A PATHOLOGY OF ASYMPTOTIC MULTIPLICITY IN THE RELATIVE SETTING

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ABSTRACT. We point out an example of a projective family $\pi : X \to S$, a $\pi$-pseudoeffective divisor $D$ on $X$, and a subvariety $V \subset X$ for which the asymptotic multiplicity $\sigma_V(D; X/S)$ is infinite. This shows that the divisorial Zariski decomposition is not always defined for pseudoeffective divisors in the relative setting.

1. Introduction

We work throughout over $\mathbb{C}$. Suppose that $X$ is a smooth projective variety and $D$ is a pseudoeffective $\mathbb{R}$-divisor on $X$. The asymptotic multiplicity of $D$ along a subvariety $V \subset X$, studied by Nakayama [11] and Ein-Lazarsfeld-Mustață-Nakamaye-Popa [5], has proved to be a fundamental tool in understanding the properties of the divisor $D$. For big divisors $D$, the definition of the asymptotic multiplicity is straightforward: roughly, one considers the linear series $|mD|$ for larger and larger values of $m$, and takes $\sigma_V(D) = \lim_{m \to \infty} \frac{1}{m} \text{mult}_V |mD|$, where the multiplicity of a linear series along a subvariety is defined to be the multiplicity of a general member.

Complications arise, however, in carrying out this construction for divisors $D$ which are pseudoeffective but not big, i.e. for divisors on the boundary of the pseudoeffective cone $\overline{\text{Eff}}(X) \subset N^1(X)$. Nakayama realized that $\sigma_V(D)$ can be extended to a lower semicontinuous function on $\overline{\text{Eff}}(X)$ by setting

$$\sigma_V(D) = \lim_{\epsilon \to 0} \sigma_V(D + \epsilon A),$$

where $A$ is a fixed ample divisor. In some applications (e.g. in the construction of Zariski decompositions), it is important to know that the limit in question takes a finite value. While it is clear that the quantity on the right is nondecreasing as $\epsilon$ is made smaller, it might a priori be unbounded in the limit. That this does not happen in the non-relative setting was observed by Nakayama.

Our aim in this note is to demonstrate by example that when asymptotic multiplicity invariants are considered in the greater generality of divisors on a projective family $\pi : X \to S$, this finiteness need not hold: for a $\pi$-pseudoeffective divisor, the limit defining $\sigma_V(D; X/S)$ can indeed be infinite. This answers a question of Nakayama [11, pg. 33]. The example itself is familiar, a divisor on the versal deformation space of a fiber of Kodaira type $I_2$, which has been considered in related contexts by Reid [12, 6.8] and Kawamata [8, Example 3.8(2)],[9, Example 9].

**Theorem 1.** There exists a projective family $\pi : X \to S$, a $\pi$-pseudoeffective divisor $D$, and a subvariety $V \subset X$ for which $\sigma_V(D; X/S)$ is infinite.

An important use of asymptotic multiplicity invariants is in the construction of the divisorial Zariski decomposition, a higher-dimensional analog of the usual Zariski decomposition.
on surfaces. The example here shows that trouble arises if one generalizes this construction to pseudoeffective classes in the relative setting: after passing to a blow-up on which the valuation corresponding to \( V \) is divisorial, we obtain an example in which the decomposition is not defined.

**Corollary 2.** Let \( \pi : X \to S \) be as in Theorem 1. If \( f : W \to X \) is the blow-up along \( V \) with exceptional divisor \( E \), then \( \tilde{D} = f^*D \) has \( \sigma_E(\tilde{D}; W/S) = \infty \) and \( N_\sigma(\tilde{D}; W/S) \) is not defined.

Moreover, the divisor \( \tilde{D} \) does not admit any Zariski decomposition in a very strong sense:

**Corollary 3.** There does not exist a birational model \( g : Z \to W \) for which \( g^*\tilde{D} \) admits a decomposition \( g^*\tilde{D} = P + N \) with \( P \) a \( g \circ (f \circ \pi) \)-movable divisor and \( N \) effective.

In Section 2 we recall the basic definitions and properties of the invariants \( \sigma_V(D; X/S) \) and \( N_\sigma(D; X/S) \) appearing in Theorem 1 and Corollary 2, before establishing the claims in Section 3. In Section 4, we describe a more general setting for making computations in a similar spirit.

## 2. Preliminaries

Suppose that \( \pi : X \to S \) is a projective, surjective morphism with connected fibers, with \( X \) and \( S \) smooth (hereafter, a *nice family*). We will find it convenient to allow the base \( S \) to be a surface germ, following [7]. The proofs of the results in this section hold either when \( S \) is a quasiprojective variety or a germ. In Section 3, it will be convenient to make computations with the base a germ. However, the germ we consider is algebraizable, and it follows that the same pathology occurs when the base is extended to be an affine scheme; this is discussed in Remark 1.

Two divisors \( D \) and \( D' \) on \( X \) are said to be numerically equivalent over \( S \), or \( \pi \)-numerically equivalent, if \( D \cdot C = D' \cdot C \) for any curve \( C \) that is contracted by \( \pi \); write \( D \equiv_\pi D' \) for the relation of numerical equivalence over \( S \), and \( N^1(X/S) \) for the vector space of \( \mathbb{R} \)-divisors on \( X \), modulo this equivalence.

The familiar cones of positive divisors on a projective variety all have analogs in the relative setting: a divisor \( D \) on \( X \) is said to be

1. \( \pi \)-ample if \( D_s \) is ample on every fiber \( X_s = \pi^{-1}(s) \);
2. \( \pi \)-strongly movable if the support of the cokernel of \( f^*f_*\mathcal{O}_X(D) \to \mathcal{O}_X(D) \) has codimension at least 2;
3. \( \pi \)-big if the restriction of \( D \) to the generic fiber is big.

