of RK shape functions on temporal stability is also studied. The predictions from von Neumann analysis are validated with the corresponding results from full-scale transient analyses. The content of this chapter has been accepted for publication as “Von Neumann Stability Analysis of
the u-p Reproducing Kernel Formulation for Saturated Porous Media” in the computational mechanics journal with my advisor, Sheng-Wei Chi, as the co-author.

This chapter is organized as follows: Von Neumann method for u-p RK formulation is presented in Section 5.1. Stability analysis of the dynamic u-p formulation is performed in Section 5.2. Section 5.3 analyzes temporal stability of the quasi-static u-p formulation. Validations of the von Neumann analyses are demonstrated by numerical study in Section 5.4. Conclusions are given in Section 5.5.

5.1 Semi-Discrete Equations and Fourier Representation

Consider a one-dimensional system of equations (Equation (4.9) and Equation (4.10) after applying approximations) in matrix forms with the absence of body forces and neglecting the boundary conditions

\[
M \dd \dot{d} + K \dd d - Q p = 0
\]  
(5.1)

\[
Q^T d + S \dd \dot{p} + H p = 0
\]  
(5.2)

where

\[
M_{ij} = \int_{\Omega} \xi_i \rho A \xi_j d\Omega
\]

\[
K_{ij} = \int_{\Omega} \xi_i \xi_j EA \xi_j \cdot \xi_j d\Omega
\]

\[
Q_{ij} = \int_{\Omega} \xi_i \alpha A \xi_j d\Omega
\]

\[
S_{ij} = \int_{\Omega} \frac{\xi_i \xi_j}{M} Ad\Omega
\]

\[
H_{ij} = \int_{\Omega} \xi_i \frac{k}{\mu w} A \xi_j d\Omega
\]  
(5.3)
Here $\xi_i$ denotes the shape function of node $I$, which can be RK shape function or standard FE shape function. $E$ is the Young’s modulus and $A$ is the cross-sectional area.

Performing domain integration of each term in Equation (5.3), semi-discrete forms with different spatial discretization and different integration schemes are obtained. The details of the procedure and components of each matrix are given in Appendix D, where a normalized support size of less than 2 is used for demonstration.

The nodal solutions of displacement and pore water pressure at $x = x_i$ and $t = t_n$ are assumed to be of the following form:

$$\begin{bmatrix}
    d^n_i \\
    p^n_i
\end{bmatrix} = e^{i\lambda t} e^{in\eta} \begin{bmatrix} d_0 \\
    p_0
\end{bmatrix}$$

(5.4)

where $i$ is the imaginary unit and $\eta$ is the wave number; $d_0$ and $p_0$ are constants. Now consider the uniform spatial discretization $x_{I+i} = x_i + \Delta x I$ with the uniform time stepping $t_{n+\bar{n}} = t_n + \Delta \bar{m}$, where $\bar{n}$ and $I$ are integers. The nodal solutions of a point at the vicinity of $x_i$ and $t_n$ can be expressed as

$$\begin{bmatrix}
    d^{n+\bar{n}}_{I+I} \\
    p^{n+\bar{n}}_{I+I}
\end{bmatrix} = \gamma^{\bar{n}} e^{i\theta} \begin{bmatrix} d^n_i \\
    p^n_i
\end{bmatrix}$$

(5.5)

where $\gamma = e^{i\lambda t}$ is the amplification factor, $\gamma^{\bar{n}}$ is the $\bar{n}$-th power of $\gamma$, and $\theta = \eta \Delta x$ is the normalized wave number ranging from 0 (infinite wavelength) to $\pi$ (shortest wavelength of the discretization).
Applying nodal solutions (5.5) into Equation (5.1) and Equation (5.2), the Fourier representation of the discrete system equations is obtained; the stability of numerical schemes can then be studied by examining the growth or decay of the Fourier representation as time marches. Taking advantage of uniformity of discretization, the analysis on the \( I \)-th row of the matrix equations is sufficient.

\[
\begin{array}{|c|c|c|}
\hline
\text{Parameter} & \text{FEM with Full Integration} & \text{RK with DNI} \\
\hline
\tilde{m} & \frac{\rho A \Delta x}{3} (2 + \cos \theta) & \rho A \Delta x \left( a_1 + 2a_2 \cos \theta + 2a_3 \cos 2\theta \right)^2 \\
\tilde{k} & \frac{2EA}{\Delta x} (1 - \cos \theta) & \frac{4EA}{\Delta x} \left( m_1 \sin \theta + m_2 \sin 2\theta \right)^2 \\
\tilde{q} & \alpha A i \sin \theta & 2\alpha A i \left( a_1 + 2a_2 \cos \theta + 2a_3 \cos 2\theta \right) \\
& & \left( m_1 \sin \theta + m_2 \sin 2\theta \right) \\
\tilde{s} & \frac{A \Delta x}{3M} (2 + \cos \theta) & \frac{A \Delta x}{M} \left( a_1 + 2a_2 \cos \theta + 2a_3 \cos 2\theta \right)^2 \\
\tilde{h} & \frac{2kA}{\mu'' \Delta x} (1 - \cos \theta) & \frac{4kA}{\mu'' \Delta x} \left( m_1 \sin \theta + m_2 \sin 2\theta \right)^2 \\
\hline
\end{array}
\]

\[ a_1 \equiv \Psi_i(x_j); \quad a_2 \equiv \Psi_j(x_{j-1}); \quad a_3 \equiv \Psi_j(x_{j-2}); \quad m_1 \equiv \Delta x \Psi_{J,x}(x_{j-1}); \quad m_2 \equiv \Delta x \Psi_{J,x}(x_{j-2}) \]

TABLE IV. FOURIER REPRESENTATION OF EACH TERM FOR FEM WITH FULL INTEGRATION AND RK WITH DNI
TABLE V. FOURIER REPRESENTATION OF EACH TERM FOR RK WITH MSCNI AND MSNNI

The coefficient of each term in the \( I \)-th row is summarized in Table IV and Table V, where \( \ddot{m} \), \( \ddot{k} \), \( \ddot{q} \), \( \ddot{s} \), and \( \ddot{h} \) represent the terms \( \Delta [M\ddot{d}] \), \( \Delta [K\ddot{d}] \), \( \Delta [Q^T\ddot{d}] \), \( \Delta [S\ddot{p}] \), and \( \Delta [H\ddot{p}] \), respectively, and \( \begin{bmatrix} \cdots \end{bmatrix}_I \) denotes the \( I \)-th row of the matrix. Table IV gives the Fourier representations of each term for FEM with full integration (to serve as a reference solution) and for RK with DNI, whereas Table V gives those for RK with MSCNI and MSNNI. It is noted that the coefficients in Table IV and Table V are dependent on the choice of RK shape function, such
as the order and the support size. The dependence on the support size for the RK formulation is very distinguishable from the FEM. In the following analysis, the linear basis is used with the support size of $\tilde{a} < 3$ for RK shape function. Nonetheless, the general form of each term for an arbitrary support size of value $N$ is also given in Table IV and Table V as a reference for readers who are interested in using larger support size.

It is also noted that the Fourier representations depend on the nodal smoothing techniques when SCNI/SNNI or MSCNI/MSNNI is used. Figure 12 shows the boundaries of nodal representative domains, where $x_{NB1}$, $x_{NB2}$, and $x_{NB3}$ denotes the boundaries of nodal representative domains for $x_i$, $x_{i-1}$, and $x_{i-2}$, respectively. In this study, the size of the nodal representative domain of SNNI is defined to be equal to the nodal volume, which in turn results in the SCNI and SNNI yielding the same discrete system in the following analysis. For MSCNI/MSNNI, two subdomains are needed for each nodal representative domain. $x_{c_1}$ and $x_{c_2}$ denote the center points of subdomains $c_1$ (left) and $c_2$ (right) of node $J$, respectively (see Figure 12).

![Figure 12. RK shape functions using linear basis with cubic spline function and a normalized support size of 2.5](image-url)
To take advantage of explicit time-stepping schemes, diagonal matrices $M$ and $S$ are preferable to avoid solving a system of simultaneous equations (Equation (5.1) and Equation (5.2)) for $\mathbf{d}^{n+1}$ and $\mathbf{p}^{n+1}$. A lumped scheme such as row-sum is usually employed for such purpose, and hence the coefficients $\hat{m}$ and $\hat{s}$ change to $\rho A \Delta x$ and $A \Delta x / M$, respectively, for all discretization methods and nodal integration schemes. The effects of the lumped scheme on temporal stability are also studied later in the following section.

5.2 Von Neumann Stability Analysis of Dynamic Saturated Biot Theory

The central difference and forward Euler method are employed for displacement and pore water pressure, respectively, which can be described as

$$d_i^{n+1} = d_i^n + \Delta t \ddot{d}_i^n + \frac{\Delta t^2}{2} \dddot{d}_i^n$$

$$\dddot{d}_i^{n+1} = \dddot{d}_i^n + \frac{\Delta t}{2} (\dddot{d}_i^n + \dddot{d}_i^{n+1})$$

$$p_i^{n+1} = p_i^n + \Delta t \dot{p}_i^n$$

Substituting Equation (5.5) into Equation (5.6) - Equation (5.8) leads to

$$(\gamma - 1) d_i^n = \Delta t d_i^n + \frac{\Delta t^2}{2} d_i^n$$

$$(\gamma - 1) \dot{d}_i^n = (1 + \gamma \frac{\Delta t}{2}) \dddot{d}_i^n$$
Applying Equation (5.9) - Equation (5.11) and spatial discretization given in Table IV and Table V into Equation (5.1) and Equation (5.2), the fully discrete system is obtained as

\[
\begin{align*}
\mathbf{A} \mathbf{d}^n &= 
\begin{bmatrix}
\bar{m} (\gamma - 1)^2 + \Delta t^2 \bar{k} \gamma & \Delta t^2 \bar{q} \gamma \\
\frac{\bar{q} (\gamma^2 - 1)}{2\gamma} & (\gamma - 1) \bar{s} + \Delta t \bar{h}
\end{bmatrix}
\begin{bmatrix}
d_i^n \\
p_i^n
\end{bmatrix} = 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\end{align*}
\] (5.12)

For a non-trivial solution of Equation (5.12), the amplification factors can be obtained by requiring

\[
\det (\mathbf{A}_i) = 0
\] (5.13)

A numerical method is considered spectrally temporarily stable [103], [106] if

\[
|\gamma_k| \leq 1
\] (5.14)

and

\[
\text{if } |\gamma_k| = |\gamma_l| = 1, \text{ then } \gamma_k \neq \gamma_l
\] (5.15)

where \(\gamma_k\) are the roots of Equation (5.13) for \(k, l = 1, 2, 3\) with \(k \neq l\) and \(|\cdot|\) denotes the complex norm. As a closed form solution of Equation (5.13) is not feasible, the temporal stability conditions for different schemes will be examined individually and graphically demonstrated in the following subsections.
5.2.1 Effects of Material Parameters on Temporal Stability

To study factors that affect temporal stability in the $u$-$p$ RK formulation, the effects from material parameters on temporal stability are first examined by considering the RK formulation with SCNI/SNNI, a normalized support size of 2.5, a nodal distance of 0.5 m, and lumped $M$ and $S$. Typical ranges of material properties for various soil types are considered [107], [108]. Saturated density $\rho$ and Biot coefficient $\alpha$ are fixed to be 2000 kg/m$^3$ and 1, respectively, since these two parameters of soil in nature vary relatively little compared to the Young’s modulus $E$ (Pa), Biot compressibility modulus, $M$ (Pa), and permeability $k/\mu_w$ ($m^2/(Pa\cdot s)$). Based on the numerical preliminary study, it was observed that $Mk/\mu_w$ has a predominant influence on temporal stability. Therefore, in Table VI, the contours of critical time step from the von Neumann estimation are given in $\log(\Delta t_{cr})$ with $\log(Mk/\mu_w)$ ranging from -4 to 4 and non-dimensional wave number $\theta/\pi$ varying from 0 to 1. Different combinations of values of $E$ and $M$ that are selected from commonly seen soil properties are included in Table VI. Noteworthy observations from the study in Table VI are summarized as follows:

1. $Mk/\mu_w$ considerably affects the temporal stability of the $u$-$p$ formulation. In the range of material properties that are studied, the maximum critical time step for all cases is in the order of $10^{-2}$ and it occurs when $Mk/\mu_w$ is close to $10^2$. When $Mk/\mu_w$ is smaller than $10^2$, the critical time step decreases with the decrease of $Mk/\mu_w$.

2. For a given $Mk/\mu_w$ less than $10^2$, the critical time step decreases with the increase of $M$. 
Note: The temporal stability is undetermined in the white areas due to round-off errors when the solution of Equation (5.13) is sought (See Appendix E). It typically happens when $\frac{Mk}{\mu^\nu}$ or $\Delta t$ is small. Full-scale numerical investigations have shown that, within the areas, the temporal stability can be either conditionally stable or unconditionally unstable.

<table>
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<th>$10^8$</th>
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<tr>
<td>$10^8$</td>
<td><img src="image7" alt="Diagram" /></td>
<td><img src="image8" alt="Diagram" /></td>
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<tr>
<td>$10^9$</td>
<td><img src="image10" alt="Diagram" /></td>
<td><img src="image11" alt="Diagram" /></td>
<td><img src="image12" alt="Diagram" /></td>
</tr>
</tbody>
</table>

**TABLE VI. EFFECTS OF MATERIAL PARAMETERS ON CRITICAL TIME STEP SIZES**
3. Similar to the temporal stability in the single-field (displacement) formulation, the critical time step decreases with the increase of $E$. However, such effect is marginal if $E$ is smaller than $M$, for instance, see the contour plots when $M = 10^9$ Pa.

4. Excluding the extreme points at 0 and 1, if $M$ is larger than $10^7$ Pa, the non-dimensional wave number $\theta/\pi$ has marginal effects on the critical time step with a variation less than one order.

Furthermore, the contours of $\log(\Delta t_{cr})$ in Table VI exhibit a similar trend, indicating that for a small $Mk/\mu^w$ the results are uncertain; for a middle value of $Mk/\mu^w$, $\log(\Delta t_{cr})$ varies subtly with $\theta/\pi$; for a large $Mk/\mu^w$, $\log(\Delta t_{cr})$ increases when $\theta/\pi$ is close to 0 or 1. Therefore, in the following sections, three sets of $k/\mu^w$: $10^{-6}$, $10^{-9}$, and $10^{-12}$ m$^2$/Pa·s, with $E=10^7$ Pa, $M = 10^9$ Pa, $\rho = 2000$ kg/m$^3$, and $\alpha = 1$, are selected to investigate how other factors affect temporal stability.

5.2.2 Effect of Discretization Methods and Nodal Integration Schemes on Temporal Stability

Critical time step sizes of FEM, RK with DNI, and RK with MSCNI and MSNNI are studied and compared in consideration of the full range of wave numbers. The nodal distance is set to be 0.5 m. Three values of stabilization parameter $\bar{\alpha}$ are considered for MSCNI/MSNNI: 0 (this recovers SCNI and SNNI), 0.5, and 1. A normalized support size of 2.5 is used for RK shape functions and the row-sum lumped scheme is employed for matrices $\mathbf{M}$ and $\mathbf{S}$. The estimation of
the critical time step size from the von Neumann method is plotted against the non-dimensional wave number $\theta/\pi$ as shown in Figure 13. The results show that RK formulations, regardless of the domain integration schemes, allow a larger critical time step than FEMs, which is particularly noticeable when $\theta/\pi$ is closer to 1. In comparison with different domain integration schemes, Figure 13 also shows that RK with DNI yields smaller critical time step size than RK with SCNI or SNNI (Figure 13a) and that the stabilization parameter $\tilde{\alpha}$ reduces the critical time step size (Figure 13b). For a small $k/\mu^\nu = 10^{-12}$ m$^2$/(Pa·s) (set 3), the temporal stability is undetermined for all discretization schemes and integration methods, which can also be seen from Table VI.

![Figure 13. Critical time step sizes of RK with different domain integration schemes for (a) $k/\mu^\nu = 10^{-6}$ m$^2$/(Pa·s) (set 1) and (b) $k/\mu^\nu = 10^{-9}$ m$^2$/(Pa·s) (set 2)](image)

5.2.3 Effect of Lumped Matrices and Nodal Support Size on Temporal Stability

The effects of using lumped or consistent matrices in conjunction with different nodal support sizes in the RK formulation are studied. Domain integration using DNI and normalized support sizes $\tilde{\alpha}$ of 1.5 and 2.5 are considered. Figure 14 plots the critical time step size versus non-dimensional wave number for different sets of parameters. The results show that lumped
matrices yield better temporal stability than consistent matrices, particularly when non-dimensional wave number is close to 1. Moreover, the increase of critical time step due to the use of lumped matrices becomes more noticeable with the decrease of $k/\mu^w$. It is also interesting to note that when lumped matrices are used, larger nodal support size results in larger critical time step size, whereas when consistent matrices are used, the trend is opposite.

