

Higher Order Painlevé Equations and their Symmetries via Reductions of a Class of Integrable Models

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ABSTRACT

Higher order Painlevé equations and their symmetry transformations belonging to extended affine Weyl groups $A_n^{(1)}$ are obtained through a self-similarity limit of a class of pseudo-differential Lax hierarchies with symmetry inherited from the underlying generalized Volterra lattice structure. In particular, an explicit example of the Painlevé V equation and its Bäcklund symmetry is obtained through a self-similarity limit of a generalized KdV hierarchy from reference [3].

1 Introduction

The aim of this work is to explore integrable origins of higher order Painlevé equations and their extended affine Weyl symmetry groups. Towards this goal this paper investigates a self-similarity limit of a special class of pseudo-differential Lax hierarchies of the constrained KP hierarchy with symmetry structure defined by Bäcklund transformations induced by a discrete structure of the Volterra type lattice. The underlying integrable hierarchy is parametrized in terms of $2M$ Lax coefficients $e_i, c_i, i = 1, \dots, M$. In terms of these coefficients the second Gelfand-Dickey bracket of the underlying constrained Kadomtsev-Petviashvili (KP) hierarchy [3] simplifies to a Heisenberg Poisson bracket algebra:

$$\{e_i(x), c_j(y)\}_2 = -\delta_{ij}\delta_x(x-y), \quad i, j = 1, 2, \dots, M, \quad (1.1)$$

where $\delta_x(x-y) = \partial\delta(x-y)/\partial x$.

It is shown that in the self-similarity limit the second t_2 -flow equations of that hierarchy reduce to the higher order Painlevé equations:

$$f_{ix} = f_i(f_{i+1} - f_{i+2} + f_{i+3} - f_{i+4} + \dots - f_{i-1}) + \alpha_i, \quad i = 0, 1, \dots, 2M \quad (1.2)$$

under a change of variables from $e_i, c_i, i = 1, \dots, M$ to $f_i, i = 1, \dots, 2M$, described below in subsection 4.1. Equation (1.2) introduced $f_0 = -\sum_{i=1}^{2M} f_i - 2x$ and constants α_i satisfying $\sum_{i=0}^{2M} \alpha_i = -2$. The system satisfies periodicity conditions $f_{2M+i} = f_{i-1}, \alpha_{2M+i} = \alpha_{i-1}, i = 0, 1, 2, \dots, 2M$. These equations are invariant under Bäcklund transformations forming the extended affine Weyl group $A_{2M}^{(1)}$. The extended affine Weyl group $A_n^{(1)}$ is generated by $n+1$ transformations s_0, s_1, \dots, s_n in addition to a cyclic permutation π . Together they satisfy relations

$$\begin{aligned} s_i s_j s_i &= s_j s_i s_j \quad (j = i \pm 1), & s_i s_j &= s_j s_i \quad (j \neq i \pm 1) \\ \pi s_i &= s_{i+1} \pi, & \pi^{n+1} &= 1, & s_i^2 &= 1. \end{aligned} \quad (1.3)$$

To obtain Painlevé systems invariant under the extended affine Weyl group $A_n^{(1)}$ for odd n we impose second-class constraints working with Dirac's modified second KP Poisson bracket structure. This procedure effectively reduces a number of $2M$ Lax coefficients of the original Lax hierarchy to $2M-1$ (or in general $2M-k$) coefficients and via the self-similarity reduction reproduces Painlevé equations with the extended affine Weyl symmetry $A_{2M-1}^{(1)}$. Here, we present results for the special case of the extended affine Weyl symmetry $A_3^{(1)}$ with the corresponding Painlevé V equation:

$$w_{zz} = \left(\frac{1}{2w} + \frac{1}{w-1} \right) w_z^2 - \frac{1}{z} w_z + \frac{(w-1)^2}{z^2} \left(\alpha w + \frac{\beta}{w} \right) + \frac{\gamma w}{z} + \delta \frac{w(w+1)}{w-1}, \quad (1.4)$$

with the parameter $\delta = -1/2$.

The symmetric Painlevé equations with their extended affine Weyl symmetry group $A_n^{(1)}$ first appeared in Adler's [1] and Veselov and Shabat's papers [19] in

the setting of periodic dressing chains and later were discussed in great details from the purely affine Weyl group symmetry point of view by Noumi and Yamada [11, 12] (see also [14] and [5] for accessible presentations). As shown in [13] (see also [17, 16]) the higher order Painlevé equation of type $A_n^{(1)}$ can be obtained by self-similarity reduction of the $n+1$ -reduced modified KP hierarchy with $(n+1) \times (n+1)$ Lax pair with construction based on a standard regular element Λ in the principal Drinfeld-Sokolov hierarchy. Equivalence of both approaches has been explained in [17] (see also [18, 10]) within the Hamiltonian framework. Our investigation provides a different origin of the higher order Painlevé systems as a self-similarity limit of a class of the constrained KP hierarchies defined in terms of the pseudo-differential Lax operators naturally connected with unconventional (not principal) gradations [3].

In section 2 we present the underlying integrable Lax hierarchy focusing on the second flow equations and the Bäcklund transformation keeping the Lax equations invariant. In Section 3 the self-similarity limit is taken and the Hamiltonians governing the t_2 flow equations are derived in the self-similarity limit. Next, in Section 4 the Hamiltonians found in Section 3 are shown to reproduce Hamiltonian structure of the higher Painlevé equations invariant under the extended affine Weyl symmetry $A_{2M}^{(1)}$ when expressed in terms of the canonical variables. This is illustrated for the special cases of $M = 1, 2$ for which the generators of the extended affine Weyl symmetry group are derived from the Bäcklund transformation of Section 2. The Dirac reduction scheme when applied on the original integrable constrained KP hierarchy leads to reduction of the model with $A_{2M}^{(1)}$ symmetry down to a model characterized by $A_{2M-1}^{(1)}$ symmetry. This is illustrated in Section 5 for $M = 2$ with the reduced model being nothing but the Painlevé V equation with its Bäcklund symmetry. Concluding comments are given in Section 6 in which we also announce future plans for obtaining solutions to the higher order Painlevé equations and generalizing these equations to the higher Painlevé hierarchies by making use of the results presented here.