Corresponding to these classes of divisors are cones inside \( N^1(X/S) \):

\[
\text{Amp}(X/S) \subseteq \text{Mov}(X/S) \subseteq \text{Big}(X/S).
\]

The cones \( \text{Amp}(X/S) \) and \( \text{Big}(X/S) \) are both open. We will also consider the closures of these three cones inside \( N^1(X/S) \), which are, respectively:

- \( \text{Nef}(X/S) \). A divisor is \( \pi \)-nef if \( D_s \) is nef on every fiber \( X_s \) (i.e. if \( D \cdot C \geq 0 \) for every curve \( C \) contracted by \( \pi \));
- \( \text{Mov}(X/S) \), the cone of \( \pi \)-movable divisors.
- \( \text{Eff}(X/S) \), the cone of \( \pi \)-pseudoeffective divisors. A divisor \( D \) is \( \pi \)-pseudoeffective if the restriction of \( D \) to the generic fiber is pseudoeffective.
The meaning of “movable” is unfortunately not entirely uniform in the literature, and we stress that here an \( \mathbb{R} \)-divisor is called \( \pi \)-movable if it lies in the closed cone \( \text{Mov}(X/S) \); this is sometimes called \( \pi \)-nef in codimension 1. We note too that the cone \( \text{Eff}(X/S) \) is not necessarily a strictly convex cone, in that it might contain an entire line through the origin; this contrasts with the familiar case when \( S \) is a point. For example, if \( D \) restricts to 0 on a general fiber of \( \pi \), then \( D \) and \( -D \) are both \( \pi \)-pseudoeffective.

For simplicity, we will assume that the base space \( S \) is either affine or a germ. This is not really necessary, but the invariants under consideration can be computed when the base is projective simply by restricting to the preimage of a suitable affine open set; we refer to [11, §3.2] for details. The existence of a \( \pi \)-ample divisor \( A \) on \( X \) is automatic in this setting.

If \( D \) is a \( \pi \)-big divisor, then \( f_*\mathcal{O}_X(mD) \neq 0 \) for sufficiently large and divisible \( m \), and so \( H^0(X, \mathcal{O}_X(mD)) = f_*(\mathcal{O}_X(mD)) \) is nonzero as well. Hence in these settings, any \( \pi \)-big class has an effective representative.

**Definition 1.** Suppose that \( X \) is smooth. Given an irreducible subvariety \( V \subset X \) and a \( \pi \)-big \( \mathbb{R} \)-divisor \( D \), set

\[
\sigma_V(D; X/S) = \inf_{D' \equiv \pi \neq 0} \text{mult}_V(D').
\]

Since \( D \) is \( \pi \)-big, there exists an effective \( \mathbb{R} \)-divisor \( D' \) that is \( \pi \)-numerically equivalent to \( D \), and this infimum is taken over a nonempty set.

When \( D \) is a big integral divisor, a sequence \( D'_m \) of effective \( \mathbb{R} \)-divisors with multiplicities converging to the infimum can be found by taking \( D'_m \in \frac{1}{m} |mD| \), where we choose a general element of the linear system \( |mD| \).

We next extend the definition of the asymptotic multiplicity from \( \pi \)-big divisors to \( \pi \)-pseudoeffective divisors.

**Definition 2.** Given a \( \pi \)-pseudoeffective \( \mathbb{R} \)-divisor \( D \), set

\[
\sigma_V(D; X/S) = \lim_{\epsilon \to 0} \sigma_V(D + \epsilon A; X/S).
\]

This is evidently a nondecreasing function as \( \epsilon \) approaches 0, but it might have infinite limit. To show that it has a finite limit, it suffices to bound \( \sigma_V(D + \epsilon A; X/S) \) above, independent of \( \epsilon \). Nakayama gives several conditions under which this can be achieved.

**Theorem 4** ([11], Lemmas 2.1.2, 3.2.6). If any of the following holds, then \( \sigma_V(D; X/S) \) is finite.

1. \( S = \text{Spec} \mathbb{C} \) is a point;
2. \( D \) is numerically equivalent over \( S \) to an effective \( \mathbb{R} \)-divisor \( \Delta \);
3. \( \text{codim} \pi(V) < 2 \).

We recall the proof in case (1), perhaps the most important in practice. Case (2) is immediate from the definition, and we refer to [11] for (3). Assume for a moment that \( V \subset X \) is an irreducible divisor; that this implies the general statement will follow from Theorem 5(2) below.

**Proof of (1).** For any \( \epsilon \), \( (D + \epsilon A) - \sigma_V(D + \epsilon A)V \) is pseudoeffective, and so

\[
((D + \epsilon A) - \sigma_V(D + \epsilon A)V) \cdot A^{n-1} \geq 0.
\]
As long as \( \epsilon < 1 \) it follows that
\[
\sigma_V(D + \epsilon A) \leq \left( \frac{D + \epsilon A}{V \cdot A^{n-1}} \right) \leq \left( \frac{D + A}{V \cdot A^{n-1}} \right)
\]
is bounded above as \( \epsilon \) decreases to 0. \( \square \)

This argument relies in a crucial way on the properness of \( X \) to carry out intersection theory, and is not applicable in the relative setting in general.

**Proposition 5** ([11], Lemmas 2.1.4, 2.2.2, 2.1.7). Suppose that \( \pi : X \to S \) is a nice family and \( V \subset X \) is an irreducible subvariety.