![Figure 14](image)

(a) Critical time step sizes of RK with lumped and consistent matrices for (a) $k/\mu^w = 10^{-6}$ m$^2$/(Pa·s) (set 1) and (b) $k/\mu^w = 10^{-9}$ m$^2$/(Pa·s) (set 2)

### 5.3 Von Neumann Stability Analysis of Quasi-Static Saturated Biot Theory

For slow speed phenomena such as consolidation, the inertial term in Equation (5.1) is negligible, which leads to the following quasi-static $u$-$p$ formulation in the matrix form

\[
Kd - Qp = 0
\]  
\[
Q^Td + Sp + Hp = 0
\]
After applying the RK formulation, the Fourier representation of preceding equations of each term is the same as in Table IV and Table V. The generalized trapezoidal scheme is employed for both displacement and pressure fields, which can be expressed as

\[ d_i^{n+1} = d_i^n + (1 - \bar{\alpha}) \Delta t \dot{d}_i^n + \bar{\alpha} \Delta t \ddot{d}_i^{n+1} \]  
(5.18)

\[ p_i^{n+1} = p_i^n + (1 - \bar{\alpha}) \Delta t \dot{p}_i^n + \bar{\alpha} \Delta t \ddot{p}_i^{n+1} \]  
(5.19)

where \( \bar{\alpha} \in [0, 1] \). It is known as the forward Euler method when \( \bar{\alpha} = 0 \) and as the backward Euler method when \( \bar{\alpha} = 1 \).

Substituting Equation (5.5) into Equation (5.18) and Equation (5.19) gives

\[ (\gamma - 1) d_i^n = (1 - \bar{\alpha} + \bar{\alpha} \gamma) \Delta t d_i^n \]  
(5.20)

\[ (\gamma - 1) p_i^n = (1 - \bar{\alpha} + \bar{\alpha} \gamma) \Delta t p_i^n \]  
(5.21)

Applying Equation (5.20) and Equation (5.21) and spatial discretization from Table IV and Table V into Equation (5.16) and Equation (5.17), the fully discrete system reads

\[
A_x d^n = \begin{bmatrix} \bar{k} & \bar{q} \\ \bar{q} (\gamma - 1) & \bar{s} (\gamma - 1) + \bar{h} \Delta t (1 - \bar{\alpha} + \bar{\alpha} \gamma) \end{bmatrix} \begin{bmatrix} d_i^n \\ p_{i}^{n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]  
(5.22)

For a non-trivial solution of Equation (5.22), the amplification factor can be obtained by requiring

\[ \det(A_x) = 0 \]  
(5.23)

The scheme is temporally stable if
\[ |\gamma| \leq 1 \text{ or } -1 \leq \gamma \leq 1 \quad (5.24) \]

From Equation (5.23), it can be shown that when \( \theta = 0 \) and \( \theta = \pi \), the temporal stability condition is automatically satisfied. The general form of the amplification factor \( \gamma \) can be obtained as

\[
\gamma = 1 - \frac{\tilde{k}h\Delta t}{\tilde{s} - \tilde{q}^2 + kh\Delta t \tilde{\alpha}} \quad (5.25)
\]

It can be shown from Equation (5.25) that \( \gamma \leq 1 \) holds regardless of the schemes since \( \tilde{k}, \tilde{s}, \tilde{h}, \) and \( -\tilde{q}^2 \) are positive. The remaining condition that needs to be considered for temporal stability is \( \gamma \geq -1 \), which leads to

\[
\tilde{\alpha} \geq 0.5 - \frac{\tilde{k}\tilde{s} - \tilde{q}^2}{\tilde{k}h\Delta t} \quad (5.26)
\]

The inequality in (5.26) indicates that the scheme is unconditionally stable when \( \tilde{\alpha} \geq 0.5 \) is used. Furthermore, \( \gamma \geq 0 \) is required for non-oscillatory solutions. From Equation (5.25), it leads to

\[
\tilde{\alpha} \geq 1 - \frac{\tilde{k}\tilde{s} - \tilde{q}^2}{\tilde{k}h\Delta t} \quad (5.27)
\]

It is noted from the above inequality that \( \tilde{\alpha} = 1 \) guarantees no temporal oscillations. Equation (5.26) and Equation (5.27) give similar temporal stability conditions as provided in [105], [109] for FEM; however, here it has been shown that they are valid for \( u-p \) RK with nodal integration methods as well.
When the implicit/explicit weight \( \bar{\alpha} \) is less than 0.5, the scheme is conditionally stable. The critical time step can be obtained from rearranging (5.26) as

\[
\Delta t \leq \frac{\bar{k} \bar{s} - \bar{q}^2}{kh (0.5 - \bar{\alpha})}
\]  

(5.28)

Substituting the coefficients from Table IV and Table V into the inequality above, the critical time step for each method is given as follows:

**FEM**

\[
\Delta t \leq \frac{2}{3M} \left( 2 + \cos \theta \right) + \frac{\alpha^2}{E} (1 + \cos \theta) \Delta x^2
\]

\[
\frac{4k}{\mu^w} \left( 1 - \cos \theta \right) (0.5 - \bar{\alpha})
\]

(5.29)

**RK with DNI**

\[
\Delta t \leq \left( \frac{a_1 + 2a_2 \cos \theta + 2a_3 \cos 2\theta}{m_1 \sin \theta + m_2 \sin 2\theta} \right)^2 \frac{1}{M} + \frac{\alpha^2}{E} \Delta x^2
\]

\[
\frac{4k}{\mu^w} (0.5 - \bar{\alpha})
\]

(5.30)

**RK with MSCNI/MSNNI**

\[
\Delta t \leq \left( \frac{a_1 + 2a_2 \cos \theta + 2a_3 \cos 2\theta}{a_{45} \sin \theta + a_{56} \sin 2\theta} \right)^2 \frac{1}{M} + \frac{4\alpha^2}{E} \left( a_{45} \sin \theta + a_{56} \sin 2\theta \right)^2
\]

\[
\frac{4k}{\mu^w} (0.5 - \bar{\alpha})
\]

(5.31)

\[
\tilde{E} = E \left[ 4 \left( a_{45} \sin \theta + a_{56} \sin 2\theta \right)^2 + \right]
\]

where

\[
\tilde{\alpha} \left( (a_{14} - (a_{42} - a_{25}) \cos \theta - (a_{53} - a_{36}) \cos 2\theta)^2 + (a_{45} \sin \theta + a_{56} \sin 2\theta)^2 \right)
\]

(5.32)
Inequalities (5.29) - (5.31) suggest that greater critical time step sizes can be achieved as \( \bar{\alpha}, \alpha \), or \( \Delta x \) increases, or \( M, E \), or \( k/\mu^w \) decreases. Notice that the effect of \( Mk/\mu^w \) on critical time step size for the quasi-static case is opposite to the dynamic case in Section 5.2.

Further analyzing inequality (5.29) for FEM, it is noted that the right-hand side of the inequality is minimum as \( \theta = \pi \) which is

\[
\Delta t \leq \frac{\mu^w}{12Mk(0.5 - \bar{\alpha})} \Delta x^2
\]

(5.33)

Inequality (5.33) can be used as a conservative estimation for the critical time step size. One can notice that the effects from \( E \) and \( \alpha \) on the critical time step size completely vanish in this case.

5.3.1 Effect of Discretization Methods and Nodal Integration Schemes on Temporal Stability

The effect of different discretization methods and nodal integration schemes on critical time step size is studied for the quasi-static case with explicit time integration. The same sets of material parameters as used in the dynamic case are considered. The nodal distance is set to be 0.5 m and a normalized support size of 2.5 is used for RK shape functions. The results in Figure 15 show that, regardless of integration schemes, the RK formulation offers larger critical time step than FEM for all \( \theta/\pi \) and the increase of critical time step of the RK formulation over FEM is particularly prominent when \( \theta/\pi \) is close to 1, with over three orders of increase. Among the RK formulations with different integration schemes, RK with strain smoothing, SCNI or SNNI,
offers better temporal stability and thus larger critical time step than RK with DNI. Nonetheless, the additional stabilization parameter in MSCNI and MSNNI marginally reduces the critical time step in comparison with DNI. The above observations are in the similar trend as in the dynamics case, indicating that discretization and domain integration has analogous effects on the temporal stability in both dynamic and quasi-static cases. On the other hand, the observation in Figure 15 that the critical time step increases with the decrease of $k/\mu^w$ is in contrast to the observation in the dynamic case (Figure 13). This is attributed to the different effect of $Mk/\mu^w$ on temporal stability as can be seen by comparing the first remark given in Section 5.2.1 with inequalities (5.29) - (5.31).

![Figure 15](image)

**Figure 15.** The critical time step size estimation from RK with DNI, SCNI/SNNI, and MSCNI/MSNNI for (a) $k/\mu^w = 10^{-6}$ m$^2$/(Pa·s) (set 1), (b) $k/\mu^w = 10^{-9}$ m$^2$/(Pa·s) (set 2), and (c) $k/\mu^w = 10^{-12}$ m$^2$/(Pa·s) (set 3)

### 5.3.2 Effect of Nodal Support Size on Temporal Stability

To study the effect of nodal support size on temporal stability, RK with DNI is employed using the same nodal distance as in Section 5.3.1. As shown in Figure 16, regardless of material constants, a smaller support size yields larger critical time step size, in particular, when $\theta/\pi$ is
close to 1. This is consistent with the observation in the dynamic case when consistent matrices are employed.

Figure 16. Comparison of critical time step sizes using RK with DNI and different nodal support sizes for (a) \( k/\mu = 10^{-6} \text{ m}^2/(\text{Pa} \cdot \text{s}) \) (set 1), (b) \( k/\mu = 10^{-9} \text{ m}^2/(\text{Pa} \cdot \text{s}) \) (set 2), and (c) \( k/\mu = 10^{-12} \text{ m}^2/(\text{Pa} \cdot \text{s}) \) (set 3)

### 5.4 Numerical Study of Stability of the \( u-p \) RK Formulation

Four numerical examples are given to validate the stability prediction from the von Neumann analysis against results from the full-scale analysis. The first numerical example considers the quasi-static \( u-p \) formulation, while the others consider dynamic \( u-p \) formulation. Unless otherwise noted, the following setups are used: RK with SCNI, a normalized support size of 1.5, linear basis function, row-sum lumped scheme for matrices \( M \) and \( S \) for dynamic \( u-p \) formulation, and uniform nodal distributions.
5.4.1 One-Dimensional Quasi-Static $u$-$p$ Formulation with a Single Wave Number

The governing equations of the one-dimensional quasi-static $u$-$p$ formulation are obtained from the general three-dimensional equations (4.1) and (4.2), where the inertial term is neglected, under the assumption of no displacement and no fluid flux in the transverse directions as

$$\bar{E}u_{,xx} - \alpha P_{,x} = 0 \quad ; \quad x \in (0, L)$$  \hspace{1cm} (5.34)

$$\alpha \ddot{u}_{,x} + \frac{\dot{P}}{M} - \frac{k}{\mu} P_{,xx} = 0 \quad ; \quad x \in (0, L)$$  \hspace{1cm} (5.35)

where $\bar{E} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$ and $\nu$ is Poisson's ratio.

The aim of this example is to validate the von Neumann study of the implicit/explicit weight effect on temporal stability as shown in Section 5.3. The problem is designed such that, for each case, the solutions possess only one wave number, which are

$$u = \bar{d}_0 e^{-\omega t} \sin(\eta x)$$  \hspace{1cm} (5.36)

$$P = \bar{p}_0 e^{-\omega t} \cos(\eta x)$$  \hspace{1cm} (5.37)

where $\omega$ is the circular frequency, $\bar{d}_0$ and $\bar{p}_0$ are constants. The following initial and boundary conditions are applied in this problem:

$$u(0, t) = u(L, t) = 0$$
$$P(0, t) = P(L, t) = \bar{p}_0 e^{-\omega t}$$
$$u(x, 0) = \bar{d}_0 \sin(\eta x)$$
$$P(x, 0) = \bar{p}_0 \cos(\eta x)$$  \hspace{1cm} (5.38)

with the same problem setup as problem 2.1 in Table VII.
Three cases of wave number are considered:

Case 1: \( \theta/\pi = \eta \Delta x/\pi = 0.2, \quad \omega = 2.6 \text{ rad/s}, \quad \bar{p}_0 = 2.09 \times 10^3 \text{ Pa} \)

Case 2: \( \theta/\pi = \eta \Delta x/\pi = 0.4, \quad \omega = 10.4 \text{ rad/s}, \quad \bar{p}_0 = 4.19 \times 10^3 \text{ Pa} \)

Case 3: \( \theta/\pi = \eta \Delta x/\pi = 0.6, \quad \omega = 24 \text{ rad/s}, \quad \bar{p}_0 = 6.47 \times 10^3 \text{ Pa} \)

In each case, the implicit/explicit weight \( \bar{\alpha} \) of 0 (explicit), 0.5, and 1 (implicit) are tested by varying time step size in full-scale analyses until critical time step sizes are found. For all of the cases, the results are always stable (tested up to time step size = 2 s) when \( \bar{\alpha} = 0.5 \) or 1 is used and no temporal oscillations are observed when \( \bar{\alpha} = 1 \), which agree with the von Neumann study in Section 5.3 and reports in [105], [109] for FEM. However if \( \bar{\alpha} = 0 \) is used, the result is conditionally stable (Figure 17 - Figure 19), where the von Neumann prediction of critical time step sizes are 0.47 s, 0.14 s, and 0.12 s, respectively. It can be seen from the figures that the von Neumann prediction of critical time step sizes agree well with the critical time step sizes observed from the full-scale analyses.

Figure 17. Normalized pore pressure histories at the middle node by stable and unstable time step sizes for \( \theta/\pi = 0.2 \)
Figure 18. Normalized pore pressure histories at the middle node by stable and unstable time step sizes for $\theta/\pi = 0.4$

Figure 19. Normalized pore pressure histories at the middle node by stable and unstable time step sizes for $\theta/\pi = 0.6$
5.4.2 One-Dimensional Dynamic Problems

The governing equations of the one-dimensional dynamic $u$-$p$ formulation with the absence of body forces are obtained from the general three-dimensional equations (4.1) and (4.2) under the assumption of no displacement and no fluid flux in the transverse directions.

\begin{equation}
\ddot{E}u_{,xx} - \alpha P_{,x} = \rho \ddot{u} \quad ; \quad x \in (0, L)
\end{equation}

\begin{equation}
\alpha \ddot{u}_{,x} + \frac{P}{M} - \frac{k}{\mu} P_{,xx} = 0 \quad ; \quad x \in (0, L)
\end{equation}

Two different problems setups, leading to two different solution characteristics, are considered to validate the predictions from von Neumann method.

5.4.2.1 One-Dimensional Dynamic $u$-$p$ Formulation with a Single Wave Number

This one-dimensional problem is designed such that, for each case, the solution possesses only one wave number. The aim of this study is to validate the effects of lumped/consistent matrices and nodal support size on the critical time step size. The problem is designed with the following initial/boundary conditions

\begin{align*}
  u(0, t) &= u(L, t) = 0 \\
P(0, t) &= P(L, t) = \tilde{p}_0 e^{-\omega t} \\
u(x, 0) &= \tilde{d}_0 \sin(\eta x) \\
\ddot{u}(0, 0) &= -\tilde{d}_0 \omega \sin(\eta x) \\
P(x, 0) &= \tilde{p}_0 \cos(\eta x)
\end{align*}

such that the solutions possess only one wave number, as given as follows:
\[ u = \tilde{d}_0 e^{-\omega t} \sin(\eta x) \] (5.42)

\[ P = \tilde{p}_0 e^{-\omega t} \cos(\eta x) \] (5.43)

The problem setup is shown in Table VII.

