Compatible Trees and Outer Automorphisms of a Free Group

by

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This thesis is dedicated to all the teachers who told me I'd never amount to nothin.

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SUMMARY

This thesis is wrapped up in the analogy among arithmetic groups, mapping class groups of surfaces, and outer automorphisms of free groups. Each of these groups satisfies Tits' alternative for subgroups, but there is disanalogy in the finer subgroup structure. The motivating question, which this thesis answers in the linearly growing case, is whether or not two generator subgroups of the outer automorphism group of a free group behave like those in arithmetic groups or like those mapping class groups. In mapping class groups, any two mapping classes have the property that a sufficient power of the two classes will generate a group that is either free of rank two or abelian; this is not true in arithmetic groups, and unknown in general for outer automorphism groups. To study the motivating question the theory of group actions on trees is expanded, and new tools for understanding when two tree actions are compatible are introduced. These tools are ultimately much stronger than necessary for answering the motivating question in the linearly growing case, but it is hoped they will provide a method for answering the motivating question in the future.

CHAPTER 1

IN THE GARDEN OF FORKING PATHS

I leave to the various futures (not to all) my

garden of forking paths.

TS'UI PÊN

Begin with the most famous group which acts on a tree, $SL(2, \mathbb{Z})$ and its cousin, $GL(2, \mathbb{Z})$. The classical theory of these groups touches on number theory, algebraic geometry, and topology in myriad ways. Topologically, $SL(2, \mathbb{Z})$ is the mapping class group $\mathrm{Mod}(\Sigma)$ of orientation-preserving homeomorphisms modulo isotopy of either the torus or the once-punctured torus. Broadening the perspective to include all homeomorphisms up to isotopy, we grow naturally into $GL(2, \mathbb{Z})$ as the extended mapping class group $\mathrm{Mod}^{\pm}(\Sigma)$ of both surfaces. Free groups enter the picture when we focus on the punctured torus $\Sigma_{1,1}$, where $\pi_1(\Sigma_{1,1}) \cong F_2$ and there are isomorphisms [24]

$$GL(2, \mathbb{Z}) \cong \mathrm{Mod}^{\pm}(\Sigma_{1,1}) \cong \mathrm{Out}(F_2).$$

This group $SL(2, \mathbb{Z})$ marks the first fork in our path through this garden, each of these three perspectives gives a different generalization. The first, to $GL(n, \mathbb{Z})$ and arithmetic groups, and the second, to $\mathrm{Mod}(\Sigma)$ for a finite-type surface Σ will not be taken in this work in any detail. This thesis focuses on the third, $\mathrm{Out}(F_r)$ for $r \geq 2$, the outer automorphisms of a free group.

These three perspectives are not wholly unrelated; the paths are similar and an analogy among the three families of groups is incredibly deep, guiding the past fifty years of research in the area. In the case of $\mathrm{Out}(F_r)$, Bestvina reminds us that these connections are not only spiritual [5]. The abelianization functor induces a surjection $\mathrm{Out}(F_r) \to GL(r, \mathbb{Z})$ and any surface Σ with fundamental group F_r gives an injection $\mathrm{Mod}(\Sigma) \to \mathrm{Out}(F_r)$.

The Tits alternative richly illustrates the parallel techniques used in the study of $\mathrm{Out}(F_r)$, $\mathrm{Mod}(\Sigma)$, and arithmetic groups [42]. In 1972 Tits [49] proved that any linear group over a field of characteristic zero either contains a non-abelian free subgroup or a finite index solvable subgroup. Such a dichotomy is now known as a Tits alternative. Specifically, a class of groups $\mathfrak{G}$ satisfies a Tits alternative when every member either contains a F_2 subgroup or a finite-index solvable subgroup. This implies the von Neumann conjecture (that a group is either amenable or has an F_2 subgroup) for $\mathfrak{G}$ and excludes Tarskii monster groups from $\mathfrak{G}$. When $\mathfrak{G}$ is the set of subgroups of a fixed group G, a Tits alternative for $\mathfrak{G}$ is an important component of the study of the subgroup structure of G.

Tits' proof exploits the dynamics of the action of a linear group G on a suitable projective space. Tits used this action to verify a criterion for freeness, the ping-pong lemma, which dates to Klein [34]. A version of the lemma states: if a two-generator group $G = \langle a, b\rangle$ acts on a set P and there are disjoint non-empty subsets $P_a, P_b \subseteq P$ such that for all $n > 0$ $a^{\pm n}(P_b) \subseteq P_a$ and $b^{\pm n}(P_a) \subseteq P_b$, then $G \cong F_2$. Subgroups of $\mathrm{Mod}(\Sigma)$ also satisfy a Tits alternative, as was shown independently by McCarthy [39] and Ivanov [31]. McCarthy's argument is analogous to Tits', with the Thurston boundary of Teichmüller space, the set of projective measured laminations, playing the role of projective space. His proof also makes use of a stronger statement for two-generator subgroups. If $\langle \sigma, \tau\rangle \leq \mathrm{Mod}(\Sigma)$, then there exists an integer $N > 0$ such that $\langle \sigma^N, \tau^N\rangle$

is either abelian or free of rank two. Moreover, $\langle \sigma^N, \tau^N \rangle$ is abelian exactly when the intersection number of certain laminations associated to σ and τ is zero. This stronger statement about two-generator subgroups is false in $GL(n, \mathbb{Z})$, due to the presence of the Heisenberg group.

Bestvina, Feighn, and Handel established a Tits alternative for subgroups of $\mathrm{Out}(F_r)$ [7, 8]. Their proof, in two papers, draws on analogies with both $\mathrm{Mod}(\Sigma)$ and $GL(n, \mathbb{Z})$. The notion of a lamination of a free group is introduced in the first paper [7], analogous to a lamination of a surface. Free group laminations are algebraic; no notion of measure is introduced. Instead, Bestvina, Feighn, and Handel develop a topological attraction theory for laminations and use this attraction theory to play ping-pong on the set of laminations. This ping-pong argument reduces the problem to the establishment of a Tits alternative for subgroups consisting entirely of unipotent outer automorphisms. A **unipotent** outer automorphism is a polynomially growing outer automorphism with unipotent image in $GL(r, \mathbb{Z})$. In the mapping class group the analogous subgroups are all virtually abelian, but $\mathrm{Out}(F_r)$ contains unipotent free subgroups. The second paper [8] addresses the case of unipotent subgroups using an analog of Kolchin's theorem on the existence of invariant flags for unipotent subgroups of linear groups. Their proof of this analog makes use of the dynamics of the $\mathrm{Out}(F_r)$ action on a space of F_r actions on trees. No general analog of McCarthy's theorem on two generator subgroups is known for $\mathrm{Out}(F_r)$. The analogy between $\mathrm{Mod}(\Sigma)$ and $GL(n, \mathbb{Z})$ does not extend to McCarthy's theorem and it is of interest to know which of the two families $\mathrm{Out}(F_r)$ follows. The situation is summarized in Table I.

TABLE I

ANALOGY AND DISANALOGY FOR THE TITS ALTERNATIVE

	$GL(n,\mathbb{Z})$	$\mathrm{Mod}(\Sigma)$	$\mathrm{Out}(F_r)$
Geometric Action	Projective space	Teichmüller Space	Outer Space
Tits Alternative	Yes	Yes	Yes
McCarthy's Theorem	No	Yes	?

Virtually solvable subgroups of both $\mathrm{Out}(F_r)$ and $\mathrm{Mod}(\Sigma)$ must be virtually abelian [9, 12], in contrast with $GL(n,\mathbb{Z})$. This fact, along with analogs of McCarthy's theorem for subgroups of $\mathrm{Out}(F_r)$ with special generators, motivates the following conjecture.

Conjecture A. Given a two generator subgroup $\langle \sigma, \tau \rangle \leq \mathrm{Out}(F_r)$ there is an N such that $\langle \sigma^N, \tau^N \rangle$ is either abelian or free of rank two.

Bestvina, Feighn, and Handel [6] prove a special case, in which the two generators are required to be fully irreducible. An outer automorphism is **fully irreducible** if it has no periodic free factors. A subgroup of $\mathrm{Out}(F_r)$ generated by two fully irreducible elements, σ and τ, is either virtually cyclic or contains a free group generated by powers of σ and τ. The subgroup is virtually cyclic exactly when the invariant lamination sets for the two generators are equal. Ghosh [25] extends this result, using Handel and Mosher's omnibus subgroup theorem, to a larger family of exponentially growing generators. Building on the work of Cohen and Lustig [18], and using a variation on the ping-pong technique used by Clay and Pettet [17], we show the conjecture holds for groups generated by Dehn twist outer automorphisms, in Chapter 5; as a corollary this implies the conjecture for all linearly growing outer automorphisms. Cohen and Lustig defined the notion of an efficient Dehn twist automorphism with respect to a graph of groups and showed that every Dehn twist outer automorphism has a unique such representative. A major theme of this thesis is that compatibility of efficient representatives substitutes for intersection number, and gives a criterion for when the group generated is free of rank two (even in cases not covered by the Clay and Pettet technique). Work in progress continues this direction, extending the techniques to a McCarthy-type result for unipotent outer automorphisms. Every unipotent outer automorphism is represented by an efficient relative Dehn twist with respect to a graph of groups [43], generalizing the efficient Dehn twist representatives of Cohen and Lustig. Unfortunately, the hierarchical nature of polynomially growing outer automorphisms makes the compatibility condition complicated to state, and it is not clear if there is a geometric invariant to divide the cases.

In each of these McCarthy-type theorems the $\mathrm{Out}(F_r)$ action on a compact space of F_r actions on trees plays a role. Culler and Vogtmann introduced Outer Space, cv_r as the space of marked metric graphs [23]. The universal cover of a marked metric graph is a metric tree with free properly discontinuous F_r action, and we take this perspective on cv_r. Restricting to those actions with covolume 1 gives the projectivized outer space CV_r. Culler and Morgan compactified CV_r using an embedding into the space of length functions on F_r [22]. It was later shown by Cohen, Lustig, Bestvina, and Feighn that each point in the closure is realized uniquely as the translation length function of a very small F_r action on an $\mathbb{R}$-tree [10, 18] .

The space of laminations of F_r and the boundary of CV_r in this compactification both serve as analogs of Thurston's boundary of Teichmüller space; but unlike in surface theory the two are not equivalent.

Guirardel [26] defined an intersection number for $\mathbb{R}$-trees that generalizes the intersection number for measured laminations of a surface. The Guirardel intersection number of $\pi_1(\Sigma)$ actions on $\mathbb{R}$-trees dual to measured laminations is equal to the lamination intersection number. The intersection number $i(A, B)$ of very small $\mathbb{R}$-trees A and B is the "covolume" of a particular subset $\mathcal{C} \subseteq A \times B$ called the core. The core $\mathcal{C}(A, B)$ is the minimal subset of $A \times B$ invariant under the diagonal action such that both projection maps have convex fibers. It is not always connected, but has a canonical connected superset, the augmented core, $\widehat{\mathcal{C}}(A, B)$ with the same covolume. When T is a simplicial tree with free action and σ a fully irreducible outer automorphism, the asymptotic behavior of $i(T, \sigma^N T)$ is governed by the growth rate of σ [4], similar to a curve under iterations of a pseudo-Anosov mapping class.

The geometry of the core provides rich information about the input trees in specific cases. Guirardel gives a topological interpretation of the core of simplicial trees, and analyzes the core of the two fixed trees of a geometric fully irreducible outer automorphism. The ability of the core to measure the compatibility of length functions on a group is taken up in Chapter 3, where it is related to the ambient vector space structure of length functions on a group. This relationship is exploited in the specific case of trees related to unipotent outer automorphisms, where the core is the key technical tool in the analysis in Chapter 5.

1.1 Plan of the thesis

Moral: always carry a map when you are in hyperbolic space!

W. P. Thurston

The many species of tree to be encountered in this thesis, and the associated theory, are grown in Chapter 2. An analysis of their compatibility as it relates to both the vector space structure of length functions and Guirardel's core is conducted in Chapter 3, where the PL structure of certain deformation spaces of trees is also described. The theory of outer automorphisms of a free group, and in particular unipotent outer automorphisms, arrives in Chapter 4. This theory is connected with the compatibility of certain trees and a McCarthy-type theorem for linearly growing outer automorphisms given in Chapter 5. Future directions for an attack on Conjecture A, as well as what avenues would be opened up by its resolution are sketched in Chapter 6.

CHAPTER 2

ARDORS AND ARBORS

I think that I shall never see

A poem lovely as a tree.

Joyce Kilmer

The theory of group actions on (real) trees has been an important aspect of the study of groups in the past fifty years. This thesis, which also studies groups, is no exception, so we review the parts of the theory that are particularly relevant.

2.1 Real trees

The key property of simplicial trees is that the geodesic joining two points is the unique arc joining the points [28]. Taking this property to a general metric space we arrive in the forest of real trees. First, some notation. In a uniquely geodesic metric space X, let $[p,q]$ denote the geodesic from p to q. If $p,q,r \in X$ and $r \in [p,q]$, we will use the notation $[p,r,q]$ for the geodesic path, for emphasis.

Definition 2.1.1. A **real tree** or $\mathbb{R}$-tree T is a complete connected uniquely geodesic metric space such that for any pair of points $p,q \in T$ there is a unique arc $[p,q]$ from p to q, and this arc is isometric to $[0,d(p,q)]$. A **subtree** of a real tree is a complete connected subset $S \subseteq T$.

Throughout this thesis, when $e \subseteq T$ is an arc in a real tree we will assume this arc is oriented, that is we have a fixed isometry $\gamma : [0, \mathrm{length}_T(e)] \to T$. We will use the notation $o(e) = \gamma(0)$ and $t(e) = \gamma(\mathrm{length}_T(e))$ for the origin and terminus of the arc, and $\bar{e}$ for the reversed orientation.

To keep the notation uncluttered we will not refer to the isometry γ unless it is desperately necessary for clarity. We will always specify an orientation when specifying an arc (or it will inherit one from context, by being a sub-arc of an oriented arc).

Example 2.1.2. Real trees abound in mathematics, Cayley graphs of a free group (with respect to a basis and its inverses) are a first natural example. Another example is $\mathbb{R}^2$ with the so-called Paris metric: if two points $x, y \in \mathbb{R}^2$ are on the same ray based at the origin, $d_P(x, y) = d_{euc}(x, y)$ the euclidean distance. Otherwise, $d_P(x, y) = ||x|| + ||y||$.

In the study of real trees, logging is a useful tool, which has a well developed vocabulary.

Definition 2.1.3. Let T be a real tree. A point $p \in T$ is a **branch point** if $T \setminus \{p\}$ has more than two connected components. The **order** of a branch point is the number of connected components of $T \setminus \{p\}$. A point $p \in T$ is a **leaf** if $T \setminus \{p\}$ is connected. A **direction** based at p, $\delta_p \subseteq T$, is a connected component of $T \setminus \{p\}$.

Not all real trees have leaves, and for present purposes those with leaves will turn out to be uninteresting. Even without leaves real trees have a useful notion of an end and a boundary.

Definition 2.1.4. The **visual boundary** of a real tree T based at $p \in T$ is the set

$$\partial_p T = \{\rho \subseteq T | \ \rho \text{ is a geodesic ray based at } p \ \}.$$

The boundary is topologized by the basis of open sets

$$V(\rho, r) = \{\gamma \in \partial_p T \mid B(p, r) \cap \gamma = B(p, r) \cap \rho\}$$

for $r > 0$ and $\rho \in \partial_p T$.

This boundary can be thought of as all the different ways a squirrel can escape a cat chasing it along the tree. This definition is also used in the more general setting of a δ-hyperbolic metric space, though in that generalization an equivalence relation is needed on geodesic rays. Different base points p give different identifications of the same boundary.

Lemma 2.1.5 ([13, Proposition II.8.8])**.** For any $p, q \in T$ $\partial_p T$ is homeomorphic to $\partial_q T$, via the map that takes a ray $\gamma \in \partial_p T$ to the geodesic tightening of the concatenation of paths $[q, p]\gamma$.

The above discussion applies entirely to simplicial trees, treating them as real trees by giving the topological realization the metric induced by a choice of metric on each one cell. A real tree that came from this procedure can be recognized,

Definition 2.1.6. A real tree T is **simplicial** if the set of branch points is discrete.

Lemma 2.1.7. A simplicial real tree T has the structure of a simplicial complex with no 2-cells; this structure is the unique such structure with no valence 2 zero cells.

Remark. The metric topology and CW-topology do not agree when T has infinite order branch points.

2.2 Lengths and actions

Definition 2.2.1. Let G be a group and $\rho : G \to Isom(T)$ be an injection, with T a real tree, so that G acts on T on the right. The triple (G, ρ, T) is a G**-tree**.

Throughout this thesis, and as is standard in the literature, the action of G will be clear from context and G will be fixed, so we suppress the notation and refer to a tree T as a G-tree. If two trees are isometric but carry different G actions they will be referred to by separate letters, or as necessary other great pains will be taken to clarify the situation. The restriction to actions where $\rho : G \to Isom(T)$ is injective is not standard in the literature, some authors allow group actions with kernel; however, tree actions with kernel are better treated as $(G/\ker\rho)$-trees. Works that treat the more general case typically call actions without kernel **effective**.

The action of G extends naturally to a continuous action on ∂T (this is true of any isometry of T) [13, Corollary II.8.9]. Following a long tradition, we study the geometry of G-tree actions via their translation length functions, which have been likened to the characters of a representation. The elements of G-tree geometry reviewed here are for the most part based on the exposition given by Culler and Morgan [22], with other developments cited as relevant.

Definition 2.2.2. The **translation length function** of a G-tree T, denoted $\ell_T : G \to \mathbb{R}$ is defined by

$$\ell_T(g) = \inf_{p \in T} d_T(p, p \cdot g).$$

Any G-tree T divides the elements $g \in G$ into **hyperbolic** elements, when $\ell_T(g) > 0$ and **elliptic** elements, when $\ell_T(g) = 0$. When an element $g \in G$ is elliptic, $\mathrm{Fix}(g)$ will denote the set of fixed points of g. Given a subtree $S \subseteq T$, $\mathrm{Stab}(S)$ will denote the set of group elements that fix S pointwise, and $\mathrm{Inv}(S)$ the set of group elements that fix S set-wise.

The metric on a G-tree T induces a G invariant measure, and the **covolume** of T is

$$\mathrm{covol}(T) = \inf_{S \subseteq T} \{\mu_T(S) | S \cdot G = T \text{ and } S \text{ is measurable}\}$$

For a simplicial G-tree, $\mathrm{covol}(T)$ is the total length of the one cells in the quotient.

2.2.1 A taxonomy

There is a well established taxonomy of G-trees for a given G, and this taxonomy is intimately related to the translation length functions.

Definition 2.2.3. A G-tree T is **minimal** if there is no proper G invariant subtree $T' \subsetneq T$.

Definition 2.2.4. A G-tree T where for all $g \in G$, $\mathrm{Fix}(g) \neq \emptyset$ is **trivial**.

For finitely generated groups this is equivalent to the condition that G has a global fixed point, but this is not true for infinitely generated groups [40, 50]. These actions are invisible to translation length functions and will not be pursued further.

Definition 2.2.5. A G-tree T is **lineal** if there is a G invariant subtree homeomorphic to the line.

Definition 2.2.6. A G-tree T is **reducible** if G fixes an end of T.

Lineal and reducible actions are uninteresting from the perspective of translation length functions, in light of the foundational work of Culler and Morgan.

Theorem 2.2.7 ([22, Theorem 2.4,2.5])**.** If T is a lineal or reducible G-tree, then there is a homomorphism $\rho : G \to Isom(\mathbb{R})$ such that $\ell_T(g) = N(\rho(g))$, where N is the translation length function of the induced action on $\mathbb{R}$.

The G-trees of interest for this thesis are the ones whose study is not an indirect study of subgroups of $Isom(\mathbb{R})$.

Definition 2.2.8. A G-tree T is **irreducible** if it is minimal and neither trivial, lineal, nor reducible.

Irreducible G-trees admit detailed study through their translation length functions, indeed the translation length function is an isometry invariant.

Theorem 2.2.9 ([22, Theorem 3.7])**.** Suppose A and B are two irreducible G trees and $\ell_A = \ell_B$. Then there is an equivariant isometry from A to B.

Chiswell, building on work of Lyndon studied based length functions $L_p : G \to \mathbb{R}$, where $L_p(g) = d(p, p \cdot g)$. These functions are useful for concrete calculations and when establishing the existence of an equivariant isometry between G-trees.

2.2.2 Axes

Definition 2.2.10. The **characteristic set** of some $g \in G$ in a G-tree T is the set

$$C_g^T = \{p \in T | d(p, p \cdot g) = \ell_T(g)\}$$

of points achieving the translation length. When T is clear from context we write C_g.

Lemma 2.2.11 ([22, Lemma 1.3])**.** For any G-tree T and $g \in G$, the characteristic set C_g^T is a closed non-empty subtree of T invariant under g. Moreover,

- If $\ell_T(g) = 0$ then $C_g = \mathrm{Fix}(g)$.
- If $\ell_T(g) > 0$ then C_g is isometric to the real line and the action of g on C_g is translation by $\ell_T(g)$. In this case we call C_g the **axis** of g.

- For any $p \in T$, $d(p, p \cdot g) = \ell_T(g) + d(p, C_g)$.

The axis of a hyperbolic element is an important subtree. When g is a hyperbolic element of a G-tree T, ∂C_g is a pair of fixed ends. The action of g on C_g gives C_g a natural orientation and we always consider an axis oriented by the element specifying it, so that $C_{g^{-1}}$ is the same set as C_g but with the opposite orientation. The point of ∂T in the equivalence class of a positive ray along C_g with the g orientation will be denoted $\omega_T(g)$. If g is elliptic we take the convention that g has no boundary at infinity, even if C_g is some interesting subtree. When g is hyperbolic, $\omega_T(g)$ is an attracting fixed point for the action of g on ∂T (this follows from the third item).

Definition 2.2.12. Let T be a G-tree. The T**-boundary** of $g \in G$, $\partial_T g$ is the empty set if g is elliptic, and the set $\{\omega_T(g), \omega_T(g^{-1})\}$ if g is hyperbolic.

The intersection of characteristic sets is deeply related to the translation length function.

Lemma 2.2.13 ([22, Lemma 1.5])**.** Let T be a G-tree. For any $g, h \in G$ such that $C_g \cap C_h = \emptyset$, we have

$$\ell(gh) = \ell(gh^{-1}) = \ell(g) + \ell(h) + 2d(C_g, C_h)$$

This lemma is also used in its contrapositive formulation, if $\ell(gh) \leq \ell(g) + \ell(h)$, then $C_g \cap C_h \neq \emptyset$. For hyperbolic isometries there is a more precise relationship between the intersection of characteristic sets and the length function.

Lemma 2.2.14 ([22, Lemma 1.8])**.** Suppose g and h are hyperbolic in a G-tree T. Then $C_g \cap C_h \neq \emptyset$ if and only if

$$\max\{\ell_T(gh), \ell_T(gh^{-1})\} = \ell_T(g) + \ell_T(h).$$

Moreover $\ell(gh) > \ell(gh^{-1})$ if and only if $C_g \cap C_h$ contains an arc and the orientations of C_g and C_h agree on $C_g \cap C_h$.

These two lemmas are proved by the construction of explicit fundamental domains. These fundamental domains are sufficiently useful that we detail them here, that they have the claimed properties is a consequence of the proofs of the previous two lemmas.

Definition 2.2.15. Let T be a G-tree and suppose g and h are such that $C_g \cap C_h = \emptyset$. Let $\alpha = [p, q]$ be the geodesic joining C_g to C_h. The **Culler-Morgan fundamental domain** for the action of gh on C_{gh} is the geodesic

$$[p \cdot g^{-1}, p, q, q \cdot h, p \cdot h].$$

Definition 2.2.16. Let T be a G-tree and suppose g and h are such that $C_g \cap C_h \neq \emptyset$, at least one of g and h is hyperbolic, and that if both g and h are hyperbolic the orientations agree. Let $\alpha = [p, q]$ be the possibly degenerate ($p = q$) common arc of intersection with the induced orientation. The **Culler-Morgan fundamental domain** for the action of gh on C_{gh} is the geodesic

$$[q \cdot g^{-1}, q, q \cdot h].$$

If gh^{-1} is also hyperbolic, then the **Culler-Morgan** fundamental domain for the action of gh^{-1} on $C_{gh^{-1}}$ is the geodesic

$$[q \cdot g^{-1}, q \cdot h^{-1}].$$

The axes of a minimal G-tree provide complete information about the G-tree.

Proposition 2.2.17 ([22, Proposition 3.1])**.** A minimal non-trivial G-tree T is equal to the union of the axes of the hyperbolic elements.

Building from this, we also get some understanding of the topology of ∂T

Proposition 2.2.18. The endpoints of axes of hyperbolic elements are dense in ∂T for a minimal G-tree T.

Proof. Fix a basepoint p and use the model $\partial_p T$ for ∂T. Suppose $\rho \subseteq T$ is a geodesic ray based at p, parameterize ρ by distance so that $\rho : [0, \infty) \to T$. If $\rho \cap C_g$ is a ray for some hyperbolic $g \in G$ then ρ represents either $\omega_T(g)$ or $\omega_T(g^{-1})$. Suppose ρ is not the end of any hyperbolic group element. Since T is covered by axes, there are group elements g_i such that $C_{g_i} \cap \rho = [p_i, q_i]$, $q_i \to \infty$ and $\rho = \bigcup [p_i, q_i]$, with orientation chosen to agree with that of ρ. Let γ_i be the geodesic ray based at q_i representing $\omega_T(g_i)$. By construction, $[p, q_i]$ is the geodesic joining p to q_i, so that $\rho_i = [p, q_i] \cup \gamma_i$ is the geodesic ray based at p representing $\omega_T(g_i)$. Since $q_i \to \infty$ we have $\omega_T(g_i) = \rho_i \to \rho$ in $\partial_p T$. $\square$

2.2.3 Axioms

For irreducible G-trees, length functions provide a complete invariant, as noted above. Culler and Morgan characterize these length functions in terms of a list of useful properties; Parry showed that any length function satisfying these axioms comes from an irreducible G-tree [41].

Definition 2.2.19. An **axiomatic length function** (or just **length function**) is a function $\ell : G \to \mathbb{R}_{\geq 0}$ satisfying the following six axioms.

1. $\ell(\mathrm{id}) = 0$.
2. For all $g \in G$, $\ell(g) = \ell(g^{-1})$.
3. For all $g, h \in G$, $\ell(g) = \ell(hgh^{-1})$.
4. For all $g, h \in G$, either

$$\ell(gh) = \ell(gh^{-1})$$

or

$$\max\{\ell(gh), \ell(gh^{-1})\} \leq \ell(g) + \ell(h).$$

5. For all $g, h \in G$ such that $\ell(g) > 0$ and $\ell(h) > 0$, either

$$\ell(gh) = \ell(gh^{-1}) > \ell(g) + \ell(h)$$

or

$$\max\{\ell(gh), \ell(gh^{-1})\} = \ell(g) + \ell(h).$$

6. There exists a pair $g, h \in G$ such that

$$0 < \ell(g) + \ell(h) - \ell(gh^{-1}) < 2\min\{\ell(g), \ell(h)\}.$$

Proposition 2.2.20 ([22])**.** If ℓ_T is the translation length function of an irreducible G-tree then ℓ_T is an axiomatic length function.

Theorem 2.2.21 ([41])**.** If ℓ is an axiomatic length function on a group G then there is an irreducible G-tree T such that $\ell = \ell_T$.

In the wider literature, axiom VI is omitted, including the consideration of all G-trees, instead of only irreducible G trees. Without this axiom, length functions are no longer a complete isometry invariant. A pair of elements witnessing Axiom VI for a given length function ℓ is called an **good pair** for ℓ. A comprehensive treatment is given in Section 2.4.

2.2.4 Deformation spaces

The above axiomatization describes a space $ILF(G) \subseteq \mathbb{R}^{\Omega}$, where Ω is the set of conjugacy classes of G. Axiom VI guarantees that every point is non-zero, so $ILF(G) \subseteq \mathbb{R}^{\Omega} \setminus \{0\}$, and the length function axioms are scale-invariant, so there is a well-defined projectivization $PILF(G) \subseteq P\mathbb{R}^{\Omega}$. Culler and Morgan show that $PILF(G)$ is compact [22, Theorem 4.5].

Requiring trees to have a particular property gives a variety of subspaces of $ILF(G)$ or their projectivization that have seen study in the literature. The following subspaces will be relevant.

- $Free(G)$ the space of free irreducible G-trees.

- $Simp(G)$ the space of simplicial G-trees.
- $SLF(G)$ the space of G-trees where no arc stabilizer contains a rank 2 free group (so-called **small** trees). $PSLF(G)$, its projectivization, is compact [22].
- $cv_r = Free(F_r) \cap Simp(F_r)$, Culler-Vogtmann Outer Space, and CV_r its projectivization [23].
- $\overline{CV}_r$ the closure of CV_r in $PILF(F_r)$. Combining the results of Cohen and Lustig with those of Bestvina and Feighn characterize the trees of $\overline{CV}_r$ in terms of certain stabilizers [10, 18].

2.3 Very small trees and bounded cancellation

The work of Cohen and Lustig combined with that of Bestvina and Feighn characterizes the F_r-trees representing projective classes in $\overline{CV}_r$ as the space of all **very small** trees [10, 18].

Definition 2.3.1. A G-tree T is **very small** if it is minimal, small, and has

- **No obtrusive powers:** for all $g \in G \setminus \{\mathrm{id}\}$ and n such that $g^n \neq e$, $\mathrm{Fix}(g) = \mathrm{Fix}(g^n)$.
- **No tripod stabilizers:** for all $a, b, c \in T$ such that the convex hull $H = Hull(a, b, c)$ is not a point or arc, $Stab(H) = \{\mathrm{id}\}$.

By virtue of their free simplicial approximability, many classical results about free groups have analogs for very small trees. One indispensable tool is Grayson and Thurston's bounded cancellation lemma, recorded by Cooper [20]. Fix a basis for the free group F_r and let $|\cdot|$ denote word length with respect to this basis. The classical bounded cancellation lemma states

Lemma 2.3.2 ([20])**.** Given an automorphism $f: F_r \to F_r$ there is a constant C such that for all $w_1, w_2 \in F_r$, if $|w_1 w_2| = |w_1| + |w_2|$ then

$$|f(w_1 w_2)| \geq |f(w_1)| + |f(w_2)| - C(f).$$

Let T be the F_r-tree given by the Cayley graph of the fixed basis, so that $|\cdot| = L_e$ is a based length function on this tree. An automorphism $f : F_r \to F_r$ induces a Lipschitz equivariant map $\tilde{f} : T \to T$; $\tilde{f}$ is the lift of some homotopy equivalence of a wedge of circles representing f. Lemma 2.3.2 implies that geodesics based at e get sent to the $\frac{C}{2}$ neighborhood of the geodesic between the endpoints. Since $\tilde{f}$ is equivariant, we conclude that for all finite geodesics $\gamma : [p, q] \to T$, $f(\gamma([p, q]))$ is in the $\frac{C}{2}$ neighborhood of the geodesic $[f(\gamma(p)), f(\gamma(q))]$. This property generalizes to equivariant maps between real trees:

Definition 2.3.3. An equivariant continuous map $f : S \to T$ between F_r trees has bounded cancellation with constant C if for all geodesics $\gamma : [p, q] \to S$, $f(\gamma)$ is in the C neighborhood of the T geodesic $[f(\gamma(p)), f(\gamma(q))]$.

In this form Bestvina, Feighn, and Handel give a bounded cancellation lemma for very small trees.

Lemma 2.3.4 ([6, Lemma 3.1])**.** Suppose T_0 is a free simplicial F_r-tree and T a very small F_r-tree, and $f : T_0 \to T$ is an equivariant Lipschitz map. Then f has a bounded cancellation constant $C(f)$ satisfying $C(f) \leq Lip(f)\,\mathrm{covol}(T_0)$.