2 The integrable hierarchy, its second flow and generalized Volterra symmetry structure

2.1 A “half-integer” lattice

It is well-known that symmetries of many continuum KP-type hierarchies are governed by discrete lattice-like structures. A standard example is provided by the AKNS hierarchy and the Toda lattice structure of its Bäcklund transformations leading to Hirota type equations for the Toda chain of tau-functions [2]. There also exists the so-called two-boson formulation of the AKNS hierarchy which is invariant under symmetry transformations on a “half-integer” lattice which generalizes Toda lattice [2]. We now present a general “half-integer” lattice (or the generalized Volterra lattice) following closely reference [3]. The foundation of this formalism

rests on two spectral equations:

$$\lambda^{1/2} \tilde{\Psi}_{n+\frac{1}{2}} = \Psi_{n+1} + \mathcal{A}_{n+1}^{(0)} \Psi_n + \sum_{p=1}^M \mathcal{A}_{n-p+1}^{(p)} \Psi_{n-p} \quad (2.1)$$

$$\lambda^{1/2} \Psi_n = \tilde{\Psi}_{n+\frac{1}{2}} + \mathcal{B}_n^{(0)} \tilde{\Psi}_{n-\frac{1}{2}} \quad (2.2)$$

and “time” evolution equations:

$$\tilde{\Psi}_{n+\frac{1}{2}} = (\partial - \mathcal{B}_n^{(0)} - \mathcal{A}_n^{(0)}) \tilde{\Psi}_{n-\frac{1}{2}} \quad ; \quad \Psi_{n+1} = (\partial - \mathcal{B}_n^{(0)} - \mathcal{A}_{n+1}^{(0)}) \Psi_n, \quad (2.3)$$

which both involve objects labeled by integers and half-integers and $\partial = \partial/\partial x$. If we remove the term $\sum_{p=1}^M \mathcal{A}_{n-p+1}^{(p)} \Psi_{n-p}$ from equation (2.1) the remaining system yields the Volterra chain equations. For that reason we will refer to equations (2.1)–(2.3) as a generalized Volterra system. As shown in [3], upon eliminating the half-integer modes, the generalized Volterra system (2.1)–(2.3) reduces to the Toda lattice equations. From (2.1)–(2.3) we find:

$$\lambda^{1/2} \tilde{\Psi}_{n+\frac{1}{2}} = \left(\partial - \mathcal{B}_n^{(0)} + \sum_{p=1}^M \mathcal{A}_{n-p+1}^{(p)} (\partial - \mathcal{B}_{n-p}^{(0)} - \mathcal{A}_{n-p+1}^{(0)})^{-1} \cdots \right. \\ \left. \cdots (\partial - \mathcal{B}_{n-1}^{(0)} - \mathcal{A}_n^{(0)})^{-1} \right) \Psi_n \quad (2.4)$$

and

$$\lambda^{1/2} \Psi_n = (\partial - \mathcal{A}_n^{(0)}) \tilde{\Psi}_{n-\frac{1}{2}}. \quad (2.5)$$

Eliminating half-integer modes from the last two relations yields a spectral equation of a form

$$\lambda \Psi_n = L_n^{(M+1)} \Psi_n \quad (2.6)$$

with Lax operator $L_n^{(M+1)}$ given by recurrence relation:

$$L_n^{(M+1)} = e^{\int \mathcal{B}_{n-1}^{(0)}} \left(\partial - \mathcal{A}_n^{(0)} + \mathcal{B}_{n-1}^{(0)} \right) L_n^{(M)} (\partial - \mathcal{A}_n^{(0)})^{-1} e^{-\int \mathcal{B}_{n-1}^{(0)}} \quad (2.7)$$

where

$$L_n^{(M)} = \partial + \sum_{p=1}^M \mathcal{A}_{n-p}^{(p)} (\partial + \mathcal{B}_{n-1}^{(0)} - \mathcal{B}_{n-p-1}^{(0)} - \mathcal{A}_{n-p}^{(0)})^{-1} \cdots (\partial + \mathcal{B}_{n-1}^{(0)} - \mathcal{B}_{n-2}^{(0)} - \mathcal{A}_{n-1}^{(0)})^{-1} \quad (2.8)$$

Using equation (2.5) it is easy to shift the spectral equation (2.6) to the half-integer lattice:

$$\lambda \tilde{\Psi}_{n-\frac{1}{2}} = \tilde{L}_n^{(M+1)} \tilde{\Psi}_{n-\frac{1}{2}}, \quad \tilde{L}_n^{(M+1)} = (\partial - \mathcal{A}_n^{(0)})^{-1} L_n^{(M+1)} (\partial - \mathcal{A}_n^{(0)}). \quad (2.9)$$

The similarity transformation responsible for transformation from integer to half-integer lattice will be shown below to play a central role as a Bäcklund transformation of the higher order Painlevé equations.