Three cases of wave number are considered:

Case 1: \( \theta / \pi = \eta \Delta x / \pi = 0.2, \quad \omega = 2.6 \text{ rad/s}, \quad \tilde{p}_0 = 2.1 \times 10^3 \text{ Pa} \)

Case 2: \( \theta / \pi = \eta \Delta x / \pi = 0.4, \quad \omega = 11 \text{ rad/s}, \quad \tilde{p}_0 = 4.24 \times 10^3 \text{ Pa} \)

Case 3: \( \theta / \pi = \eta \Delta x / \pi = 0.6, \quad \omega = 24 \text{ rad/s}, \quad \tilde{p}_0 = 6.47 \times 10^3 \text{ Pa} \)

<table>
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<tr>
<th>Problem</th>
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<th>2.2</th>
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<td>Young's Modulus (Pa), ( E )</td>
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<td>( 2.01 \times 10^7 )</td>
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<tr>
<td>Poisson's Ratio, ( \nu )</td>
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<td>0.2</td>
</tr>
<tr>
<td>Saturated Density (kg/m(^3)), ( \rho )</td>
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<td>( 1.67 \times 10^3 )</td>
</tr>
<tr>
<td>Biot Coefficient, ( \alpha )</td>
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<td>1</td>
</tr>
<tr>
<td>Biot Compressibility Modulus (Pa), ( M )</td>
<td>( 1 \times 10^8 )</td>
<td>( 1 \times 10^{10} )</td>
</tr>
<tr>
<td>Permeability ( \left( \text{m}^2/(\text{Pa} \cdot \text{s}) \right) ), ( k/\mu^\nu )</td>
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<td>( 1 \times 10^{-6} )</td>
</tr>
<tr>
<td>Porosity, ( n )</td>
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<td>0.33</td>
</tr>
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<td>Length (m), ( L )</td>
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<td>10</td>
</tr>
<tr>
<td>Nodal Distance (m), ( \Delta x )</td>
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<td>0.5</td>
</tr>
<tr>
<td>( \tilde{d}_0 ) (m)</td>
<td>0.001</td>
<td>-</td>
</tr>
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</table>

**TABLE VII. PROBLEM SETUPS OF THE 1-D PROBLEMS**

In each case, critical time step sizes observed from the full-scale analyses are collected by varying normalized support sizes between 1 and 3 along with lumped and consistent schemes. The accuracy and stability of the solutions for each case, when the stable time step size is employed, are demonstrated in Figure 20a - Figure 22a by plotting normalized pore pressure at the middle of the domain against time. Unstable results due to the use of time step size larger than
the critical time steps are also shown in Figure 20b - Figure 22b. It is noted that when \( \theta/\pi \) is large the pressure solutions from the full-scale dynamic analysis are oscillatory due to the transient effects at the initial stage, which is particularly noticeable when an explicit time integration is employed (see [110]). Nonetheless, when a solution is stable, the oscillatory tapers off and the numerical solution converges to the analytical one.

Figure 20. Normalized pore pressure histories at the middle node by stable and unstable time step sizes for \( \theta/\pi = 0.2 \)

Figure 21. Normalized pore pressure histories at the middle node by stable and unstable time step sizes for \( \theta/\pi = 0.4 \)
Figure 22. Normalized pore pressure histories at the middle node by stable and unstable time step sizes for $\theta/\pi = 0.6$

Figure 23. Comparison of the results from von Neumann and full-scale analyses at various normalized support size for (a) case 1, (b) case 2, and (c) case 3

Figure 23 demonstrates the comparison of critical time step sizes between von Neumann predictions and results from full-scale analyses. It can be seen that the stability estimations from von Neumann analysis agree with the results from the full-scale dynamic analysis for both lumped and consistent matrices. For the range of the support sizes that has been tested, the von Neumann analysis offers a tight, conservative estimation of the critical time step. The results also
validate the observation in Section 5.2.3 that lumped matrices yield larger critical time step size than consistent matrices and larger critical time step size can be achieved by increasing and decreasing the support size for lumped and consistent schemes, respectively.

5.4.2.2 One-Dimensional Soil Column with a Sinusoidal Loading

The same numerical example as in Section 4.3.2 is revisited to analyze critical time step size of the explicit scheme for lumped and consistent matrices. The critical time step sizes predicted from the von Neumann are approximately $4.2 \times 10^{-5}$ s and $2.6 \times 10^{-5}$ s for lumped and consistent schemes, respectively, whereas the corresponding critical time step sizes from the full-scale analyses are approximately $3 \times 10^{-5}$ s and $2.2 \times 10^{-5}$ s. Solid displacement histories of the top node are plotted in Figure 24 to demonstrate the stability and accuracy when stable/unstable time step sizes are used. The results of this problem support the conclusions made in the first problem that the von Neumann method yields acceptable predictions and the lumped scheme has better temporal stability than consistent scheme.

![Figure 24](image)

(a) (b)

Figure 24. Displacement histories at the top of soil column by stable and unstable time step sizes
5.4.3 Three-Dimensional Adobe

A three-dimensional adobe brick in drained condition \((P = 0)\) subjected to unconfined compression as shown in Figure 25 is considered. The aim of this example is to demonstrate the accuracy of the proposed numerical framework, as well as the conservative von Neumann estimation of the critical time step size, for three-dimensional problem under finite deformation. The accuracy of the \(u-p\) RK is verified and validated by comparing the result obtained from the method with the results from peridynamics [111] and static FEM [112] as well as the experimental data [112]. The same Drucker-Prager constitutive model as in [112] with material parameters as shown in Table VIII taken directly from [112] and [111]), is employed to describe elastoplastic material response. The adobe is fixed in the vertical direction on the bottom and subjected to prescribed downward displacement on the top with strain rate equal to 0.0333 per second [111]. The adobe is discretized by 150 (5x5x6) nodes.

![Figure 25. Schematic of an adobe prism under unconfined compression](image)

Figure 25. Schematic of an adobe prism under unconfined compression
<table>
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<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s Modulus (Pa), $E$</td>
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</tr>
<tr>
<td>Poisson's Ratio, $\nu$</td>
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<tr>
<td>Density (kg/m$^3$), $\rho$</td>
<td>$1.3 \times 10^3$</td>
</tr>
<tr>
<td>Biot Coefficient, $\alpha$</td>
<td>1</td>
</tr>
<tr>
<td>Biot Compressibility Modulus (Pa), $M$</td>
<td>$4.4 \times 10^{10}$</td>
</tr>
<tr>
<td>Permeability ($m^2/(Pa \cdot s)$), $k/\mu^w$</td>
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</tr>
<tr>
<td>Cohesion (Pa)</td>
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</tr>
<tr>
<td>Friction Angle (°)</td>
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</tr>
</tbody>
</table>

**TABLE VIII. MATERIAL PARAMETERS OF AN ADOBE PRISM**

Figure 26. Stress-strain curves of an adobe prism from numerical methods and experiment

The vertical stress and vertical strain results of each method including experimental data are plotted in Figure 26. From Figure 26, although there are small oscillations in the dynamic $u-p$ RK result due to the transient effect from $0.0333$ (1/s) strain rate, the result agrees well with the static FEM result. The results also show that the dynamic $u-p$ RK formulation is much more accurate than peridynamics under the same loading condition. As for the comparison with experimental data, it should be noted that the focus of [111] is to use Drucker-Prager with no hardening/softening (i.e., perfectly plastic) to represent material response from experimental data,
in which this study directly follows as stated. The $u$-$p$ RK result shown in the Figure 26 is the best response that such constitutive model can represent, which is evident by the result from static FEM.

Additionally, the observed critical time step size of this problem is about $3.7 \times 10^{-4}$ s, while the von Neumann estimation of critical time step size for this problem is $5 \times 10^{-5}$ s. This shows that von Neumann yields slightly conservative result.

### 5.5 Conclusions

In order to investigate the temporal stability of the $u$-$p$ RK formulation, this chapter introduces the von Neumann method and provides the estimation of the critical time step for explicit time integration schemes. Transient analyses are performed to validate the predictions from von Neumann analysis. Despite neglecting the boundary effects and considering a uniformly nodal distribution in one dimension, the von Neumann analysis is shown to be effective in offering a temporal stability assessment without the need of constructing a full-scale model.

The numerical studies using von Neumann analysis show that the $u$-$p$ RK formulation with nodal domain integration schemes in the weak form offers better temporal stability in comparison to FEM reference solutions. This is particularly prominent when the spatial discretization is relatively coarse compared to the wavelength (the non-dimensional wave number is close to 1). The von Neumann analysis is also used to study the influences of material parameters, domain integration techniques, and the support size of the RK in the $u$-$p$ RK formulation on the temporal stability. Important conclusions are summarized as follows.
1. The RK formulations with strain-smoothing stabilized nodal integration schemes show better temporal stability than the counterpart with DNI. However, the gradient stabilization terms added in the strain energy (MSCNI and MSNNI) reduce the critical time step size, especially when a large stabilization parameter is used.

2. Using lumped matrix schemes in the dynamic case yields a larger critical time step size than using consistent matrices, for all discretization methods and nodal integration schemes considered in the study.

3. The support size of RK shape functions greatly influences the temporal stability; the effect is also dependent on the use of lumped or consistent matrices. For lumped matrices, larger nodal support size results in greater critical time step size; however, for consistent matrices, smaller nodal support size has better temporal stability.

4. The permeability has opposite effect on temporal stability depending on whether the quasi-static or dynamic formulation is considered. To have a larger critical time step size, permeability is required to be smaller in the quasi-static case but larger in dynamic case.

5. The implicit/explicit weight in the quasi-static analysis is suggested to be at least 0.5 to ensure unconditionally stable solutions. To guarantee no numerical temporal oscillations, the implicit/explicit weight must be equal to 1 (backward Euler method).
CHAPTER 6

STRONG-FORM COLLOCATION METHOD

Due to the flexibility of meshfree methods that allows constructing continuously differentiable approximation functions up to a desired order, an alternative way to solve Partial Differential Equations (PDEs) is feasible. Instead of solving the problems in the Galerkin weak formulation, the PDEs can directly be solved in the strong form with point collocation methods, in which domain integration is not required and imposition of Dirichlet boundary conditions is straightforward. Highly smoothed approximation functions are required in strong-form collocation method, and hence the infinite order of continuity function, such as the radial basis function (RBF), is commonly employed. RBF was first introduced by Hardy [113] to interpolate scattered data and was also shown to be excellent for interpolating irregular data distribution [114]. It was shown by Madych and Nelson [115] that when interpolating a function with RBFs, the interpolation converges exponentially. In 1990, the RBF was first adopted by Kansa [116], [117] to solve PDEs with the point collocation method. However, due to the nonlocality of the function, the RBF approximation yields a dense discrete system and suffers ill-conditioned issues when the size of the system increases. On the other hand, local approximations, such as Reproducing Kernel (RK) approximation, yield a sparse discrete system, thus has better conditioning. It was introduced by Aluru [118] to solve PDEs using the strong-form collocation method, which is later known as the Reproducing Kernel Collocation Method (RKCM). The convergence of the RKCM was shown by Hu et al. [119] to be algebraic when the RK shape function has a fixed normalized support size. To combine the advantages of RBF and RK
approximations, the localized RBF approximation, constructed under the partition of unity using RK basis functions, was proposed by Chen et al. [120] for the solution of PDEs. The localized RBF approximation yields a spare discrete system and possesses the exponential convergence property.

In this chapter, the RK approximation is introduced to analyze both static and dynamic problems in strong formulation. The review of RBF is given in Section 6.1. The implementation of strong-form collocation method is demonstrated in Section 6.2. The performance of RKCM and RBCM for static hyperelasticity is illustrated in Section 6.3. In Section 6.4, the RKCM is introduced for the \( u-p \) formulation. Conclusions are given in Section 6.5.

6.1 RBF

One of the advantages of RBFs is straightforward construction since each of RBF depends only on the distance between evaluation points and the source point, which can be viewed as the discretization point inserted in the domain of the problem. The function \( u_i \) can be discretized and approximated by a RBF as

\[
u_i^h(x) = \sum_{I=1}^{NS} g_{I}^{RBF}(x)d_I
\]

where \( NS \) is the number of source points. The function \( g_{I}^{RBF}(x) \) is expressed in term of the radial distance from source point \( x_I \) as

\[
g_{I}^{RBF}(x) = g_{I}^{RBF}(\|x - x_I\|)
\]
Some of the commonly used forms of RBFs are multiquadrics (MQ) RBF

\[ g_i^{RBF}(\mathbf{x}) = \left( r_i^2 + c^2 \right)^{-n} \quad ; \quad n = 1, 2, \ldots \]  \hspace{1cm} (6.3)

and Gaussian RBF

\[ g_i^{RBF}(\mathbf{x}) = e^{-\frac{r_i^2}{c^2}} \]  \hspace{1cm} (6.4)

where \( r_i = \| \mathbf{x} - \mathbf{x}_i \| \) and \( c \) is the shape parameter, which affects the rate of convergence, linear dependency and condition number of the system [115]. An example of an Inverse MQ RBF (MQ RBF when \( n = 1 \)) and the effect of parameter \( c \) can be viewed in Figure 27.

![Figure 27](image-url)

(a) \hspace{1cm} (b)

Figure 27. Inverse MQ RBF in 2-D using shape parameter (a) \( c = 6 \) and (b) \( c = 13 \)

## 6.2 Strong-Form Collocation Method

Consider the following boundary value problem

\[ Lu_i(\mathbf{x}) = f_i(\mathbf{x}) \quad \text{in} \quad \Omega \]  \hspace{1cm} (6.5)
\[ B^h u_i(x) = h_i(x) \quad \text{on} \quad \Gamma_h \] 

\[ B^g u_i(x) = u_i^g(x) \quad \text{on} \quad \Gamma_g \] 

where \( L \) is the differential operator in \( \Omega \), \( f_i(x) \) is the force, \( B^h \) is the differential operator on \( \Gamma_h \), \( h_i(x) \) is the prescribed traction, \( B^g \) is the operator on \( \Gamma_g \), \( u_i^g(x) \) is the prescribed displacement, and \( \Gamma = \Gamma_h \cup \Gamma_g \) is the total boundary enclosing \( \Omega \).

As the RBF is continuously differentiable and the continuity of the RK shape function can be constructed as desired using a smooth kernel, \( u_i(x) \) can be approximated by the RBF or RK approximation as

\[ u_i(x) \approx \sum_{I=1}^{NS} g_I(x) d_{I} \] 

where

\[ g_I(x) = \begin{cases} 
\Phi_I(x) & \text{for RKCM} \\
\varphi_I(x) & \text{for RBCM} 
\end{cases} \] 

Let \( \overline{P} \) be a set of collocation points \( \overline{P} = \{ \overline{P}_j \}_{j=1}^{NC} = \{ \overline{P}_1, \overline{P}_2, ..., \overline{P}_{NC} \} \subseteq \Omega \cup \Gamma \), where \( NC \) is the number of collocation points. By enforcing the residual to be zero at the collocation points and applying approximation of \( u_i(x) \) using either RBF or RK, Equation (6.5) - Equation (6.7) can be rewritten as

\[ \sum_{I=1}^{NS} L g_I(\overline{P}_j) d_{I} = f_i(\overline{P}_j) \quad \forall \overline{P}_j \in \Omega \] 

\[ \sum_{I=1}^{NS} B^h g_I(\overline{P}_j) d_{I} = h_i(\overline{P}_j) \quad \forall \overline{P}_j \in \Gamma_h \]
Although a $C^2$ approximation can be used in the collocation method for second order PDEs, higher-order continuity often offers better accuracy when the RK approximation is used [7]. Therefore, the quintic spline function is used as the kernel function in the RKCM. The 1-D kernel function with the quintic spline function is described as

\[
\Phi^{1-D}(z_i) = \begin{cases} 
0 & ; \quad z_i \geq 1 \\
\frac{-81z_i^5}{40} + \frac{81z_i^4}{8} - \frac{81z_i^3}{4} + \frac{81z_i^2}{8} - \frac{81z_i}{40} & ; \quad \frac{2}{3} \leq z_i < 1 \\
\frac{-81z_i^5}{8} + \frac{243z_i^4}{4} - \frac{135z_i^3}{4} + \frac{63z_i^2}{8} - \frac{15z_i}{40} + \frac{17}{3} & ; \quad \frac{1}{3} \leq z_i < \frac{2}{3} \\
\frac{-81z_i^5}{4} + \frac{81z_i^4}{4} - \frac{9z_i^2}{2} + \frac{11}{20} & ; \quad 0 \leq z_i < \frac{1}{3}
\end{cases}
\]  

(6.13)

A multi-dimensional kernel function can be constructed by the tensor product similar to the Equation (3.16).

For optimal solutions of the system of equations (6.10) - (6.12), more collocation points than source points are usually required, which leads to an over-determined system. The least-squares method can be employed to solve the system; however, the errors between the domain and boundaries need to be weighted properly [121]. By applying the weights derived from [121] into the system of equations (6.10) - (6.12), the weighted strong-form collocation method reads

\[
\sum_{I=1}^{NS} B^g I g_I(\bar{P}_I) d_I = f_I(\bar{P}_I) \quad ; \quad \forall \bar{P}_I \in \Omega
\]  

(6.14)

\[
\sqrt{\alpha^h} \sum_{I=1}^{NS} B^h h_I(\bar{P}_I) d_I = \sqrt{\alpha^h} h_I(\bar{P}_I) \quad ; \quad \forall \bar{P}_I \in \Gamma_h
\]  

(6.15)
\[
\sqrt{\alpha^g} \sum_{j=1}^{NS} B^g g_I(\bar{P}_j) d_g = \sqrt{\alpha^e} u^e_i(\bar{P}_j) \quad \forall \bar{P}_j \in \Gamma_g
\] (6.16)

where
\[
\sqrt{\alpha^h} \approx O(1)
\] (6.17)

and
\[
\sqrt{\alpha^g} \approx \begin{cases} 
O(NS) & \text{for Poisson problems} \\
O(NS \times \max(\lambda, \mu)) & \text{for Elasticity problems}
\end{cases}
\] (6.18)

where \( \lambda \) and \( \mu \) are Lamé parameters.