Their proof uses free simplicial approximation to bootstrap this result from Lemma 2.3.2. This lemma in turn implies a form of bounded cancellation for length functions of very small trees, reminiscent of the form of Lemma 2.3.2

Lemma 2.3.5. Suppose T is a very small F_r tree and Λ a basis for F_r. There is a constant $C(\Lambda, T)$ such that for all $g, h \in F_r$, if $|gh|_\Lambda = |g|_\Lambda + |h|_\Lambda$ and gh is cyclically reduced with respect to Λ, then

$$\ell_T(gh) \geq \ell_T(g) + \ell_T(h) - C(\Lambda, T)$$

Proof. Let S_Λ be the universal cover of the rose marked by the basis Λ where all edges have length one. Suppose $f : S_\Lambda \to T$ is an equivariant Lipschitz surjection. (Such maps always exist: pick $* \in S_\Lambda$ and $\star \in T$, define $f : S_\Lambda^0 \to T$ on the zero skeleton by $f(* \cdot g) = \star \cdot g$ and extend linearly and equivariantly over edges. Since S_Λ has finitely many edge orbits, this extension is Lipschitz. Moreover, f is surjective since T is minimal.) By Lemma 2.3.4, f has bounded cancellation. Let $B = C(f)$ be the bounded cancellation constant. Suppose $g, h \in F_r$ satisfy $|gh|_\Lambda = |g|_\Lambda + |h|_\Lambda$ and gh is cyclically reduced. We will show that there is a constant C such that for all $q \in S_\Lambda$,

$$d(f(q), f(q \cdot gh)) \geq \ell_T(g) + \ell_T(h) - C \tag{$\dagger$}$$

Since f is equivariant and surjective, this implies the conclusion.

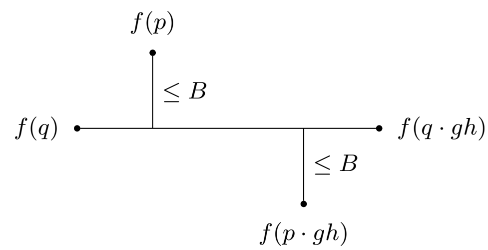


Figure 1. A convex hull in T.

We will establish Equation † by showing that for any $q \in S_\Lambda$ there is a $p \in C_{gh}^{S_\Lambda}$ so that

$$d(f(q), f(q \cdot gh)) \geq d(f(p), f(p \cdot gh)) - C', \tag{2.1}$$

and for all $p \in C_{gh}^{S_\Lambda}$,

$$d(f(p), f(p \cdot gh)) \geq \ell_T(g) + \ell_T(h) - C'' \tag{2.2}$$

Proof of Equation 2.1. Let p be the point of $C_{gh}^{S_\lambda}$ closest to q. The geodesic $[q, q \cdot gh]$ contains the points p and $p \cdot gh$. Consider the convex hull of the image in T of $q, p, q \cdot gh$, and $p \cdot gh$ (Figure 1). Since the map f has bounded cancellation, both $f(p)$ and $f(p \cdot gh)$ are in the B neighborhood of the geodesic $[f(q), f(q \cdot gh)] \subset T$, and we have

$$d(f(q), f(q \cdot gh)) \geq d(f(p), f(p \cdot gh)) - 2B.$$

Proof of Equation 2.2. Suppose now that $p \in C_{gh}^{S_\Lambda}$. We may assume that p is a vertex of S_Λ, indeed let p' be the closest vertex to p. Since f is equivariantly Lipschitz we have

$$d(f(p), f(p')) \leq \frac{1}{2} Lip(f)$$
$$d(f(p \cdot gh), f(p' \cdot gh)) \leq \frac{1}{2} Lip(f)$$

and so

$$d(f(p), f(p \cdot gh)) \geq d(f(p'), f(p' \cdot gh)) - Lip(f)$$

The action of F_r is transitive on the vertices of S_Λ, so we may further assume that p is the initial point of the Culler-Morgan fundamental domain for the action of gh on $C_{gh}^{S_\Lambda}$. Therefore, since gh is reduced and cyclically reduced, the word length equals the translation length of gh on S_Λ. Hence $|gh|_\Lambda = |g|_\Lambda + |h|_\Lambda$ implies that $p \cdot g \in C_{gh}^{S_\Lambda}$, and $p \cdot g$ is on the geodesic $[p, p \cdot gh]$. Consider the image of $p, p \cdot g$, and $p \cdot gh$ in T and the geodesic triangle they span. Let $x \in T$ be the midpoint of this triangle (Figure 2). The bounded cancellation of f implies that $d(x, f(p \cdot g)) \leq B$. We have

$$\begin{aligned} d(f(p), f(p \cdot gh)) &= d(f(p), f(p \cdot g)) + d(f(p \cdot g), f(p \cdot gh)) - 2d(x, f(p \cdot g)) \\ &\geq \ell_T(g) + \ell_T(h) - 2B \end{aligned}$$

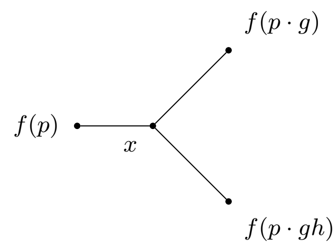


Figure 2. The triangle $f(p), f(p \cdot g), f(p \cdot gh)$ in T.

establishing Equation 2.2 with $C'' = 2B + Lip(f)$.

Combining Equation 2.1 and Equation 2.2, we have for all $q \in S_\Lambda$

$$d(f(q), f(q \cdot gh)) \geq \ell_T(g) + \ell_T(h) - 4B - Lip(f)$$

and therefore,

$$\ell_T(gh) \geq \ell_T(g) + \ell_T(h) - 4B - Lip(f).$$

Finally, we note that this proof holds for all equivariant Lipschitz surjections $f : S^\Lambda \to T$, and $B \leq Lip(f) \cdot \mathrm{covol}(S_\Lambda) = Lip(f) \cdot r$. Taking an infimum over equivariant Lipschitz surjections $f : S_\Lambda \to T$ define $C(S_\Lambda, T) = \inf\{Lip(f)\}$. We conclude

$$\ell_T(gh) \geq \ell_T(g) + \ell_T(h) - (4r + 1)C(S_\Lambda, T)$$

where the constant C depends only on the basis and the very small tree T. $\square$

2.4 Good pairs

Culler and Morgan used good pairs in the proof of their uniqueness statement for G-trees coming from a given length function. They give a geometric definition.

Definition 2.4.1. Let T be a G-tree. A pair of elements $g, h \in G$ is a **good pair** for T if

- the elements g and h are hyperbolic;
- the axes C_g and C_h meet in an arc of positive length;
- the orientations of C_g and C_h agree on the intersection;
- $\text{length}(C_g \cap C_h) < \min\{\ell(g), \ell(h)\}$.

Proposition 2.4.2 ([22, Lemma 3.6])**.** A pair of elements $g, h \in G$ is a good pair for a G-tree T if and only if g and h witness Axiom VI for ℓ_T.

Lemma 2.4.3. Suppose $g, h \in G$ is a pair of hyperbolic elements of a G-tree T whose axes intersect in an arc of finite length and the induced orientations agree. Then there are integers $A, B > 0$ so that for all $a \geq A$ and $b \geq B$, g^a, h^b is a good pair.

Proof. By hypothesis g and h satisfy the first three points of the geometric definition of a good pair. Let $N = \text{length}(C_g \cap C_h)$. It is immediate that $A = \lceil N/\ell_T(g) \rceil$ and $B = \lceil N/\ell_T(h) \rceil$ are the desired integers. □

The axes of a pair of group elements satisfying the hypotheses of Lemma 2.4.3 have distinct endpoints; this is a form of independence seen by the tree, and closely related to the algebraic independence of group elements in the subgroup generated by a good pair.

Definition 2.4.4. Let T be a G-tree. Two hyperbolic elements $g, h \in G$ are T**-independent** when

$$\partial_T g \cap \partial_T h = \emptyset.$$

Lemma 2.4.5. Suppose $g, h \in G$ is a good pair for a G-tree T. Then $\langle g, h\rangle \cong F_2$, the action of F_2 on T is free and properly discontinuous, and all $x, y \in \langle g, h\rangle$ are algebraically independent if and only if x and y are T-independent.

Proof. This proof is essentially given by Culler and Morgan [22, Lemma 2.6] however we also understand the ends of axes of elements.

Let H be the union of a fundamental domain for the action of g on C_g containing $C_g \cap C_h$ and a fundamental domain for the action of h on C_h containing $C_g \cap C_h$. It is evident from the construction that $H \cdot g^{\pm}$ and $H \cdot h^{\pm}$ meet H in its endpoints. One can show by induction that for any reduced word in g and h the interior of H is disjoint from $H \cdot w$. Let S be the subtree of T which is the orbit of H under $\langle g, h\rangle$. The action of $\langle g, h\rangle$ is free and properly discontinuous on S with fundamental domain H, which implies $\langle g, h\rangle \cong F_2$. Moreover, there is a homeomorphism from the Gromov boundary ∂F_2 to ∂S induced by the quasi-isometry given by the Milnor-Švarc lemma [13, Proposition I.8.19, Theorem III.H.3.9]; the set $\partial_T x$ is the image of the endpoints in the Gromov boundary of the axis of x acting on F_2, which are the limits of $x^{\pm n}$. Thus, if $x, y \in \langle g, h\rangle$ are algebraically independent, they are T-independent.

Conversely, if $x, y \in \langle g, h\rangle$ are T-independent, applying the inverse of the homeomorphism used above we see that x and y are algebraically independent. $\square$

Good pairs abound and are readily created from nice enough T-independent pairs.

Lemma 2.4.6. Suppose $g, h \in G$ act hyperbolically on an irreducible G-tree T with length function ℓ. Suppose further that $\ell(h) < \ell(g)$, g and h are T-independent, and that if $C_g \cap C_h \neq \emptyset$ the orientations induced by g and h agree. In this case $C_{gh} \cap C_{gh^{-1}}$ is an arc of finite length.

Corollary 2.4.7. With $g, h \in G$, T as above, there exist $J, K > 0$ such that $(gh)^j, (gh^{-1})^k$ is a good pair for T, for all $j \geq J$ and $k \geq K$.

Proof. Combine the lemma with Lemma 2.4.3. □

Proof of Lemma 2.4.6. We analyze the cases for $C_g \cap C_h$. Since g, h are T-independent, the only possibilities are

- $C_g \cap C_h = \emptyset$
- $C_g \cap C_h$ is a bounded and non-empty arc.

Case $C_g \cap C_h = \emptyset$. Let α denote the oriented geodesic from C_g to C_h, and set $o(\alpha) = p$ and $t(\alpha) = q$. The axis of $C_{gh^{-1}}$ contains the geodesic

$$[p \cdot g^{-1}hg^{-1}, p \cdot hg^{-1}, q \cdot hg^{-1}, q \cdot g^{-1}, p \cdot g^{-1}, p, q, q \cdot h^{-1}, p \cdot h^{-1}].$$

The axis of C_{gh} contains the geodesic

$$[p \cdot g^{-1}h^{-1}g^{-1}, p \cdot h^{-1}g^{-1}, q \cdot h^{-1}g^{-1}, q \cdot g^{-1}, p \cdot g^{-1}, p, q, q \cdot h, p \cdot h].$$

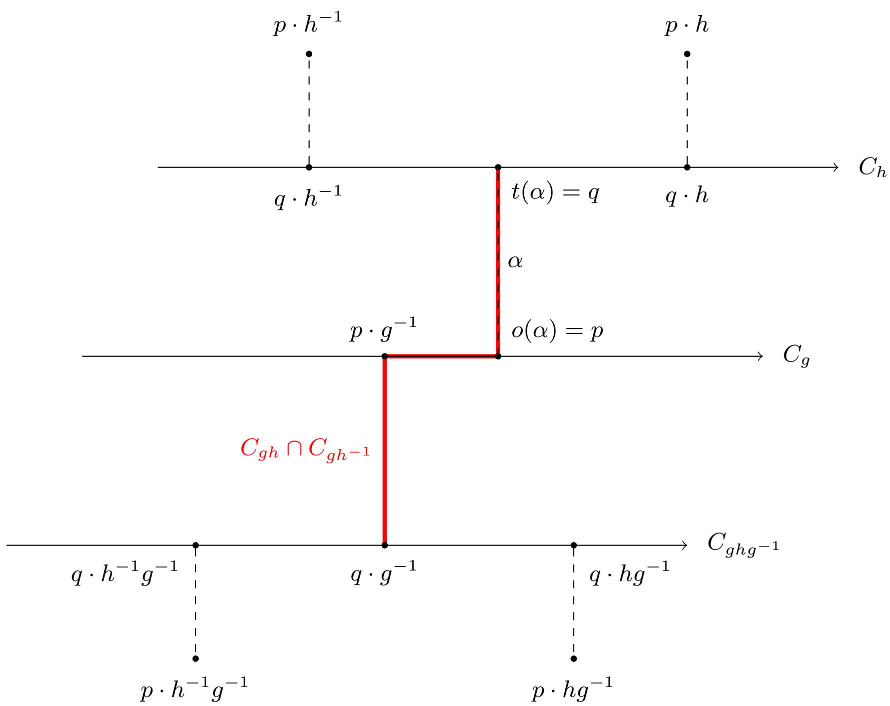


Figure 3. Creating a good pair from elements with disjoint axes.

These geodesics are two copies of the respective Culler-Morgan fundamental domains, and are given in the order of their their induced orientations. Hence $C_{gh} \cap C_{gh^{-1}}$ is the geodesic

$$[q \cdot g^{-1}, p \cdot g^{-1}, p, q],$$

illustrated in Figure 3, and the induced orientations from C_{gh} and $C_{gh^{-1}}$ agree.

Case $C_g \cap C_h = \alpha$. In this case we take the common induced orientation on α, with the convention $\alpha = o(\alpha) = t(\alpha)$ when α is a point, and set $o(\alpha) = p$ and $t(\alpha) = q$. As before, the Culler-Morgan fundamental domains for gh and gh^{-1} (note that, since $\ell(g) \neq \ell(h)$ both products are hyperbolic) concatenated with their images under $(gh)^{-1}$ and $(gh^{-1})^{-1}$ are the oriented geodesics

$$[q \cdot g^{-1}h^{-1}g^{-1}, q \cdot h^{-1}g^{-1}, q \cdot g^{-1}, q, q \cdot h]$$

and

$$[q \cdot g^{-1}hg^{-1}, q \cdot hg^{-1}, q \cdot g^{-1}, q \cdot h^{-1}]$$

respectively. Since $\ell(h) < \ell(g)$, $d(q \cdot h^{-1}, q) < d(q \cdot g^{-1}, q)$. If $q \cdot h^{-1} \in \alpha$ we see that $C_{gh} \cap C_{gh^{-1}}$ is the path $q \cdot g^{-1}, q \cdot h^{-1}$. Otherwise, $q \cdot g^{-1} \notin \alpha$ and $C_{gh} \cap C_{gh^{-1}} = [q \cdot g^{-1}, p]$. The situations are illustrated in Figure 4. $\square$

2.5 A core sampler

The Guirardel core of two trees was introduced by Guirardel to give a geometric unification of several intersection phenomena in group theory, including the intersection of curves on surfaces, Scott's intersection number for splittings, and Culler, Levitt, and Shalen's core of trees dual to measured laminations on a surface. Guirardel gives two definitions, one which makes it easy to understand the geometry of the object and its algebraic implications, and one for which calculations (and seeing that the core is even non-empty!) are easier.

Definition 2.5.1 ([26])**.** The **core** of two G-trees A and B, $\mathcal{C}(A, B)$ is the minimal closed subset of $A \times B$ with convex fibers invariant under the diagonal action of G. The **augmented**

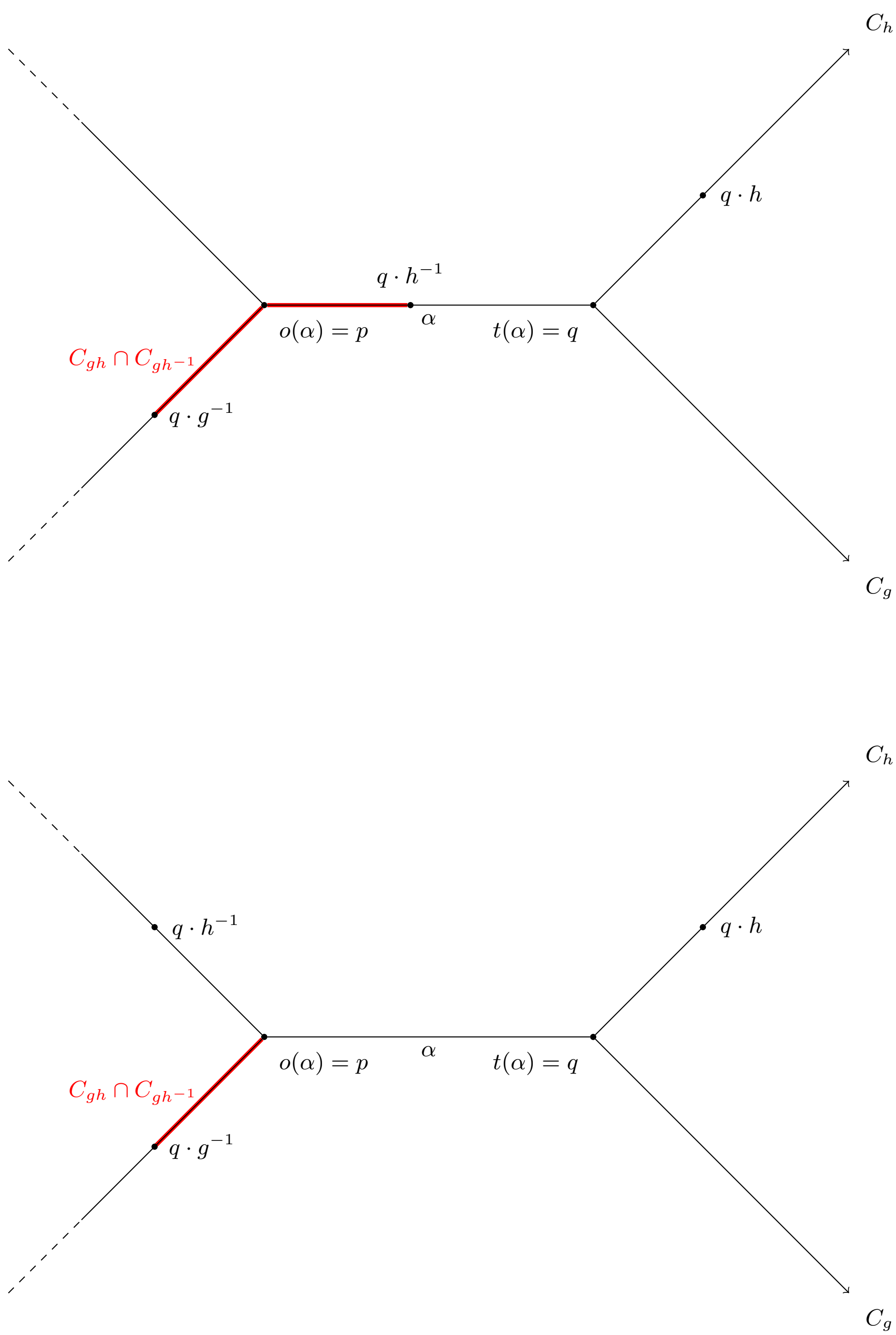


Figure 4. Creating a good pair from elements with intersecting axes.

core $\widehat{\mathcal{C}}(A,B) \supseteq \mathcal{C}(A,B)$ is the minimal closed connected superset with convex fibers invariant under the diagonal action.

Remark. If A and B have minimal subtrees A' and B' then the core must be contained in $A' \times B'$.

To construct the core in a more concrete fashion, given two G-trees A and B, consider products of directions, called **quadrants** $Q = \delta_A \times \delta_B$.

Definition 2.5.2. A quadrant $Q = \delta_A \times \delta_B \subseteq A \times B$ in a product of G-trees is **heavy** if there is a basepoint $* = (\star, \bullet) \in A \times B$ and a sequence $g_k \in G$ so that

- For all k, $* \cdot g_k \in Q$,
- The sequences $d_A(\star, \star \cdot g_k)$ and $d_B(\bullet, \bullet \cdot g_k)$ both diverge.

Otherwise, we say that Q is **light**.

Definition 2.5.3 ([26])**.** The **core** of two G-trees A and B is the subset

$$\mathcal{C}(A,B) = A \times B \setminus \left[\bigcup_{Q \text{ light}} Q \right].$$

The choice of basepoint is not important for the definition. Guirardel works in a very general setting, necessitating the somewhat awkward definition for light and heavy quadrants. In our setting, a simpler definition is available.

Definition 2.5.4. A quadrant $Q = \delta_A \times \delta_B \subseteq A \times B$ in a product of G-trees is **made heavy by a hyperbolic element** if there is a $g \in G$ and a sequence of the form $g_k = g^k$ that witnesses Q being heavy.

Lemma 2.5.5 ([26, Corollary 3.8])**.** Suppose A and B are irreducible G-trees. Then every heavy quadrant is made heavy by a hyperbolic element.

For irreducible trees, the core is always non-empty, though it is not always connected.

Theorem 2.5.6 ([26, Main Theorem])**.** Definitions 2.5.1 and 2.5.3 are equivalent.

The augmented core also has a definition in terms of quadrants. Two directions $\delta, \eta \subseteq T$ a G-tree are **facing** if $\delta \cup \eta = T$, two quadrants $Q, P \subseteq A \times B$ are **facing** if their constituent directions are facing in both A and B.

Lemma 2.5.7 ([26, Section 4.1])**.** The intersection of two facing light quadrants $Q \cap P$ is the product of arcs. Such an intersection is contained in a maximal rectangle $R = Q' \cap P'$, and $\overline{R} \cap \mathcal{C}(A, B)$ is the two corners that are not the basepoints of the defining quadrants, these rectangles are called **twice light rectangles**. The augmented core is the union of $\mathcal{C}(A, B)$ and the diagonal of each maximal twice light rectangle joining the two points of the core in that twice light rectangle.

The (augmented) core has excellent geometry. Guirardel shows that the core is a deformation retract of the product, $CAT(0)$ in the induced path metric, and contractible. A theme in this thesis is that the core retains certain good properties of input trees. Guirardel proves a proposition in this direction, if A and B are simplicial G-trees then $\mathcal{C}(A, B)$ is a simplicial subcomplex of $A \times B$ [26, Proposition 2.6].

The diagonal action of G on $\mathcal{C}(A, B)$ induces a notion of covolume, though this notion is not well behaved in general. The measures on A and B induce a product measure, and the covolume of any invariant subset $C \subseteq A \times B$ is

$$\operatorname{covol}(C) = \inf\{\mu_A \times \mu_B(E) | C \subseteq E \cdot G \text{ and } E \text{ is measurable}\}.$$

Definition 2.5.8. The **intersection number** of two G-trees A and B is

$$i(A, B) = \operatorname{covol}(\mathcal{C}(A, B)).$$

The intersection number is often a useful quantity, for example when A and B are simplicial it gives the metric area of the quotient $\mathcal{C}(A, B)/G$. Unfortunately for general real trees there is no relationship between the topological dimension of $\mathcal{C}(A, B)$ and the area.

Example 2.5.9. Fix algebraically independent irrational numbers $\alpha, \beta, \gamma, \delta$. We look at two $\mathbb{Z}^4 = \langle a, b, c, d\rangle$ actions on $\mathbb{R}$. Let T_x denote translation by x, and A and B be the $\mathbb{Z}^4$ trees coming from the actions:

$$\rho_A(a) = T_\alpha \quad \rho_A(b) = T_\beta \quad \rho_A(c) = T_\gamma \quad \rho_A(d) = \delta$$

$$\rho_B(a) = T_\alpha \quad \rho_B(b) = T_{-\beta} \quad \rho_B(c) = T_\gamma \quad \rho_B(d) = T_{-\delta}$$

The core $\mathcal{C}(A,B) = A \times B = \mathbb{R}^2$, and the orbit of any point under the diagonal action is dense, so $i(A,B) = 0$.

2.6 Bass and Serre's arboretum

A group action on a tree provides information about its algebraic structure. In the case of a simplicial tree, Bass and Serre [46] developed a detailed structure theory, relating the tree action to a generalization of amalgamated products known as a graph of groups. Below we recall key results of the theory and fix notation.

A **graph** Γ is a collection of vertices $V(\Gamma)$, edges $E(\Gamma) \subseteq V \times V$, so that $e = (o(e), t(e))$, and an involution $\bar{\cdot} : E \to E$, satisfying $\bar{e} \neq e$ and $o(\bar{e}) = t(e)$. These edges are referred to as **oriented edges**, and a graph Γ has a topological realization as a CW-complex by taking a zero cell for each vertex, and attaching a one cell to $o(e), t(e)$ for a set of representatives for the orbits of the involution $\bar{\cdot}$. An **orientation** of a graph Γ is a set of orbit representatives for the involution.

A simplicial tree T can be given a graph structure by taking branch points as vertices, and for each pair of vertices $p, q \in T$ such that the only vertices on the oriented arc $[p,q]$ are p, q the edge (p,q). The involution is given by reversing the orientation, so that $\overline{[p,q]} = [q,p]$. The tree T is the topological realization of this graph structure (though we use the metric topology on T which is not equivalent to the CW topology when there are branch points with infinite order). When it is important to do so we will distinguish between a simplicial tree and a graph structure arising from a simplicial tree by calling the latter a **graphical tree**. A group G acting on T by isometries naturally acts on this graph structure, and we say this action is **without**

inversion if for all $e \in E(T)$ and $g \in G$, $e \cdot g \neq \bar{e}$. An action with inversion can be turned into an action without inversion by subdividing the graph structure of T.

Definition 2.6.1. A **graph of groups** is a pair (G, Γ) where Γ is a connected graph, and G is an assignment of groups to the vertices and edges of Γ with injections $\iota_e : G_e \to G_{t(e)}$, satisfying $G_e = G_{\bar{e}}$. We will often suppress the assignment G and write Γ_e, Γ_v, etc.

The fundamental theorem of Bass-Serre theory gives an equivalence between actions on graphical trees and graphs of groups. Given a group G acting on a graphical tree T, the quotient graph $\bar{T}$ has a graph of groups structure as follows. Pick a maximal subtree $S \subseteq \bar{T}$ and an orientation Y of Γ. Define a section $j : \bar{T} \to T$ by first fixing a lift of S, and then for each $e \in Y \setminus E(\bar{T})$, define $j(e)$ so that $o(j(e)) = j(o(e))$; also choose elements $\gamma_e \in G$ so that $t(je) = \gamma_e j(t(e))$ for these edges. The assignment of γ_e is extended to all of $E(\bar{T})$ by $\gamma_{\bar{e}} = \gamma_e^{-1}$ and $\gamma_e = 1$ for $e \in E(S)$. Let χ be the indicator function for $E(\bar{T}) \setminus Y$. The graph of groups structure on $\bar{T}$ is given by $G_v = \mathrm{Stab}(j(v))$, $G_e = \mathrm{Stab}(j(e))$ and the inclusion maps by $a^e = \gamma_e^{\chi(e)-1} a \gamma_e^{1-\chi(e)}$. Different choices of lift and maximal tree give isomorpic graphs of groups structures on the quotient, with the underlying graph isomorphism the identity.

Starting from a graph of groups Γ there is an inverse operation, which recovers the group G as the **fundamental group** of the graph of groups, and the tree T that G acts on so that the quotient is Γ. This is the **Bass-Serre tree** of Γ, the construction depends on a choice of maximal tree, but different choices of maximal tree give equivariantly isometric trees. We will denote the quotient graph of groups by $\bar{T}$ and its tree T. When working with properties that are not conjugacy invariants the fundamental domain used will be specified.

The construction of the fundamental group of a graph of groups sits naturally in the context of the **fundamental groupoid** of a graph of groups, introduced by Higgins [29].

Definition 2.6.2. The **fundamental groupoid** $\pi_1(\Gamma)$ of a graph of groups Γ is the groupoid with vertex set $V(\Gamma)$, generated by the path groupoid of Γ and the groups G_v subject to the following conditions. First we required that the groups G_v are sub-groupoids based at the vertex v and the group and groupoid structures agree. Further for all $e \in E(\Gamma)$ and $g \in G_e$, we have

$$\bar{e}\iota_{\bar{e}}(g)e = \iota_e(g)$$

In particular this implies $\bar{e}$ and e are inverse in $\pi_1(\Gamma)$.

By taking the vertex subgroup of $\pi_1(\Gamma)$ at a vertex v, we get the fundamental group of $\pi_1(\Gamma, v)$. Changing basepoint results in an isomorphic group. The group $\pi_1(\Gamma, v)$ can also be described in terms of maximal trees. Fix a maximal tree T, and take the quotient of $\pi_1(\Gamma)$ by first identifying all vertices and then collapsing all edges of T. As explained by Higgins, it follows from standard results in groupoid theory that the result is isomorphic to $\pi_1(\Gamma, v)$.

Let $e = (e_1, e_2, \ldots, e_n)$ be a possibly empty edge path starting at v and $g = (g_0, g_1, \ldots, g_n)$ a sequence of elements $g_i \in G_{t(e_i)}$ with $g_0 \in G_v$. These data represents an element of $\pi_1(\Gamma)$ from v to $t(e_n)$ by the groupoid product

$$g_0 e_1 g_1 \cdots e_n g_n.$$

A non-identity element of $\pi_1(\Gamma)$ expressed this way is **reduced** if either $n = 0$ and $g_0 \neq \mathrm{id}$, or $n > 0$ and for all i such that $e_i = \bar{e}_{i+1}$, $g_i \notin G_{e_i}^{e_i}$. By fixing appropriate left transversals, a normal form for arrows of $\pi_1(\Gamma)$ is obtained. For each edge $e \in E(\Gamma)$, fix a left transversal S_e of the image of G_e in $G_{o(e)}$ containing the identity; by inductively applying the defining relations a reduced arrow is equivalent to a reduced arrow of the form

$$s_0 e_0 s_1 \cdots e_n h$$

with each $s_i \in S_{e_i}$ and $h \in G_{t(e_n)}$. This representation is unique [29]. By specializing to $\pi_1(\Gamma, v)$ we obtain the **Bass-Serre normal form** for elements of the fundamental group based at v, with $h \in G_v$. This normal form depends on the choice of left-transversal, but the edges used do not.

For a conjugacy class $[g] \in \pi_1(\Gamma, v)$, a representative g is cyclically reduced if it is reduced, $s_0 = \mathrm{id}$, and g has no sub-arrow g' based at v such that $g = cg'c^{-1}$ for $c \in \pi_1(\Gamma, v)$. In particular, if $o(e_0) = t(e_0) = v$, we have that if $\bar{e}_n = e_0$, then $h \notin \iota_{e_n}(G_{e_n})$.

When $\pi_1(\Gamma, v)$ is free, all vertex and edge groups are also free. By fixing a basis for $\pi_1(\Gamma, v)$, the Nielsen-Schreier theorem gives a preferred basis for each G_v, and a unique left Schreier transversal for each image G_e with respect to the preferred basis of $G_{o(e)}$. Further, using the right Schreier transversal R_e of G_e in $G_{t(e)}$ with respect to the preferred basis, we obtain a unique expression of the form

$$x_0 r_0 e_1 x_1 r_1 \cdots e_n x_n r_n$$

where $x_0 \in \iota_{e_n}(G_{e_n})$, each $x_i \in \iota_{e_i}(G_{e_i})$, and $r_i \in R_{e_i}$.

The combinatorial structure of a graph of groups reflects the G-tree structure of its Bass Serre tree to an extent.

Definition 2.6.3. A graph of groups Γ is **minimal** if for every connected proper subgraph Γ' and $v \in V(\Gamma')$ the induced map $\pi_1(\Gamma', v) \to \pi_1(\Gamma, v)$ is not surjective.

Remark. This implies that $v \in V(\Gamma)$ for a minimal graph of groups Γ has valence one then $f_e(G_e)$ is not surjective, with $v = t(e)$. As long as $\pi_1(\Gamma) \ncong \mathbb{Z}$ or D_∞, the resulting tree T is then an irreducible $\pi_1(\Gamma)$-tree.

Proposition 2.6.4 ([18, Proposition 9.2])**.** A graph of groups Γ is minimal if and only if its Bass-Serre tree T is a minimal $\pi_1(\Gamma)$ tree.

