Dade’s Conjecture

in the Finite Special Unitary Groups

BY

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THESIS

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SUMMARY

We prove Dade’s ordinary conjecture (DOC) for the finite Special Unitary groups in the defining characteristic. We base our approach on the following existing work. Olsson and Uno proved DOC for the finite General Linear groups. Sukizaki then modified their approach in order to show that DOC holds for the finite Special Linear groups. The key aspect of his approach is to reformulate the alternating sum for $\text{SL}_n(q)$ in terms of an alternating sum for $\text{GL}_n(q)$ via Clifford theory. Chao Ku modified Olsson and Uno’s work to show DOC for the finite Unitary groups in his Caltech Ph.D. thesis. In this thesis, we take Sukizaki’s method and apply it to Ku’s work. Some modifications are necessary as the structure of the finite unitary groups is more complicated than that of the finite general linear groups. In particular the cancellations, in the alternating sum in the statement of DOC, in the unitary case are very different from the cancellations that occur in the general linear case.

The combinatorial details involved in counting characters of parabolic subgroups are fairly involved. Hence we devote a chapter by way of example to the manageable case when $n = 4$. This is the smallest case which is just large enough to be non trivial with respect to the cancellation in the sum.
CHAPTER 1

INTRODUCTION

Let $G$ be a finite group. An ordinary representation of $G$ is a group homomorphism

$$
\rho : G \rightarrow \text{GL}(V)
$$

where $V$ is a finite dimensional vector space over a field $K$ of characteristic zero. Then $V$ is the $G$-module, or $KG$-module, afforded by $\rho$. If we fix a basis for $V$, then $\text{GL}(V)$ is isomorphic to $\text{GL}_n(K)$. Then the degree of the representation $\rho$ is $n$. We define the character $\chi$ of $\rho$ by $\chi(g) = \text{Tr}(\rho(g))$. In general, though we may consider $K$ to be the field of complex numbers, in practice we may take $K$ to be a sufficiently large extension of $\mathbb{Q}$. A character $\chi$ is irreducible if the associated vector space $V$ afforded by $\rho$ has no proper nonzero $KG$-submodules. We will denote the ordinary irreducible characters of $G$ by $\text{Irr}(G)$. The set $\text{Irr}(G)$ forms an orthonormal basis for the $K$-space of class functions on $G$.

Throughout this thesis we assume that $p$ is a fixed prime number. A modular representation of $G$ is defined similarly, the key difference being that the module afforded is a vector space over a field of characteristic $p$. A modular representation of $G$ is a group homomorphism

$$
\rho : G \rightarrow \text{GL}(W)
$$
where $W$ is a vector space over a field $k$ of characteristic $p$. Then $W$ is the $G$-module, or $kG$-module, afforded by $\rho$. Restricting our attention to the elements of $G$ with order prime to $p$, we may define a so called Brauer character, a complex valued class function on elements with order prime to $p$.

The philosophy of modular representations goes back to Brauer and in part relates “global information” to “$p$-local information”. By global we mean the group $G$, and by $p$-local we mean subgroups related to $p$-subgroups of $G$, for example normalizers of $p$-subgroups. Information can refer to a variety of things including numbers of characters or character values.

Modular representations of $G$ give rise to a partition of $\text{Irr}(G)$ into blocks. Dade’s Conjecture was first presented in a series of papers entitled *Counting Characters in Blocks*. The interplay between global and local blocks is informative in both directions. The conjecture involves an alternating sum involving characters of $p$-local subgroups. In order to state Dade’s fairly elaborate conjecture we must first assemble some concepts.

1 Modular Representations

Let $K$ be an algebraic number field, a splitting field for $G$ and its subgroups. Let $\mathcal{O}$ be the ring of algebraic integers in $K$. Let $\mathcal{P}$ be a prime ideal in $\mathcal{O}$ containing $p$. Let $\mathcal{R}$ be the ring of $\mathcal{P}$-integral elements of $K$, i.e. the localization of $\mathcal{O}$ at $\mathcal{P}$. Let $\pi\mathcal{R}$ be the unique maximal ideal in $\mathcal{R}$ with respect to a valuation associated with the prime ideal $\mathcal{P}$. We define $k$ to be the residue field $\mathcal{R}/\pi\mathcal{R}$ which is isomorphic to $\mathcal{O}/\mathcal{P}$. Then $k$ has characteristic $p$. The triple
$(K, \mathcal{R}, k)$ is called a $p$-modular system. We may look at the following group algebras: $KG$, $\mathcal{R}G$, and $kG$.

We use $\mathcal{R}G$ in order to pass from ordinary to $p$-modular representations of $G$. As an $\mathcal{R}$-module, the $\mathcal{R}$-algebra $\mathcal{R}G$ is free and of finite rank. We have

$$KG = K \otimes_\mathcal{R} \mathcal{R}G$$

and

$$kG = k \otimes_\mathcal{R} \mathcal{R}G = \mathcal{R}G / \pi \mathcal{R}G.$$ 

A finitely generated $\mathcal{R}$-free $\mathcal{R}G$-module is called an $\mathcal{R}G$-lattice. If $V$ is a $KG$-module then there exists an $\mathcal{R}G$-lattice $M$ with

$$K \otimes_\mathcal{R} M \cong V.$$ 

Note that this is not unique as it depends on our choice of basis for $V$. Set

$$\overline{M} = k \otimes_\mathcal{R} M = M / \pi M$$

as $kG$-module.

Then $\overline{M}$ is a modular representation, or module, of $G$. The composition factors of $\overline{M}$ are unique up to isomorphism and do not depend on the choice of $M$.

The $K$-algebra $KG$ is semi-simple and thus completely reducible. However $kG$ is not semi-simple if $p \mid |G|$. Rather it can only be written as a sum of indecomposable two sided ideals.
These indecomposable subalgebras of $kG$ are called the $p$-blocks of $kG$. A decomposition of $kG$ into blocks

$$kG = B_1 \oplus B_2 \oplus \cdots \oplus B_s$$

corresponds to a decomposition of the identity $1 = e_1 + e_2 + \cdots + e_s$ where the $e_i$ are orthogonal primitive central idempotents in $kG$. This is given by $B_i = e_i kG$. The class sums of elements in $G$ form a basis for both $Z(\mathcal{R}G)$ and $Z(kG)$. Hence reducing mod $\pi \mathcal{R}$ is a surjective map $Z(\mathcal{R}G) \rightarrow Z(kG)$. We may lift the $e_i$ to orthogonal primitive central idempotents $f_i$ in $\mathcal{R}G$. Then the decomposition $1 = f_1 + f_2 + \cdots + f_s$ in $\mathcal{R}G$ corresponds to a decomposition of $\mathcal{R}G$ into two sided ideals also called the $p$-blocks of $\mathcal{R}G$.

$$\mathcal{R}G = \hat{B}_1 \oplus \hat{B}_2 \oplus \cdots \oplus \hat{B}_s$$

where $\hat{B}_i = f_i \mathcal{R}G$ and $B_i = k \otimes_{\mathcal{R}} \hat{B}_i$.

If $V$ is an irreducible $KG$-modules then $f_i V = V$ for a unique $f_i$ and $f_j V = 0$ for all $j \neq i$. We say that $V$ belongs to the block $B = f_i KG$. If $V$ affords the character $\chi$, we also say that $\chi$ is in $B$. This gives rise to a partition of the set of ordinary characters of $G$ into blocks. We may informally think of a $p$-block $B$ of $G$ as simultaneously being all of the following related objects:

- an indecomposable two-sided ideal $kG$-module $e_i kG$ for primitive idempotent $e_i \in Z(kG)$
- an indecomposable two-sided ideal $\mathcal{R}G$-module $f_i \mathcal{R}G$ for primitive idempotent $f_i \in Z(\mathcal{R}G)$

where $f_i$ is the lift of $e_i$. 
• the set of irreducible $KG$-modules $V$ for which $f_i V = V$

• the set of ordinary characters of the $KG$-modules $V$ as above

The two-sided ideal summands of $kG$ (or $RG$) are the same as the direct summands of $kG$ (or $RG$) as a $k(G \times G)$ (or $R(G \times G)$) module where the action of $k(G \times G)$ (or $R(G \times G)$) on $kG$ (or $RG$) is given by $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$.

Let $B$ be a $p$-block of $G$. Then $B$ has associated to it a $p$-group $D$ called a defect group of $B$ and a non-negative integer called the defect of $B$. The subgroup $D$ is a minimal subgroup of $G$ such that every $B$-module is a direct summand of an induced module from $D$ ((19), p. 122). If $D$ is such a defect group and $|D| = p^d$ then $B$ has defect $d$. We define the defect of a character $\chi$ to be the maximum power of $p$ dividing $\frac{|G|}{\chi(1)}$. Clearly, the defect of a character is inversely related to the power of the $p$-part of its degree. If $B$ has defect $d$, then $B$ contains a character of defect $d$ and the defect of all other characters in $B$ is less than or equal to $d$.

We have two extremes. Write $|G| = p^e m$ where $p \nmid m$. If $B$ contains a linear character then the defect of $B$ is $e$. In this case we say $B$ has full defect. For example the block containing the trivial module $K$, equivalently containing the trivial character, is called the principal block and has full defect. If $B$ contains a character of degree divisible by $p^e$, then $B$ has zero defect.

It turns out that a block $B$ of defect zero contains exactly one character ((7), Proposition 56.31).

Brauer’s First Main Theorem states that if $D$ is a $p$-subgroup of $G$ then there exists a bijection between blocks $B$ of $G$ with defect group $D$ and blocks $b$ of $N_G(D)$ with defect group $D$. Let $H \leq G$ satisfy $DC_G(D) \leq H \leq N_G(D)$. Let $B$ be a block of $G$ and $b$ be a block of
H. We say \( b \) induces to \( B \) and write \( b^G = B \) if \( b \), as a \( k(H \times H) \)-module, is a summand in the restriction \( B_{H \times H} \) of the \( k(G \times G) \)-module \( B \) to \( H \times H \) and that \( B \) is the only block for which this holds ((1), p.101). For \( H \) as above, \( b^G \) is always defined.

2 Statement of Dade’s Ordinary Conjecture

Let \( G \) be a finite group and \( p \) a prime. Given a chain of \( p \)-subgroups \( C : U_0 < U_1 < \cdots < U_l \) in \( G \) we define the length of \( C \), \( |C| = l \). We say that \( C \) is radical if \( U_0 = O_p(G) \), the maximal normal \( p \)-subgroup of \( G \) and \( U_i = O_p(\cap_{j=0}^i N_G(U_j)) \) for \( 1 \leq i \leq l \). Let \( N_G(C) \) denote \( \cap_{j=0}^l N_G(U_j) \). Observe that if two chains \( C_1 \) and \( C_2 \) are conjugate to one another, then \( N_G(C_1) \cong N_G(C_2) \). If \( b \) is a \( p \)-block of \( N_G(C) \), then \( b^G = B \) is defined. Let

\[
\text{Irr}(N_G(C), B, d) = \{ \psi \in \text{Irr}(N_G(C)) \mid \psi \in b \text{ where } b^G = B \text{ and } \psi \text{ has defect } d \}.
\]

We will set \( k(N_G(C), B, d) = |\text{Irr}(N_G(C), B, d)| \).

**Conjecture 1.2.1** Dade’s Ordinary Conjecture (DOC) ((8), Conjecture 6.3). Let \( G \) be a finite group with \( O_p(G) = 1 \) so that all radical chains in \( G \) begin with the trivial group \( U_0 = 1 \). Let \( B \) be a block of \( G \) of nonzero defect. Then the following holds:

\[
\sum' (-1)^{|C|} k(N_G(C), B, d) = 0, \quad \forall d \geq 0
\]

where \( |C| = l \) is the length of \( C \) and \( \sum' \) indicates the sum over a set of representatives of conjugacy classes of radical chains in \( G \).
3 Refinement of DOC for Certain Finite Reductive Groups

DOC reduces nicely for certain finite reductive groups in the defining characteristic. Let $G$ be a finite reductive group of characteristic $p$. Then $G$ is the group of fixed points of a Frobenius endomorphism of a connected reductive algebraic group. We consider the $p$-blocks of $G$. Let $I$ be an index set for the distinguished generators of the Weyl group $W$ of $G$. Let $B$ be a Borel subgroup of $G$. In this thesis $P_J$ will denote the parabolic subgroup $BW_{I \setminus J}B$. For example, $P_I = B$ is the Borel subgroup of $G$ (rather than $P_\emptyset$). It is also useful to think of parabolics indexed in the following way: If $\{P_j \mid j \in I\}$ is a complete set of maximal parabolic subgroups in $G$, then

$$P_J = \bigcap_{j \in J} P_j.$$

Let $C : U_0 < U_1 < \cdots < U_l$ be a radical chain of $p$-subgroups in $G$. Then $U_0 = O_p(G) = 1$. Moreover, $U_1 = O_p(N_G(U_1))$ and hence, by ((4), Corollary), $U_1$ must be the unipotent radical $U_J$ of a parabolic subgroup $P_J$ of $G$ with $N_G(U_J) = P_J$. We have the familiar Levi decomposition $P_J = L_J U_J$. It is obvious that $U_1 \subseteq B$. Notice that $P_J / U_J \cong L_J$ is itself a finite group of Lie type with Borel subgroup isomorphic to $B \cap L_J$. The quotient $U_2 / U_J$ is isomorphic to a $p$-group of $B \cap L_J$ and hence is isomorphic to a unipotent radical of a parabolic subgroup of $L_J$. Since $U_2 = O_p(P_J \cap N_G(U_2)) = O_p(N_{P_J}(U_2))$, we must have $U_2 = U_{J'}$ where $J' \supset J$. Hence $C$ is a chain of unipotent radicals and $N_G(C)$ is equal to $N_G(U_l)$ the normalizer of the last term so that $N_G(C) = P_J$ for suitable $J$ depending only on the last term of the chain $C$. 
It turns out that there is considerable cancellation amongst the $G$-conjugacy classes of chains of unipotent radicals for $G$. The collection of all such chains $C$ which terminate with a fixed $U_J$ and thus have $N_G(C) = P_J$ cancels almost entirely, due to the alternating parity of the involved chains. One uncancelled chain remains of maximal length $J$. By a standard argument ((14), p.58),

$$
\sum_{C} (-1)^{|C|} k(N_G(C), B, d) = \sum_{J \subseteq I} (-1)^{|J|} k(P_J, B, d)
$$

where the sum on the left is taken over a set of representatives of $G$-conjugacy classes of chains of unipotent radicals.

The possible defect of $p$-blocks is well known for finite groups of Lie type, otherwise known as finite reductive groups, of characteristic $p$ ((13)). The only possibilities are blocks of zero defect and blocks of full defect. In Humphreys’ concluding remarks he notes that the number of blocks of zero defect is equal to the index of the derived subgroup $G'$ in $G$ and that the number of block of full defect is equal to the order of the center of $G$.

Let us now restrict our attention to $G = \text{GL}_n(q)$, $\text{SL}_n(q)$, $\text{U}_n(q)$, or $\text{SU}_n(q)$, where $q$ is a power of $p$. In fact we have a bijection from the set of $p$-blocks of $P_J$ to the set of $p$-blocks of $G$ of full defect. Indeed, the center of $G$ is a torus and hence has order prime to $p$. Hence $O_p(Z(G)) = Z(G)$ and $O_p(Z(G)) = 1$. Let $Z(G) = Z$. The group $Z$ centralizes $U_J$ so $U_JZ \subseteq U_JC_G(U_J)$ certainly holds. It happens that $U_JC_G(U_J) \subseteq U_JZ$ holds for these four families of groups. Thus by ((16), Lemma 2.1), $\psi$ and $\psi'$ lie in the same block $b$ of $P_J$ if and
only if their restrictions to \( Z \) have the same constituent. In other words, \( P_J \) has \(|Z|\) blocks and a block \( b \) of \( P_J \) is determined by a unique character \( \rho \in \text{Irr}(Z) \). The induced block \( b^G = B \) is defined. \( B \) has full defect and is determined by the same \( \rho \). The proof for \( U_n(q) \) is analogous to the proof for \( \text{GL}_n(q) \) in ((16), Lemma 2.1). If \( \psi \in \text{Irr}(P_J) \) restricted to \( Z \) contains \( \rho \) we will say that \( \psi \) lies over \( \rho \).

Write \(|G| = p^e m\). Each parabolic subgroup \( P_J \) contains \( U_I \) the unipotent radical of the Borel subgroup of \( G \). This is a Sylow \( p \)-subgroup of \( G \). Hence \(|P_J|\) is divisible by \( p^e \) for every \( J \subseteq I \). As noted above the defect of a character is inversely related to the power of \( p \) dividing its degree. If the \( p \)-part of \( \psi(1) \) is \( p^a \) for \( \psi \in \text{Irr}(P_J) \), then the defect of \( \psi \) is \( e - a \). Hence it is equivalent to count characters by their so called \( p \)-height rather than their defect.

**Definition 1.3.1** We define the \( p \)-height of \( \psi \) to be \( d \) if \( p^d \| \psi(1) \). Similarly, we define the \( q \)-height \( \psi \) to be \( d \) if \( q^d \| \psi(1) \).

**Remark:** This definition is not entirely standard. In the literature \( p \)-height is generally defined with reference to the defect of the block containing the character. For example Brauer’s definition of height in his Height Conjecture is more standard. However if \( \psi \) is in a block of full defect, then the \( p \)-height as it is usually defined is equal to the maximal power of \( p \) dividing \( \psi(1) \) and hence our definition coincides with the standard.

Let \( \rho \) be an irreducible character of the center of \( G \) and define

\[
k_d(P_J, \rho) = \left| \{ \psi \in \text{Irr}(P_J) \mid \psi \text{ lies over } \rho \text{ and } p^d \| \psi(1) \} \right|.
\]
Then DOC is equivalent to the following:

**Conjecture 1.3.2** Let $q = p^a$. Let $G = \text{GL}_n(q)$, $\text{SL}_n(q)$, $\text{U}_n(q)$, or $\text{SU}_n(q)$, with parabolic subgroups $P_J$ indexed by subsets $J \subseteq I$. Let $Z$ be the center of $G$. Let $|G| = p^e m$ where $p \nmid m$.

Then $\forall \rho \in \text{Irr}(Z)$

$$\sum_{J \subseteq I} (-1)^{|J|} k_d(P_J, \rho) = 0 \forall d \leq d < e.$$ 

If $G$ is either of $\text{GL}_n(q)$ or $\text{U}_n(q)$, then the $p$ part of the degree of characters in $\text{Irr}(G)$ are powers of $q$. Indeed, this is well known for $\text{GL}_n(q)$ and follows for $\text{U}_n(q)$ by replacing $q$ in the $\text{GL}_n(q)$-character theory by $-q$ in the $\text{U}_n(q)$-theory, by Ennola’s conjecture, now proved. For $G = \text{GL}_n(q)$, or $\text{U}_n(q)$, let $S = \text{SL}_n(q)$, or $\text{SU}_n(q)$ respectively. The group $S$ is the kernel of the determinant map on $G$. Moreover the quotient $G/S$ is cyclic of order $q - 1$ or $q + 1$ respectively, in either case prime to $p$. Take any $\psi \in \text{Irr}(S)$, then there exists $\chi \in \text{Irr}(G)$ such that $\psi$ is a constituent of the restriction of $\chi$ to $S$. By Frobenius reciprocity, we may choose any irreducible $\chi$ appearing in the induction of $\psi$ from $S$ to $G$. Then, by 2.4.1 and 2.4.4 in the next chapter,

$$\chi|_S = \psi_1 + \psi_2 + \cdots + \psi_r$$ 

where the $\psi_i \in \text{Irr}(S)$ are $G$-conjugates of $\psi$

and $r$ divides $|G/S|$. Thus

$$\chi(1) = \psi_1(1) + \psi_2(1) + \cdots + \psi_r(1).$$
Since $r$ is prime to $p$, it follows from Clifford theory that the $p$-height of $\psi$ is equal to the $p$-height of $\chi$ and hence is also a power of $q$. Suppose $\chi \in \text{Irr}(G)$, or $\text{Irr}(S)$ has $p$-height $d$. Then $d$ is certainly divisible by $a$ so $\chi$ has $q$-height $d/a$. It turns out that from Olsson and Uno’s construction for $\text{GL}_n(q^2)$ and Ku’s construction for $\text{U}_n(q)$ the characters of parabolic subgroups of $G$ also have degrees with $p$ part equal to a power of $q$. As we will see in chapter 4, parabolic subgroups of $S$ are in fact the kernel of the determinant map restricted to parabolic subgroups of $G$. Thus, by the same reasoning as above, they also have degrees with $p$ part equal to a power of $q$. Hence in statement 1.3.2 of DOC, for $d$ not divisible by $a$ the left hand side of the sum is empty and so vacuously true. This allows us to simplify our notation by counting characters via $q$-height rather than $p$-height. Henceforth and for the rest of this thesis we redefine the subscript $d$ so that it indicates $q$-height so for example $\text{Irr}_d(P_J, \rho)$ will denote irreducible characters of $P_J$ lying over $\rho$ with $q$-height $d$.

4 Some Results for Dade’s Conjecture and Implications

We summarize the cases for which some version of Dade’s conjecture has been shown, including the result of this thesis. References for this section are ((12), Section 5) and on the web at ((17)).
1. Classical Groups:

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<td>ord., $p \mid q$</td>
<td>Olsson, Uno</td>
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<tr>
<td>$U_n(q)$</td>
<td>ord., $p \mid q$</td>
<td>Ku</td>
</tr>
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<td>$GL_n(q), U_n(q)$</td>
<td>invar., $p \nmid q$</td>
<td>An</td>
</tr>
<tr>
<td>$Sp_{2n}(q), SO_{2n}^+(q)$</td>
<td>ord., $p \nmid q, p, q$ odd</td>
<td>An, Dade</td>
</tr>
<tr>
<td>$L_2$</td>
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<td>Dade</td>
</tr>
<tr>
<td>$L_3$</td>
<td>final, $p \mid q$</td>
<td>Sukizaki</td>
</tr>
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<td>$L_n$</td>
<td>ord., $p \mid q$</td>
<td>An, Himstedt, Huang</td>
</tr>
<tr>
<td>$Sp_4(2^n)$</td>
<td>final, $p = 2$</td>
<td>An, Himstedt, Huang</td>
</tr>
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<td>$SU_4(2^{2n})$</td>
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<tr>
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<tr>
<td>$SU_n(q)$</td>
<td>ord., $p \mid q$</td>
<td>Bird</td>
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2. Sporadic Simple Groups:

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<td>final</td>
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<td>$M_{22}$</td>
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<td>Huang</td>
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<tr>
<td>$M_{23}$, $M_{24}$</td>
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<td>Schwartz, An, Conder</td>
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<td>Kotlica</td>
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<td>$McL$</td>
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<td>Murray, Entz, Pahlings</td>
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<td>$Ru$</td>
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<td>Dade, An, O’Brien</td>
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<td>$He$</td>
<td>final</td>
<td>An</td>
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<tr>
<td>$HS$</td>
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<td>Hassan, Horváth</td>
</tr>
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<td>$Co_1$</td>
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<td>$Suz$</td>
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<td>$O’N$</td>
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<td>Uno</td>
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<td>$Fi_{23}$</td>
<td>final</td>
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<td>$Fi_{22}$</td>
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<td>$B$</td>
<td>$p$ odd</td>
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<td>$Fi'_{24}$</td>
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3. Exceptional Groups:

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<tr>
<td>$2B_2(2^{2n+1})$</td>
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<td>Dade</td>
</tr>
<tr>
<td>$2G_2(3^{2n+1})$</td>
<td>final</td>
<td>An, Eaton</td>
</tr>
<tr>
<td>$G_2(q)$</td>
<td>final, $2, 3 \mid q$, $p \nmid q$, $q \neq 3, 4$</td>
<td>An</td>
</tr>
</tbody>
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| $G_2(q)$        | final, $p 
mid q(p \geq 5)$, $q = 3, 4$ | Huang        |
| $^3D_4(q)$      | final, $p \nmid q$    | An           |
| $^3D_4(q)$      | final, $p \mid q$     | An, Himstedt, Huang |
| $^2F_4(2^{2n+1})$ | ord., $p \neq 2$     | An           |
| $^2F_4(2^{2n+1})$ | final, $p = 2$       | Himstedt, Huang |
| $^2F_4(2)'$    | final                  | An           |

4. Other cases:

<table>
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<tbody>
<tr>
<td>$S_n$</td>
<td>ord., $p \neq 2$</td>
<td>Olsson, Uno</td>
</tr>
<tr>
<td>$S_n$</td>
<td>ord., $p = 2$</td>
<td>An</td>
</tr>
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<td>$A_n$, abelian defect</td>
<td>ord.</td>
<td>Fong, Harris</td>
</tr>
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<td>Cyclic defect group</td>
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<td>Dade</td>
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<td>Tame block</td>
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<td>Uno</td>
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<td>Abelian defect unipotent blocks</td>
<td>ord.</td>
<td>Broué, Malle, Michel</td>
</tr>
<tr>
<td>Abelian defect principal blocks</td>
<td>ord., $p = 2$</td>
<td>Fong, Harris</td>
</tr>
<tr>
<td>Abelian defect some cases</td>
<td>ord.</td>
<td>Ping, Usami</td>
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<td>$p$-solvable</td>
<td>proj.</td>
<td>Robinson</td>
</tr>
<tr>
<td>$O_p(G)$ cyclic, $O_p(G)/G$ $p$-Sylow TI</td>
<td>proj.</td>
<td>Eaton</td>
</tr>
<tr>
<td>Nilpotent blocks</td>
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</table>

The sequence of most interest with respect to this thesis is the following: Olsson and Uno proved DOC for $GL_n(q)$ in the defining characteristic (16). Sukizaki proved it for $SL_n(q)$ also in the defining characteristic (22). Chao Ku verified DOC in his doctoral thesis for $U_n(q)$.

Assuming that Dade’s Ordinary Conjecture is true for all finite groups implies a number of other conjectures. In this sense DOC encodes a variety of information. DOC implies Alperin’s Weight Conjecture which counts Brauer characters. DOC implies the Alperin-McKay Conjecture which is a refinement of the McKay Conjecture. DOC also implies one direction of Brauer’s Height Conjecture which involves abelian defect groups.
CHAPTER 2

PRELIMINARIES

We begin these preliminaries with our definition of the finite unitary and special unitary groups. Throughout this thesis $q$ is a fixed power of the prime $p$. Let $K = \overline{\mathbb{F}}_q$, and $\tilde{G} = \text{GL}_n(K)$.

Define the matrix

$$M = \begin{pmatrix}
0 & \ldots & 0 & 1 \\
0 & \ldots & 1 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
1 & \ldots & 0 & 0
\end{pmatrix}.
$$

Define the following Frobenius map $F$ on $\tilde{G}$ by:

$$F(a_{i,j}) = M(a_{j,i}^q)^{-1}M^{-1}.$$

The group of fixed points $\tilde{G}^F$ is the finite unitary group $U_n(q)$, i.e.

$$U_n(q) = \{(a_{i,j}) \mid M = (a_{i,j})M(a_{j,i}^q)\}.$$
Clearly $U_n(q) \leq GL_n(q^2)$. The advantage of this definition is that $F$ fixes the subgroup of upper triangular matrices in $GL_n(q^2)$. We can define the special unitary groups in two equivalent ways. On the one hand, the group of fixed points of $SL_n(K)$ under $F$ is $SU_n(q)$. On the other hand,

$$SU_n(q) = \{ A \in U_n(q) \mid \det(A) = 1 \}.$$ 

The Weyl group $W$ of $U_n(q)$ is of type $B_m$, where $n = 2m$, or $2m + 1$, and is isomorphic to the wreath product $C_2 \wr S_m$. The symmetric group on $m$ elements is generated by reflections indexed by $\{1, 2, \ldots, m - 1\}$ and the cyclic group of order 2 is generated by the reflection indexed by $\{m\}$. With this identification, the distinguished generators of $W$ may be indexed by $I = \{1, 2, \ldots, m\}$ denoted by $[m]$.