6.3 **Strong-Form Collocation Method for Hyperelasticity**

For finite deformation elasticity problems, consider the following governing equation and boundary conditions in the initial configuration

\[
P_{\dot{g},j} + b^0_i = 0 \quad \text{in} \quad \Omega^0
\] (6.19)

\[
P_{\dot{g}} n_j = h^0_i \quad \text{on} \quad \Gamma^0_h
\] (6.20)

\[
u_i = u^g_i \quad \text{on} \quad \Gamma^0_g
\] (6.21)

where \( \dot{} \) denotes spatial derivative with respect to the initial configuration. \( b^0_i \), \( h^0_i \), and \( u^g_i \) are the body force, prescribed traction, and prescribed displacement in the initial configuration, respectively. \( n_j \) is the unit outward normal vector on \( \Gamma^0_h \). \( P_{\dot{g}} \), denoting the first PK stress, can be obtained from the strain energy density function \( W \) as
\[ P_{ij} = \frac{\partial W}{\partial F_{ij}} \]  

(6.22)

where \( F_{ij} \) is the deformation gradient tensor.

### 6.3.1 Linearization

Since Equation (6.19) - Equation (6.21) involve material nonlinearity and geometric nonlinearity, the Newton-Raphson method will be used to solve the equations. Employing Equation (6.22) and applying linearization to Equation (6.19) and Equation (6.20), the system of equations (6.19) - (6.21) can be written in the incremental form as

\[
\frac{\partial^3 W}{\partial F_{ij} \partial F_{kl} \partial F_{mn}} F_{m,n,j} \Delta u_{k,i} + \frac{\partial^3 W}{\partial F_{ij} \partial F_{kl}} \Delta u_{k,j} = -\frac{\partial^2 W}{\partial X_{j} \partial F_{ij}} F_{k,l} - b_{ij}^0 
\]  

(6.23)

\[
\frac{\partial^2 W}{\partial F_{ij} \partial F_{kl}} \Delta u_{k,l} n_j - L[ h_i^0 ] = -\frac{\partial W}{\partial F_{ij}} n_j + h_i^0 
\]  

(6.24)

\[
\Delta u_i = -u_i + u_i^g 
\]  

(6.25)

The linearization of the traction \( L[ h_i^0 ] \) in Equation (6.24), is expressed depending on the function \( h_i^0 \). The Mooney-Rivlin model will be employed, and the strain energy density function \( W \) is expressed in a decoupled form as

\[
W = W(I_1, I_2) + W(J) = A_{10}(I_1 - 3) + A_{01}(I_2 - 3) + \frac{K}{2} (J - 1)^2 
\]  

(6.26)
where $\bar{W}$ and $\tilde{W}$ are the deviatoric and volumetric parts of the strain energy density function, respectively. $J$ is the Jacobian. $K$ is the bulk modulus. $A_{i0}$ and $A_{01}$ are material parameters, which can be related to Lamé parameters in the initial state by

$$\mu = 2(A_{i0} + A_{01})$$

(6.27)

$$\lambda = K - \frac{2}{3} \mu$$

(6.28)

$\bar{I}_1$ and $\bar{I}_2$ are defined by

$$\bar{I}_1 = I_1 J^{-1/3}$$

(6.29)

$$\bar{I}_2 = I_2 J^{-2/3}$$

(6.30)

where $I_1$, $I_2$, and $I_3$ are the first, second, and third invariants of the right Cauchy-Green deformation tensor, respectively. The derivatives of $I_1$, $I_2$, $I_3$, and $J$ with respect to $F_{ij}$, can be denoted as

$$\frac{\partial I_1}{\partial F_{ij}} = 2F_{ij}$$

(6.31)

$$\frac{\partial I_2}{\partial F_{ij}} = 2(I_1 F_{ij} - F_{ij} C_{ij})$$

(6.32)

$$\frac{\partial I_3}{\partial F_{ij}} = 2J^2 F^{-1}_{ij}$$

(6.33)
\[
\frac{\partial J}{\partial F_{ij}} = JF_{ii}^{-1}
\]  

(6.34)

where \(C_{ij}\) is the Right Cauchy-Green deformation tensor which can be described by

\[
C_{ij} = F_{ij}F_{ij}
\]  

(6.35)

### 6.3.2 Spatial Discretization

Applying approximations \(u_i^h = \sum_{l=1}^{NS} g_l(x) d_{il}\) and \(\Delta u_i^h = \sum_{l=1}^{NS} g_l(x) \Delta d_{il}\), and introducing the weighted strong-form collocation method into Equation (6.23) - Equation (6.25), the final equations can be written in matrix forms as

\[
\begin{bmatrix}
G_1^T \left( g^T d^v \right) \\
\sqrt{\alpha^h} G_2^T \left( g^T d^v \right)
\end{bmatrix} \Delta d = \begin{bmatrix}
R_1 \left( g^T d^v \right) \\
\sqrt{\alpha^v} R_2 \left( g^T d^v \right)
\end{bmatrix}
\] \tag{6.36}

or

\[
G^T d = R
\] \tag{6.37}

where

\[
\sqrt{\alpha^h} \approx O(1)
\] \tag{6.38}

and

\[
\sqrt{\alpha^v} \approx O(\kappa NS) \quad ; \quad \kappa = \max(\lambda^v, \mu^v)
\] \tag{6.39}

\(\lambda^v\) and \(\mu^v\) are the equivalent Lamé parameters at iteration step \(\nu\).

A summary of the procedures can be viewed from the flowchart in Figure 28.
6.3.3 Numerical Examples

The RBCM and RKCM are tested for hyperelasticity problems. The Mooney-Rivlin model with $A_{10} = 10$, $A_{01} = 10$ is employed for hyperelastic material. Both compressible and nearly incompressible cases are analyzed. The inverse MQ RBF is used for RBCM. The quintic...
spline function is employed for the RK shape function with quadratic basis and normalized support size equal to 3, unless otherwise noted. Body force for all examples will be neglected, i.e., $b_i^0 = 0$. For the sake of generality, a self-consistent dimensionless unit system is adopted in the numerical examples.

### 6.3.3.1 Uniaxial Tension

The first example is a uniaxial tension with finite deformation under compressible and nearly incompressible conditions. The schematic and boundary conditions are illustrated in Figure 29a. The bulk modulus $K = 100$ and $K = 10000$ are used to represent compressible and nearly incompressible materials, respectively. The problem is discretized by $4 \times 4$ source points with $7 \times 7$ collocation points, as shown in Figure 29b. The shape parameter $\tilde{c} = 12$ is used for the inverse MQ RBF.

![Figure 29](image.png)

(a) Schematic and boundary conditions of the uniaxial tension problem (b) Discretization of the uniaxial tension problem
Displacement in 1-direction is prescribed on the right side to stretch the problem up to 100% (i.e., \( u^x_1 = 1 \)). The Cauchy stress \( \sigma_{11} \) is plotted against the displacement for both compressible and nearly incompressible cases. As shown in Figure 30, the solutions obtained by the RBCM and RKCM are in well agreement with the analytical solutions in both cases. It is interesting to note that the solution from the RKCM is better than that from the RBCM in both compressible and nearly incompressible cases in this example.

![Graphs](image1)

Figure 30. Comparison of the solutions from RBCM and RKCM in the uniaxial tension test for (a) compressible (\( K = 100 \)) and (b) nearly incompressible (\( K = 10000 \))

### 6.3.3.2 Simple Shear

This example tests the performance of the RBCM and RKCM under simple shear condition for nearly incompressible material. The schematic and boundary conditions of the problem is depicted in Figure 31a. The shape parameter \( \tilde{c} = 8 \) is employed for the inverse MQ RBF. The discretization of the problem is shown in Figure 31b with \( 3 \times 5 \) source points and \( 5 \times 9 \) collocation points. The bulk modulus \( K = 10000 \) is employed for nearly incompressible material.
The specified displacement of 1 in 2-direction ($u_2^\alpha = 1$) is applied on the right boundary for shear strain. Figure 32 shows the plot of shear stress $\sigma_{12}$ versus engineering shear strain $\gamma_{12}$. The numerical results from both methods are in good agreement with the analytical solution, with slightly in favor of RBCM.

![Figure 31](image1.png)  
(a) Schematic and boundary conditions of the simple shear test  
(b) Discretization of the simple shear test

![Figure 32](image2.png)  
Figure 32. Comparison of the solutions from RBCM and RKCM in the simple shear test
6.3.3.3 Infinite-Long Cylinder with Internal Pressure

An infinite long cylinder is modeled by only a quarter of its cross section due to the symmetry condition and the plane strain assumption, as shown in Figure 33a. Prescribed displacement of 7 unit \( u_r^* = 7 \) is applied in the radial direction inside of the cylinder, as depicted in Figure 33a. Both compressible and nearly incompressible materials are considered by using bulk modulus \( K = 100 \) and \( K = 10000 \), respectively. The inverse MQ RBF with shape parameter \( \hat{c} = 12 \) is employed. The normalized support size of RK shape function in this problem is 3.5. RBCM uses \( 6 \times 10 \) source points with \( 11 \times 19 \) collocation points (Figure 33b) for both cases. However, RKCM requires more source points \( (12 \times 20) \) and collocation points \( (23 \times 39) \) for compressible case and suffers from incompressible locking for incompressible case. Finite element method with mixed formulation using four-node quadrilateral with constant pressure (Q4P1) is employed for comparison with the solutions from RBCM and RKCM.

Figure 33. (a) Schematic and boundary conditions of the cylinder with internal pressure (b) RBCM Discretization of the cylinder
For compressible material (Figure 34a), the solutions from both RBCM and RKCM agree well with the analytical solution. Particularly, the result from RBCM with only 60 source points is more accurate than solution from the mixed FEM with 240 nodes. For the incompressible case (Figure 34b), the RBCM with much less number of discretization points demonstrates a better solution accuracy over the mixed FEM.

Figure 34. Comparison of the solutions from RBCM, RKCM, and FEM in infinite-long cylinder with internal pressure test. (a) Compressible (\( K = 100 \)) (b) Nearly incompressible (\( K = 10000 \))

6.4 Reproducing Kernel Collocation Method for Elastodynamic

In this section, the strong-form collocation method using RK approximation is introduced to elastodynamic problems. The method is first tested in the single-field formulation by considering determined system and over-determined system. Two approaches are presented in this section. The first approach is to use similar technique as described in Section 6.2 and solve the domain and boundary equations together, whereas the other approach is to employ the static
condensation technique to remove all boundary condition terms [122]. Subsequently, the RKCM is extended for the solution of the $u$-$p$ formulation for small deformation problems.

### 6.4.1 Single-Field Formulation

Consider the balance of momentum equation

$$\nabla \cdot \sigma + b = \rho \ddot{u} \quad \text{in} \quad \Omega \quad (6.40)$$

and the corresponding boundary and initial conditions

$$\begin{align*}
\sigma \cdot n &= h \quad \text{on} \quad \Gamma_h \\
u &= u^k \quad \text{on} \quad \Gamma_g \\
u(x,t)\big|_{t=0} &= u^0(x) \quad \text{in} \quad \Omega \\
u(x,t)\big|_{t=0} &= \dot{u}^0(x) \quad \text{in} \quad \Omega \end{align*} \quad (6.41)$$

By applying the strong-form collocation method with RK approximation, Equation (6.40) and Equation (6.41) can be written as

$$\begin{bmatrix} M & 0 \\ 0 & \text{K}_h \\ 0 & \text{K}_g \end{bmatrix} \begin{bmatrix} d \\ f_h \\ f_g \end{bmatrix} = \begin{bmatrix} f \\ \dot{f}_h \\ \dot{f}_g \end{bmatrix} \quad (6.42)$$

and

$$\begin{align*}
\Psi(P_j) d \big|_{t=0} &= u^0(P_j) \\
\Psi(P_j) d \big|_{t=0} &= \dot{u}^0(P_j) \quad ; \quad \forall P_j \in \Omega \quad (6.43)
\end{align*}$$
with

\[
M = \begin{bmatrix}
\rho \Psi(\overline{P}_j) \\
\vdots \\
\rho \Psi(\overline{P}_{N_1})
\end{bmatrix}, \quad K = \begin{bmatrix}
L(\Psi(\overline{P}_j)) \\
\vdots \\
L(\Psi(\overline{P}_{N_1}))
\end{bmatrix}, \quad f = \begin{bmatrix}
b(\overline{P}_j) \\
\vdots \\
b(\overline{P}_{N_1})
\end{bmatrix}; \quad \forall \overline{P}_j \in \Omega \quad (6.44)
\]

\[
K_h = \begin{bmatrix}
C^e \nabla \Psi(\overline{P}_{N_1+1}) \cdot n(\overline{P}_{N_1+1}) \\
\vdots \\
C^e \nabla \Psi(\overline{P}_{N_1+N_2}) \cdot n(\overline{P}_{N_1+1})
\end{bmatrix}, \quad f_h = \begin{bmatrix}
h(\overline{P}_{N_1+1}, t) \\
\vdots \\
h(\overline{P}_{N_1+N_2}, t)
\end{bmatrix}; \quad \forall \overline{P}_j \in \Gamma_h \quad (6.45)
\]

\[
K_g = \begin{bmatrix}
\Psi(\overline{P}_{N_1+N_2+1}) \\
\vdots \\
\Psi(\overline{P}_{N_1+N_2+N_3})
\end{bmatrix}, \quad f_g = \begin{bmatrix}
u^g(\overline{P}_{N_1+1}, t) \\
\vdots \\
u^g(\overline{P}_{N_1+N_2+N_3}, t)
\end{bmatrix}; \quad \forall \overline{P}_j \in \Gamma_g \quad (6.46)
\]

where \( C^e \) is the elastic material tangent tensor and \( L \) is the spatial differential operator associated with elastodynamic. \( N_1, N_2, \) and \( N_3 \) are the number of collocation points in \( \Omega \), on \( \Gamma_h \), and on \( \Gamma_g \), with \( N_1 + N_2 + N_3 = NC \).

The central difference scheme is employed for explicit temporal integration of Equation (6.42), the fully discrete equations, considering an over-determined system, read

\[
\begin{bmatrix}
M \\
\sqrt{\alpha^b} K_h \\
\sqrt{\alpha^g} K_g
\end{bmatrix} d^{n+1} = \begin{bmatrix}
\Delta t^2 f + \Delta t^2 Kd^n + 2Md^n - Md^{n-1} \\
\sqrt{\alpha^b} f_h^{n+1} \\
\sqrt{\alpha^g} f_g^{n+1}
\end{bmatrix} \quad (6.47)
\]

with

\[
\Psi(\overline{P}_j) d^n = u^0(\overline{P}_j) \quad (6.48)
\]

\[
\Psi(\overline{P}_j) d^i = \Delta u^0(\overline{P}_j) + \Psi(\overline{P}_j) d^0 \quad ; \quad \forall \overline{P}_j \in \Omega
\]
where \( \sqrt{\alpha^h} \approx O(1) \) and \( \sqrt{\alpha^s} \approx O(NS \times \max(\lambda, \mu)) \) \hspace{1cm} (6.49)

At each time step, \( \mathbf{d}^{n+1} \) can be solved directly from Equation (6.47) similar to the approach shown in Section 6.2. For simplicity, this approach is called “regular RKCM” hereafter.