1. If \( F \) is any \( \pi \)-pseudoeffective divisor on \( X \), then
   \[
   \lim_{\epsilon \to 0} \sigma_V(D + \epsilon F; X/S) = \sigma_V(D; X/S).
   \]
2. Let \( f : W \to X \) be the normalized blow-up of \( X \) along \( V \), and let \( E \) be the exceptional divisor over \( V \). Then \( \sigma_E(f^*D; W/S) = \sigma_V(D; X/S) \).
3. The number of prime divisors \( \Gamma \) for which \( \sigma_\Gamma(D; X/S) > 0 \) is finite.

The first of these shows that Definition 1 is independent of the choice of \( \pi \)-ample divisor \( A \), while the second completes the proof of Theorem 4(1) in the case that \( V \) has codimension greater than 1.

**Definition 3** ([11], [4]). Suppose that \( \pi : X \to S \) is a nice family and that \( D \) is a \( \pi \)-pseudoeffective divisor such that \( \sigma_\Gamma(D; X/S) \) is finite for every prime divisor \( \Gamma \). Then set
\[
N_\sigma(D; X/S) = \sum_\Gamma \sigma_\Gamma(D; X/S) \Gamma,
\]
\[
P_\sigma(D; X/S) = D - N_\sigma(D; X/S).
\]

It follows from Proposition 5(3) that there are only finitely many nonzero terms in the sum defining \( N_\sigma(D; X/S) \).

We refer to \( N_\sigma(D; X/S) \) as the negative part of the divisorial Zariski decomposition, and \( P_\sigma(D; X/S) \) as the positive part. The negative part is a rigid, effective divisor. The positive part might not be nef, but it lies in the cone \( \overline{\text{Mov}}(X/S) \) of \( \pi \)-movable divisors. Corollary 2 shows that without the finiteness hypothesis on \( \sigma_\Gamma(D; X/S) \), the definition is not always applicable in the relative setting.

In the non-relative setting, the divisorial Zariski decomposition is defined for any pseudoeffective class \( D \), but it lacks certain useful properties of the classical Zariski decomposition in dimension 2: most importantly, the positive part \( P_\sigma(D) \) is not nef in general. In many cases one may construct a birational model \( f : W \to X \) on which \( P_\sigma(f^*D) \) is actually nef, even if \( P_\sigma(D) \) is not. However, a basic example of Nakayama shows that even this is not always possible [11, Theorem 5.2.6].

Another variant on Zariski decomposition in higher dimensions, the weak Zariski decomposition of Birkar, imposes fewer conditions and so exists for a larger class of divisors, including that of Nakayama’s example. In this decomposition, the positive part \( P \) is allowed to be any relatively nef divisor, not necessarily the positive part \( P_\sigma(f^*D) \) of the divisorial Zariski decomposition.
Definition 4 ([2]). Suppose that $\pi : X \to S$ is a nice family and that $D$ is a pseudoeffective divisor on $X$. We say that $D$ admits a weak Zariski decomposition over $S$ if there exists a birational map $f : Y \to X$ and a decomposition $f^*D = P + N$, where $P$ is $(f \circ \pi)$-nef and $N$ is effective.

This condition is fairly unrestrictive, but there nevertheless exist pseudoeffective $\mathbb{R}$-divisors on smooth threefolds which do not admit a weak Zariski decomposition [10]. Corollary 3 asserts that the divisor $\tilde{D}$ provides another such example. Indeed, $\tilde{D}$ admits no Zariski decomposition in a still stronger sense: even after pulling back to a higher model, it cannot be decomposed as the sum of an effective divisor and a relatively movable divisor. The example is qualitatively rather different from that of [10]: there, a certain pseudoeffective divisor $D_\lambda$ has negative intersection with infinitely many curves; here, there is a single curve on which $D$ is negative, but the multiplicity of $D$ along this curve is infinite.

3. Main example

The claimed pathology follows from a few calculations on an example that has been studied by Reid [12, 6.8] and Kawamata [8, Example 3.8(2)]. Let $\pi : X \to S$ be the versal deformation space of a fiber of Kodaira type $I_2$. The base $S$ is smooth, 2-dimensional germ. The fiber over the central point $0 \in S$ consists of two smooth rational curves $C_1$ and $C_2$, meeting transversely at two points $p_1$ and $p_2$. Let $C = \pi^{-1}(0)$ be the union of these two curves.

There are two divisors $\Gamma_1, \Gamma_2 \subset S$ corresponding to the smoothings of the two nodes of $C$. The fiber of $\pi$ over a general point of $\Gamma_i$ is a nodal rational curve, while the fiber over a general point of $S$ is a smooth curve of genus 1.

![Figure 1. The family $\pi : X \to S$](image)

Remark 1. The computations that follow will give an example in which some $\sigma_V(D;X/S)$ is infinite, in the case where $S$ is a germ. The calculations rely on the fact that $\pi : X \to S$ is a versal deformation space. However, the local analytic results imply that the same pathological behavior occurs even when the base $S$ is an affine surface. Indeed, we will see in
Lemma 7 below that there is a projective family $\bar{\pi} : \bar{X} \to \bar{S}$ where $\bar{S}$ is an affine surface, such that the restriction of $\bar{\pi}$ to the germ at a point $0 \in \bar{S}$ coincides with the map $\pi : X \to S$.

If $G$ is a $\bar{\pi}$-big divisor, with restriction $\bar{G}$ to the germ, then $\sigma_{C_1}(\bar{G} ; \bar{X}/\bar{S}) \geq \sigma_{C_1}(G ; X/S)$: indeed, if $\bar{G}$ is an effective divisor on $\bar{X}$ which is $\bar{\pi}$-numerically equivalent to $G$, its restriction to the central germ is an effective divisor on $X$ which is $\pi$-numerically equivalent to $G$.