1 **Some Notation**

Throughout this thesis we will make use of the following notation. Let $q$ be a fixed power of prime $p$. We consider the finite field $F_{q^2}$ and its group of units $F_{q^2}^*$. For divisors $h$ of $q^2 - 1$, let $C_h$ denote the cyclic subgroup of order $h$ in $F_{q^2}^*$. So in particular $C_{q+1}$ denotes the cyclic subgroup of order $q + 1$ in $C_{q^2 - 1}$.

2 **On radical $p$-chains**

In order to reformulate DOC for the finite special unitary groups we will need the following proposition due to Sukizaki.
Proposition 2.2.1  ((22), Proposition 2.1) Let \( G \) be a finite group and let \( H \) be a subgroup of \( G \). If \( H \) contains all \( p \)-subgroups of \( G \) and satisfies \( O_p(G) = O_p(H) \), then any radical \( p \)-chain of \( H \) is a radical \( p \)-chain of \( G \).

3 Certain functions on partitions

In this section we are following the development of Olsson and Uno ((16)), Sukizaki ((22)), and Ku ((15)). To that end we discuss partitions. Further we define two important functions \( \alpha \) and \( \beta \) on pairs \((\mu, a)\) where \( \mu \) is a partition and \( a \) is a field element. The function \( \alpha \) was defined in ((16), p.363). The function \( \beta \) was introduced as a unitary version of \( \alpha \) and was defined in ((15), p.16). These functions are involved in expressing the \( q \)-height of characters. Further, it turns out that they are also involved in the splitting of characters upon restriction to certain subgroups. We assert some combinatorial facts about the behavior of these functions.

Let \( \mu = (a^1_1, a^2_2, \ldots, a^r_r) \vdash n \), where \( a_1 > a_2 \cdots > a_r > 0 \). We define \( |\mu| = \sum_{i=1}^r l_i a_i = n \).

Let the number of distinct parts of the partition be \( \delta(\mu) = r \) and the length of the partition \( l(\mu) = \sum_{i=1}^r l_i \). We define \( \gamma(\mu) = \gcd(a_1, a_2, \ldots, a_r) \) and \( \lambda(\mu) = \gcd(l_1, l_2, \ldots, l_r) \).

Given \( \mu_1 \vdash n_1 \) and \( \mu_2 \vdash n_2 \) we can define \( 2\mu_1 \cup \mu_2 \vdash n = 2n_1 + n_2 \). In order to define this new partition write \( \mu_i = (1^{m_{i1}}, 2^{m_{i2}}, \ldots, n_i^{m_{in_i}}) \), so that for nonzero \( m_{it} \), the integer \( t \) appears in \( \mu_i \) with multiplicity \( m_{it} \). Then define \( 2\mu_1 \cup \mu_2 = (1^{2m_{11}+m_{21}}, 2^{2m_{12}+m_{22}}, \ldots, n^{2m_{1n}+m_{2n}}) \).

Definition 2.3.1 Let \( a \in \mathbb{C}_{q^2-1} \) and \( \mu = (a^1_1, a^2_2, \ldots, a^r_r) \vdash n \). We define a function \( \alpha \) by

\[
\alpha(\mu, a) = |\{(x_1, x_2, \ldots, x_r) \in (\mathbb{C}_{q^2-1})^r \mid (-1)^n x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r} = a\}|.
\]
Lemma 2.3.2 With $\mu$ defined as above and $a \in \mathbb{C}_{q^2-1}$ we have

$$\alpha(\mu, a) = (q^2 - 1)^{r-1} \alpha(\gamma, a)$$

where $\gamma = \gamma(\mu)$ and

$$\alpha(\gamma, a) = \begin{cases} 
\gcd(q^2 - 1, \gamma), & \text{if } a \in \mathbb{C}_{q^2-1}/\gcd(q^2-1, \gamma); \\
0, & \text{otherwise.} 
\end{cases}$$

See ((16), p.363) for the proof.

Definition 2.3.3 Let $b \in \mathbb{C}_{q+1}$ and $\mu = (a_1^1, a_2^1, \ldots, a_r^1) \vdash n$. We define a function $\beta$

$$\beta(\mu, b) = |\{(x_1, x_2, \ldots, x_r) \in (\mathbb{C}_{q+1})^r \mid (-1)^n x_1^{a_1^1} x_2^{a_2^1} \ldots x_r^{a_r^1} = b\}|.$$ 

Lemma 2.3.4 With $\mu$ defined as above and $b \in \mathbb{C}_{q+1}$ we have

$$\beta(\mu, b) = (q + 1)^{r-1} \beta(\gamma, b)$$

where $\gamma = \gamma(\mu)$ and

$$\beta(\gamma, b) = \begin{cases} 
\gcd(q + 1, \gamma), & \text{if } b \in \mathbb{C}_{q+1}/\gcd(q+1, \gamma); \\
0, & \text{otherwise.} 
\end{cases}$$

The proof is similar to the proof of Lemma 2.3.2.
In practice we will be restricting our attention to elements $b$ in $\mathbb{C}_{q+1}$ and hence have need of the following modification of our $\alpha$ function which was defined in ((15), p.16).

**Definition 2.3.5** For $b \in \mathbb{C}_{q+1}$ and $\mu = (a_1^{l_1}, a_2^{l_2}, \ldots, a_r^{l_r}) \vdash n$ we define

$$\beta(\mu, b) = \sum_{a \in \mathbb{C}_{q^2-1}} \alpha(\mu, a).$$

Some important technical facts from ((15), (16), and (22)) regarding $\alpha$, $\beta$, and $\beta$ are summarized in the following lemmas.

**Lemma 2.3.6** If $(k) \vdash k$ and $\mu = (a_1^{l_1}, a_2^{l_2}, \ldots, a_r^{l_r}) \vdash n$, then

$$\sum_{b_1, b_2 \in \mathbb{C}_{q+1}} \beta((k), b_1) \beta(\mu, b_2) = \beta(\lambda, b)$$

where $\lambda = ((a_1 + k)^{l_1}, (a_2 + k)^{l_2}, \ldots, (a_r + k)^{l_r}, k^x) \vdash (n + (l(\mu) + x)k)$.

**Lemma 2.3.7** If $\mu_i \vdash n_i$, for $i = 1, 2$, and $\mu = 2\mu_1 \cup \mu_2 \vdash n = 2n_1 + n_2$, then

$$\sum_{b_1, b_2 \in \mathbb{C}_{q+1}} \beta(\mu_1, b_1) \beta(\mu_2, b_2) = (q - 1)^{\delta(\mu_1)}(q + 1)^{c(\mu_1, \mu_2)} \beta(\mu, b)$$

where $c(\mu_1, \mu_2)$ is the number of distinct entries that $\mu_1$ and $\mu_2$ have in common.
Lemma 2.3.8 If $\mu \vdash n$, then

$$\sum_{(\mu_1, \mu_2) \atop \mu = 2\mu_1 \cup \mu_2} q^{2(l(\mu) - \delta(\mu))} (q - 1)^{\delta(\mu_1)} (q + 1)^{c(\mu_1, \mu_2)} = q^{l(\mu) - \delta(\mu)}.$$ 

Notice that this sum is taken over all pairs of partitions $(\mu_1, \mu_2)$ such that $\mu = 2\mu_1 \cup \mu_2$.

These results are proved in ((15)). We mention that the last 2.3.8 is proved by associating to the pair $(\mu_1, \mu_2)$ a matrix and its shadow which are defined as follows:

**Definition 2.3.9** For two partitions $\mu_i = (t^m_i)$, $i = 1, 2$, with $\mu = (2\mu_1 \cup \mu_2) \vdash n$ we define the $2 \times n$ matrix

$$A(\mu_1, \mu_2) = A = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \end{pmatrix}$$

Given such an $A$ we define the shadow of $A$ to be the $2 \times n$ matrix $(c_{ij})$ where the $ij$-entry

$$c_{ij} = \begin{cases} 1, & \text{if } m_{ij} \text{ is nonzero;} \\ 0, & \text{otherwise}. \end{cases}$$

Thus taking the sum over pairs $(\mu_1, \mu_2)$ in Lemma 2.3.8 is equivalent to taking the sum over possible matrices $A(\mu_1, \mu_2)$. We also remark that with this definition of the shadow, the number of entries that $\mu_1$ and $\mu_2$ have in common $c(\mu_1, \mu_2) = \sum_{t=1}^{n} c_{1t} c_{2t}$.

Suppose that we have a pair $(\mu_1, \mu_2)$ with $\mu = 2\mu_1 \cup \mu_2$ a partition of $n$ and all nonzero multiplicities of $\mu_i$, $i = 1, 2$, are divisible by a fixed integer $j$, i.e. $j | \gcd(\lambda(\mu_1), \lambda(\mu_2))$. First of
all it is clear that $j|\lambda(\mu)$. Write $\mu = (tm_t)$ so that $2m_{1t} + m_{2t} = m_t$. Observe that $tm_t \leq n$ must hold. In particular $(n/j)m_{n/j} \leq n$ implies that $m_{n/j}$ is the last possibly nonzero exponent in $\mu$. In other words the matrix $A(\mu_1, \mu_2)$ must have zero entries to the right of the $n/j$-column. Furthermore, $A$ may be decomposed:

$$A = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \end{pmatrix} = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} \begin{pmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \end{pmatrix} = jB.$$ 

If we remove the zero columns to the right of the $n/j$-column of $B$, we obtain $A(\kappa_1, \kappa_2)$, the $2$ by $n/j$ matrix associated to $\kappa_1 = (t^{k_{1t}})$ and $\kappa_2 = (t^{k_{2t}})$ for $1 \leq t \leq n/j$. If $\mu = (t^{2m_{1t}+m_{2t}})$, then $\kappa = (t^{2k_{1t}+k_{2t}}) = 2\kappa_1 \cup \kappa_2$, a partition of $n/j$. We have the following equalities:

$$l(\mu_1)/j = l(\kappa_1); \quad \delta(\mu_1) = \delta(\kappa_1);$$

$$c(\mu_1, \mu_2) = c(\kappa_1, \kappa_2); \quad l(\mu)/j = l(\kappa); \quad \delta(\mu) = \delta(\kappa).$$

Thus for a fixed partition $\mu \vdash n$

$$\sum_{\substack{\mu_1, \mu_2 \\
\mu \vdash 2\mu_1 \cup \mu_2 \\
j | \gcd(\lambda(\mu_1), \lambda(\mu_2))}} q^{2l(\mu_1)/j - \delta(\mu_1)}(q-1)^{\delta(\mu_1)}(q+1)^{c(\mu_1, \mu_2)}$$

$$= \sum_{\kappa_1, \kappa_2 \vdash n/j} q^{2l(\kappa_1) - \delta(\kappa_1)}(q-1)^{\delta(\kappa_1)}(q+1)^{c(\kappa_1, \kappa_2)}$$

Hence we have the following important corollary to 2.3.8:
Corollary 2.3.10 If $\mu \vdash n$, then

$$\sum_{\substack{(\mu_1, \mu_2) \\ \mu = \mu_1 \cup \mu_2 \\ j | \gcd(\lambda(\mu_1), \lambda(\mu_2))}} q^{2(\ell(\mu_1)/j - \delta(\mu_1))}(q - 1)^{\delta(\mu_1)}(q + 1)^c(\mu_1, \mu_2) = q^{\ell(\mu)/j - \delta(\mu)}.$$

4 Applications of the Clifford Theory

We will make abundant use of Clifford Theory. A reference for this section is ((7), Chapter 11). We summarize here the results that we need. In this section we assume that $G$ is a finite group with a normal subgroup $H$. For $\psi \in \text{Irr}(H)$ and $g \in G$ we define the character $^g\psi$ by

$$^g\psi(h) = \psi(g^{-1}hg)$$

for all $h \in H$. Let $T_G(\psi)$ denote the stabilizer of $\psi$ in $G$ so that

$$T_G(\psi) = \{ g \in G \mid ^g\psi = \psi \}.$$

We define a subset of characters in $\text{Irr}(G)$

$$\text{Irr}(G, \psi) = \{ \chi \in \text{Irr}(G) \mid (\chi|_H, \psi)_H \neq 0 \}.$$

If $\chi \in \text{Irr}(G, \psi)$ we will say that $\chi$ corresponds to $\psi$.

We define a subset of characters in $\text{Irr}(H)$

$$\text{Irr}(H, \chi) = \{ \psi \in \text{Irr}(H) \mid (\chi|_H, \psi)_H \neq 0 \}.$$
This definition is equivalent to saying that the \( \chi \) appear in the induced character of \( \psi \) to \( G \).

Thus we will say that \( \psi \) corresponds to \( \chi \) if \( \psi \in \text{Irr}(H, \chi) \).

**Theorem 2.4.1** ((7), Proposition 11.4) Let \( \psi \in \text{Irr}(H) \) and \( \chi \in \text{Irr}(G, \psi) \). Then

\[
\chi|_H = e \left( \sum_{x \in G/T_G(\psi)} x \psi \right)
\]

where \( e \) is a positive integer.

**Theorem 2.4.2** ((7), Theorem 11.5) Let \( \psi \in \text{Irr}(H) \) and suppose that \( \psi = \tilde{\psi}|_H \) for some character \( \tilde{\psi} \) of \( T_G(\psi) \), that is, suppose that \( \psi \) can be extended to a character \( \tilde{\psi} \) of \( T_G(\psi) \). Write \( T = T_G(\psi) \) then

\[
\text{Irr}(T, \psi) = \{ \theta \tilde{\psi} \mid \theta \in \text{Irr}(T/H) \}, \quad \text{and}
\]

\[
\text{Irr}(G, \psi) = \{ (\theta \tilde{\psi})^G \mid \theta \in \text{Irr}(T/H) \}.
\]

Here we regard \( \theta \) as a character of \( T \).

We have the following corollary which is a simple consequence of the transitivity of character induction.

**Corollary 2.4.3** If \( \psi \in \text{Irr}(H) \) and \( P \) is any subgroup of \( G \) that contains \( T_G(\psi) \) then

\[
|\text{Irr}(P, \psi)| = |\text{Irr}(G, \psi)|.
\]
Moreover, there is a 1-1 correspondence between characters in each of these sets given by

\[ \phi = (\theta \tilde{\psi})^P \leftrightarrow (\theta \tilde{\psi})^G = \chi. \]

If \( \phi \in \text{Irr}(P, \psi) \) corresponds to \( \chi \in \text{Irr}(G, \psi) \) and \( \chi(1) \) has \( p \)-part equal to \( p^d \) then the \( p \)-part of \( \phi(1) \) is \( p^{d-d'} \) where \( |P:G| \) has \( p \)-part \( p^{d'} \).

**Lemma 2.4.4** (23), Lemma 2.5) If \( G/H \) is cyclic then the following hold:

1. Characters of \( G \) restricted to \( H \) are multiplicity free. In other words \( e = 1 \) in 2.4.1.
2. Two characters of \( G \) either restrict to the same character of \( H \) or have disjoint irreducible components.
3. If \( \psi \in \text{Irr}(H) \) and \( \chi \in \text{Irr}(G, \psi) \) then \( |\text{Irr}(H, \chi)| = |G|/(|H||\text{Irr}(G, \psi)|) \).

We have another important consequence of Clifford Theory concerning when an extension of a character exists when \( G \) is a semi-direct product.

**Lemma 2.4.5** (22), Theorem 2.5) Let \( G \) be a finite group with \( G = P \ltimes M \).

1. If \( \tau \in \text{Irr}(M) \) is linear, then \( \tau \) extends to an irreducible character \( \tilde{\tau} \) of \( T = T_G(\tau) \).

Moreover

\[ \text{Irr}(G, \tau) = \{ (\theta \tilde{\tau})^G \mid \theta \in \text{Irr}(T/M) \}. \]

2. Let \( H \) be a normal subgroup of \( G \) containing \( M \) and suppose that \( G/H \) is cyclic. If \( \theta \in \text{Irr}(T/M) \), then

\[ \text{Irr}(H, (\theta \tilde{\tau})^G) = \text{Irr}(T_H(\tau), \theta). \]
We will be interested in the existence of extensions of non-linear characters of certain normal subgroups. The following result of Dade’s on the extendibility of characters of normal extraspecial $p$-subgroups is certainly relevant.

**Lemma 2.4.6** ((9)) Let $E$ be an extra special $p$-group and $G = H \ltimes E$ with $Z(E) \leq Z(G)$. Assume that for each normal $p'$-subgroup $K$ of $H$, the commutator subgroup $[K, E] = 1$. If $\psi \in \text{Irr}(E)$ is non-linear, then $\psi$ is extendible to $G$.

5 On a Product of Groups

We will be examining the splitting of characters of direct products upon restriction to certain normal subgroups. We note that if $G = G_1 \times G_2$, then an irreducible character of $G$ is of the form $\chi_1 \chi_2$ where $\chi_i \in \text{Irr}(G_i)$. We will have need of the following result.

**Lemma 2.5.1** ((22)) Let $G = G_1 \times G_2$ where the group homomorphism $\phi_i : G_i \rightarrow F_{q^2}^\times$ has image $C_{h_i}$ for $i = 1, 2$. Set

$$H = \{(g_1, g_2) \in G \mid \phi_1(g_1)\phi_2(g_2) = 1\}.$$

If $\chi_i$ has $m_i$ irreducible constituents upon restriction to $\ker \phi_i$, then $\chi = \chi_1 \chi_2$ restricted to $H$ has $m$ irreducible constituents, where

$$m = \frac{\gcd(m_1(q^2 - 1)/h_1, m_2(q^2 - 1)/h_2)}{\gcd((q^2 - 1)/h_1, (q^2 - 1)/h_2)}$$
6 Restriction of Characters to the Kernel of the Determinant Map

In this section we present results for $\text{GL}_n(q^2)$ and $\text{U}_n(q)$. Sukizaki’s result is proved using G.I. Lehrer’s work which uses an earlier parametrization of the characters of $\text{GL}_n(q)$. We will use the more modern approach of Deligne-Lusztig theory. However, our goal remains the same in that we construct sequences of polynomials corresponding to characters and use them to count the number of characters.

**Definition 2.6.1** Let $G = \text{GL}_n(q^2)$ or $\text{U}_n(q)$. Given a homomorphism $\phi : G \to (F_q^*)^*$ and $\rho \in \text{Irr}(Z(G))$, we define the following:

1. Let $\text{Irr}_d(G, \rho, \phi, j)$ be the set of irreducible ordinary characters $\chi$ of $G$ with $q$-height $d$ and lying over $\rho$ such that the restriction of $\chi$ to the kernel of the map $\phi$ has $j$ irreducible components.

2. Let $k_d(G, \rho, \phi, j)$ denote the number of irreducible ordinary characters $\chi$ of $G$ with $q$-height $d$ and lying over $\rho$ such that the restriction of $\chi$ to the kernel of the map $\phi$ has $j'$ irreducible components, where $j$ divides $j'$, i.e.

$$k_d(G, \rho, \phi, j) = \sum_{j' | j} |\text{Irr}_d(G, \rho, \phi, j')|.$$ 

We will be considering the determinant map on $\text{U}_n(q)$ and certain subgroups. For a matrix element $A$, $\det(A)$ denotes the usual matrix determinant. We will consider subgroups of $\text{U}_n(q)$
whose elements are block matrices. If $A$ is a block matrix with block matrices $A_1, A_2, \ldots, A_s$ down its diagonal then $\det(A) = \det(A_1) \det(A_2) \cdots \det(A_s)$. Moreover, if certain of the $A_i$ are repeated then $\det(A)$ may involve powers of $\det(A_i)$. Define $\det^h(A) = (\det(A))^h$. We apply definition 2.6.1 below with $\phi = \det$.

We now fix an isomorphism between $(\overline{F}_q)^*$ and $\text{Irr}((\overline{F}_q)^*)$ and consider it fixed for the rest of this thesis. In practice, we are primarily interested in the subgroup $F_{q^2}^*$. The group $Z = Z(\text{GL}_n(q^2)) \cong F_{q^2}^* = \mathbb{C}_{q^2-1}$. Further, assume that the induced isomorphism of $\mathbb{C}_{q^2-1}$ with $\text{Irr}(\mathbb{C}_{q^2-1})$ is given by the following. Let $\varepsilon$ generate $\mathbb{C}_{q^2-1}$. Define the isomorphism via

$$
\varepsilon \mapsto \rho_\varepsilon \text{ where } \rho_\varepsilon(\varepsilon) = e^{(2\pi i)/(q^2-1)}.
$$

Under this isomorphism, $\rho \in \text{Irr}(\mathbb{C}_{q^2-1})$ corresponds to $a_\rho \in \mathbb{C}_{q^2-1}$. Equivalently $a \in \mathbb{C}_{q^2-1}$ corresponds to $\rho_a \in \text{Irr}(\mathbb{C}_{q^2-1})$. This induces an isomorphism of $\text{Irr}(Z(U_n(q)))$ with $\mathbb{C}_{q+1}$.

The following integer valued function on partitions of $n$ is involved in the $q$-height of characters for $\text{GL}_n(q^2)$ and $U_n(q)$.

**Definition 2.6.2** We define $n'(\mu)$:

$$
n'(\mu) = \sum_{i=1}^r l_i \left( \binom{a_i}{2} \right).
$$
Our first proposition is a slight reformulation of Sukizaki’s result in (22). This is needed as our ground field is $F_{q^2}$ rather than $F_q$. For $\mu \vdash n$, recall $\lambda(\mu)$, $l(\mu)$, and $\delta(\mu)$ defined on page 16.

**Proposition 2.6.3** ((22), Lemma 4.1) Let $\rho \in \text{Irr}(\mathbb{C}_{q^2-1})$. Then

$$k_{2d}(\text{GL}_n(q^2), \rho, \det, j) = \sum_{\mu \vdash n \atop n'(\mu)=d \atop j|\gcd(q^2-1, \lambda(\mu))} q^{2l(\mu)/(j-\delta(\mu))} \alpha(\mu, a_\rho).$$

For $\rho \in \text{Irr}(\mathbb{C}_{q+1})$ we define

$$k_{2d}(\text{GL}_n(q^2), \rho, \det^{1-q}, j) = \sum_{\rho' \in \text{Irr}(\mathbb{C}_{q^2-1}) \atop \rho'|_{q+1}=\rho} k_{2d}(\text{GL}_n(q^2), \rho', \det^{1-q}, j).$$

Then since $(q^2 - 1)/(q - 1) = q + 1$, by Sukizaki’s equation following equation 3-5 in (22) we have a disjoint union

$$\text{Irr}_{2d}(\text{GL}_n(q^2), \rho', \det^{1-q}, j) = \bigsqcup_{j'=\gcd(q+1, j')} \text{Irr}_{2d}(\text{GL}_n(q^2), \rho', \det, j').$$

This together with our definition of $\overline{\beta}$ from earlier in this chapter implies the following corollary:
Corollary 2.6.4 Let $\rho \in \text{Irr}(C_{q+1})$. Then

$$k_{2d}(\text{GL}_n(q^2), \rho, \det^{1-q}, j) = \sum_{\substack{\mu \vdash n \mu' = d \\mu = d \\text{gcd}(q+1, \lambda(\mu))}} q^{2(l(\mu)/j - \delta(\mu))} \beta(\mu, a_\rho).$$

We present the case for $U_n(q)$ now for completeness. We will prove this in the next chapter.

Proposition 2.6.5 Let $\rho \in \text{Irr}(C_{q+1})$. Then

$$k_d(U_n(q), \rho, \det, j) = \sum_{\substack{\mu \vdash n \mu' = d \\mu = d \\text{gcd}(q+1, \lambda(\mu))}} q^{l(\mu)/j - \delta(\mu)} \beta(\mu, a_\rho).$$

Remark: Let $\chi$ be an irreducible character of $\text{GL}_n(q^2)$. If $\chi |_{\ker \det}$ has $j$ irreducible constituents, then $j$ divides $\gcd(q^2 - 1, n)$. ($(20)$, Theorem 4.7). If $\chi |_{\ker \det^{1-q}}$ has $j$ irreducible constituents, then $j$ divides $\gcd(q + 1, n)$. Now let $\chi$ be an irreducible character of $U_n(q)$. If $\chi |_{\ker \det}$ has $j$ irreducible constituents, then $j$ divides $\gcd(q + 1, n)$. An identical theorem for the unitary case may be obtained making the following simple modifications to the proof of ($(20)$, Theorem 4.7): Change Definition 4.6 by defining

$$M(d) = \{ A \in U_n(q) \mid \text{det } A = \xi^{dk}, k = 1, \ldots, (q + 1)/d \}$$

where $d = \gcd(n, q + 1)$ and $\xi = \epsilon^{1-q}$, a generator of the subgroup $C_{q+1}$ in $F_{q^2}^*$. Then in Lemma 4.6 and Theorem 4.7 replace $\text{GL}_n(q)$ with $U_n(q)$ and $\text{SL}_n(q)$ with $\text{SU}_n(q)$. 
In this chapter we prove Proposition 2.6.5. We start by parameterizing irreducible characters \( \chi \) of \( U_n(q) \) via pairs \((s, \lambda)\), and construct a unique sequence of polynomials \((h_1(x), h_2(x), \ldots)\) corresponding to \( \chi \). The subgroup \( SU_n(q) \) is normal in \( U_n(q) \) with cyclic quotient isomorphic to \( \mathbb{C}_{q+1} \) which acts naturally on \( \text{Irr}(SU_n(q)) \) via \( U_n(q) \)-conjugation. In the last chapter, we fixed an isomorphism \( \mathbb{C}^{q^2-1} \cong \text{Irr}(\mathbb{C}^{q^2-1}) \) and hence we have an isomorphism \( \mathbb{C}_{q+1} \cong \text{Irr}(\mathbb{C}_{q+1}) \). The group \( \text{Irr}(\mathbb{C}_{q+1}) \) acts on \( \text{Irr}(U_n(q)) \). Indeed, if \( \rho \in \text{Irr}(\mathbb{C}_{q+1}) \) then we have a corresponding linear character of \( U_n(q) \) also denoted by \( \rho \). Then \( \rho \in \text{Irr}(\mathbb{C}_{q+1}) \) acts on \( \text{Irr}(U_n(q)) \) by

\[ \chi \mapsto \rho \otimes \chi \text{ abbreviated by } \rho \chi. \]

Let \( \chi \in \text{Irr}(U_n(q)) \). By Clifford Theory, \( \chi \) restricted to \( SU_n(q) \) is multiplicity free. If

\[ \chi|_{SU_n(q)} = \psi_1 + \psi_2 + \cdots + \psi_j \quad \text{where } \psi_i \in \text{Irr}(SU_n(q)) \]

then the \( \psi_i \) are \( U_n(q) \)-conjugates of one another.
The following lemma uses the well-known fact on characters of finite groups that if \( G \) is a finite group and \( H \leq G \), and \( \vartheta, \eta \) are characters of \( H \), \( G \) respectively, then

\[
\eta \text{Ind}_H^G(\vartheta) = \text{Ind}_H^G(\eta|_H \vartheta).
\]

**Lemma 3.0.6** Let \( \chi, \chi' \in \text{Irr}(U_n(q)) \). Then \( \chi, \chi' \) have the same restriction to \( SU_n(q) \) if and only if \( \chi' = \rho \chi \) for some \( \rho \in \text{Irr}(C_{q+1}) \).

**Proof:** Let \( \chi' = \rho \chi \). Since \( SU_n(q) \) is in the commutator subgroup of \( U_n(q) \), \( \rho \) is trivial on \( SU_n(q) \) and hence \( \chi, \chi' \) have the same restriction to \( SU_n(q) \).