Another approach to solve Equation (6.47) is to follow the idea from [122] to condense the source/collocation points on boundary conditions out and then perform temporal integration; however, here RK approximation is used instead of RBF. For the remainder of this section, this approach is called “condensed RKCM” for simplicity. For demonstration purpose, the matrix in Equation (6.47) is rewritten as

\[
\begin{bmatrix}
\tilde{K}_{DD} & \tilde{K}_{DB} \\
\tilde{K}_{BD} & \tilde{K}_{BB}
\end{bmatrix}
\begin{bmatrix}
\mathbf{d}_D^{n+1} \\
\mathbf{d}_B^{n+1}
\end{bmatrix} =
\begin{bmatrix}
\tilde{f}_D \\
\tilde{f}_B
\end{bmatrix}
\]

where subscript D and B indicate the points that are in the domain and on the boundaries, respectively. From Equation (6.50), \( \mathbf{d}_B^{n+1} \) can be written as

\[
\mathbf{d}_B^{n+1} = \tilde{K}_{BB}^{-1} \left( \tilde{f}_B - \tilde{K}_{BB} \mathbf{d}_D^{n+1} \right)
\]

Using Equation (6.50) and Equation (6.51), \( \mathbf{d}_D^{n+1} \) can be solved by

\[
\mathbf{d}_D^{n+1} = \left( \tilde{K}_{DD} - \tilde{K}_{DB} \tilde{K}_{BB}^{-1} \tilde{K}_{BD} \right)^{-1} \left( \tilde{f}_D - \tilde{K}_{DB} \tilde{K}_{BB}^{-1} \tilde{f}_B \right)
\]

For each time step, \( \mathbf{d}_D^{n+1} \) is solve by using Equation (6.52), then \( \mathbf{d}_B^{n+1} \) is calculated from Equation (6.51).
To test the performance of the aforementioned approaches, consider the following one-dimensional equation of motion

\[ Eu_{xx} = \rho \ddot{u} \quad \text{in} \quad x \in (0, \pi) \]  

(6.53)

with boundary conditions described as

\[
\begin{align*}
    u(0,t) &= 0 \\
    h(\pi,t) &= -E \sin(\omega t)
\end{align*}
\]

(6.54)

and initial conditions described as

\[
\begin{align*}
    u(x,0) &= 0 \\
    \dot{u}(x,0) &= \omega \sin(x)
\end{align*}
\]

(6.55)

where \( \omega = \sqrt{E/\rho} \) is the elastic wave speed.

The analytical solution of the problem is

\[ u(x,t) = \sin(\omega t)\sin(x) \]  

(6.56)

The problem is uniformly discretized by 21 source points (and 81 collocation points if over-determined approach is used). Young’s modulus \( E \) of 10 MPa and density \( \rho \) of 2000 kg/m\(^3\) are employed. The quadratic basis with quintic spline function and a normalized support size of 2.5 are used for RK shape functions.
Figure 35. Results from regular RKCM considering determined system

Figure 36. Results from regular RKCM considering over-determined system with weights on Neumann boundary $\sqrt{\alpha^h} = 1$ and Dirichlet boundary $\sqrt{\alpha^g} = 10^7$

Figure 37. Results from regular RKCM considering over-determined system without weights on boundaries (i.e., $\sqrt{\alpha^h} = 1$ and $\sqrt{\alpha^g} = 1$)

The results in Figure 35 and Figure 36 demonstrate that the over-determined system yields more accurate solutions than the determined system, when the regular RKCM is employed. However, proper weights on boundaries must be used for the over-determined system to obtain
optimum results (see Figure 36 and Figure 37). This conclusion is the same as in the static analysis [121].

Figure 38. Results from condensed RKCM considering determined system

Figure 39. Results from condensed RKCM considering over-determined system with weights on Neumann boundary $\sqrt{\alpha^h} = 1$ and Dirichlet boundary $\sqrt{\alpha^g} = 10^7$

Figure 40. Results from condensed RKCM considering over-determined system without weights on boundaries (i.e., $\sqrt{\alpha^h} = 1$ and $\sqrt{\alpha^g} = 1$)
From Figure 38 - Figure 40, results from condensed RKCM agree well with the analytical solution. Over-determined system yields slightly more accurate results than determined system. The results also demonstrate that, when over-determined system is used, the effect of boundary weights on result accuracy is not significant for condensed RKCM. This conclusion is similar to the case when RBCM is employed [122]. Additionally, condensed RKCM yields exact solution on Dirichlet boundary condition unlike regular RKCM, where the Dirichlet boundary condition is weakly satisfied, (see Figure 35 - Figure 40).

6.4.2 Two-Field Formulation

Consider the quasi-static $u$-$p$ formulation

\[
\begin{align*}
\nabla \cdot \mathbf{\sigma} - \alpha \nabla \mathbf{P} \cdot \mathbf{I} &= 0 & \text{in } & \Omega \\
\alpha \nabla \cdot \mathbf{u} + \frac{\hat{P}}{M} + \nabla \cdot \mathbf{q} &= 0 & \text{in } & \Omega 
\end{align*}
\]

(6.57)

with its corresponding boundary and initial conditions

\[
\begin{align*}
\mathbf{\sigma} \cdot \mathbf{n} = h & \quad \text{on } & \Gamma_h \\
\mathbf{u} = \mathbf{u}^i & \quad \text{on } & \Gamma_g \\
-\mathbf{q} \cdot \mathbf{n} = \mathbf{v}^w & \quad \text{on } & \Gamma_s \\
\mathbf{P} = \mathbf{P}_r & \quad \text{on } & \Gamma_r \\
\mathbf{u}(\mathbf{x},t) \big|_{t=0} = \mathbf{u}^0(\mathbf{x}) & \quad \text{in } & \Omega \\
\mathbf{P}(\mathbf{x},t) \big|_{t=0} = \mathbf{P}_0(\mathbf{x}) & \quad \text{in } & \Omega 
\end{align*}
\]

(6.58)

where $\mathbf{I}$ is the identity tensor and $\mathbf{q} = -\left(\frac{k}{\mu_w}\right)(\nabla \mathbf{P} - \rho_w \mathbf{g})$ is the superficial velocity of water.

Applying RKCM to Equation (6.57) and Equation (6.58), the semi-discrete equations read
\[
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
\end{bmatrix}
= 
\begin{bmatrix}
K & -Q \\
K_h & 0 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
f_h \\
\end{bmatrix}
\begin{bmatrix}
d \\
Q^* \\
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
H \\
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
H_s \\
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
H_r \\
\end{bmatrix}
\begin{bmatrix}
f_g \\
f_j \\
\end{bmatrix}
\begin{bmatrix}
f_r \\
\end{bmatrix}
\]

(6.59)

\[
\Psi (\bar{P}_j) d_{l=0} - u^0 (\bar{P}_j) \}
\quad \forall \bar{P}_j \in \Omega \text{ and } \forall \bar{P}_j^* \in \Omega
\]

(6.60)

with

\[
Q = \begin{bmatrix}
\alpha \nabla \Psi (\bar{P}_1) \\
\vdots \\
\alpha \nabla \Psi (\bar{P}_{N_1})
\end{bmatrix}
; \quad \forall \bar{P}_j \in \Omega
\]

(6.61)

\[
Q^* = \begin{bmatrix}
\alpha \nabla \Psi (\bar{P}_1) \\
\vdots \\
\alpha \nabla \Psi (\bar{P}_{N_1})
\end{bmatrix}
, \quad S = \begin{bmatrix}
\Psi (\bar{P}_1) \\
\vdots \\
\Psi (\bar{P}_{N_1})
\end{bmatrix}
, \quad H = \begin{bmatrix}
-\frac{k}{\mu_w} \Delta \Psi (\bar{P}_1) \\
\vdots \\
-\frac{k}{\mu_w} \Delta \Psi (\bar{P}_{N_1})
\end{bmatrix}
, \quad f' = \begin{bmatrix}
-\frac{k}{\mu_w} \rho \cdot g \\
\vdots \\
-\frac{k}{\mu_w} \rho \cdot g
\end{bmatrix}
; \quad \forall \bar{P}_j \in \Omega
\]

(6.62)

\[
H_s = \begin{bmatrix}
\frac{k}{\mu_w} \nabla \Psi (\bar{P}_{N_1}^{*+1}) \cdot n^w (\bar{P}_{N_1}^{*+1}) \\
\vdots \\
\frac{k}{\mu_w} \nabla \Psi (\bar{P}_{N_1}^{*+2}) \cdot n^w (\bar{P}_{N_1}^{*+1})
\end{bmatrix}
, \quad f_s = \begin{bmatrix}
v^w_s (\bar{P}_{N_1}^{*+1}, t) + \frac{k}{\mu_w} \rho \cdot g \cdot n^w (\bar{P}_{N_1}^{*+1}) \\
\vdots \\
v^w_s (\bar{P}_{N_1}^{*+2}, t) + \frac{k}{\mu_w} \rho \cdot g \cdot n^w (\bar{P}_{N_1}^{*+1})
\end{bmatrix}
; \quad \forall \bar{P}_j \in \Gamma_s
\]

(6.63)
\[
H_r = \begin{bmatrix}
\Psi \left( \overline{P}^{*}_{N_{1r}^{*}+N_{2r}^{*}+N_{3r}^{*}} \right) \\
\vdots \\
\Psi \left( \overline{P}^{*}_{N_{3r}^{*}+N_{2r}^{*}+N_{1r}^{*}} \right)
\end{bmatrix}, \quad f_r = \begin{bmatrix}
P_r \left( \overline{P}^{*}_{N_{3r}^{*}+N_{2r}^{*}+N_{1r}^{*}} \right) \\
\vdots \\
P_r \left( \overline{P}^{*}_{N_{1r}^{*}+N_{2r}^{*}+N_{3r}^{*}} \right)
\end{bmatrix}; \quad \forall \overline{P}^{*} \in \Gamma_r \quad (6.64)
\]

where $\Delta$ is the Laplace operator and $N_{1r}^{*}, N_{2r}^{*},$ and $N_{3r}^{*}$ are the number of collocation points in $\Omega$, on $\Gamma_s$, and on $\Gamma_r$, respectively, with $N_{1r}^{*} + N_{2r}^{*} + N_{3r}^{*} = NC^{*}$ and superscript $*$ indicates that the corresponding terms are associated with the continuity equation. One may notice that $u$ and $P$ are discretized by the same RK approximation similar to the Galerkin weak form approach in Chapter 4. For efficiency and simplicity, the condensed RKCM with determined system is employed (see Section 6.4.1 for details) resulting in $NC = m_{sd}^{*} NC^{*}$, $NS = m_{sd}^{*} NS^{*}$, and $NC + NC^{*} = NS + NS^{*}$, where $m_{sd}$ is the number of dimension.

The backward Euler is employed for implicit temporal discretization of both displacement and pore pressure in Equation (6.59), the fully discrete equations read

\[
\begin{bmatrix}
K & -Q \\
K_h & 0 \\
K_g & 0 \\
Q^{*} S + \Delta t H & 0 \\
0 & H_s \\
0 & H_r
\end{bmatrix}
\begin{bmatrix}
d^{n+1} \\
p^{n+1}
\end{bmatrix} =
\begin{bmatrix}
0 \\
f_{h}^{n+1} \\
f_{g}^{n+1} \\
Q^{*} d^{n} + Sp^{n} + \Delta f^{*} \\
f_{s}^{n+1} \\
f_{r}^{n+1}
\end{bmatrix} \quad (6.65)
\]

Equation (6.65) can be solved by condensed RKCM in a similar manner as Equation (6.50). For each time step, nodal coefficients of displacement and pore pressure on domain are solved first and then the terms on boundaries are calculated.
The condensed $u$-$p$ RKCM is tested with 1-D consolidation problem, which has the same problem statement as shown in Section 4.3.1. The problem is uniformly discretized by 61 source points for each field (total of 122 source points). Quadratic basis with the quintic spline function and a normalized support size of 2.5 are used for RK shape functions. The pore pressure history at the point 18-m deep from the surface is plotted in Figure 35. As can be seen from the figure, the result from $u$-$p$ RKCM agrees well with analytical solution. No oscillation is observed in the result due to the use of backward Euler for implicit temporal discretization on both displacement and pore pressure fields.

![Graph showing pore pressure history](image)

**Figure 41.** Comparison of the result from condensed $u$-$p$ RKCM and analytical solution

### 6.5 Conclusions

RK and RBF are introduced in the strong-form collocation method for the solutions of hyperelasticity under finite deformation. RBCM and RKCM yield more accurate solutions compared to the solutions obtained from the FEM. In particular, RBCM with single-field formulation is effective for nearly incompressible problem; however, RKCM may suffer from
incompressible locking similar to what has been observed in FEM, and hence require more source points. Additionally, RKCM is introduced to elastodynamic problems. The condensed RKCM demonstrates excellent performance in both single-field elasticity and two-field poromechanics problems. In general, the strong form collocation method exhibits some promising features and can be employed as an alternative approach to solve PDEs. Nevertheless, the third order derivative of the strain energy density function with respect to strain tensors has to be computed, which can pose a difficulty to the method when a complex constitutive model is employed.
CHAPTER 7

GALERKIN MESHFREE METHODS FOR EXTREME GEOTECHNICAL EVENTS

Prediction and assessment of the environmental impact caused by landslide activities requires accurate slope stability analysis and run-out simulation. Due to distinctive characteristics in each landslide process, different numerical techniques are usually employed to model different landslide processes. Continuum-based methods with Lagrangian formulation, such as finite element methods (FEMs), are preferable before crack initiation in slope stability analysis [67]–[69], [123]. On the other hand, discontinuum-based methods, such as discrete element method (DEM) [82] and smoothed particle hydrodynamics (SPH) [22], [23], are common for addressing fluid-like behaviors in run-out simulation [71]–[81]. However, it is ineffective or inaccurate to use one of these numerical methods to model the entire landslide processes [87]–[90].

This chapter presents the performance of the frameworks described in previous chapters to accurately and effectively handle landslide simulations within one mathematical framework. Three numerical examples are demonstrated in Section 7.1 to verify and validate the results from the single-field semi-Lagrangian RK with stabilized nodal integration methods. Section 7.2 illustrates and validates the performance of the $u$-$p$ semi-Lagrangian RK by an experimental result and field data of an actual landslide. Section 7.3 shows another type of application of the $u$-$p$ semi-Lagrangian RK framework by using the method for a munitions penetration problem. Conclusions are given in Section 7.4.


7.1 **Single-Field Formulation**

In this section, the framework described in Chapter 3 is applied for the simulations of landslide activities. The first two numerical examples are slope stability analyses verified with FEM results, whereas the last numerical example is used to validate the run-out simulation of the method with experimental data. Unless otherwise noted, the following setups are used for all numerical examples: semi-Lagrangian RK with VC-MSNNI (MSNNI and VCI) and 8 subdomains per node with stabilization parameter of 1; linear basis with cubic spline function and a normalized support size of 1.5 for semi-Lagrangian RK approximation functions; Drucker-Prager with damage model and non-associated flow rule with dilatancy angle of 0 (see Appendix A).

7.1.1 **Slope Stability Analysis I**

Stability of a slope (Figure 42a) is analyzed and verified with FEM result from Zienkiewicz et al. [124]. Drucker-Prager with associated flow rule is employed, which is the same as in [124]. The critical value of cohesion, when slope is unstable as reported in [124], is employed (Table IX) to compare the slip surface results. Other parameters used in this study are identical to that in [124] with only difference on the use of damage model to determine the slip surface instead of the slip circle analysis used in [124]. The problem is discretized by 25,588 nodes. This study also demonstrates the effects of using different basis orders and domain integration methods on the accuracy and stability of the results.
**Figure 42.** Schematic of the slope

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young's Modulus (Pa), $E$</td>
<td>$2 \times 10^8$</td>
</tr>
<tr>
<td>Poisson's Ratio, $\nu$</td>
<td>0.25</td>
</tr>
<tr>
<td>Cohesion (Pa), $c$</td>
<td>2000</td>
</tr>
<tr>
<td>Friction Angle (°), $\phi$</td>
<td>20</td>
</tr>
<tr>
<td>Damage Parameter: Initiation, $c_2$</td>
<td>0.05</td>
</tr>
<tr>
<td>Damage Parameter: Critical, $c_1$</td>
<td>1</td>
</tr>
<tr>
<td>Density (kg/m$^3$), $\rho$</td>
<td>2039</td>
</tr>
</tbody>
</table>

**TABLE IX.** MATERIAL PROPERTIES OF THE SLOPE

**Figure 43.** Results from semi-Lagrangian RK with Drucker-Prager and damage model using (a) constant basis with MSNNI, (b) linear basis with DNI, (c) linear basis with MSNNI, and (d) linear basis with VC-MSNNI
Figure 44. Comparison between the results from FEM with slip circle analysis (black) and semi-Lagrangian RK with damage model using VC-MSNNI and linear basis function (color)

Figure 43b illustrates the instability occurred due to the rank deficiency when DNI is used as discussed in Section 3.5, whereas the result in Figure 43c is stable due to the use of MSNNI. Further, the accuracy of the result in Figure 43c is much higher compared to the result from Figure 43a, this is because of the use of linear basis function when there are enough neighboring nodes covering an evaluation point. Even higher accuracy can be obtained when the integration is variationally consistent, as can be seen in Figure 43d. The slip surface and displacement contour of the result from FEM [124] is illustrated in Figure 44 by black solid lines and arrows, which is put on top of the result from semi-Lagrangian RK with VC-MSNNI (from Figure 43d). It can be seen from Figure 44 that the slip surface from semi-Lagrangian RK agrees well with the result from FEM, which verifies the capability of the framework to effectively analyzing slope stability.