Thus the infimum defining $\sigma_{C_1}(\bar{G} ; \bar{X}/\bar{S})$ in is taken over a subset of the infimum defining $\sigma_{C_1}(G ; X/S)$ in Definition 1, giving the claimed inequality. It follows that in the limit at the pseudoeffective boundary, $\sigma_{C_1}(\bar{D} ; \bar{X}/\bar{S}) \geq \sigma_{C_1}(D ; X/S)$, and it must be that $\sigma_{C_1}(\bar{D} ; \bar{X}/\bar{S})$ is infinite as well. The claims about Zariski decomposition follow in the same way.

**Lemma 6.** The normal bundle $N_{C_i/X}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

**Proof.** Suppose for instance that $i = 1$. There is an exact sequence

$$0 \longrightarrow N_{C_1/X} \longrightarrow (N_{C/X})_{C_1} \longrightarrow T_{C_2,p_1} \oplus T_{C_2,p_2} \longrightarrow 0$$

with the property that a first-order deformation, determined by a section $s \in H^0(C, N_{C_i/X})$ smooths the node at $p_i$ if and only if $s$ has nonzero image in $T_{C_2,p_i}$ [6, Lemma 2.6]. The sheaf in the middle is the trivial $\mathcal{O}_C \oplus \mathcal{O}_C$. In one direction $p_1$ is smoothed, and in another $p_2$ is, so the map sends $(1,0)$ to $(1,0)$ and $(0,1)$ to $(0,1)$ with respect to the direct sum decompositions. It follows that the kernel is $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.}$

Lemma 6 shows in particular that $K_X$ has intersection 0 with both $C_1$ and $C_2$ and so is relatively numerically trivial.

**Lemma 7.** There exists a flop $\tau : X \dashrightarrow X^+/S$ with flopping curve $C_1$. Let $C_1^+ \subset X^+$ be the flopped curve, and $C_2' \subset X^+$ be the strict transform of $C_2$. There exists an isomorphism $\sigma : X^+ \to X/S$ which sends $C_1^+$ to $C_1$ and $C_2'$ to $C_2$. Furthermore, there exists an involutive automorphism $\iota : X \to X/S$ which exchanges the two curves $C_1$ and $C_2$.

**Proof.** The arguments here are due to Kawamata [8, Example 3.8(2)]. We make some aspects of the proof explicit by working with local defining equations given by Reid [12]. In what follows, we use the notation $\bar{\tau}$ to denote objects on a family $\bar{\pi} : \bar{X} \to \bar{S}$ over an affine base, while objects with no bar will be the restrictions to a certain germ.

Let $\bar{S} = \mathbb{A}^2$, with coordinates $t_1$ and $t_2$. Fix two distinct complex numbers $a_1$ and $a_2$ and define $\bar{X}_0 \subset (\mathbb{A}^1 \times \mathbb{A}^1) \times \bar{S}$ by the equation

$$x_1^2 = ((x_2 - a_1)^2 - t_1)((x_2 - a_2)^2 - t_2).$$

The closure $\bar{X} \subset (\mathbb{P}^1 \times \mathbb{P}^1) \times \bar{S}$ is smooth, and the second projection $\bar{\pi} : \bar{X} \to \bar{S}$ is proper. The fiber of $\bar{\pi}$ over a general point $(t_1, t_2)$ is a smooth curve of genus 1. If exactly one of $t_1$ and $t_2$ is zero, the fiber is nodal, while if $t_1 = t_2 = 0$, the fiber is given by $x_1^2 = (x_2 - a_1)^2(x_2 - a_2)^2$. This central fiber has two components, the rational curves $C_1$ defined by $x_1 = -(x_2 - a_1)(x_2 - a_2)$ and $C_2$ defined by $x_1 = (x_2 - a_1)(x_2 - a_2)$. The restriction of $\bar{\pi} : \bar{X} \to \bar{S}$ to the germ at $(0,0) \in \bar{S}$ is the versal deformation space $\pi : X \to S$ considered above. The involution $\iota : X \to X/S$ defined by $\iota(x_1, x_2) = (-x_1, x_2)$ exchanges the two components of the central fiber.

There is a section $\bar{\sigma} : \bar{S} \to \bar{X}$ given by

$$x_2(t_1, t_2) = \frac{a_1 + a_2}{2} - \frac{t_1 - t_2}{2(a_1 - a_2)},$$
$$x_1(t_1, t_2) = (x_2(t_1, t_2) - a_1)^2 - t_1.$$
This has $\bar{\sigma}(0, 0) = \left(\frac{(a_2 - a_1)^2}{4}, \frac{1}{2}(a_1 + a_2)\right)$, which lies on $C_1$ and is disjoint from $C_2$.

Let $\Sigma_1$ be the divisor $\sigma(\bar{S})$. Since $\Sigma_1 \cdot C_1 = 1$ and $\Sigma_1 \cdot C_2 = 0$, the curves $C_1$ and $C_2$ have distinct classes in $N_1(\bar{X}/\bar{S})$. Since all other fibers of $\bar{\pi}$ are irreducible, it must be that $N^1(\bar{X}/\bar{S})$ has dimension $2$. The divisor $2\iota_*(\Sigma_1) - \Sigma_1$ has positive degree on general fibers, and so is $\bar{\pi}$-big. Since $S$ is affine, there is an effective divisor $\Delta$ representing this class. For sufficiently small $\epsilon$, the pair $(X, \epsilon\Delta)$ is klt. Since $\Delta \cdot C_1 < 0$, there exists a $(K_{\bar{X}/\bar{S}} + \epsilon\Delta)$-flip $\tau : \bar{X} \rightarrow \bar{X}^+$, which is a $K_{\bar{X}/\bar{S}}$-flop. The map $\bar{\pi}^+ : \bar{X}^+ \rightarrow \bar{S}$ is a minimal model of $\bar{X}^+$.