Suppose \( \psi \) is a common constituent of \( \chi, \chi' \) restricted to \( SU_n(q) \). Let \( T \) be the stabilizer of \( \psi \) in \( SU_n(q) \). Then \( \psi \) extends to \( \tilde{\psi} \in \text{Irr}(T) \), and we have

\[
\chi = \text{Ind}_{T}^{U_n(q)}(\tilde{\psi}\phi), \quad \chi' = \text{Ind}_{T}^{U_n(q)}(\tilde{\psi}\phi')
\]

where \( \phi, \phi' \) are lifts to \( T \) of characters of \( T/SU_n(q), \) denoted \( \phi_1, \phi'_1 \). Then \( \phi_1, \phi'_1 \) can be extended to characters \( \xi_1, \xi'_1 \) of \( U_n(q)/SU_n(q), \) which can be lifted to characters \( \xi, \xi' \) of \( U_n(q). \) Then we have

\[
\chi = \text{Ind}_{T}^{U_n(q)}(\tilde{\psi})\xi, \quad \chi' = \text{Ind}_{T}^{U_n(q)}(\tilde{\psi})\xi'.
\]

Thus \( \chi, \chi' \) differ by a linear character. Since every linear character of \( U_n(q) \) is of the form \( \rho z \) for some \( z \in \mathbb{C}_{q+1} \) we have \( \chi' = \rho \chi \).
Let $\mathcal{E} = \{\psi_1, \psi_2, \ldots, \psi_j\}$ and let $\mathcal{F} = \{\chi_1, \chi_2, \ldots, \chi_r\}$ where the $\chi_i$ are the constituents of the induced character $\text{Ind}_{\text{SU}_n(q)}^{\text{U}_n(q)}(\psi)$ for any $\psi \in \mathcal{E}$. Then $\mathcal{E}$ is a $C_{q+1}$-stable subset of $\text{Irr}(\text{SU}_n(q))$ and $\mathcal{F}$ is a $\text{Irr}(C_{q+1})$-stable subset of $\text{Irr}(U_n(q))$. Hence $r = (q+1)/j$ and our original character $\chi$ is stabilized by $\rho_z \in \text{Irr}(C_{q+1})$ where $z \in C_{q+1}$ is a primitive $j$-th root of unity, i.e.

$$\chi = \rho_z \chi.$$ 

This forces certain conditions on the coefficients in the polynomials in $(h_1(x), h_2(x), \ldots)$ corresponding to $\chi$ and allows us to count how many $\chi$ of a fixed $q$-height are fixed by a $j$-th root of unity.

1 **Pairs $(s, \lambda)$**

Let $K = \overline{F}_q$. Consider the algebraic group $\tilde{G} = \text{GL}_n(K)$ with Frobenius endomorphism defined in the last chapter, $F : \tilde{G} \to \tilde{G}$ by $F((a_{ij})) = M(a_{ji}^q)^{-1} M^{-1}$. Then let $G = \tilde{G}^F = U_n(q)$.

A reference for the following is ((11), section 1). A subgroup $L$ of $G$ is Levi if $L = \tilde{L}^F$ for some $F$-stable Levi subgroup $\tilde{L}$ of a parabolic subgroup $\tilde{P}$ of $\tilde{G}$. For a Levi subgroup $L$ of $G$, let $R_L^G$ be the additive operator from $X(L)$ to $X(G)$ defined in the Deligne-Lusztig theory, where $X(L)$ and $X(G)$ are the character rings of representations of $L$ and $G$ over $\overline{\mathbb{Q}}_l$, an algebraic closure of the $l$-adic field $\mathbb{Q}_l$ ($l \neq p$). Recall, in the previous chapter we fixed an
isomorphism between \((F_q)^*\) and \(\text{Irr}((F_q)^*)\). Providing a coherent choice of roots of unity (via monomorphisms of multiplicative groups) has been made, this leads to an isomorphism

\[ Z(L) \cong \text{Irr}(Z(L)) = \text{Hom}(Z(L), \overline{Q}_l) \]

as in ((5), Section 8.2). Recall, \(\rho_s\) is the linear character of \(L\) corresponding to \(s \in Z(L)\). We have a Jordan decomposition of characters of \(G\). Namely the set of ordinary irreducible characters of \(G\) is in one-to-one correspondence with the set of pairs \((s, \lambda)\). In our case, this means

\[ \chi \leftrightarrow (s, \lambda) \]

where \(s\) is a representative of a semi-simple conjugacy class of \(G\) and \(\lambda\) is a unipotent character of \(L = C_G(s)\), i.e. \(\lambda\) appears as a constituent of \(R^T_L(1)\) for some maximal torus \(T\) of \(L\).

Let \(\epsilon_L = (-1)^d\) where \(d\) is the dimension of a maximal \(F_q^2\) split torus of \(L\). Then

\[ \chi = \epsilon_G \epsilon_L R^G_L(\rho_s \lambda) \text{ by ((11), p.116).} \]

**Proposition 3.1.1** For \(\rho_z \in \text{Irr}(Z(G))\), and \(\chi \in \text{Irr}(G)\)

\[ \chi \leftrightarrow (s, \lambda) \leftrightarrow \rho_z \chi \leftrightarrow (zs, \lambda). \]

where \((s, \lambda)\) is the Jordan decomposition of \(\chi\).
\textbf{Proof:} Let $L = C_G(s) = C_G(sz)$. Then

$$\chi = \epsilon_G \epsilon_L R_L^G(\rho_s \lambda).$$

Moreover

$$\rho_z \chi = \epsilon_G \epsilon_L R_L^G(\rho_z(\rho_s \lambda)), \text{ by ((5), Proposition 8.20)}$$

$$= \epsilon_G \epsilon_L R_L^G((\rho_z \rho_s) \lambda),$$

which corresponds to the pair $(zs, \lambda)$.

\textbf{Remark:} The above discussion also holds for the finite group $\text{GL}_n(q) = \tilde{G}^{F'}$ where the Frobenius map is given by $F'((a_{i,j})) = (a_{i,j}^q)$.

\section{Sequences of polynomials}

In order to count efficiently we make use of polynomial sequences. We construct certain sequences which correspond to the irreducible characters of $U_n(q)$. This identification arises naturally out of Deligne-Lusztig Theory. These polynomials encode information about both $s$ and $\lambda$. This procedure is known in the case of $\text{GL}_n(q)$ where if $\chi$ corresponds to the pair $(s, \lambda)$ then $\chi$ corresponds to a sequence $(h_1(x), h_2(x), \ldots)$. The $h_i(x)$ are products of powers of irreducible polynomials over $F_q$ which are elementary divisors of $s$; the powers of these divisors come from $\lambda$. In precisely the same spirit, irreducible characters of $U_n(q)$ can be identified with sequences $(h_1(x), h_2(x), \ldots)$ where the $h_i(x)$ are products of powers of polynomials over $F_q^2$,.
appropriate for $U_n(q)$, which are elementary divisors of $s$. We proceed with this identification.

It is well known that the conjugacy class of an element in $\text{GL}_n(q^2)$ may be described by the elementary divisors of the rational canonical form. These divisors are powers of monic irreducible polynomials in $F_{q^2}[x]$ with non-zero roots. View $U_n(q)$ as a subgroup of $\text{GL}_n(q^2)$. Let $g \in \text{GL}_n(q^2)$ have $\text{GL}_n(q^2)$-conjugacy class $[g]$. The intersection $[g] \cap U_n(q)$ is either a $U_n(q)$-conjugacy class or is empty ((11), p.111). Let $f$ be a monic irreducible polynomial in $F_{q^2}[x]$ of degree $d$ with nonzero roots $\{\omega\}$. We define $\tilde{f}$ to be the polynomial in $F_{q^2}[x]$ with roots $\{\omega^{-q}\}$. Let $m_{f_i}(g)$ denote the multiplicity of $f_i$ as an elementary divisor of $g$. Then $[g] \cap U_n(q)$ is nonempty precisely when $m_{f_i}(g) = m_{\tilde{f}_i}(g)$ holds $\forall f$ and $\forall i$. Hence the conjugacy class of an element in $U_n(q)$ is given by the elementary divisors of its rational canonical form and these divisors are powers of polynomials in a subset $\mathfrak{F}$ of $F_{q^2}[x]$.

**Definition 3.2.1** Let $\mathfrak{F}_1 = \{f | f \neq x \text{ is monic, irreducible and } f = \tilde{f}\}$ and let $\mathfrak{F}_2 = \{f \tilde{f} | f \neq x \text{ is monic, irreducible and } f \neq \tilde{f}\}$. Let $\mathfrak{F} = \mathfrak{F}_1 \cup \mathfrak{F}_2$.

Notice that for every polynomial $f \in \mathfrak{F}$, $f = \tilde{f}$. Members of $\mathfrak{F}_1$ have odd degree and members of $\mathfrak{F}_2$ have even degree. The latter fact is obvious. The former can be observed by noting that since $f$ is irreducible the roots are the Galois conjugates of $\omega$. Suppose that $d = 2k$. If $f = \tilde{f}$, then $\{\omega, \omega^{-q}, \ldots, \omega^{(-q)^{d-1}}\} = \{\omega^{-q}, \omega q^2, \ldots, \omega^{(-q)^d}\}$. Hence $\omega = \omega^{(-q)^n} = \omega^{q^d}$ so $\omega \in F_{q^d}$. But $F_{q^d}$ is an extension of $F_{q^2}$ of degree $k$, hence $f$ is reducible, a contradiction.
An element \( g \in U_n(q) \) is semi-simple if and only if \( m_{f_i}(g) = 0 \) for all \( i > 0 \). Given a semi-simple \( s \in U_n(q) \), we want to describe its centralizer \( C_{U_n(q)}(s) \). Let \( s \) have primary decomposition \( s = \prod_{f \in \mathcal{F}} s_f \) where \( s_f \) is the primary component corresponding to elementary divisor \( f \). Let \( s \) have minimal polynomial \( \min(x) = \prod_{f \in \mathcal{F}} f \) and characteristic polynomial \( ch(x) = \prod_{f \in \mathcal{F}} f^{m_f(s)} \). Then \( s \) has rational canonical form

\[
s = \bigoplus_{f \in \mathcal{F}} m_f(s)c(f)
\]

where \( c(f) \) denotes the \( d_f \times d_f \) companion matrix of the polynomial \( f \) with degree \( d_f \) and for nonzero multiplicity \( m_f(s) \), \( m_f(s)c(f) \) denotes the \( m_f(s)d_f \times m_f(s)d_f \) matrix with \( m_f(s) \) copies of \( c(f) \).

**Proposition 3.2.2** ((11), Proposition 1A) Let \( s \) have primary decomposition \( s = \prod_{f \in \mathcal{F}} s_f \) and rational canonical form

\[
s = \bigoplus_{f \in \mathcal{F}} m_f(s)c(f).
\]

The structure of the centralizer of \( s \) is given by

\[
C_{U_n(q)}(s) = \prod_{f \in \mathcal{F}} C(s_f), \text{ where }
\]

1. If \( f \in \mathcal{F}_1 \), then \( C(s_f) = U_{m_f(s_f)}(F_f) \), where \( |F_f : F_{q^2}| = \deg(f) \).
2. If \( f \in \mathcal{F}_2 \), then \( C(s_f) = GL_{m_f(s_f)}(F_f) \), where \( |F_f : F_{q^2}| = \frac{1}{2}\deg(f) \).
Hence the centralizer of an element is a product of general linear and unitary groups. A unipotent character of such a product is a product of unipotent characters. Moreover, the unipotent characters of both the general linear and unitary groups are indexed by partitions of the dimension of the underlying vector space. In particular, the unipotent characters of $U_n(q^m)$ and $GL_n(q^m)$ are given by partitions of $n$, for any exponent $m$.

Let $\chi \in \text{Irr}(U_n(q))$ correspond to the pair $(s, \lambda)$. Since $\lambda$ is a unipotent character of $C_{U_n(q)}(s)$ it is a product of unipotent characters of the $C(s_f)$ which are general linear or unitary groups each of which corresponds to a partition $\mu_f \vdash m_f(s)$. Let $\mathcal{P}$ denote the set of all partitions including the empty partition. We define the map

$$\Lambda : \mathcal{F} \rightarrow \mathcal{P}$$

$$f \mapsto \mu_f.$$ 

Notice that $\sum_{f \in \mathcal{F}} d_f m_f(s) = n$.

Our construction is summarized in the following often quoted proposition which originates with Green’s important paper on general linear characters, and has been modified for $U_n(q)$ by several authors. Here we use the notation of Ku ((15)). For a partition $\mu$ recall the definitions of $|\mu|$ on page 16 and $n'(\mu)$ on page 26 in the previous chapter.
Proposition 3.2.3 (15), Proposition 4.2.2 and Lemma 4.2.3) Let $\mathcal{P}$ denote the set of all partitions of all integers $n > 0$, together with the empty partition. The irreducible characters of $U_n(q)$ are in one-to-one correspondence with maps $\Lambda$ from $\mathfrak{S}$ to $\mathcal{P}$ which satisfy the following:

$$\sum_{f \in \mathfrak{S}} |\Lambda(f)|d_f = n.$$  

If $\chi \in \text{Irr}(U_n(q))$ corresponds to such a map $\Lambda$, then the following hold:

1. The $q$-height of $\chi$ is $\sum_{f \in \mathfrak{S}} d_f n'(\Lambda(f)')$ where $\Lambda(f)'$ is the conjugate partition of $\Lambda(f)$.

2. The character $\chi$ lies over $\rho \in \text{Irr}(Z(U_n(q)))$ where $a_\rho$ is the product of the roots of $\prod_{f \in \mathfrak{S}} f^{\Lambda(f)}$.

Let $\chi \in \text{Irr}(G)$ be associated to the pair $(s, \lambda)$ which is in turn associated to the map $\Lambda : \mathfrak{S} \to \mathcal{P}$. For each $f \in \mathfrak{S}$, write the conjugate partition $\Lambda(f)' = (t_{m,f})$. Using these exponents, we may now define for $\chi$ a unique sequence of polynomials $(h_1(x), h_2(x), \ldots)$ by letting

$$h_i(x) = \prod_{f \in \mathfrak{S}} f^{m_{f,i}}.$$  

We will be concerned with examining classes of irreducible characters which share certain properties. We want to group characters by their $q$-height and also by their splitting upon restriction to certain subgroups. To that end we make the following definition which will be of utmost importance in this endeavor.
Definition 3.2.4 If $\chi$ determines the sequence $(h_1(x), h_2(x), \ldots)$, we will say that $\chi$ is of $\mu$-type where $\mu = (\mu \deg(h_i(x))) \vdash n$.

3 Proof of Proposition 2.6.5

In this section we verify Proposition 2.6.5. Recall $k_d(U_n(q), \rho, \det, j)$ is the number of $\chi \in \text{Irr}(U_n(q))$ of $q$-height $d$ lying over $\rho$ such that $\chi|_{SU_n(q)}$ has $j'$ irreducible constituents where $j | j'$. Let $\rho \in \text{Irr}(\mathbb{C}_{q+1})$ and $\mu = (a_1^{l_1}, a_2^{l_2}, \ldots, a_r^{l_r}) \vdash n$. Let $\text{Irr}(U_n(q), \mu, \rho)$ denote the irreducible characters of $U_n(q)$ of $\mu$-type lying over $\rho$. Let $\chi \in \text{Irr}(U_n(q), \mu, \rho)$ correspond to $(s, \lambda)$ and $(h_1, h_2, \ldots)$. Suppose $\rho_z$ is the linear character of $U_n(q)$ corresponding to $z \in \mathbb{C}_{q+1}$, a primitive $j$-th root of unity and that

$$\chi = \rho_z \chi.$$ 

By 3.1.1 $\rho_z \chi$ corresponds to $(zs, \lambda)$. If $h_i$ in $(h_1, h_2, \ldots)$ has roots $\{\omega\}$ then $(zs, \lambda)$ corresponds to $(g_1, g_2, \ldots)$ where $g_i$ has roots $\{z\omega\}$. Then we have

$$(h_1, h_2, \ldots) = (g_1, g_2, \ldots).$$

Since $\chi$ is of $\mu$-type, $h_{a_i}(x)$ is a polynomial of degree $l_i$ and hence $g_{a_i}(x)$ also has degree $l_i$. Let

$$\{\omega_{i,k} | 1 \leq k \leq l_i\}$$

denote the roots of $h_{a_i}(x)$. 

Then
\[ h_{a_i}(x) = \prod_{k=1}^{l_i} (x - \omega_{i,k}) \quad \quad g_{a_i}(x) = \prod_{k=1}^{l_i} (x - z \omega_{i,k}) \]
\[ = x^{l_i} + \cdots + b_{i,1}x + b_{i,0} \quad \quad = x^{l_i} + \cdots + z^{l_i-1}b_{i,1}x + z^lb_{i,0}. \]

Recall \( \lambda(\mu) \) was defined on page 16 and is equal to \( \text{gcd}(l_1, l_2, \ldots, l_r) \). Our first observation is that \( b_{i,0} \) is nonzero. Hence \( z^{l_i} = 1 \) for each \( i = 1, 2, \ldots, r \) thus \( j \) divides \( \lambda(\mu) \). Secondly, we must have \( b_{i,j} \) is divisible by \( j \), i.e. \( j \) doesn’t divide \( k \), the coefficient \( b_{i,k} = 0 \). This reduces the possible number of nonzero coefficients.

If \( \chi \) lies over \( \rho \) then \( (-1)^n \prod_{i=1}^{r} (b_{i,0})^{a_i} = a_\rho \) by construction. The \( b_{i,k} \) are symmetric functions of the roots. Simplifying notation for a moment, since \( h(x) = x^m + \cdots + b_1x + b_0 \) is a product of polynomials in \( \mathfrak{F} \), the coefficients satisfy \( b_{m-i} = b_0 b_i^q \). If \( m \) is even \( b_1^{1-q} = b_0 \). Hence we have \( (l_i/j - 1)/2 \) degrees of freedom in the nonconstant coefficients, i.e. \( q^{2(l_i/j-1)/2} \) choices for the \( b_{i,k} \) and thus
\[ |\text{Irr}(U_n(q), \mu, \rho)| = q^{l(\mu)/j-\delta(\mu)} \beta(\mu, a_\rho). \]

The left hand side of the sum in Proposition 2.6.5 can now be evaluated.

\[ k_d(U_n(q), \rho, \det, j) = \sum_{j' \mid j} |\text{Irr}_d(U_n(q), \rho, \det, j')| \]
\[ = \sum_{j' \mid j} \sum_{\mu \vdash n \atop n' = d} \sum_{j' \mid \text{gcd}(q+1, \lambda(\mu))} |\text{Irr}(U_n(q), \mu, \rho)| \]
\[ = \sum_{\mu \vdash n \atop n' = d} \sum_{j' \mid \text{gcd}(q+1, \lambda(\mu))} q^{l(\mu)/j-\delta(\mu)} \beta(\mu, a_\rho). \]
CHAPTER 4

THE FINITE SPECIAL UNITARY GROUPS

1 A Reduction of DOC for the Finite Special Unitary Groups

In this chapter we make use of Clifford theory in the manner of Sukizaki ((22)) to reformulate DOC for the Special Unitary group SU\(_n(q)\) in terms of U\(_n(q)\). In this section we will distinguish between subgroups of U\(_n(q)\) and SU\(_n(q)\) with superscripts as indicated. Recall \(I = [m]\) is our index set for the distinguished generators of the Weyl group for U\(_n(q)\). We have \(n = 2m\) or \(2m+1\). In keeping with established notation, let \(K = \overline{F}_q\). Let \(\mathring{B}\) be the Borel subgroup of upper triangular matrices of the linear algebraic group \(\mathring{G} = \text{GL}_n(K)\). A Frobenius endomorphism on \(\mathring{G}\) was defined by

\[
F(a_{ij}) = M(a_{ji}^q)^{-1}M^{-1},
\]

and the unitary group U\(_n(q)\) = \(\mathring{G}^F\). The special unitary group is defined

\[
\text{SU}_n(q) = \{ g \in U_n(q) \mid \det g = 1 \}.
\]

Note that except for the cases \(n = 2\) and \(q \leq 3\), the derived subgroup \(\mathring{G}' = \text{SL}_n(K)\) so \(\text{SU}_n(q) = \mathring{G}'^F\). The group of fixed points of \(\mathring{B}\) under \(F\) is a Borel subgroup for U\(_n(q)\). Let \(B^U\) be this subgroup. Notice that \(B^U\) is upper triangular. We will fix a Borel subgroup \(B^{SU} = B^U \cap \text{SU}_n(q)\) for SU\(_n(q)\). Notice that \(B^{SU}\) is the group of fixed points of the Frobenius
restricted to $\text{SL}_n(K)$ and is also upper triangular. We have corresponding Levi decompositions 
$B^U = T \ltimes U$ and $B^{SU} = S \ltimes U$ where $S = T \cap \text{SU}_n(q)$. By standard parabolic subgroups we mean 
subgroups containing $B^U$ or $B^{SU}$. For $J \subseteq I$ let $P^U_J$ or $P^{SU}_J$ be the standard parabolic group of 
$U_n(q)$ or $\text{SU}_n(q)$ respectively corresponding to $J$. For fixed $J \subseteq I$, we have $P^U_J = N_{U_n(q)}(U_J)$ and $P^{SU}_J = N_{\text{SU}_n(q)}(U_J)$, both containing the same upper triangular unipotent radical, i.e. 
$O_p(P^U_J) = O_p(P^{SU}_J) = U_J$

The group $\text{SU}_n(q)$ contains every $p$-subgroup of $U_n(q)$ and $O_p(\text{SU}_n(q)) = O_p(U_n(q)) = 1$. Thus any radical $p$-chain of $\text{SU}_n(q)$ is a radical $p$-chain of $U_n(q)$ by Proposition 2.2.1. Conversely, let 

$C : U_0 < U_1 < \cdots < U_l$

be a radical $p$-chain of $U_n(q)$. The $U_i$ are unipotent radicals of parabolic subgroups of $U_n(q)$. Hence each $U_i$ is conjugate to a standard unipotent radical $U_{J_i}$, i.e. for each $i$ there exists $g_i \in U_n(q)$ such that 

$U_i = g_i U_{J_i} g_i^{-1}$. 
For all $x \in F_{q^2}^\times$, the matrix

$$
\begin{pmatrix}
  x & 0 & \ldots & 0 \\
  0 & 1 & & \\
  & & \ddots & \\
  0 & \ldots & 0 & x^{-q}
\end{pmatrix}
$$

stabilizes all standard $U_{J_i}$. In particular, this holds for $\overline{x_i}$ such that $x_i^{1-q} = (\det(g_i))^{-1}$. Moreover, $\forall g_i \in U_n(q)$ there exists such an $x_i \in F_{q^2}^\times$. Thus

$$
U_i = g_i(\overline{x_i}U_{J_i}\overline{x_i}^{-1})g_i^{-1} = (g_i\overline{x_i})U_{J_i}(g_i\overline{x_i})^{-1}
$$

where $g_i\overline{x_i} \in SU_n(q)$. Thus $C$ is $SU_n(q)$-conjugate to a radical $p$-chain of $SU_n(q)$.

For an irreducible character $\rho$ of $Z^{SU}$, the center of $SU_n(q)$, recall $k_d(P^SU_{J_i}, \rho) = |\text{Irr}_d(P^SU_{J_i}, \rho)|$ is the number of irreducible characters of $P^SU_{J_i}$ lying over $\rho$ (i.e. in the $p$-block corresponding to $\rho$) with $q$-height $d$. The $p$-part of $|SU_n(q)|$ is equal to $q^{\binom{n}{2}}$. As we saw in the introduction DOC can be written:

$$
\sum_{J \subseteq I} (-1)^{|J|} k_d(P^SU_{J_i}, \rho) = 0, \quad \text{for all } \rho \in \text{Irr}(Z^{SU}) \text{ and nonnegative integers } d < \binom{n}{2}. \quad (4.1)
$$
We now reformulate this statement using Clifford Theory. Let det be the determinant map on $U_n(q)$. Then $\det(U_n(q)) = \mathbb{C}_{q+1}$ and $\ker \det = SU_n(q)$. Moreover, restricting the determinant map to parabolic subgroups $P^U_j$ we have $\det(P^U_j) = \mathbb{C}_{q+1}$ and $\ker \det|_{P^U_j} = P^{SU}_j$. The group $P^{SU}_j$ is normal in $P^U_j$ and hence $P^U_j$ acts on the set $\text{Irr}(P^{SU}_j)$ in the natural way. For $g \in P^U_j$ and $\phi \in \text{Irr}(P^{SU}_j)$, $g \cdot \phi = \phi g\phi$ where $g\phi(x) = \phi(gxg^{-1})$ as defined in Section 4 of Chapter 2. The quotient group is cyclic

$$P^U_j/P^{SU}_j \cong \mathbb{C}_{q+1}.\]$$

Let $\text{Irr}_d(P^{SU}_j, \rho, j)$ denote the irreducible characters $\phi \in \text{Irr}_d(P^{SU}_j)$ such that $\phi$ lies over $\rho$, has $q$-height $d$, and the $P^U_j$ orbit of $\phi$ contains $j$ characters. Then the following implies Equation 4.1:

For integers $0 \leq d < \binom{n}{2}$, $1 \leq j$ and any $\rho \in \text{Irr}(Z^{SU})$

$$\sum_{J \subseteq I} (-1)^{|J|} |\text{Irr}_d(P^{SU}_j, \rho, j)| = 0. \quad (4.2)$$

Rather than counting characters of $P^{SU}_j$ we count characters of $P^U_j$. Let $\chi \in \text{Irr}(P^U_j)$. By Clifford Theory, $\chi$ restricted to $P^{SU}_j$ is multiplicity free. The restrictions of two irreducible characters $\chi$ and $\chi'$ of $P^U_j$ to $P^{SU}_j$ have the same irreducible constituents or are disjoint (Lemma 2.4.4). If $\phi \in \text{Irr}_d(P^{SU}_j, \rho, j)$, then the $P^U_j$-orbit of $\phi$ contains $j$ characters. For $\rho \in \text{Irr}(Z^{SU})$, let $\text{Irr}_d(P^U_j, \rho, \det, j)$ denote the subset of $\text{Irr}_d(P^U_j)$ consisting of characters such that their restrictions to $\ker \det$ belong to $\text{Irr}_d(P^{SU}_j, \rho, j)$. 

A character $\phi \in \text{Irr}_d(P^U_J, \rho, j)$ extends to $\tilde{\phi} \in \text{Irr}(T_{P^U_J}(\phi))$. The induced character $(\tilde{\phi}\theta)_{P^U_J}$ is irreducible where $\theta$ is the lift to $T_{P^U_J}(\phi)$ of an irreducible character of $T_{P^U_J}(\phi)/P^U_J$. Then

$$|T_{P^U_J}(\phi)/P^U_J| = \frac{q + 1}{j} \text{ since } |P^U_J/T_{P^U_J}(\phi)| = j.$$ 

Let $k_J$ be the number of $P^U_J$-orbit representatives in $\text{Irr}_d(P^U_J, \rho, j)$. Then

$$|\text{Irr}_d(P^U_J, \rho, j)| = j \cdot k_J$$
$$|\text{Irr}_d(P^U_J, \rho, \det, j)| = \frac{q+1}{j} \cdot k_J.$$

Then

$$\sum_{J \subseteq I} (-1)^{|J|} |\text{Irr}_d(P^U_J, \rho, j)| = \sum_{J \subseteq I} (-1)^{|J|} j \cdot k_J = j \cdot \sum_{J \subseteq I} (-1)^{|J|} k_J = 0$$

holds if and only if $\sum_{J \subseteq I} (-1)^{|J|} k_J = 0$ if and only if

$$0 = \frac{q+1}{j} \cdot \sum_{J \subseteq I} (-1)^{|J|} k_J = \sum_{J \subseteq I} (-1)^{|J|} \frac{q+1}{j} \cdot k_J = \sum_{J \subseteq I} (-1)^{|J|} |\text{Irr}_d(P^U_J, \rho, \det, j)|.$$ 

Hence the following equation is equivalent to Equation 4.2:

For integers $0 \leq d < \binom{n}{2}$, $1 \leq j$ and any $\rho \in \text{Irr}(Z^SU)$:

$$\sum_{J \subseteq I} (-1)^{|J|} |\text{Irr}_d(P^U_J, \rho, \det, j)| = 0. \quad (4.3)$$

We will need the case where $d = \binom{n}{2}$. Recall from Chapter 2, the isomorphism of $(F_{q^2})^*$ and $\text{Irr}((F_{q^2})^*)$ so that $a_\rho \in (F_{q^2})^*$ corresponds to $\rho \in \text{Irr}((F_{q^2})^*)$ under this isomorphism.
This induces an isomorphism of the cyclic subgroups \( Z^U \) and \( Z^{SU} \) with \( \text{Irr}(Z^U) \) and \( \text{Irr}(Z^{SU}) \) respectively. Also recall from Chapter 2 the definition (2.3.3) of \( \beta(\mu, a_\rho) \).