Proof. Cohen and Lustig leave this proof to the reader. We include it here. Suppose $\Gamma' \subseteq \Gamma$ is a connected proper subgraph and $\pi_1(\Gamma', v) \to \pi_1(\Gamma, v)$ is surjective. Take a lift of T' (the tree of Γ') to T. This is a $\pi_1(\Gamma')$ invariant subtree by construction, and the action of $\pi_1(\Gamma$ is induced by inclusion, so $T_{\Gamma'}$ is a $\pi_1(\Gamma)$ invariant subtree, since the inclusion is surjective. Conversely, if $T' \subseteq T$ is proper and $\pi_1(\Gamma)$ invariant, then $T'/\pi_1(\Gamma)$ is a connected proper subgraph with graph-of-groups fundamental group $\pi_1(\Gamma)$, the induced inclusion map is an isomorphism. $\square$

To ensure that two minimal graphs of groups with equivariantly isometric Bass-Serre trees are isomorphic as graphs of groups a certain pathology must be excluded.

Definition 2.6.5. Let Γ be a graph of groups. A valence two vertex $v \in V(\Gamma)$ with $v = t(e_1) = t(e_2)$ is **invisible** if f_{e_1} and f_{e_2} are isomorphisms. If Γ has no invisible vertices it is a **visible** graph of groups.

Invisible vertices are readily created by barycentric subdivision of edges and result in non-isomorphic simplicial structures on the Bass-Serre tree without changing the equivariant isometry class.

2.6.1 Topological models

Several authors give, in varying stages of development, an approach to building a topological model of a graph of groups [3, 14, 44, 51]. The treatment given by Scott and Wall is the popular reference [44], though Tretkoff's account includes a significantly more extensive discussion of the topological basis of normal forms [51]. The definitions given by the various authors are equivalent in the cellular category, though the language is quite variable. This section will most closely follow Tretkoff's account.

Definition 2.6.6. A **graph of spaces** $\mathcal{X}$ over a graph Γ is a collection of cell complexes $\mathcal{X}$ indexed by the vertices and edges of gamma, such that $\mathcal{X}_e^m = \mathcal{X}_{\bar{e}}^m$, and cellular inclusions $\iota_e : \mathcal{X}_e^m \to \mathcal{X}_{t(e)}$. The **total space** of $\mathcal{X}$, denoted X is the quotient of the disjoint union

$$\sqcup_{v \in V(\Gamma)} \mathcal{X}_v \sqcup_{e \in E(\Gamma)} \mathcal{X}_e \times [0,1]$$

by the identifications

$$\mathcal{X}_e^m \times [0,1] \to \mathcal{X}_{\bar{e}}^m \times [0,1] \qquad (x,t) \mapsto (x, 1-t)$$

$$\mathcal{X}_e^m \times 1 \to \mathcal{X}_v \qquad (x,1) \mapsto \iota_e(x)$$

The total space X of a graph of spaces over Γ comes with a map $q : X \to \Gamma$ to the topological realization of Γ by $q(\mathcal{X}_v) = v$ and $q(\mathcal{X}_e^m \times \{t\}) = e(t)$, the point of e at coordinate t realizing e as the one-cell $[0,1]$. If X is a cell complex with cellular map $q : X \to \Gamma$ such that the preimages of vertices and midpoints of edges gives a graph-of-spaces structure with X as the total space, we say q induces a graph of space structure on X. Note that the image of $\mathcal{X}_e^m \times [0,1]$ in X is the double mapping cylinder on the two inclusion maps, we denote this image $\mathcal{X}_e$. (Indeed, some authors only require the maps be π_1 injective and construct the total space with the double mapping cylinder.) The spaces $\mathcal{X}_e^m$ naturally include into the total space X via the map $\mathcal{X}_e^m \to \mathcal{X}_e^m \times \{\frac{1}{2}\}$, hence the superscript m for midpoint.

By taking fundamental groups of the vertex and edge spaces of a graph of spaces we obtain an associated graph of groups assignment G on Γ, and $\pi_1(X) \cong \pi_1(\Gamma, v)$. This operation of course has an inverse, given a graph of groups Γ a natural graph of spaces over Γ can be constructed from $K(\Gamma_v, 1)$ and $K(\Gamma_e, 1)$ spaces. The action on the universal cover $\tilde{X}$ gives a basepoint-free definition of the fundamental group of the graph of groups.

Tretkoff gives a topological normal form for the homotopy class of a path relative to the endpoints in a graph of spaces, taking advantage of a classification of edges in the one skeleton.

For graph of spaces structure $\mathcal{X}$ with total space X, an edge in $X^{(1)}$ is **$\mathcal{X}$-nodal** if it lies in a vertex space, and **$\mathcal{X}$-crossing** otherwise. Tretkoff's form makes use of a fixed topological realization of the left transversals to ensure uniqueness, we need only the topological taxonomy of edges in the path, as formulated by Bestvina, Feighn, and Handel [8]. As a technical convenience when dealing with normal forms, we require that the cellular structure of the $\mathcal{X}_e$ be of the form $\mathcal{X}_e^m \times [0, 1]$.

Lemma 2.6.7 ([8, Section 2.7; 51])**.** Every path in a graph of spaces X is homotopic relative to the endpoints to a path of the form (called **normal form**)

$$v_0 H_1 v_1 H_2 \cdots H_n v_n$$

where each v_i is a (possibly trivial) tight edge path of $\mathcal{X}$-nodal edges, each H_i is $\mathcal{X}$-crossing, and for all $1 \leq i \leq n-1$, $H_i v_i H_{i+1}$ is not homotopic relative to the endpoints to an $\mathcal{X}$-nodal edge path. Any two representatives of the homotopy class of a path in normal form have the same n. A similar statement holds for free homotopy classes of loops.

The proof of this lemma also illustrates that an edge path can be taken to normal form by iteratively **erasing a pair of crossing edges**; if $H_i v_i H_{i+1}$ is homotopic relative to the endpoints to a nodal edge path v_i' then the subpath $v_{i-1} H_i v_i H_{i+1} v_{i+1}$ is homotopic relative to endpoints to $v_{i-1} v_i' v_{i+1}$ which can subsequently be tightened. Note that a path is in normal form if and only if every sub-path is. This should be compared to the normal form for arrows in the fundamental groupoid of a graph of groups, indeed one proof of the groupoid normal form is to prove this normal form and then apply the natural map from the fundamental groupoid of the total space X to the fundamental groupoid of the graph of groups in question.

CHAPTER 3

GETTING ALONG

A brotherhood of venerable Trees.

William Wordsworth

The Guirardel intersection number measures the incompatibility of two tree actions. Suppose A and B are two $\pi_1(\Sigma)$-trees dual to measured laminations λ and μ on a surface Σ. The intersection number $i(A, B)$ is equal to $i(\lambda, \mu)$ [26]; incompatibility comes from geometric intersection. If

$$i(A, B) = i(\lambda, \mu) = 0,$$

then the leaves of the two laminations are either disjoint or equal, and $\lambda \cup \mu$ is also a measured lamination. The tree T dual to $\lambda \cup \mu$ has length function $\ell_A + \ell_B$ and equivariant Lipschitz surjections $T \to A, T \to B$. Guirardel's intersection number captures this compatibility in a broader setting.

Definition 3.0.1. A G-tree T is a **common refinement** of G-trees A and B if there are equivariant Lipschitz surjections $T \to A$ and $T \to B$.

Theorem 3.0.2 ([26, Theorem 6.1])**.** Suppose A and B are two minimal G-trees such that $\mathcal{C}(A, B) \neq \emptyset$. Then A and B have a common refinement if and only if $\mathcal{C}(A, B)$ is one-dimensional. In this case $\widehat{\mathcal{C}}(A, B)$ with the ℓ_1 metric is a common refining tree.

This chapter relates Guirardel's compatibility condition for G-trees to a compatibility condition for their respective length functions, in the case that both trees are irreducible.

Theorem 3.0.3. Suppose G is a finitely generated group. Suppose A and B are irreducible G-trees with length functions ℓ and m respectively. The core $\mathcal{C}(A, B)$ is one dimensional if and only if $\ell + m$ is a length function on G. In this case, $\widehat{\mathcal{C}}(A, B)$ is the irreducible G-tree with length function $\ell + m$.

Remark. Applying Guirardel's theorem naïvely shows only that

$$\ell_{\widehat{\mathcal{C}}(A,B)} \geq \ell + m,$$

and does not give a converse when $\ell + m$ gives a G-tree.

Theorem 3.0.3 characterizes the PL structure of certain deformation spaces of G-trees.

Definition 3.0.4. A property P of G-trees is **additive** if for all pairs (A, B) of compatible G-trees with property P, the augmented core $\widehat{\mathcal{C}}(A, B)$ also has property P.

We will show in Section 3.7 that stability and smallness are additive properties, but very smallness is not. An immediate corollary of Theorem 3.0.3 describes the PL structure of a deformation space of trees with an additive property.

Corollary 3.0.5. Suppose G is a finitely generated group and $X \subseteq PLF(G)$ is a space of G-trees with an additive property. The space X has a decomposition into simplices, where two points $[\ell], [m] \in X$ are in a common simplex if they have compatible representatives.

The proof of Theorem 3.0.3 is in three parts. In the first, we characterize the additivity of length functions in terms of combinatorial compatibility conditions. Secondly we show that these compatibility conditions are equivalent to the absence of rectangles in the core. Finally we compute the based length function of the augmented core when it is a tree, and show that it is equal to the sum of based length functions in the input trees. A careful choice of basepoints gives the desired conclusion.

3.1 Tree ends, boxes, and length function combinatorics

Consider a measured geodesic lamination λ of a closed hyperbolic surface Σ. Lifting λ to the universal cover, $\mathbb{H}^2$ gives a dual $\pi_1(\Sigma)$-tree T [21]. Corresponding to an oriented arc $e \subseteq T$ there is a subset of the boundary of $\mathbb{H}^2$. For each point of e coming from a leaf $\gamma \subseteq \lambda$, $t(e)$ determines a side of γ in $\mathbb{H}^2$, and so picks a connected component of $\mathbb{H}^2 \setminus \gamma$. The intersection of the boundaries of these connected components is the subset of the boundary corresponding to e, as in Figure 5. Endpoints of axes of the $\pi_1(\Sigma)$ action on $\mathbb{H}^2$ are dense in the boundary, as are endpoints of the dual tree T, so this subset can be described entirely in terms of the group.

The description of this subset in terms of the group generalizes to G-trees. Note that for each $p \in e^\circ$, the orientation of e picks a unique direction δ_p^e based at p such that $t(e) \in \delta_p^e$. The subset of ends of the tree corresponding to e is then

$$\bigcap_{p \in e^\circ} \omega_T(\delta_p^e).$$

In the sequel we will be more concerned with describing this directly from the group.

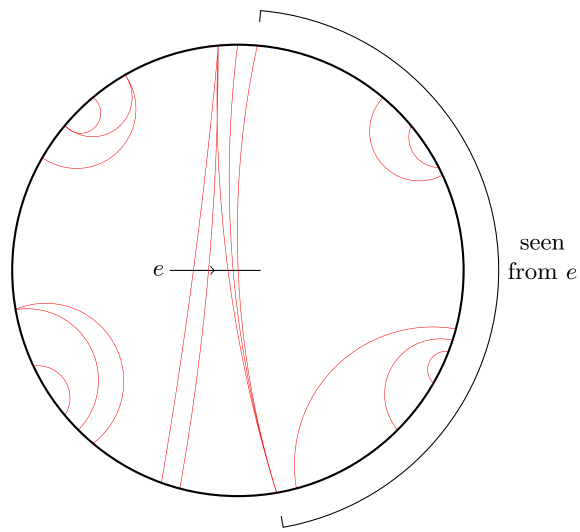


Figure 5. The part of the boundary "seen" from an arc e in the tree dual to a lamination.

Definition 3.1.1. The **group ends** of a direction $\delta \subseteq T$ is the set of group elements

$$\delta(G) = \{g \in G | \omega_T(g) \in \omega_T(\delta)\}.$$

Definition 3.1.2. The **asymptotic horizon** of an oriented arc $e \subseteq T$ of a G-tree is

$$[\![e]\!] = \bigcap_{p \in e^\circ} \delta_p^e(G),$$

where δ_p^e is the unique direction based at p such that $t(e) \in \delta_p^e$.

Remark. In some figures $[\![e]\!]$ will be used to indicate the set $\{\omega(g) | g \in [\![e]\!]\} \subseteq \partial X$ where X is hyperbolic. This abuse of notation is used only in illustrative figures, and the set of group elements will play the important role in the text. Proposition 2.2.18 implies this is not a misleading practice.

The asymptotic horizon of an oriented arc e is all hyperbolic group elements whose axes have an endpoint visible from e, when looking in the direction specified by the orientation. The visibility of group ends is sufficient to find group elements whose axes either contain e or are disjoint from e, exercises in the calculus of axes that recorded in the next two lemmas.

To fix notation, for an oriented arc $e \subseteq T$ in a G-tree, let R_e^- be the connected component of $T \setminus e^\circ$ containing $o(e)$ and R_e^+ the component containing $t(e)$.

Lemma 3.1.3. Suppose $e \subseteq T$ is an oriented arc in a G-tree T. Suppose $g \in [\![e]\!]$ and $h \in [\![\bar{e}]\!]$. Then there is an $N > 0$ such that for all $n \geq N$, $f = h^{-n}g^n$ is hyperbolic and $e \subseteq C_f$. Moreover the orientation of e agrees with the orientation on C_f induced by f.

Proof. Consider the intersection $C_g \cap C_h$. There are three cases.

Case 1: $C_g \cap C_h = \emptyset$**.** Let a be the unique shortest oriented arc joining C_g to C_h with $t(a) \in C_g$. Take

$$N > \frac{d_T(e, a) + \mathrm{length}(e)}{\min\{\ell_T(g), \ell_T(h)\}}$$

and suppose $n \geq N$. Consider the Culler-Morgan fundamental domain for the action of $f = h^{-n}g^n$ on its axis: the geodesic path b in T passing through the points

$$[o(a) \cdot h^n, o(a), t(a), t(a) \cdot g^n, o(a) \cdot g^n].$$

By hypothesis, the axis C_h meets R_e^- in at least a positive ray and $hR_e^- \subseteq R_e^-$. If $o(a) \in T \setminus R_e^-$, then the ray of C_h based at $o(a)$ directed at $\omega_T(h)$ must pass through $o(a)$. By the choice of N, $o(a) \cdot h^n \in R_e^-$. Similarly, $t(a) \cdot g^n \in R_e^+$. The arc e is the unique geodesic in T joining R_e^- to R_e^+, hence $e \subseteq b$. Moreover, the action of f takes $o(b) = o(a) \cdot h^n$ to $t(b) = o(a) \cdot g^n$, so the orientations of e and b agree, as required.

Case 2: $C_g \cap C_h = a \neq \emptyset$, a a point or arc. Orient a according to the orientation of g. (When a is a point, orientation does not matter; we use the convention $o(a) = a = t(a)$.) Take

$$N > \frac{d_T(e, a) + \text{length}(e) + \text{length}(a)}{\min\{\ell_T(g), \ell_T(h)\}}$$

and suppose $n \geq N$. Again consider the Culler-Morgan fundamental domain for the action of $f = h^{-n}g^n$ on its axis. It contains (regardless of the agreement between the orientations of h and a) the geodesic path b in T passing through the points

$$[t(a) \cdot h^n, t(a), t(a) \cdot g^n].$$

As in the previous case, we find $t(a) \cdot h^n \in R_e^-$ and $t(a) \cdot g^n \in R_e^+$. We conclude $e \subseteq b$ and the orientations agree.

Case 3: $C_g \cap C_h$ contains a ray. If $C_g = C_h$ then $e \subseteq C_{h^{-1}g} = C_g = C_h$ and $N = 1$ suffices. So suppose $C_g \neq C_h$. Let $p \in C_g \cap C_h$ be the basepoint of the common ray. Take

$$N > \frac{d_T(p, e) + \text{length}(e)}{\min\{\ell_T(g), \ell_T(h)\}}$$

and suppose $n \geq N$. Once more, a fundamental domain for the action of $f = h^{-n}g^n$ on its axis can be described. It contains the geodesic path b in T passing through the points

$$[p \cdot h^n, p, p \cdot g^n].$$

By the choice of n, we find $p \cdot h^n \in R_e^-$ and $p \cdot g^n \in R_e^+$. We conclude $e \subseteq b$ and the orientations agree. □

Lemma 3.1.4. Suppose $e \subseteq T$ is an oriented arc in a G-tree T. Suppose $g, h \in [\![e]\!]$ and $\omega_T(g) \neq \omega_T(h)$. Then there is an $N > 0$ such that for all $n \geq N$, $f = h^{-n}g^n$ is hyperbolic and $C_f \subseteq R_e^+$.

Proof. As in the proof of the previous lemma, there are three cases depending on $C_g \cap C_h$.

Case 1: $C_g \cap C_h = \emptyset$. Let a be the oriented geodesic from C_h to C_g, so that $t(a) \in C_g$. Let C_g^+ and C_h^+ be the positive rays of C_g and C_h based at $t(a)$ and $o(a)$ respectively. The infinite geodesic $C_h^+ \cup a \cup C_g^+$ has both endpoints in ∂R_e^+, so must be contained in R_e^+, therefore $a \subseteq R_e^+$.

At this point it is tempting to take $N = 1$, however we must exercise care to ensure that the axis of the product is contained in R_e^+, as this axis is not the infinite geodesic previously mentioned.

Since $g, h \in [\![e]\!]$, there is an integer $N_1 > 0$ such that for all $n \geq N_1$ we have

$$d(t(a) \cdot g^n, t(e)) > d(t(a), t(e))$$

and

$$d(o(a) \cdot h^n, t(e)) > d(o(a), t(e)).$$

Let α_g and α_h be the geodesics from $t(e)$ to C_g and C_h respectively, oriented such that $o(\alpha_g) = o(\alpha_h) = o(e)$. Since g acts by translation on its axis in the direction of $\omega_T(g)$, there is an N_2 such that for all $n \geq N_2$, $t(a) \cdot g^n > t(\alpha_g)$ (in the orientation on C_g induced by the action of g). Similarly there is an N_3 such that for all $n \geq N_3$ $o(a) \cdot h^n > t(\alpha_h)$. Take $N = \max\{N_1, N_2, N_3\}$.

Suppose $n \geq N$. As in the previous lemma, we use the Culler-Morgan fundamental domain for the action of $f = h^{-n}g^n$ on C_f: the geodesic b passing through the points

$$[o(a) \cdot h^n, o(a), t(a), t(a) \cdot g^n].$$

By construction, $b \subseteq R_e^+$. Further, the geodesic from $t(\alpha_g)$ to $t(\beta_g)$ is a proper subarc of b. Therefore, the center u of the geodesic triangle $t(\alpha_g), t(\alpha_h), t(e)$ is in the interior of b. This point is, by construction, the unique closest point of b to $o(e)$. Since u is in the interior of b, u is also the unique closest point of C_f to $o(e)$, whence $e \nsubseteq C_f$ and so $C_f \subseteq R_e^+$ as required.

Case 2: $C_g \cap C_h = a \neq \emptyset$**,** a **an arc or point.** Orient a so that it agrees with the orientation of C_g induced by the action of g (again with the convention that if a is a point, $o(a) = a = t(a)$). If the orientations of C_g and C_h disagree on a, then with C_g^+ and C_h^+ defined as in the previous case, the previous argument applies. If the orientations of C_g and C_h agree on a, let C_g^+ be as before and instead take C_h^+ to be the infinite ray of C_h based at $t(a)$. The infinite geodesic $C_g^+ \cup C_h^+$ has both endpoints in ∂R_e^+, so $t(a) \in R_e^+$. The argument from the previous case then applies, *mutatis mutandis*, with $t(a)$ in place of $o(a)$.

Case 3: $C_g \cap C_h$ **contains a ray.** In this case, since $\omega_T(g) \neq \omega_T(h)$, $C_g \neq C_h$. Let p be the basepoint of the common ray $C_g \cap C_h$. Since $g, h \in [\![e]\!]$, we must have $p \in R_e^+$. The argument from case one then applies, *mutatis mutandis*, with p in place of $o(a)$. $\square$

3.2 Combinatorial compatibility conditions

Consider two measured geodesic laminations λ and μ on a closed hyperbolic surface Σ. Suppose λ and μ have leaves that intersect transversely. This intersection produces arcs $a \subseteq A$ and $b \subseteq B$ in the dual trees to λ and μ such that the horizons of a and b, with both orientations, all intersect, as in Figure 6.

This intersection is also detected by certain pairs of elements of $\pi_1(\Sigma)$. The hyperbolic structure on Σ gives an action of $\pi_1(\Sigma)$ on the hyperbolic plane $\mathbb{H}^2$, and elements of $\pi_1(\Sigma)$ act hyperbolically. Given two elements $x, y \in \pi_1(\Sigma)$, if the axes of x and y are separated by a set of leaves of positive measure of the lift $\tilde{\lambda} \subseteq \mathbb{H}^2$ then the axes of x and y in the dual tree A will be disjoint. On the other hand, if the axes of x and y cross a common set of leaves of $\tilde{\mu}$ (with

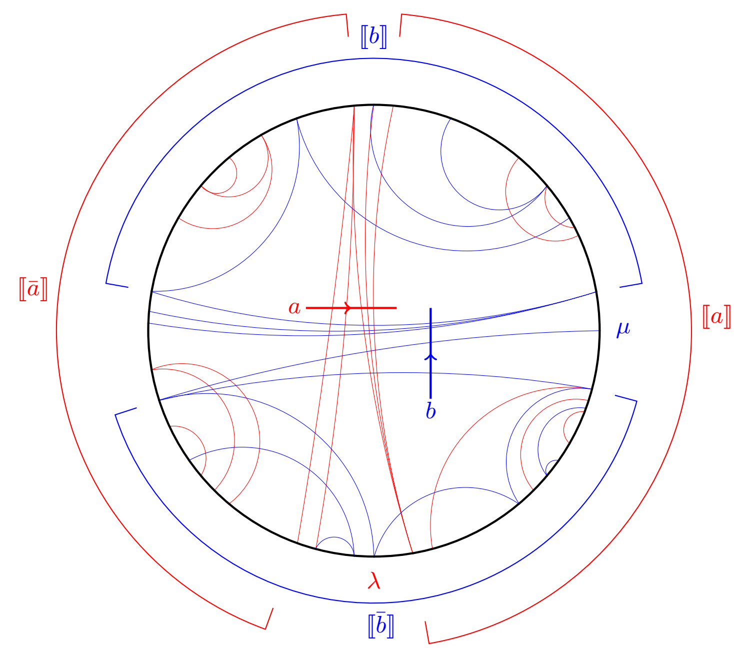


Figure 6. Intersecting measured laminations produce intersecting horizons.

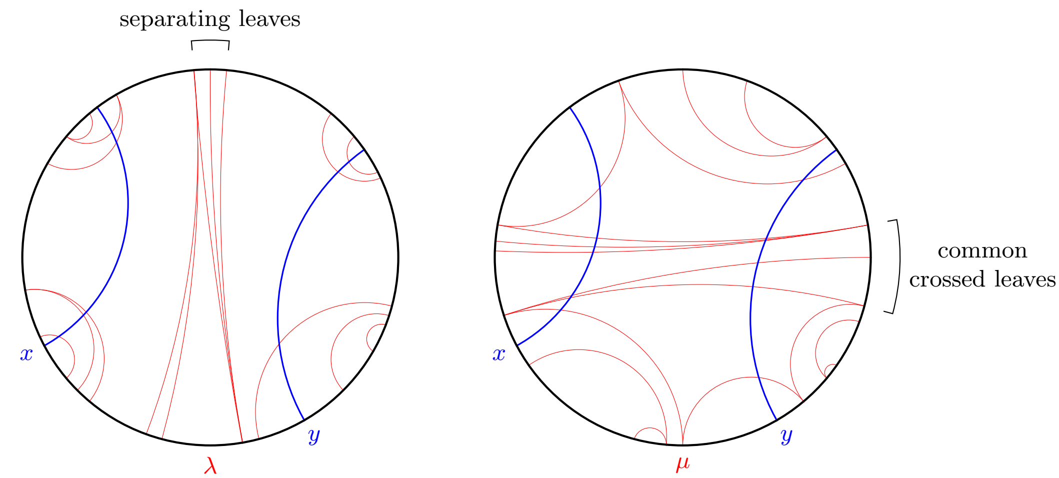


Figure 7. Detecting the intersection of laminations from the intersection pattern of the axes of two fundamental group elements with each lamination.

positive measure) then their axes in the dual tree B will overlap in an arc. The presence of a pair of such elements detects the intersection of λ and μ, as illustrated in Figure 7.

Another situation in which a pair of fundamental group elements $x, y \in \pi_1(\Sigma)$ detect the intersection of λ and μ occurs when the axes of x and y cross a common set of leaves of positive measure in both $\tilde{\lambda}$ and $\tilde{\mu}$. In this case, if the axes of x and y cross their common leaves of $\tilde{\lambda}$ with differing orientations and their common leaves of $\tilde{\mu}$ with the same orientation, then $\tilde{\lambda}$ and $\tilde{\mu}$ must intersect, as in Figure 8.

Each of these situations has a natural generalization to the setting of G-trees.

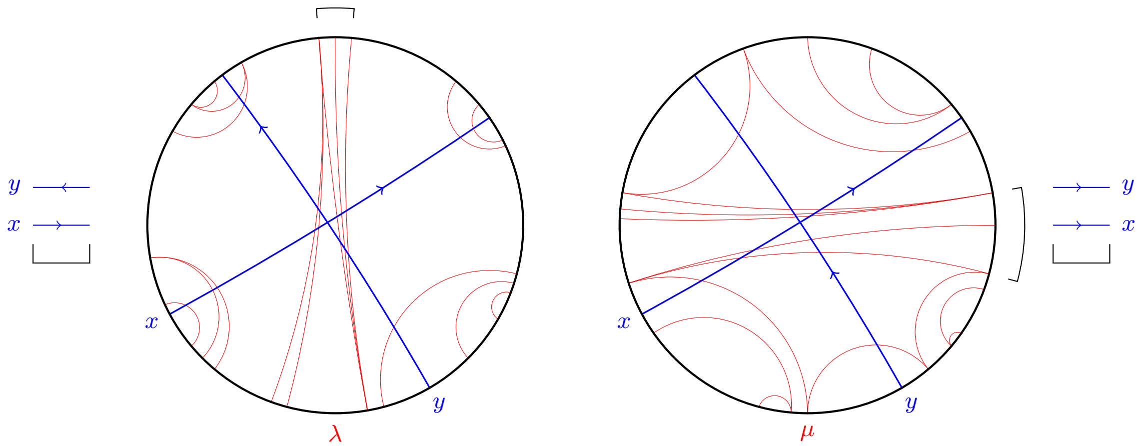


Figure 8. Detecting the intersection of laminations from the intersection orientation of the axes of two fundamental group elements.

Definition 3.2.1. Two G-trees A and B are **incompatible** if there are oriented arcs $a \subseteq A$ and $b \subseteq B$ such that the four sets

$$[\![a]\!] \cap [\![b]\!] \quad [\![\bar{a}]\!] \cap [\![b]\!] \quad [\![a]\!] \cap [\![\bar{b}]\!] \quad [\![\bar{a}]\!] \cap [\![\bar{b}]\!]$$

are non empty. Trees that are not incompatible are **compatible**.

Remark. Behrstock, Bestvina, and Clay [4] consider a similar collection of sets when giving a criterion for the presence of a rectangle in the Guirardel core of two free simplicial F_r trees. This connection will be elaborated on in Section 3.4.

Pairs of group elements with either overlapping or disjoint axes for a given action capture the situations in Figure 7 and Figure 8. Let $P(G) = G \times G \setminus \Delta$ be the set of all distinct pairs of elements in our group.

Definition 3.2.2. For a G-tree T the **overlap set**, $\mathsf{O}^T \subseteq P(G)$, is all pairs $(g, h) \in P(G)$ such that g and h are hyperbolic and $C_g \cap C_h$ contains an arc.

The **disjoint set**, $\mathsf{D}^T \subseteq P(G)$, is all pairs $(g, h) \in P(G)$ such that $C_g \cap C_h = \emptyset$.

This definition can also be stated for length functions.

Definition 3.2.3. For a length function $\ell : G \to \mathbb{R}_{\geq 0}$ the **overlap set**, $\mathsf{O}^\ell \subseteq P(G)$ is all pairs $(g, h) \in P(G)$ such that

$$\ell(gh) \neq \ell(gh^{-1}).$$

The **disjoint set**, $\mathsf{D}^\ell \subseteq P(G)$ is all pairs $(g, h) \in P(G)$ such that

$$\ell(gh) = \ell(gh^{-1}) > \ell(g) + \ell(h).$$

In the definition for a tree, the hyperbolicity requirement for membership in O^T is necessary, but the length function requirement implies that O^ℓ consists of pairs of hyperbolic elements.

Lemma 3.2.4. Suppose ℓ is a length function on G. If $(g, h) \notin \mathsf{D}^\ell$ satisfies $\ell(g) = 0$, then

$$\ell(gh) = \ell(gh^{-1}) = \ell(h).$$

In particular all pairs in O^ℓ are pairs of hyperbolic elements.

Proof. First, suppose $\ell(h) = 0$ also. Since $(g,h) \notin \mathsf{D}^\ell$, length function axiom IV implies

$$\max\{\ell(gh), \ell(gh^{-1})\} \leq \ell(g) + \ell(h) = 0$$

and we are done. So suppose $\ell(h) > 0$. Let T be the irreducible tree realizing ℓ. It must be the case that $C_g \cap C_h$ is non-empty, by Lemma 2.2.13. Consider $p \in T$ and α the shortest arc from p to $C_g \cap C_h$. Let q be the endpoint of α in $C_g \cap C_h$. Since g is elliptic, $\alpha \cdot g \cap C_g \cap C_h$ contains q, as does $\alpha \cap \alpha \cdot g \cap C_h$. The element h is hyperbolic, therefore

$$d_T(p, p \cdot gh) \geq d_T(q, q \cdot gh) = d_T(q, q \cdot h) = \ell(h)$$
$$d_T(p, p \cdot gh^{-1}) \geq d_T(q, q \cdot h^{-1}) = \ell(h),$$

and we conclude $\ell(gh) = \ell(gh^{-1}) = \ell(h)$ as required. □

The equivalence of definitions 3.2.2 and 3.2.3 can not be expected in general. However, for irreducible trees, which are determined by their length functions, the two definitions are equivalent.

Proposition 3.2.5. Suppose T is an irreducible G-tree with length function ℓ. Then $\mathsf{O}^T = \mathsf{O}^\ell$ and $\mathsf{D}^T = \mathsf{O}^\ell$, that is, definitions 3.2.2 and 3.2.3 are equivalent.

Proof. It is immediate from the definitions that $\mathsf{O}^T \subseteq \mathsf{O}^\ell$ and similarly $\mathsf{D}^T \subseteq \mathsf{D}^\ell$.

To demonstrate the reverse inclusions, suppose $(g,h) \in \mathsf{O}^\ell$. By Lemma 3.2.4, g and h are hyperbolic. If, for a contradiction, $(g,h) \notin \mathsf{O}^T$ then either $C_g \cap C_h = \emptyset$ or $C_g \cap C_h = \{*\}$. In either case we have

$$\ell(gh) = \ell(gh^{-1}) = \ell(g) + \ell(h) + d_T(C_g, C_h),$$

a contradiction.

If $(g,h) \in \mathsf{D}^\ell$ but $(g,h) \notin \mathsf{D}^T$ then $C_g \cap C_h$ is non-empty, and so

$$\max\{\ell(gh), \ell(gh^{-1})\} \leq \ell(g) + \ell(h),$$

a contradiction. $\square$

Note that the definitions of O^ℓ and D^ℓ depend only on the projective class of ℓ; the overlap condition is a topological property of a tree, so this is expected. Also be aware that $\mathsf{O}^\ell \cup \mathsf{D}^\ell \neq P(G)$; pairs such that $\ell(gh) = \ell(gh^{-1}) = \ell(g) + \ell(h)$ exist.

The interaction of overlap and disjoint sets captures the situations pictured in Figure 7 and Figure 8. We state the definitions in terms of length functions; equivalent formulations in terms of irreducible G-trees are possible but not useful in the sequel.

Definition 3.2.6. Two length functions ℓ and m on a group G have **compatible combinatorics** if

$$\mathsf{O}^\ell \cap \mathsf{D}^m = \mathsf{D}^\ell \cap \mathsf{O}^m = \emptyset.$$

Remark. The equivalent definition for trees is trivial for abelian actions. For an abelian action the tree is a line, and the disjoint set is empty.