2.2 Basic facts about the $2M$ -bose constrained KP hierarchy

The recurrence relation (2.7) is realized by the Lax operators:

$$L_M = (\partial - e_M) \prod_{k=M-1}^1 \left(\partial - e_k - \sum_{l=k+1}^M c_l \right) \left(\partial - \sum_{l=1}^M c_l \right) \\ \times \prod_{k=1}^M \left(\partial - e_k - \sum_{l=k}^M c_l \right)^{-1}, \quad M = 1, 2, \dots, \quad (2.10)$$

of the $2M$ -bose constrained KP hierarchy build of M pairs $(c_k, e_k)_{k=1}^M$. Recall that the KP hierarchy is endowed with the bi-Hamiltonian Poisson bracket structure resulting from two compatible Hamiltonian structures on the algebra of pseudo-differential operators. Remarkably, for the above Lax hierarchy the second bracket of the hierarchy is realized in terms of $(c_k, e_k)_{k=1}^M$ as the Heisenberg Poisson bracket algebra (1.1). The Lax operator (2.10) realizes the recursive relation (2.7) rewritten in this context as follows:

$$L_M = e^{\int c_M} (\partial + c_M - e_M) L_{M-1} (\partial - e_M)^{-1} e^{-\int c_M}, \quad L_0 \equiv \partial \quad (2.11)$$

for $M = 1, 2, \dots$. The corresponding second flow equations can be obtained from the second bracket structure through:

$$\frac{\partial f}{\partial t_2} = \{f, H_2\}_2, \quad (2.12)$$

where the Hamiltonian H_2 is an integral of the coefficient $u_1(M)$ appearing in front of ∂^{-2} in the Lax operator (2.11) when cast in a conventional KP form:

$$L_M = \partial + u_0(M)\partial^{-1} + u_1(M)\partial^{-2} + \dots$$

As a consequence of equation (2.11) we obtain

$$u_0(M) = \sum_{i=1}^M \left(\frac{\partial e_i}{\partial x} + e_i c_i \right) \\ u_1(M) = \sum_{i=1}^{M-1} (M-i) \frac{\partial}{\partial x} \left(\frac{\partial e_i}{\partial x} + e_i c_i \right) + 2 \sum_{i=1}^{M-1} u_0(i) c_{i+1} + \sum_{i=1}^M \left(\frac{\partial e_i}{\partial x} + e_i c_i \right) (e_i + c_i)$$

The Darboux-Bäcklund transformation of the Lax operator L_M defined in equation (2.10) takes a form

$$L_M \rightarrow (\partial - e_M)^{-1} L_M (\partial - e_M)$$

and since $e_M \sim \mathcal{A}_n^{(0)}$ we see from equation (2.5) that it represents transformation on the Volterra lattice from integer modes to half-integer modes. For coefficients with

highest indices this transformation results in:

$$\begin{aligned}
g(e_M) &= e_{M-1} + c_M, & g(c_M) &= -e_{M-1} + e_M - \frac{c_M x}{c_M} \\
g(e_{M-1}) &= e_{M-2} + e_{M-1} - e_M + c_M + c_{M-1} + \frac{c_M x}{c_M} \\
g(c_{M-1}) &= -e_{M-2} + e_{M-1} - \frac{\left(-e_{M-1} + e_M - c_M - c_{M-1} - \frac{c_M x}{c_M}\right)_x}{\left(-e_{M-1} + e_M - c_M - c_{M-1} - \frac{c_M x}{c_M}\right)}
\end{aligned} \tag{2.13}$$

in addition to

$$g\left(e_k + \sum_{l=k+1}^M c_l\right) = e_{k-1} + \sum_{l=k}^M c_l, \quad 2 \leq k \leq M, \quad g\left(e_1 + \sum_{l=2}^M c_l\right) = \sum_{l=1}^M c_l. \tag{2.14}$$

For a special example of the so-called two Bose system with $M = 1$ (which below will be shown to correspond to the symmetric Painlevé IV equation) the Lax operator is:

$$L_1 = (\partial - e_1)(\partial - c_1)(\partial - e_1 - c_1)^{-1}.$$

The Lax operator L_1 possesses a Darboux-Bäcklund symmetry:

$$L_1 \rightarrow (\partial - e_1)^{-1} L_1 (\partial - e_1) = (\partial - c_1)(\partial - e_1 + c_{1x}/c_1)(\partial - e_1 - c_1 + c_{1x}/c_1)^{-1},$$

which keeps its form unchanged while transforming e_1, c_1 as follows:

$$g(e_1) = c_1, \quad g(c_1) = e_1 - c_{1x}/c_1. \tag{2.15}$$

3 Hamiltonians and the t_2 -flows in the self-similarity limit of the $2M$ -bose constrained KP hierarchy

The second flow equation (2.12) results in the following expressions for the Lax coefficients:

$$\begin{aligned}
\frac{\partial c_j}{\partial t_2} &= \frac{\partial}{\partial x} \left(\frac{\partial c_j}{\partial x} - c_j^2 - 2e_j c_j + 2 \sum_{i=j+1}^M \frac{\partial c_i}{\partial x} - 2 \sum_{i=j}^{M-1} c_i c_{i+1} \right), \quad j = 1, \dots, M \\
\frac{\partial e_j}{\partial t_2} &= \frac{\partial}{\partial x} \left(-\frac{\partial e_j}{\partial x} - e_j^2 - 2e_j c_j - 2u_0(j-1) - 2e_j \sum_{i=j+1}^M c_i \right), \quad j = 1, \dots, M
\end{aligned} \tag{3.1}$$

Effectively, the action of the self-similarity reduction replaces $\partial f / \partial t_2$ with $-(xf)_x / 2$. Integrating all equations obtained by taking the self-similarity limit we find

$$e_{jx} + 2 \sum_{i=1}^{j-1} e_{ix} = \frac{\partial \mathcal{H}_M}{\partial c_j}, \quad c_{jx} + 2 \sum_{i=j+1}^M c_{ix} = -\frac{\partial \mathcal{H}_M}{\partial e_j} \tag{3.2}$$

for $j = 1, \dots, M$ with

$$\mathcal{H}_M = - \sum_{j=1}^M e_j c_j (e_j + c_j - 2x) - 2 \sum_{1 \leq j < i \leq M} e_j c_j c_i + \sum_{j=1}^M \bar{\kappa}_j c_j - \sum_{j=1}^M \kappa_j e_j \quad (3.3)$$

where $\kappa_j, \bar{\kappa}_j$ are integration constants.