For \( \rho' \in \text{Irr}(Z^U) \) let \( \text{Irr}(P^U_j, \rho') \) be the set of characters \( \chi \in \text{Irr}(P^U_j) \) such that \( \chi \) lies over \( \rho' \). For \( J \neq \emptyset \), \( \text{Irr}(P^U_j, \rho') \) consists of a unique \( p \)-block corresponding to \( \rho' \). However if \( J = \emptyset \) then \( P^U_j = U_n(q) \) and \( \text{Irr}(U_n(q), \rho') \) consists of two \( p \)-blocks one of zero defect the other of full defect. Observe \( |\text{Irr}(\binom{n}{2}(U_n(q), \rho'))| \) is one, the number of irreducible characters of \( U_n(q) \) of full \( q \)-height lying over \( \rho' \). For \( \rho \in \text{Irr}(Z^{SU}) \), let \( \text{Irr}(P^U_j, \rho) \) denote the set of irreducible characters of \( P^U_j \) that lie over \( \rho' \in \text{Irr}(Z^U) \) where \( \rho' \) lies over \( \rho \). Then for \( \rho \in \text{Irr}(Z^{SU}) \) we have the following disjoint union

\[
\text{Irr}(P^U_j, \rho) = \bigsqcup_{\rho' \in \text{Irr}(Z^U)} \text{Irr}(P^U_j, \rho').
\]

We focus now on irreducible characters in \( \text{Irr}(Z^U) \) and so switch the roles of \( \rho \) and \( \rho' \). The following implies Equation 4.3:

For integers \( 0 \leq d, 1 \leq j \) and any \( \rho \in \text{Irr}(Z^U) \),

\[
\sum_{J \subseteq I} (-1)^{|J|} |\text{Irr}_d(P^U_j, \rho, \det, j)| = \begin{cases} 
\beta((n), a_\rho), & \text{if } d = (n) \text{ and } j = 1; \\
0, & \text{otherwise.}
\end{cases}
\]  

(4.4)
Given $\rho \in \text{Irr}(Z^U)$, let $k_d(P^U_J, \rho, \det, j)$ denote the number of irreducible characters $\chi \in \text{Irr}(P^U_J)$ such that $\chi$ lies over $\rho$, has $q$-height $d$ and $\chi|_{\ker \det}$ has $j'$ irreducible constituents where $j$ divides $j'$. Observe that

$$\ker(\det) = P^U_J \cap SU_n(q) = P^{SU}_J$$

as mentioned.

Then

$$k_d(P^U_J, \rho, \det, j) = \sum_{j|j'} |\text{Irr}_d(P^U_J, \rho, \det, j')|.$$

We may now drop the superscript notation and restrict our attention to irreducible characters of parabolic subgroups of $U_n(q)$.

Taking into account that Equation 4.4 implies Equation 4.3 which implies Equation 4.2 which implies Equation 4.1, we will have proved DOC for $SU_n(q)$ if we prove the following theorem, which is the main result in this thesis.

**Theorem 4.1.1 (Main)** Let $Z = Z(U_n(q))$ and $\{P_J|J \subseteq I\}$ the set of standard parabolic subgroups in $U_n(q)$. For any $\rho \in \text{Irr}(Z)$, any positive integer $j$, and all nonnegative integers $d$ we have

$$\sum_{J \subseteq I} (-1)^{|J|} k_d(P_J, \rho, \det, j) = \begin{cases} 
\beta((n), a_\rho), & \text{if } d = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } j=1; \\
0, & \text{otherwise.}
\end{cases}$$
In order to prove Theorem 4.1.1 we first break the left hand side into two sub-sums, the second of which will reduce quite spectacularly. Let us differentiate between characters \( \chi \) counted by \( k_d(P_J, \rho, \det, j) \) for which ker \( \chi \) contains \( U_J \) or not.

**Definition 4.1.2** Let \( k_0^d(P_J, U_J, \rho, \det, j) \) be the number of characters counted by \( k_d(P_J, \rho, \det, j) \) which contain \( U_J \) in their kernel and let \( k_1^d(P_J, U_J, \rho, \det, j) \) count those characters which do not contain \( U_J \) in their kernel.

Then

\[
\sum_{J \subseteq I} (-1)^{|J|} k_d(P_J, \rho, \det, j) = \sum_{J \subseteq I} (-1)^{|J|} (k_0^d(P_J, U_J, \rho, \det, j) + k_1^d(P_J, U_J, \rho, \det, j)) \\
= \sum_{J \subseteq I} (-1)^{|J|} k_0^d(P_J, U_J, \rho, \det, j) + \sum_{J \subseteq I} (-1)^{|J|} k_1^d(P_J, U_J, \rho, \det, j)
\]  

(4.5)

We will show the following:

**Proposition 4.1.3** For any \( \rho \in \text{Irr}(Z) \), any positive integer \( j \), and all nonnegative integers \( d \)

\[
\sum_{J \subseteq I} (-1)^{|J|} k_0^d(P_J, U_J, \rho, \det, j) = \sum_{\mu \vdash n \atop n'(\mu) = d \atop j | \gcd(q+1, \lambda(\mu))} \beta(\mu, a_\rho)  \\
\sum_{J \subseteq I} (-1)^{|J|} k_1^d(P_J, U_J, \rho, \det, j) = -\sum_{\mu \vdash n \atop n'(\mu) = d \atop j | \gcd(q+1, \lambda(\mu))} \beta(\mu, a_\rho).
\]  

(4.6a, 4.6b)

Clearly Proposition 4.1.3 implies Theorem 4.1.1.
This chapter is dedicated to proving Equation 4.6a, the first half of Proposition 4.1.3. Our first observation is that if $\chi$ is an irreducible character of $P_J$ containing $U_J$ in its kernel then we may consider $\chi$ as an irreducible character of the Levi subgroup $L_J$ of $P_J$. We must be careful in applying the determinant map.

Let $J = \{j_1, j_2, \ldots, j_s\}$. Then $L_J$ can be written as the following direct product:

$$L_J = \text{GL}_{n_1}(q^2) \times \text{GL}_{n_2}(q^2) \times \cdots \times \text{GL}_{n_s}(q^2) \times U_{n-2j_s}(q)$$

where $n_1 = j_1$, $n_i = j_i - j_{i-1}$ for $2 \leq i \leq s$. Then since $P_J = L_J U_J$ for $x \in P_J$, $x = lu$ where $l \in L_J$ and $u \in U_J$, so the determinant $\det(x) = \det(lu) = \det(l) \det(u) = \det(l)$ since $u$ is unipotent.

Recall our definition

$$U_n(q) = \{(a_{i,j}) \in \text{GL}_n(q^2) \mid M = (a_{i,j}) M (a_{i,j}^T)\}$$

where $M$ is the $n \times n$ matrix with ones down the reverse diagonal.
With this definition the fixed Borel subgroup of $U_n(q)$ is upper triangular. Thus for $x \in P_J$ we may write $x$ as a block matrix:

$$
\begin{pmatrix}
A_1 & * & \cdots & & \cdots & * & * \\
0 & A_2 & \cdots & & \cdots & * & * \\
\vdots & \vdots & \ddots & & \ddots & \vdots & \vdots \\
A_s & & & B & & & \\
\vdots & \vdots & & \ddots & & \vdots & \vdots \\
0 & 0 & \cdots & & \cdots & \tilde{A}_2 & * \\
0 & 0 & \cdots & & \cdots & 0 & \tilde{A}_1 \\
\end{pmatrix}
$$

where $A_k \in \text{GL}_{n_k}(q^2)$, $B \in U_{n_{s+1}}(q)$, and if $A_k = (a_{i,j})$, then $\tilde{A}_k = M(a_{qj,i})^{-1}M^{-1}$.

The determinant of $x$ as an element in $P_J$ which is embedded in $U_n(q)$ may be defined in terms of the determinant map on the component factors of the Levi subgroup $L_J$ in $P_J$. We have $\det(x) = (\det(A_1) \det(A_2) \cdots \det(A_s))^{1-q} \det(B)$ since $\det(\tilde{A}_k) = \det(A_k)^{-q}$.

Thus $k_d^0(P_J, U_J, \rho, \det, j) = k_d(L_J, \rho, \det | L_J, j)$ where the determinant map $\det | L_J$ is as indicated. Hence we are proving the equivalent statement:

$$
\sum_{J \subseteq I} (-1)^{|J|} k_d(L_J, \rho, \det, j) = \sum_{\substack{\mu \vdash n \\mu | n \vdash d \\gcd(n, \lambda(\mu)) = d \\gcd(q+1, \lambda(\mu))}} \beta(\mu, a_\rho). \tag{5.1}
$$
We make the following observation. Suppose $L_J = G_1 \times G_2$ where $G_1 \cong \text{GL}_{n_1}(q^2)$ and $G_2 \cong \text{U}_{n_2}(q)$ with $2n_1 + n_2 = n$. As a subgroup embedded in $\text{U}_n(q)$ the determinant map on elements in $L_J$ is defined in terms of the determinant map on the factors $G_1$ and $G_2$. For $x = g_1 g_2$ in $L_J$ with $g_i \in G_i$ $\det(x) = \det(g_1)^{1-q} \det(g_2)$. Recall $\det(g_1)^{1-q}$ is denoted $\det^1_q(g_1)$ Then 

$$\ker(\det) = \{(g_1, g_2) \mid \det^1_q(g_1) \det(g_2) = 1\}, \det^1_q(G_1) = \mathbb{C}_{q+1}, \text{ and } \det(G_2) = \mathbb{C}_{q+1}.$$

If $\chi_i$ lies over $\rho_i \in \text{Irr}(\mathbb{C}_{q+1})$, then $\chi = \chi_1 \chi_2$ lies over $\rho = \rho_1 \rho_2$ since for $z \in Z(\mathbb{U}_n(q))$

$$\chi(z) = \chi_1(1) \chi_2(1) \rho_1(z) \rho_2(z).$$

Hence 2.5.1 implies that

$$k_d(L_J, \rho, \det, j) = \sum_{\begin{smallmatrix} 2d_1 + d_2 = d \\ \rho_1 \rho_2 = \rho \end{smallmatrix}} k_{2d_1}(G_1, \rho_1, \det^1_q, j) k_{d_2}(G_2, \rho_2, \det, j). \quad (5.2)$$

We proceed by induction on $n$ or equivalently by induction on $m$, where $n = 2m$ or $2m + 1$.

1. **Small Case**

   Let $n = 1$ so that $m = 0$ and $I$ is empty. Then we have but one levi subgroup, $U_1(q)$ itself which is equal to its center. The determinant map is just the identity and hence $\ker \det$ is trivial so that the left hand side of Equation 5.1 is

   $$k_d(U_1(q), \rho, \det, j) = \begin{cases} 1, & \text{if } d = 0 \text{ and } j = 1; \\ 0, & \text{otherwise}. \end{cases}$$

   Certainly this is equal to the right hand side of Equation 5.1 since we have but one partition of 1 and $\beta((1), a) = 1$. 

Let $m = 1$ so that $n = 2$ or $n = 3$. In either case we have but two levi subgroups $U_n(q)$ and the borel levi subgroup $L_I$. First suppose that $n = 2$. Then $L_I = GL_1(q^2)$ and we may apply 2.6.5 and 2.6.4 directly. The left hand side of Equation 5.1 is

$$
k_d(U_2(q), \rho, \det, j) - k_d(GL_1(q^2), \rho, \det^{1-q}, j) = \begin{cases} 
\beta((2), a_{\rho}) - 0, & \text{if } d = 1 \text{ and } j = 1; \\
\beta((1^2), a_{\rho}) - 0, & \text{if } d = 0 \text{ and } j = 2; \\
q\beta(1^2, a_{\rho}) - \overline{3}(1, a_{\rho}), & \text{if } d = 0 \text{ and } j = 1; \\
0, & \text{otherwise.}
\end{cases}
$$

This is equal to the right hand side of Equation 5.1.

We continue to assume that $m = 1$. Now suppose that $n = 3$. Then $L_I = GL_1(q^2) \times U_1(q)$ so that Equation 5.2 implies

$$
k_d(GL_1(q^2) \times U_1(q), \rho, \det, j) = \sum_{2d_1 + d_2 = d, \rho_1, \rho_2 = \rho} k_{2d_1}(GL_1(q^2), \rho_1, \det^{1-q}, j) k_{d_2}(U_1(q), \rho_2, \det, j)
$$
which is nonzero only for $d = 0$. Then the left hand side of Equation 5.1 is

$$k_d(U_3(q), \rho, \det, j) - k_d(GL_1(q^2) \times U_1(q), \rho, \det, j)$$

$$= \begin{cases} 
\beta((3), a_\rho) - 0, & \text{if } d = 3 \text{ and } j = 1; \\
\beta((2, 1), a_\rho) - 0, & \text{if } d = 1 \text{ and } j = 1; \\
\beta(1^3, a_\rho) - 0, & \text{if } d = 0 \text{ and } j = 3; \\
q^2\beta(1^3, a_\rho) - (q - 1)(q + 1), & \text{if } d = 0 \text{ and } j = 1; \\
0, & \text{otherwise.}
\end{cases}$$

This is equal to the right hand side of (Equation 5.1).

2 Inductive Case

We assume that Equation 5.1 holds for all dimensions strictly less than $n$. Our first observation is that for fixed $J$ with minimal element $j_1 = k$ we may write $L_J = GL_k(q^2) \times L_{J'}$ where $L_{J'}$ is a levi subgroup in $U_{n-2k}(q)$ and $J' = \{j_i - k \mid 2 \leq i \leq s\}$. Note $|J'| = |J| - 1$. We will use superscripts to indicate the dimension of the ambient group when necessary. So for example $L_{J'}$ will be written as $L_{J'}^{n-2k}$. For such a $J$ we have

$$k_d(GL_k(q^2) \times L_{J'}^{n-2k}, \rho, \det, j) = \sum_{2d_1 + d_2 = d, \rho_1 \rho_2 = \rho} k_{2d_1}(GL_k(q^2), \rho_1, \det^{1-q}, j)k_{d_2}(L_{J'}^{n-2k}, \rho_2, \det, j).$$

We remark that $k_{2d_1}(GL_k(q^2), \rho_1, \det^{1-q}, j) = 0$ for $k$ not divisible by $j$. We could eliminate from our sum all $J$ whose smallest members are not all multiples of $j$. This isn’t necessary
though since the contribution is just zero. In fact we may discard all $J$s not contained in
\{j, 2j, 3j, \ldots \} \subseteq I$ but again this isn’t necessary for our induction.

**Definition 5.2.1** Fix $k \geq 1$ and let $J_k$ be the collection of all $J \subseteq I$ with minimal member
$k$.

We have

$$
\sum_{J \in J_k} (-1)^{|J|} k_d(L, \rho, \det, j) = \sum_{J \subseteq I^{m-k}} (-1)^{|J|} k_d(\GL_k(q^2) \times L_{|J|^2, \rho, \det, j})
$$

$$
= \sum_{J \subseteq I^{m-k}} \sum_{|J| = d} (-1)^{|J| + 1} k_{2d_1}(\GL_k(q^2), \rho_1, \det^{1-q}, j) k_{d_2}(L_{|J|^2, \rho_2, \det, j})
$$

$$
= -\sum_{|J| = d} \left( k_{2d_1}(\GL_k(q^2), \rho_1, \det, j) \sum_{J \subseteq I^{m-k}} (-1)^{|J|} k_{d_2}(L_{|J|^2, \rho_2, \det, j}) \right)
$$

$$
= -\sum_{|J| = d} \left( \sum_{\mu_1 \equiv k} q^{2(l(\mu_1)/j - \delta(\mu_1))} \beta(\mu_1, a_{\rho_1}) \times \sum_{\mu_2 \equiv (n-2k)} q^{2(l(\mu_2)/j - \delta(\mu_2))} \beta(\mu_2, a_{\rho_2}) \right)
$$

by our inductive assumption, for indeed $n - 2k$ is strictly less than $n$. 

(5.3)
Summing over all possible values for \( k \) and switching the order of summation we have

\[
\sum_{k=1}^{m} \sum_{J \in J_k} (-1)^{|J|} k_d(L, J, \rho, \text{det}, j)
\]

\[
= - \sum_{k=1}^{m} \sum_{\rho_1 \rho_2 = \rho} \sum_{(\mu_1, \mu_2) \mu_1 \equiv k \mod \mu_2 \equiv (n-2k) \mod n'_{(\mu_1)} = d_3 \mod n'_{(\mu_2)} = d_2} q^{2(\ell(\mu_1)/j - \delta(\mu_1))} \beta(\mu_1, a_{\rho_1}) \beta(\mu_2, a_{\rho_2})
\]

\[
= - \sum_{\mu \vdash n \atop n'(\mu) = d} \sum_{\mu_1 \cup \mu_2} q^{2(\ell(\mu_1)/j - \delta(\mu_1))} \beta(\mu_1, a_{\rho_1}) \beta(\mu_2, a_{\rho_2})
\]

Now we remind the reader that from 2.3.10 we have the following for each \( \mu \vdash n \)

\[
\sum_{(\mu_1, \mu_2) \mu = \mu_1 \cup \mu_2 \atop j \mid \text{gcd}(\lambda(\mu_1), \lambda(\mu_2))} q^{2(\ell(\mu)/j - \delta(\mu))} (q - 1)^{\delta(\mu)} (q + 1)^{c(\mu_1, \mu_2)} = q^{\ell(\mu)/j - \delta(\mu)}.
\]

Notice that this sum includes the pair \((\mu_1, \mu_2) = (\emptyset, \mu)\) and that for this particular pair

\[
q^{2(\ell(\mu)/j - \delta(\mu))} (q - 1)^{\delta(\mu)} (q + 1)^{c(\mu_1, \mu_2)} = 1.
\]
We are now ready to prove Equation 5.1:

\[
\sum_{J \subseteq I} (-1)^{|J|} k_d(L_J, \rho, \det, j) = k_d(U_n(q), \rho, \det, j) + \sum_{\emptyset \neq J \subseteq I} (-1)^{|J|} k_d(L_J, \rho, \det, j)
\]

\[
= k_d(U_n(q), \rho, \det, j) + \sum_{k=1}^{m} \sum_{J \subseteq I_k} (-1)^{|J|} k_d(L_J, \rho, \det, j)
\]

\[
= \sum_{\mu \vdash n} q^{l(\mu)/j - \delta(\mu)} \beta(\mu, a_\rho) - \left( \sum_{\mu \vdash n} q^{l(\mu)/j - \delta(\mu)} \beta(\mu, a_\rho) - \sum_{\mu \vdash n} \beta(\mu, a_\rho) \right)
\]

\[
= \sum_{\mu \vdash n} \beta(\mu, a_\rho).
\]

And we are done.
CHAPTER 6

MODULES FOR PARABOLIC SUBGROUPS

Recall the set up in Chapter 5. Let \( J \subset I = [m] \), an index set for the distinguished generators of the Weyl group of type \( B_m \). Let \( P_J \) be the standard parabolic subgroup of \( U_n(q) \) corresponding to \( J \). Then \( P_J = \bigcap_{j \in J} P_j \) where \( P_j \) is the maximal parabolic subgroup corresponding to \( j \). We have the usual Levi decomposition of \( P_J = L_J U_J \) where \( L_J \) is a levi subgroup and \( U_J \) is the unipotent radical of \( P_J \). Thus far we have seen a reduction of DOC in the finite special unitary case to Proposition 4.1.3 which involves two sub-sums of certain characters of parabolic subgroups in \( U_n(q) \). Having proved Equation 4.6a we now devote the rest of this thesis to proving Equation 4.6b.

The first step in the process of finding a formula for \( k_1^d(P_J, U_J, \rho, \det, j) \) is to examine orbits in submodules afforded by quotient groups in the normal series for \( U_J \) discussed below. We are faced with the task of describing certain irreducible characters of parabolic subgroups \( P_J \) in \( U_n(q) \) which do not contain \( U_J \) in their kernels. In general this is not an easy task as \( \text{Irr}(P_J) \) is not known. We avoid this difficulty by examining \( P_J \) orbits of irreducible \( U_J \) characters.

In order to distinguish between parabolic subgroups in general linear and in unitary groups of varying dimensions, we make the following definition.

**Definition 6.0.2**

1. Let \( P_J^{+n} \) denote a parabolic subgroup in \( \text{GL}_n(q^2) \), where \( J \subseteq [n-1] \).

2. Let \( P_J^n \) denote a parabolic subgroup in \( U_n(q) \), where \( n = 2m \), or \( 2m + 1 \) and \( J \subseteq [m] \).
Write \( J = \{j_1, j_2, \ldots, j_s\} \) in increasing order.

**Definition 6.0.3** The following subsets of \( J \) are defined

1. \( J(\geq j_i) = \{j_i, j_{i+1}, \ldots, j_s\} \) and \( J(> j_i) = \{j_{i+1}, \ldots, j_s\} \)
2. \( J(\leq j_i) = \{j_1, j_2, \ldots, j_i\} \) and \( J(< j_i) = \{j_1, j_2, \ldots, j_{i-1}\} \).

The unipotent radical \( U_J \) has the following normal series:

\[
U_J = U_{J(\geq j_1)} > U_{J(\geq j_2)} > \cdots > U_{J(\geq j_s)} = U_{j_s} \geq Z(U_{j_s}) > 1 \tag{6.1}
\]

The quotient groups in this series are elementary abelian. The set of all irreducible characters of \( P_J \) divides into subsets of characters in the following way. Let \( \chi \) be an irreducible character of \( P_J \) then there exits a group \( N \) in the above series such that the kernel of \( \chi \) contains \( N \) but does not contain the previous subgroups in the series. Since \( N \leq P_J \) and \( N \subseteq \ker(\chi) \) for \( \chi \in \text{Irr}(P_J) \), we may consider \( \chi \) as an irreducible character of the quotient group \( P_J/N \). If \( N' \) is the previous subgroup in the normal series, then \( N'/N \) is elementary abelian and hence may be regarded as a \( P_J/N \)-module over \( F_p \). There are four types of characters originally found by KU which we now briefly describe. What distinguishes these four types is the action induced by conjugation by \( P_J \) on the abelian quotients in Equation 6.1.

1. **Levi Characters**: Suppose \( \chi \in \text{Irr}(P_J) \) and \( U_J \) is contained in \( \ker(\chi) \). Then we may consider \( \chi \) a character of the quotient group \( P_J/U_J \) which is of course isomorphic to \( L_J \).

Such a character \( \chi \) is trivial on \( U_J \) and has already been accounted for in Equation 4.6a.
2. **General Linear Characters:** Suppose $\chi \in \text{Irr}(P_J)$ and for fixed $i$ satisfying $1 \leq i < s$ we have $U_{j(\geq j_i+1)} \subseteq \ker(\chi)$ but $U_{j(>j_i)} \not\subseteq \ker(\chi)$. We may consider $\chi$ as a character of $P_J/U_{j(\geq j_i+1)}$. Then

$$P_J/U_{j(\geq j_i+1)} \cong P_{j(\leq j_i)}^{+j_i+1} \times L_{j'} \quad (15), 7.1.2.2$$

where $J' = \{ j - j_i+1 | j \in J(>j_i+1) \}$ and $L_{j'}$ is a levi subgroup in $U_{n-2j_i+1}(q)$. Let $V(j_i,j_i+1)$ denote the quotient group $U_{j(\geq j_i)}/U_{j(\geq j_i+1)}$.

$$P_{j(\leq j_i)}^{+j_i+1} \cong \left( P_{j(< j_i)}^{+j_i} \times GL_{j_{i+1}-j_i}(q) \right) \rtimes V(j_i,j_i+1).$$

We call $V(j_i,j_i+1)$ a general linear module to indicate that a general linear group is acting on the module. The character $\chi$ restricted to $V(j_i,j_i+1)$ is nontrivial. Hence $\chi = \chi_1\chi_2$ where $\chi_1$ corresponds to some irreducible character of $V(j_i,j_i+1)$ and $\chi_2$ is an irreducible character of the factor $L_{j'}$.

3. **Unitary Linear Characters:** Now we suppose that $\chi \in \text{Irr}(P_J)$ with $Z(U_{j_s}) \subseteq \ker(\chi)$ but $U_{j_s} \not\subseteq \ker(\chi)$. We may consider $\chi$ as a character of

$$P_J/Z(U_{j_s}) \cong \left( P_{j(< j_s)}^{+j_s} \times U_{n-2j_s}(q) \right) \rtimes (U_{j_s}/Z(U_{j_s})) \quad (15), 7.1.2.3.$$

Then $\chi$ corresponds to an irreducible character of the quotient $U_{j_s}/Z(U_{j_s})$, a unitary linear module to indicate that a unitary group is acting on the module.
4. **Central Characters:** Finally we make take \( \chi \in \text{Irr}(P_J) \) with only the trivial subgroup contained in \( \ker(\chi) \) so that \( Z(U_{j_s}) \not\subset \ker(\chi) \). In this case \( \chi \) corresponds to a non trivial character of \( Z(U_{j_s}). \)

As mentioned these four types are outlined by Ku in (15). The main difference in this thesis is the introduction of two extra parameters in the manner of Sukizaki's approach (22) to the special linear case. Amazingly, it turns out that parabolic characters as parameterized by Ku are very well behaved with regard to their splitting upon restriction to the kernel of the determinant map. This is very convenient and not an obvious fact. One important result of this fact is that Ku's parametrization of the character \( q \)-heights is sufficient.

Conjugation by \( P_J \) on the abelian quotients in Equation 6.1 gives rise to the aforementioned internal modules with non-trivial action by a group \( H_J \) in a quotient group \( P_J/N \) of \( P_J \), where \( N \) appears in the series Equation 6.1. Hence we need to examine what occurs at the level of \( H_J \).

1 **Parabolic Actions**

In this section we study the modules which arise in the following situation. Fix positive integers \( n_1 \) and \( n_2 \). Let \( V_1 \) be the natural module for \( \text{GL}_{n_1}(q^2) \) and \( V_2 \) the dual of the natural module for \( \text{GL}_{n_2}(q^2) \). Fix a basis for \( V_1, n_1 \)-dimensional column vectors \( \{e_1, e_2, \ldots, e_{n_1}\} \) where \( e_i \) has a 1 in the \( i \)-position and zeros elsewhere. Fix a basis for \( V_2, n_2 \)-dimensional row vectors \( \{e^1, e^2, \ldots, e^{n_2}\} \) where \( e^j \) has a 1 in the \( j \)-position and zeros elsewhere. Set \( V = V_1 \otimes V_2 \cong M_{n_1,n_2}(q^2) \). A basis for \( V \) is given by \( \{E_{i,j}\} \) where \( E_{i,j} = e_i \otimes e^j \) the \( n_1 \times n_2 \)-matrix with \( (i,j) \)-entry 1 and zeros elsewhere. Then \( \text{GL}_{n_1}(q^2) \times \text{GL}_{n_2}(q^2) \) acts on \( V \) in the natural way
via left multiplication by $GL_{n_1}(q^2)$ and right multiplication by inverses in $GL_{n_2}(q^2)$. Let $G_i$ be a subgroup of $GL_{n_i}(q^2)$. Then $G = G_1 \times G_2$ acts on $V$ and hence $V$ is a module for $G$. In subsequent sections we will be considering the following cases:

1. The group $G_1 = P_j^{+n_1}$ and $G_2 = GL_{n_2}(q^2)$,

2. the group $G_1 = P_j^{+n_1}$ and $G_2 = U_{n_2}(q)$, and

3. the group $G_1 = P_j^{+n_1}$ and $G_2$ is an isomorphic copy of $G_1$. In this case, as we will see, the module we consider is a subgroup of $V$ isomorphic to $M_{n_1,n_2}(q)$. We will discuss this in the central module section.