Definition 3.2.7. Two length functions ℓ and m on a group G are **coherently oriented** if for all $(g, h) \in \mathsf{O}^{\ell} \cap \mathsf{O}^{m}$

$$\ell(gh^{-1}) < \ell(gh) \Leftrightarrow m(gh^{-1}) < m(gh).$$

The figures in the motivating discussion of this section strongly suggest that these three compatibility definitions are equivalent, at least for irreducible G-trees. Further motivation is provided by the following lemma, which produces pairs of group elements with distinct axes, mirroring the pictures.

Lemma 3.2.8. Suppose A and B are irreducible G-trees that are incompatible in the sense of definition 3.2.1. Let $a \subseteq A$ and $b \subseteq B$ be arcs witnessing this fact. Then there exist group elements $g \in [\![a]\!] \cap [\![b]\!]$ and $\alpha \in [\![a]\!] \cap [\![\bar{b}]\!]$ such that $C^A_g \cap C^A_\alpha$ is bounded; and elements $h \in [\![\bar{a}]\!] \cap [\![\bar{b}]\!]$ and $\beta \in [\![\bar{a}]\!] \cap [\![b]\!]$ such that $C^B_h \cap C^B_\beta$ is bounded.

Proof. The argument is symmetric so we give the construction of g and α. Since A and B are incompatible the relevant sets are non-empty. Take any $g \in [\![a]\!] \cap [\![b]\!]$ and $\alpha \in [\![a]\!] \cap [\![\bar{b}]\!]$. If $C^A_g \cap C^A_\alpha$ is bounded we are done. Suppose $C^A_g \cap C^A_\alpha$ contains a ray. Let $s \in G$ be any A-hyperbolic element such that $C^A_s \cap C^A_\alpha$ is bounded. Such an element exists since A is irreducible (see Proposition 2.2.17). If s is elliptic in B then αs is hyperbolic in both A and B and $C^A_{\alpha s} \cap C^A_\alpha$ is bounded, so we may suppose that s is hyperbolic in both A and B. Since $g \in [\![a]\!] \cap [\![b]\!]$ there is some $N > 0$ such that $g^N s g^{-N} \in [\![a]\!] \cap [\![b]\!]$. Take $g' = g^N s g^{-N}$. By construction $C^A_{g'} \cap C^A_\alpha$ is bounded, so g', α is the desired pair. $\square$

Corollary 3.2.9. The group elements g and α are A-independent, and the group elements h and β are B-independent.

For irreducible G-trees, the three definitions of compatibility are equivalent. The strategy suggested by the pictures is to use boundary points to pick suitable elements of G. This philosophy guides the proof below.

Lemma 3.2.10. Suppose ℓ and m are length functions on G corresponding to the irreducible G-trees A and B respectively. The following are equivalent.

1. The length functions ℓ and m **do not** have compatible combinatorics.
2. The length functions ℓ and m **are not** coherently oriented.
3. The trees A and B **are not** compatible.

Proof. We will show $1 \Leftrightarrow 3$ and $2 \Leftrightarrow 3$.

($1 \Rightarrow 3$.) Suppose, without loss of generality, $(g, h) \in \mathsf{D}^\ell \cap \mathsf{O}^m$. In A, by definition $C^A_g \cap C^A_h = \emptyset$; let $a \subseteq A$ be the geodesic joining C^A_g and C^A_h, oriented so that $t(a) \in C^A_g$. We have $g^\pm \in [\![a]\!]$ and $h^\pm \in [\![\bar{a}]\!]$. In B, again by definition there is an arc $b = C^B_g \cap C^B_h$. Without loss of generality we assume g and h induce the same orientation on b, and use this orientation. Then $g, h \in [\![b]\!]$ and $g^{-1}, h^{-1} \in [\![\bar{b}]\!]$. We conclude the four sets

$$[\![a]\!] \cap [\![b]\!] \quad [\![\bar{a}]\!] \cap [\![b]\!] \quad [\![a]\!] \cap [\![\bar{b}]\!] \quad [\![\bar{a}]\!] \cap [\![\bar{b}]\!],$$

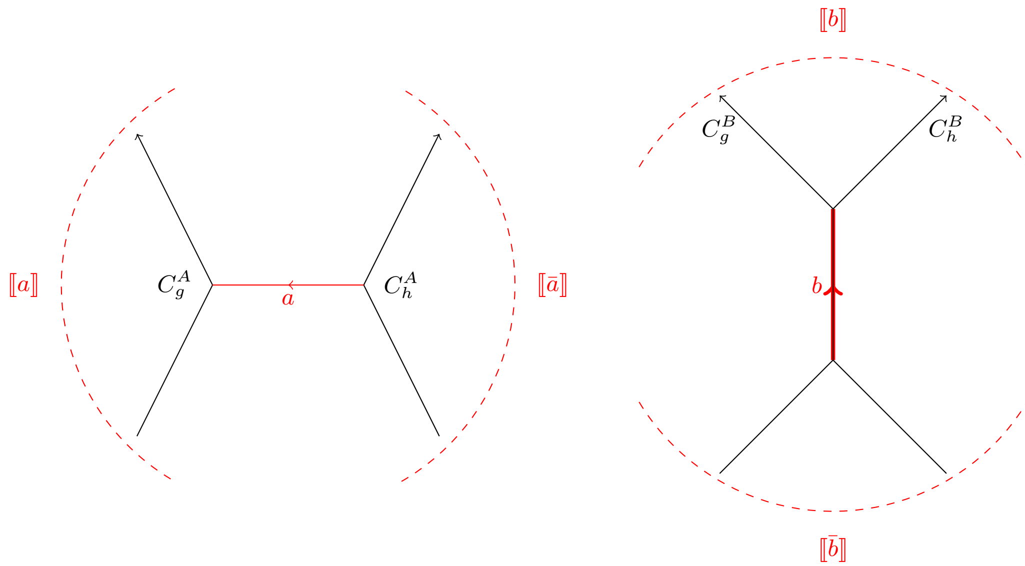


Figure 9. Incompatible combinatorics implies incompatible trees.

are all non-empty. (See Figure 9 for an illustration.)

(2 ⇒ 3.) Let $g, h \in G$ witness the incoherent orientation of ℓ and m, so that $\ell(gh^{-1}) < \ell(gh)$ but $m(gh^{-1}) > m(gh)$. Let $a = C_g^A \cap C_h^A$ and $b = C_g^B \cap C_h^B$. Since $(g, h) \in \mathsf{O}^\ell \cap \mathsf{O}^m$, both a and b are arcs. Orient a according to the orientation induced by g on C_g^A, and similarly orient b. The inequality implies that the orientation on a induced by h agrees with the orientation on a; thus $g, h \in [\![a]\!]$ and $g^{-1}, h^{-1} \in [\![\bar{a}]\!]$. Similarly, the inequality $m(gh^{-1}) > m(gh)$ implies $g, h^{-1} \in [\![b]\!]$ and $g^{-1}, h \in [\![\bar{b}]\!]$. We conclude the four sets

$$[\![a]\!] \cap [\![b]\!] \quad [\![\bar{a}]\!] \cap [\![b]\!] \quad [\![a]\!] \cap [\![\bar{b}]\!] \quad [\![\bar{a}]\!] \cap [\![\bar{b}]\!],$$

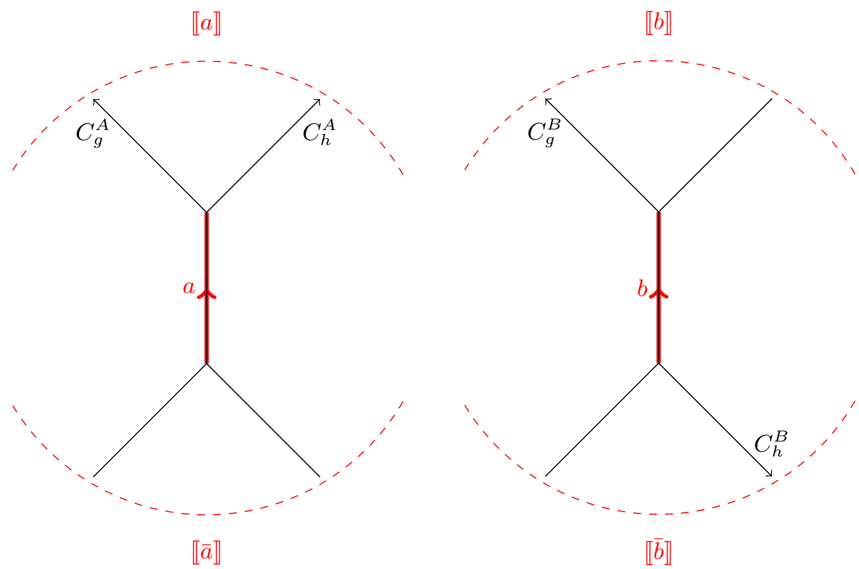


Figure 10. Incoherent orientation implies incompatible trees.

are all non-empty. (See Figure 10 for an illustration.)

(3 $\Rightarrow$ 1 and 2.) Let $a \subseteq A$ and $b \subseteq B$ be arcs witnessing the incompatibility of A and B. Fix group elements $g \in [\![a]\!] \cap [\![b]\!], h \in [\![\bar{a}]\!] \cap [\![\bar{b}]\!], \alpha \in [\![a]\!] \cap [\![\bar{b}]\!]$, and $\beta \in [\![\bar{a}]\!] \cap [\![b]\!]$ using Lemma 3.2.8; by Corollary 3.2.9 the ends of g and α in A are distinct, and the ends of h and β in B are distinct.

Let N_B be the integer guaranteed by Lemma 3.1.3 applied to g and α in B, and N_A be the integer supplied by Lemma 3.1.4 applied to g and α in A. (Note that the hypothesis of Lemma 3.1.4 on the ends of g and α is satisfied.) Set $N = \max\{N_A, N_B\}$ and consider $\rho = \alpha^{-N} g^N$. Lemma 3.1.3 implies $b \subseteq C_\rho^B$, and Lemma 3.1.4 implies $C_\rho^A \subseteq R_a^+$. Choose M by a similar process applied to h and β, so that $\sigma = \beta^{-M}h^{M}$ satisfies $b \subseteq C^B_\sigma$ and $C^A_\sigma \subseteq R^-_a$. By construction $C^B_\rho \cap C^B_\sigma \supseteq b$, so $(\rho, \sigma) \in \mathsf{O}^m$; and $C^A_\rho \cap C^A_\sigma = \emptyset$, so $(\rho, \sigma) \in \mathsf{D}^\ell$. Hence $\mathsf{D}^\ell \cap \mathsf{O}^m \neq \emptyset$ and ℓ and m do not have compatible combinatorics, as required.

Continuing the theme, let J_a be the integer given by Lemma 3.1.3 applied to g, h and $a \subseteq A$, J_b be the integer given by the application to g, h and $b \subseteq B$, and $J = \max\{J_a, J_b\}$. Similarly, let K_a be the integer given by Lemma 3.1.3 applied to α, β and a, K_b be the integer given by the application to α, β and $\bar{b} \subseteq B$ (note the reversed orientation), and $K = \max\{K_a, K_b\}$. Consider $c = h^{-J}g^{J}$ and $\gamma = \beta^{-K}\alpha^{K}$. By Lemma 3.1.3 $a \subseteq C^A_a \cap C^A_\gamma$ and all three orientations agree; however $b \subseteq C^B_c \cap C^B_\gamma$, but the orientation of C^B_c induced by c agrees with b, while that of C^B_γ induced by γ agrees with $\bar{b}$. Translating this to the length functions ℓ and m we find $(c, \gamma) \in \mathsf{O}^\ell \cap \mathsf{O}^m$ and $\ell(c\gamma^{-1}) < \ell(c\gamma)$ but $m(c\gamma^{-1}) > m(c\gamma)$, hence ℓ and m are not coherently oriented, as required. $\square$

In light of this lemma a single definition of compatible will be used throughout the remainder of this thesis.

Definition 3.2.11. Two irreducible G-trees A and B with length functions ℓ and m are **compatible** if, equivalently

- The length functions ℓ and m have compatible combinatorics.
- The length functions ℓ and m are coherently oriented.
- The trees A and B are compatible in the sense of Definition 3.2.1.

Any of the three characterizations will be used as convenient, without further explicit reference to Lemma 3.2.10. Note that this definition applies equally well to projective classes of trees. The first two points depend only on the projective class, so if ℓ and m are compatible then so are $s\ell$ and tm for all $s, t \in \mathbb{R}_{>0}$.

3.3 Compatibility is equivalent to additivity

When two measured laminations λ and μ are compatible the transverse measures may be summed. More is true, the compatibility of the supports implies every point in the convex cone spanned by λ and μ is a measure on $\lambda \cup \mu$. This compatibility generalizes to length functions, and Lemma 3.3.1 forms part of the proof of Theorem 3.0.3.

Lemma 3.3.1. Suppose ℓ and m are length functions on a group G. The sum $\ell + m$ is a length function on G if and only if ℓ and m are compatible.

Proof. First observe that $\ell + m$ always satisfies length function axioms I–III. We will focus on IV–VI.

For the forward implication, suppose $\ell + m$ is a length function on G. For a contradiction suppose that ℓ and m do not have coherent orientation, and there is some pair $(g, h) \in \mathsf{O}^\ell \cap \mathsf{O}^m$ such that $\ell(gh^{-1}) < \ell(gh)$ and $m(gh) < m(gh^{-1})$. By Lemma 3.2.4, g and h are hyperbolic with respect to both ℓ and m, so both g and h must be hyperbolic in $\ell + m$. Length function axiom V implies that for ℓ and m respectively,

$$\ell(gh) = \ell(g) + \ell(h)$$

and

$$m(gh^{-1}) = m(g) + m(h).$$

Taking a sum we have

$$\ell(gh) + m(gh^{-1}) = \ell(g) + m(g) + \ell(h) + m(h).$$

By hypothesis, both

$$\ell(gh) + m(gh) < \ell(gh) + m(gh^{-1})$$

$$\ell(gh^{-1}) + m(gh^{-1}) < \ell(gh) + m(gh^{-1}).$$

We conclude that

$$\max\{(\ell + m)(gh), (\ell + m)(gh^{-1})\} < \ell(gh) + m(gh^{-1})$$

$$= (\ell + m)(g) + (\ell + m)(h).$$

This is a contradiction, since $\ell + m$ satisfies length function axiom V, which implies the above strict inequality must be equality. We conclude that ℓ and m are compatible.

For the converse, suppose ℓ and m are compatible. As remarked previously, $\ell + m$ satisfies length function axioms I–III. We will show $\ell + m$ satisfies the remaining axioms.

Axiom IV. Suppose $g, h \in G$. We will proceed through the following cases:

- $(g, h) \in \mathsf{O}^\ell$,
- $(g, h) \in \mathsf{O}^m$,
- $(g, h) \in P(G) \setminus (\mathsf{O}^\ell \cup \mathsf{O}^m)$.

Case $(g, h) \in \mathsf{O}^\ell$**.** Since ℓ satisfies axiom IV,

$$\max\{\ell(gh), \ell(gh^{-1})\} \leq \ell(g) + \ell(h).$$

Since ℓ and m have compatible combinatorics, $(g, h) \in P(G) \setminus \mathsf{D}^m$, which implies that

$$\max\{m(gh), m(gh^{-1})\} \leq m(g) + m(h).$$

Hence we may calculate

$$\begin{aligned}\max\{(\ell(gh) + m(gh), \ell(gh^{-1}) + m(gh^{-1})\} &\leq \max\{\ell(gh), \ell(gh^{-1})\} + \max\{m(gh), m(gh^{-1})\} \\ &\leq \ell(g) + \ell(h) + m(g) + m(h)\end{aligned}$$

and conclude that in this case $\ell + m$ satisfies axiom IV.

Case $(g, h) \in \mathsf{O}^m$**.** The proof is symmetric with the previous case.

Case $(g, h) \in P(G) \setminus (\mathsf{O}^\ell \cup \mathsf{O}^m)$**.** In this case, by hypothesis both

$$\ell(gh) = \ell(gh^{-1})$$

and

$$m(gh) = m(gh^{-1}).$$

Adding, we conclude

$$\ell(gh) + m(gh) = \ell(gh^{-1}) + m(gh^{-1})$$

as required.

Axiom V. Suppose $g, h \in G$ satisfy $\ell(g) + m(g) > 0$ and $\ell(h) + m(h) > 0$. This implies that g and h are both hyperbolic in at least one of ℓ and m. We proceed through the same cases.

- $(g, h) \in \mathsf{O}^\ell$,
- $(g, h) \in \mathsf{O}^m$,
- $(g, h) \in P(G) \setminus (\mathsf{O}^\ell \cup \mathsf{O}^m)$.

Case $(g, h) \in \mathsf{O}^\ell$**.** In this case, since ℓ and m are compatible, $(g, h) \notin \mathsf{D}^m$ and we argue by subcases.

- $m(g) > 0$ and $m(h) > 0$,
- $m(g) = 0$ and $m(h) \geq 0$,

- $m(g) \geq 0$ and $m(h) = 0$.

Subcase $m(g) > 0$ and $m(h) > 0$. Since ℓ and m are coherently oriented we have, without loss of generality,

$$\ell(gh^{-1}) < \ell(gh)$$

and

$$m(gh^{-1}) \leq m(gh)$$

Appealing to axiom V for ℓ and m we have,

$$\ell(gh) = \ell(g) + \ell(h)$$

and

$$m(gh) = m(g) + m(h).$$

Summing, we conclude

$$\begin{aligned} \ell(gh^{-1}) + m(gh^{-1}) &\leq \ell(gh) + m(gh) \\ &= \ell(g) + m(g) + \ell(h) + m(h). \end{aligned}$$

Therefore in this subcase $\ell + m$ satisfies axiom V.

Subcase $m(g) = 0$ and $m(h) \geq 0$. By Lemma 3.2.4, $m(gh^{-1}) = m(gh) = m(h)$, so axiom V for $\ell + m$ follows immediately from axiom V for ℓ.

Subcase $m(g) \geq 0$ and $m(h) = 0$. This subcase is symmetric with the previous one.

Case $(g, h) \in \mathsf{O}^m$. This case is symmetric with the previous case.

Case $(g, h) \in P(G) \setminus (\mathsf{O}^\ell \cup \mathsf{O}^m)$. In this case we have

$$\ell(gh) = \ell(gh^{-1}) = \ell(g) + \ell(h) + \Delta_\ell$$

$$m(gh) = m(gh^{-1}) = m(g) + m(h) + \Delta_m$$

for real numbers $\Delta_\ell, \Delta_m \geq 0$. Immediately we have that

$$\ell(gh) + m(gh) = \ell(gh^{-1}) + m(gh^{-1})$$

and from

$$\ell(gh) + m(gh) = \ell(g) + m(g) + \ell(h) + m(h) + \Delta_\ell + \Delta_m$$

we conclude that axiom V is satisfied by $\ell + m$.

Axiom VI. Finally we confirm that $\ell + m$ has a good pair of elements. Let (g, h) be a good pair of elements for ℓ, so that

$$0 < \ell(g) + \ell(h) - \ell(gh^{-1}) < 2 \min\{\ell(g), \ell(h)\}.$$

We check the following cases

- $(g,h) \in \mathsf{O}^m$,
- $(g,h) \notin \mathsf{O}^m$.

Case $(g,h) \in \mathsf{O}^m$**.** In this case, since ℓ and m are coherently oriented, $m(gh^{-1}) < m(gh)$. By Lemma 3.2.4, g and h are hyperbolic in m. Therefore, by Lemma 2.4.3, there are positive integers a and b so that (g^a, h^b) is a good pair for m. Further by Lemma 2.4.3 the property of being a good pair is preserved under taking positive powers, so (g^a, h^b) is a good pair for ℓ also. Adding the good pair inequalities, we calculate

$$\begin{aligned} 0 &< \ell(g^a) + m(g^a) + \ell(h^b) + m(h^b) - \ell(g^a h^{-b}) - m(g^a h^{-b}) \\ &< 2(\min\{\ell(g^a), \ell(h^b)\} + \min\{m(g^a), m(h^b)\}) \\ &\leq 2\min\{\ell(g^a) + m(g^a), \ell(h^b) + m(h^b)\}. \end{aligned}$$

Hence (g^a, h^b) is a good pair for $\ell + m$.

Case $(g,h) \notin \mathsf{O}^m$**.** In this case, since ℓ and m have compatible combinatorics, $(g,h) \notin \mathsf{D}^m$, and we have

$$m(gh^{-1}) = m(gh) = m(g) + m(h).$$

Adding this to the ℓ good pair inequality for (g,h), we have

$$0 < \ell(g) + \ell(h) - \ell(gh^{-1}) = \ell(g) + m(g) + \ell(h) + m(h) - \ell(gh^{-1}) - m(gh^{-1})$$

Since

$$2\min\{\ell(g), \ell(h)\} \leq 2\min\{\ell(g) + m(g), \ell(h) + m(h)\},$$

we conclude (g, h) is again a good pair for $\ell + m$.

This concludes the case analysis. We have verified axioms IV–VI for $\ell + m$, and conclude that $\ell + m$ is a length function, as required. □

Corollary 3.3.2. Every point in $\mathrm{Cone}(\ell, m)$, the convex cone spanned by ℓ and m in the space of length functions, is a length function if and only if ℓ and m are compatible.

3.4 Compatibility and the core

In the setting of free simplicial F_r-trees, Behrstock, Bestvina, and Clay give a definition of edge boxes similar to our asymptotic horizons, and show that intersections of edge boxes can characterize the presence of rectangles in the core [4]. Using our definition of asymptotic horizon we generalize this characterization to the setting of arbitrary irreducible G-trees. This characterization connects the notions of compatibility discussed so far with Guirardel's core.

Lemma 3.4.1. Suppose A and B are irreducible G-trees and $a \subseteq A, b \subseteq B$ are open oriented arcs. The rectangle $a \times b \subseteq \mathcal{C}(A, B)$ if and only if for all closed subarcs $a' \subseteq a$ and $b' \subseteq b$ the four sets

$$[\![a']\!] \cap [\![b']\!] \quad [\![\bar{a'}]\!] \cap [\![b']\!] \quad [\![a']\!] \cap [\![\bar{b'}]\!] \quad [\![\bar{a'}]\!] \cap [\![\bar{b'}]\!]$$

are non-empty.

Corollary 3.4.2. Two irreducible G-trees are compatible if and only if their core does not contain a rectangle.

Remark. The argument below can be readily adapted to the following other conditions for core membership when taking the core of irreducible G-trees.

- A point $(p, q) \in \mathcal{C}(A, B)$ if and only if for every quadrant $Q = \delta \times \eta$ containing (p, q),

$$\delta(G) \cap \eta(G) \neq \emptyset.$$

- For an open arc $b \subseteq B$ and a point $p \in A$, $\{p\} \times b \subseteq \mathcal{C}(A, B)$ if and only if for every direction $\delta \subseteq A$ containing p and every closed subarc $b' \subseteq b$,

$$\delta(G) \cap [\![b']\!] \neq \emptyset$$

  A symmetric condition also holds for open subarcs $a \subseteq A$ and points $q \in B$.

- The rectangle $a \times b$ is twice light, with main diagonal from $(o(a), o(b))$ to $(t(a), t(b))$ if and only if for all closed subarcs $a' \subseteq a$ and $b' \subseteq b$ the sets

$$[\![a']\!] \cap [\![\bar{b}']\!] = [\![\bar{a}']\!] \cap [\![\bar{b}]\!] = \emptyset$$

These conditions collectively are referred to as **many horizon conditions**. When applying the conditions we will often use the fact that $[\![a]\!] \subseteq [\![a']\!]$ for any subarc $a' \subseteq a$.

Proof. Suppose the four sets are non-empty for all closed subarcs $a' \subseteq a$ and $b' \subseteq b$ (orient subarcs with the same orientation as their parent arc). We will show every quadrant meeting $a \times b$ is heavy, so that by definition $a \times b \subseteq \mathcal{C}(A, B)$. Suppose $\delta \times \sigma$ is a quadrant and $\delta \times \sigma \cap a \times b \neq \emptyset$. Further suppose $t(a) \in \delta$ and $t(b) \in \sigma$. Since $a \times b \cap \delta \times \sigma$ is non-empty there is a point $(p, q) \in a \times b \cap \delta \times \sigma$, and we have $\delta_p^+ \times \delta_q^+ \subseteq \delta \times \sigma$. Let a' and b' be closed subarcs containing p and q respectively. The set $[\![a']\!] \cap [\![b']\!]$ is non-empty by hypothesis, and by definition any $g \in [\![a']\!] \cap [\![b']\!]$ is a hyperbolic element that makes $\delta_p^+ \cap \delta_q^+$ heavy. The other three possible orientations of $\delta \times \sigma$ are seen to be heavy similarly.

Now suppose $a \times b \subseteq \mathcal{C}(A, B)$. Let $a' \subseteq a$ and $b' \subseteq b$ be any closed subarcs. We will show $[\![a']\!] \cap [\![b']\!]$ is non-empty; the other three cases are handled symmetrically. Let δ_p^a be the direction based at $p = t(a')$ containing $t(a)$ and δ_q^b be the direction based at $q = t(b')$ containing $t(b)$. Since $a \times b \cap \delta_p^a \times \delta_q^b \neq \emptyset$ and $a \times b \subseteq \mathcal{C}(A, B)$, the quadrant $\delta_p^a \times \delta_q^b$ is made heavy by some $g \in G$, so $\omega_A(g) \in \omega_A(\delta_p^a)$ and $\omega_B(g) \in \omega_B(\delta_q^b)$. For any points $p' \in a'$ and $q' \in b'$, the directions based at p' and q' containing $t(a')$ and $t(b')$ respectively contain δ_p^a and δ_q^b; we conclude $g \in [\![a']\!] \cap [\![b']\!]$. $\square$

3.5 Adding based length functions

Culler and Morgan used based length functions to show that the irreducible tree realizing a length function is unique up to isometry. To understand the isometry type of a tree realizing $\ell + m$ when ℓ and m are compatible we also analyze based length functions. The following lemma is used in the proof of Theorem 3.0.3 to conclude that the core of two compatible trees realizes the length function $\ell + m$. Its proof will take the remainder of the section.

Lemma 3.5.1. Suppose ℓ and m are compatible length functions coming from compatible G-trees A and B respectively. Let T be the irreducible G-tree realizing $\ell + m$. Then there are points $p \in A, q \in B$, and $r \in T$ such that for all $g \in G$

$$P_r(g) = L_p(g) + M_q(g),$$

where L_p, M_q, and P_r, are the based length functions on G coming from the pairs $(A, p), (B, q)$, and (T, r) respectively.

The basepoints will be determined by a pair (g, h) that is simultaneously a good pair for ℓ, m, and $\ell + m$. Culler and Morgan used a good pair to give a formula for a based length function L_* at a specific basepoint $*$ in the tree realizing ℓ. Specifically, if $g, h \in G$ are a good pair for ℓ a length function realized by irreducible G-tree A, then

$$C_g^A \cap C_h^A \cap C_{gh^{-1}}^A = \{*\}$$

and

$$L_*(k) = \max\{d_A(C, D)\}, \tag{$\dagger$}$$

where C ranges over $\{C_g^A, C_h^A, C_{gh^{-1}}^A\}$ and D ranges over $C \cdot k$.

The first tool in the proof of Lemma 3.5.1 is the existence of a simultaneous good pair for two compatible length functions.

Lemma 3.5.2. Suppose ℓ and m are compatible length functions on a group G. Then there is a pair of elements g, h that is a good pair for both ℓ and m.

Remark. If (g, h) is a good pair for ℓ and m then it is a good pair for $\ell + m$. This is an immediate consequence of Axiom VI.

Proof. Let A and B be the irreducible G-trees realizing ℓ and m respectively. The good pair lemma (Lemma 2.4.6) and its corollary reduce the problem to finding a pair of elements that satisfy the hypotheses of the good pair lemma in both A and B simultaneously. Indeed, if $x, y \in G$ is such a pair, then Corollary 2.4.7 implies there are numbers $J_\ell, K_\ell, J_m, K_m > 0$ such that for all $j \geq \max\{J_\ell, J_m\}$ and $k \geq \max\{K_\ell, K_m\}$, $(xy)^j, (xy^{-1})^k$ is a good pair for both ℓ and m as required.

We will now find such a pair $x, y \in G$. Let $g, h \in G$ be a good pair for ℓ and $a, b \in G$ be a good pair for m. Recall (Lemma 2.4.5) that a good pair generates a rank two free subgroup consisting of hyperbolic elements, and that algebraically independent elements of such a subgroup are T-independent.

Consider the group $H = \langle g, h \rangle \cap \langle a, b \rangle \leq G$. We consider the following cases:

- $H \geq F_2$,
- H is an infinite cyclic group,
- H is the trivial group.

Case $H \geq F_2 \cong \langle x, y \rangle$. Since ℓ and m are coherently oriented we can choose the generators x and y so that they satisfy the orientation hypothesis of the good pair lemma for ℓ and m

simultaneously. After passing to a power we may further assume $\ell(y) < \ell(x)$ and $m(y) < m(x)$. Every algebraically independent pair of elements in H both A and B independent by Lemma 2.4.5, so x, y is the desired pair.

Case $H \cong \langle z \rangle$ an infinite cyclic group. Pick any $g' \in \langle g, h \rangle$ and $a' \in \langle a, b \rangle$ algebraically independent from z. Note that g' is hyperbolic in ℓ and a' is hyperbolic in m. Since a' is infinite order and the action of G on A is effective, there are finitely many values $n_{A,i} > 0$ such that $\partial_A z \cdot g' a'^{n_{A,i}} \cap \partial_A z \neq \emptyset$. Since a' is hyperbolic in m and independent of z there is an N_B such that for $n > N_B$ the intersection $\partial_B z \cdot g' a'^{n} \cap \partial_B z = \emptyset$. Fix $N \geq \max_i\{N_{A_i}, N_B\}$, sufficiently large so that $y = g' a'^N$ is hyperbolic in both ℓ and m. By construction we have

$$\partial_A y \cap \partial_A z = \partial_B y \cap \partial_B z = \emptyset.$$

Since ℓ and m are coherently oriented, we can find a (possibly negative) integer K such that the pair z^K, y satisfies the orientation and magnitude hypotheses of the good pair lemma in both ℓ and m.

Case H is trivial. Pick $N > 0$ such that ga^N is hyperbolic in both ℓ and m, and $\partial_B ga^N \cap \partial_B b = \emptyset$ (this last condition is possible since a and b have distinct fixed end sets). Using an argument similar to the previous case, first pick $K > 0$ such that

$$\partial_A ga^N \cdot h^K \cap \partial_A b = \emptyset$$

$$\partial_B ga^N \cdot h^K \cap \partial_B b = \emptyset$$

and then $M > 0$ such that $h^K b^M$ is hyperbolic in both ℓ and m, and

$$\partial_A ga^N \cdot h^K b^M \cap \partial_A ga^N = \emptyset$$

$$\partial_B ga^N \cdot h^K b^M \cap \partial_B ga^N = \emptyset.$$

With $x = ga^N$ and $y = h^K b^M$, we can again use the coherent orientation of ℓ and m to find a power J such that x^J, y is the desired pair. $\square$

The next lemma allows us to analyze the maximum in Equation † in a simultaneous manner for compatible trees.

Lemma 3.5.3. Suppose ℓ and m are compatible length functions on G corresponding to G-trees A and B. If $f, g, h \in G$ are hyperbolic in both ℓ and m and $C_f^T \cap C_g^T \cap C_h^T = \{*^T\}$ for $T = A, B$, then for each $y \in G$ there is an $x \in \{f, g, h\}$ so that the geodesic $[*^T, *^T \cdot y]$ intersects C_x^T in a point, for $T = A, B$.

Corollary 3.5.4. With ℓ, m, A, B and $f, g, h \in G$ as above, for all $z \in G$ there is a pair $(x, y) \in \{f, g, h\} \times \{z^{-1}fz, z^{-1}gz, z^{-1}hz\}$ such that the geodesic $[*^T, *^T \cdot z]$ meets C_x^T and C_y^T in a point for $T = A, B$.