By Lemma 6, the map $\tau$ is the flop of a rational curve with normal bundle $O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}(-1)$. It follows by looking at a resolution of $\tau$ (discussed in the proof of Lemma 8 below) that the map $\tau|_{\bar{S}}$ simply blows up the point $\Sigma_1 \cap C_1$ and that the strict transform of $\Sigma_1$ on $\bar{X}^+$ is smooth. This strict transform contains the curve $C_1^+$, and satisfies $\tau_* \Sigma_1 \cdot C_2^+ = 2$.

Now, the varieties $X^+$ and $\bar{S}$ are smooth and $\bar{\pi}^+ : X^+ \rightarrow \bar{S}$ has all fibers 1-dimensional, so $\bar{\pi}^+$ is flat. Since $\pi : X \rightarrow S$ is a versal deformation space, there exists an isomorphism $\beta : X^+ \rightarrow X$ over some automorphism of $S$. However, $\beta$ might not be defined over the identity map on $S$. The divisor $\Sigma_2 = \beta_*(\tau_*(\Sigma_1))$ is a smooth divisor on $X$, containing $C_1$, and meeting $C_2$ at two points. There is a translation on the smooth fibers of $\pi$ sending $\Sigma_1$ to $\Sigma_2$, which defines a birational self-map $\gamma : X \dasharrow X$ over the identity on $S$. The map $\pi \circ \gamma : X \rightarrow S$ must be isomorphic to some minimal model of $X$ over $S$, and indeed must be isomorphic to $\pi^+ : X^+ \rightarrow S$ since the strict transforms of $\Sigma_1$ under $\gamma$ and $\tau$ have the same numerical classes. It follows that there exists an isomorphism $\sigma : X^+ \rightarrow X$ over the identity of $S$. Replacing $\sigma$ with $\sigma \circ \iota$ if necessary, we may assume that $\sigma(C_1^+) = C_1$ and $\sigma(C_2^+) = C_2$, as required. \hfill $\square$

Each of the maps $\sigma \circ \tau$ and $\iota$ is a birational involution of $X$ over $S$, but we will soon see that the composition $\phi = (\sigma \circ \tau) \circ \iota$ is of infinite order. Since $\iota(C_2) = C_1$, the effect of repeatedly applying $\phi$ is to flop $C_1$, then $C_2$, then $C_1$ again, and so on. We will denote by $\phi_* D$ the strict transform of a divisor $D$ under a birational map $\phi$, and use the same notation for the induced map on numerical groups when confusion seems unlikely.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Resolution of the flop $\tau$}
\end{figure}
To an effective divisor \( D \) on \( X \), associate the 4-tuples

\[
\begin{align*}
  v_D &= (D \cdot C_1, D \cdot C_2, \text{mult}_{C_1}(D), \text{mult}_{C_2}(D)), \\
  \sigma_D &= (D \cdot C_1, D \cdot C_2, \sigma_{C_1}(D); X/S), \sigma_{C_2}(D; X/S)).
\end{align*}
\]

**Lemma 8.** Suppose that \( D \) is a divisor on \( X \), and let \( \tilde{D} \) denote the strict transform of \( D \) under the flop \( \tau : X \dashrightarrow X^+ \). Then

1. \( \tilde{D} \cdot C_1^+ = -D \cdot C_1 \),
2. \( \tilde{D} \cdot C_2^+ = D \cdot C_2^+ + 2(D \cdot C_1) \),
3. \( \text{mult}_{C_1^+}(\tilde{D}) = \text{mult}_{C_1}(D) + D \cdot C_1^+ \),
4. \( \text{mult}_{C_2^+}(\tilde{D}) = \text{mult}_{C_2}(D) \).

In matrix form, we have \( v_{\phi,D} = Mv_D \) and \( \sigma_{\phi,D} = M\sigma_D \) where

\[
M = \begin{pmatrix}
2 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0
\end{pmatrix}
\]

**Proof.** Let \( W \) be the graph of the flop \( \tau \):

\[
\begin{tikzcd}
W \\
X \\
\tau \\
X^+
\end{tikzcd}
\]

Since \( \tau \) is the flop of a rational curve with normal bundle \( \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \) by Lemma 6, the map \( f \) is simply the blow-up of \( X \) along \( C_1 \), while \( g \) is the blow-up of \( X^+ \) along \( C_1^+ \). There is a single \( f \)-exceptional divisor \( E \) on \( W \), which is isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \) and has normal bundle of bidegree \((-1, -1)\). Let \( \tilde{C}_1 \) be a ruling of \( E \) contracted by \( g \), so that \( f \) sends \( \tilde{C}_1 \) isomorphically to \( C_1 \). Similarly, let \( \tilde{C}_1^+ \) be a ruling of \( E \) contracted by \( f \), so that \( g \) maps \( \tilde{C}_1^+ \) isomorphically onto \( C_1^+ \). Lastly, let \( \tilde{C}_2^+ \) be the strict transform of \( C_2 \) on \( W \), a curve which meets \( E \) transversely at 2 points. This resolution is illustrated in Figure 2.