The vector space $V$ is an elementary abelian group. We have the following $G$-isomorphisms of abelian groups ((15), 6.1.2):

$$\text{Irr}(V) = \text{Hom}(V, \mathbb{C}^*) \cong \text{Hom}(V, \mathbb{C}_p) \cong \text{Hom}_{F_p}(V, F_p) \cong \text{Hom}_{F_{q^2}}(V_1 \otimes V_2, F_{q^2}) \cong \text{Hom}_{F_{q^2}}(V_1, V_2^*)$$

where now $V_2^*$ is the restriction to $G_2$ of the natural module for $GL_{n_2}(q^2)$.

The first isomorphism is clear since complex characters of $V$ take values in $\mathbb{C}_p$. The second isomorphism is also clear since the multiplicative group $\mathbb{C}_p$ can be identified with the additive group $F_p$. The last isomorphism is also clear by adjoint associativity of the tensor product. The penultimate isomorphism is less clear. The field $F_{q^2}$ is a finite extension of $F_p$. Let $\theta$ generate
so that \( \{1, \theta, \theta^2, \ldots \} \) is a basis for \( F_{q^2} \) as a vector space over \( F_p \). Note that \( \theta^0 = 1 \). Define the projection

\[
\pi : F_{q^2} \rightarrow F_p, \quad \sum_i a_i \theta^i \mapsto a_0.
\]

We fixed a basis \( \{E_{i,j}\} \) for \( V \) over \( F_{q^2} \) above. Let \( U \) be the \( F_p \)-span of \( \{E_{i,j}\} \). Then

\[
V = F_{q^2} \otimes_{F_p} U
\]

so that \( \{\theta^k \otimes E_{i,j}\} \) is an \( F_p \)-basis of \( V \). The following is a well-defined isomorphism

\[
\text{Hom}_{F_{q^2}}(V, F_{q^2}) \rightarrow \text{Hom}_{F_p}(V, F_p), \quad f \mapsto \pi \circ f.
\]

Hence we have \( \text{Hom}_{F_p}(V, F_p) \cong \text{Hom}_{F_{q^2}}(V_1 \otimes V_2, F_{q^2}) \).

The action of \( G \) on \( V \) gives rise to a parallel action of \( G \) on \( \text{Irr}(V) \) and hence on \( \text{Hom}_{F_{q^2}}(V_1, V_2^*) \).

Since we fixed a basis for \( V_2 \), the dual \( V_2^* \) has basis \( \{(e^1)^*, (e^2)^*, \ldots, (e^{n_2})^*\} \) where \( (e^j)^* \) is the \( n_2 \)-dimension row vector with 1 in the \( j \)-position and zeros elsewhere. Take \( \tau \in \text{Irr}(V) \). Then \( \tau \) corresponds to \( f \in \text{Hom}_{F_{q^2}}(V_1, V_2^*) \) and

\[
V_1 / \ker(f) \cong f(V_1) \cong (V_2 / \text{Ann}(f))^*.
\]
where

$$\ker(f) = \{ v \in V_1 \mid f(v)(w) = 0 \ \forall w \in V_2 \}$$

is the kernel of $f$, and

$$\Ann(f) = \{ w \in V_2 \mid f(v)(w) = 0 \ \forall v \in V_1 \}$$

is the annihilator of $f(V_1)$ in $V_2$.

The codimension of both of these subspaces is the same. If $r$ is the codimension of $\ker(f)$ and $\Ann(f)$ we will say that $f$ has rank $r$. In this way if $\tau \in \Irr(V)$ corresponds to $f \in \Hom_{F_q^2}(V_1, V_2^*)$ then $\tau$ is said to have rank $r$. We remark that via the isomorphism between $\Irr(V)$ and $V$, if $\tau$ corresponds to the matrix $v$ then the rank of $\tau$ is the row rank of $v$, which is invariant under the action of $G$.

We are interested in describing representatives of $G$-orbits in $\Irr(V)$ and their stabilizers in $G$. Given $\tau \in \Irr(V)$, let $\ker(\tau)$ and $\Ann(\tau)$ denote $\ker(f)$ and $\Ann(f)$, respectively, where $\tau$ corresponds to the element $f \in \Hom_{F_q^2}(V_1, V_2^*)$. Write $\overline{V}_1 = V_1 / \ker(\tau)$ and $\overline{V}_2 = V_2 / \Ann(\tau)$. The dimension of both quotient spaces is $r$. Let

$$C_{G_i}(\overline{V}_i) = \{ g \in G_i \mid g \cdot \overline{v} = \overline{v} \ \forall \overline{v} \in \overline{V}_i \}$$

where as noted above $g \cdot \overline{v}$ indicates left or right multiplication depending on $i$ equal to 1 or 2, respectively. As stated by Ku ((15), p.67) we have

$$C_{G_1}(\overline{V}_1) \times C_{G_2}(\overline{V}_2) \leq T_G(\tau) \leq T_{G_1}(\ker(\tau)) \times T_{G_2}(\Ann(\tau)).$$
Before describing these orbits and stabilizers we pause to consider some cancellation which occurs in the alternating sum of DOC.

2 Some Useful Cancellation

An alternating sum may reduce to a smaller alternating sum in an advantageous way. One may approach this from a combinatorial perspective in which case, like terms that appear with opposite parity cancel one another. We will make use of this approach later in the thesis. One may also approach the reduction of an alternating sum from a topological perspective. We discuss this now and apply the results in order to make a first reduction of Equation 4.6b.

Consider the Burnside ring $b(G)$ of a finite group $G$ the free abelian group on equivalence classes $[G/H]$, where $[G/H]$ is equal to $[G/K]$ if and only if $H$ and $K$ are conjugate subgroups of $G$. A typical element of $b(G)$ is of the form

$$a_1[G/H_1] + a_2[G/H_2] + \cdots + a_N[G/H_N]$$

where $a_i \in \mathbb{Z}$ and the $H_i$ are representatives of conjugacy classes of subgroups in $G$. Multiplication in $b(G)$ is given by

$$[G/H] \cdot [G/K] = [G/(H \cap K)].$$
Let $G$ act on a finite poset $P$ ordered by inclusion. The simplicial complex $\mathcal{O}(P)$ consists of chains, 

$$c : x_0 < x_1 < \cdots < x_k \quad x_i \in P,$$

where we require strict inclusion. The chain $c$ as above has length $k + 1$. The chains of length $k + 1$ form the $k$-simplices of $\mathcal{O}(P)$. By convention the $-1$-simplex is $1$ the trivial $G$-set. Let $\Delta(P) = \{1 < x_0 < x_1 < \cdots < x_k \mid x_i \in P\}$. Every chain in $\Delta(P)$ in (including the trivial chain) begins with the trivial $G$-set 1. For $c : 1 < x_0 < x_1 < \cdots < x_k \in \Delta(P)$ define the absolute value $|c| = k + 1$. The reduced Lefshetz element of $P$ in $G$ is an element of $b(G)$ and is defined

$$\Lambda_G(P) = \sum_{c \in \Delta(P)/G} (-1)^{|c|}[G/G_c]$$

where $c$ runs over a set of $G$-orbit representatives in $\Delta(P)$ and $G_c$ is the stabilizer $T_G(c)$ of $c$ in $G$.

We may assign topological concepts to posets in the following sense. When we say a poset $P$ has a certain property we mean that its associated simplicial complex has the property. See ((18)) for a discussion of this.

Let $G$ act on posets $P$ and $Q$. A poset map $f : P \rightarrow Q$ is a map that preserves ordering, i.e. $x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$. The poset map $f$ gives rise to a simplicial map $f_\mathcal{O} : \mathcal{O}(P) \rightarrow \mathcal{O}(Q)$. This map is $G$-equivariant if $gf_\mathcal{O}(c) = f_\mathcal{O}(gc)$ for all $c \in \mathcal{O}(P)$. Two simplicial maps $f_\mathcal{O}$, $g_\mathcal{O} : \mathcal{O}(P) \rightarrow \mathcal{O}(Q)$ are homotopic if there exists a continuous map $H : \mathcal{O}(P) \times [0,1] \rightarrow \mathcal{O}(Q)$
such that $H(c,0) = f_O(c)$ and $H(c,1) = g_O(c)$ for all $c \in O(P)$. If $f, g : P \to Q$ are poset maps such that $f(x) \leq g(x)$, for all $x \in P$, then $f_O$ and $g_O$ are homotopic.

The spaces $O(P)$ and $O(Q)$ are said to be $G$-homotopy equivalent if there exist $G$-equivariant maps $f_O : O(P) \to O(Q)$ and $g_O : O(Q) \to O(P)$ such that $f_O \circ g_O, id_{O(P)}$ are $G$-equivariant homotopic and $g_O \circ f_O, id_{O(Q)}$ are $G$-equivariant homotopic (25, p.352). We may say that the posets $P$ and $Q$ are $G$-homotopy equivalent, by which we mean that $O(P)$ and $O(Q)$ are $G$-homotopy equivalent. Given our interest in the reduced Lefshetz element we may also say that that $\Delta(P)$ and $\Delta(Q)$ are $G$-homotopy equivalent by which we mean again that $O(P)$ and $O(Q)$ are $G$-homotopy equivalent.

It is well known that, if $O(P)$ and $O(Q)$ are $G$-homotopy equivalent, then $\Lambda_G(P) = \Lambda_G(Q)$.

Let $S(G)$ denote the set of subgroups of $G$. Let $f : S(G) \to \mathbb{Z}$ be a $G$-stable map, i.e. constant on conjugate subgroups of $G$. Set $f(\lfloor G/G_c \rfloor) = f(G_c)$ and extend linearly to $b(G)$. We may apply $f$ to $\Lambda_G(P)$,

$$f(\Lambda_G(P)) = \sum_{c \in \Delta(P)/G} (-1)^{|c|} f(G_c).$$

This is an integer. Moreover if $P$ and $Q$ are $G$-homotopy equivalent then $f(\Lambda_G(P)) = f(\Lambda_G(Q))$, i.e.

$$\sum_{c \in \Delta(P)/G} (-1)^{|c|} f(G_c) = \sum_{c \in \Delta(Q)/G} (-1)^{|c|} f(G_c).$$

Given a $G$-poset $P$, we may make use of a contractible sub-poset in order to find a smaller complex which is $G$-homotopy equivalent to $O(P)$. A poset $Q$ is $G$-contractible if $Q$ is itself
$G$-homotopy equivalent to a point. Let $Q$ contain a single element $x$. Then $\Delta(Q)$ consists of two chains

$$c_1 : 1 \text{ with } |c_1| = 0 \text{ and } c_2 : 1 < x \text{ with } |c_2| = 1,$$

which are both necessarily stabilized by $G$. Observe that $\Lambda_G(Q) = [G/G] - [G/G] = 0$ and thus for any $G$-stable map $f$, $f(\Lambda_G(Q)) = 0$

A reference for more material on $G$-homotopy is ((21), Chapters 3 and 4).

**An important example of homotopy equivalence:** Let $V$ be a vector space over a finite field, and let $P$ be the poset of non-trivial proper subspaces of $V$ ordered by inclusion. Let $w$ be a non-trivial proper subspace of $V$. Let $Q$ be the sub-poset of $P$ whose elements are subspaces which are not complements in $V$ to $w$. Then $\Delta(P)$ is the set of chains of subspaces of $V$ beginning with the zero subspace and $\Delta(Q)$ is the subset of chains which do not contain a complement in $V$ to $w$. Let $G = T_{GL(V)}(w)$. The group $G$ certainly acts on $P$ and $Q$. The vector space $w$ is in $Q$ and is fixed by $G$. It turns out that $Q$ is $G$-contractible ((24), Corollary 1.9). Thus $\Lambda_G(Q) = 0$. Let $\Delta(P,w)$ denote the sub-set of $\Delta(P)$ consisting of chains which contain complements to $w$. Then $\Delta(P) = \Delta(Q) \cup P, w)$ ((15), Lemma 5.2.13). Since $Q$ is $G$-contractible, $\Lambda_G(P) = \Lambda_G(P, w)$.

We are now ready to apply this discussion to our situation. For the rest of this section let $G = \text{GL}_{n_1}(q^2) \times G_2$ where $G_2$ is a subgroup of $\text{GL}_{n_2}(q^2)$. For now, we will be considering $G_2$
to be either $\text{GL}_{n_2}(q^2)$ or $\text{U}_{n_2}(q)$. Let $H = G \ltimes V$ where $V = V_1 \otimes V_2$ where $V_1$ is the natural module for $\text{GL}_{n_1}(q^2)$ and $V_2$ is the dual of the natural module for $\text{GL}_{n_1}(q^2)$, as above. For each $J \subseteq [n_1 - 1]$, we let $G_J$ denote $P_J^{+n_1} \times G_2$ and $H_J$ denote $G_J \ltimes V$.

**Definition 6.2.1** Let $\text{Irr}(V, r)$ denote the subset of characters in $\text{Irr}(V)$ of rank $r$, where we recall that the rank of $\tau$ is the codimension of $\ker(\tau)$ in $V_1$, viewing $\tau$ as a homomorphism from $V_1$ to $V_2^*$.

Set $X = \text{Irr}(V, r)$. Let $Y = \{ y \leq V_1 \mid \text{codim}(y) = r \}$ and for $y \in Y$ let $X(y) = \{ \tau \in X \mid \ker(\tau) = y \}$. We extend the action of $\text{GL}_{n_1}(q^2)$ on $V_1$ to $H$ by declaring that $G_2 \ltimes V$ act trivially on $V_1$. In this way $H$ acts on $X$ and $Y$ but also on the poset $P = P(V_1)$ of subspaces of $V_1$ ordered by inclusion. Then $\Delta(P)$ is the set of flags in $P$ together with the empty flag and so $H$ also acts on $\Delta(P) \times X$. A set of orbit representatives for the action of $\text{GL}(V_1)$ and hence $H$ on $\Delta(P)$ is given by $\{ c_J \mid J \subseteq [n_1 - 1] \}$, where if $J = \{ j_1, j_2, \ldots, j_s \}$ then $c_J$ is the flag

$$0 < V_{j_1} < V_{j_2} < \cdots < V_{j_s}$$

where $V_{j_i}$ is the subspace of $V_1$ spanned by the vectors $\{ e_1, e_2, \ldots, e_{j_i} \}$. We also note that $P_J^{+n_1}$ is the stabilizer of the flag $c_J$.

The proposition below is a modification of ((15), Proposition 6.2.1), where we obtain a first reduction for equation Equation 4.6b. Ku’s proof makes use of some fairly elaborate abstractions involving alternating sums. We present a more direct proof with the necessary adjustments.
which include extra parameters in the definition of an $H$-stable function which lead to the desired reduction. The significance of this result is that for fixed $r$ we need only describe orbit representatives for the subset $X(w) = \{\tau | \ker \tau = w\}$ in Irr($V, r$), where $w$ is a complement in $V_1$ of the $r$-dimensional subspace stabilized by $P_{r+n_1}$. In fact, since $T_{H_J}(\tau) \leq T_{H_J}(w) \leq H_J$ and $T_{H_J}(w) = T_{G_J}(w) \ltimes V$ by Corollary 2.4.3 it is sufficient to find representatives for $T_{G_J}(w)$-orbits in $X(w)$. What is more we only need the stabilizers in $H_J$ where $J \cup \{n_1\}$ contains the element $r$.

We will be considering the group $H_J = G_J \ltimes V$ as embedded in $U_n(q)$.

**Definition 6.2.2** Define the map on $G_J$,

$$D: P_{J+n_1} \times G_2 \rightarrow F_{q^2}^*, \quad (x, y) \mapsto \det(x)^i(1-q) \det(y)^k,$$

for some positive integer $i$ where the exponent of $\det(y)$ depends on $G_2$ in the following way:

$$k = \begin{cases} 
1 - q, & \text{if } G_2 = \text{GL}_{n_2}(q^2), \\
1, & \text{if } G_2 = U_{n_2}(q). 
\end{cases}$$

Extend $D$ to $H_J$ by letting $D(v) = 1$ for all $v \in V$. Observe that $\ker(D)$ is normal in $H_J$, contains $V$, and the quotient $H_J/\ker(D)$ is cyclic. The map $D$ is constructed to be the restriction to $H_J$ of the determinant map on $U_n(q)$. The value of the integer $i$ depends on the embedding of $H_J$ in $U_n(q)$. 
Definition 6.2.3 For a subset \( X \subseteq \text{Irr}(V) \), let \( k_d(H_J, X, \rho, D, j) \) denote the number of irreducible characters \( \chi \) of \( H_J \) of \( q \)-height \( d \), lying over \( \rho \), and corresponding to \( \tau \in X \) such that \( \chi \) restricted to the kernel of the map \( D \) is a sum of \( j' \) irreducible characters where \( j \) divides \( j' \).

Proposition 6.2.4 Let \( Z \leq Z(G) \) and \( \rho \in \text{Irr}(Z) \). Fix \( 1 \leq r \leq \min(n_1, n_2) \). Let \( X = \text{Irr}(V, r) \) and Let \( Y = \{ y \leq V_1 \mid \text{codim}(y) = r \} \). Let \( w \) be a complement in \( V_1 \) to the \( r \)-dimensional subspace stabilized by \( P_r^{+n_1} \), in its action on the natural module for \( \text{GL}_{n_1}(q^2) \). Let \( X(w) = \{ \tau \in X \mid \ker(\tau) = w \} \). Let \( D \) be the map on \( H_J \) defined above. For \( d \geq 0 \) the following holds

1. If \( r = n_1 \), then \( Y = \{ 0 \} \) and \( X = X(0) \).

2. If \( r < n_1 \), then

\[
\sum_{J \subseteq [n_1-1]} (-1)^{|J|} k_d(H_J, X, \rho, D, j) = \sum_{J \subseteq [n_1-1]} \sum_{r \in J} (-1)^{|J|} k_d(H_J, X(w), \rho, D, j).
\]

Proof: The first part is immediate since the trivial subspace is the only subspace of \( V_1 \) of codimension \( n_1 \). The proof of the second part is based on the exposition on homotopy equivalences which begins this section.

Let \( r < n_1 \). We define a function

\[
f : \Delta(P) \times Y \rightarrow Z \, , \quad (c_J, y) \mapsto k_d(H_J, X(y), \rho, D, j).
\]
This is a well defined $H$-stable function on $\Delta(P) \times Y$, i.e. constant on the $H$-orbits. Indeed, the action of $H$ on $\Delta(P)$ preserves chain type and thus a set of $H$-orbit representatives is given by

$$\{ c_J \mid J \subseteq [n_1 - 1] \}.$$ 

Moreover, $k_d(H_J, \tau, \rho, \mathcal{D}, j) = k_d(H_J, h\tau, \rho, \mathcal{D}, j)$ for any $h \in H$. If $y' = gy$ for $g \in P_j^{+n_1}$, then

$$k_d(H_J, X(y'), \rho, \mathcal{D}, j) = \sum_{\ker \tau = y'} k_d(H_J, \tau, \rho, \mathcal{D}, j)$$

$$= \sum_{\ker \tau = gy} k_d(H_J, \tau, \rho, \mathcal{D}, j)$$

$$= \sum_{g^{-1} \ker \tau = y} k_d(H_J, g^{-1}\tau, \rho, \mathcal{D}, j)$$

$$= \sum_{\ker \tau = y} k_d(H_J, \tau, \rho, \mathcal{D}, j)$$

$$= k_d(H_J, X(y), \rho, \mathcal{D}, j).$$

The function $f$ induces the following projection functions. For fixed $c_J \in \Delta(P)$

$$f_{c_J} : Y \longrightarrow \mathbb{Z}, \quad y \mapsto k_d(H_J, X(y), \rho, \mathcal{D}, j)$$

is $T_H(c_J) = H_J$-stable on $Y$; for fixed $y \in Y$

$$f_y : \Delta(P) \longrightarrow \mathbb{Z}, \quad c_J \mapsto k_d(H_J, X(y), \rho, \mathcal{D}, j)$$

is $T_H(y)$-stable on $\Delta(P)$. 
For fixed \( \tau \in X(y) \), \( T_{H_J}(\tau) \leq T_{H_J}(y) \leq H_J \). Thus by Corollary 2.4.3

\[
k_d(H_J, X(y), \rho, D, j) = \sum_{\tau \in X(y)/H_J} k_d(H_J, \tau, \rho, D, j)
= \sum_{\tau \in X(y)/T_{H_J}(y)} k_{d-d'}(T_{H_J}(y), \tau, \rho, D, j')
= k_{d-d'}(T_{H_J}(y), X(y), \rho, D, j')
\]

where \( d' \) is the exponent of \( q \) in \( |T_{H_J}(y)\backslash H_J| \) and \( j' \) is the least positive integer such that

\[
j | j' \cdot |T_{H_J}(y) \ker(D)\backslash H_J|.
\]

In all three cases which we will consider \( j' = j \) will hold. Indeed, if \( G_2 = \text{GL}_{n_2}(q^2) \), or \( U_{n_2}(q) \) then this is certainly true since \( n_2 \neq 0 \) so

\[
T_{H_J}(y) = (T_{P_j^{+n_1}} \times G_2) \ltimes V,
\]

hence \( \mathcal{D}(T_{H_J}(y)) = \mathcal{D}(H_J) \). We will be considering the central case where \( G_2 \) is an isomorphic copy of \( P_j^{+n_1} \). In this instance, the fact that \( r < n_1 \) is sufficient to imply that \( j' = j \).
Let $d_0$ be the exponent of $q$ in $|H|$. Let $w$ be a subspace of $V_1$ as defined in the statement of the proposition. Define the following function on subgroups of $T_H(w)$

$$g : S(T_H(w)) \longrightarrow \mathbb{Z}$$

$$K \mapsto \begin{cases} 
k_{d-(d_0-d(K))}(K, X(w), \rho, D, j), & \text{if } V \leq K; \\
0, & \text{otherwise.}
\end{cases}$$

where $d(K)$ is the exponent of $q$ in $|K|$. This is a $T_H(w)$-stable function. For $c_J \in \Delta(P)$, $T_H(c_J) = H_J$. Since the $q$-height of $H$ is equal to the $q$-height of $H_J$,

$$d_0 - d(K) = |T_H(w) \setminus H_J|.$$ 

Hence $g(H_J) = f_w(c_J)$. Applying $g$ to the reduced Lefshetz element we have

$$g(\Lambda_{T_H(w)}(P)) = g(\Lambda_{T_H(w)}(P,w)).$$
We put the above together

\[ \sum_{J \subseteq [n_1 - 1]} (-1)^{|J|} k_d(H_J, X, \rho, \mathcal{D}, j) \]

\[ = \sum_{c_J \in \Delta(P)/H} (-1)^{|J|} \sum_{y \in Y/H_J} k_d(H_J, X(y), \rho, \mathcal{D}, j) \]

\[ = \sum_{y \in Y/H} \sum_{c_J \in \Delta(P)/T_H(y)} (-1)^{|J|} k_d(H_J, X(y), \rho, \mathcal{D}, j) \]

\[ = \sum_{c_J \in \Delta(P)/T_H(y)} (-1)^{|J|} k_d(H_J, X, \rho, \mathcal{D}, j), \text{ since there is a unique } H\text{-orbit in } Y \]

\[ = \sum_{c_J \in \Delta(P)/T_H(w)} (-1)^{|J|} k_{d-d'}(T_{H_J}(w), X(w), \rho, \mathcal{D}, j) \]

\[ = \sum_{c_J \in \Delta(P,w)/T_H(w)} (-1)^{|J|} k_d(H_J, X(w), \rho, \mathcal{D}, j) \]

\[ = \sum_{J \subseteq [n_1 - 1]} (-1)^{|J|} k_d(H_J, X(w), \rho, \mathcal{D}, j), \]

and we are done.

### 3 General Linear Modules

In this section we first summarize Ku’s results and then examine a map which is related to the usual determinant map. In this section \( G_1 \) is \( P_{j+n_1}^+ \), a parabolic subgroup of \( \text{GL}_{n_1}(q^2) \) for fixed \( J \subset [n_1 - 1] \). Let \( G_2 \) be \( \text{GL}_{n_2}(q^2) \). The module \( V_1 \) is the restriction to \( P_{j+n_1}^+ \) of the natural module for \( \text{GL}_{n_1}(q^2) \) and \( V_2 \) is the dual of the natural module for \( \text{GL}_{n_2}(q^2) \). Recall,
in this context \( G_J = P_J^{+n_1} \times GL_{n_2}(q^2) \). Fix \( r \in J \) or \( r = n_1 \). If \( r < n_1 \) then let \( w \leq V_1 \) be a complement to the \( r \)-dimensional subspace stabilized by \( P_r^{+n_1} \). If \( r = n_1 \) then let \( w \leq V_1 \) be the trivial subspace. Let \( \tau \in X(w) \) so that \( \tau \) has rank \( r \) and \( \ker \tau = w \). We can use \( r \) to divide the set \( J \) into two subsets:

\[
\{ j \in J \mid j < r \} \quad \text{and} \quad \{ j \in J \mid r < j \}.
\]

Let us set \( J_2 = \{ j \mid j \in J \text{ and } j < r \} \) and \( J_1 = \{ j - r \mid j \in J \text{ and } r < j \} \). Notice that we have the containment \( J_2 \subseteq [r-1] \) and \( J_1 \subseteq [n_1 - r] \).

**Proposition 6.3.1** ((15), Lemma 6.2.2) The group \( T_{G,J}(w) = P_{J_1}^{+(n_1-r)} \times P_{J_2}^{+r} \times GL_{n_2}(q^2) \) and is transitive on \( X(w) \). The stabilizer of \( \tau \) in \( G_J \) is

\[
T_{G,J}(\tau) = \begin{cases} 
P_{J_1}^{+(n_1-r)} \times P_{J_2}^{+n_2}, & \text{if } r = n_2; \\
P_{J_1}^{+(n_1-r)} \times P_{J_2 \cup \{r\}}^{+n_2}, & \text{if } r < n_2. 
\end{cases}
\]

(6.2)

### 3.1 The determinant map in the general linear module context

We would like to determine the determinant map on

\[
G_J = P_J^{+n_1} \times GL_{n_2}(q^2),
\]
as embedded in a larger unitary group. At this stage we consider the map

\[
\mathcal{D} : P_J^{+n_1} \times \text{GL}_{n_2}(q^2) \rightarrow F^*_{q^2}
\]

\[
(x, y) \mapsto (\det(x)^i \det(y))^{1-q},
\]

for some positive integer \(i\). As we will see \(\mathcal{D}\) is constructed to be the restriction to \(G_J\) of the usual determinant map on \(U_n(q)\). The integer \(i\) depends on the embedding of \(G_J\) in \(U_n(q)\). For the rest of this subsection let \(K = \ker(\mathcal{D})\).

Our first observation is that the image of \(G_J\) under this map is all of \(C_{(q+1)}\) since \(n_2 \neq 0\). We need to examine the map \(\mathcal{D}\) restricted to the subgroup \(T_{G_J}(\tau)\) and calculate \(|T_{G_J}(\tau)K \backslash G_J|\).

The cases depend on \(r\), \(n_1\), and \(n_2\). In order to calculate \(|T_{G_J}(\tau)K \backslash G_J|\) we need to find the image of \(T_{G_J}(\tau)\) under \(\mathcal{D}\). If \(\mathcal{D}(T_{G_J}(\tau)) = C_{(q+1)/h}\) then \(\mathcal{D}(T_{G_J}(\tau)K) = C_{(q+1)/h}\) so that

\[
|T_{G_J}(\tau)K \backslash G_J| = |C_{(q+1)/h}| / |C_{(q+1)}| = h.
\]

We examine the four cases:

1. If \(r = n_1 = n_2\) then \(T_{G_J}(\tau) = P_J^{+r}\). If \((x, y) \in G_J\) stabilizes \(\tau\) then \(x = y = B\) and

\[
\mathcal{D} : P_J^{+r} \rightarrow F^*_{q^2}
\]

\[
B \mapsto (\det(B)^{i+1})^{1-q}.
\]
Also notice that \( \mathcal{D}(P_j^{+r}) = \mathbb{C}_{(q+1)/\gcd(q+1,i+1)} \).