Proof of Corollary. Apply the lemma to z and z^{-1} and translate. $\square$

Proof of Lemma 3.5.3. Fix $y \in G$. Let $\alpha = [*^A, *^A \cdot y]$ and $\beta = [*^B, *^B \cdot y]$. Suppose, for a contradiction, that for each $x \in \{f, g, h\}$ either $\alpha \cap C_x^A$ or $\beta \cap C_x^B$ is an arc. Since $C_f^A \cap C_g^A \cap C_h^A$ is a point, up to relabeling we may assume $\alpha \cap C_f^A$ is a point and $\beta \cap C_f^B$ is a positively

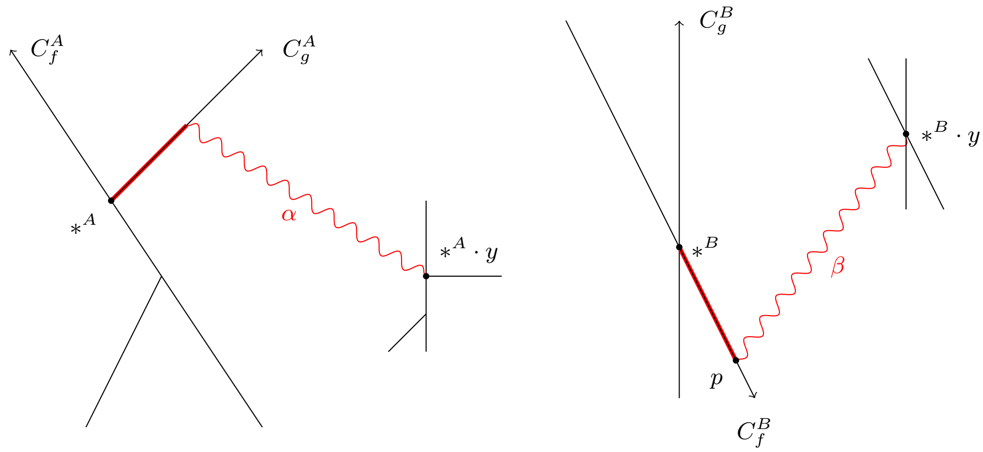


Figure 11. Schematic of the troublesome construction.

oriented arc. Further, up to relabeling g and h, we can assume that β meets C_g^B in a point, as $C_f^B \cap C_g^B \cap C_h^B$ is also a point. By our supposition, $\alpha \cap C_g^A$ must be an arc, and we relabel so that the intersection is positively oriented. Figure 11 gives a schematic.

Since f is hyperbolic in ℓ and $C_y^A \cap C_f^A$ is at most a point, $y^{-1}f$ is hyperbolic in A. The axis of $C_{y^{-1}f}^A$ contains α (this can be seen by considering the Culler-Morgan fundamental domain), so $C_{y^{-1}f}^A \cap C_g$ is an arc, and $(y^{-1}f, g) \in \mathsf{O}^\ell$. Consider $\beta \cap C_f^B = [*^B, p]$. The characteristic set of $C_{y^{-1}f}^B$ contains $[*^B \cdot y, p, p \cdot f]$, and the shortest path from this characteristic set to the axis C_g^B is the arc $[*^B, p]$, hence $C_{y^{-1}f}^B \cap C_g^B = \emptyset$ and $(y^{-1}f, g) \in \mathsf{D}^m$. This is a contradiction, as ℓ and m have compatible combinatorics. $\square$

We are now well-situated to use a simultaneous good pair to establish Lemma 3.5.1.

Proof of Lemma 3.5.1. By Lemma 3.5.2 and the following remark, there is a pair $g, h \in G$ that is a good pair for ℓ, m, and $\ell + m$. Let p, q, and r be the triple intersection point

$$C_g^S \cap C_h^S \cap C_{gh^{-1}}^S$$

for S equal to A, B, and T respectively. Using Equation † we have for all $y \in G$

$$P_r(y) = \max\{d_T(C, D)\},$$

where C ranges over $\{C_g^T, C_h^T, C_{gh^{-1}}^T\}$ and D ranges over $\{C_g^T \cdot y, C_h^T \cdot y, C_{gh^{-1}}^T \cdot y\}$.

Calculating the distance between particular choices of $C = C_x^T$ and $D = C_z^T$, $z = y^{-1}x'y$ we have

$$\begin{aligned} d_T(C_x^T, C_z^T) &= \frac{1}{2}\max\{0, (\ell+m)(xz) - (\ell+m)(x) - (\ell+m)(z)\} \\ &= \frac{1}{2}\max\{0, \ell(xz) - \ell(x) - \ell(z) + m(xz) - m(x) - m(z).\} \end{aligned}$$

Since ℓ and m are coherently oriented,

$$\ell(xz) - \ell(x) - \ell(z) \geq 0 \Leftrightarrow m(xz) - m(x) - m(z) \geq 0.$$

Therefore

$$\begin{aligned} d_T(C_x^T, C_z^T) &= \frac{1}{2}\max\{0, \ell(xz) - \ell(x) - \ell(z)\} + \frac{1}{2}\max\{0, m(xz) - m(x) - m(z)\} \\ &= d_A(C_x^A, C_z^A) + d_B(C_x^B, C_z^B). \end{aligned}$$

Hence, it suffices to show that the maxima in Equation † for $L_p(y)$ and $M_q(y)$ occur for the same pair in $\{g, h, gh^{-1}\} \times \{y^{-1}gy, y^{-1}hy, y^{-1}gh^{-1}y\}$. This is the exact content of Corollary 3.5.4. $\square$

Remark. In applications of Lemma 3.5.1 it will be important to use that the points p, q, and r come from a mutually good pair (g, h).

3.6 Proof of Theorem 3.0.3

We are now in a position to give a proof of the main theorem of this chapter.

Theorem 3.0.3. Suppose G is a finitely generated group. Suppose A and B are irreducible G-trees with length functions ℓ and m respectively. The core $\mathcal{C}(A, B)$ is one dimensional if and only if $\ell + m$ is a length function on G. In this case, $\widehat{\mathcal{C}}(A, B)$ is the irreducible G-tree with length function $\ell + m$.

Proof. By Lemma 3.3.1 $\ell + m$ is a length function if and only if ℓ and m are compatible; by Corollary 3.4.2 this is also equivalent to $\mathcal{C}(A, B)$ being one-dimensional. Together, this implies $\ell + m$ is a length function if and only if $\mathcal{C}(A, B)$ is one-dimensional. It remains to compute the length function of $\widehat{\mathcal{C}}(A, B)$ in this case.

Guirardel shows that when $\mathcal{C}(A,B)$ is one-dimensional, $\widehat{\mathcal{C}}(A,B)$ with the metric d_1, the restriction of the ℓ_1 metric on $A \times B$ is a minimal G-tree [26, Theorem 6.1]. To complete the proof we will show that the length function of $(\widehat{\mathcal{C}}(A,B), d_1)$ is $\ell + m$.

Let T be the minimal G-tree realizing $\ell + m$. Let $g, h \in G$ be a good pair for both ℓ and m, so that the points $p = C_g^A \cap C_h^A \cap C_{gh^{-1}}^A$, $q = C_g^B \cap C_h^B \cap C_{gh^{-1}}^B$, and $r = C_g^T \cap C_h^T \cap C_{gh^{-1}}^T$ satisfy the based length function identity certified by Lemma 3.5.1; that is

$$P_r(y) = L_p(y) + M_q(y)$$

where L_p, M_q, and P_r are based length functions for A, B, and T respectively. Suppose $(p,q) \in \widehat{\mathcal{C}}(A,B)$, and that $\mathfrak{C}_{(p,q)}$ is the based length function for $\widehat{\mathcal{C}}(A,B)$. We have, for all $y \in G$

$$\mathfrak{C}_{(p,q)}(y) = L_p(y) + M_q(y) = P_r(y),$$

and so by classical results of Alperin and Moss [2] or Imrich [30], T is equivariantly isometric to $\widehat{\mathcal{C}}(A,B)$, and the length function on $\widehat{\mathcal{C}}(A,B)$ is $\ell + m$ as required.

Why is $(p,q) \in \widehat{\mathcal{C}}(A,B)$? Suppose $\delta_x \times \delta_y$ is a quadrant based at (x,y) containing (p,q). Since the axes in A of the three elements g, h, and gh^{-1} intersect in a point, at most two of

$$\{\omega_A(g), \omega_A(h), \omega_A(gh^{-1}), \omega_A(g^{-1}), \omega_A(h^{-1}), \omega_A(hg^{-1})\}$$

are not in $\omega_A(\delta_x)$, and similarly at most two of

$$\{\omega_B(g), \omega_B(h), \omega_B(gh^{-1}), \omega_B(g^{-1}), \omega_B(h^{-1}), \omega_B(hg^{-1})\}$$

are not in $\omega_B(\delta_y)$, so there is an $f \in \{g^{\pm}, h^{\pm}, (gh^{-1})^{\pm}\}$ with

$$\omega_A(f) \in \omega_A(\delta_x) \quad \text{and} \quad \omega_B(f) \in \omega_B(\delta_y),$$

that is the quadrant $\delta_x \times \delta_y$ is made heavy by f. Hence $(p, q) \in \widehat{\mathcal{C}}(A, B)$ and we are done. $\square$

Remark. The proof suggests an alternate construction of the augmented core of compatible trees. Use Chiswell's construction of a tree from a based length function on $L_p(y) + M_q(y)$. The sum decomposition gives a way to define the projection map. Verifying minimality and convexity of fibers would then show that the resulting tree is the augmented core, giving an alternate proof. However, this approach quickly gets technical and obscures the importance of simultaneous good pairs, so we do not develop it further.

3.7 Additive properties of G-trees

Throughout this section suppose that A and B are compatible irreducible G-trees, so that $\widehat{\mathcal{C}}(A, B)$ is also an irreducible G-tree. When A and B are stable or small, $\widehat{\mathcal{C}}(A, B)$ shares this property, suggesting a meta-principle: minimal common refinements of G-trees should retain algebraic properties shared by the base trees. The case of very small trees illustrates that this principle is false in general, and that only arcwise properties are shared.

The main tool in understanding the global structure of $\widehat{\mathcal{C}}(A,B)$ is the following local structure lemma.

Lemma 3.7.1. Suppose A, B and $\widehat{\mathcal{C}}(A,B)$ are G-trees. Let $\pi_T : \widehat{\mathcal{C}}(A,B) \to T$ denote the projection maps to A and B.

No collapses. For every arc $e \subseteq \widehat{\mathcal{C}}(A,B)$

$$\text{length}_A(\pi_A(e)) + \text{length}_B(\pi_B(e)) > 0.$$

No local folds. For every pair of geodesic arcs $\alpha, \beta : [0,\epsilon) \to \widehat{\mathcal{C}}(A,B)$ with $\alpha(0) = \beta(0)$, if $\pi_T \circ \alpha([0,\epsilon)) \cap \pi_T \circ \beta([0,\epsilon))$ for T equal to either A or B, then there is a $0 < \delta < \epsilon$ such that

$$\alpha([0,\delta)) = \beta([0,\delta)).$$

Proof. Suppose $S \subseteq \widehat{\mathcal{C}}(A,B)$ such that

$$\text{diam}_A(\pi_A(S)) = \text{diam}_B(\pi_B(S)) = 0.$$

Then $\pi_A(S) = \{p\}$ and $\pi_B(S) = \{q\}$, hence $S \subseteq \{(p,q)\}$, a point. Therefore if e is an arc

$$\text{length}_A(\pi_A(e)) + \text{length}_B(\pi_B(e)) > 0.$$

Now suppose $\alpha, \beta : [0, \epsilon) \to \widehat{\mathcal{C}}(A, B)$ are geodesic arcs with $\alpha(0) = \beta(0) = \tilde{p}$ and $\alpha((0, \epsilon)) \cap \beta((0, \epsilon)) = \emptyset$. For a contradiction, suppose that $e = \pi_A \circ \alpha([0, \epsilon)) \cap \pi_A \circ \beta([0, \epsilon))$ contains an arc. Let $q \in e$ be a point not equal to $\pi_A(\tilde{p})$. By construction, there are points

$$\tilde{q}_\alpha \in \pi_A^{-1}(q) \cap \alpha([0, \epsilon))$$

and

$$\tilde{q}_\beta \in \pi_A^{-1}(q) \cap \beta([0, \epsilon))$$

and $\tilde{q}_\alpha \neq \tilde{q}_\beta$. Further, the unique arc joining $\tilde{q}_\alpha$ to $\tilde{q}_\beta$ is the geodesic $[\tilde{q}_\alpha, \tilde{q}_\beta]$, which contains $\tilde{p}$ by construction. Since $\tilde{p} \notin \pi_A^{-1}(q)$, we conclude that the fiber over q is not convex, a contradiction. $\square$

Corollary 3.7.2. If A and B are compatible simplicial G-trees then $\widehat{\mathcal{C}}(A, B)$ is a simplicial G-tree and for each edge $e \in \widehat{\mathcal{C}}^{(1)}(A, B)$, either $\pi_A|_e$ or $\pi_B|_e$ is a homeomorphism.

Proof. Guirardel shows the augmented core of simplicial trees is always a simplicial complex [26, Proposition 2.6], and the projection maps are simplicial. By the lemma, for every edge e, either $\pi_A(e)$ or $\pi_B(e)$ is an edge. $\square$

The local structure of $\widehat{\mathcal{C}}(A, B)$ immediately forces $\widehat{\mathcal{C}}(A, B)$ to inherit restrictions on the arc stabilizers of A and B.

Lemma 3.7.3. If A and B are compatible small G-trees, then $\widehat{\mathcal{C}}(A, B)$ is also a small G-tree.

Proof. Suppose $e \subseteq \widehat{\mathcal{C}}(A, B)$ is an arc. From the local structure lemma we may assume, without loss of generality, that $\pi_A(e)$ is an arc. Since π_A is equivariant, we have $\operatorname{Stab}(e) \leq \operatorname{Stab}(\pi_A(e))$, hence $\operatorname{Stab}(e)$ is cyclic or trivial, as required. (Note that, though we assume that $\pi_A(e)$ is non-trivial, since we are relabeling we do require that both A and B be small.) $\square$

Recall that the G-trees encountered in practice are often "stable" trees, and that stable trees admit a detailed analysis via the Rips machine [11]. The definition of stability is often stated in terms of subtrees, however it is essentially a condition on arc stabilizers.

Definition 3.7.4. Let T be a G-tree. A non-degenerate subtree $S \subset T$ is a **stable subtree** if for every non-degenerate $S' \subseteq S$, $\operatorname{Stab}(S) = \operatorname{Stab}(S')$. A tree T is **stable** if for every non-degenerate $T' \subseteq T$ there is a stable subtree $S \subseteq T'$.

Note that if S_1 and S_2 are stable subtrees of T with non-degenerate intersection then $S_1 \cup S_2$ is stable, so that every stable subtree is contained in a maximal subtree. Note also that Definition 3.7.4 could be phrased in terms of stable arcs.

Lemma 3.7.5. Suppose A and B are compatible stable G-trees, then $\widehat{\mathcal{C}}(A, B)$ is also a stable G-tree.

Proof. Let $\{A_i\}$ and $\{B_j\}$ be the families of maximal stable subtrees of A and B respectively. Since A and B are stable, $A = \cup A_i$ and $B = \cup B_j$. Define $C_{ij} = \widehat{\mathcal{C}}(A, B) \cap A_i \times B_j$, and note that $\widehat{\mathcal{C}}(A, B) = \cup C_{ij}$.

Suppose $T \subseteq \widehat{\mathcal{C}}(A, B)$ is a non-degenerate subtree. The intersection $T_{ij} = T \cap C_{ij}$ must be non-degenerate for some i and j. There is a non-degenerate subtree $S \subseteq T_{ij}$ such that either

- $S = \{*\} \times S_B$
- $S = S_A \times \{*\}$
- or for every non-degenerate $S' \subseteq S$ both $\pi_A(S')$ and $\pi_B(S')$ is non-degenerate.

In the first case, for all $S' \subseteq S$, $\pi_A(S') = \pi_A(S)$, and in the second $\pi_B(S') = \pi_B(S)$. We claim that S is the desired stable subtree of T. Indeed, $\pi_A(S) \subseteq A_i$ and $\pi_B(S) \subseteq B_j$, so for any non-degenerate $S' \subseteq S$, either $\pi_T(S') = \pi_T(S)$ or $\pi_T(S')$ is a non-degenerate subtree of a stable subtree for both $T = A$ and $T = B$, and we have

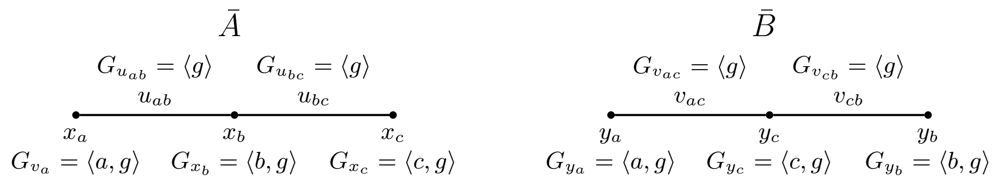
$$\begin{aligned}
\mathrm{Stab}_{\widehat{\mathcal{C}}(A,B)}(S') &= \mathrm{Stab}_A(\pi_A(S')) \cap \mathrm{Stab}_B(\pi_B(S')) \\
&= \mathrm{Stab}_A(\pi_(S)) \cap \mathrm{Stab}_B(\pi_B(S)) \\
&= \mathrm{Stab}_{\widehat{\mathcal{C}}(A,B)}(S).
\end{aligned}$$

$\square$

Remark. The above proof is significantly more general, and applies to any equivariant embedding $T \to A \times B$, but we do not need this generality in the sequel.

Very small trees demonstrate the limitations of the naïve meta-principle suggested in the introduction of this section. In the definition of very small, in addition to requiring that arc stabilizers be cyclic, tripod stabilizers are required to be trivial. This second condition is not an arc-wise condition, and is not preserved by the core, even in the nicest possible setting.

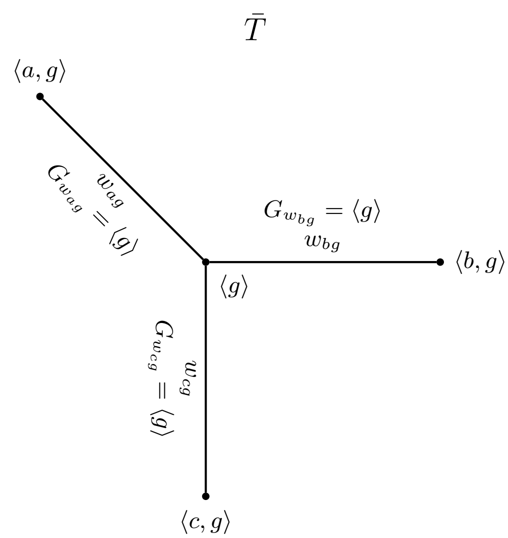
Example 3.7.6. Let $F_4 = \langle a, b, c, g\rangle$ and identify F_4 with the fundamental groups of the following two graphs of groups via the given marking.



Assign all edges length one, and treat $\bar{A}$ and $\bar{B}$ as metric graphs. Observe that the Bass-Serre trees A and B are simplicial very small F_4-trees.

Claim. The trees A and B are compatible, and the augmented core $\widehat{\mathcal{C}}(A, B)$ is not very small.

Proof of Claim. recisely, we will show $\widehat{\mathcal{C}}(A, B) = T$ where T is the Bass-Serre tree of the graph of groups



Define $\tilde{f}_A : T \to A$ by the equivariant collapse induced by the map $f_A : \bar{T} \to \bar{A}$

$$f_A(w_{ag}) = u_{ab}$$

$$f_A(w_{bg}) = x_b$$

$$f_A(w_{cg}) = u_{bc}$$

and similarly define $\tilde{f}_B : T \to B$ by

$$f_B(w_{ag}) = v_{ac}$$

$$f_B(w_{bg}) = v_{cb}$$

$$f_B(w_{cg}) = y_c.$$

By construction, the product map $\tilde{f} : T \to A \times B$ has connected fibers an its image is connected and F_4 invariant, so $\widehat{\mathcal{C}}(A, B) \subseteq \tilde{f}(T)$. We calculate $\tilde{f}(T)$ explicitly for a fundamental domain of T which covers fundamental domains for A and B:

$$\tilde{f}(\tilde{w}_{ag}) = \Delta(\tilde{u}_{ab} \times \tilde{v}_{ac})$$

$$\tilde{f}(\tilde{w}_{bg}) = \{\tilde{x}_b\} \times \tilde{v}_{cb}$$

$$\tilde{f}(\tilde{w}_{cg}) = \tilde{u}_{bc} \times \{\tilde{y}_c\}.$$

Therefore $\tilde{f}(T) \cong T$.

To show $\widehat{\mathcal{C}}(A,B)=T$ it suffices to show the three edges of a fundamental domain for T are either in $\mathcal{C}(A,B)$ or are diagonals of twice light rectangles. Orient $\tilde{w}_{ag}, \tilde{u}_{ab}$, and $\tilde{v}_{ac}$ consistently. By construction $[\![\tilde{u}_{ab}]\!]=[\![\tilde{v}_{ac}]\!]$ and $[\![\overline{\tilde{u}_{ab}}]\!]=[\![\overline{\tilde{v}_{ac}}]\!]$, which implies that $\tilde{u}_{ab}\times\tilde{v}_{ac}$ is twice light with the positively oriented diagonal the main diagonal, by the remark following Lemma 3.4.1. Hence $f(\tilde{w}_{ag})\subseteq\widehat{\mathcal{C}}(A,B)$.

We will also use the many horizon condition to show $\tilde{f}(\tilde{w}_{bg})\subseteq\widehat{\mathcal{C}}(A,B)$. Orient $\tilde{v}_{cb}$ so that $t(\tilde{v}_{cb})=\tilde{y}_b$. Every direction $\delta_p^{\tilde{x}_b}\subseteq A$ containing $\tilde{x}_b$ faces some b conjugate of a hyperbolic element, and $[\![\tilde{v}_{cb}]\!]$ contains all b conjugates, so all sets $[\![\delta_p^{\tilde{x}_b}]\!]\cap[\![\tilde{v}_{cb}]\!]$ are non-empty. Every direction containing $\tilde{x}_b$ also faces either some a conjugate of some hyperbolic element or some c conjugate; the set $[\![\overline{\tilde{v}_{cb}}]\!]$ contains all a and c conjugates, so all sets $[\![\delta_p^{\tilde{x}_b}]\!]\cap[\![\overline{\tilde{v}_{cb}}]\!]$ are non-empty, whence $\tilde{f}(\tilde{w}_{bg})\subseteq\widehat{\mathcal{C}}(A,B)$.

The argument to show $\tilde{f}(\tilde{w}_{cg})\subseteq\widehat{\mathcal{C}}(A,B)$ is symmetric. Therefore $\widehat{\mathcal{C}}(A,B)\cong T$ as claimed, and this tree is evidently not very small. $\diamond$

3.8 The Bass-Serre case

While not all useful stabilizer restrictions are retained by the core of compatible trees, when A and B are compatible Bass-Serre trees for graph of groups decompositions of G the structure theory of the core still permits a very explicit description of the augmented core.

Lemma 3.8.1. Suppose $\bar{A}$ and $\bar{B}$ are minimal visible graphs of groups with fundamental group $G\ncong\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}*\mathbb{Z}/2\mathbb{Z}$, and compatible Bass-Serre trees A and B. The augmented core $\widehat{\mathcal{C}}(A,B)$ is then then the Bass-Serre tree for a graph of groups Γ with fundamental group G, and the edge groups of Γ are in the set of conjugacy classes of the edge groups of $\bar{A}$ and $\bar{B}$. Moreover, $\bar{A}$ and $\bar{B}$ are isomorphic (via inner automorphisms of G) to graphs of groups $\bar{A}'$ and $\bar{B}'$ so that

$$\begin{tikzcd} & \Gamma \arrow[dl, "\pi_{\bar{A}'}"'] \arrow[dr, "\pi_{\bar{B}'}"] & \\ \bar{A}' & & \bar{B}' \end{tikzcd}$$

where $\pi_{\bar{A}'}$ and $\pi_{\bar{B}'}$ are quotient maps that collapse edges.

Proof. By Corollary 3.7.2, $\widehat{\mathcal{C}}(A, B)$ is a simplicial tree, with edges of three forms

$$\{v_A\} \times e_B, e_A \times \{v_B\}, \text{ or } \Delta \subseteq e_A \times e_B.$$

As in the proof of Lemma 3.7.5 we find, for each edge $e \in \widehat{\mathcal{C}}(A, B)$

$$\mathrm{Stab}_{\widehat{\mathcal{C}}(A,B)}(e) = \mathrm{Stab}_A(\pi_A(e)) \cap \mathrm{Stab}_B(\pi_B(e)).$$

Suppose $\pi_A(e) = a \in E(A)$. We claim

$$\mathrm{Stab}_{\widehat{\mathcal{C}}(A,B)}(e) = \mathrm{Stab}_A(a).$$

Indeed, suppose there is some $g \in \mathrm{Stab}_A(e)$ but not in $\mathrm{Stab}_B(e)$. Let $p \in a$ be the midpoint and let $q \in \pi_B(e)$ be any point. The point (p, q) is in the interior of e, and since g is not in the stabilizer, $(p \cdot g, q \cdot g) = (p, q \cdot g)$ is disjoint from e. Both $(p, q), (p, g \cdot q) \in \pi_A^{-1}(p)$, which is convex. However, the path in $\widehat{\mathcal{C}}(A,B)$ must pass through $o(e)$ or $t(e)$, neither of which is in $\pi_A^{-1}(p)$, a contradiction. Symmetrically, if $\pi_B(e) = b \in E(B)$ we find

$$\mathrm{Stab}_{\widehat{\mathcal{C}}(A,B)}(e) = \mathrm{Stab}_B(b).$$

The remainder of the lemma is then immediate from standard facts in Bass-Serre theory, with $\widehat{\mathcal{C}}(A,B)$ the Bass-Serre tree of the desired graph of groups Γ. $\square$

Remark. This characterizes the edge groups of compatible graphs of groups: An edge group $\bar{A}_e$ is either conjugate to some $\bar{B}_e$ or contained within a conjugate of some $\bar{B}_v$, and vise-versa.

CHAPTER 4

OUTER AUTOMORPHISMS

In these days the angel of topology and the

devil of abstract algebra fight for the soul of

each individual mathematical domain.

Hermann Weyl

By definition, the outer automorphism group $\mathrm{Out}(F_r) = \mathrm{Aut}(F_r)/\mathrm{Inn}(F_r)$ of a free group F_r is the automorphism group modulo the inner automorphisms. We briefly review various topological perspectives on elements of $\mathrm{Out}(F_r)$, the classification by growth, and some details about representatives of outer automorphisms of linear growth.

4.1 Topological representatives and growth

Let Γ be a topological graph with $\pi_1(\Gamma) = F_r$. An immersed path $\gamma : [0,1] \to \Gamma$ is **tight** if any lift $\tilde{\gamma} : [0,1] \to \tilde{\Gamma}$ is an embedding. Since $\tilde{\Gamma}$ is a tree, it is immediate that every immersed path is homotopic relative to the endpoints to a unique tight path, called its **tightening**. Given a path γ we denote the tightening $[\gamma]$. Similarly, a closed loop is tight if it is tight for every choice of basepoint, and is freely homotopic to a unique tightening (a fundamental domain for the action of $\gamma_* \in \pi_1(\Gamma)$ on the universal cover $\tilde{\Gamma}$, with basepoint chosen on the axis of γ_*), the tightening of a loop γ is denoted $[[\gamma]]$. Two paths γ and δ are composable if the end of γ equals the start of δ, and their composition is denoted $\gamma\delta$; if γ is a based loop γ^{-1} denotes its reverse and γ^m its m-fold concatenation for $m \in \mathbb{Z}$ (when $m = 0$ this is a constant path at the basepoint of γ). A loop γ is primitive if there is no γ' such that $[\gamma] = [\gamma'^m]$ for some $m > 1$. We will assume from here on that all paths have endpoints at the vertices of Γ.

Given an outer automorphism $\sigma \in \mathrm{Out}(F_r)$, we can realize σ as a homotopy equivalence $\hat{\sigma} : \Gamma \to \Gamma$. Such a realization is referred to as a **topological representative**; particularly nice topological representatives are indispensable in the analysis of outer automorphisms.

The growth of an outer automorphism is measured in terms of an action on a topological representative. We say σ is **exponentially growing** if there is some loop $\gamma \subseteq \Gamma$ such that $\ell_\Gamma([[\hat{\sigma}^n(\gamma)]])$ is bounded below by an exponential function, and that σ is **polynomially growing** if there is some d such that $\ell_\Gamma([[\hat{\sigma}^n(\gamma)]]) \in O(n^d)$ for all loops $\gamma \subseteq \Gamma$. This classification does not depend on the choice of topological representative, as demonstrated by Bestvina, Feighn, and Handel [7]; the choice does matter for the details of the exponent in the exponentially growing case, however we are not concerned with exponentially growing outer automorphisms in this thesis.

Polynomially growing outer automorphisms can exhibit a certain amount of finite-order periodic behavior which results in significant technical headaches. These phenomena can be removed by passing to a uniform power. A polynomially growing outer automorphism σ is **unipotent** if the induced action on the first homology $H_1(F_r, \mathbb{Z})$ is a unipotent matrix. Bestvina, Feighn, and Handel proved that any polynomially growing outer automorphism that acts trivially on $H_1(F_r, \mathbb{Z}/3\mathbb{Z})$ is unipotent [8, Proposition 3.5], so all polynomially growing outer automorphisms have a unipotent power.

4.2 Upper triangular representatives and the Kolchin theorem

Unipotent polynomially growing outer automorphisms have particularly nice topological representatives. A homotopy equivalence $\hat{\sigma} : \Gamma \to \Gamma$ is **filtered** if there is a filtration $\emptyset = \Gamma_0 \subsetneq \Gamma_1 \subsetneq \cdots \subsetneq \Gamma_k = \Gamma$ preserved by $\hat{\sigma}$.

Definition 4.2.1. A filtered homotopy equivalence $\hat{\sigma}$ is **upper triangular** if

- $\hat{\sigma}$ fixes the vertices of Γ,
- Each stratum of the filtration $\Gamma_i \setminus \Gamma_{i-1} = E_i$ is a single topological edge,
- Each edge E_i has a preferred orientation and with this orientation there is a tight closed path $u_i \subseteq \Gamma_{i-1}$ based at $t(E_i)$ so that $\hat{\sigma}(E_i) = E_i u_i$.

The path u_i is called the suffix associated to u_i, and when working with an upper triangular homotopy equivalence we will always refer to edges of the filtered graph with the preferred orientation. A filtration assigns to each edge a **height**, the integer i such that $E \in \Gamma_i \setminus \Gamma_{i-1}$, and by taking a maximum this definition extends to tight edge paths. An upper-triangular homotopy equivalence preserves the height of each edge path.

Every upper triangular homotopy equivalence of a fixed filtered graph evidently induces a unipotent outer automorphism, and using relative train tracks Bestvina, Feighn, and Handel show the converse, every unipotent polynomially growing outer automorphism has an upper triangular representative [7, Theorem 5.1.8]. Moreover, for a given filtered graph Γ the upper-triangular homotopy equivalences taken up to homotopy relative to the vertices form a group under composition. The suffixes for the inverse are defined inductively up the filtration by $\hat{\sigma}^{-1}(E_i) = E_i v_i$ where $v_i = \overline{\hat{\sigma}^{-1}(u_i)}$.

A nontrivial path $\gamma \subseteq \Gamma$ is a **periodic Nielsen path** for $\hat{\sigma}$ if for some $m > 0$, we have $[\hat{\sigma}^m(\gamma)] = [\gamma]$. If $m = 1$ we call γ a **Nielsen path**. An **exceptional path** in Γ is a path of the form $E_i \gamma^m \bar{E}_j$, where γ is a primitive Nielsen path, and $\hat{\sigma}(E_i) = E_i \gamma^p$ and $\hat{\sigma}(E_j) = E_j \gamma^q$ for $p, q > 0$ and any m. For a unipotent automorphism, every closed periodic Nielsen path is Nielsen [8, Proposition 3.16]. If $p \neq q$ we say the exceptional path is a **linearly growing**, otherwise it is an **exceptional Nielsen path**.

Every path $\gamma \subseteq \Gamma$ has a **canonical decomposition** with respect to an upper triangular $\hat{\sigma}$ into single edges and maximal exceptional paths [8, Lemma 4.26].

For all of the terms in the previous two paragraphs, when we are dealing with more than one upper-triangular homotopy equivalence we will specify which homotopy equivalence is involved, e.g. a path γ is $\hat{\sigma}$-Nielsen or consider the $\hat{\tau}$-canonical decomposition of $\gamma = \gamma_1 \gamma_2 \cdots \gamma_k$.