It follows that equations (3.2) can be rewritten as

$$\begin{aligned} e_{jx} &= 2xe_j + 4x \sum_{k=1}^{j-1} (-1)^{j-k} e_k - e_j \left(e_j + 2 \sum_{k=j}^M c_k \right) \\ &+ \sum_{k=1}^{j-1} (-1)^{j-k+1} e_k \left(2e_k + 2c_k + 4 \sum_{l=k+1}^M c_l \right) + \bar{k}_j \end{aligned} \quad (3.4)$$

$$\begin{aligned} c_{jx} &= -2xc_j + 4x \sum_{k=j+1}^M (-1)^{j-k+1} c_k + c_j \left(c_j + 2e_j + 2 \sum_{k=j+1}^M c_k \right) \\ &+ \sum_{k=j+1}^M (-1)^{j-k} c_k \left(2c_k + 4e_k + 4 \sum_{l=k+1}^M c_l \right) + k_j \end{aligned} \quad (3.5)$$

for $j = 1, \dots, M$ with appropriately redefined constants k_j, \bar{k}_j :

$$\kappa_j = k_j + 2 \sum_{i=j+1}^M k_i, \quad \bar{\kappa}_j = \bar{k}_j + 2 \sum_{i=1}^{j-1} \bar{k}_i, \quad j = 1, \dots, M.$$

Equations (3.4)-(3.5) can be reproduced through

$$e_{jx} = \{e_j, \mathcal{H}_M\}, \quad c_{jx} = \{c_j, \mathcal{H}_M\}, \quad (3.6)$$

with a Poisson bracket structure:

$$\{e_j, c_i\} = \delta_{j,i} + 2E_{j,i}, \quad (3.7)$$

where $E_{j,i}$ is an element of strictly lower-triangular matrix equal to:

$$E_{j,i} = \begin{cases} (-1)^{j-i} & j > i, \\ 0 & j \leq i. \end{cases}$$

4 Connection to higher order Painlevé equations

4.1 General Construction

Let $q_i, p_i, i = 1, \dots, M$ be canonical coordinates satisfying the canonical brackets

$$\{q_i, p_j\} = -\delta_{ij}, \quad \{q_i, q_j\} = 0 = \{p_i, p_j\}, \quad i = 1, \dots, M.$$

Relations

$$q_i = f_{2i}, \quad p_i = \sum_{k=1}^i f_{2k-1}, \quad i = 1, \dots, M \quad (4.1)$$

define new variables f_k , $k = 1, \dots, 2M$ and map the canonical brackets into the following Poisson brackets:

$$\{f_i, f_{i+1}\} = 1, \quad \{f_i, f_{i-1}\} = -1, \quad i = 1, \dots, 2M.$$

We now propose a conversion table between e_i, c_i $i = 1, \dots, M$ and a special set of canonical coordinates and equivalently Painlevé variables f_k , $k = 1, \dots, 2M$ that will satisfy the higher order Painlevé equations (1.2).

First, we list the result for e_i , $i = 1, \dots, M$:

$$\begin{aligned} e_M &= q_M + p_M + 2x + \frac{k_M}{p_M - p_{M-1}}, & e_{M-1} &= -p_{M-1} \\ e_{M-2k} &= -q_k - q_{k+1} - \dots - q_{M-k-1}, & k &= 1, 2, \dots \\ e_{M-2k-1} &= -p_{M-k-1} + p_k, & k &= 1, 2, \dots \end{aligned} \quad (4.2)$$

and next for c_i , $i = 1, \dots, M$:

$$\begin{aligned} c_M &= -p_M + p_{M-1}, & c_{M-1} &= p_M - p_{M-1} - q_M - q_{M-1} \\ c_{M-2k} &= p_{k-1} + p_{M-k-1} + q_k + \dots + q_{M-k-1} + 2q_{M-k} + 2q_{M-k+1} + \dots \\ &+ 2q_{M-1} + 2q_M + 2x, & k &= 1, 2, 3, \dots \\ c_{M-2k-1} &= -(p_k + p_{M-k-1} + q_k + \dots + q_{M-k-2} + 2q_{M-k-1} + \dots \\ &+ 2q_{M-1} + 2q_M + 2x), & k &= 1, 2, 3, \dots \end{aligned} \quad (4.3)$$

The Hamiltonian \mathcal{H}_M defined in (3.3) reads in terms of q_i, p_i , $i = 1, \dots, M$ defined in equations (4.2)-(4.3) as follows

$$\mathcal{H}_M = \sum_{j=1}^M p_j q_j (p_j + q_j + 2x) + 2 \sum_{1 \leq j < i \leq M} p_j q_j q_i - \sum_{j=1}^M \alpha_{2j} p_j + \sum_{j=1}^M q_j \left(\sum_{k=1}^j \alpha_{2k-1} \right) \quad (4.4)$$

in agreement with reference [15]. The corresponding Hamilton equations:

$$\begin{aligned} p_{ix} &= \frac{\partial \mathcal{H}_M}{\partial q_i} = p_i \left(p_i + 2 \sum_{j=i}^M q_j + 2x \right) + 2 \sum_{j=1}^{i-1} p_j q_j + \sum_{j=1}^i \alpha_{2j-1} \\ q_{ix} &= -\frac{\partial \mathcal{H}_M}{\partial p_i} = -q_i \left(2p_i + q_i + 2 \sum_{j>i}^M q_j + 2x \right) + \alpha_{2i} \end{aligned}$$

are equivalent to the higher Painlevé equations as given in equation (1.2) with identification of variables provided by relation (4.1).