**2.** If \( r = n_1 \) but \( r < n_2 \) then \( T_{GJ}(\tau) = P_{J_1^{+n_2}}^{+r} = (P_j^{+r} \times \text{GL}_{n_2-r}(q^2)) \ltimes M_{r,n_2-r}(q^2) \). If \( (x,y) \in G_J \) stabilizes \( \tau \) then

\[
x = B \quad \text{and} \quad y = \begin{pmatrix} B & * \\ 0 & C \end{pmatrix}.
\]

Since \( \mathcal{D} \) is trivial on the normal factor \( M_{r,n_2-r}(q^2) \) we have

\[
\mathcal{D} : \quad (P_j^{+r} \times \text{GL}_{n_2-r}(q^2)) \rightarrow F_{q^2}^* \quad \begin{pmatrix} B & * \\ 0 & C \end{pmatrix} \rightarrow (\det(B)^{i+1} \det(C))^{1-q}.
\]

Observe that \( \mathcal{D}(P_j^{+r} \times \text{GL}_{n_2-r}(q^2)) \ltimes M_{r,n_2-r}(q^2) = \mathbb{C}_{(q+1)} \) since \( n_2 - r \neq 0 \).

**3.** If \( r < n_1 \) but \( r = n_2 \), then \( T_{GJ}(\tau) = P_{J_1^{+n_1-r}}^{+r} \times P_{J_2^{+r}}^{+r} \). If \( (x,y) \in G_J \) stabilizes \( \tau \), then

\[
x = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \quad \text{and} \quad y = B.
\]

We have

\[
\mathcal{D} : \quad P_{J_1^{+n_1-r}}^{+r} \times P_{J_2^{+r}}^{+r} \rightarrow F_{q^2}^* \quad (A,B) \quad \rightarrow \quad (\det(A)^{i+1}\det(B))^{1-q}.
\]
Take any element $\alpha$ of $F_{q^2}^*$. Then the diagonal matrix $A = (\alpha^{-1}, 1, 1, \cdots, 1)$ is in $P_{J_1}^+((n_1 - r))$ and the diagonal matrix $B = (\alpha, 1, 1, \cdots, 1)$ is in $P_{J_2}^+$. Then

$$D(A, B) = (\alpha^{-i} \alpha^{i+1})^{1-q} = \alpha^{1-q}$$

and thus $D(P_{J_1}^+((n_1 - r)) \times P_{J_2}^+)$ is all of $C(q+1)$.

4. Finally we suppose that $r < n_1$ and $r < n_2$. We have

$$T_{G_J}(\tau) = P_{J_1}^+((n_1 - r)) \times P_{J_2}^{+n_2} = P_{J_1}^+((n_1 - r)) \times \left( P_{J_2}^+ \times \operatorname{GL}_{n_2-r}(q^2) \right) \times M_{r,n_2-r}(q^2).$$

If $(x, y) \in G_J$ stabilizes $\tau$, then

$$x = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} B & * \\ 0 & C \end{pmatrix}.$$  

As above, $D$ is trivial on the normal factor $M_{r,n_2-r}(q^2)$ and

$$D : \quad P_{J_1}^+((n_1 - r)) \times P_{J_2}^+ \times \operatorname{GL}_{n_2-r}(q^2) \quad \rightarrow \quad F_{q^2}^*$$

$$\quad (A, B, C) \quad \mapsto \quad (\det(A)^i \det(B)^{i+1} \det(C))^{1-q}.$$  

In this case we also have $D(T_{G_J}(\tau))$ equal to all of $C(q+1)$ since $n_2 - r \neq 0$. 

Summarizing these results, we get the following proposition which is a step in computing the number of irreducible characters of $P_J$ (in an appropriate $p$-block, of appropriate $q$-height) which split as desired upon restriction to $P_J \cap SU_n(q)$:

**Proposition 6.3.2**

\[
|T_{G_J}(\tau)K \backslash G_J| = \begin{cases} 
\gcd(q + 1, i + 1), & \text{if } r = n_1 = n_2; \\
1, & \text{otherwise.}
\end{cases}
\]

4 Unitary Modules

In this section we first summarize Ku’s results and then examine a map which is related to the usual determinant map. We make use of the following notations which are due to Ku: $S^u(V, J, r)$, $S^{nu}(V, J, r)$, and $S^{nu}(V, J, r)$. In this section $G_1$ is $P_J^{+n_1}$ a parabolic subgroup of $GL_{n_1}(q^2)$ for fixed $J \subset [n_1 - 1]$ as above. In the previous section we had $G_2 = GL_{n_2}(q^2)$ which is transitive on subspaces of the same dimension in the dual space $V_2^*$. Now we consider $G_2 = U_{n_2}(q)$. Recall, in this context $G_J = P_J^{+n_1} \times U_{n_2}(q)$. The module $V_1$ is still the restriction to $P_J^{+n_1}$ of the natural module for $GL_{n_1}(F_{q^2})$. However, now $V_2$ is the dual of the natural module for $U_{n_2}(q)$ and thus has a unitary structure. In this case, $G_2$ is not transitive on subspaces of the same dimension of $V_2^*$.

Recall if $U$ is a unitary vector space, a subspace $W$ is totally isotropic if $<v, w> = 0$ for all vectors $v, w$ in $W$, where $<,>$ indicates the hermitian form on $U$. A totally isotropic subspace $W$ is degenerate since its radical $rad(W) = W \cap W^\perp = W$. A chain of totally isotropic subspaces will be called a singular chain. A chain of subspaces which are not all totally isotropic
will be called a non-singular chain. A subspace $W$ is non-degenerate if $rad(W) = 0$ in which case $V = W \oplus W^\perp$.

Let $W$ be a non-degenerate subspace of $V_2^*$ of dimension $r$. Then

$$T_{U_{n_2}}(q)(W) = U_{n_2 - r}(q) \times U_r(q).$$

Now consider $W$ a totally isotropic subspace. A basis for $V_2^*$ has already been fixed. To simplify notation, denote $(e_j)^*$ by $e_j$ so that the basis is $\{e_1, e_2, \ldots, e_{n_2}\}$. Further suppose that with respect to the inner product on $V_2^*$ we have

$$<e_i, e_j> = \begin{cases} 
1, & \text{if } i + j = n_2 + 1; \\
0, & \text{otherwise.}
\end{cases}$$

Let $W$ be the totally isotropic subspace equal to $<e_1, e_2, \ldots, e_r>$. Note that $r$ must be less than or equal to $[n_2/2]$. Then the stabilizer is a maximal parabolic subgroup

$$T_{U_{n_2}}(q)(W) = P_r^{n_2}.$$ 

Just as in the previous section we fix $r$ where either $r \in J$ or $r = n_1$. If $r = n_1$ then let $w \leq V_1$ be the trivial subspace. If $r < n_1$ then let $w \leq V_1$ be a complement to the $r$-dimensional subspace stabilized by $P_r^{+n_1}$. Let $\tau \in X(w)$ so that $\tau$ has rank $r$ and $\ker \tau = w$. Ku parameterizes the $T_{GJ}(w)$ orbits on $X(w)$ with chains of subspaces in $V_2^*$. These are so-
called normal flags of fixed type depending on $J$ and reflect the unitary structure in $V_2^*$ which we define now.

**Definition 6.4.1** A normal flag in $V_2^*$ is a chain of subspaces

$$c : 0 < V_1 < V_2 < \cdots < V_s$$

satisfying the following. There exists $0 = i_0 < i_1 < \cdots i_k \leq s$, $k \geq 0$, such that for all $0 \leq j \leq k$

1. $V_{i_j}$ is either a non-degenerate subspace in $V_2^*$ or the zero subspace and

2. for any $i_j < i < i_{j+1}$ we have $V_i = V_{i_j} \oplus \text{rad}(V_i)$, where we assume $i_{k+1} = s + 1$ and $V_{s+1} = V_2^*$.

Take linear maps $f, g : V_1 \to V_2^*$ with kernel $w$. The stabilizer $T_{G_J}(w) = P_{J_1}^{r_{n_1}} \times P_{J_1(<r)}^{r_{n_2}} \times U_{n_2}(q)$. Under its action on $X(w)$, $f$ and $g$ are in the same orbit if and only if $f(c_J)$ and $g(c_J)$ are isomorphic as flags in $V_2^*$. Observe that $f(c_J)$ and $g(c_J)$ are both flags of type $\{J(<r) \cup \{r\}\} \backslash \{n_2\}$ because we can choose a basis so that $w = < e_{r+1}, \ldots, e_{n_1} >$ and $f(w) = g(w) = 0$ the trivial subspace. Let $P(V_2^*)$ be the poset of subspaces in $V_2^*$ ordered by inclusion.

If $r = n_1$ then $X(w) = X$ and $T_{G_J}(w) = G_J$. The $G_J$-orbits in $X$ are in 1-1 correspondence with the $U_{n_2}(q)$-orbits on chains of type $\{J \cup \{r\}\} \backslash \{n_2\}$ in $P(V^*_2)$. If $\tau$ corresponds to chain $c$ of such type then

$$T_{G_J}(\tau) = T_{U_{n_2}(q)}(c).$$
See ((15), Lemma 6.3.1).

If \( r < n_1 \) then \( w \) is non-trivial and

\[
T_{G_J}(w) = P_{J_1}^{+n_1-r} \times P_J^{+r} \times U_{n_2}(q)
\]

where \( J_1 = \{ j - r \mid j \in J(>r) \} \). The group \( P_{J_1}^{+n_1-r} \) acts trivially on the quotient space \( \mathbf{V}_1 = V_1/w \) and hence acts trivially on \( X(w) \) which is isomorphic to \( \text{Irr} (\mathbf{V}_1 \otimes V_2, r) \). By the above discussion for the case \( r = n_1 \) the \((P_{J_1}^{+r} \times U_{n_2}(q))\)-orbits in \( \text{Irr} (\mathbf{V}_1 \otimes V_2, r) \) are in 1-1 correspondence with the \( U_{n_2}(q) \)-orbits on chains of type \( \{ J(>r) \cup \{ r \} \} \{ n_2 \} \) in \( \mathcal{P}(V_2^*) \).

**Definition 6.4.2** Let \( S^\text{u}(V, J, r) \) denote the set of \( T_{G_J}(w) \)-orbits in \( X(w) \) labeled by normal flags of type \( \{ J(>r) \cup \{ r \} \} \{ n_2 \} \) in \( \mathcal{P}(V_2^*) \).

The set \( S^\text{u}(V, J, r) \) is defined for \( r \in J \cup \{ r \} \) and is in 1-1 correspondence with the \( U_{n_2}(q) \)-orbits on normal chains of type \( \{ J(>r) \cup \{ r \} \} \{ n_2 \} \). Let us examine the possible flags of such type in \( \mathcal{P}(V_2^*) \). Let \( c \) be such a flag. Since \( V_2^* \) has a unitary structure, subspaces fall into two categories. They are either totally isotropic or not. The action of \( U_{n_2}(q) \) on \( V_2^* \) preserves this structure.

**Definition 6.4.3** If \( c \) is a flag of totally isotropic subspaces we will say that \( c \) is singular. If \( c \) contains a non-degenerate subspace we will say that \( c \) is a non-singular flag. Moreover we define the non-singular rank of \( c \) to be the dimension of the minimal non-degenerate subspace.
If \( \tau \in X(w) \) has rank \( r \) and corresponds to a non-singular flag \( c \) with non-singular rank \( r' \), then \( \tau \) itself is said to have non-singular rank \( r' \). Moreover \( r' \in J \cup \{r\} \) since \( r' \) is the dimension of a subspace in the flag \( c \) corresponding to \( \tau \).

Suppose \( c \) is a flag of type \( \{J(<r) \cup \{r\}\}\}\setminus\{n_2\} \) of non-singular rank \( r' \) with \( 1 \leq r' \leq r \). We have \( r', r \in J \cup \{r\} \). Let \( \tilde{J} = \{k_1, k_2, \ldots, k_s\} \) be the type of \( c \). The element \( r' \) divides \( \tilde{J} \) into two subsets

\[ \tilde{J}_1 = \{ k \mid k \in \tilde{J}, k < r' \} \quad \text{and} \quad \tilde{J}_2 = \{ k \mid k \in \tilde{J}, r' < k \}. \]

Write

\[ c : 0 < V_{k_1} < \cdots < V_{r'} < \cdots < V_{k_s}. \]

We assign to \( c \) a pair \((c_1, c_2)\) of shorter flags in the following way. The subspace \( V_{r'} \) is the minimal non-degenerate subspace. We define

\[ c_1 : 0 < V_{k_1} < \cdots < V_{\max(\tilde{J}_1)} \]

\[ c_2 : 0 < V_{\min(\tilde{J}_2)} \cap V_{r'}^\perp < \cdots < V_{k_s} \cap V_{r'}^\perp. \]

The flag \( c_1 \) is a singular flag in the unitary space of dimension \( r' \). The flag \( c_2 \) is a flag in the unitary space of dimension \( n_2 - r' \).

**Definition 6.4.4** We define the following subsets of \( S^u(V, J, r) \):

1. Let \( S^{su}(V, J, r) \) be the subset of \( S^u(V, J, r) \) labeled by singular flags.
2. Let $S_{nu}(V,J,r)$ be the subset of $S_u(V,J,r)$ labeled by non-singular flags.

3. Finally, let $S_{nu}^{r'}(V,J,r)$ be the subset of $S_{nu}^{r''}(V,J,r)$ with non-singular rank $r'$.

Observe that the set $S_{nu}(V,J,r)$ is non-empty if and only if $r \leq n_2/2$, in which case it consists of a single member. Notice also that $S_{nu}^{r'}(V,J,r)$ is nonempty if and only if $r' \in J(< r)$ and $J(< r') \subseteq [n_2/2]$. We have the following structure of stabilizers of characters in these sets given by Ku.

**Proposition 6.4.5** ((15), Remark 6.3.12)

1. For $\tau \in S_{nu}(V,J,r)$ we have $T_{G_J}(\tau) = P_{J_1}^{+(n_1-r)} \times P_{\{j \in J(<r)\} \setminus \{n_2\}}^{r_j}$

2. For $\tau \in S_{nu}^{r'}(V,J,r)$ we have $T_{G_J}(\tau) = P_{J_1}^{+(n_1-r)} \times P_{J(<r')}^{r'} \times T_{U_{n_2-r'(q)}}(c_2)$

where $J_1 = \{j - r \mid j \in J(>r)\}$ and $c_2$ is obtained as above from $c$ which corresponds to $\tau$.

4.1 The determinant map in the unitary linear module context

We would like to determine the determinant map on

$$G_J = P_J^{+n_1} \times U_{n_2}(q),$$

as embedded in a larger unitary group. At this stage we consider the map

$$D : P_J^{+n_1} \times U_{n_2}(q) \longrightarrow F_{q^2}^*$$

$$(x, y) \mapsto \det(x)^{(1-q)} \det(y).$$
for some positive integer \( i \). The map \( D \) is constructed to be the restriction to \( G_J \) of the usual determinant map on \( U_n(q) \). The integer \( i \) depends on the embedding of \( G_J \) in \( U_n(q) \). For the rest of this subsection let \( K = \ker(D) \)

Observe the image of \( G_J \) under this map is all of \( \mathbb{C}_{(q+1)} \) since \( n_2 \neq 0 \). We need to examine the map \( D \) restricted to the subgroup \( T_{G_J}(\tau) \) and calculate \( |T_{G_J}(\tau)K\backslash G_J| \).

Let \( \tau \in S^{su}(V, J, r) \) so that

\[
T_{G_J}(\tau) = P^{+/(n_1-r)}_{J_1} \times P^{n_2}_{J(\langle r \rangle \cup \{\tau\}) \setminus \{n_2\}}
= P^{+/(n_1-r)}_{J_1} \times \left( \left( P^{+r}_{J(\langle r \rangle)} \times U_{n_2-2r}(q) \right) \rtimes U^r_{n_2} \right)
\]

where \( U^r_{n_2} \) is the unipotent radical in the maximal parabolic subgroup \( P^{n_2}_r \) in \( U_{n_2}(q) \). If \((x, y) \in G_J \) stabilizes \( \tau \) then

\[
x = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} B & * \\ 0 & C \end{pmatrix}.
\]

The map \( D \) restricted to \( T_{G_J}(\tau) \) is trivial on the normal factor \( U^r_{n_2} \) and

\[
D : P^{+/(n_1-r)}_{J_1} \times P^{+r}_{J(\langle r \rangle)} \times U_{n_2-2r}(q) \rightarrow F^*_{q^2}
(A, B, C) \mapsto (\det(A)^i \det(B)^{i+1})^{1-q} \det(C)
\]
where it is understood that if any of these dimensions are zero, we assume \( \det(A) = 1 \) for \( A \) in any group of dimension zero. The image of \( D \) on \( T_{GJ}(\tau) \) depends on \( r, n_1, \) and \( n_2 \). There are three cases:

1. If \( r = n_1 = n_2/2 \), then the image of \( T_{GJ}(\tau) \) under \( D \) is \( \mathbb{C}_{(q+1)/\gcd(q+1,i+1)} \).

2. If \( r < n_2/2 \), then \( D(T_{GJ}(\tau)) = \mathbb{C}_{(q+1)} \).

3. If \( r < n_1 \) but \( r = n_2/2 \), then for any \( \alpha \in F_q^* \), the diagonal matrix \( A = (\alpha^{-1}, 1, \ldots, 1) \) is in \( P_{J_1}^{+(n_1-r)} \) and the diagonal matrix \( B = (\alpha, 1, \ldots, 1) \) is in \( P_{J(<r)}^{+r} \). We have \( D(AB) = (\alpha^{-i}\alpha^{i+1})^{1-q} = \alpha^{1-q} \) and thus \( D(T_{GJ}(\tau)) \) is all of \( \mathbb{C}_{(q+1)} \).

Summarizing these results, we get the following proposition which is another step in computing the number of irreducible characters of \( P_J \) (in an appropriate \( p \)-block, of appropriate \( q \)-height) which split as desired upon restriction to \( P_J \cap \text{SU}_n(q) \):

**Proposition 6.4.6** For \( \tau \in S^{\text{su}}(V, J, r) \)

\[
|T_{GJ}(\tau)K \backslash G_J| = \begin{cases} 
\gcd(q + 1, i + 1), & \text{if } r = n_1 = n_2/2; \\
1, & \text{otherwise.}
\end{cases}
\]

Now take \( \tau \in S^{\text{nu}}(V, J, r) \) with non-singular rank \( r' \neq 0 \). We have

\[
T_{GJ}(\tau) = P_{J_1}^{+(n_1-r)} \times P_{J(<r')}^{+r'} \times T_{U_{n_2-r'(q)}(c_2)}
\]
where \( \tau \) corresponds to \( c \) which corresponds to the pair \((c_1, c_2)\) as above so that \( c_2 \) is a flag in the unitary space of dimension \( n_2 - r' \). If \((x, y) \in G_J\) stabilizes \( \tau \) then

\[
x = \begin{pmatrix}
B & 0 & 0 \\
0 & C' & 0 \\
0 & 0 & A
\end{pmatrix}
\quad \text{and} \quad
y = \begin{pmatrix}
B & 0 & 0 \\
0 & C' & * \\
0 & 0 & C''
\end{pmatrix}.
\]

Let

\[
C = \begin{pmatrix}
C' & * \\
0 & C''
\end{pmatrix}.
\]

Then

\[
\mathcal{D} : \ P_{J_1}^{n_1-r} \times P_J^{r'}(<r') \times T_{U_{n_2-r'(q)}}(c_2) \rightarrow F^*_{q^2} \\
(A, B, C) \mapsto \det(A)^{(1-q)} \det(B)^{2i+1} \mathcal{D}'(C).
\]

where \( \mathcal{D}'(C) \) is the determinant of \( C \) as an element embedded in \( U_n(q) \). In other words \( \mathcal{D}'(C) \) depends on the embedding of \( C' \). We take note of the restriction of \( \mathcal{D} \) to the factor \( P_{J(<r')}^r \) in the stabilizer \( T_{G_J}(\tau) \). For \( B \in P_{J(<r')}^r \), \( \mathcal{D}(B) = \det(B)^{2i+1} \) since \(-q \equiv 1 \mod (q + 1)\). Notice also that \( T_{U_{n_2-r'(q)}}(c_2) \) is a stabilizer in a unitary group of smaller dimension than \( n_2 \) and as mentioned \( \mathcal{D}' \) depends on the embedding in \( U_n(q) \). Later we will make use of an inductive argument.
5 Central Modules

In this section we first summarize Ku’s results and then briefly examine a map which is related to the usual determinant map. At the end of this section we also examine how the useful cancellation applies to the central module case. We will be examining parabolic subgroups $P_J$ in $U_{n'}(q)$, where $n' \leq n$, as embedded in $U_n(q)$. Let $l$ be the maximal member of $J$. Write $Z_l = Z(U_l)$. Then the normal series for $U_J$ terminates with $U_l \geq Z_l > 1$. Thus far the modules we have considered arise when $\chi \in \text{Irr}(P_J)$ contains $Z(U_l)$ in its kernel. We will consider now the case when $\chi \in \text{Irr}(P_J)$ does not contain $Z(U_l)$ in its kernel.

The parabolic group $P_J$ has the following decomposition

$$P_J = \left( P_{J(<l)}^{\perp} \times U_{n'-2l}(q) \right) \ltimes U_l$$

and acts on $U_l$ by conjugation. This induces an action of the quotient group $P_J/U_l$ on $Z_l$ which is isomorphic as an abelian group to $M_{l,l}(q^2)$ which we denote by $V$. Indeed, since $P_J$ is upper triangular we may write elements of $V$ as matrices

$$\begin{pmatrix}
    a_{1,1} & a_{1,2} & d_1 \\
    a_{2,1} & d_2 \\
    & & -a_{1,2}^q \\
    d_l & -a_{2,1}^q & -a_{1,1}^q
\end{pmatrix} \text{ in } M_{l,l}(q^2) \text{ where } d_l^q + d_l = 0.$$
Let $V^l$ be the usual module for $GL_l(q^2)$ then $V^l \otimes (V^l)^* \cong M_{l,l}(q^2)$ is a module for $P_{l<l}^{+l}$. Observe that $V \leq M_{l,l}(q^2)$. The quotient $P_{l}/U_l$ acts on $V$ as follows. For matrix $A \in P_{l<l}^{+l}$, $B \in U_{n'-2l}(q)$, and $v \in V$, $(A, B) \cdot v = Av(\tilde{A})^{-1}$ where $\tilde{A}$ is as defined at the beginning of Chapter 5, i.e. if $A = (a_{i,j})$, then $\tilde{A} = M((a_{j,i})^q)^{-1}M^{-1}$ where $M$ is the matrix with ones on the reverse diagonal. This induces an action on the subset $\text{Irr}(V) \leq \text{Irr}(V^l \otimes (V^l)^*) \cong \text{Hom}(V^l, V^l)$ which is invariant on the rank of $\tau \in \text{Irr}(V)$ which recall is the co-dimension of the kernel of $\tau$ viewed as a homomorphism from $V^l$ to itself. Notice that the unitary factor acts trivially.

**Case 1.** We begin with the special case $n' = 2m$ and $J = \{m\}$ so that $P_{J} = GL_m(q^2) \ltimes U_m$ and $U_m = Z_m \cong M_{m,m}(q)$.

Let $X = \text{Irr}(V, r)$ the subset of characters in $\text{Irr}(V)$ of rank $r$. Fix non-zero $\epsilon$ in the algebraic closure of $F_q$ satisfying $\epsilon^q + \epsilon = 0$ and define the $m \times m$ matrix

$$x_r = (a_{i,j}) \text{ where } a_{i,j} = \begin{cases} \epsilon, & j - i = m - r; \\ 0, & \text{otherwise.} \end{cases}$$

As a matrix $x_r$ has rank $r$.

**Proposition 6.5.1** ((15), Chapter 7) The group $GL_m(q^2)$ is transitive on $X$. The set $\{0, x_r \mid 1 \leq r \leq m\}$ is a complete set of representatives for the $GL_m(q^2)$-orbits on $\text{Irr}(V)$, where $0$ denotes the zero matrix. Moreover

$$T_{GL_m(q^2)}(\tau_r) = T_{GL_m(q^2)}(x_r) = (U_r(q) \times GL_{m-r}(q^2)) \ltimes M_{r,m-r}(q^2)$$
as in ((15), Lemma 7.3.1).

Case 2. Now consider the case where \( U_l \neq Z_l \). Let \( J = \{ l \} \) so that

\[
P_J = P_l \cong (\text{GL}_l(q^2) \times U_{n'-2l}(q)) \rtimes U_l
\]

where \( U_l/Z_l \cong M_{l,n'-2l}(q^2) \) and \( Z_l \cong M_{l,l}(q) \). Let \( \tau_r \) be identified with \( x_r \). Since \( U_{n'-2l}(q) \) acts trivially on \( Z_l \), by the first special case the set \( \{ 1, \tau_r \mid 1 \leq r \leq l \} \) is a complete set of representatives for the \( P_J \)-orbits on \( \text{Irr}(Z_l) \), where \( 1 \) is the trivial character. Let

\[
\text{Irr}(U_l, \tau_r) = \{ \phi \in \text{Irr}(U_l) \mid \phi \text{ lies over } \tau_r \}.
\]

If \( \chi \in \text{Irr}(P_l) \) does not contain \( Z_l \) in its kernel then \( \chi \) lies over \( \phi \in \text{Irr}(U_l, \tau_r) \) where \( \phi \) restricted to \( Z_l \) is a multiple of \( \tau_r \), for some \( 1 \leq r \leq l \). In this case, \( \tau_r \) is not extendible to its stabilizer in \( P_l \). However, it turns out that \( \phi \) is extendible to \( T_{P_l}(\phi) \).

Given \( \phi \in \text{Irr}(U_l, \tau_r) \), \( \ker(\tau_r) \leq \ker(\phi) \). We may consider \( \phi \) as a character of the quotient group \( U_l/\ker(\tau_r) \). We may consider \( \tau_r \) as a character of the quotient group \( Z_l/\ker(\tau_r) \). The center \( Z_l \) is elementary abelian so \( \text{Irr}(Z_l) \cong \text{Hom}_{F_p}(Z_l, F_p) \). Thus \( Z_l/\ker(\tau_r) \) is cyclic of order \( p \). The group \( U_l/\ker(\phi) \) has an irreducible faithful representation and hence has cyclic center. Moreover, \( Z(U_l/\ker(\phi)) \) is a homomorphic image of \( Z(U_l/\ker(\tau_r)) \) which is elementary abelian ((15), p.101). Thus \( Z(U_l/\ker(\phi)) \) must have order \( p \). We have

\[
Z(U_l/\ker(\phi)) \geq Z_l \ker(\phi)/\ker(\phi) \cong Z_l/(\ker(\phi) \cap Z_l) = Z_l/\ker(\tau).
\]
Hence
\[ Z(U_l/\ker(\phi)) \cong Z_l/\ker(\tau). \]

Moreover
\[ U_l/\ker(\phi)/Z(U_l/\ker(\phi)) \] is an elementary abelian p-group.

Hence \( U_l/\ker(\phi) \) is an extraspecial p-group. The ordinary character theory of such groups is well known. In summary, we have the following:

1. The order of \( U_l/\ker(\phi) \) is \( p^{1+2a} \) for some integer \( a \).

2. There are exactly \( p^{2a} \) linear characters, each corresponding to a character of the quotient
\[ U_l/\ker(\phi)/Z(U_l/\ker(\phi)). \]

3. There are exactly \( p - 1 \) non-linear characters each of dimension \( p^a \). There is one of these characters \( \chi \) for each non-trivial irreducible character \( \theta \) of \( Z(U_l/\ker(\phi)) \) with character values given by \( \chi(x) = p^a\theta(x) \) for \( x \) in \( Z(U_l/\ker(\phi)) \) and \( \chi(x) = 0 \) for \( x \) not in \( Z(U_l/\ker(\phi)) \).