Then write

\[
f^*D + aE = g^*\tilde{D},
\]

for some constant \( a \). Taking the intersection of both sides with \( \tilde{C}_1 \) yields \( D \cdot C_1 + a(E \cdot \tilde{C}_1) = 0 \). Since \( E \cdot \tilde{C}_1 = -1 \), we obtain \( a = D \cdot C_1 \). Intersecting with \( \tilde{C}_1^+ \), we have \( -a = \tilde{D} \cdot C_1^+ \). Similarly, intersecting with \( \tilde{C}_2^+ \), we have \( D \cdot C_2 + a(E \cdot \tilde{C}_2^+) = \tilde{D} \cdot C_2^+ \), and since \( E \cdot C_2^+ = 2 \), we have (2). It is clear that \( \text{mult}_{C_2^+}(\tilde{D}) = \text{mult}_{C_2}(D) \), since \( \tau \) is an isomorphism at the generic point of \( C_2 \). Finally,

\[
\text{mult}_{C_1^+}(\tilde{D}) = \text{mult}_E(g^*\tilde{D}) = \text{mult}_E(f^*D) + a = \text{mult}_{C_1}(D) + a.
\]

These calculations immediately yield \( v_{\phi,D} = Mv_D \), since the second map \( \iota \) exchanges the two curves \( C_1 \) and \( C_2 \). Write \( D_m \) for a general divisor linearly equivalent to \( mD \), and then

\[
\begin{align*}
\sigma_{C_1}(\phi_*D) &= \lim_{m \to \infty} \frac{1}{m} \text{mult}_{C_1}(\phi_*D_m) = \lim_{m \to \infty} \frac{1}{m} (\text{mult}_{C_1}D_m + D_m \cdot C_1) \\
&= \left( \lim_{m \to \infty} \frac{1}{m} \text{mult}_{C_1}D_m \right) + D \cdot C_1 = \sigma_{C_1}(D) + a.
\end{align*}
\]

We are now in position to make the main computation.
**Theorem 9.** Let $\pi : X \to S$ be the versal deformation space of a singular fiber of Kodaira type $I_2$, and let $C_1$ be a component of the central fiber. There exists a divisor $D$ on the boundary of the cone $\overline{Eff}(X/S)$ with $\sigma_{C_1}(D; X/S) = \infty$.

**Proof.** Fix a $\pi$-ample effective $\mathbb{Q}$-divisor $H = H_0$ on $X$ with $H \cdot C_1 = H \cdot C_2 = 1$. Then $\sigma_{C_1}(H) = 0$ for each $i$. For example, we might take $H = \Sigma_1 + i_*(\Sigma_1)$ in the notation of Lemma 7.

Let $H_n = \phi^*(nH)$ be the strict transform of $H$ on $X$ under $n$ applications of $\phi$. Using the last part of Lemma 8, and the Jordan decomposition of $M$, which has a $3 \times 3$ block associated to the eigenvalue 1, we compute $\sigma_{H_n} = (2n + 1, -2n + 1, n(n-1)/2, n(n+1)/2)$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$H_n \cdot C_1$</th>
<th>$H_n \cdot C_2$</th>
<th>$\sigma_{C_1}(H_n)$</th>
<th>$\sigma_{C_2}(H_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>-3</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>-5</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$n$</td>
<td>$2n+1$</td>
<td>$-2n+1$</td>
<td>$\frac{n(n-1)}{2}$</td>
<td>$\frac{n(n+1)}{2}$</td>
</tr>
</tbody>
</table>

The key feature of the example is that while $H_n \cdot C_1$ grows linearly in $n$, the multiplicity $\sigma_{C_1}(H_n)$ grows quadratically. Let $D$ be the divisor class on the boundary of $\overline{Eff}(X/S)$ with $D \cdot C_1 = 1$ and $D \cdot C_2 = -1$. Since $C_1$ and $C_2$ span $N_1(X/S)$, we see that

$$H_n \equiv \pi (2n)D + H_0.$$ 

It follows that $\frac{1}{2n}H_n \equiv \pi D + \frac{1}{2n}H_0$ is a sequence of divisors converging to $D$, whose multiplicities along the curves is known. By Definition 1, we compute

$$\sigma_{C_1}(D; X/S) = \lim_{n \to \infty} \sigma_{C_1}(D + \frac{1}{2n}H_0) = \lim_{n \to \infty} \frac{1}{2n} \sigma_{C_1}H_n = \lim_{n \to \infty} \frac{n-1}{4} = \infty. \quad \square$$

Note that $\text{codim} \pi(C_1) = 2$, so there is no contradiction with Theorem 4(3).

**Corollary 10.** If $f : W \to X$ is the blow-up along $C_1$ with exceptional divisor $E$, then $\overline{D} = f^*D$ has $\sigma_{\overline{D}}(\overline{D}; W/S) = \infty$ and $N_*(\overline{D}; W/S)$ contains the divisor $D$ with infinite coefficient. In particular, there does not exist a birational model $g : Z \to W$ for which $g^*\overline{D}$ admits a decomposition $g^*\overline{D} = P + N$ with $P$ a $g \circ (f \circ \pi)$-movable divisor and $N$ effective.

**Proof.** By Theorem 5(2), if $f : W \to X$ is the blow-up along $C_1$, with exceptional divisor $E$, we have $\sigma_{\overline{D}}(f^*D; W/S) = \infty$. Now, suppose that $g : Y \to W$ is any birational map, and that $g^*f^*D = P + N$, where $P$ is a $(g \circ f \circ \pi)$-movable divisor and $N$ effective. Let $\tilde{E}$ denote the strict transform of $E$ on $Y$. Then

$$\sigma_{\tilde{E}}(g^*f^*D; Y/S) \leq \sigma_{\tilde{E}}(P; Y/S) + \sigma_{\tilde{E}}(N; Y/S) = \sigma_{\tilde{E}}(N; Y/S).$$

The last of these is finite since $N$ is effective, while the first is infinite, a contradiction. This completes the proof. \quad \square

## 4. A general set-up

The key feature that made possible the computation of the preceding example is that if the four numbers $D \cdot C_i$ and $\sigma_{C_i}(D)$ are all known, then the same four invariants can be computed for the strict transform of $D$ under $\phi$ using Lemma 8. In this section, we give an
John Lesieutre

Figure 3. Chambers in $N^1(X/S)$

explanation for this, and describe how to make analogous computations in a more general setting.