The analogy between unipotent outer automorphisms and unipotent matrices stretches beyond having an upper-triangular basis. The classical Kolchin theorem for linear groups states [36] that if a subgroup $H \leq GL(n, \mathbb{C})$ consists of unipotent matrices then there is a basis so that with respect to this basis every element of H is upper triangular with 1's on the diagonal. There is an analogous theorem for unipotent polynomially growing outer automorphisms, due to Bestvina, Feighn, and Handel.

Theorem 4.2.2 ([8, Main Theorem])**.** Suppose $H \leq \mathrm{Out}(F_n)$ is a finitely generated subgroup with every element unipotent. Then there is a filtered graph Γ and a fixed preferred orientation such that every $\sigma \in H$ is upper triangular with respect to Γ.

Remark. Bestvina, Feighn, and Handel use a different definition of upper-triangular, allowing that $\sigma(E_i) = v_i E_i u_i$, however our definition can be obtained by subdividing each edge and doubling the length of the filtration.

4.3 Dehn twists and linear growth

Let Σ be a closed hyperbolic surface. Given $\gamma \subseteq \Sigma$ an essential simple closed curve, consider a homeomorphism $\tau_\gamma : \Sigma \to \Sigma$ that is the identity outside an annular neighborhood of γ and performs a twist of 2π on the annulus. Such a homeomorphism is known as a **Dehn twist**. The induced map $\tau_{\gamma *} : \pi_1(\Sigma) \to \pi_1(\Sigma)$ can be expressed in terms of the graph of groups decomposition of $\pi_1(\Sigma)$ induced by γ, and this expression motivates the following definition for general graphs of groups.

Definition 4.3.1. Suppose Γ is a graph-of-groups. Given a fixed collection of edges $\{e_i\} \subseteq E(\Gamma)$ closed under the edge involution and $z_{e_i} \in Z(G_{e_i})$ satisfying $z_{\bar{e}_i} = z_{e_i}^{-1}$, the **Dehn twist about**

$\{e_i\}$ **by** $\{z_i\}$, $D_z \in \mathrm{Out}(\pi_1(\Gamma, v))$, is the outer automorphism induced by $\tilde{D}_z$ on the fundamental groupoid of Γ, given by

$$\begin{aligned} \tilde{D}_z(e_i) &= e_i z_i^{e_i} \\ \tilde{D}_z(g) &= g, && g \in G_v, v \in V(\Gamma) \\ \tilde{D}_z(e) &= e, && e \notin \{e_i\} \end{aligned}$$

The induced outer automorphism does not depend on the choice of basepoint.

Note that $D_z^n = D_{z^n}$, defining $z^n = \{z_{e_i}^n\}$ for any n, and that any two twists on a fixed graph of groups Γ commute. The requirement that each $z_{e_i} \in Z(G_{e_i})$ is necessary to ensure that the defining relations of the fundamental groupoid are respected. In turn, when $\pi_1(\Gamma)$ is free a Dehn twist can only twist around edges with cyclic stabilizers.

Example 4.3.2. Let Γ be the graph of groups associated to the amalgamated product $A *_C B$ and $z \in Z(C)$. The twist of Γ about its edge by z can be represented by $D_z(a) = z^{-1}az$, $a \in A$, $D_z(b) = b, b \in B$. Since $A \cup B$ generates $\pi_1(\Gamma, v)$ this fully specifies the automorphism.

Let $\mathcal{H}$ be the graph of groups associated to the HNN extension $A*_C$ and pick $z \in Z(C)$. The twist of $\mathcal{H}$ about its one edge by z is represented by $D_z(a) = a$ and $D_z(t) = tz$ with $a \in A$ and t the edge of the extension.

Specializing these examples to splittings of $\pi_1(\Sigma)$ given by an essential closed curve in a closed hyperbolic surface $\gamma \subseteq \Sigma$, this gives the previously mentioned algebraic representation of $\tau_{\gamma*}$ as the Dehn twist about the edge of the splitting corresponding to γ by $\gamma_* \in \pi_1(\Sigma)$.

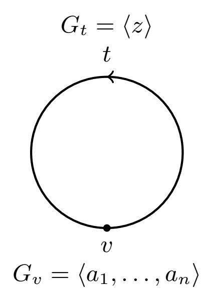


Figure 12. The graph of groups used to represent Nielsen automorphisms.

Example 4.3.3 (Nielsen automorphisms of F_r)**.** Consider the graph of groups Γ in Figure 12. The edge morphims for the single edge are given by $\iota_t(z) = a_j$ and $\iota_{\bar{t}}(z) = a_k$. The map $F : \langle x_1, \ldots, x_n \rangle \to \pi_1(\Gamma, v)$ given by $F(x_i) = a_i$, $i \neq j$, and $F(x_j) = t$ gives a realization of the Nielsen automorphism $\phi(x_i) = x_i, \phi(x_j) = x_k x_j$ as the Dehn twist about the single edge by z.

A Dehn twist outer automorphism has many graph of groups representatives, most of which are not well suited to analysis using the Guirardel core, due to lots of extra information. Certain ill-behaved stabilizers, non-minimal graphs, invisible vertices, and unused edges all cause trouble. Cohen and Lustig identified a particularly useful class of representatives, called efficient twists.

Definition 4.3.4. A Dehn twist D on a graph of groups Γ is **efficient** if

- Γ is minimal, small, and visible
- D twists about every edge (every $z_e \neq \mathrm{id}$).

- (**No positively bonded edges**) There is no pair of edges $e_1, e_2 \in E(\Gamma)$ such that $v = t(e_1) = t(e_2)$, and integers $m, n \neq 0$ with $mn > 0$, such that $z_{e_1}^m$ is conjugate in G_v to $z_{e_2}^n$.

Cohen and Lustig remark that it is a consequence of these three properties that Γ is necessarily very small. Returning our attention to $\mathrm{Out}(F_r)$ a Dehn twist outer automorphism $D \in \mathrm{Out}(F_r)$ is one that can be represented as a Dehn twist of some graph of groups decomposition of F_r (such a decomposition necessarily only twists about those edges with cyclic edge groups). These outer automorphisms have linear growth (and all outer automorphisms with linear growth are roots of Dehn twists [38]).

By assigning each edge of a graph of groups Γ a positive length, the Bass-Serre tree T of Γ becomes a metric F_r tree. Given a very small graph of groups Γ with fundamental group F_r, the collection of projective classes of all choices of metric on T determines an open simplex $\Delta(\Gamma) \subseteq \overline{CV}_r$ in projectivized outer space. If Γ is visible and minimal, this simplex is of dimension $|E(\Gamma)| - 1$. When D is an efficient Dehn twist on Γ, the simplex $\Delta(\Gamma)$ is essential to understanding the dynamics of the action of D on CV_r, and characterizes these dynamics completely, as shown by Cohen and Lustig.

Theorem 4.3.5 ([18, Theorem 13.2])**.** Suppose D is a Dehn twist in $\mathrm{Out}(F_r)$ with an efficient representative on a graph of groups Γ. Then for all $[T] \in CV_r$,

$$\lim_{n\to\infty} D^n([T]) = \lim_{n\to\infty} D^{-n}([T]) \in \Delta(\Gamma).$$

Corollary 4.3.6. If $D \in \mathrm{Out}(F_r)$ has an efficient Dehn twist representative, then the simplicial structure of the Bass-Serre tree of the representative is unique.

Proof. Suppose D has efficient representatives D_1 on Γ_1 and D_2 on Γ_2. By the theorem, $\Delta(\Gamma_1) = \Delta(\Gamma_2)$ since two open simplices which share a point are equal. This completes the claim. $\square$

An efficient graph-of-groups representative can be constructed from an upper-triangular representation. Bestvina, Feighn, and Handel give this construction in the metric category, using a particular upper-triangular representation that permits them to compute metric information about the limit in $\overline{cv}_r$, but the uniqueness of the algebraic structure permits the calculation from any upper-triangular representation. First note that an upper-triangular homotopy equivalence grows linearly if and only if each suffix is Nielsen, and that each edge is either fixed or grows linearly.

To construct the efficient representative from an upper-triangular representative we need the notion of folding in a tree or graph, due to Stallings [47]. In a simplicial G-tree T, a **fold** of two edges $u, v \in T$ with $o(u) = o(v)$ for a linear homeomorphism $\phi : u \to v$ is the quotient of T by the smallest equivalence relation satisfying $x \sim \phi(x)$ for all points $x \in u$ and if $x \sim y$ and $g \in G$ then $x.g \sim y.g$. The quotient map of this equivalence $\tilde{f} : T \to T/\sim$ is called the **folding map**, and the resulting space $T/\sim$ is a G-tree (it may be necessary to subdivide to ensure that the action is without inversions). When the action on the folded tree $T/\sim$ is without inversions, we get a graph of groups morphism on the quotient $f : \bar{T} \to \overline{T/\sim}$.

Let $q : T \to \bar{T}$ be a graph of groups quotient map. There is a particular type of fold we treat in detail. Suppose there is an element $g \in G$ such that the folding homeomorphism $\phi : u \to v$ is induced by the g action. In this case $g \in \bar{T}_{o(u)}$ and g conjugates Stab(u) to Stab(v). The folded graph of groups $\overline{T/\sim}$ has the same combinatorial structure as $\bar{T}$, however Stab$(u/\sim) = \langle \text{Stab}(u), g \rangle$. This is referred to as "pulling an element in a vertex group over an edge".

By subdividing an edge we may perform a **partial fold** of the first half of u over v. (Partial folding can be discussed in much greater generality; we require only the midpoint version.) We will often specify a fold by a pair of edges u and v with $o(u) = o(v)$ in the quotient graph of groups, it is understood that we mean the equivariant fold of all pairs of lifts $\tilde{u}, \tilde{v}$ with $o(\tilde{u}) = o(\tilde{v})$. The definition of folding generalizes to allow v to be an edge path, and we use this more general definition.

Lemma 4.3.7. Suppose $\hat{\sigma} : \Gamma \to \Gamma$ is a linearly growing upper-triangular homotopy equivalence of a filtered graph Γ. Then there is a graph of groups fold $f : \Gamma \to \mathcal{G}$ which realizes the outer automorphism represented by $\hat{\sigma}$ as an efficient Dehn twist.

Proof. The strategy of the proof is to collapse every fixed edge; in the resulting graph of groups, the suffix of the lowest linear edge is in a vertex group, and so the suffix can be folded over that edge. Working up the filtration in this fashion the result is a graph of groups with cyclic edge stabilizers, and by twisting on every edge by the twister specified by its suffix; the result is a Dehn twist on this graph which represents $\hat{\sigma}$.

The problem with this construction, as just described, is that the result may not be efficient: there may be obtrusive powers, and there may be positively bonded edges. The first problem is solved by using the primitive root of the suffix, but the second requires some work. One could use Cohen and Lustig's algorithm to remove positive bonding, however we give a different construction similar to that of Bestvina, Feighn, and Handel [8] useful when considering more than one Dehn twist.

We assume without loss of generality Γ is minimal (that is, the quotient of a minimal tree under the F_r action).

Step 1: Fold Conjugates. We construct a series of folds by working up the filtration from lowest edge to highest. Start with $\Gamma^0 = \Gamma$. If the suffix u_i of E_i is of the form $\gamma_i \eta_j^k \bar{\gamma}_i$, where $u_j = [\eta_j^{k'}]$ so that η_j is the primitive Nielsen path associated to u_j, $j < i$ and γ_i a closed path, fold the terminal half of E_i over $\bar{\gamma}_i$. Let $f_i : \Gamma^{i-1} \to \Gamma^i$ be the folding map in this step. We claim the induced homotopy equivalence $\hat{\sigma}_i = f_i \hat{\sigma}_{i-1} f_i^{-1}$ is upper triangular. Let E_i' denote the unfolded initial half of E_i, and filter Γ^i by the filtration of Γ^{i-1} where the ith stratum is now E_i'. It suffices to check that $\hat{\sigma}_i(E_i') = E_i' u_i'$. Indeed, using the equation

$$f_i \hat{\sigma}_{i-1}(E_i \gamma_i) = \hat{\sigma}_i(E_i')$$

we have for some $m \in \mathbb{Z}$

$$f_i \hat{\sigma}_{i-1}(E_i \gamma_i) = f_i(E_i \gamma_i \eta_j^k \bar{\gamma}_i \gamma_i \eta_j^m) = E_i' \bar{\gamma}_i \gamma_i \eta_j^k \bar{\gamma}_i \gamma_i \eta_j^m$$

and so we can take $\hat{\sigma}_i(E'_i) = E'_i[u_j^{m+1}]$. If the suffix u_i of E_i is not of the above form, take $\Gamma^i = \Gamma^{i-1}$ and $f_i = \mathrm{id}$.

Denote the total folding map $f_k \cdots f_0 = f' : \Gamma \to \Gamma'$, and the induced automorphism $\hat{\sigma}'$. By construction $\hat{\sigma}'$ is upper triangular and has the property that for every two edges E_i and E_j with common terminal vertex, if their suffixes have conjugate roots then they are of the form $u_i = [\eta^{k_i}]$, $u_j = [\eta^{k_j}]$ for positive powers of a primitive Nielsen path η.

Step 2: Fold Linear Families. Starting now with $\hat{\sigma}'$, we perform another sequence of folds to ensure that twisters will not be positively bonded. For a primitive Nielsen path η, the linear family associated to η is all edges of Γ' with suffix $[\eta^k]$ for some $k \neq 0$. We now work down the filtration of Γ'. Set $\Gamma'_k = \Gamma'$. If E'_i is in some linear family associated to η, let E_j be the next edge lower than E'_i in the linear family, and fold half of E'_i over all of E'_j. Denote the fold $f'_i : \Gamma'_i \to \Gamma'_{i-1}$ in this case; otherwise set $\Gamma'_{i-1} = \Gamma_i$ and $f'_i = \mathrm{id}$. Let $\Gamma'' = \Gamma'_0$ be the total result of this folding, with total folding map $f'_0 \cdots f'_k = f'' : \Gamma' \to \Gamma''$, and denote the unfolded halves of edges by E''_i. (If an edge is not folded we will also use E''_i for the edge as an edge of Γ''). The graph Γ'' is naturally filtered, with the filtration induced by f''. We claim that the induced homotopy equivalence $\hat{\sigma}'' = f''\hat{\sigma}'f''^{-1}$ is again upper triangular. Indeed, as in the previous case we can calculate the suffixes. For E'_i denote by $E'_{i_1}, \ldots E'_{i_l}$ the edges in the linear family of E'_i below E'_i in descending order, so that $f''(E'_i) = E''_i E''_{i_1} \cdots E''_{i_l}$. Working inductively up the linear family, we see that $\hat{\sigma}''(E''_i) = E''_i E''_{i_1} \cdots E''_{i_l}[f''(\eta)^{k_i - k_{i-1}}]\bar{E}''_{i_l} \cdots \bar{E}'_{i_1}$, and the associated primitive Nielsen path to E''_i is $\eta''_i = E''_{i_1} \cdots E''_{i_l}[f''(\eta)]\bar{E}''_{i_l} \cdots \bar{E}''_{i_1}$.

Step 3: Collapse and Fold Edge Stabilizers. From $\hat{\sigma}''$ and Γ'' we can now construct a graph of groups; the previous two steps will ensure that no twisters in the result are positively bonded. We work up the filtration once more. Let $\mathcal{G}^0$ be the graph of groups constructed from Γ'' by collapsing all edges with trivial suffix. Obtain $\mathcal{G}^i$ from $\mathcal{G}^{i-1}$ as follows. If $\hat{\sigma}''(E_i'') = E_i''$, set $\mathcal{G}^i = \mathcal{G}^{i-1}$. If $\hat{\sigma}''(E_i'') = E_i''[\eta_i''^{k_i''}]$ then obtain $\mathcal{G}^i$ from $\mathcal{G}^i$ by pulling η_i'' over E_i''. By construction η_i'' represents an element in a vertex group at $t(E_i'')$. The result is $\mathcal{G}$. The composition of folding maps $f'' : \Gamma'' \to \mathcal{G}$ induces a Dehn twist $\tilde{\sigma}$ on $\mathcal{G}$ where the system of twisters is given by $z_{E_i''} = \eta_i''^{k_i''}$. By construction, this twist represents $\hat{\sigma}''$ and so $\hat{\sigma}$; moreover the edge stabilizers are not conjugate in the vertex groups; the resulting twist is efficient except for the possibility of invisible vertices. Invisible vertices are an artifact of the graph of groups; removing them gives the desired efficient twist. $\square$

Remark. It is possible that $\hat{\sigma}$ is upper triangular with respect to several different filtrations of Γ. By fixing a filtration a choice is being made, but the choices made do not matter because of Corollary 4.3.6.

CHAPTER 5

POLYNOMIALLY GROWING DIPLOMACY

I had never expected that the China

initiative would come to fruition in the form

of a Ping-Pong team.

Richard M. Nixon

McCarthy's theorem for two-generator subgroups of the mapping class group of a surface Σ can be viewed through the lens of a compatibility condition for geometric invariants associated to a pair of mapping classes. Recall that a mapping class $\sigma \in \mathrm{Mod}(\Sigma)$ is **rotationless** if the Thurston normal form has no non-trivial permutation components. Associated to a rotationless mapping class is a decomposition of Σ into invariant surfaces of negative Euler characteristic Σ_i and annuli A_j, so that (up to isotopy) $\sigma|_{\Sigma_i}$ is either identity or pseudo-Anosov, and $\sigma|_{A_j}$ is some power of a Dehn twist about the core curve of A_j. The supporting lamination λ of σ is the union of the core curves of the non-trivial Dehn twist components (thought of as measured laminations with atomic measure equal to the absolute value of the twist power on the core curve) and the attracting measured laminations of the pseudo-Anosov components.

Theorem 5.0.1 (McCarthy)**.** Suppose $\sigma, \tau \in \mathrm{Mod}(\Sigma)$ are mapping classes of a closed hyperbolic surface Σ. Then there is an N such that $\langle \sigma^N, \tau^N \rangle$ is either abelian or free of rank two. Moreover, $\langle \sigma^N, \tau^N \rangle \cong F_2$ exactly when $i(\lambda, \mu) > 0$, where λ and μ are the supporting measured laminations of rotationless powers of σ and τ respectively.

Using algebraic laminations an analogous result can be obtained for two generator subgroups of $\mathrm{Out}(F_r)$ when both generators are exponentially growing; this was first done by Bestvina, Feighn, and Handel [6] for pairs of fully irreducible outer automorphisms (with a novel proof by Kapovich and Lustig [32]), and for exponentially growing outer automorphisms satisfying certain technical hypotheses by Taylor [48] and Ghosh [25]. The techniques involved depend, in one way or another, on the existence of an attracting lamination for both generators. These approaches therefore do not apply to polynomially growing outer automorphisms, which have no laminations. Unlike the surface setting, there is no one-to-one correspondence between trees and laminations [42]. While this fact complicates the dynamical picture it provides new avenues for understanding polynomially growing outer automorphisms.

The notion of tree compatibility from Chapter 3 provides insight for linearly growing outer automorphisms. Once more, the issue of periodic behavior poses an obstacle to providing good tree representatives, passing to a rotationless power gives a generalization of McCarthy's theorem to the linearly growing setting.

Theorem 5.0.2. Suppose σ and τ are linearly growing outer automorphisms. Then there is an N such that $\langle \sigma^N, \tau^N \rangle$ is either abelian or free of rank two. Moreover, the latter case holds exactly when the core of the efficient representatives of Dehn-twist powers of σ and τ contains a rectangle.

To motivate the development of the tools needed in the proof, we will first turn to a series of guiding examples, treating the case of commuting twists and a slight generalization of the setting considered by Clay and Pettet [17]. The theme of the proof is to use the augmented core: when it is a tree, it is a small tree mutually fixed by both automorphisms, and gives a commuting realization of the automorphisms. Should it fail to be a tree it will provide length bounds needed to play ping-pong and find a power generating a free group.

5.1 Guiding examples

When the Guirardel core of two Bass-Serre trees has no rectangles, its quotient provides a simultaneous resolution of the two graphs-of-groups. This construction immediately gives us a sufficient condition for two Dehn twists to commute.

Lemma 5.1.1. Suppose $\tilde{\sigma}, \tilde{\tau}$ are efficient Dehn twists based on graphs of groups $\bar{A}$ and $\bar{B}$ covered by F_r-trees A and B respectively, representing $\sigma, \tau \in \mathrm{Out}(F_r)$. If $i(A, B) = 0$ then $[\sigma, \tau] = 1$ in $\mathrm{Out}(F_r)$.

Proof. Since A and B are simplicial, $i(A, B) = 0$ implies that $\widehat{\mathcal{C}}(A, B)$ is a tree. Therefore, by Lemma 3.8.1 and the subsequent remarks, $\widehat{\mathcal{C}}(A, B)$ is the Bass-Serre tree of a graph of groups Γ, and without changing the outer automorphism class of σ and τ we can assume that the identifications of F_r with $\pi_1(\Gamma, (u, v))$, $\pi_1(\bar{A}, u)$, and $\pi_1(\bar{B}, v)$ are such that in the following diagram $\pi_{\bar{A}}$ and $\pi_{\bar{B}}$ are quotient graph of groups morphisms that collapse edges.

$$\begin{tikzcd} & \Gamma \arrow[dl, "\pi_{\bar{A}}"'] \arrow[dr, "\pi_{\bar{B}}"] & \\ \bar{A} & & \bar{B} \end{tikzcd}$$

Moreover (and this is still the content of Lemma 3.8.1), the edge groups of Γ are edge groups of either $\bar{A}$ or $\bar{B}$.

Define $\hat{\sigma}$ on Γ by the system of twisters

$$z_e = \begin{cases} z_{\pi_{\bar{A}}(e)} & \pi_{\bar{A}}(e) \in E(\bar{A}) \\ 1 & \text{otherwise} \end{cases}$$

By construction, $\pi_{\bar{A}}\hat{\sigma} = \tilde{\sigma}\pi_{\bar{A}}$ at the level of the fundamental groupoid, so that $\hat{\sigma}$ is also a representative of σ. (The induced automorphism on the fundamental group coming from a graph of groups collapse is the identity [18].) Similarly define $\hat{\tau}$, thus simultaneously realizing σ and τ as Dehn twists on Γ, whence $[\sigma, \tau] = 1$. $\square$

Towards a converse, Clay and Pettet give a partial result, using the notion of a filling pair of Dehn twists [17].

Definition 5.1.2. Let X be a finitely generated group and T a simplicial X-tree. The **free T volume of** X, $\mathrm{covol}_T(X)$ is the number of edges with trivial stabilizer in the graph of groups associated to the minimal subtree $T^X \subset T$.

Note that $\mathrm{covol}_T(\langle g \rangle) = \ell_T(g)$ for $g \in X$.

Definition 5.1.3. Two graphs of groups $\bar{A}$ and $\bar{B}$ associated to F_r-trees A and B **fill** if for every free factor $X \leq F_r$,

$$\mathrm{covol}_A(X) + \mathrm{covol}_B(X) > 0.$$

Definition 5.1.4. Suppose $\tilde{\sigma}, \tilde{\tau}$ are representatives of Dehn twists based on $\bar{A}$ and $\bar{B}$, where both graphs of groups have one edge and fundamental group F_r. If $\bar{A}$ and $\bar{B}$ fill then we call the induced outer automorphisms σ and τ a **filling pair**.

This definition is a close parallel to the notion of a pair of filling simple closed curves, and Clay and Pettet strengthen this parallel to a theorem.

Theorem 5.1.5 ([17, Theorem 5.3])**.** Suppose $\sigma, \tau \in \mathrm{Out}(F_r)$ are a filling pair of Dehn twists. Then there is an N such that

- $\langle \sigma^N, \tau^N \rangle \cong F_2$
- If $\phi \in \langle \sigma^N, \tau^N \rangle$ is not conjugate to a generator then ϕ is an atoroidal fully irreducible outer automorphism.

In developing their definition of free volume, Clay and Pettet use the Guirardel core as motivation, but give a form suited explicitly to the proof of their theorem. The definition of filling is indeed noticed by the core.

Proposition 5.1.6. Suppose $\bar{A}$ and $\bar{B}$ are one-edge cyclic splittings without obtrusive powers associated to F_r trees A and B that fill. Then $i(A, B) > 0$ and the action of F_r on $\mathcal{C}(A, B)$ is free.

Proof. First, for any $(p, q) \in \mathcal{C}$, and $x \neq \mathrm{id} \in F_r$, since $\bar{A}$ and $\bar{B}$ fill, we have

$$\ell_A(x) + \ell_B(x) > 0$$

and therefore $(p, q) \cdot x \neq (p, q)$.

To see that the core contains a rectangle we will show that the two trees have incompatible combinatorics. To fix notation let e be the edge of $\bar{A}$ and f be the edge of $\bar{B}$. Let $G_e = \langle c \rangle$. If

$o(e) \neq t(e)$, let $a \in \bar{A}_{o(e)}$ be an element not conjugate into $\iota_{\bar{e}}(\bar{A}_e)$, and $b \in \bar{A}_{t(e)}$ be an element not conjugate into $\iota_e(\bar{A}_e)$. Set $\alpha = a$ and $\beta = ebe^{-1}$ in $\pi_1(\bar{A}, o(e))$. If $o(e) = t(e)$ take α as before and $\beta = e$ in $\pi_1(\bar{A}, o(e))$.

Since $\bar{A}$ and $\bar{B}$ fill and $\ell_A(\alpha) = 0$, we have $\ell_B(\alpha) > 0$. By construction, $\ell_A(\alpha\beta) > 0$, and so $\alpha\beta$ is not conjugate to $\iota_{\bar{e}}(c)$. Again, by the filling property, since $\ell_A(c) = 0$, $\ell_B(c) > 0$. Since $\alpha\beta$ and c are not conjugate, the characteristic sets of $\alpha\beta$ and c in B meet in at most a finite number of edges of C_c^B, since B is small. Thus there is some $n > 0$ such that $C_{\alpha\beta}^B \cap C_{c^{-n}\alpha\beta c^n}^B = \emptyset$. However, by construction $\mathcal{C}_{\alpha\beta}^A$ contains the arc in A stabilized by c, so $\mathcal{C}_{\alpha\beta}^A \cap \mathcal{C}_{c^{-n}\alpha\beta c^n}^A$ contains this arc for all n. Therefore the two Bass-Serre trees are incompatible, the core contains a rectangle, and since both trees are simplicial this implies that the intersection number is positive, as required. □

This proposition motivates a variation of Clay and Pettet's result, in pursuit of a converse to Lemma 5.1.1. This variation cannot make the stronger assertion that the generated group contain an atoroidal fully irreducible element. Indeed, take σ and τ to be a filling pair of Dehn twists for F_k and consider the automorphism $\sigma * \mathrm{id}_m$ and $\tau * \mathrm{id}_m$ acting on $F_k * F_m$. This is a pair of Dehn twists of F_{k+m} that has powers generating a free group, but does not fill, and every outer automorphism $\langle \sigma * \mathrm{id}_m, \tau * \mathrm{id}_m \rangle$ fixes the conjugacy class of the complementary F_m free factor, so all elements of the group generated are reducible. Nevertheless, there is a partial converse to Lemma 5.1.1, finding free groups generated by hyperbolic-hyperbolic pairs of Dehn twists based on one-edge graphs of groups using a variation on their argument.

Definition 5.1.7. Suppose $\bar{A}$ and $\bar{B}$ are minimal, visible, small graphs of groups with one edge and associated F_r-trees A and B. The pair is **hyperbolic-hyperbolic** if both for the edge $e \in E(\bar{A})$, a generator z_e of $\bar{A}_e$ acts hyperbolically on B; and for the edge $f \in E(\bar{B})$, a generator z_f of $\bar{B}_f$ acts hyperbolically on A.

Proposition 5.1.8. If $\bar{A}$ and $\bar{B}$ are minimal, visible, small graphs of groups with one edge. If $\bar{A}$ and $\bar{B}$ are hyperbolic-hyperbolic, then $i(A, B) > 0$.

Proof. The proof of Proposition 5.1.6 applies immediately to show that the two Bass-Serre trees are not compatible. The construction used only the positive translation length of $\ell_B(c)$ for a generator c of an edge group of $\bar{A}_e$ and that $\bar{B}$ is small. $\square$

Remark. The above proposition, as noted in the proof, is much more general, giving a sufficient condition for incompatibility: for any two minimal, visible, small graphs of groups, if there is an edge of one with a generator hyperbolic in the other then the core of the Bass-Serre trees has a rectangle.

The hyperbolic-hyperbolic condition is sufficient to give a length function ping-pong argument similar to Clay and Pettet's.

Lemma 5.1.9. Suppose $\tilde{\sigma}$ and $\tilde{\tau}$ are efficient Dehn twist representatives of $\sigma, \tau \in \mathrm{Out}(F_r)$, on one-edge graphs of groups $\bar{A}$ and $\bar{B}$ respectively. If $\bar{A}$ and $\bar{B}$ are hyperbolic-hyperbolic, then there is an N such that $\langle \sigma^N, \tau^N \rangle \cong F_2$.

Proof. Let e denote the edge of $\bar{A}$, $\bar{A}_e = \langle a \rangle$, f the edge of $\bar{B}$ and $\bar{B}_f = \langle b \rangle$. Let s, t be nonzero integers so that the twisters of $\tilde{\sigma}$ and $\tilde{\tau}$ are $z_e = a^s$ and $z_f = b^t$ respectively. We will conduct a ping-pong argument similar to Clay and Pettet's free factor ping pong technique. Consider the subset of conjugacy classes of F_r $P = \{[w] \in [F_r] | \ell_A(w) + \ell_B(w) > 0\}$. Note that by hypothesis $[a], [b] \in P$ so P is non-empty. Moreover, by considering the normal form of b with respect to $\bar{A}$ and a with respect to $\bar{B}$, we see that there are powers n, m so that $a^{\pm n} b^{\pm m}$ has positive length in both trees; as a result $[\langle a^n, b^m \rangle] \setminus \{[\mathrm{id}]\} \subseteq P$. Partition $P = P_\sigma \sqcup P_\tau$,

$$P_\sigma = \{[w] \in P | \sqrt{2}\ell_A(w) < \ell_B(w)\}$$

$$P_\tau = \{[w] \in X | \ell_B(w) < \sqrt{2}\ell_A(w)\}$$

The use of $\sqrt{2}$ is arbitrary, any irrational will ensure that this is a partition, as ℓ_A and ℓ_B are integer valued. This is a non-trivial partition, $a \in P_\tau$ and $b \in P_\sigma$ by hypothesis.

Our goal then is to find a power N such that for all $n \geq N$, $\sigma^{\pm n}(P_\tau) \subseteq P_\sigma$ and $\tau^{\pm n}(P_\sigma) \subseteq P_\tau$. By a variation on the ping-pong lemma, this implies $\langle \sigma^N, \tau^N \rangle \cong F_2$, as required. The argument will be symmetric.