Considered as a character of \( U_l/\ker(\tau_r) \), \( \phi \) is uniquely determined by its kernel. We summarize the properties of \( \phi \).

**Proposition 6.5.2** \((15), \text{Lemma 7.2.4})\ Let \( 1 \leq r \leq l \) and \( \phi \in \text{Irr}(U_l, \tau_r) \).

1. \( \phi \) is extendible to \(TP_l(\phi)\)
2. $\phi(1) = q^{r(n-2l)}$

3. If $r = l$, then $\text{Irr}(U_l, \tau_l)$ contains a unique member and

$$T_{F_l}(\phi) = T_{F_l}(\tau_l) \cong (U_l(q) \times \text{GL}_{n'-2l}(q^2)) \ltimes U_l.$$  

4. If $1 \leq r < l$, then $\phi$ is uniquely determined by its kernel.

$$T_{F_l}(\phi) = T_{L_i}(\phi) \ltimes U_l = (T_{L_i}(\tau_r) \cap T_{L_i}(\ker(\phi))) \ltimes U_l.$$  

For a non-negative integer $k$, let $V^k$ be the natural module for $\text{GL}_k(q^2)$. Let $R \leq V^l$ be a subspace of codimension $r$ stabilized by $T_{L_i}(\tau_r)$. Then

$$V^{l-r} \otimes (V^{n'-2l})^*$$

is a module for $\text{GL}_{l-r}(q^2) \times U_{n'-2l}(q)$. Moreover, there is a $1 - 1$ correspondence

$$\text{Irr}(U_l, \tau_r) \longleftrightarrow \text{Irr}(V^{l-r} \otimes (V^{n'-2l})^*).$$
Case 3. In order to count characters of $P_J$ that do not contain $Z_l$ in their kernel, we observe that

$$k_d^1(P_J, Z_l, \rho, \det, j) = \sum_{\tau \in Z_l/P_J} k_d(P_J, \tau, \rho, \det, j)$$

$$= \sum_{\tau \in Z_l/P_J} \sum_{\phi \in \text{Irr}(U_1, \tau)} k_d(P_J, \phi, \rho, \det, j)$$

$$= \sum_{\tau \in Z_l/P_J} \sum_{\phi \in \text{Irr}(U_1, \tau)} k_{d-d'}(T_{P_J}(\phi)/U_1, \rho, \det, j').$$

where $d' = r(n' - 2l) - d''$ for $d''$ equal to the exponent of $q$ in $|T_{P_J}(\phi)\backslash P_J|$ and $j'$ is the least positive integer such that

$$j \text{ divides } j' \cdot |T_{P_J}(\phi)\ker(\det)\backslash P_J|.$$

We use Proposition 6.5.2 to describe $P_J$ orbits on $X = \text{Irr}(V, r)$ for the general case where $J$ is any subset in $I$ with maximal element $l$.

Definition 6.5.3 For fixed $r$, let $K = T_{\text{GL}_l(q^2)}(\tau_r)$.

Let $w$ be zero if $r = l$, or if $r < l$ a complement in $V^l$ to the $r$ dimensional subspace stabilized by $P_r^{+l}$. We have the structure of $K$ given above by Proposition 6.5.1.

Proposition 6.5.4 There is a 1-1 correspondence between the $P_J$-orbits on $X$ and the $K$-orbits on the set of chains of type $J(<l)$ in $\mathcal{P}(V^l)$. If $\tau$ corresponds to $c$ then up to conjugation

$$T_{P_J^{+l}}(\tau) = T_K(c).$$
Let $r = l$ so that $K = T_{\GL_l(q^2)}(\pi) = U_l(q)$.

**Definition 6.5.5** Let $S^z(V, J, l)$ denote the $P_J$-orbits in $X$ labeled by a normal chain in $\mathcal{P}(V^l)$ of type $J(< l)$.

Let $r < l$ so that $w$ is a complement in $V^l$ to the $r$ dimensional subspace stabilized by $P_r^+$. We can assume that $K \leq T_{\GL_l(q^2)}(w)$. Then $K$ is transitive on complements of $w$. Moreover if $\tau \in X$ corresponds to $c$ of type $J(< l)$ with

$$c : 0 < V_1 < \cdots < V_i < \cdots < V_s$$

where $V_i$ is a complement to $w$ then we may assign to $c$ a pair $(c_1, c_2)$ of shorter chains much as we did in the previous section. As $(V_i)^\perp = w$ we may define

$$c_1 : 0 < V_1 < \cdots < V_{i-1}$$

$$c_2 : 0 < (V_{i+1} \cap w) < \cdots < (V_s \cap w).$$

Notice that $c_1$ is a chain of type $J(< r)$ in $\mathcal{P}(V^r)$ and $c_2$ of of type $\{j-r|j \in J(> r)\}$ in $\mathcal{P}(V^{l-r})$. We have the stabilizer of $c$ in $K$ given by

$$T_K(c) = T_{U_r(q)}(c_1) \times T_{\GL_{l-r}(q^2)}(c_2).$$

With this identification of $c$ with the pair $(c_1, c_2)$, we make the following definition.
**Definition 6.5.6** Let $S^z(V, J, r)$ denote the $P_J$-orbits in $X$ labeled by a normal chain $c_1$ of type $J(<r)$ in $P(V^r)$.

By construction $S^z(V, J, r)$ is in 1-1 correspondence with the $U_r(q)$-orbits on the set of chains of type $J(<r)$ in $P(V^r)$.

**Definition 6.5.7** We define the following subsets of $S^z(V, J, r)$.

1. Let $S^z_r(V, J, r)$ be the subset labeled by a singular normal chain $c_1$ of type $J(<r)$ in $P(V^r)$.

2. For $r'<r$ let $S^z_{r'}(V, J, r)$ be the subset labeled by a normal chain $c_1$ of type $J(<r)$ in $P(V^r)$ with non-singular rank $r'$.

We may now assign to $c_1$ a pair of even shorter chains based on the dimension of the minimal non-degenerate subspace in $c_1$. In the manner of the previous section $c_1$ corresponds to the pair $(c_{11}, c_{12})$ where $c_{11}$ is a totally isotropic chain in $P(V^{r'})$ and $c_{12}$ is a normal chain in $P(V^{r'-r'})$.

In summary for $J$ with maximal member $l$, $r \in J$ and $\tau \in X = \text{Irr}(V, r)$ corresponding to $(c_1, c_2)$ where $c_1$ corresponds to $(c_{11}, c_{12})$ we have

$$T_{P_J}(\tau) = \left( T_{P_J}(\tau) \times U_{n'-2l}(q) \right) \rtimes U_l$$

$$\left( T_{U_r(q)(c_1)} \times T_{GL_{l-r}(q^2)}(c_2) \times U_{n'-2l}(q) \right) \rtimes U_l$$

$$\left( T_{U_{r'}(q)(c_{11})} \times T_{U_{r'-r'}(q)(c_{12})} \times T_{GL_{l'-r'}(q^2)}(c_2) \times U_{n'-2l}(q) \right) \rtimes U_l$$

$$\left( P^r_{J(<r')} \times T_{U_{r'-r'}(q)(c_{12})} \times T_{GL_{l'-r'}(q^2)}(c_2) \times U_{n'-2l}(q) \right) \rtimes U_l.$$
where if \( r = l \) then we take \( c_2 = 0 \). Now let \( \phi \in \text{Irr}(U_l, \tau_r) \) correspond to \( \psi \in \text{Irr}(V^{l-r} \otimes (V^{n'-2l})*) \) and let

\[
D = T_{\text{GL}_{l-r}(q^2)}(c_2) \times U_{n'-2l}(q).
\]

Then

\[
T_{P_J}(\phi) = (T_{P_J}(\tau) \times T_D(\psi)) \ltimes U_l.
\]

### 5.1 The determinant map in the central module context

We would like to determine the determinant map on quotients of the form \( P_J/U_l \leq U_{n'}(q) \), as embedded in the larger unitary group \( U_{n}(q) \). Here \( P_J \) is a parabolic subgroup of \( U_{n'}(q) \). We have

\[
P_J/U_l \cong P_{J_{(l)}}^{+l} \times U_{n'-2l}(q).
\]

At this stage we consider the same map \( \mathcal{D} \) from section 4.1 on unitary linear modules.

\[
\mathcal{D} : \quad P_{J_{(l)}}^{+l} \times U_{n'-2l}(q) \to F^*_q,
\]

\[
(x, y) \mapsto \det(x)^{i(1-q)} \det(y),
\]

for some positive integer \( i \). The map \( \mathcal{D} \) is constructed to be the restriction to \( P_J/U_l \) of the usual determinant map on \( U_n(q) \). The integer \( i \) depends on the embedding of \( P_J/U_l \) in \( U_n(q) \).

Observe the image of \( P_J/U_l \) under this map is all of \( C_{(q+1)} \) if \( n'-2l \neq 0 \). On the other hand if \( n' = 2l \) then the image is \( C_{(q+1)}/\text{gcd}(q+1,i) \).
We briefly remark on the restriction of the map $D$ to the factor $P_{J(\langle r' \rangle)}$ in the stabilizer $T_{J}(\tau)$. For $A \in P_{J(\langle r' \rangle)}$, $D(A) = \det(A)^{2i}$ since $-q \equiv 1 \mod (q + 1)$.

**Remark:** A version of the useful cancellation discussed in section 2 applies to the central modules in the following way: Let $G$ be the subgroup of $\text{GL}_l(q^2) \times \text{GL}_l(q^2)$ defined

$$G = \{(A, \tilde{A})| A \in \text{GL}_l(q^2)\}.$$  

Let $V^l$ be the natural module for $\text{GL}_l(q^2)$. Set $V = V^l \otimes (V^l)^*$. Let $G$ act on $V$ via $A \cdot v = Av\tilde{A}^{-1}$.

In keeping with the notation used in section 1, $l = n_1 = n_2$ and $G_2$ is an isomorphic copy of $G_1$. As an $F_{q^2}$-vector space, recall a basis for $V$ was given by $\{E_{i,j}\}$. View $F_{q^2}$ as an extension of $F_q$. Let $\vartheta$ be a root of the irreducible polynomial

$$x^2 - (\vartheta + \vartheta^q) + \vartheta^{q+1} \text{ in } F_q[x]$$

so that $F_{q^2} = F_q(\vartheta)$. Let $V$ be the $F_q$-subspace of $V$ with basis given by

$$\{E_{i,j} - E_{l-j+1,l-i+1}\} \bigcup \{\vartheta E_{i,j} - \vartheta^q E_{l-j+1,l-i+1}\}.$$
The subspace $V$, and hence $\text{Irr}(V)$, is closed under the action of $G$. Let $X = \text{Irr}(V, r)$. Then $X$ is a subset of $\text{Irr}(V, r)$. For $J \subseteq [l - 1]$, define the subgroup

$$G_J = \{(A, \tilde{A}) | A \in P^{+l}_J\} \leq G.$$ 

Let $H = G \ltimes V$ and $H_J = G_J \ltimes V$. In this case for $(A, \tilde{A}) \in G_J$,

$$D(A, \tilde{A}) = \det(A)^i(1-q)$$

for some integer $i$.

Then $Y = \{y \leq V^l | \text{codim}(y) = r\}$ is a transitive $H$-set. If $J$ has maximal member $l$ then

$$P_J/U_l \cong G_{J(<l)} \times U_{n'-2l}(q)$$

and $Z_l \cong V$.

If $r = l$ then $X(0) = X$ and $T_{G_J}(0) = G_J$. Now suppose that $r < l$. Set $w$ to be a complement to the $r$-dimensional subspace $R$ of $V^l$ stabilized by $P_r^{+l}$. The vector space $V^l = R \oplus w$. Take $\tau \in X(w)$ so that viewed as a linear transformation of $V^l$, $\tau$ has kernel equal to $w$. Now even if $r$ is not a member of $J(<l)$ we have

$$T_{G_J}(w) = M_{r,l-r}(q^2) \ltimes \left( P_{J(<r)}^{+r} \times P_{J_1}^{+(l-r)} \right)$$
where $J_1 = \{ j - r | j \in J(> r) \}$. For $y \in Y$, $y = Aw$ for some $A \in \text{GL}_l(q^2)$, hence $T_{G_J}(y)$ is $\text{GL}_l(q^2)$-conjugate to $T_{G_J}(w)$.

The main difference in applying the useful cancellation to the central module context is in the definition of a $G$-stable function. We define

$$f : \Delta(P) \times Y \longrightarrow Z$$

$$(c_J, y) \mapsto k_d(P_{J \cup \{1\}}, X(y), \rho, \mathcal{D}, j).$$

Notice that $P_{J \cup \{1\}}$ is not $H_J$. Also notice that $l - r \neq 0$ in this case so that

$$|TP_j(y) \ker(\mathcal{D}) \setminus P_J| = 1$$

holds.

With these changes in the proof of Proposition 6.2.4 the useful cancellation applies to the the central module case.

**Proposition 6.5.8** Let $Z \leq Z(G)$ and $\rho \in \text{Irr}(Z)$. Fix $1 \leq r \leq l$. Let $V = Z_l$ and let $X = \text{Irr}(V, r)$. Let $w$ be a complement in $V^l$ to the $r$-dimensional subspace stabilized by $P_{r+1}^l$, in its action on the natural module for $\text{GL}_l(q^2)$. Let $X(w) = \{ \tau \in X | \ker(\tau) = w \}$. Let $\mathcal{D}$ be the map on $P_J$ defined above. For $d \geq 0$ the following holds

1. If $r = l$, then $Y = 0$ and $X = X(0)$. 

2. If $r < l$, then

$$
\sum_{J \subseteq [l-1]} (-1)^{|J|} k_d(P_{J \cup \{l\}}, X, \rho, \mathcal{D}, j) = \sum_{J \subseteq [l-1]} (-1)^{|J|} k_d(P_{J \cup \{l\}}, X(w), \rho, \mathcal{D}, j).
$$

Representatives of $T_G(w)$-orbits in $X(w)$ are given by chains of type $J(\prec l) \cup \{r\}$ in $\mathcal{P}(V^l)$ where $c$ corresponds to $(c_1, c_2)$ as described earlier in this section.

Having examined the orbits, stabilizers, and maps $\mathcal{D}$ at the level of $H_J$ in the linear cases (both general and unitary) and $P^n_J$ in the central case, we are ready to proceed up to the level of $P^n_J$. We will do so in the following chapter where we begin a systematic codification of the alternating sum that occurs on the left hand side of Equation 4.6b.
CHAPTER 7

COUNTING CHARACTERS OF PARABOLIC SUBGROUPS NOT TRIVIAL ON THE UNIPOTENT RADICAL

For fixed $J \subset I$ write $J = \{j_1, j_2, \ldots, j_s\}$ in increasing order. Then $U_J$ has the following normal series:

$$U_J = U_J(\geq j_1) > U_J(\geq j_2) > \cdots > U_J(\geq j_s) = U_{j_s} \geq Z(U_{j_s}) > 1 \quad (7.1)$$

The quotient groups in this series are abelian. These are the modules described as general linear, unitary linear, and central modules in the previous chapter where recall we examined their $H_J$ orbits. If $\chi \in \text{Irr}(P_J)$ does not contain $U_J$ in its kernel then there exists a term in the above series which is contained in the kernel of $\chi$, but the previous term is not in the kernel of $\chi$.

Recall $V(j_i, j_{i+1})$ denotes the quotient group $U_J(\geq j_i)/U_J(\geq j_{i+1})$. We note that in the following definition, 1. is a slight modification of definition 4.1.2 but that 2. has already been defined and is only restated for clarity.

**Definition 7.0.9** For $\rho \in \text{Irr}(Z)$:

1. Let $k^1_d(P_J, V(j_i, j_{i+1}), \rho, \text{det}, j)$ be the number of $\chi \in \text{Irr}(P_J)$ which are trivial on $U_J(\geq j_{i+1})$ but not trivial on $U_J(\geq j_i)$, lie over $\rho$, and upon restriction to $\ker(\text{det})$ have $j'$ irreducible constituents where $j \mid j'$.
2. Let $k_d^1(P_J, U_{j,\rho,\det, j})$ be the number of $\chi \in \text{Irr}(P_J)$ which are not trivial on $U_{j,\rho,\det, j}$, lie over $\rho$, and upon restriction to $\ker(\det)$ have $j'$ irreducible constituents where $j \mid j'$.

**Case 1.** If $\chi \in \text{Irr}(P_J)$ is trivial on $U_{j(\geq j_i+1)}$ but not trivial on $U_{j(\geq j_i)}$, then we may consider $\chi$ as a character of

$$
\overline{P_J} = P_J/U_{j(\geq j_i+1)} \cong P_{J(\leq j_i)}^{j+1} \times L_{j'}
$$

where $J' = \{ j - j_{i+1} | j \in J(> j_{i+1}) \}$ and $L_{j'}$ is isomorphic to a Levi subgroup in $U_{n-2j_i+1}(q)$.

Then

$$P_{J(\leq j_i)}^{j+1} \cong \left( P_{J(\leq j_i)}^{j+1} \times \text{GL}_{j_{i+1}-j_i}(q) \right) \rtimes V(j_i,j_{i+1}), \quad (7.2)
$$

as in section 3 of chapter 6, on general linear modules, where $P_{J(\leq j_i)}^{j+1}$ plays the role of $H_J$.

Hence $\chi$ is of the following form:

$$\chi = \chi'(\tilde{\tau} \psi)_{P_{J(\leq j_i)}^{j+1}}
$$

where $\tau \in \text{Irr}(V(j_i,j_{i+1}))$ is linear and hence extendible to $\tilde{\tau} \in \text{Irr}(T)$ and $\psi \in \text{Irr}(T/V(j_i,j_{i+1}))$ where $T$ is the stabilizer of $\tau$ in $P_{J(\leq j_i)}^{j+1}$ and $\chi'$ is an irreducible character of the factor $L_{j'}$. It follows that $\chi$ corresponds to $\tau$ and that

$$k_d^1(P_J, V(j_i,j_{i+1}), \rho, \det, j) = \sum_{\tau} k_d(P_J, \tau, \rho, \det, j)
$$

where this sum is taken over representatives $\tau$ of $P_J$-orbits in $\text{Irr}(V(j_i,j_{i+1}))$. 
**Case 2.** Suppose that $\chi \in \text{Irr}(P_J)$ is not trivial on $U_{js}$. There are two possibilities: $\chi$ is trivial on $Z(U_{js}) = Z_{js}$ or not.

**Case 2a.** If $\chi$ is trivial on $Z_{js}$, then we may consider $\chi$ as a character of $P_J/Z_{js}$ as in section 4 of chapter 6, on unitary linear modules, where $P_J/Z_{js}$ plays the role of $H_J$. Hence $\chi$ is of the following form:

$$\chi = (\tilde{\tau}\psi)^{P_J/Z_{js}},$$

where $\tau \in \text{Irr}(U_{js}/Z_{js})$ linear and hence extendible to $\tilde{\tau} \in \text{Irr}(T)$ and $\psi \in \text{Irr}(T/(U_{js}/Z_{js}))$ where $T$ is the stabilizer of $\tau$ in $P_J/Z_{js}$. It follows that $\chi$ corresponds to an irreducible character of the quotient $U_{js}/Z_{js}$, and that

$$k_d(P_J, U_{js}/Z_{js}, \rho, \text{det}, j) = \sum_{\tau} k_d(P_J, \tau, \rho, \text{det}, j)$$

where this sum is taken over nontrivial representatives $\tau$ of $P_J$-orbits in $\text{Irr}(U_{js}/Z_{js})$.

**Case 2b.** Suppose $\chi$ is not trivial on $Z_{js}$. As discussed in section 5 of chapter 6, $\chi$ corresponds to $\phi \in \text{Irr}(U_{js})$ where $\phi$ lies over a non-trivial character $\tau_r \in \text{Irr}(Z_{js})$. Set $N = \ker(\phi)$ then $K = N \cap Z_{js}$ is non-trivial. Then $U_{js}/N$ is an extra special $p$-group and hence the
non-linear $\phi \in \text{Irr}(U_{js}/N)$ is extendible to $\tilde{\phi} \in \text{Irr}(T)$ where $T$ is the stabilizer of $\phi$ in $P_J/N$.

Thus $\chi$ is of the form:

$$\chi = (\tilde{\phi}\psi)^{P_J}$$

where $\phi \in \text{Irr}(U_{js}/N)$ is the unique character whose restriction to $Z_{js}/K$ is a multiple of non-trivial $\tau \in \text{Irr}(Z_{js}/K)$. The character $\psi$ is the lift to $T$ of an irreducible character of $T/(U_{js}/N)$. The character $\phi$ lifts to an irreducible character in $\text{Irr}(U_{js})$ and $\tau$ lifts to an irreducible character in $\text{Irr}(Z_{js})$. It follows that $\chi$ corresponds to an irreducible character of $Z_{js}$, and that

$$k_d^1(P_J, Z_{js}, \rho, \text{det}, j) = \sum_{\tau} \sum_{\phi \in \text{Irr}(U_{js}, \tau)} k_d(P_J, \phi, \rho, \text{det}, j)$$

where this sum is taken over nontrivial representatives $\tau$ of $P_J$ orbits in $\text{Irr}(Z_{js})$.

We have the following decomposition:

$$k_d^1(P_J, U_J, \rho, \text{det}, j) = \sum_{i=1}^{s-1} k_d^1(P_J, V(j_i, j_i+1), \rho, \text{det}, j) + k_d^1(P_J, U_{js}, \rho, \text{det}, j). \quad (7.4)$$

For fixed $J$ containing adjacent elements $l$ and $l'$, let $V = V(l, l') = U_{jl(l)}/U_{jl(l')} \cong V_1 \otimes V_2$, where we recall that $V_1$ is the natural module for $\text{GL}_l(q^2)$ and $V_2$ is the dual of the natural module for $\text{GL}_{l'-l}(q^2)$. Then

$$\overline{P}_J = P_J/U_{jl(l')}$$ contains a submodule isomorphic to $V$ (see Equation 7.2).
The useful cancellation from chapter 6 applies in this situation. We will sum over all $J \subseteq I$ of the form $J = J' \cup J''$ where $J' \subseteq [l-1]$ varies and $J'' \subset I$ is fixed with minimal member $l'$. We will only be concerned with calculating $T_{H_J}(\tau)$ where $\tau$ has rank $r \in J$ and $\ker(\tau)$ is a complement $w$ in $V_1$ to the $r$-dimensional subspace stabilized by $P^r_{l}$. Then, for $r \in J$ and $\tau \in \text{Irr}(V,r)$ with $\ker(\tau) = w$

$$k_d(P_J, \tau, \rho, \det, j) = k_d(P_{J'}/V, \tau, \rho, \det, j) = k_{d-d'}(P_{J'}/V(\tau), \rho, \det, j')$$

(7.5)

where $d'$ is the power of $q$ in the index of $P_{J'}/V(\tau)$ in $P_{J'}/V$ and $j'$ is the smallest positive integer such that

$$j | j' \cdot |P_{J'}/V(\tau) \cdot \ker(\det) \setminus (P_{J'}/V)|.$$

It turns out that $P_{J'}/V(\tau)$ contains a subgroup which itself contains a submodule isomorphic to a general linear module and hence we may further expand Equation 7.5.

Let $J$ have maximal element $l$. Let $V = U_l/Z_l \cong V_1 \otimes V_2$, where recall $V_1$ is the natural module for $GL_l(q^2)$ as above and $V_2$ is the dual of the natural module for $U_{n-2l}(q)$. Then

$$P_J = P_{J}/Z_l$$

contains a submodule isomorphic to $V$ (see Equation 7.3).

As above, the useful cancellation from chapter 6 applies to this situation. We will when sum over all $J \subseteq I$ with maximal element $l$. We will only be concerned with calculating $T_{H_J}(\tau)$
where $\tau$ has rank $r \in J$ and $\ker(\tau)$ is a complement $w$ in $V_1$ to the $r$-dimensional subspace stabilized by $P^r$. Then, for $r \in J$ and $\tau \in \text{Irr}(V,r)$ with $\ker(\tau) = w$,

$$k_d(P_J, \tau, \rho, \det, j) = k_d(\overline{P}_J/V, \tau, \rho, \det, j) = k_{d-d'}(T_{P^l_J/V}(\tau), \rho, \det, j')$$  (7.6)

where $d'$ and $j'$ are as given above in Equation 7.5. It turns out that when $\tau$ corresponds to a singular chain in the unitary vector space $V_2^*$, $T_{P_J/V}(\tau)$ contains a subgroup which itself contains submodule isomorphic to a (unitary) quotient module and hence we may further expand Equation 7.6.

The main goal in this chapter is to unravel Equation 7.4 via Equation 7.5 and Equation 7.6. Ku has introduced two sets of triples $E$ and $F$ with related objects which codify this unraveling for the unitary case but without regard to the determinant map or any splitting. For an $e \in E$ or an $f \in F$ we will present Ku’s objects including a length, a parity, a group, and a normal subgroup. For our purposes we will also define a map related to the determinant and an integer related to splitting. We remark that while the definitions of these two new objects are very natural extensions of Ku’s existing objects, the computations involved in producing their definitions are far from trivial. The set $E$ will codify parabolic characters corresponding to internal general linear modules, i.e. those counted in the alternating sum

$$\sum_{J \subseteq I} (-1)^{|J|} \sum_{i=1}^{s-1} k_d(P_J, V(j_i, j_{i+1}), \rho, \det, j),$$
whereas the set $F$ will codify the parabolic characters corresponding to internal unitary linear or central modules, i.e. those characters counted in the alternating sum

$$
\sum_{J \subseteq I} (-1)^{|J|} k_{d_j}^1(P_{J,\rho, \det, j}).
$$

Before proceeding to the rather technical definitions, we present the idea. Fix nonempty $J \subseteq I$.

- For each pair of adjacent members $l$ and $l'$ of $J$, we will define an initial triple $e$ so that its associated group

$$
P(e) \cong P_J/U_\nu \cong \left(\left( P_{J(\leq l)}^{+l} \times \text{GL}_{n-l}(q^2) \right) \rtimes V(l, l') \right) \times L_J',
$$

where $J' = \{ j - l' | j \in J(> l') \}$ and $L_J'$ is a Levi subgroup in $U_{n-2l'}(g)$. Moreover, for $\tau \in \text{Irr}(V(l, l'))$ of rank $r$ where $r \in J$ and $\text{ker}(\tau)$ is a complement $w$ to the $r$-dimensional subspace stabilized by $P_r^{+l}$, we will define a subsequent triple $e'$ related to $e$ such that

$$
T_{P(e)}(\tau) = P(e') \rtimes V(l, l').
$$

If $P(e')$ itself contains a non trivial general linear module we will define a further triple $e''$ related to $e'$ in a similar way. In this way $J, l,$ and $l'$ give rise to a sequence $e, e', e'', \ldots$ of elements in $E$. 
Now let \( l \) be the maximal member of \( J \). We will define an initial triple \( f \) so that its associated group

\[
P(f) \cong P_J \cong \left( P_{j<l}^{+l} \times U_{n-2l}(q) \right) \ltimes U_l
\]

Moreover, for \( \tau \in \text{Irr}(U_l/Z_l) \) of rank \( r \) where \( r \in J \), \( \ker(\tau) \) is a complement \( w \) to the \( r \)-dimensional subspace stabilized by \( P_{r+l}^{+l} \), and \( \tau \) corresponds to a singular flag in a unitary space of suitable dimension, we will define a subsequent triple \( f' \) related to \( f \) such that

\[
T_{P(f)}(\tau) = P(f') \ltimes U_l.
\]

If \( P(f') \) itself contains a non trivial unitary linear module with an irreducible singular character (i.e. corresponds to a singular flag in a unitary space of suitable dimension) we will define a further triple \( f'' \) related to \( f' \) in a similar way. In this way \( J \) and \( l \) give rise to a sequence \( f, f', f'', \ldots \) of elements in \( F \).