7.1.2 Slope Stability Analysis II

Another verification of the semi-Lagrangian RK result is performed by comparing slip surface automatically identified by semi-Lagrangian RK and damage model with FEM result
The dimensions of the landscape and its boundary conditions are depicted in Figure 45. The problem is discretized by 151,532 nodes with material parameters as shown in Table X [125].

![Figure 45. Schematic of the landscape](image)

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
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<tbody>
<tr>
<td>Young's Modulus (Pa), $E$</td>
<td>$1 \times 10^8$</td>
</tr>
<tr>
<td>Poisson's Ratio, $\nu$</td>
<td>0.3</td>
</tr>
<tr>
<td>Cohesion (Pa), $c$</td>
<td>7490</td>
</tr>
<tr>
<td>Friction Angle ($^\circ$), $\phi$</td>
<td>15.3</td>
</tr>
<tr>
<td>Damage Parameter: Initiation, $c_2$</td>
<td>0.1</td>
</tr>
<tr>
<td>Damage Parameter: Critical, $c_1$</td>
<td>1</td>
</tr>
<tr>
<td>Density (kg/m$^3$), $\rho$</td>
<td>2040.8</td>
</tr>
</tbody>
</table>

**TABLE X. MATERIAL PROPERTIES OF THE LANDSCAPE**

From Figure 47, the slip surface calculated by semi-Lagrangian RK with damage model agrees well with the FEM deformation result (black line), which is amplified by 25 times due to the difficulty related to mesh distortion in FEM when modeling very large deformation problem [125]. On the other hand, semi-Lagrangian RK with damage model can model run-out simulation with ease (Figure 46). This example shows that the method not only accurately analyzes slope stability but also can effectively simulation landslide propagation.
Figure 46. Slip surface and landslide propagation from semi-Lagrangian RK

Figure 47. Comparison between the results from FEM (black) and semi-Lagrangian RK (color)

7.1.3 Run-Out Simulation

Previous example demonstrates the capability of the semi-Lagrangian RK with damage model to effectively simulate landslide propagation, this example provides a validation for such simulation by comparing the run-out simulation from the method with the experimental data from [76]. The geometry and boundary conditions of the problem are illustrated in Figure 48a with material parameters as described in Table XI [76]. The problem is modeled by 11,518 nodes using two bodies, where one body represents the fixed boundaries and another is soil (Figure 48a).
Figure 48. (a) Schematic of the slope and (b) comparison between semi-Lagrangian RK simulation (color) and experiment (black).

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
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<td>Young’s Modulus (Pa), $E$</td>
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<tr>
<td>Poisson’s Ratio, $\nu$</td>
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<tr>
<td>Cohesion (Pa), $c$</td>
<td>0</td>
</tr>
<tr>
<td>Friction Angle (°), $\phi$</td>
<td>19.8</td>
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<tr>
<td>Damage Parameter: Initiation, $c_2$</td>
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<tr>
<td>Damage Parameter: Critical, $c_1$</td>
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<tr>
<td>Density (kg/m$^3$), $\rho$</td>
<td>2650</td>
</tr>
</tbody>
</table>

**TABLE XI. MATERIAL PROPERTIES OF THE SLOPE**

Figure 48b shows the run-out simulation from semi-Lagrangian RK with damage model (color) and experimental data (black), where the undeformed region is indicated by the stripe area. It can be seen from the figure that the deposition of the simulation agrees well with the experiment and the non-damage region (blue) from damage model matches well with the undeformed region from the experiment. This result shows that semi-Lagrangian RK with damage model can effectively model propagation and deposition stages of landslides.

### 7.2 Two-Field Formulation

The performance of the explicit framework described in Chapter 4 is tested with two landslide simulations. The first example is the centrifugal test and the second example is the field
data from an actual landslide. From the numerical study of these two numerical examples, it has been tested that only the use of SNII is enough to provide accurate and stable results, which is much more efficient than VC-MSNNI used in previous section; therefore, SNII is used for the numerical examples in this section for efficiency and simplicity. Unless otherwise noted, the following setups are used for all numerical examples: semi-Lagrangian RK with SNII; linear basis with cubic spline function and a normalized support size of 1.5 for semi-Lagrangian RK approximation functions; Drucker-Prager with damage model and non-associated flow rule with dilatancy angle of 0 (see Appendix A). for the behaviors of solid phase of porous media.

7.2.1 Vertical-Cut Slope

A validation of a landslide simulation using the explicit $u-p$ semi-Lagrangian RK is conducted by comparing the simulation of a three-dimensional vertical-cut slope (Figure 49) with the experimental result from a centrifuge test [126]. The vertical-cut slope is made of soft clay and is put under the centrifugal acceleration of 150 times of the gravitational acceleration. Material parameters are given in Table XII. The boundary conditions for displacement are shown in Figure 49, while the bottom, left, and rightmost boundaries are impervious and the other boundaries are pervious. The slope is discretized by 97,776 nodes.
Figure 49. Schematic of the vertical-cut slope

<table>
<thead>
<tr>
<th>Property</th>
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</thead>
<tbody>
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<td>Young's Modulus (Pa), $E$</td>
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<tr>
<td>Poisson's Ratio, $\nu$</td>
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<tr>
<td>Density (kg/m$^3$), $\rho$</td>
<td>2000</td>
</tr>
<tr>
<td>Biot Coefficient, $\alpha$</td>
<td>1</td>
</tr>
<tr>
<td>Biot Compressibility Modulus (Pa), $M$</td>
<td>$3.33 \times 10^7$</td>
</tr>
<tr>
<td>Permeability ($\frac{m^2}{Pa \cdot s}$), $k$</td>
<td>$1 \times 10^{-6}$</td>
</tr>
<tr>
<td>Cohesion (Pa), $c$</td>
<td>$3 \times 10^4$</td>
</tr>
<tr>
<td>Friction Angle (°), $\phi$</td>
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<tr>
<td>Damage Parameter: Initiation, $c_2$</td>
<td>0.1</td>
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<tr>
<td>Damage Parameter: Critical, $c_1$</td>
<td>1</td>
</tr>
</tbody>
</table>

**TABLE XII. MATERIAL PROPERTIES OF THE VERTICAL-CUT SLOPE**

From Figure 50 and Figure 51, the simulation from the explicit $u$-$p$ semi-Lagrangian RK has similar failure pattern and deposition as in the experimental result from a centrifuge test. This problem validates the capability of the proposed framework to determine accurate slip surface and predict landslide propagation.
Figure 50. Simulation of the vertical-cut slope from the $u$-$p$ semi-Lagrangian RK

Figure 51. Comparison between (a) the result from $u$-$p$ semi-Lagrangian RK and (b) sketch of the experimental result
7.2.2 Landslide at the Reservoir Area of Xiangjiaba, China

A three-dimensional landslide simulation with problem setup as shown in Figure 52 and Table XIII is modeled to validate the result with the landslide occurred at a reservoir area in Xiangjiaba, China. The occurrence of the landslide is due to the excavation at the toe of the slope (Figure 52) under rainfall conditions [127]. The problem setup and material parameters used in the simulation is taken directly from the information given in [127], where the gaps on the top of

![Figure 52. Problem setup of the simulation of the landslide in China](image)

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young's Modulus (Pa), $E$</td>
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<tr>
<td>Poisson's Ratio, $\nu$</td>
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<td>Density (kg/m$^3$), $\rho$</td>
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<tr>
<td>Biot Coefficient, $\alpha$</td>
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<tr>
<td>Biot Compressibility Modulus (Pa), $M$</td>
<td>$3.33 \times 10^6$</td>
</tr>
<tr>
<td>Permeability ($\frac{m^2}{Pa \cdot s}$), $k/\mu^w$</td>
<td>$1 \times 10^{-6}$</td>
</tr>
<tr>
<td>Cohesion (Pa), $c$</td>
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<td>Damage Parameter: Initiation, $c_2$</td>
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<tr>
<td>Damage Parameter: Critical, $c_1$</td>
<td>1</td>
</tr>
</tbody>
</table>

TABLE XIII. MATERIAL PARAMETERS OF THE SIMULATION OF THE LANDSLIDE IN CHINA
the slope in Figure 52 are used to represent initial soil cracks, which were observed before the landslide occurred. The top boundary is pervious, whereas other boundaries are impervious. The problem is discretized by 99,426 nodes.

Figure 53. Simulation of the landslide in Xiangjiaba from the $u$-$p$ semi-Lagrangian RK

Figure 54. Comparison between (a) simulation from the $u$-$p$ semi-Lagrangian RK and (b) sketch of the landslide at a reservoir area in Xiangjiaba, China

The results from the explicit $u$-$p$ semi-Lagrangian RK (Figure 53 and Figure 54) demonstrate the capability of the framework for the simulation of full-scale landslide processes. The main slip surface from the simulation (Figure 54a) also matches well with the sketch of the actual landslide (Figure 54b) [127]. It should be noted that the simulation cannot capture the sub-
slip surfaces, which is understandable since uniform material parameters are employed in the numerical analysis, whereas the soil slope in the actual site can have non-uniform or locally weaker soil mechanical properties.

7.3 Other Application

Due to the abilities to handle extreme deformation, arbitrary crack initiations, arbitrary contacts, and material separations, as shown in previous section, the \( u-p \) semi-Lagrangian RK with damage model can also be applied to other extreme geotechnical events besides landslides, for instance, this section applies the framework for the simulation of munitions penetrating into the ground. The dimension of the soil is 12x12x6 m modeled by Drucker-Prager with non-associated flow rule (dilatancy angle of 0) and damage model (Table XIV). The body of munitions is a cylinder shape with 1.75-m long and radius of 0.25 m, while the tip of the munitions is in a spherical shape with 0.25 m in radius. The material properties for the munitions are that of the steel (i.e., Young's modulus \( E = 2 \times 10^{11} \) Pa, poisson's ratio \( \nu = 0.25 \), and density \( \rho = 8050 \) kg/m\(^3\)). The initial velocity of the munitions is 150 m/s downward. The \( u-p \) semi-Lagrangian RK with SNNI is employed with 133,291 discretization nodes. Linear basis with cubic spline function and a normalized support size of 1.5 is used for the semi-Lagrangian RK shape functions. All boundaries of the soil are fixed and impervious except the top boundary that is free and pervious. The result of the munitions penetrating into the ground is demonstrated in Figure 55. The figure shows the huge impact from the munitions sends the soil particles flying and makes the ground level slightly higher around the entering point.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s Modulus (Pa), ( E )</td>
<td>( 2 \times 10^8 )</td>
</tr>
<tr>
<td>Poisson’s Ratio, ( \nu )</td>
<td>0.2</td>
</tr>
<tr>
<td>Density (kg/m³), ( \rho )</td>
<td>2000</td>
</tr>
<tr>
<td>Biot Coefficient, ( \alpha )</td>
<td>1</td>
</tr>
<tr>
<td>Biot Compressibility Modulus (Pa), ( M )</td>
<td>( 3.33 \times 10^7 )</td>
</tr>
<tr>
<td>Permeability ( \left( \frac{m^2}{Pa \cdot s} \right) ), ( \frac{k}{\mu^w} )</td>
<td>( 1 \times 10^{-6} )</td>
</tr>
<tr>
<td>Cohesion (Pa), ( c )</td>
<td>( 2 \times 10^4 )</td>
</tr>
<tr>
<td>Friction Angle (°), ( \phi )</td>
<td>20</td>
</tr>
<tr>
<td>Hardening Modulus (Pa), ( H )</td>
<td>( 1 \times 10^8 )</td>
</tr>
<tr>
<td>Damage Parameter: Initiation, ( c_2 )</td>
<td>0.05</td>
</tr>
<tr>
<td>Damage Parameter: Critical, ( c_1 )</td>
<td>1</td>
</tr>
</tbody>
</table>

**TABLE XIV. MATERIAL PARAMETERS OF THE GROUND**

Figure 55. Simulation of the munitions penetrating the soil from the \( u-p \) semi-Lagrangian RK
7.4 Conclusions

The semi-Lagrangian RK with the damage model is introduced for the simulations of the entire landslide activities considering both single-field and two-field formulations. Verification and validation are provided for the single-field framework, which is shown to be effective and accurate for determining slip surface and simulating landslide propagation. For stability and accuracy of the results, linear basis with VC-MSNNI is suggested. The VC-MSNNI is, however, computationally challenging, especially when using it in the \( u-p \) semi-Lagrangian RK framework; therefore, to reduce computational cost, the SNNI is used in the numerical examples of the two-field formulation. This is feasible due to the preliminary numerical studies of these particular numerical examples, which suggests that acceptable stability and accuracy can be obtained just by using SNNI. The \( u-p \) semi-Lagrangian RK framework can also be employed for other geotechnical applications such as munitions penetration into the soil as demonstrated in Section 7.3.
CHAPTER 8

CONCLUSIONS AND FUTURE WORK

8.1 Conclusions

The semi-Lagrangian RK with VC-MSNNI is introduced for a single-field formulation to handle large deformation, material defragmentation, and material separation. The Drucker-Prager with associated and non-associated flow rule and the damage model are developed to represent material behaviors.

To consider the poromechanics of geomaterials, the $u$-$p$ formulation following the Biot theory for saturated porous media is introduced under the semi-Lagrangian RK framework. The VCI and MSNNI are extended into the two-field frameworks to help improve stability and accuracy. Although the equal-order interpolation is employed for displacement and pore pressure, no significant pressure oscillations are observed due to the violation of LBB condition. The backward Euler method is adopted for temporal discretization of quasi-static analysis. For this case, the numerical performance of the framework is studied with benchmark problems and shown to have no oscillations, unlike the result from FEM [100]. In addition, the central difference scheme and forward Euler method are employed for the temporal discretization of dynamic saturated case, in which, for certain sets of material parameters, it suffers from temporal oscillations. These oscillations can either taper off and lead to accurate solutions or increasingly oscillate and ultimately become unstable. This calls for a detailed investigation on the temporal stability of the framework.
The von Neumann stability analysis is employed to analyze the temporal stability of saturated Biot theory. Numerical studies show that the $u-p$ RK with nodal integration schemes yield better temporal stability than FEM, particularly when the non-dimensional wave number is close to 1. The effects of nodal support size, discretization methods, nodal integration schemes, and the stabilization parameter, are studied and shown to influence critical time step size.

The closed-forms of the amplification factor for quasi-static analysis are derived for FEM and RK with nodal integration methods. Unconditionally stable can be achieved when the weight of explicit/implicit Euler method is at least 0.5. No temporal oscillations are guaranteed when the weight is equal to 1. Critical time step sizes for different discretization methods and nodal integration schemes are derived when the weight is less than 0.5.

On the other hand, the closed-forms of the amplification factor for dynamic saturated Biot theory cannot be determined, and hence the stability conditions are analyzed by different combinations within the typical ranges of Young’s modulus, Biot compressibility modulus, and permeability. Due to the limitation of computer precision, the von Neumann result suffers from round-off error when the time step size, permeability, or Biot compressibility modulus is too small. Nevertheless, aside from the aforementioned conditions, the von Neumann results are shown to be accurate and effective in analyzing temporal stability. The stabilized nodal integration schemes offer larger critical time step size than DNI, but high value of stabilization parameter in MSCNI and MSNNI can deteriorate the temporal stability. Regardless of discretization methods and domain integration schemes, lumped matrix schemes yield better temporal stability than consistent matrices. The support size of the RK shape functions also significantly affects temporal stability, which has different trends depending on the use of lumped
or consistent matrices. Larger nodal support size yields higher critical time step size for lumped schemes, whereas smaller support size offers better temporal stability if consistent matrices are used. Smaller permeability yields smaller critical time step size in the dynamic case; however, this effect is opposite if quasi-static formulation is considered (i.e., smaller permeability offers better temporal stability in quasi-static analysis).

As an alternative approach, the RK approximation is introduced in the strong-form collocation method (RKCM) for hyperelasticity problems. Numerical results show that the single-field RKCM yields more accurate solutions in comparison to the FEM using mixed formulation. However, the single-field RKCM suffers from locking in incompressible cases. Additionally, the RKCM is introduced to elastodynamic and poroelasticity problems. The numerical studies demonstrate the promising performance of the RKCM for dynamic problems, which show the potential of the approach as an alternative to the Galerkin-weak-form methods.