Suppose $[w] \in P_\tau$, so that $\ell_A(w) > 0$. Fix a cyclically reduced representative in Bass-Serre normal form with respect to a fixed basis of F_r based at a vertex of $\bar{A}$:

$$w = e_1 a^{k_1} w_1 e_2 a^{k_2} w_2 \cdots e_\ell a^{k_\ell} w_\ell$$

where $\ell = \ell_A(w)$, $e_i \in \{e, \bar{e}\}$, we are suppressing the different edge morphisms sending a into relevant vertex groups, and each w_i is in the right transversal of the image of a in the vertex group involved. Let C be the bounded cancellation constant for the fixed basis of F_r basis into B. With respect to this basis, after an appropriate conjugation we have the cyclically reduced conjugacy class representative w' satisfying

$$|w'| = |a^{k'_1}| + \cdots + |w'_{\ell-1}| + |a^{k'_\ell}| + |w'_\ell|$$

where w'_i is the reduced word in this basis for the group element represented by joining $a^{\pm 1} w_i e_{i+1} a^{\pm 1}$ to the basepoint and including the appropriate conjugating transversal elements coming from each instance of an a, and $k'_i = sgn(k_i)(|k_i| - 2)$ where each w'_i might have disturbed at most two cyclically adjacent copies of conjugates of a depending on the particular spelling (this follows from the normal form). We have

$$\sqrt{2}\ell_A(w) > \ell_B(w) \geq \left(\sum |k'_i|\right) \ell_B(a) - C\ell_A(w).$$

Re-writing, we conclude

$$\sum |k'_i| < \left(\frac{\sqrt{2} + C}{\ell_B(a)}\right) \ell_A(w). \tag{\dagger}$$

Using the Dehn twist representative of σ, we calculate

$$\tilde{\sigma}^n(w) = e_1 a^{\epsilon_1 sn} a^{k_1} w_1 e_2 a^{\epsilon_2 sn} a^{k_2} w_2 \cdots e_\ell a^{\epsilon_\ell sn} a^{k_\ell} w_\ell$$

where $\epsilon_i \in \{\pm 1\}$ according to the orientation of e represented by e_i. Reducing these words, and applying bounded cancellation in the same fashion we have

$$\begin{aligned}\ell_B(\tilde{\sigma}^n(w)) &\geq \sum_{i=1}^{\ell}(|\epsilon_i sn + k_i'|)\ell_B(a) - C\ell_A(w) \\ &\geq \left(|sn|\ell_A(w) - \sum |k_i'|\right)\ell_B(a) - C\ell_A(w) \\ &\geq \left(|sn| - \frac{\sqrt{2}+C}{\ell_B(a)}\right)\ell_A(w)\ell_B(a) - C\ell_A(w)\end{aligned}$$

with the last step following from Equation †. Thus we have

$$\frac{\ell_B(\tilde{\sigma}^n(w))}{\ell_A(\tilde{\sigma}^n(w))} = \frac{\ell_B(\tilde{\sigma}^n(w))}{\ell_A(w)} \geq \left(|sn| - \frac{\sqrt{2}+C}{\ell_B(a)}\right)\ell_B(a) - C.$$

Therefore, to ensure $\sigma^n(w) \in P_\sigma$ we require

$$\left(|sn| - \frac{\sqrt{2}+C}{\ell_B(a)}\right)\ell_B(a) - C > \sqrt{2}$$

that is,

$$|n| > \frac{2\sqrt{2}+2C}{|s|\ell_B(a)}.$$

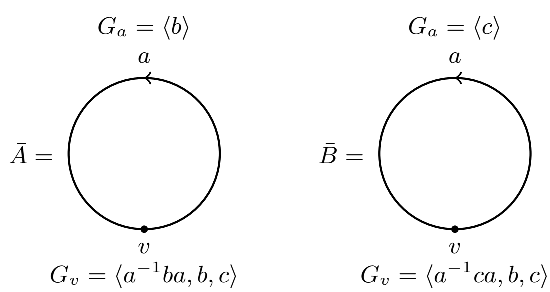
Define N_σ to be the least positive integer satisfying the above inequality. For any $n \geq N_\sigma$, $\sigma^{\pm n}(P_\tau) \subseteq P_\sigma$. Similarly, work out a number N_τ depending on $\ell_A(b)$ and the bounded cancellation constant for the chosen basis and A that ensures for all $n \geq N_\tau$, $\tau^{\pm n}(P_\sigma) \subseteq P_\tau$. The integer

$N = \max\{N_\sigma, N_\tau\}$ is then large enough to ensure that σ^N, τ^N acting on $P = P_\sigma \sqcup P_\tau$ satisfies the hypotheses of the ping-pong lemma, and we conclude $\langle \sigma^N, \tau^N \rangle \cong F_2$ as required. $\square$

Remark. The reader familiar with Cohen and Lustig's skyscraper lemma and parabolic orbits theorem may wonder why I did not use these tools in lieu of the above proof. Both of these tools are not strong enough to give the uniform convergence necessary to carry out a ping-pong type argument on $\overline{CV}_r$; the skyscraper lemma has constants that depend on the particular skyscraper involved, and the parabolic orbits theorem gives pointwise convergence of length functions on conjugacy classes but does not control the rate of convergence. A priori, this rate could be very bad, as demonstrated by the examples of Bestvina, Feighn, and Handel [8, Remark 4.24].

Together Lemmas 5.1.1 and 5.1.9 come very close to a proof of Theorem 5.0.2. Nature is not so kind, and there are incompatible graphs-of-groups that are not hyperbolic-hyperbolic and do have rectangles in their core.

Example 5.1.10. Let A and B be the Bass-Serre trees of the graphs of groups decompositions of F_3



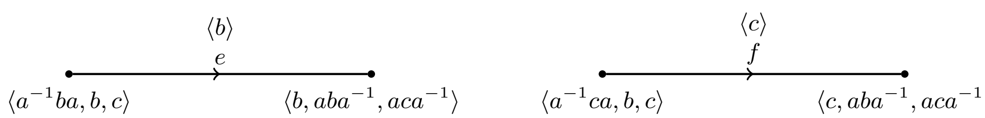
Let σ and τ be the Nielsen transformations represented by Dehn twists about $\bar{A}$ and $\bar{B}$ by b and c respectively, so that

$$\sigma(a) = ba \qquad \tau(a) = ca$$

$$\sigma(b) = b \qquad \tau(b) = b$$

$$\sigma(c) = c \qquad \tau(c) = c$$

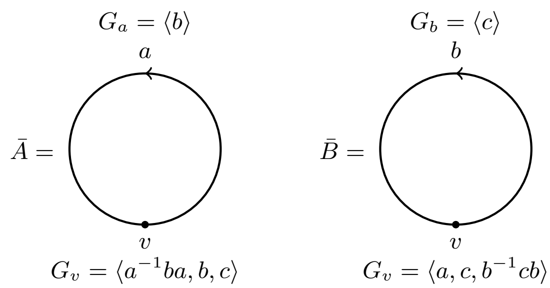
We claim that $\mathcal{C}(A, B)$ has a rectangle, so that $i(A, B) > 0$. Indeed, focus on the edges $e \subseteq A$ and $f \subseteq B$, each on the axis of a with the induced orientation and the given edge stabilizers:



Note that $a \in [\![e]\!] \cap [\![f]\!]$ and $a^{-1} \in [\![\bar{e}]\!] \cap [\![\bar{f}]\!]$. Further, investigation of the diagrams shows that $b^{-1}ab \in [\![e]\!] \cap [\![\bar{f}]\!]$ and $c^{-1}ac \in [\![\bar{e}]\!] \cap [\![f]\!]$, so by Lemma 3.4.1, $e \times f \subseteq \mathcal{C}(A, B)$.

This example is not hyperbolic-hyperbolic; $\ell_A(c) = \ell_B(b) = 0$. Nevertheless $\langle \sigma, \tau \rangle \cong F_2$. For a ping-pong set take $P = \{wa \in F_3 | w \in \langle b, c \rangle\}$ the reduced words ending in a, and for the disjoint decomposition take $P_\sigma = \{wb^{\pm}a\}$, reduced words ending in $b^{\pm}a$, and $P_\tau = \{wc^{\pm}a\}$ reduced words ending in $c^{\pm}a$. It is immediate that for all $N \neq 0$, $\sigma^N(P_\tau) \subseteq P_\sigma$ and the symmetric condition holds for τ, satisfying the hypotheses of the ping-pong lemma.

Example 5.1.11. Let A and B be the Bass-Serre trees of the graphs of group decompositions of F_3



Let σ and τ be the Nielsen transformations represented by Dehn twists about $\bar{A}$ and $\bar{B}$ by b and c respectively, so that

$$\sigma(a) = ba \qquad \tau(a) = a$$

$$\sigma(b) = b \qquad \tau(b) = cb$$

$$\sigma(c) = c \qquad \tau(c) = c$$

Again we have a rectangle in $\mathcal{C}(A, B)$. Consider $g = a, h = bab^{-1}$. Calculating with length functions we have

$$\ell_A(g) = \ell_A(h) = 1 \qquad \ell_B(g) = \ell_B(h) = 0$$

and also

$$\ell_A(gh) = 2 \neq 0 = \ell_A(gh^{-1})$$

$$\ell_B(gh) = \ell_B(gh^{-1}) = 2 > 0 = \ell_B(g) + \ell_B(h).$$

Therefore A and B do not have compatible combinatorics, so by Corollary 3.4.2 $\mathcal{C}(A,B)$ has a rectangle.

This example is also not hyperbolic-hyperbolic, $\ell_B(b) = 1$ but $\ell_A(c) = 0$. Again, however, $\langle \sigma^3, \tau^3 \rangle \cong F_2$. For a ping-pong set we again use $P = \{wa \in F_3 | w \in \langle b, c \rangle\}$ reduced words ending in a, and ping-pong sets $P_\sigma = \{wb^{\pm 2}a\}$ and $P_\tau = P \setminus P_\sigma$. For all $N \neq 0$, we again have $\sigma^{3N}(P_\tau) \subseteq P_\sigma$ and $\tau^{3N}(P_\sigma) \subseteq P_\tau$. Note that it is only out of an aesthetic desire to use the same power of N on both generators that we use τ^3, it is the case that $\tau^N(P_\sigma) \subseteq P_\tau$ for all $N \neq 0$.

Both of these examples are presented with respect to a particularly nice basis, and by taking the associated homotopy equivalence of the wedge of three circles marked by the given basis, we see that all automorphisms in the above example are upper triangular (and with a bit more work, that every element in the subgroup generated is polynomially growing). Both ping-pong arguments rely on the interaction between the suffixes in this particular upper triangular setting. This suggests a dichotomy, either length function ping-pong is possible, or the group generated by a pair of Dehn twists is polynomially growing. To analyze the growth of elements in a subgroup of $\mathrm{Out}(F_r)$ generated by a pair of Dehn twists we will follow the cue of Bestvina, Feighn, and Handel, and understand the growth in topological models associated to the Dehn twists.

5.2 Simultaneous graphs of spaces and normal forms

Guirardel gives a topological interpretation of the core of two simplicial F_r trees.

Theorem 5.2.1 ([26, Theorem 7.1])**.** Given two non-trivial simplicial F_r trees A and B there exists a cell complex X with $\pi_1(X) \cong F_r$ and two 2-sided subcomplexes $Y_A, Y_B \subset X$ intersecting transversely such that $i(A, B) = |\pi_0(Y_A \cap Y_B)|$.

The space X is constructed as follows. Let $\tilde{X} = \widehat{\mathcal{C}}(A, B) \times T$ where $T \cong \tilde{R}_r$ is the universal cover of a fixed wedge of r circles. Let M_A be the set of midpoints of edges of A and M_B be the set of midpoints of edges of B. The spaces $\tilde{Y}_A = \pi_A^{-1}(M_A) \times T$ and $\tilde{Y}_B = \pi_B^{-1}(M_B) \times T$ are a family of two-sided subcomplexes of $\tilde{X}$. The connected components of $\tilde{Y}_A \cap \tilde{Y}_B$ are of the form $x \times T$ where x is a point in the interior of a 2-cell of $\widehat{\mathcal{C}}(A, B)$ or a midpoint of an edge in $\widehat{\mathcal{C}} \setminus \mathcal{C}$. The intersections of the form $x \times T$ when x is a midpoint of an edge in the augmented core are not transverse, indeed $x \times T$ is a connected component of both $\tilde{Y}_A$ and $\tilde{Y}_B$ in this case. A transverse intersection can be obtained by instead using M'_B and equivariant choice of points in the interior of the edges of B none of which are the midpoints, denote this perturbation of $\tilde{Y}_B$ by $\tilde{Y}'_B$. The intersection components of $\tilde{Y}_A$ and $\tilde{Y}'_B$ are in one-to-one correspondence with the 2-cells of $\widehat{\mathcal{C}}(A, B)$. The quotients by the diagonal F_r action, denoted X, Y_A, and Y'_B respectively, are the desired spaces.

These quotient spaces can be viewed through the lens of model spaces for graphs of groups, discussed in Section 2.6. Let $\bar{A}$ and $\bar{B}$ be the graphs of groups covered by A and B respectively. The compositions $\pi_A \circ \pi_{\widehat{\mathcal{C}}}$ and $\pi_B \circ \pi_{\widehat{\mathcal{C}}}$ of projection maps descend to the quotient and give maps $q_A : X \to \bar{A}$ and $q_B : X \to \bar{B}$. These maps make X a graph of spaces over $\bar{A}$ and $\bar{B}$ simultaneously, with the connected components of Y_A and Y_B in the role of edge spaces. Denote by $\mathcal{A}$ and $\mathcal{B}$ the graphs of spaces structures on X induced by q_A and q_B respectively, with $\mathcal{A}_v = q_A^{-1}(v)$ the vertex space over $v \in V(\bar{A})$, $\mathcal{A}_e = q^{-1}(e)$ the mapping cylinder over the midpoint space $\mathcal{A}_e^m = q^{-1}(m_e)$ of an edge $e \in E(\bar{A})$, and similar notation for $\mathcal{B}$. The goal of this section is to establish a normal form for paths and circuits in a simultaneous graph of spaces structure. This behavior of the core is captured in the following definition.

Definition 5.2.2. Let $\bar{A}$ and $\bar{B}$ be two F_r graphs of groups. A complex X is a **simultaneous graph of spaces resolving $\bar{A}$ and $\bar{B}$** if there are maps $q_A : X \to \bar{A}$ and $q_B : X \to \bar{B}$ making X a graph of spaces for $\bar{A}$ and $\bar{B}$ respectively (the induced structures denoted $\mathcal{A}$ and $\mathcal{B}$), and the following conditions on subspaces are satisfied:

- The midpoint spaces $\mathcal{A}_e^m$ and $\mathcal{B}_f^m$ are either equal or intersect transversely for all edges $e \in E(\bar{A})$ and $f \in E(\bar{B})$.
- The intersection $\mathcal{A}_v \cap \mathcal{B}_e$ is the mapping cylinder for the maps of $\mathcal{A}_v \cap \mathcal{B}_e^m$ into $\mathcal{A}_v \cap \mathcal{B}_{o(e)}$ and $\mathcal{A}_v \cap \mathcal{B}_{t(e)}$ as a sub-mapping cylinder of $\mathcal{B}_e$.

The **core** of X is the subcomplex

$$\bigcup_{\substack{e \in E(\bar{A}) \\ f \in E(\bar{B})}} \mathcal{A}_e \cap \mathcal{B}_f$$

A subcomplex $Y = \mathcal{A}_e \cap \mathcal{B}_f$ of the core is **twice-light** if $\mathcal{A}_e^m = \mathcal{B}_f^m$.

Corollary 5.2.3. For any two F_r graphs of groups $\bar{A}$ and $\bar{B}$ there is a simultaneous graph of spaces resolving them.

Proof. The space X constructed in the proof of Theorem 5.2.1 from the core of the Bass-Serre trees covering $\bar{A}$ and $\bar{B}$ is the desired space. $\square$

Remark. When $X = \widehat{\mathcal{C}} \times_{F_r} T$, the core of X is the closure of the preimages of the interiors of the 2-cells of $\widehat{\mathcal{C}}$ and the edges of $\widehat{\mathcal{C}} \setminus \mathcal{C}$. The latter are the twice-light subcomplexes.

Edges $e \subseteq X^{(1)}$ in the 1-skeleton of a simultaneous graph of spaces fall into a taxonomy given by the two decomopositions. Recall that in a single graph of spaces structure $\mathcal{X}$, an edge in $X^{(1)}$ is **$\mathcal{X}$-nodal** if it lies in a vertex space, and **$\mathcal{X}$-crossing** otherwise. We extend this terminology to a simultaneous graph of spaces.

Definition 5.2.4. Let $e \subseteq X^{(1)}$ be an edge in the 1-skeleton of a simultaneous graph of spaces resolving $\bar{A}$ and $\bar{B}$. We say e is

nodal if it is both $\mathcal{A}$ and $\mathcal{B}$ nodal,

$\mathcal{A}$-crossing if it is $\mathcal{A}$-crossing but $\mathcal{B}$-nodal,

$\mathcal{B}$-crossing if it is $\mathcal{B}$-crossing but $\mathcal{A}$-nodal,

double-crossing if it is both $\mathcal{A}$-crossing and $\mathcal{B}$-crossing.

The possible ambiguity of terminology will be avoided by always making clear whether we are considering a single graph of spaces structure or a simultaneous graph of spaces structure.

For a single graph of spaces, based paths have a normal form that gives a topological counterpart to the Bass-Serre normal form for the fundamental groupoid. Recall Lemma 2.6.7, that every path based in the zero skeleton of a graph of spaces is homotopic relative to the endpoints to a path

$$v_0 H_1 v_1 H_2 \cdots H_n v_n$$

where each v_i is a (possibly trivial) tight edge path of $\mathcal{X}$-nodal edges, each H_i is $\mathcal{X}$-crossing, and for all $1 \leq i \leq n-1$, $H_i v_i H_{i+1}$ is not homotopic relative to the endpoints to an $\mathcal{X}$-nodal edge path. A similar normal form is possible in a simultaneous graph of spaces.

Lemma 5.2.5. Every path in X, a simultaneous graph of spaces resolving $\bar{A}$ and $\bar{B}$, is homotopic relative to the endpoints to a path of the form (called **simultaneous normal form**)

$$W_{0,0}K_{0,1}W_{0,1}\cdots K_{0,n_0}W_{0,n_0}H_1W_{1,0}\cdots H_mW_{m,0}K_{m,1}\cdots K_{m,n_m}W_{m,n_m}$$

where the $W_{i,j}$ are (possibly trivial) tight edge paths of nodal edges, the $K_{i,j}$ are $\mathcal{B}$-crossing edges, and the H_i are either $\mathcal{A}$-crossing or double-crossing edges. Further this path is in normal form for both $\mathcal{A}$ and $\mathcal{B}$, so that the number of $\mathcal{B}$-crossing edges plus double-crossing edges and the number of $\mathcal{A}$-crossing edges plus double-crossing edges are both invariants of the relative homotopy class of the path. A similar statement holds for free homotopy classes of loops.

Proof. Throughout this proof all homotopies will be homotopies of paths relative to the endpoints. Suppose γ is a path in X. First, by Lemma 2.6.7, γ is homotopic to a path of the form

$$v_0H_1v_1H_2v_2\cdots H_mv_m$$

with each v_i a $\mathcal{A}$-nodal path and each H_i either $\mathcal{A}$-crossing or double-crossing. With respect to $\mathcal{B}$, each v_i is an edge path, not necessarily in normal form, of the form

$$W_{i,0}K_{i,1}W_{i,1}\cdots K_{i,n_i}W_{i,n_i}$$

where each $W_{i,j}$ is $\mathcal{B}$-nodal (and so nodal in the simultaneous graph of spaces) and each $K_{i,j}$ is $\mathcal{B}$-crossing (in the simultaneous graph of spaces sense).

We can take this path to $\mathcal{B}$-normal form by erasing pairs of crossing edges, but we must do so without introducing $\mathcal{A}$-crossing edges. Suppose for some i the path $K_{i,j}W_{i,j}K_{i,j+1}$ is homotopic to a path W'_{ij} that is $\mathcal{B}$-nodal. Suppress the common index i. Let p be the vertex of $\bar{A}$ such that $K_jW_jK_{j+1} \subseteq \mathcal{A}_p$, e the edge of $\bar{B}$ such that $K_jW_jK_{j+1} \subseteq \mathcal{B}_e$, so that $W_j \subseteq \mathcal{B}_{t(e)}$ and $W'_j \subseteq \mathcal{B}_{o(e)}$. Since $K_jW_jK_{j+1} \subseteq \mathcal{A}_p \cap \mathcal{B}_e$, this is a path in the mapping cylinder for the inclusions of $\mathcal{A}_p \cap \mathcal{B}_e^m$ into the endpoints, and W_j is a fiber of this cylinder. Thus $K_jW_jK_{j+1}$ is homotopic to a path $W''_j \subseteq \mathcal{A}_p \cap \mathcal{B}_{o(e)}$. Using W''_j to erase the pair of crossing edges, we see that each v_i can be expressed in $\mathcal{B}$ normal form and remain $\mathcal{A}$-nodal. Thus γ is homotopic to a path of the form

$$W_{0,0}K_{0,1}W_{0,1}\cdots K_{0,n_0}W_{0,n_0}H_1W_{1,0}\cdots H_mW_{m,0}K_{m,1}\cdots K_{m,n_m}W_{m,n_m}$$

This path may not be in $\mathcal{B}$-normal form. There are two possible cases, and in both we will show that it is possible to erase a pair of $\mathcal{B}$-crossing edges without destroying $\mathcal{A}$-normal form.

First, suppose this path is not $\mathcal{B}$-normal because there is some i such that $K_{i,n_i}W_{i,n_i}H_{i+1}$ (or symmetrically $H_iW_{i,0}K_{i,1}$) is homotopic to a path W'_i that is $\mathcal{B}$-nodal. In this case H_{i+1} must be double-crossing. Note that this path is already in $\mathcal{A}$-normal form. Again suppress the common index, and take K_nW_nH to a path in the $\mathcal{B}$-vertex space $\mathcal{B}_q$. This path will have some number of $\mathcal{A}$-crossing edges, but similar to the previous paragraph, this path is homotopic to one in $\mathcal{A}$-normal form via a homotopy inside $\mathcal{B}_q$, so that by Lemma 2.6.7 $K_n W_n H$ is homotopic to a path of the form $W'_n H' W'$ with exactly one $\mathcal{A}$-crossing edge, and W'_n and W' are nodal.

Second, suppose the resulting path is not $\mathcal{B}$-normal because there is some i such that $K_{i,n_i} W_{i,n_i} H_{i+1} W_{i+1,0} K_{i,0}$ is homotopic to a path W'_i that is $\mathcal{B}$-nodal. In this case H_{i+1} must be $\mathcal{A}$-crossing. As before, the path $W'_i \subseteq \mathcal{B}_q$ is homotopic to a path in $\mathcal{A}$-normal form contained in $\mathcal{B}_q$.

Therefore, a path γ is homotopic to a path in simultaneous normal form, and can be taken to this normal form by composing the following homotopies

1. Take γ to $\mathcal{A}$-normal form.
2. Take each $\mathcal{A}$-nodal sub-path to $\mathcal{B}$-normal form within the appropriate $\mathcal{A}$ vertex space.
3. Erase remaining pairs of $\mathcal{B}$-crossing edges, maintaining $\mathcal{A}$-normal form.

The homotopy invariance of the number of crossing edge types follows immediately from Lemma 2.6.7. $\square$

5.3 Twisting in graphs of spaces

A Dehn twist on a graph of groups can be realized by an action on based homotopy classes of paths in a graph of spaces. Let Γ be a graph of groups modeled by the graph of spaces X, and D a Dehn twist based on Γ. Each crossing edge $H \in X^{(1)}$ lies over some edge $e \in E(\Gamma)$. For each crossing edge H pick a loop γ_H in $\mathcal{X}_{t(e)}$, contained in the image of $\mathcal{X}_e \times \{1\}$ representing z_e and based at $t(H)$. The action of D on a crossing edge is the concatenation

$$D(H) = H\gamma_H$$

The action is extended to an action on all paths in $X^{(1)}$ by concatenation and $D(v) = v$ for every nodal path, and to based homotopy classes by taking one-skeleton representatives. That this action is well-defined and represents the Dehn twist D faithfully follows from noting that the below diagram of fundamental groupoids commutes.

$$\begin{tikzcd}
\pi_1(X, X^{(0)}) \arrow[r, "D"] \arrow[d] & \pi_1(X, X^{(0)}) \arrow[d] \\
\pi_1(\Gamma) \arrow[r, "D"'] & \pi_1(\Gamma)
\end{tikzcd}$$

Also from this diagram we see that if a path γ is in normal form, then so is $D(\gamma)$, with the same crossing edges.

Extending this to the setting of a simultaneous graph of spaces resolving $\bar{A}$ and $\bar{B}$, and twists $\tilde{\sigma}$ based on $\bar{A}$ and $\tilde{\tau}$ based on $\bar{B}$, we see that $\tilde{\sigma}$ preserves $\mathcal{A}$-normal form (though we can make no comment on the $\mathcal{B}$-normal form) and a symmetric statement holds for $\tilde{\tau}$. To understand the behavior of paths in simultaneous normal form we must track the extent to which $\tilde{\sigma}$ alters the number of $\mathcal{B}$-crossing edges and vise-versa. This interaction is contained entirely in the graphs of groups, and applies to all twists based on the graphs.

Definition 5.3.1. The **edge twist digraph** $\mathcal{ET}(\bar{A}, \bar{B})$ of two small graphs of groups is a directed graph with vertex set

$$V(\mathcal{ET}) = \{(e, \bar{e}), | e \in E(\bar{A})\} \cup \{(f, \bar{f}) | f \in E(\bar{B})\}$$

directed edges $((e, \bar{e}), (f, \bar{f}))$ $e \in E(\bar{A}), f \in E(\bar{B})$ when a generator $\bar{A}_e = \langle z_e \rangle$ or its inverse uses f or $\bar{f}$ in cyclically reduced normal form with respect to $\bar{B}$, and directed edges $((f, \bar{f}), (e, \bar{e}))$ $f \in E(\bar{B}), e \in E(\bar{A})$ when a generator $\bar{B}_f = \langle z_f \rangle$ or its inverse uses e or $\bar{e}$ in cyclically reduced normal form with respect to $\bar{A}$.

This definition is made somewhat cumbersome by the presence of orientation. The vertex set is the unoriented edges of the two graphs of groups, and the property of crossing an unoriented edge in normal form is shared by the generator and its inverse. We encapsulate the resulting awkwardness here, so that subsequent arguments about paths in simultaneous normal form are clear.

The edge-twist structure controls the growth rate of elements in any group generated by a twist $\tilde{\sigma}$ on $\bar{A}$ and $\tilde{\tau}$ on $\bar{B}$.

Lemma 5.3.2. Suppose $\bar{A}$ and $\bar{B}$ are minimal, visible, small graphs of groups with free fundamental group F_r and $\mathcal{ET}(\bar{A}, \bar{B})$ is acyclic. Then for any pair of Dehn twists $\sigma, \tau \in \mathrm{Out}(F_r)$ represented by $\tilde{\sigma}$ based on $\bar{A}$ and $\tilde{\tau}$ based on $\bar{B}$, every element of $\langle \sigma, \tau \rangle \leq \mathrm{Out}(F_r)$ is polynomially growing. Moreover, the growth degree is at most the length of the longest directed path in $\mathcal{ET}(\bar{A}, \bar{B})$.

Proof. Let $X = \widehat{\mathcal{C}}(A, B) \times_F T$ be the simultaneous graph of spaces constructed from the augmented core of the Bass-Serre trees A and B for $\bar{A}$ and $\bar{B}$, with $\bar{T}$ a wedge of circles, equipped with the ℓ_1 metric. Note that $\tilde{X}$ has an equivariant Lipschitz surjection to T given by projection and that this descends to a Lipschitz homotopy equivalence on the quotient, denoted $\rho : X \to \bar{T}$. Further, if γ is a loop in $X^{(1)}$ representing a conjugacy class $[g]$ of $\pi_1(\bar{T})$,

$$\ell_T(g) \leq |\rho(\gamma)|_{\bar{T}} \leq Lip(\rho) \cdot |\gamma|_X$$

where $|\cdot|$ is the arclength. Further, for any $w \in \langle \sigma, \tau \rangle$, by expressing w as a word in the generators we get an action on paths $\tilde{w}$, with the property that $w(g)$ is represented by $\tilde{w}(\gamma)$.

Therefore, it suffices to give a polynomial bound on the growth of paths in X under the topological representatives of σ and τ. Moreover, for any edge path γ, the growth under the action of $\tilde{\sigma}$ and $\tilde{\tau}$ is bounded by the number of $\mathcal{A}$-crossing edges of γ times the growth of $\mathcal{A}$-crossing edges plus the similar quantity for $\mathcal{B}$-crossing edges. So it suffices to bound the growth of crossing edges. (Note, this is an upper bound, we make no attempt to understand cancellation that might happen, as a result these bounds could be quite bad.)

First, as a technical convenience, replace $\bar{A}$ and $\bar{B}$ by the isomorphic graphs of groups constructed from A and B using a fundamental domain in each that is the image under projection of a fundamental domain for $\widehat{\mathcal{C}}(A, B)$, so that the edge stabilizers of edges covered by diagonals of the core are not just conjugate, but equal on the nose. This does not change the outer automorphism class of the Dehn twists under consideration, nor does it change the edge twist graph.

It follows that if D is a double-crossing edge of $X^{(1)}$ lying over $e \in E(A)$ and $f \in E(B)$, the vertices $(e, \bar{e})$ and $(f, \bar{f})$ of $\mathcal{ET}(\bar{A}, \bar{B})$ have no outgoing edges. Moreover, we can choose a loop representing a generator of $\bar{A}_e = \langle z \rangle = \bar{B}_f$ that is nodal and based at $t(D)$, and alter the topological representatives of $\tilde{\sigma}$ and $\tilde{\tau}$ so that $\tilde{\sigma}(D) = \gamma^a$ and $\tilde{\tau}(D) = \gamma^b$, concatenations of either γ or its reverse, according to the expression of the twisters of $\tilde{\sigma}$ about e and $\tilde{\tau}$ about f in terms of the generator z. Thus,

$$\tilde{\sigma}^{s_n}\tilde{\tau}^{t_n}\cdots\tilde{\sigma}_1^{s}\tilde{\tau}^{t_1}(D) = D\gamma^{a\sum s_i + b\sum t_i}$$

which has edge length at most linear in $\sum |s_i| + \sum |t_i|$.

Suppose H is an $\mathcal{A}$ or $\mathcal{B}$-crossing edge of $X^{(1)}$ lying over $(e, \bar{e}) \in V(\mathcal{ET})$. Let d_e be the length of the longest directed path in $\mathcal{ET}(\bar{A}, \bar{B})$ starting at e. We will use the notation $\mathrm{poly}_d(x)$ to stand for some polynomial of degree d in x, as we are looking for an upper bound and making no attempt to estimate coefficients.

Claim. The length of $\tilde{\sigma}^{s_n}\tilde{\tau}^{s_n}\cdots\tilde{\sigma}^{s_1}\tilde{\tau}^{t_1}(H)$ is at most $\mathrm{poly}_{d_e+1}(\sum |s_i| + \sum |t_i|)$.

Proof of Claim. he proof is by induction on d_e. As the argument is symmetric, we will suppose H is $\mathcal{A}$-crossing, so that $e \in E(A)$.

Base Case: $d_e = 0$. If the edge H is double-crossing, then the growth is at most linear, as shown in the discussion of double-crossing edges. Let γ_H be a loop representing a generator

z_e of $\bar{A}_e$ based at $t(H)$ and in simultaneous normal form. Since $(e, \bar{e})$ has no outgoing edges in $\mathcal{ET}$, the loop γ_H is $\mathcal{B}$-nodal. Let a be the power so that z_e^a is the e twister of $\tilde{\sigma}$. Use γ_e^a in the topological representative of $\tilde{\sigma}$. Then for any s, $\tilde{\sigma}^s(H) = H\gamma_e^{as}$ is a $\mathcal{B}$-nodal path, and we have

$$\tilde{\sigma}^{s_n}\tilde{\tau}^{t_n}\cdots\tilde{\sigma}^{s_1}\tilde{\tau}^{t_1}(H) = H\gamma_H^{a\sum s_i}$$

which has edge length at most linear in $\sum|s_i| + \sum|t_i|$, as required.