Suppose that $X$ is normal and $\mathbb{Q}$-factorial and that $\phi: X \to X$ is a pseudoautomorphism over $S$ (i.e. a birational map for which neither $\phi$ nor $\phi^{-1}$ contracts any divisors). We will say that a birational morphism $f: Y \to X$ from a normal $\mathbb{Q}$-factorial variety $Y$ is a small lift of $\phi$ if the induced map $\psi: Y \to Y$ is also a pseudoautomorphism.

\[
\begin{array}{ccc}
Y & \xrightarrow{\psi} & Y \\
\downarrow f & & \downarrow f \\
X & \xrightarrow{\phi} & X
\end{array}
\]

Observe that if $f: Y \to X$ is a small lift, then the map $\psi$ must permute the exceptional divisors of $f$.

**Example 11.** Suppose that $\phi: X \to X$ is a pseudoautomorphism and $x$ is a point not contained in indet $\phi$. The blow-up $f: \text{Bl}_x X \to X$ is a small lift of $\phi$ if and only if $x$ is a fixed point of $\phi$. If $x$ is not a fixed point, then the induced map $\psi: Y \to Y$ contracts the exceptional divisor $E$, while if $x$ is fixed, then $\psi|_E: E \to E$ is an automorphism.

The more interesting examples are those in which $f$ contracts a divisor lying over indet $\phi$.

**Example 12.** Next we construct a small lift of the map $\phi: X \to X/S$ from Section 3. Let $f: W \to X$ be the blow-up along $C_1$ as before, with exceptional divisor $E_1$, and let $h: Y \to W$ be the blow-up along $C'_2$, with exceptional divisor $E_2$. The two exceptional divisors $E_1$ and $E_2$ are swapped by the induced map $\psi: Y \to Y$, and $h \circ f$ is a small lift.

\[
\begin{array}{ccc}
Y & \xrightarrow{\psi} & Y \\
\downarrow h & & \downarrow h \\
W & \xrightarrow{\psi} & W \\
\downarrow f & & \downarrow f \\
X & \xrightarrow{\phi} & X
\end{array}
\]

The curves $C_1$ and $C_2$ could have been blown up in the opposite order, yielding a different small lift $f': Y' \to X$. This makes no real difference: the threefolds $Y$ and $Y'$ differ only by
flops, and strict transform induces an identification $N^1(Y) \sim N^1(Y')$ with respect to which the maps $\psi_*$ and $\psi'_*$ coincide.

If $f : Y \to X$ is a small lift, it follows from the negativity lemma [1, Lemma 3.6.2] and the $\mathbb{Q}$-factoriality of $X$ that there is a decomposition $N^1(Y) = f^*N^1(X) \oplus V_E$, where $V_E = \bigoplus \mathbb{R} \cdot [E_i]$. If $D$ is a divisor class on $X$, it is not necessarily true that $f^*\phi_*D = \psi_*f^*D$. However, the difference $f^*\phi_*D - \psi_*f^*D$ is an $f$-exceptional divisor, since

$$f_*(f^*\phi_*D - \psi_*f^*D) = \phi_*D - f_*(\psi_*f^*D) = \phi_*D - \phi_* f_*f^*D = \phi_*D - \phi_*D = 0.$$ Define $K : N^1(X) \to V_E$ by $K = f^*\phi_* - \psi_*f^*$. The next lemma characterizes the action of the strict transform $\psi_* : N^1(Y) \to N^1(Y)$ with respect to this decomposition.

**Lemma 13.** Suppose that $X$ and $Y$ are $\mathbb{Q}$-factorial and $f : Y \to X$ is a small lift of a pseudoautomorphism $\phi : X \dasharrow X$. With respect to the decomposition $N^1(Y) \cong f^*N^1(X) \oplus V_E$, $\psi_*$ is given in block form as

$$\psi_* = \left( \begin{array}{c|c} \phi_* & 0 \\ \hline -K & P \end{array} \right),$$

where $P$ is the permutation matrix for the action of $\psi_*$ on the $E_i$. The eigenvalues of $\psi_*$ are the union of those of $\phi_*$ and those of $P$, which are roots of unity. Its eigenvectors are

1. $f^*v_i - (\lambda_i I - P)^{-1}Kv_i$, where $v_i$ are the eigenvectors of $\phi_*$, with eigenvalues $\lambda_i$;
2. $E_i$, the exceptional divisors of $f$, with eigenvalues that are roots of unity.

**Proof.** For a divisor $D$ on $X$, $\psi_*f^*D = f^*\phi_*D - KD$, while the exceptional divisors $E_i$ are simply permuted by $\psi$; this gives the block form of the map. The eigenvectors follow from elementary linear algebra. \hfill \square

Note that if $\psi_*$ fixes the exceptional divisors, which can always be arranged by replacing $\phi$ and $\psi$ by suitable iterates, the permutation matrix $P$ is the identity.

A rational map $\phi : X \dasharrow Y$ between $\mathbb{Q}$-factorial varieties is said to be $D$-non-negative for an $\mathbb{R}$-divisor $D$ if on some common resolution $p : W \to X$, $q : W \to Y$, we have $p^*D + E = q^*(\phi_*D)$, where $E$ is an effective $q$-exceptional divisor. If $\phi : X \dasharrow X$ is a pseudoautomorphism with a small lift $f : Y \to X$, then we may consider a resolution of the form

\[
\begin{array}{c}
W \\
\downarrow p \\
Y \\
\downarrow f \\
X
\end{array}
\begin{array}{c}
\downarrow q \\
Y \\
\downarrow f \\
X
\end{array}
\]