For the Galerkin weak formulation, the single-field and two-field frameworks are introduced to the simulations of extreme geotechnical events, such as landslides and munitions penetration to the soil. For single-field formulation, the verifications of the slope stability analyses are given by comparing the results from the method with the FEM results from [124] and [125]. The semi-Lagrangian RK with the damage model is shown to be an accurate method for determination of the slip surfaces. In addition, experimental data of the soil collapse is provided as a validation of the run-out simulation from the method. The result also shows the capability of the framework in simulating landslide propagation. For saturated porous media, experimental data and field data from an actual landslide are used to validate the result from the $u$-$p$ semi-Lagrangian RK. The method shows good performance in accurately determining slip
surface and effectively predicting landslide propagation. In addition, the capability of the framework to other type of geotechnical applications is also demonstrated by a munitions penetration problem.

8.2 Future Work

The future work that can be extended from this thesis should focus on the numerical issues that require further studies and enhancements of the current numerical frameworks for broader applications of the extreme geotechnical events with various conditions. It can be summarized as follows:

1. Unsaturated multi-phase deformable porous media models are suggested to incorporate in the framework of semi-Lagrangian RK. The model takes into account the degree of saturation of soil, which affects the capillary pressure, permeability, and void ratio. This can help expand the applications of the framework to the events when saturation degree of soil is changed (e.g., under rainfall condition).

2. Nodal integration methods can still be improved to enhance stability and efficiency, while sustaining accuracy. Other less computationally expensive stabilized nodal integration schemes, such as the Naturally Stabilized Nodal Integration [128], may be used instead of MSNNI, which is computationally challenging. The VCI can also be improved to maintain the symmetric bilinear form instead of the non-symmetric one, due to the modification of the test function to satisfy the integration constraints. The integration constraints should be inherited when the RK shape functions are constructed, without any
help from conforming domain to be able to apply to extreme geotechnical applications (otherwise, it would be similar to SCNI that requires conforming nodal representative domain and is not preferable for this type of applications).

3. Since the RKCM shows promising results in elastic and hyperelastic regions, considering both static and dynamic analyses, it is strongly recommend to formulate and incorporate inelastic material constitutive models into the single-field and two-field RKCM to be able to apply the method with more sophisticated applications. Plasticity models, such as Drucker-Prager, and the damage model developed herein should be formulated and incorporated under the framework of RKCM to better represent geomaterial behaviors, which can ultimately model extreme geotechnical events similar to the Galerkin-weak-form approaches shown in Chapter 7.

4. Further stability analysis should be performed to confirm the observation about the non-oscillatory result when an equal-order interpolation is employed in the $u$-$p$ semi-Lagrangian RK with stabilized nodal integration, which is unlike the result from standard FEM that exhibits pressure oscillations due to violation of the LBB condition.
APPENDICES
Appendix A

**DRUCKER-PRAGER PLASTICITY WITH DAMAGE MODEL**

One of the most widely adopted constitutive model for granular materials is Drucker-Prager plasticity model [129]–[131]. The yield surface of this isotropic elastoplastic model is dependent on hydrostatic pressure, as can be seen in Figure 56a. The yield function of the model is expressed as

\[
 f = \sqrt{2J_2 + BI_1} - k
\]

where \( B \) and \( k \) are material constants. \( I_1 \) is the first invariant of the stress. \( J_2 \) is the second invariant of the deviatoric stress \( s_{ij} \) denoted by

\[
 J_2 = \frac{1}{2}s_{ij}s_{ji}
\]

![Figure 56. Drucker-Prager yield surface in (a) principal stress space and (b) \( \pi \)-plane](image-url)
Appendix A (continued)

The model has a cone-shape yield surface, which is an approximate yet smooth (except its vertex) version of Mohr-Coulomb yield surface. The yield surface of Drucker-Prager can either inscribe or circumscribe the yield surface of Mohr-Coulomb (Figure 56b). Generally, the latter case is more common; therefore, the material parameters $B$ and $k$ can be related to cohesion $c$ and friction angle $\phi$ as follows:

$$B = \frac{2\sqrt{6} \sin \phi}{3 \left(3 - \sin \phi \right)}$$  \hspace{1cm} (A.3)

$$k = \frac{2\sqrt{6} c \cos \phi}{3 - \sin \phi}$$  \hspace{1cm} (A.4)

In elastic region, Equation (A.1) is always less than zero ($f < 0$); however, when the material undergoes plastic deformation, the following conditions are satisfied:

$$f = 0$$  \hspace{1cm} (A.5)

and

$$\dot{f} = 0$$  \hspace{1cm} (A.6)

Equation (A.6) is regarded as the consistency condition for plastic loading, which ensures the stress to be on the yield surface. The flow rule is employed to determine plastic deformation, it is defined as

$$\dot{\varepsilon}^p = \dot{\lambda} \frac{\partial f}{\partial \sigma_{ij}} ; \hspace{0.5cm} \text{Associated flow rule}$$  \hspace{1cm} (A.7)

$$\dot{\varepsilon}^n = \dot{\lambda} \frac{\partial g}{\partial \sigma_{ij}} ; \hspace{0.5cm} \text{Non-associated flow rule}$$
Appendix A (continued)

where $\dot{\varepsilon}^p$ is the plastic strain rate decomposed from the total strain rate $\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p$ and $\dot{\varepsilon}^e$ is the elastic strain rate. $\dot{\lambda}$ is the rate of Lagrange multiplier (plastic multiplier) defining the magnitude of plastic strain rate, and hence $\dot{\lambda} \geq 0$. When the material is in elastic region, $\dot{\lambda} = 0$ and $f < 0$; however, when it is plastic, $\dot{\lambda} > 0$ and $f = 0$. This satisfies the Kuhn-Tucker condition

$$\dot{\lambda} \geq 0 \quad , \quad f \leq 0 \quad , \quad \dot{\lambda} f = 0$$  \hspace{1cm} (A.8)

The Kuhn-Tucker condition (A.8) is used to prove the consistency condition (A.6).

For non-associated flow rule, the plastic potential function has different form from the yield function. In this study, the plastic potential function takes the following form

$$g = \sqrt{2J_2} + \beta I_1 - k$$  \hspace{1cm} (A.9)

where $\beta$ can be related to dilatancy angle $\psi$ as

$$\beta = \frac{2\sqrt{6}\sin\psi}{3(3-\sin\psi)}$$  \hspace{1cm} (A.10)

Nonlinear hardening is considered in this model by evolving $k$ and $B$ with $\dot{\lambda}$. The sample procedures to compute stress by return mapping algorithm considering the incremental form of Equation (A.7) (using associated flow rule) and Equation (A.9) are listed as follows:

1. Compute $\hat{\sigma}_{ij}^{n+1} = \sigma_{ij}^n + C_{ijkl}^e \Delta \varepsilon_{kl}$

2. If $\hat{f} = \sqrt{2J_2(\hat{\sigma}_{ij}^{n+1}) + B^n(\dot{\lambda}) I_1(\hat{\sigma}_{ij}^{n+1}) - k^n(\dot{\lambda})} < 0$, go to (2.1). Otherwise, go to (2.2)
(2.1) \( \sigma_{ij}^{n+1} = \hat{\sigma}_{ij}^{n+1} \), \( B^{n+1}(\lambda) = B^n(\lambda) \), \( k^{n+1}(\lambda) = k^n(\lambda) \), then exit.

(2.2) Use Newton-Raphson to solve for \( \Delta \lambda \) that will give \( \hat{f}(\Delta \lambda) = 0 \), then go to (3)

\[
(\text{If } \hat{f} \text{ is linear hardening, no iteration is required and } \Delta \lambda = \frac{\hat{f}}{2\mu + H + 9\left(B^n(\lambda)\right)^2 K})
\]

(3) Update \( \sigma_{ij}^{n+1} = \frac{1}{3}\left(I_1(\hat{\sigma}_{ij}^{n+1}) - 9KB^n(\lambda)\Delta \lambda\right)\delta_{ij} + s_{ij}\left(\hat{\sigma}_{ij}^{n+1}\right) - 2\mu \Delta \lambda \frac{s_{ij}\left(\hat{\sigma}_{ij}^{n+1}\right)}{\sqrt{s_{ij}\left(\hat{\sigma}_{ij}^{n+1}\right)s_{jk}\left(\hat{\sigma}_{ij}^{n+1}\right)}}
\]

(4) Update \( B^{n+1}(\lambda) = B^n(\lambda) + \frac{\partial B(\lambda)}{\partial \lambda} \Delta \lambda \) and \( k^{n+1}(\lambda) = k^n(\lambda) + \frac{\partial k(\lambda)}{\partial \lambda} \Delta \lambda \), then exit.

Here \( \hat{f} \) is the trial yield function. \( J_2(\hat{\sigma}_{ij}^{n+1}) \) is the second invariant of the trial deviatoric stress \( s_{ij}(\hat{\sigma}_{ij}^{n+1}) \). \( I_1(\hat{\sigma}_{ij}^{n+1}) \) is the first invariant of the trial stress \( \hat{\sigma}_{ij}^{n+1} \). \( \mu \) is the Lamé parameter. \( H \) is the hardening parameter. \( K \) is the bulk modulus. \( C_{ijkl}^e \) is the elastic tangent modulus. \( \Delta \epsilon_{kl} \) and \( \Delta \lambda \) are the incremental strain and plastic multiplier, respectively.

Nevertheless, the previously mentioned plasticity model cannot represent the separation of materials. Therefore, the plasticity model is incorporated with the damage model adapted from [94], [132], [133] to capture material behavior when it starts to lose strength and ultimately reaches point of rupture. The damage model is proposed to have the following form

\[
\sigma_{ij}^{\text{damage}} = (1-d)\sigma_{ij}^{\text{dev}} + \frac{1}{3}(1-d)(\sigma_{kk}^+ + \sigma_{kk}^-)\delta_{ij}
\]

(A.11)
where $\sigma_{ij}^{\text{dmg}}$ is the stress after taking into the account of damage and $d$ is the damage parameter. Superscript dev, +, and – indicate deviatoric, tensile, and compressive parts of the corresponding stress terms, respectively. The damage parameter is defined by

$$d = \frac{c_1 \left( \eta - c_2 \right)}{\eta (c_1 - c_2)} ; \quad \eta > c_2$$  

(A.12)

where $\eta$ is the norm of the deviatoric strain used as a means to identify material damage, which is described as

$$\eta = \sqrt{\varepsilon_{ij}^{\text{dev}} \varepsilon_{ji}^{\text{dev}}}$$  

(A.13)

The damage parameter $c_2$ specify the initiation point, when material starts to damage (i.e. $d = 0$). The damage parameter $c_1$ specify the critical point, when material is fully damaged (i.e. $d = 1$). For $u$-$p$ formulation, this is also the point when pore water pressure is enforced to be equal to atmospheric pressure ($P_{\text{damage}} = P_{\text{atm}}$). One may notice from Equation (A.11) that only one damage parameter is considered for simplicity. There are also other damage models that have more than one parameters, for instance, the damage model from [134] separates the damage in tension and compression by using two different damage parameters.
Appendix B

UPDATE OF CAUCHY STRESS

The numerical integration of incremental constitutive equation following [91] are carried out to calculate the Cauchy stress at time step \( n+1 \),

\[
\sigma_{ij}^{n+1} = T_{ik}^{n+1} \sigma_{kl}^{n} T_{lj}^{n+1} + C_{ijkl} \Delta \varepsilon_{kl}^{n+0.5}
\]  

(B.1)

where \( T_{ik}^{n+1}, T_{0j}^{n+1} \) are the transformation matrices denoted by

\[
T_{ij}^{n+1} = \delta_{ij} + \left( \delta_{ik} - \frac{1}{2} W_{ik}^{0.5} \right)^{-1} W_{kj}^{n+0.5}
\]  

(B.2)

and

\[
W_{ij}^{n+0.5} = \Delta u_{i,j}^{n+0.5} = \frac{1}{2} \left( \frac{\partial \Delta u_i}{\partial x_j^{n+0.5}} - \frac{\partial \Delta u_j}{\partial x_i^{n+0.5}} \right)
\]  

(B.3)

The incremental strain \( \Delta \varepsilon_{kl}^{n+0.5} \) can be expressed as

\[
\Delta \varepsilon_{kl}^{n+0.5} = \Delta u_{i,j}^{n+0.5} = \frac{1}{2} \left( \frac{\partial \Delta u_i}{\partial x_j^{n+0.5}} + \frac{\partial \Delta u_j}{\partial x_i^{n+0.5}} \right)
\]  

(B.4)

where

\[
\frac{\partial \Delta u_i}{\partial x_j^{n+0.5}} = \frac{\partial \Delta u_i}{\partial X_j} \frac{\partial X_j}{\partial x_j^{n+0.5}} = \frac{\partial \Delta u_i}{\partial X_j} F_{ij}^{-1} \quad \text{for RK}
\]  

(B.5)

and

\[
\frac{\partial \Delta u_i}{\partial x_j^{n+0.5}} = \frac{\partial \Delta u_i}{\partial x_j} \frac{\partial x_j}{\partial x_j^{n+0.5}} = \frac{\partial \Delta u_i}{\partial x_j} F_{ij} \quad \text{for semi-Lagrangian RK}
\]  

(B.6)
Appendix B (continued)

The superscript $n + 0.5$ denotes that the corresponding variables are evaluated from the configuration at time step $n + 0.5$, which can minimize the error [91].
Appendix C

ANALYTICAL SOLUTION OF FLUID-SATURATED INCOMPRESSIBLE POROUS MEDIA

An analytical solution for the displacement of a fluid-saturated incompressible porous media subjected to a loading function \( T_0(t) \) on top is given as [99]

\[
u(x,t) = -\frac{1}{\sqrt{a} \left( \lambda^S + 2\mu^S \right)} \int_0^t T_0(t-\tau) e^{-\frac{b}{2a} I_0 \left( \frac{b\sqrt{\tau^2 - ax^2}}{2a} \right)} U(\tau - \sqrt{ax}) d\tau \quad \text{(C.1)}
\]

with

\[
a = \frac{\left( \rho^F \right)^2 \left( n^F \right)^2 + \left( \rho^S \right)^2 \left( n^S \right)^2}{\left( \lambda^S + 2\mu^S \right)}
\]

\[
b = \frac{S_v}{\left( \lambda^S + 2\mu^S \right)}
\]

\[
S_v = \frac{\left( n^F \right)^2 \gamma^{FR}}{k^F}
\]

where \( I_0 \) is the modified Bessel function of the first kind of zero order and \( U(t) \) is the Heaviside function. The parameters used in above equations are taken from [99]. The relationships of the parameters used herein and those used in [99] can be found in Table XV, which has been used for the numerical example in Section 4.3.2 and Section 5.4.2.2 to convert material parameters from [99] (Table XVI) into material parameters used in this study (Table III).
Appendix C (continued)

<table>
<thead>
<tr>
<th>Parameters in [99]</th>
<th>Parameters in this study</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n^S )</td>
<td>( 1-n )</td>
</tr>
<tr>
<td>( n^F )</td>
<td>( n )</td>
</tr>
<tr>
<td>( \rho^S + \rho^F )</td>
<td>( \rho )</td>
</tr>
<tr>
<td>( E )</td>
<td>( E/(1-n) )</td>
</tr>
<tr>
<td>( \nu )</td>
<td>( \nu )</td>
</tr>
<tr>
<td>( k^F )</td>
<td>( (k/\mu^w)\rho^w g )</td>
</tr>
<tr>
<td>( \gamma^{FR} )</td>
<td>( \rho^w g )</td>
</tr>
</tbody>
</table>

TABLE XV. RELATIONSHIPS OF PARAMETERS

<table>
<thead>
<tr>
<th>( n^S = 0.67 )</th>
<th>( n^F = 0.33 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho^S = 1.34\times 10^3 ) kg/m(^3)</td>
<td>( \rho^F = 0.33\times 10^3 ) kg/m(^3)</td>
</tr>
<tr>
<td>( E = 3\times 10^7 ) Pa</td>
<td>( \nu = 0.2 )</td>
</tr>
<tr>
<td>( \lambda^S = 5.5833\times 10^6 ) Pa</td>
<td>( \mu^S = 8.375\times 10^6 ) Pa</td>
</tr>
<tr>
<td>( k^F = 1\times 10^{-2} ) m/s</td>
<td>( \gamma^{FR} = 1\times 10^4 ) N/m(^3)</td>
</tr>
</tbody>
</table>

TABLE XVI. PARAMETERS USED IN [99]
Appendix D

EXAMPLE OF PROCEDURES TO OBTAIN FOURIER REPRESENTATION

For demonstration purpose, consider the case when normalized support size $\tilde{a}$ is less than 2. By applying spatial discretization and selecting the $I-th$ row of each matrix in Equation (5.1) and (5.2), the equations become

$$\{ M_{IJ} \}_{J=1}^{J=NP} \{ d \}_{J=1}^{J=NP} + \{ K_{IJ} \}_{J=1}^{J=NP} \{ d \}_{J=1}^{J=NP} - \{ Q_{IJ} \}_{J=1}^{J=NP} \{ p \}_{J=1}^{J=NP} = 0 \quad (D.1)$$

$$\{ Q_{IJ}^{T} \}_{J=1}^{J=NP} \{ d \}_{J=1}^{J=NP} + \{ S_{IJ} \}_{J=1}^{J=NP} \{ \dot{p} \}_{J=1}^{J=NP} + \{ H_{IJ} \}_{J=1}^{J=NP} \{ p \}_{J=1}^{J=NP} = 0 \quad (D.2)$$