Inductive Step: $d_e > 0$. In this case, the edge H cannot be double-crossing. Since $d_e > 0$, $(e, \bar{e})$ has neighbors $(f_1, \bar{f}_1), \ldots, (f_k, \bar{f}_k)$. As before, use a simultaneous normal form representative γ_H for a generator z_e of $\bar{A}_e$ based at $t(H)$, so that $\sigma(H) = \gamma_H^a$. Since γ_H has an $\mathcal{A}$-nodal representative by definition, we have in simultaneous normal form

$$\gamma_H = W_0K_1\cdots K_mW_m$$

where K_i lies over either f_{k_i} or $\bar{f}_{k_i}$ by the definition of the edge twist graph. Further, for each f_i, the longest path in $\mathcal{ET}$ based at f_i, $d_{f_i} \leq d_e - 1$. Calculating, we have

$$\begin{aligned}\tilde{\sigma}^{s_n}\tilde{\tau}^{t_n}\cdots\tilde{\sigma}^{s_1}\tilde{\tau}^{t_1}(H) = H\gamma_H^{as_n}&\tilde{\sigma}^{s_n}\tilde{\tau}^{t_n}(\gamma_H^{as_{n-1}})\\ &\cdot\tilde{\sigma}^{s_n}\tilde{\tau}^{t_n}\tilde{\sigma}^{s_{n-1}}\tilde{\tau}^{t_{n-1}}(\gamma_H^{as_{n-2}})\\ &\vdots\\ &\cdot\tilde{\sigma}^{s_n}\tilde{\tau}^{t_n}\cdots\tilde{\tau}^{t_2}(\gamma_H^{as_1})\end{aligned}$$

By the induction hypothesis, the length of each K_{v_i} under a composition of powers of $\tilde{\sigma}$ and $\tilde{\tau}$ is bounded by a polynomial of degree at most d_e. Hence $\tilde{\sigma}^{s_n}\tilde{\tau}^{t_n}\cdots\tilde{\sigma}^{s_2}\tilde{\tau}^{t_2}(\gamma^{as_1})$ has length at most

$$|as_1| \cdot \mathrm{poly}_{d_e}\left(\sum_{i\geq 2}|s_i| + |t_i|\right)$$

Similarly, we calculate

$$|\tilde{\sigma}^{s_n}\tilde{\tau}^{t_n}\cdots\tilde{\sigma}^{s_1}\tilde{\tau}^{t_1}(H)| \leq \sum_{i=1}^{n}|as_i| \cdot \mathrm{poly}_{d_e}\left(\sum_{j>i}|s_j| + |t_j|\right)$$

and this quantity is in turn at most $\mathrm{poly}_{d_e+1}\left(\sum|s_i| + |t_i|\right)$. This completes the claim. $\diamond$

Finally, suppose $w = \sigma^{s_n}\tau^{s_n}\cdots\sigma^{s_1}\tau^{t_1} \in \langle\sigma,\tau\rangle$. For any $g \in F$, let γ be a loop in simultaneous normal form representing the conjugacy class of g in $X^{(1)}$. The length $\ell_T(w^N(g))$ is bounded by the length in X of $\tilde{w}^N(\gamma)$, which by the claim is at most

$$\mathrm{poly}_{d+1}(N \cdot \left(\sum|s_i| + |t_i|\right)$$

where d is the length of the longest directed path in $\mathcal{ET}$. This is a polynomial of degree $d+1$ in N, which completes the lemma. $\square$

An interesting question, which we do not answer here but will discuss again in the conclusion, is whether or not Lemma 5.3.2 is sharp. That is, if $\mathcal{ET}(\bar{A},\bar{B})$ contains a cycle, is there some pair of twists σ,τ with representatives based on $\bar{A}$ and $\bar{B}$ respectively so that the group generated contains an outer automorphism with an exponentially growing stratum? In the setting of one-edge splittings, Clay and Pettet's result is in this direction: two one-edge graphs of groups that fill have a directed cycle of length two in their edge-twist graphs.

5.4 Dehn twists on incompatible graphs generate free groups

We are now in a position to give a full converse to Lemma 5.1.1. The proof is by two cases, decided by the structure of the edge-twist graph. When the edge-twist graph contains a cycle, this cycle enables a length function ping-pong argument that is almost identical to the proof of Lemma 5.1.9. When the edge-twist graph is acyclic, the group generated by the pair of twists is polynomially growing and we analyze its structure using the Kolchin theorem for $\mathrm{Out}(F_r)$ of Bestvina, Feighn, and Handel. As the two arguments are significantly different, we present them as two lemmas.

Lemma 5.4.1. Suppose σ and τ are Dehn twists with efficient representatives $\tilde{\sigma}$ and $\tilde{\tau}$ on graphs of groups $\bar{A}$ and $\bar{B}$ such that $\mathcal{ET}(\bar{A},\bar{B})$ contains a cycle. Then there is an integer N such that $\langle \sigma^N, \tau^N \rangle \cong F_2$.

Proof. Let $(u_1,\bar{u_1}),\ldots,(u_c,\bar{u}_c)$ and $(v_1,\bar{v}_1),\ldots,(v_c,\bar{v}_c)$ be the vertices of a primitive cycle in $\mathcal{ET}$, with $u_i \in E(\bar{A})$ and $v_i \in E(\bar{B})$. (It is psychologically unfortunate to use u and v for edges, but is only for this proof.) The index c is the same for both sets as $\mathcal{ET}$ is bipartite, and no vertex $(u_i,\bar{u}_i)$ or $(v_i,\bar{v}_i)$ is repeated. For each edge u_i fix a generator $\langle a_{u_i}\rangle = \bar{A}_{u_i}$ and $s_{u_i} \neq 0$ so that the twister of $\tilde{\sigma}$ about u_i is $z_{u_i} = a_{u_i}^{s_{u_i}}$, and similarly fix $\langle b_{v_i}\rangle = \bar{B}_{v_i}$ and $t_i \neq 0$. (The s_u and t_v are nonzero as both $\tilde{\sigma}$ and $\tilde{\tau}$ twist on every edge of their respective graphs.) Let $\bar{A}'$ and $\bar{B}'$ be the quotient graphs of groups obtained by collapsing $E(\bar{A}) \setminus \{u_i,\bar{u}_i\}$ and $E(\bar{B}) \setminus \{v_i,\bar{v}_i\}$ in $\bar{A}$ and $\bar{B}$ respectively.

We will again use conjugacy class ping-pong. Let $P = \{[w] \in F_r | \ell_{A'}(w) + \ell_{B'}(w) > 0\}$. Note that, from the structure of the edge twist graph, we have $\{a_u, b_v\} \subseteq P$, indeed $\ell_{A'}(b_v) > 0$ and $\ell_{B'}(a_u) > 0$ for each $u \in \{u_i, \bar{u}_i\}$ and $v \in \{v_i, \bar{v}_i\}$, so that P is non-empty. Partition $P = P_\sigma \sqcup P_\tau$

$$P_\sigma = \{[w] \in P | \sqrt{2}\ell_{A'}(w) < \ell_{B'}(w)\}$$

$$P_\tau = \{[w] \in P | \ell_{B'}(w) < \sqrt{2}\ell_{A'}(w)\}$$

Again the use of $\sqrt{2}$ is arbitrary and any irrational will ensure the decomposition is a partition. Moreover, this partition is non-trivial, the $a_u \in P_\tau$ and $b_v \in P_\sigma$.

Once more we will find an N so that for all $n \geq N$, $\sigma^{\pm n}(P_\tau) \subseteq P_\sigma$ and $\tau^{\pm n}(P_\sigma) \subseteq P_\tau$, so that, by the ping-pong lemma $\langle \sigma^N, \tau^N \rangle \cong F_2$. The argument will be symmetric, and almost identical to that of Lemma 5.1.9.

Suppose $[w] \in P_\tau$, so that $0 < \ell_{A'}(w)$. Fix a cyclically reduced representative of w in Bass-Serre normal form with respect to a fixed basis of F_r and $\bar{A}'$,

$$w = e_1 a_{e_1}^{k_1} w_1 e_2 a_{e_2}^{k_2} w_2 \cdots e_\ell a_{e_\ell}^{k_\ell} w_\ell$$

where we are suppressing the different edge morphisms, using $\ell = \ell_{A'}(w)$ for legibility, $e_i \in \{u_i, \bar{u}_i\}$, and each w_i is in the right transversal of the image of a_{e_i} in the vertex group involved. Let C be the bounded cancellation constant for the fixed basis of F_r basis into B'. With respect to this basis, after an appropriate conjugation we have the cyclically reduced conjugacy class representative w' satisfying

$$|w'| = |a_{e_1}^{k_1'}| + \cdots + |w'_{\ell-1}| + |a_{e_\ell}^{k_\ell'}| + |w'_\ell|$$

where w_i' is the reduced word for the group element obtained by joining $a_{e_i}^{\pm 1} w_i e_{i+1} a_{e_{i+1}}^{\pm 1}$ to the basepoint and including the appropriate transversal elements coming from a_{e_i} and $a_{e_{i+1}}$, and $k_i' = sgn(k_i)(|k_i| - 2)$. (This follows from normal form as in the proof of Lemma 5.1.9.) Let $\alpha = \min_i\{\ell_{B'}(a_{u_i})\}$. Since each $(u_i, \bar{u}_i)$ is joined to some $(v_i, \bar{v}_i)$ by an edge in $\mathcal{ET}$ as they are all vertices of a cycle, $\alpha > 0$. We have, by bounded cancellation,

$$\begin{aligned} \sqrt{2}\ell_{A'}(w) > \ell_{B'} &\geq \sum_{i=1}^{\ell} |k_i| \ell_{B'}(a_{e_i}) - C\ell_{A'}(w) \\ &\geq \left(\sum_{i=1}^{p} |k_i'| \right) \alpha - C\ell_{A'}(w). \end{aligned}$$

We conclude

$$\sum |k_i'| < \left(\frac{\sqrt{2} + C}{\alpha} \right) \ell_{A'}(w). \tag{$\dagger$}$$

Calculating with the induced relative Dehn twist on $\bar{A}'$ and abusing notation to also call it $\tilde{\sigma}$, we have

$$\tilde{\sigma}^n(w) = e_1 a_{e_1}^{s_{e_1} n} a_{e_1}^{k_1} \tilde{\sigma}^n(w_1) e_2 a_{e_2}^{s_{e_2} n} a_{e_2}^{k_2} \cdots e_\ell a_{e_\ell}^{s_{e_\ell} n} a_{e_\ell}^{k_\ell} \tilde{\sigma}^n(w_\ell)$$

When reducing this word there is the possibility that $\tilde{\sigma}^n(w_i)$ is of the form $a_{e_i}^{\epsilon n} x_i a_{e_{i+1}}^{\delta n}$, however the no positive bonding condition of the efficient representative forces ϵn and $s_{e_i} n$ to have the same sign, and also δn and $s_{e_{i+1}} n$. So, reducing and applying bounded cancellation in the same fashion, we have, with $s = \min_i\{|s_i|\}$

$$\begin{aligned}
\ell_{B'}(\sigma^n(w)) &\geq \sum_{i=1}^{\ell} |s_{e_i} n + k_i'| \ell_{B'}(a_{e_i}) - C\ell_{A'}(w) \\
&\geq \left(|sn| \ell_{A'}(w) - \sum_{i=1}^{p} |k_i'| \right) \alpha - C\ell_{A'}(w) \\
&\geq \left(|sn| - \frac{\sqrt{2} + C}{\alpha} \right) \alpha \ell_{A'}(w) - C\ell_{A'}(w)
\end{aligned}$$

with the last step following from Equation †. Thus we have

$$\frac{\ell_{B'}(\sigma^n(w))}{\ell_{A'}(\sigma^n(w))} = \frac{\ell_{B'}(\sigma^n(w))}{\ell_{A'}(w)} \geq \left(|sn| - \frac{\sqrt{2} + C}{\alpha} \right) \alpha - C$$

Therefore, to ensure $\sigma^n(w) \in P_\sigma$ we require

$$\left(|sn| - \frac{\sqrt{2} + C}{\alpha} \right) \alpha - C > \sqrt{2}$$

that is,

$$|n| > \frac{2\sqrt{2} + 2C}{s\alpha}.$$

Define N_σ to be the least positive integer satisfying the above inequality. For any $n \geq N_\sigma$, $\sigma^{\pm n}(P_\tau) \subseteq P_\sigma$. Similarly, find N_τ depending on $\beta = \min_i\{\ell_{A'}(b_{v_i})\}$, $t = \min_i\{|t_i|\}$ and the bounded cancellation constant for the fixed basis into A' that ensures for all $n \geq N_\tau$,

$\tau^{\pm n}(P_\sigma) \subseteq P_\tau$. The integer $N = \max\{N_\sigma, N_\tau\}$ is the desired power, and $\langle \sigma^N, \tau^N \rangle \cong F_2$ by the ping-pong lemma. □

The presence of a cycle in $\mathcal{ET}(\bar{A}, \bar{B})$ is essential in the above proof; it guarantees there is some subset of twisters and edges where the growth of one restricted length function is linear in the value of the other restricted length function. Without a cycle, this kind of uniform control is unavailable, as illustrated by Examples 5.1.10 and 5.1.11. Fortunately, this is the exact case where the generated group is polynomially growing and the Kolchin theorem can be applied. Using the simultaneous upper triangular representatives a different form of ping-pong can be effected.

First we require a lemma relating the core of two efficient twists and the structure of their simultaneous upper triangular representatives.

Lemma 5.4.2. Suppose σ and τ are Dehn twist outer automorphisms with upper-triangular relative train-track representatives $\hat{\sigma}$ and $\hat{\tau}$ with respect to a filtered graph $\emptyset = \Gamma_0 \subsetneq \Gamma_1 \subsetneq \cdots \subsetneq \Gamma_k = \Gamma$, and efficient representatives $\tilde{\sigma}$ and $\tilde{\tau}$ on graphs of groups $\bar{A}$ and $\bar{B}$ covered by A and B respectively. If

- Every suffix of $\hat{\sigma}$ is $\hat{\tau}$-Nielsen,
- Every suffix of $\hat{\tau}$ is $\hat{\sigma}$-Nielsen,
- For every edge $E_i \in \Gamma_i \setminus \Gamma_{i-1}$ if E_i is a linear edge of both $\hat{\sigma}$ and $\hat{\tau}$ the associated primitive Nielsen paths are equal (up to a reverse),

then $\mathcal{C}(A, B)$ does not contain a rectangle.

Proof. The construction of efficient representatives in Lemma 4.3.7 from a relative train-track involves first folding conjugates, then a series of folding edges in linear families, and finally a series of graph of groups Stallings folds; it follows from Cohen and Lustig's parabolic orbits theorem that the simplicial structure of the resulting tree is unique (Theorem 4.3.5 and Corollary 4.3.6). We carry out the same construction, using both $\hat{\sigma}$-linear edges and $\hat{\tau}$-linear edges. A **joint linear family** is a collection of single edges $\{E_i\}$ which have either $\hat{\sigma}$ or $\hat{\tau}$ suffixes that are a power of a fixed primitive Nielsen path γ. By hypothesis, if two edges E_i and E_j are in the same linear family for one of the maps, then they are in the same joint linear family. As in the construction of efficient representatives, we first fold conjugates and then linear families; the hypotheses ensure that this can be done in a compatible fashion. The resulting folded graph and folded representatives, $\hat{\sigma}', \hat{\tau}' : \Gamma' \to \Gamma'$ are still upper triangular, represent σ and τ respectively, and have the property that every linear family contains one edge.

We now construct a tree C that resolves the trees A and B. First, recall that the efficient representative of $\hat{\sigma}'$ on a tree A can be constructed from Γ' as follows. Start with A_0 obtained from the universal cover of Γ' by collapsing all $\hat{\sigma}'$ fixed edges of Γ'. We then work up the remaining orbits of edges of A_0 by the filtration of Γ'. If $\hat{\sigma}'(E_i) = E_i$ then set $A_i = A_{i-1}$, otherwise $\hat{\sigma}'(E_i) = E_i u_i$, and each lift of u_i by construction represents an element in the vertex group based at a lift of $t(E_i)$; the tree A_i is obtained from A_{i-1} by folding the associated primitive Nielsen path γ_i over E_i (the details are in Lemma 4.3.7), and the result A_k is A. To construct the resolving tree, we start with C_0, obtained from the universal cover of Γ' by collapsing all edges that are fixed by both $\hat{\sigma}'$ and $\hat{\tau}'$. Then, working up the hierarchy of Γ, if

E_i is both $\hat{\sigma}'$ and $\hat{\tau}'$ fixed, set $C_i = C_{i-1}$, otherwise $\hat{\sigma}'(E_i) = E_i\gamma_i^s$ and $\hat{\tau}'(E_i) = E_i\gamma_i^t$ for a primitive Nielsen path γ_i (allowing the possibility s or t is zero); in this case by construction lifts of γ_i represent elements in the vertex stabilizers of lifts of $t(E_i)$, so we obtain C_i from C_{i-1} by folding γ_i over E_i. The desired resolving tree is $C = C_k$. It is readily apparent from this construction that C maps to A and B by collapse maps: collapse any remaining σ fixed edges of C to obtain A, and any remaining τ fixed edges of C to obtain B.

Since collapse maps have convex fibers, by the universal property of the core [26, Corollary 5.3], the core is contained in the image of $C \mapsto A \times B$, which has no rectangles because C is a tree. □

The contrapositive of this lemma will be used to find paths suitable for ping-pong, after applying the Kolchin theorem.

Lemma 5.4.3. Suppose $\tilde{\sigma}, \tilde{\tau}$ are efficient Dehn twists based on $\bar{A}$ and $\bar{B}$ respectively. If $\mathcal{ET}(\bar{A}, \bar{B})$ is acyclic and $\mathcal{C}(A, B)$ contains a rectangle, then there is an $N > 0$ such that $\langle \sigma^N, \tau^N \rangle \cong F_2$.

For the proof we require some notation. For two paths $\gamma, \delta \subseteq \Gamma$ with the same initial point, the **overlap length** is defined by $\theta(\gamma, \delta) = \frac{1}{2}(\text{length}_\Gamma([\gamma]) + \text{length}_\Gamma([\delta]) - \text{length}([\bar{\gamma}\delta]))$, where we use the metric on Γ induced by assigning each edge length one. We will often understand the overlap length by calculating the **common initial segment** of two tight paths, this is the connected component of the intersection of lifts of γ and δ based at the common intersection point containing that point. The length of this segment is equal to the overlap length.

Proof. By Lemma 5.3.2, the group $\langle \sigma, \tau \rangle$ is a polynomially growing subgroup of $\mathrm{Out}(F_r)$, so it has a finite index subgroup that is unipotent polynomially growing, and by passing to a power we can ensure that the group generated is UPG [8, Proposition 3.5]. Therefore, by the Kolchin theorem for $\mathrm{Out}(F_r)$ (see Theorem 4.2.2) there is a filtered graph $\emptyset = \Gamma_0 \subsetneq \Gamma_1 \subsetneq \cdots \Gamma_k = \Gamma$ with each step in the filtration a single edge, so that $\langle \sigma, \tau \rangle$ is realized as a group of upper-triangular homotopy equivalences of Γ with respect to the filtration. Let $\hat{\sigma}$ and $\hat{\tau}$ be the realizations of the generators. Note that since σ and τ are UPG, every periodic Nielsen path of $\hat{\sigma}$ and $\hat{\tau}$ is Nielsen, so that if a path is not fixed (up to homotopy rel endpoints) it must grow linearly.

Since $\mathcal{C}(A, B)$ contains a rectangle, the contrapositive of Lemma 5.4.2 implies that either (up to relabeling) $\hat{\sigma}$ has a linear edge E_i with suffix u_i that grows under $\hat{\tau}$ (as in Example 5.1.11), or there is an edge E_i so that the $\hat{\sigma}$ and $\hat{\tau}$ suffixes are powers of primitive Nielsen paths which generate non-equal cyclic subgroups (as in Example 5.1.10), and both suffixes do not grow under either automorphism. This gives two cases.

Case 1. Let E_i be the lowest edge in the filtration such that its suffix under one automorphism grows linearly under the other, and without loss of generality suppose that the $\hat{\sigma}$ suffix u_i grows linearly under $\hat{\tau}$. We will use as a ping-pong set

$$P = \{[\omega(E_i)] | \omega \in \langle \hat{\sigma}, \hat{\tau} \rangle\}$$

the orbits of (the based homotopy class of) E_i under tightening after applying elements of the group generated by $\hat{\sigma}$ and $\hat{\tau}$. Since a tight path is a unique representative of a based homotopy class the proof will focus on the tight representatives and the homotopy class will be suppressed. All of these classes have tight representatives of the form $E_i w$ with $w \subseteq \Gamma_{i-1}$ a tight path based at $t(E_i)$, since the group is upper triangular with respect to this filtration. Let

$$P_\sigma = \{p \in P | \theta([E_i u_i^3], p) \geq \theta([E_i u_i^3], [E_i u_i^2]) \text{ or } \theta([E_i \bar{u}_i^3], p) \geq \theta([E_i \bar{u}_i^3], [E_i \bar{u}_i^2])\}$$

and $P_\tau = P \setminus P_\sigma$ be a partition of P. It is clear that P and P_σ are non-empty, and we will show in the course of the proof that P_τ is non-empty. Let γ_k be the common initial segment of $[u_i^k]$ and $[u_i^{k+1}]$, and γ_{-k} the common initial segment of $[\bar{u}_i^k]$ and $[\bar{u}_i^{k+1}]$. Note that $[\bar{u}_i^k \gamma_1] = \gamma_{-(k-1)}$, the paths γ_j are an increasing sequence of paths, and that $[\hat{\sigma}(\gamma_j)] = \gamma_j u'$ where u' is the Nielsen path associated to an exceptional Nielsen subpath of the primitive Nielsen path associated to u_i if one exists.

We claim $\hat{\sigma}^{\pm 3N}(P_\tau) \subseteq P_\sigma$ for $N \neq 0$. Suppose $E_i w \in P_\tau$, we calculate

$$[\hat{\sigma}^{3N}(E_i w)] = E_i[[u^{3N}]\hat{\sigma}^{3N}(w)].$$

Suppose $[\hat{\sigma}^{3N}(w)] = \gamma_{-k} w'$ for $k > 3N - 1$. Consider the $\hat{\sigma}$-canonical decomposition of $\gamma_{-k} w'$. Either this agrees with the $\hat{\sigma}$-canonical decomposition of γ_{-k}, or the last edge of γ_{-k} participates in a maximal exceptional subpath of w', so that the decomposition of $\gamma_{-k} w'$ is obtained from

$\gamma_{-(k-1)}$ and some w''. In either case, since every edge of w is lower than the linear family associated to u_i, $[\hat{\sigma}^{-3N}(w'')]$ does not overlap $[u_i^k]$ in γ_k, and we have

$$w = [\hat{\sigma}^{-3N}(\gamma_{-(k-1)}w'')] = \gamma_{-(k-1)}[\hat{\sigma}^{-3N}(w'')].$$

Since $k > 3N-1$, this implies $E_i w \in P_\sigma$, but we supposed $E_i w \notin P_\sigma$. Therefore, $E_i[u^{3N}\hat{\sigma}^{3N}(w)]$ has $E_i\gamma_2$ as an initial segment, so that $\hat{\sigma}^{3N}(E_i w) \in P_\sigma$. The argument for negative powers is symmetric.

Next we claim $\hat{\tau}^{\pm N}(P_\sigma) \subseteq P_\tau$ for $N \neq 0$. Let v_i be the $\hat{\tau}$ suffix of E_i (possibly trivial). Since u_i grows linearly under τ, γ_1 must contain a $\hat{\tau}$-linear edge or $\hat{\tau}$-linear exceptional path in its $\hat{\tau}$ decomposition. Neither v_i nor $\bar{v}_i$, which are $\hat{\tau}$-Nielsen, can contain a $\hat{\tau}$-linear component in their $\hat{\tau}$-canonical decomposition as v_i is a $\hat{\tau}$ suffix. A similar statement holds for γ_{-1}. Thus v_i and $\bar{v}_i$ do not have γ_2 or γ_{-2} as an initial segment. Consider the highest $\hat{\tau}$-linear edge of γ_1; since $\hat{\tau}$ is upper-triangular this edge cannot be canceled when tightening $\hat{\tau}^N(\gamma_1)$, so $[\hat{\tau}^{\pm N}(\gamma_2)]$ has at most γ_1 in common with γ_2 (and similarly at most γ_{-1} in common with γ_{-2}). Finally, suppose $[E_i\gamma_2 w] \in P_\sigma$ is a tight representative. By the minimality in the choice of E_i, the highest $\hat{\tau}$-linear edge of w is of the same height as that in γ_2, so these edges do not cancel in the tightening of $\hat{\tau}^{\pm N}(\gamma_2)\hat{\tau}^{\pm N}(w)$. Putting this all together, the result $[\hat{\tau}^{\pm N}E_i\gamma_2 w]$ has at most $E_i\gamma_1$ in common with $E_i[u^3]$. Applying similar reasoning to $E_i\gamma_{-2}w'$, we conclude $\hat{\tau}^{\pm N}(P_\sigma) \subseteq P_\tau$ (this shows in particular that P_τ is non-empty). So by the ping-pong lemma $\langle \sigma^3, \tau^3 \rangle \subseteq \langle \sigma^3, \tau \rangle \cong F_2$ as required.

Case 2. Suppose no σ suffix is τ-growing and vise-versa, and that there is an edge E such that $\hat{\sigma}(E) = Eu$ and $\hat{\tau}(E) = Ev$, and the associated primitive Nielsen paths u' and v' do not generate isomorphic subgroups of $\pi_1(\Gamma, t(E))$. Since v is not $\hat{\sigma}$ growing it is $\hat{\sigma}$-periodic, thus $[\hat{\sigma}(v)] = v$; similarly $[\hat{\tau}(u)] = u$. By hypothesis, $u_*, v_* \in \pi_1(\Gamma, t(E))$ generate a rank two free group G. Further, for $\omega \in \langle \hat{\sigma}, \hat{\tau} \rangle$, $\omega(E) = Ew$ for some path w so that $w_* \in \langle u_*, v_* \rangle$. It is immediate that $\omega \mapsto w_*$ is an isomorphism, hence $\langle \sigma, \tau \rangle \cong F_2$. $\square$

The culmination of this effort is a proof of a McCarthy type theorem in the linearly growing case.

Theorem 5.0.2. Suppose σ and τ are linearly growing outer automorphisms. Then there is an N such that $\langle \sigma^N, \tau^N \rangle$ is either abelian or free of rank two. Moreover, the latter case holds exactly when the core of the efficient representatives of Dehn-twist powers of σ and τ contains a rectangle.

Proof. First, using train tracks Cohen and Lustig show that linearly growing automorphisms have Dehn twist powers [19], so let σ^S and τ^T be Dehn twists, with efficient representatives on graphs of groups $\bar{A}$ and $\bar{B}$. If $\mathcal{C}(A, B)$ contains a rectangle then by either Lemma 5.4.1 or 5.4.3, there is a power such that $\langle \sigma^{STN}, \tau^{STN} \rangle \cong F_2$; otherwise by Lemma 5.1.1, σ^{ST}, τ^{ST} commute. $\square$

CHAPTER 6

NEW HORIZONS

And in a way I'm yearning

To be done with all this weighing of the truth
Nicholas Edward Cave

In this thesis, we resolve Conjecture A for linearly growing outer automorphisms, using techniques very different from those used to resolve Conjecture A for exponentially growing outer automorphisms. The tools developed provide hope that a resolution of Conjecture A in general is approachable. In the course of the work, we prove Lemma 5.3.2 which gives a quantitative condition for a subgroup of $\mathrm{Out}(F_r)$ to be polynomially growing. The technique of the proof, and the proof that when the edge-twist graph contains a cycle the group generated contains a free group, suggest that a quantitative version of a subgroup decomposition theorem should be pursued; at the very least one might hope that that Lemma 5.3.2 is sharp, in the sense that if the edge-twist graph contains a cycle then the subgroup of $\mathrm{Out}(F_r)$ generated contains an exponentially growing element.

This work started as an attempt to show that a collection of polynomially growing outer automorphisms and outer automorphisms fully irreducible on some free factor has a power so that the group generated by that power of the collection is a right-angled Artin group (RAAG) on an appropriate graph. This statement contains Conjecture A as a subcase, and the proof of the analogous statement for the mapping class group relies on McCarthy's theorem. The techniques of this thesis alone do not seem sufficient to prove such a result, but there is hope that using them in a resolution of Conjecture A will clarify the situation and pave the way to a similar result.

6.1 Approaches to Conjecture A

The first step to resolving Conjecture A is resolving Conjecture A for two polynomially growing outer automorphisms. Ongoing work runs along the lines of Chapter 5. Using Rodenhausen's normalized higher Dehn twist representatives, the edge twist graph is generalized to a higher edge-twist graph, which accounts for the higher degree growth. This object is no longer an invariant of the graph of groups, only of the pair of automorphisms involved. When it is acyclic, the arguments of Lemma 5.3.2 readily show that the group generated is polynomially growing.

Unfortunately, the combinatorics of the hierarchies associated to two polynomially growing outer automorphisms make a length function ping-pong argument significantly more difficult; the Kolchin case is equally frustrated. Nevertheless, a resolution of Conjecture A using a generalization of the method of proof in Chapter 5 will be completed in the immediate future.

Building a bridge between polynomially growing outer automorphisms, which do not have laminations, and exponentially growing outer automorphisms, which do, is the next step to pursue. Here the generality of Theorem 3.0.3 is an excellent starting point. Exponentially growing outer automorphisms fix projective classes of trees in $\overline{cv}_r$. When an exponentially growing outer automorphism fixes a compatible tree fixed by a polynomially growing outer automorphism one expects to find a fixed projective line in $\overline{cv}_r$. The absence of compatible fixed trees, or of a fixed projective line, should provide enough geometric information to effect a ping-pong argument and find powers generating a free group. Otherwise, Conjecture A is reduced to analyzing the algebraic structure of stabilizers of lines in $\overline{cv}_r$.

6.2 Quantitative subgroup decomposition

Given two Dehn twists, Lemma 5.3.2 states that the edge twist graph being acyclic is a sufficient condition for the generated group to be polynomially growing. This lemma has a suitable generalization to unipotent polynomially growing outer automorphisms given as higher Dehn twists in Rodenhausen's sense. A natural question, which Clay and Pettet answer in the affirmative for hyperbolic-hyperbolic one-edge Dehn twists (these have cycles of length two), is whether or not Lemma 5.3.2 is sharp. Specifically,

Question. If $\tilde{\sigma}$ and $\tilde{\tau}$ are efficient Dehn twists of F_r based on $\bar{A}$ and $\bar{B}$, and $\mathcal{ET}(\bar{A}, \bar{B})$ contains a cycle, does $\langle \tilde{\sigma}, \tilde{\tau} \rangle$ contain an exponentially growing outer automorphism?

The combinatorial information of the cycles in the edge twist graph might provide still more information. A careful analysis of the group elements that have representatives that participate in the cycle might identify the free factor where this exponential growth occurs. As the edge twist graph is algorithmic, this invites the development of a quantitative theory of subgroup decomposition in the sense of Handel and Mosher [27] in this setting.

6.3 RAAGs in $\mathrm{Out}(F_r)$

Given a graph G, the **right-angled Artin group** (RAAG) associated to G is generated by the vertices of G with the relation $[u, v] = 1$ for every edge $(u, v) \in E(G)$. RAAGs are of interest partly because they interpolate between $\mathbb{Z}^r$ and F_r [15]. Recent work has shown that many important classes of groups virtually embed in RAAGs. These virtual embeddings play a significant role in Agol's resolution of the virtual Haken conjecture for 3-manifolds following Wise's program [1,52]. Describing RAAG subgroups of a group G gives quantitative information about the complexity of subgroups of G.

RAAG subgroups of $\mathrm{Mod}(S)$ are abundant and well-studied. The Nielsen-Thurston classification tells us that an infinite order mapping class has a power that admits a decomposition into a product of commuting Dehn twists and pseudo-Anosov-on-a-subsurface classes. Given a finite collection of mapping classes consisting of independent Dehn twists and pseudo-Anosov-on-a-subsurface classes, Koberda [35] shows that up to passing to powers of the generators such a group is the RAAG whose graph is the intersection graph of the attracting laminations of the generators. The intersection graph has the attracting laminations as vertices and edges when $i(u, v) = 0$. Koberda's proof is dynamical. Restricting to mapping classes that are pseudo-Anosov-on-a-subsurface, Clay, Leininger, and Mangahas [16] prove that not only do high powers generate the RAAG associated to the intersection graph, the inclusion map is a quasi-isometric embedding. Their proof uses the geometry of the curve complex as reflected by Masur and Minsky's subsurface projections [37]. Subsequent work of Kim and Koberda connects RAAG subgroups to the combinatorics of the curve complex [33].

Outer automorphisms of a free group have a decomposition analogous to, but more complicated than, the Nielsen-Thurston classification [45]. The analog of a pseudo-Anosov-on-a-subsurface in this setting is an outer automorphism σ of the form $\sigma' * \mathrm{id}$ with respect to some splitting $X * Y$ of F_r, where σ' is fully irreducible. In this case we say that σ is **fully irreducible on a free factor**. This decomposition provides a parallel question about $\mathrm{Out}(F_r)$.

Question. Given a finite collection of outer automorphisms of a free group that are either unipotent or fully irreducible on a free factor, what kind of subgroup do they generate?

Taylor [48] achieves an analog of the Clay, Leininger, and Mangahas result using the geometry of the free factor complex. He proves that suitable powers of outer automorphisms that are fully irreducible on a free factor give a quasi-isometrically embedded RAAG. The precise statement includes a technical admissibility condition on the free factor supports of the outer automorphisms involved, which defines the graph. A resolution of Conjecture A provides a general answer to this question for two-generator RAAGs. A good enough resolution of Conjecture A should also provide tools to approach this question.

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