If $D$ is a divisor on $X$ for which $\phi$ is $D$-non-negative, then we have $p^*f^*D + E = q^*f^*\phi_*D$ with $E \geq 0$. Pushing forward both sides by $q$, this gives

$$q_*p^*f^*D + q_*E = f^*\phi_*D$$

$$\psi_*f^*D + E' = f^*\phi_*D,$$

where $E'$ is an effective $f$-exceptional divisor. In particular, $KD = f^*\phi_*D - \psi_*f^*D = E'$ is effective.
Next we observe that if the divisorial Zariski decomposition of $f^*D$ is known for some divisor $D$, the decomposition of $f^*\phi_s D$ can often be computed. Although in earlier sections we assumed that $X$ was smooth, in what follows $X$ need only be normal and $\mathbb{Q}$-factorial, as we will only consider asymptotic multiplicities along divisors, rather than higher-dimensional subvarieties (which might lie entirely in the singular locus of $X$). We assume for simplicity that $\psi$ fixes each of the $f$-exceptional divisors $E_i$; this can always be arranged by replacing $\phi$ by a suitable iterate. This assumption implies that the permutation matrix $P$ is the identity, and that $\psi_s(KD) = KD$ since $KD$ is $f$-exceptional.

**Lemma 14.** Suppose $\phi : X \rightarrow X$ is a pseudoautomorphism over $S$, and that $D$ is a class in $N^1(X/S)$ with $\sigma_\Gamma(D; X/S)$ finite for all divisors $\Gamma$. Then $N_\sigma(\phi_s D; X/S) = \phi_s N_\sigma(D; X/S)$ and $P_\sigma(\phi_s D; X/S) = \phi_s P_\sigma(D; X/S)$. If $\phi$ is $D$-non-negative and $N_\sigma(f^*D; Y/S)$, then $P_\sigma(f^*\phi_s D; Y/S) = \psi_s P_\sigma(f^*D; Y/S)$.

**Proof.** Since $\phi$ neither contracts nor extracts any divisors, for any prime divisor $E$ we have $\sigma_E(D; X/S) = \sigma_{\phi \circ E} (\phi_s D; X/S)$. The claim for $N_\sigma(\phi_s D; X/S)$ follows, and that for $P_\sigma(\phi_s D; X/S)$ is immediate.

Now, by the $D$-non-negativity hypothesis on $\phi$, $KD$ is an effective $f$-exceptional divisor. By [11, Lemma 3.5.1], if $E$ is an effective $f$-exceptional divisor, we have $N_\sigma(f^*D + E) = N_\sigma(f^*D) + E$. This means that

$$N_\sigma(f^*\phi_s D) = N_\sigma(\psi_s f^* D + KD) = N_\sigma(\psi_s (f^*D + KD)) = \psi_s N_\sigma(f^*D + KD)$$

$$= \psi_s N_\sigma(f^*D) + \psi_s KD = \psi_s N_\sigma(f^*D) + KD.$$

We have made use of the fact that $E$ is effective by the non-negativity hypothesis on $D$. It is now simple to compute the positive part of the decomposition:

$$P_\sigma(f^*\phi_s D) = f^*\phi_s D - N_\sigma(f^*\phi_s D) = f^*\phi_s D - \psi_s N_\sigma(f^*D) - KD$$

$$= \psi_s f^* D - N_\sigma(\psi_s f^* D) = P_\sigma(\psi_s f^* D) = \psi_s P_\sigma(f^*D). \quad \square$$

**Remark 2.** The example of Section 3 can be interpreted as an instance of the calculations in this section. A small lift of the map $\phi$ is constructed in Example 12. Let $F_1, F_2$ be a basis for $N^1(X/S)$ dual to $C_1$ and $C_2$. A basis for $N^1(Y/S)$ is given by the four classes $(h \circ f)^* F_1, (h \circ f)^* F_2, E_1$, and $E_2$. The vector $\nu_\rho$ gives the coefficients for the class of the strict transform of $D$ on $Y$ with respect to this above basis. Lemma 8 is nothing more than the calculation of the induced map $\psi_s$ of Lemma 13. The final calculation in Theorem 9 can then be carried out as a repeated application of Lemma 14.

Suppose now that $S = \text{Spec} \mathbb{C}$ and $\phi : X \rightarrow X$ is a pseudoautomorphism whose action $\phi_s : N^1(X) \rightarrow N^1(X)$ has a unique largest eigenvalue $\lambda > 1$, and that $f : Y \rightarrow X$ is a small lift of $\phi$. Since $S = \text{Spec} \mathbb{C}$, the pseudoeffective cone is strictly convex, and by a version of the Perron-Frobenius theorem [3] there exists a $\lambda$-eigenvector $D_\phi$ which is contained in the pseudoeffective cone. Lemma 13 then provides the existence of a $\lambda$-eigenvector $D_\psi$ for $\psi_s$.

We are then able to compute the Zariski decomposition of the divisor $f^*D_\phi$ using Lemma 14.

**Corollary 15.** Let $D_\phi$ be the dominant eigenvector of $\phi_s : N^1(X) \rightarrow N^1(X)$, and $D_\psi$ be the dominant eigenvector of $\psi_s : N^1(Y) \rightarrow N^1(Y)$. If $D$ is $\phi$-non-negative, then $P_\sigma(f^*D_\phi) = D_\psi$.

**Proof.** If $D$ is any pseudoeffective divisor on $X$, then for every $n$ we have

$$P_\sigma(f^*(\lambda^{-n}\phi^*_s D)) = \lambda^{-n}\psi^*_s P_\sigma(f^*D).$$
Take $D = D_\phi + D_{\phi^{-1}}$, so that the above reduces to

$$P_\sigma(f^*(D_\phi + \lambda^{-2n}D_{\phi^{-1}})) = \lambda^{-n}\psi^n P_\sigma(f^*D).$$

The left hand side converges to $P_\sigma(f^*D_\phi)$ by Proposition 5(1). With a suitable choice of scaling, the right hand side converges to $D_\psi$. □

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