In the following subsections of this section we will illustrate the above general result for $M = 1$ and $M = 2$.

4.2 The case of $M = 1$ and the Painlevé IV Equation

For $M = 1$ the equations (3.4)-(3.5) simplify to

$$e_{1x} = 2xe_1 - (e_1 + 2c_1)e_1 + \bar{k}_1, \quad c_{1x} = -2xc_1 + (c_1 + 2e_1)c_1 + k_1, \quad (4.5)$$

and agree with the self-similar limit of the so-called two-boson formulation of the AKNS hierarchy [2, 4]. By elimination of one of the variables e_1 or c_1 equations (4.5) reduce, as is well-known, to the Painlevé IV equation. Equations (4.5) are kept invariant under transformations (2.15) when accompanied by transformations $g(k_1) = 2 - \bar{k}_1$, $g(\bar{k}_1) = -k_1$ of the integration constants. Note that eqs. (4.5) are Hamiltonian as they follow from (3.2) with:

$$\mathcal{H}_1 = 2xe_1c_1 - e_1^2c_1 - e_1c_1^2 + \bar{k}_1c_1 - k_1e_1, \quad \{e_1, c_1\} = 1, \quad (4.6)$$

obtained by setting $M = 1$ in the appropriate places in the previous subsection. It follows by simple differentiation and use of equations of motion that

$$\mathcal{H}_{1x} = 2e_1c_1, \quad \mathcal{H}_{1xx} = 2e_{1x}c_1 + 2e_1c_{1x} = 2e_1^2c_1 - 2c_1^2e_1 + 2\bar{k}_1c_1 + 2k_1e_1 + \bar{k}_1c_1 - k_1e_1$$

which leads to the Jimbo-Miwa equation [9] of Painlevé IV system:

$$(\mathcal{H}_{1xx} + 2(\mathcal{H}_1 - x\mathcal{H}_{1x}))(\mathcal{H}_{1xx} - 2(\mathcal{H}_1 - x\mathcal{H}_{1x})) = 2\mathcal{H}_{1x}(\mathcal{H}_{1x} - 2\bar{k}_1)(\mathcal{H}_{1x} + 2k_1).$$

for \mathcal{H}_1 . Connection of $M = 1$ example (4.5) to the $A_2^{(1)}$ manifestly symmetric Painlevé IV set of equations:

$$f_{0x} = f_0(f_1 - f_2) + \alpha_0, \quad f_{1x} = f_1(f_2 - f_0) + \alpha_1, \quad f_{2x} = f_2(f_0 - f_1) + \alpha_2$$

with $\alpha_0 + \alpha_1 + \alpha_2 = -2$ can be made explicit by setting

$$f_i = -c_i, \quad f_{i+1} = -e_i + \frac{c_{1x}}{c_1}, \quad f_{i+2} = e_i + c_1 - \frac{c_{1x}}{c_1} - 2x, \quad \alpha_i = k_1, \quad \alpha_{i+2} = -k_1 - \bar{k}_1 \quad (4.7)$$

for $i = 0, 1, 2$. For each of the values 0, 1, 2 of index i we denote by g_i the Darboux-Bäcklund transformation derived from g from relation (2.15) by replacing variables e_i, c_i by f_i according to relation (4.7). It is easy to check that each such g_i maps $f_i \rightarrow f_{i+1}$, $\alpha_i \rightarrow -\alpha_{i+1}$ and $\alpha_{i+2} \rightarrow \alpha_i + \alpha_{i+1}$ and is realized as πs_i in terms of the extended affine Weyl operators from (1.3). Note, that the generators s_i of the affine Weyl group $A_2^{(1)}$ act as $s_i(\alpha_{i+2}) = \alpha_i + \alpha_{i+2}$. The above construction together with the idea of introducing permutation symmetry of the extended affine Weyl group $A_2^{(1)}$ by associating f_i 's to any of the solutions of the self-similar limit of the so-called two-boson formulation of the AKNS hierarchy was discussed in [4].

4.3 The Four-Bose system and $A_4^{(1)}$ Painlevé Equations

We now consider a four-boson case with $M = 2$ and $(c_k, e_k)_{k=1}^2$ subject to equations

$$\begin{aligned} e_{1x} &= 2xe_1 - (e_1 + 2c_1 + 2c_2)e_1 + \bar{k}_1 \\ e_{2x} &= 2xe_2 - 4xe_1 - (e_2 + 2c_2)e_2 + (2c_1 + 2e_1 + 4c_2)e_1 + \bar{k}_2 \\ c_{1x} &= -2xc_1 + 4xc_2 + (c_1 + 2e_1)c_1 + (2c_1 - 2c_2 - 4e_2)c_2 + k_1 \\ c_{2x} &= -2xc_2 + (c_2 + 2e_2)c_2 + k_2 \end{aligned} \quad (4.8)$$

as follows from equations (3.4)-(3.4) for $M = 2$. The equations of motion (4.8) are derived from the Hamilton equations (3.6) with \mathcal{H}_2 obtained from definition (3.3) by setting $M = 2$ and with the Poisson brackets:

$$\{e_1, c_1\} = 1, \quad \{e_1, c_2\} = 0, \quad \{e_2, c_1\} = -2, \quad \{e_2, c_2\} = 1. \quad (4.9)$$