Then, performing the domain integration of Equation (D.1) and Equation (D.2), the expression of each term becomes as shown in Table XVII, where all of the definitions are the same as described in the main section. Applying the nodal solutions (5.5) into the terms in Table XVII, the coefficient of each term is obtained in Table XVIII.
### Table XVII. Expression of Each Term by Different Discretization Methods and Domain Integration Schemes

<table>
<thead>
<tr>
<th>Term</th>
<th>FEM</th>
<th>RK with DNI</th>
<th>RK with MSCNI/MSNNI</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\left[ \mathbf{Md} \right]_I$</td>
<td>$\frac{\rho \Delta x}{6} \left[ \begin{array}{c} d_{I-1} \ d_I \ d_{I+1} \end{array} \right] \left[ \begin{array}{c} a_1^2 \ 2a_1a_2 \ a_2^2 + 2a_1^2 \end{array} \right]^T$</td>
<td>$\rho A \Delta x \left[ \begin{array}{c} a_1^2 \ 2a_1a_2 \ a_2^2 + 2a_1^2 \end{array} \right]$</td>
<td>$\rho A \Delta x \left[ \begin{array}{c} a_1^2 \ 2a_1a_2 \ a_2^2 + 2a_1^2 \end{array} \right]$</td>
</tr>
<tr>
<td>$\left[ \mathbf{Kd} \right]_I$</td>
<td>$\frac{EA}{\Delta x} \left[ \begin{array}{c} 1 \ 2 \ -1 \end{array} \right] \left[ \begin{array}{c} d_{I-1} \ d_I \ d_{I+1} \end{array} \right]$</td>
<td>$\frac{EA m_1}{\Delta x} \left[ \begin{array}{c} 1 \ 0 \ 2 \ 0 \ -1 \end{array} \right] \left[ \begin{array}{c} d_{I-2} \ d_{I-1} \ d_I \ d_{I+1} \ d_{I+2} \end{array} \right]$</td>
<td>$\frac{EA}{\Delta x} \left[ \begin{array}{c} -a_{45}^2 - \tilde{\alpha}a_{42}a_{25} \ -\tilde{\alpha}a_{41}(a_{42} - a_{25}) \ 2a_{45}^2 + \tilde{\alpha} \left( a_{42}^2 + a_{25}^2 + a_{14}^2 \right) \ -\tilde{\alpha}a_{14}(a_{42} - a_{25}) \ -a_{45}^2 - \tilde{\alpha}a_{42}a_{25} \end{array} \right]^T$</td>
</tr>
<tr>
<td>$\left[ \mathbf{Qp} \right]_I$</td>
<td>$\frac{\alpha A}{2} \left[ \begin{array}{c} 1 \ 0 \ -1 \end{array} \right] \left[ \begin{array}{c} p_{I-1} \ p_I \ p_{I+1} \end{array} \right]$</td>
<td>$\alpha A m_1 \left[ \begin{array}{c} a_1 \ a_2 \ 0 \ -a_1 \ -a_2 \end{array} \right]$</td>
<td>$\alpha A a_{45} \left[ \begin{array}{c} a_1 \ a_2 \ 0 \ -a_1 \ -a_2 \end{array} \right]$</td>
</tr>
<tr>
<td>$\left[ \mathbf{Q'd} \right]_I$</td>
<td>$\frac{\alpha A}{2} \left[ \begin{array}{c} 1 \ 0 \ 1 \end{array} \right] \left[ \begin{array}{c} \dot{d}_{I-1} \ \dot{d}<em>I \ \dot{d}</em>{I+1} \end{array} \right]$</td>
<td>$\alpha A m_1 \left[ \begin{array}{c} -a_2 \ -a_1 \ 0 \ a_1 \ a_2 \end{array} \right]$</td>
<td>$\alpha A a_{45} \left[ \begin{array}{c} -a_2 \ -a_1 \ 0 \ a_1 \ a_2 \end{array} \right]$</td>
</tr>
<tr>
<td>$\left[ \mathbf{Sp} \right]_I$</td>
<td>$\frac{A \Delta x}{6M} \left[ \begin{array}{c} 1 \ 4 \ 1 \end{array} \right] \left[ \begin{array}{c} \dot{p}_{I-1} \ \dot{p}<em>I \ \dot{p}</em>{I+1} \end{array} \right]$</td>
<td>$\frac{A \Delta x}{M} \left[ \begin{array}{c} a_1^2 \ 2a_1a_2 \ a_2^2 + 2a_1^2 \end{array} \right]$</td>
<td>$\frac{A \Delta x}{M} \left[ \begin{array}{c} a_1^2 \ 2a_1a_2 \ a_2^2 + 2a_1^2 \end{array} \right]$</td>
</tr>
<tr>
<td>$\left[ \mathbf{Hp} \right]_I$</td>
<td>$\frac{k A}{\mu^w \Delta x} \left[ \begin{array}{c} 1 \ 2 \ -1 \end{array} \right] \left[ \begin{array}{c} p_{I-1} \ p_I \ p_{I+1} \end{array} \right]$</td>
<td>$\frac{k A m_1^2}{\mu^w \Delta x} \left[ \begin{array}{c} 1 \ 0 \ 2 \ 0 \ -1 \end{array} \right]$</td>
<td>$\frac{k A a_{45}^2}{\mu^w \Delta x} \left[ \begin{array}{c} 1 \ 0 \ 2 \ 0 \ -1 \end{array} \right]$</td>
</tr>
</tbody>
</table>
Appendix D (continued)

<table>
<thead>
<tr>
<th>( \tilde{m} )</th>
<th>FEM ( \frac{\rho A \Delta x}{3} (2 + \cos \theta) )</th>
<th>RK with DNI ( \rho A \Delta x (a_1 + 2a_2 \cos \theta)^2 )</th>
<th>RK with MSCNI/MSNNI ( \rho A \Delta x (a_1 + 2a_2 \cos \theta)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{k} )</td>
<td>( \frac{2EA}{\Delta x} (1 - \cos \theta) )</td>
<td>( \frac{4EA m_i^2 \sin^2 \theta}{\Delta x} )</td>
<td>( \frac{EA}{\Delta x} \left[ 4a_{15}^2 \sin^2 \theta + \tilde{a} \left( (a_{14} - (a_{42} - a_{25}) \cos \theta \right)^2 + a_{45}^2 \sin^2 \theta) \right] )</td>
</tr>
<tr>
<td>( \ddot{q} )</td>
<td>( \alpha A i \sin \theta )</td>
<td>( 2\alpha A m_i (a_1 + 2a_2 \cos \theta) i \sin \theta )</td>
<td>( 2\alpha A a_{45} (a_1 + 2a_2 \cos \theta) i \sin \theta )</td>
</tr>
<tr>
<td>( \ddot{s} )</td>
<td>( \frac{A \Delta x}{3M} (2 + \cos \theta) )</td>
<td>( \frac{A \Delta x}{M} (a_1 + 2a_2 \cos \theta)^2 )</td>
<td>( \frac{A \Delta x}{M} (a_1 + 2a_2 \cos \theta)^2 )</td>
</tr>
<tr>
<td>( \ddot{h} )</td>
<td>( \frac{2kA}{\mu^w \Delta x} (1 - \cos \theta) )</td>
<td>( \frac{4kAm_i^2 \sin^2 \theta}{\mu^w \Delta x} )</td>
<td>( \frac{4kAa_{45}^2 \sin^2 \theta}{\mu^w \Delta x} )</td>
</tr>
</tbody>
</table>

**TABLE XVIII. FOURIER REPRESENTATION OF EACH TERM**
ROUND-OFF ERROR OF THE VON NEUMANN RESULTS

Equation (5.13) can be explicitly expressed as

\[ m\bar{s}\gamma^3 + \left( \frac{\bar{k}\bar{s} - \bar{q}^2}{2} \right) \Delta t^2 + \bar{m}\bar{h}\Delta t - 3\bar{m}\bar{s} \] \gamma^2

(E.1)

\[ + \left( \bar{k}\bar{h}\Delta t^3 - \bar{k}\bar{s}\Delta t^2 - 2\bar{m}\bar{h}\Delta t + 3\bar{m}\bar{s} \right) \gamma + \frac{\bar{q}^2}{2} \Delta t^2 + \bar{m}\bar{h}\Delta t - \bar{m}\bar{s} = 0 \]

Rewriting Equation (E.1) into the following form

\[ m\bar{s} (\gamma - 1)^3 + \left( \frac{\bar{k}\bar{s} - \bar{q}^2}{2} \right) \Delta t^2 + \bar{m}\bar{h}\Delta t \] \( (\gamma - 1)^2 + \left( \bar{k}\bar{h}\Delta t^3 + \left( \bar{k}\bar{s} - \bar{q}^2 \right) \Delta t^2 \right) (\gamma - 1) + \bar{k}\bar{h}\Delta t^3 = 0 \)

(E.2)

Since \( E \in [10^6, 10^8] \) and \( \frac{k}{\mu} \in [10^{-13}, 10^{-2}] \), the order of the term \( \bar{k}\bar{h}\Delta t^3 \) can be written as

\[ \bar{k}\bar{h}\Delta t^3 \approx O(E \frac{k}{\mu} \Delta t^3) \in [10^{-7}\Delta t^3, 10^6\Delta t^3] \]

(E.3)

It can be seen from Equation (E.3) that as \( k/\mu \) and \( \Delta t \) are small, the term \( \bar{k}\bar{h}\Delta t^3 \) approaches 0 and Equation (E.2) reduces to

\[ m\bar{s} (\gamma - 1)^2 + \left( \frac{\bar{k}\bar{s} - \bar{q}^2}{2} \right) \Delta t^2 + \bar{m}\bar{h}\Delta t \] \( (\gamma - 1) + \left( \bar{k}\bar{s} - \bar{q}^2 \right) \Delta t^2 \) \( (\gamma - 1) = 0 \)

(E.4)

To further investigate Equation (E.4), notice that \( M \in [10^6, 10^9] \), thus
Appendix E (continued)

\[ \tilde{k}\tilde{s}\Delta t^2 \approx \mathcal{O}(\frac{E}{M}\Delta t^2) \in [10^{-3} \Delta t^2, 10^2 \Delta t^2] , \quad \tilde{q}^2\Delta t^2 \approx \mathcal{O}(\Delta t^2) \]  \hspace{1cm} (E.5)

If \( \Delta t \) is small (e.g., \( \Delta t < 10^{-7} \)), the terms \( \tilde{k}\tilde{s}\Delta t^2 \) and \( \tilde{q}^2\Delta t^2 \) in Equation (E.4) will be close to 0 resulting in

\[ \left( \tilde{s}(\gamma - 1) + \frac{\tilde{h}}{\tilde{s}}\Delta t \right)(\gamma - 1)(\gamma - 1) = 0 \]  \hspace{1cm} (E.6)

which has the roots of 1, 1, and \( 1 - \frac{\tilde{h}}{\tilde{s}}\Delta t \), where \( \frac{\tilde{h}}{\tilde{s}}\Delta t \approx \mathcal{O}(M \frac{k}{\mu_w}\Delta t) \). These roots violate the stability condition (5.15), and hence the von Neumann method always estimate the white areas in Table VI as unstable, which may not be true.

In summary, the white area in Table VI is deemed undetermined because of the inaccuracy of von Neumann results due to the round-off errors occurred when \( \Delta t \), \( k/\mu_w \), and \( M \frac{k}{\mu_w} \) are too small. One can also notice from Table VI that the von Neumann method cannot predict the results when \( \Delta t \) is smaller than \( 10^{-7} \), which agrees with the above demonstration.
CITED LITERATURE


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AREAS OF INTEREST

Finite element methods, Galerkin meshfree methods, domain integration schemes, strong-form collocation methods, nonlinear modeling, extreme deformation modeling, constitutive models for geomaterial behaviors, mixed formulations, damage models, poromechanics, contact algorithms

EDUCATION

2016  Civil and Materials Engineering  Doctor of Philosophy (Ph.D.)
University of Illinois at Chicago
Chicago, IL

2011  Civil and Environmental Engineering  Master of Engineering (M.E.)
Lamar University
Beaumont, TX

2009  Civil Engineering  Bachelor of Science (B.S.)
Chulalongkorn University
Bangkok, Thailand
EXPERIENCE

Research Assistant, August 2013 - October 2016

Department of Civil and Materials Engineering, University of Illinois at Chicago

- Developing meshfree methods based on the Reproducing Kernel (RK) approximation for landslide simulations. The multi-field semi-Lagrangian RK with variationally consistent stabilized nodal integration is developed to handle extreme deformation and properly represent geomaterial behaviors.

- Formulating plasticity models (e.g., Drucker-Prager and $J_2$) with damage models into the semi-Lagrangian RK frameworks. One-variable damage model, where damage occurs only on the deviatoric stresses, and two-variable damage model, where damage can separately occur on tension and compression, are developed as a “cap” for Drucker-Prager or $J_2$ to capture material failure behaviors.

- Analyzing temporal stability of the developed methods. The von Neumann analysis is employed to analyze the temporal stability of the multi-field semi-Lagrangian RK with different nodal integration schemes and study the effects of material parameters on critical time step size, when explicit temporal discretization is employed.

- Participating in a munitions penetration project. Penetration depths of munitions, especially rockets, for various soil types are evaluated and predicted by simulations from the multi-field semi-Lagrangian RK and single-field semi-Lagrangian RK frameworks.

- Formulating hyperelasticity material model in the strong formulation framework. To account for large deformation of hyperelastic material, the Mooney-Rivlin hyperelastic model is formulated in the framework of the strong form collocation method using the
Reproducing Kernel approximations (RKCM).

- Analyzing elastodynamic and poroelasticity problems using RKCM. The RKCM is introduced for elastodynamic problems by considering single-field and two-field formulations.

**Student**, September 2014

Landslide Risk Assessment and Mitigation school, University of Salerno, Italy

- Attending the two-week workshop among the 40 selected Ph.D. students around the world to develop an insight education on assessing, forecasting and mitigating landslide risks over large area by mathematical modeling, monitoring, GIS techniques, etc.

**Research Assistant**, May 2012 - August 2012

Department of Civil and Materials Engineering, University of Illinois at Chicago

- Participating in a high-speed railroad project of the mid-west US by focusing on the soil beneath the rail using finite element software, ANSYS. Research on material constitutive models, specifically in multi-field, was performed to find adequate models to represent geomaterial behaviors beneath the railroad.

**Graduate Teaching Assistant**, January 2011 - May 2011

Department of Civil and Environmental Engineering, Lamar University, TX

- Assisting in Mechanics of Solids course and Geo-technical Engineering course including being a lab assistant.
Student, August 2008 - March 2009

Civil Engineering Department, Chulalongkorn University, Bangkok, Thailand

- Conducting a study on the life cycles of the carbon fiber reinforce plastic (CFRP) under cyclic loadings. A damage criterion to define the failure of CFRP was developed and performed in experiment to study number of loading cycles until the CFRP fails.

Intern, March 2008 - May 2008

Roge and Associates Co.Ltd., Bangkok, Thailand

- In charge of cost estimation, analyzing, and designing for two-story buildings. The analyses and designs of the buildings were performed by using STAAD PRO 2006.

SKILLS & EXPERTISE

MS Visual Studio, Fortran, HPCC, MPI, ANSYS, Abaqus, Matlab, AutoCAD, SketchUp, ParaView, STAAD PRO, MS Office

AWARDS & RECOGNITION

- Student Presenter Award (UIC), 2016
- Graduate Student Council Travel Award (UIC), 2016
- Student Presenter Award (UIC), 2015
- Chicago Consular Corps Scholarship (UIC), 2015
• Student Presenter Award (UIC), 2014

• Named the student obtaining the highest Fundamentals of Engineering (FE) exam score in the school history (LU), 2011

TEST SCORES

NCEES Fundamentals of Engineering (FE) Exam

October 2010  Score:89/100

JOURNAL PUBLICATIONS

• Siriaksorn, T. and Chi, S.W. (under preparation), "Reproducing Kernel Collocation Method for Elastodynamic."

• Siriaksorn, T., Chi, S.W., and Foster, C.D. (under preparation), "Meshfree Methods for Geotechnical Disaster Simulations and Predictions."


CONFERENCE PUBLICATIONS


CONFERENCE PRESENTATIONS


ADVANCED COURSES TAKEN