The symmetry transformations (2.13)-(2.14) read here:

$$\begin{aligned} g(e_2) &= e_1 + c_2, & g(e_1) &= e_1 - e_2 + c_2 + c_1 + \frac{c_2 x}{c_2} \\ g(c_2) &= -e_1 + e_2 - \frac{c_2 x}{c_2}, & g(c_1) &= e_1 - \frac{\left(-e_1 + e_2 - c_2 - c_1 - \frac{c_2 x}{c_2}\right)_x}{\left(-e_1 + e_2 - c_2 - c_1 - \frac{c_2 x}{c_2}\right)}, \end{aligned} \quad (4.10)$$

and keep equations (4.8) invariant for:

$$\begin{aligned} g(k_1) &= -2 + \bar{k}_1 + 2\bar{k}_2, & g(\bar{k}_1) &= 2 - \bar{k}_1 - \bar{k}_2 - k_1 - 3k_2, \\ g(k_2) &= 2 - \bar{k}_1 - \bar{k}_2, & g(\bar{k}_2) &= -4 + 3\bar{k}_1 + 2\bar{k}_2 + 2k_1 + 5k_2. \end{aligned} \quad (4.11)$$

In order to see the meaning of this transformation from the group theoretic point of view we cast equations (4.8) into the symmetric $A_4^{(1)}$ Painlevé equations:

$$f_{ix} = f_i(f_{i+1} - f_{i+2} + f_{i+3} - f_{i+4}) + \alpha_i, \quad i = 0, \dots, 4 \quad (4.12)$$

with conditions $f_i = f_{i+5}$ and $\sum_{i=0}^4 \alpha_i = -2$. We propose the following identification

$$\begin{aligned} f_1 &= -e_1 \\ f_2 &= g(f_1) = -\left(e_1 - e_2 + c_2 + c_1 + \frac{c_2 x}{c_2}\right) = -e_1 - e_2 - c_1 - 2c_2 + 2x - \frac{k_2}{c_2} \\ f_3 &= -c_2 \\ f_4 &= g(f_3) = -\left(-e_1 + e_2 - \frac{c_2 x}{c_2}\right) = e_1 + e_2 + c_2 - 2x + \frac{k_2}{c_2} \\ f_0 &= -f_1 - f_2 - f_3 - f_4 - 2x = e_1 + c_1 + 2c_2 - 2x \end{aligned} \quad (4.13)$$

and

$$\begin{aligned} \alpha_1 &= -\bar{k}_1, & \alpha_2 &= 2 - \bar{k}_1 - \bar{k}_2 - k_1 - 3k_2 \\ \alpha_3 &= k_2, & \alpha_4 &= -2 + \bar{k}_1 + \bar{k}_2 \\ \alpha_0 &= -2 + \sum_{i=1}^4 \alpha_i = -2 + \bar{k}_1 + k_1 + 2k_2. \end{aligned}$$

Alternatively we can write relations between $e_i, c_i, i = 1, 2$ and $f_i, i = 0, 1, \dots, 4$ as

$$\begin{aligned} e_1 &= -f_1, & e_2 &= -f_1 - f_4 + f_{3x}/f_3 = -f_0 - f_2 + \frac{\alpha_3}{f_3} \\ c_1 &= f_3 - f_2 - f_4, & c_2 &= -f_3 \end{aligned} \quad (4.14)$$

in agreement with equations (4.2) and (4.3). Accordingly, one can rewrite the g -transformation from (4.10) as

$$\begin{aligned}
g(f_1) &= f_2, & g(f_3) &= f_4 \\
g(f_2) &= -f_3 + f_2 + f_4 - \frac{f_{4x}}{f_4} + \frac{f_{2x}}{f_2} \\
&= f_3 - \frac{\alpha_4}{f_4} + \frac{\alpha_2}{f_2} \\
g(f_4) &= f_1 + f_3 - f_2 + \frac{f_{4x}}{f_4} = f_0 + \frac{\alpha_4}{f_4}.
\end{aligned} \tag{4.15}$$

Comparing with definitions of transformations s_i , $i = 1, 2, 3, 4$ (see for instance [14]) we see that expression for transformation g from (4.10)-(4.11) agrees with

$$g = \pi s_1 + \pi s_3 - \pi \tag{4.16}$$

as applied on both f 's and α 's.

Generalizing relation (4.13), we next associate $-e_1$ and $-c_2$ to f_i and f_{i+2} , respectively, for each index i value of 1, 2, 3, 4. In that way we obtain the following realizations of the Darboux-Bäcklund transformation g defined in relation (4.10):

$$g_i = \pi s_i + \pi s_{i+2} - \pi, \quad i = 1, 2, 3, 4$$

by replacing e_i, c_i in relation (4.10) with f_i . In the above expression s_i are generators of the affine Weyl group $A_4^{(1)}$. Note, that obviously g_1 agrees with g as given in equation (4.16) for $f_1 = -e_1, f_3 = -c_2$. Thus, in a close analogy to a reasoning presented for the Painlevé IV system at the end of Subsection 4.2 as well as discussion in [4], by employing different associations of the symmetric Painlevé system variables f_i and f_{i+2} with $i = 1, 2, 3, 4$ with variables $-e_1$ and $-c_2$ of the underlying integrable model we are able to recover all of the affine Weyl $A_4^{(1)}$ generators s_i from a Darboux-Bäcklund transformation g defined in relation (4.10). For instance, $s_1 = (-I + \pi(g_1 + g_2 + g_3 - g_0 - g_4))/2$.

5 Reduction of $M = 2$ case. Painlevé V equation and its symmetries

We will now follow reference [3] and perform a Dirac reduction of $M = 2$ case (see subsection 4.3), by redefining variables as follows:

$$(e_1, c_1, e_2, c_2) \rightarrow (\tilde{e}_1 = e_1, \tilde{c}_1 = c_1, \tilde{e}_2 = (e_2 - c_2)/2, \tilde{c}_2 = (c_2 - e_2)/2),$$

which is equivalent to setting a second-class constraint

$$c = c_2 = -e_2$$

with the Dirac bracket:

$$\{c(x), c(y)\} = \frac{1}{2} \delta_x(x - y). \tag{5.1}$$

Upon imposing this constraint the Hamiltonian $H_2 = \int dy u_1(M)$ appearing in equation (2.12) reduces to

$$\tilde{H}_2 = \int dy \left((e_{1y} + e_1 c_1)_y + (e_{1y} + e_1 c_1) (e_1 + c_1 + 2c) \right)$$

and with the bracket structure (1.1) modified by the Dirac bracket (5.1) for $i = j = 2$ this Hamiltonian generates according to equation (2.12) the t_2 evolution equations:

$$\begin{aligned} \frac{\partial c_1}{\partial t_2} &= \frac{\partial}{\partial x} (c_{1x} + 2c_x - c_1^2 - 2e_1 c_1 - 2c_1 c) \\ \frac{\partial e_1}{\partial t_2} &= \frac{\partial}{\partial x} (-e_{1x} - e_1^2 - 2e_1 c_1 - 2e_1 c) \\ \frac{\partial c}{\partial t_2} &= \frac{\partial}{\partial x} (e_{1x} + e_1 c_1) \end{aligned}$$

of a so-called $SL(3, 1)$ KdV hierarchy from reference [3].

The self-similarity reduction of the above equation yields

$$-2xc_1 = c_{1x} + 2c_x - c_1^2 - 2e_1 c_1 - 2c_1 c + k_1 \quad (5.2)$$

$$-2xe_1 = -e_{1x} - e_1^2 - 2e_1 c_1 - 2e_1 c + \bar{k}_1 \quad (5.3)$$

$$-2xc = e_{1x} + e_1 c_1 + k. \quad (5.4)$$

Eliminating c from eq. (5.4) and plugging it into eq. (5.3) yields the following expression for c_1

$$c_1 = \frac{2x^2 - xe_1 - x \frac{e_{1x}}{e_1} + e_{1x} + x \frac{\bar{k}_1}{e_1} + k}{2x - e_1} \quad (5.5)$$

in terms of e_1 and its derivative.

Next we plug the above expression into eq. (5.2) together with the expression for c in terms of e_1 and c_1 to obtain the second order equation for $y = e_1/2x$:

$$\begin{aligned} y_{xx} &= -\frac{1}{x}y_x + \left(\frac{1}{2y} + \frac{1}{2(y-1)} \right) y_x^2 + \frac{y^5}{y(y-1)} 4x^2 - \frac{y^4}{y(y-1)} (2(1+k+k_1) + 10x^2) \\ &+ \frac{y^3}{y(y-1)} (4(1+k+k_1) + 8x^2) - \frac{y^2}{x^2 y(y-1)} \left[2x^4 + 2x^2(1+k+k_1) \right. \\ &\left. + \frac{1}{2}(k+1)(k+\bar{k}_1+1) \right] - \frac{y}{x^2 y(y-1)} \frac{\bar{k}_1^2}{4} + \frac{1}{x^2 y(y-1)} \frac{\bar{k}_1^2}{8} \end{aligned}$$

A change of coordinate from x to z such that $z = \sigma x^2$ yields

$$\begin{aligned} y_{zz} &= -\frac{1}{z}y_z + \left(\frac{1}{2y} + \frac{1}{2(y-1)} \right) y_z^2 - \frac{\alpha y}{z^2(y-1)} - \frac{\beta(y-1)}{z^2 y} \\ &- \frac{\gamma}{z}y(y-1) - \delta y(y-1)(2y-1) \end{aligned} \quad (5.6)$$

with constants

$$\begin{aligned}\alpha &= \frac{1}{8}(k+1)(k+\bar{k}_1+1) + \frac{\bar{k}_1^2}{32} = \frac{1}{8}\left(k+1 + \frac{\bar{k}_1}{2}\right)^2 \\ \beta &= -\frac{\bar{k}_1^2}{32} = -\frac{1}{8}\left(\frac{\bar{k}_1}{2}\right)^2, \quad \gamma = \frac{k+k_1+1}{2\sigma}, \quad \delta = -\frac{1}{2\sigma^2}\end{aligned}\tag{5.7}$$

The above equation takes on a conventional form of the Painlevé V equation (1.4) for $w = y/(y-1)$ and $\delta = -1/2$ or $\sigma^2 = 1$ (see [20]).

To study the Darboux-Bäcklund transformation of the Painlevé V system we follow the method of equation (2.9) and perform the similarity transformation

$$L \rightarrow (\partial + c)^{-1} L (\partial + c)$$

on the Lax operator for the reduced 4-boson system [3]:

$$L = (\partial + c) \left(\partial + c_1 - \tilde{B}_1 \right) (\partial + c_1 - c) \left(\partial - \tilde{B}_1 \right)^{-1},$$

with $\tilde{B}_1 = e_1 + c_1 + c$. Note that L can be rewritten in a generalized KdV form $L = u_1(\partial - u_2)^{-1} + u_3 + \partial^2$ of unconventional gradation for appropriate coefficients $u_i, i = 1, 2, 3$ [3].

This induces the following transformations for variables of the reduced subspace:

$$\begin{aligned}g(e_1) &= e_1 + c_1 + 2c \\ g(c_1) &= e_1 - \frac{(c_1 + e_1 + 2c)_x}{c_1 + e_1 + 2c} \\ g(c) &= -e_1 - c\end{aligned}\tag{5.8}$$

It follows that g must transform the constants k, k_1, \bar{k}_1 as

$$g(k) = -k - \bar{k}_1, \quad g(\bar{k}_1) = k_1 + \bar{k}_1 + 2k, \quad g(k_1) = \bar{k}_1 - 2\tag{5.9}$$

to keep equations (5.2)-(5.4) invariant. In terms of Painlevé parameters α, β, γ defined in (5.7) the above transformation becomes:

$$\begin{aligned}g(\alpha) &= \frac{1}{8}\left(\frac{1}{2}k_1 - \frac{1}{2}\bar{k}_1 + 1\right)^2 = \frac{1}{8}\left(\frac{1}{2}\left(k+1 + \frac{\bar{k}_1}{2}\right) + \frac{1}{2}\frac{\bar{k}_1}{2} - \sigma\gamma - 1\right)^2 \\ g(\beta) &= -\frac{1}{8}\left(k + \frac{1}{2}k_1 + \frac{1}{2}\bar{k}_1\right)^2 = -\frac{1}{8}\left(\frac{1}{2}\left(k+1 + \frac{\bar{k}_1}{2}\right) + \frac{1}{2}\frac{\bar{k}_1}{2} + \sigma\gamma - 1\right)^2 \\ g(\gamma) &= -\frac{\sigma}{2}(k+1) = \sigma\left(\frac{1}{2}\frac{\bar{k}_1}{2} - \frac{1}{2}\left(k+1 + \frac{\bar{k}_1}{2}\right)\right)\end{aligned}\tag{5.10}$$

Recalling that $(k+1 + \frac{\bar{k}_1}{2})/2$ and $\frac{\bar{k}_1}{2}/2$ are such that

$$\left(\frac{1}{2}\left(k+1 + \frac{\bar{k}_1}{2}\right)\right)^2 = 2\alpha, \quad \left(\frac{1}{2}\frac{\bar{k}_1}{2}\right)^2 = -2\beta$$

we see a complete agreement with references [7, 8].

Recalling transformations (5.8) we find for $y = e_1/2x$:

$$g(y) = \frac{e_1 + c_1 + 2c}{2x}$$

Substituting c by

$$c = -\frac{1}{2x}(e_{1x} + e_1c_1 + k) = -\frac{1}{2x}(2xy_x + 2y + 2xyc_1 + k)$$

and c_1 , as follows from formula (5.5), by:

$$c_1 = \frac{1}{2x(1-y)} \left(2x^2(1-y) - x\frac{y_x}{y} + 2xy_x + 2y + \frac{1}{2}\frac{\bar{k}_1}{y} + k - 1 \right)$$

yields:

$$g(y) = \frac{1}{2} + \frac{1}{4x^2(y-1)} \left(x\frac{y_x}{y} - \frac{1}{2}\frac{\bar{k}_1}{y} + k + \bar{k}_1 + 1 \right) \quad (5.11)$$

or

$$g(y(z)) = \frac{1}{2} + \frac{1}{4\sigma z(y-1)} \left(2z\frac{y_z}{y} - \frac{1}{2}\frac{\bar{k}_1}{y} + k + \bar{k}_1 + 1 \right) \quad (5.12)$$

after a change of variable $z = \sigma x^2$ with $\sigma^2 = 1$. Applying this transformation to solution $w = y/(y-1)$ of the Painlevé V equation (1.4) gives

$$g(w) = 1 - \frac{2z\sigma w}{F} \quad (5.13)$$

$$F = +zw_z - \frac{1}{2}w^2 \left(k + 1 + \frac{\bar{k}_1}{2} \right) + w \left(\frac{1}{2}(k + 1) + z\sigma \right) + \frac{1}{4}\bar{k}_1$$

In terms of quantities

$$c_g = \frac{1}{2} \left(k + 1 + \frac{\bar{k}_1}{2} \right), \quad a_g = \frac{1}{4}\bar{k}_1$$

with properties $c_g^2 = 2\alpha$, $a_g^2 = -2\beta$ the function F from relation (5.13) can be rewritten as

$$F = +zw_z - w^2c_g + w(c_g - a_g + z\sigma) + a_g$$

in complete agreement with expression for the Bäcklund transformation obtained in [7, 8]. Thus, this construction establishes Painlevé V equation with its Bäcklund symmetry structure as a limit of an integrable pseudo-differential Lax hierarchy. To the best of our knowledge, this is the first time such explicit derivation of a general Painlevé V equation with three independent Painlevé coefficients was carried out by taking a self-similarity limit of an integrable model defined by the pseudo-differential Lax operator $L = u_1(\partial - u_2)^{-1} + u_3 + \partial^2$.

6 Outlook

We have here derived the higher order Painlevé equations by taking a self-similarity limit of the special class of integrable models and showed how the extended affine Weyl groups $A_n^{(1)}$ symmetries are induced by the Bäcklund transformations generated by translations on the underlying “half-integer” Volterra lattice. The Hamiltonians of the integrable model reduced by the self-similarity procedure have been explicitly shown to transform under change of variables into the Hamiltonians for the higher Painlevé equations. In the forthcoming publication we plan to provide explicit proof for formulas governing such change of variables and include in the formalism the Painlevé equations with the extended affine Weyl groups $A_{2n-1}^{(1)}$ symmetries for $n > 2$. We will also employ a link between on the one hand integrable hierarchies and on the other hand the higher order Painlevé equations to derive the corresponding higher order Painlevé hierarchies generalizing the construction of the Painlevé IV hierarchy in [6] to higher orders. Another future goal is to use our approach to provide solutions of the higher order Painlevé equations by taking appropriate limits of soliton solutions to the original integrable models similarly to what was accomplished in [4] for the Painlevé IV equation.

Acknowledgments

JFG and AHZ thank CNPq and FAPESP for partial financial support. Work of HA was partially supported by FAPESP. HA thanks Nick Spizzirri for discussions. The authors thank Danilo Virges Ruy for discussions.

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