In this fashion, we will use the elements of \( E \) and \( F \) to reformulate Equation 4.6b in a systematic way by unraveling Equation 7.5 and Equation 7.6 which leads to a second reduction of Equation 4.6b.

The notation used in the following sections is primarily due to Ku. In particular he defines the following: \( E, P(e), V(e), l(e), |e|, \) and \( d(e) \) used in section 1; \( F, P(f), U(f), Z(f), V(f), l(f), |f|, \) and \( d(f) \) used in section 2; \( S^u(f), S^{vu}(f), S^{nu}(f), S^z(f) \) used in section 3.
1 The elements of $E$ and their related objects

Let $e$ be an ordered triple $(J, C, (l, l'))$, where either $e = (\emptyset, \emptyset, \emptyset)$ or $J$, $C$, and $(l, l')$ satisfy the following conditions:

1. $J = \{j_1, j_2, \ldots, j_r\}$ is a subset of $I = \mathbb{I}$. We will assume that $J$ is enumerated in increasing order.

2. $(l, l')$ is a pair of consecutive members of $J$.

3. $C = \{l_1, l_2, \ldots, l_s\}$ is a subset of $I = \mathbb{I}$ also enumerated in increasing order. The sequence $C$ must be convex. By this we mean that the related sequence $\partial C = (l_1, l_2 - l_1, \ldots, l_s - l_{s-1})$ is non-increasing. We require that the members of this related sequence appear in $J$. We further require that $C$ and $(l, l')$ are related via $l = l_1 < l_2 < \ldots < l_s \leq l'$.

Let $E$ be the set of all such triples $e$. The length of $e$ is $l(e) = |C| = s$ and the parity of $e$ is $|e| = |J|$. Before we proceed to the rest of the objects related to $e \in E$, we make some observations.

**Definition 7.1.1** Given $C = \{l_1, l_2, \ldots, l_s\}$, we define $\partial l_1 = l_1$ and $\partial l_i = l_i - l_{i-1}$ for $2 \leq i \leq s$ so that $\partial C$ is the partition $(\partial l_1, \partial l_2, \ldots, \partial l_s)$.

Observe that a convex sequence $C$ corresponds to a unique partition $\partial C = (\partial l_1, \partial l_2, \ldots, \partial l_s) \vdash l_s$. Notice that the $\partial l_i$’s, where $l = \partial l_1 = l_1$, and $l'$ are contained in $J$. Moreover, $l$ and $l'$ are
consecutive members of $J$. Thus the sequence $\partial l_s \leq \partial l_{s-1} \leq \cdots \leq \partial l_2 \leq \partial l_1 = l < l'$ divides $J$ into a collection of $s + 1$ disjoint subsets

\[ \{ j \in J | j < \partial l_s \}, \ldots, \{ j \in J | \partial l_2 < j < \partial l_1 \}, \{ j \in J | l' < j \}. \]

**Definition 7.1.2** We define

1. $J_s = \{ j \in J \text{ and } j < \partial l_s \}$,
2. $J_i = \{ j - \partial l_{i+1} \in J \text{ and } \partial l_{i+1} < j < \partial l_i \}$ for $1 \leq i < s$, and
3. $J_0 = \{ j - l' \in J \text{ and } l' < j \}$.

Notice that we have the containment $J_s \subset [\partial l_s - 1]$, $J_i \subset [\partial l_i - \partial l_{i+1} - 1]$, and $J_0 \subset [m - l']$.

We are now ready to define the groups related to the triple $e = (J, C, t)$.

**Definition 7.1.3** For $e \in E$ we define the following objects.

1. Let

\[
P(e) = L_{j_0(e)}^{n_0(e)} \times P_{j_1(e)}^{n_1(e)} \times \cdots \times P_{j_{s-1}(e)}^{n_{s-1}(e)} \times P_{j_s(e)}^{n_s(e)}
\]

where

(a) $n_0(e) = n - 2l'$, $J_0(e) = J_0$

(b) $n_i(e) = \partial l_i - \partial l_{i+1}$ and $J_i(e) = J_i$ for $1 \leq i < s$

(c) $n_s(e) = l' - \partial l_{s-1}$ and $J_s(e) = \begin{cases} J_s, & \text{if } l_s = l'; \\ J_s \cup \{ \partial l_s \}, & \text{if } l_s < l'. \end{cases}$
Notice that the first factor is a Levi subgroup of a unitary group whereas the remaining factors are parabolic subgroups of general linear groups.

2. We define an abelian normal subgroup of $P(e)$ in the following way. If $l_s < l'$ then the group $P^{+n_s(e)}_{J_s(e)}$ is contained in the maximal parabolic $P^{+n_s(e)}_{\partial l_s}$ which has unipotent radical $U^{+n_s(e)}_{\partial l_s}$, a normal subgroup of $P^{+n_s(e)}_{J_s(e)}$ and hence of $P(e)$. Set

$$V(e) = \begin{cases} 
U^{+n_s(e)}_{\partial l_s}, & \text{if } l_s < l'; \\
1, & \text{if } l_s = l'.
\end{cases}$$

Observe that

$$P^{+n_s(e)}_{J_s(e)} \cong \left( P^{+\partial l_s}_{J_s} \times \text{GL}_{l'-l_s}(q^2) \right) \rtimes V(e). \quad (7.7)$$

Moreover $V(e)$ is a general linear module for $P^{+\partial l_s}_{J_s} \times \text{GL}_{l'-l_s}(q^2)$ as discussed in Chapter 6 Section 3.

3. We define

$$d(e) = 2 \sum_{i=1}^{s-1} \left( \left( \partial l_i \right)_2 - \left( \partial l_i - l_{i+1} \right)_2 \right).$$

Notice that $d(e)$ depends only on the sequence $C$. 

4. Finally we define a map \( \phi_e : P(e) \to F_{q^2} \). For \( g \in P(e) \), write \( g = A_0 A_1 \cdots A_{s-1} A_s \) where \( A_0 \in L_{J_0(e)}^{n_0(e)} \) and \( A_i \in P_{J_i(e)}^{+n_i(e)} \). We have a decomposition of \( A_s \in P_{J_s(e)}^{+n_s(e)} \). Write \( A_s \equiv A_{s,1} A_{s,2} \mod (V(e)) \) where \( A_{s,1} \in P_{J_s}^{+\partial l_s} \) and \( A_{s,2} \in \text{GL}_{l'_s-l_s}(q^2) \). Let

\[
\phi_e(g) = \det A_0 \left[ \prod_{i=1}^{s-1} (\det A_i)^i \right] (\det A_{s,1})^s \det A_{s,2} \right]^{1-q}
\]

where \( \det \) denotes the usual determinant map. Notice \( \phi_e(P(e)) \leq \mathbb{C}_{q+1} \).

In the context of general linear modules, the map \( D \) defined Chapter 6 Section 3 on \( P_{J_s(e)}^{+n_s(e)} \) corresponds to \( i = s \) and is the restriction of \( \phi_e \) to \( P_{J_s(e)}^{+n_s(e)} \).

An example of \( E \): Fix \( J \) with adjacent members \( l \) and \( l' \). Set the initial \( e = (J, \{l\}, (l, l')) \).

Then \( \partial C = (l) \)

\[
P(e) \cong P_{J,J'(l')} / U_{J,J'(l')} \cong L_{J_0}^{n-2l'} \times P_{J,J'(l')}^{+l'}
\]

\[
\cong L_{J_0}^{n-2l'} \times \left( P_{J,J'(l')}^{+l} \times \text{GL}_{l'-l}(q^2) \right) \times V(e)
\]

and \( V(e) \cong V(l, l') \cong M_{l,l'-l}(q^2) \). Under this isomorphism the map \( \phi_e \) is the determinant map on \( P_{J,J'(l')} / U_{J,J'(l')} \).
For $\tau \in \text{Irr}(V(l, l'))$ of rank $r$ where $r \in J$ and $\ker(\tau) = w$, where $w$ is a complement to the $r$-dimensional subspace stabilized by $P^+_r$, there is a subsequent $e' = (J, \{l, l + r\}, (l, l'))$ with $\partial C' = (l, r)$. Then by definition

$$P(e') \cong L^{n-2l'}_{J_0} \times P^+(l-r) \times P^+(l'-l)_{J_{(\leq r)}}$$

$$\cong L^{n-2l'}_{J_0} \times P^+(l-r) \times (P^+_r \times \text{GL}_{l'-(l+r)}(q^2)) \ltimes V(e')$$

We have

$$T_{P(e)}(\tau) \cong P(e') \ltimes V(e).$$

As a block subgroup embedded in $P(e)/V(e)$, the $P^+_r$ term in the last factor $P^+(l'-l)_{J_{(\leq r)}}$ of $P(e')$ occurs twice. Thus the map $\phi_{e'}$ is the determinant on $P(e')$ as a subgroup in $P_J$.

If $l+r < l'$, then $P^+_r_{J_{(\leq r)}}$ contains a nontrivial submodule isomorphic to $V(e') \cong M_{r,l'-(l+r)}(q^2)$.

Suppose $J_{(\leq r)}$ is nonempty. Then there exists $s \in J_{(\leq r)}$ and $\tau' \in \text{Irr}(V(e'))$ of rank $s$ and $\ker(\tau')$ is a complement $w'$ to the $s$-dimensional subspace stabilized by $P^+_s$. There is a further triple $e'' = (J, \{l, l + r, l + r + s\}, (l, l'))$ with $\partial C'' = (l, r, s)$. By definition

$$P(e'') \cong L^{n-2l'}_{J_0} \times P^+(l-r) \times P^+(r-s) \times P^+(l'-(l+r))_{J_{(\leq s)}}$$

$$\cong L^{n-2l'}_{J_0} \times P^+(l-r) \times P^+(r-s) \times (P^+_s \times \text{GL}_{l'-(l+r+s)}(q^2)) \ltimes V(e'').$$

We have

$$T_{P(e)}(\tau') \cong P(e'') \ltimes V(e').$$
As a block subgroup embedded in \( P(e)/V(e) \), the \( P_{J(\leq r)}^{l(l'-l)} \) term in the last factor \( P_{J(\leq r)}^{l(l'-l)} \) of \( P(e'') \) occurs three times. Thus the map \( \phi_e^{e''} \) is the determinant on \( P(e'') \) as a subgroup in \( P_J \).

**Remark:** The collection of all \( e \in E \) with fixed \( J \) and \((l,l')\) unravels the alternating sum involving characters \( \chi \in \text{Irr}(P_J) \) that correspond to characters in \( \text{Irr}(V(l,l')) \). Informally, we describe the method as taking stabilizers in stabilizers in stabilizers, etc. At each step we mod out the involved interior (general linear) module until the only interior (general linear) module is trivial. This occurs when the last element of \( C \) is \( l' \). Of course this description misses all the important details. The length of \( C \) keeps track of how many times we iterate this process. The partition \( \partial C \) keeps track of the rank of the characters for which we calculate stabilizers.

### 2 The elements of \( F \) and their related objects

Let \( f \) be an ordered triple \((J,C,l)\) where either \( f = (\emptyset,\emptyset,0) \) or \( J, C, \) and \( l \) satisfy the following conditions:

1. The sequence \( J = \{j_1,j_2,\ldots,j_r\} \) is a subset of \( I = [m] \). We will assume that \( J \) is enumerated in increasing order.
2. The sequence \( C = \{l_1,l_2,\ldots,l_s\} \) is a subset of \( I = [m] \) also enumerated in increasing order. The sequence \( C \) must satisfy the same conditions as listed in the previous section.

So \( C \) must be convex and the members of the related sequence \( \partial C \) appear in \( J \).
3. The integer \( l = j_r \) and \( l = l_1 \), so that \( l \) is the maximal member of \( J \) and the minimal member of \( C \).
Notice that $J$ and $C$ are defined as they were for the triples $e \in E$. Let $F$ be the set of all such $f$. The length of $f$ is $l(f) = |C| = s$ and the parity of $f$ is $|f| = |J|$ just as we did for $e \in E$.

Write $\partial C = (\partial l_1, \partial l_2, \ldots, \partial l_s)$ as in definition 7.1.1. Notice that the $\partial l_i$’s are contained in $J$ with $l = \partial l_1 = l_1$. Moreover $l$ is the largest member of $J$. Thus the sequence $\partial l_s \leq \partial l_{s-1} \leq \cdots \leq \partial l_2 \leq \partial l_1 = l$ divides $J$ into a collection of $s$ disjoint subsets

$$\{j \in J | j < \partial l_s\}, \cdots, \{j \in J | \partial l_2 < j < \partial l_1 = l\}.$$

**Definition 7.2.1** We define

1. $J_s = \{j \mid j \in J \text{ and } j < \partial l_s\}$, and

2. $J_i = \{j - \partial l_{i+1} \mid j \in J \text{ and } \partial l_{i+1} < j < \partial l_i\}$ for $1 \leq i < s$.

Notice that we have the containment $J_s \subset [\partial l_s - 1]$ and $J_i \subset [\partial l_i - \partial l_{i+1} - 1]$ for $1 \leq i < s$. We are now ready to define the groups related to the triple $f = (J, C, l)$.

**Definition 7.2.2** For $f \in F$ we define the following objects.

1. Let

$$P(f) = P^{n_1(f)}_{J_1(f)} \times \cdots \times P^{n_{s-1}(f)}_{J_{s-1}(f)} \times P^{n_s(f)}_{J_s(f)}$$

where

(a) $n_i(f) = \partial l_i - \partial l_{i+1}$ and $J_i(f) = J_i$ for $1 \leq i < s$.
(b) $n_s(f) = n - 2l_{s-1}$ and $J_s(f) = J_s \cup \{\partial l_s\} = J(\leq \partial l_s)$

Notice that the last factor is a parabolic subgroup of a unitary group, whereas the first $s - 1$ factors are parabolic subgroups of general linear groups.

2. We define a normal subgroup of $P(f)$. The group $P_{J_s(f)}^{n_s}$ is contained in the maximal parabolic $P_{\partial l_s}^{n_s}$ which has unipotent radical $U_{\partial l_s}^{n_s}$, a normal subgroup of $P_{J_s(f)}^{n_s}$ and hence of $P(f)$. We set $U(f) = U_{\partial l_s}^{n_s}$. Let $Z(f) = Z(U(f))$ and the quotient $V(f) = U(f)/Z(f)$. Observe that

$$
P_{J_s(f)}^{n_s} \cong \left( P_{J_s}^{+\partial l_s} \times U_{n-2l_s}(q) \right) \rtimes U(f) \tag{7.9}
$$

Moreover $V(f)$ is a unitary linear module for $P_{\partial l_s}^{+\partial l_s} \times U_{n-2l_s}(q)$ as discussed in Chapter 6 Section 4 and $Z(f)$ is a central module for $P_{J_s}^{+\partial l_s} \times U_{n-2l_s}(q)$ as discussed in Chapter 6 Section 5.

3. We define

$$
d(f) = 2 \sum_{i=1}^{s-1} \left( \binom{\partial l_i}{2} - \binom{\partial l_i - l_{i+1}}{2} \right).
$$

Notice that $d(f)$ depends only on the sequence $C$. 
4. Finally we define a map \( \phi_f : P(f) \to F_q^2 \). For \( g \in P(f) \), write \( g = A_1 \cdots A_{s-1} A_s \) where \( A_i \in P^{+n_i(f)}_{J_i(f)} \) for \( 1 \leq i < s \) and \( A_s \in P^{n_s(f)}_{J_s(f)} \). We have a decomposition of \( A_s \). Write \( A_s \equiv A_{s,1}A_{s,2} \mod(U(f)) \) where \( A_{s,1} \in P^{+\partial l_s}_{J_s(f)} \) and \( A_{s,2} \in U_{n-2l}(q) \). Let

\[
\phi_f(g) = \left[ \prod_{i=1}^{s-1} (\det A_i)^i \right] (\det A_{s,1})^s \quad \det A_{s,2}
\]

(7.10)

where \( \det \) denotes the usual determinant map. Notice \( \phi_f(P(f)) \leq C_{q+1} \).

In the context of unitary modules (both linear and central) the map \( D \) defined in Chapter 6 Sections 4 and 5 on \( P^{n_s(f)}_{J_s(f)} \) corresponds to \( i = s \) and is the restriction of \( \phi_f \) to \( P^{n_s(f)}_{J_s(f)} \).

An example of \( F \): Fix \( J \) with maximal member \( l \). Set the initial \( f = (J, \{l\}, l) \). Then \( \partial C = (l) \) and

\[
P(f) \cong P_J \cong (P_J/U_l) \ltimes U_l \\
\cong \left( P^{+l}_{J(<l)} \times U_{n-2l}(q) \right) \ltimes U(f)
\]

where \( U(f) \cong U_l \), \( V(f) \cong U_l/Z_l \), \( Z(f) \cong Z_l \), and under this isomorphism \( \phi_f \) is the determinant map on \( P_J \).

If \( n \neq 2l \) then \( U_l/Z_l \) is not trivial. For \( \tau \in \text{Irr}(U_l/Z_l) \cong \text{Irr}(V(f)) \) of rank \( r \) and \( \ker(\tau) = w \), where \( w \) is a complement to the \( r \)-dimensional space stabilized by \( P^{+l}_r \), corresponding to a
singular chain of type $J(< r)$ where $r \in J$, there is a subsequent $f' = (J, \{l, l + r\}, l)$ with $\partial C' = (l, r)$. Then by definition

$$P(f') \cong P_{J_1(f')}^{+(l-r)} \times P_{J \leq r}^{n-2l}$$

$$\cong P_{J_1(f')}^{+(l-r)} \times \left( P_{J(<r)}^{+r} \times U_{n-2(l+r)}(q) \right) \rtimes U(f').$$

We have

$$T_{P(f)}(\tau) \cong P(f') \rtimes U(f).$$

As a block subgroup embedded in $P(f)/U(f)$, the $P_{J(<r)}^{+r}$ term in the last factor $P_{J \leq r}^{n-2l}$ of $P(f')$ occurs twice. Thus the map $\phi_{f'}$ is the determinant on $P(f')$ as a subgroup in $P_J$.

If $2(l + r) < n$, then $P_{J \leq r}^{n-2l}$ contains a submodule isomorphic to $U(f')$. If $V(f')$ is not trivial and $\text{Irr}(V(f'))$ contains a character $\tau'$ with $\ker(\tau') = w'$, where $w'$ is a complement to the $s$-dimensional space stabilized by $P_{s}^{+r}$, and $\tau'$ corresponds to a singular chain of rank $s$ and type $J(< s)$ then if $s \in J$ we have $f'' = (J, \{l, l + r, l + r + s\}, l)$ with $\partial C'' = (l, r, s)$. Checking with the definition for $f''$ we have

$$P(f'') \cong P_{J_1}^{+(l-r)} \times P_{J_2}^{+(r-s)} \times P_{J \leq s}^{n-2(l+r)}$$

$$\cong P_{J_1}^{+(l-r)} \times P_{J_2}^{+(r-s)} \times \left( P_{J(<s)}^{+s} \times U_{n-2(l+r+s)}(q) \right) \rtimes U(f'')$$

We have

$$T_{P(f')}(\tau') \cong P(f'') \rtimes U(f').$$
As a block subgroup embedded in $P(f)/V(f)$, the $P_{J(<s)}^{+s}$ term in the last factor $P_{J(\leq s)}^{n-2t(s+r)}$ of $P(f'')$ occurs three times. Thus the map $\phi_{f''}$ is the determinant on $P(f'')$ as a subgroup in $P_J$.

**Remark:** The collection of all $f \in F$ with fixed $J$ with maximal element $l$ unravels the alternating sum involving characters $\chi \in \text{Irr}(P_J)$ that correspond to characters in $\text{Irr}(U_l)$. Informally, we describe the method as taking stabilizers of singular flags in stabilizers of singular flags in stabilizers of singular flags, etc. At each step we mod out the involved interior (unitary linear) module until the remaining interior (unitary linear) module is trivial. This occurs when the last element of $C$ is $n$. Of course this description misses all the important details. The length of $C$ keeps track of how many times we iterate this process. The partition $\partial C$ keeps track of the rank of the characters corresponding to singular flags for which we calculate stabilizers.

### 3 Results concerning members of $E$ and $F$

There are a number of results that we need regarding members of $E$ and $F$. These results lead to the unraveling of our alternating sum which in turn leads to some very nice cancellation. These are primarily modifications of Ku’s results ((15), Chapter 8); however we have the added parameters det and $j$. To that end in this section we will define integers $j_e$ and $j_f$ which codify the splitting of characters upon restriction to the kernel of the determinant map. Regarding convex chains $C$ and $C'$, if $C = C'\setminus\{\max C'\}$ we say that $C'$ covers $C$ and write $C \prec C'$. 


We begin with members of the set $E$. Let $C$ be a sequence with $|C| = s$ and $l_s < l'$. Fix $e = (J, C, (l, l'))$ and $e' = (J, C \cup \{l_s + r\}, (l, l'))$ in $E$, where $1 \leq r \leq \min\{\partial l_s, l' - l_s\}$, $r \in J$, and $l_s + r \leq l'$. Then $C \prec C \cup \{l_s + r\}$. Notice $l(e) = s$ and $l(e') = s + 1$. Take $\tau \in \text{Irr}(V(e), r)$ where $\ker(\tau)$ is a complement $w$ to the $r$-dimensional space stabilized by $P^+\partial l_s$. Then

$$TP(e)(\tau) = L_{j_0(e)}^{n_0(e)} \times P^{+n_1(e)}_{J_1(e)} \times \cdots \times P^{+n_s(e)}_{J_{s-1}(e)} \times TP^{+n_s(e)}_{J_s(e)}(\tau)$$

and $TP^{+n_s(e)}_{J_s(e)}(\tau) \cong \left(P^{+n_s(e')}_{J_s(e')} \times P^{+n_{s+1}(e')}_{J_{s+1}(e')}\right) \ltimes V(e)$. Thus

$$TP(e)(\tau) \cong P(e') \ltimes V(e)$$

and $\phi_e$ restricted to $TP(e)(\tau)$ is $\phi_{e'}$. Hence

$$k_{d-d(e)}(P(e), \tau, \rho, \phi_e, j) = k_{d-d(e)}(P(e'), \rho, \phi_{e'}, j') \quad (7.11)$$

where $j'$ the least positive integer such that $j$ divides $j' \cdot |TP(e)(\tau)\ker(\phi_e)\backslash P(e)|$. Write $T = TP(e)(\tau)$, $K = \ker(\phi_e)$. Then $K \trianglelefteq P(e)$ so

$$TK/K \cong T/T \cap K \cong T/V(e) / (T \cap K)/V(e) \cong P(e') / \ker(\phi_{e'})$$

Moreover $P(e)/K \cong \phi_e(P(e))$ and $P(e')/\ker(\phi_{e'}) \cong \phi_{e'}(P(e'))$ and hence

$$|TK\backslash P(e)| = \left|P(e) / TK\right| = \left|P(e) / K / TK / K\right| = \left|\phi_e(P(e)) / \phi_{e'}(P(e'))\right|$$
Since \( l_s < l' \), i.e. \( l' - l_s \neq 0 \) the image of \( P(e) \) under \( \phi_e \) is \( C_{q+1} \). Thus we have

\[
|TK\setminus P(e)| = \frac{|\phi_e(P(e))|}{|\phi_{e'}(P(e'))|} = \frac{q + 1}{|\phi_{e'}(P(e'))|}.
\]

**Definition 7.3.1** Let \( j_e \) be the smallest positive integer such that \( j \) divides \( j_e \cdot q + 1 \)\(|\phi_e(P(e))|.

**Remark:** Observe that for \( e \) of length 1, \( j_e = j \) certainly holds. In general, for fixed \( e \in E \), \(|\phi_e(P(e))|\) is almost always equal to \( q + 1 \) and hence \( j_e = j \). The image

\[
\phi_e(P(e)) = \prod_{i=0}^{s}(C_{h_i})
\]

where \( C_{h_i} \) is the image of the \( i \)-the factor and so depends on the \( n_i(e) \). If any of \( n_0(e), n_1(e), \) or \( l' - l_s \) are nonzero, then \(|\phi_e(P(e))| = q + 1 \). If all three are zero then \( C \) cannot be covered and

1. \( n = 2m, l' = m, \) and \( l_s = m \)
2. \( C = \{l, 2l, \ldots, m\} \) so that \( \partial C \) begins \((l, l, \ldots)\).

On the other hand if \( C \) can be covered, and \( e = (J, C, (l, l')) \) and \( e' = (J, C \cup \{l_s + r\}, (l, l')) \) are members of \( E \) as in Equation 7.11, then

\[
k_{d-d(e)}(P(e), \tau, \rho, \phi_e, j) = k_{d-d(e)}(P(e), \tau, \rho, \phi_e, j_e) = k_{d-d(e')}(P(e'), \rho, \phi_{e'}, j_{e'}).\]
Finally, for fixed \( e \) with \( l(e) \geq 2 \)

\[
k_{d-d(e)}(P(e), \rho, \phi_e, j_e) = k_{d-d(e)}^0(P(e), V(e), \rho, \phi_e, j_e) + k_{d-d(e)}^1(P(e), V(e), \rho, \phi_e, j_e)
\]

where \( k_{d-d(e)}^1(P(e), V(e), \rho, \phi_e, j_e) = 0 \) if \( V(e) = 1 \) i.e. \( l_s = l' \).

By the first reduction in Proposition 6.2.4, when summing over all \( e \) of length 1 and \( J \subseteq I \) of the form \( J = J' \cup J'' \) where \( J' \subseteq [l-1] \) varies and \( J'' \subset I \) is fixed with minimal member \( l' \), we need only sum over \( \tau \in \text{Irr}(V(e), r) \) with \( r \in J' \) and \( \ker(\tau) = w \), where \( w \) is a complement to the space stabilized by \( P_r^{+l} \). By repeated application of this reasoning it follows that

**Proposition 7.3.2**

\[
\sum_{e \in E} (-1)^{|e|} k_{d-d(e)}^1(P(e), V(e), \rho, \phi_e, j_e) = \sum_{e \in E, l(e) \geq 2} (-1)^{|e|} k_{d-d(e)}^0(P(e), V(e), \rho, \phi_e, j_e).
\]

Now we turn to results concerning members of \( F \). Let \( C \) be a sequence with \( |C| = s \) and \( l_s < m \). Fix \( f = (J, C, l) \) and \( f' = (J, C \cup \{l_s + r\}, l) \) in \( F \), where \( 1 \leq r \leq \min\{\partial l_s, n - 2l_s\} \), \( r \in J \), and \( l_s + r \leq m \). Then \( C \prec C \cup \{l_s + r\} \). Notice \( l(f) = s \) and \( l(f') = s + 1 \). Take \( \tau \in \text{Irr}(V(f)) \) corresponding to a singular chain of rank \( r \). Then

\[
T_{P(f)}(\tau) = P_{j_1(f)}^{+n_1(f)} \times \cdots \times P_{j_{s-1}(f)}^{+n_{s-1}(f)} \times T_{P_{j_s(f)}}^{n_s(f)}(\tau)
\]
and $T_{P_{n_s(f)}}(\tau) \cong \left(P_{n_s(f')}^{+n_s(f')} \times P_{n_{s+1}(f')}^{n_{s+1}(f')}\right) \ltimes U(f)$. Thus

$$T_{P_{f}}(\tau) \cong P(f') \ltimes U(f)$$

and $\phi f$ restricted to $T_{P_{f}}(\tau)$ is $\phi f'$. Hence

$$k_{d-d(f)}(P(f), \tau, \phi f, j) = k_{d-d(f')}(P(f'), \rho, \phi f', j') \quad (7.12)$$

where $j'$ the least positive integer such that $j$ divides $j' \cdot |T_{P_{f}}(\tau) \ker(\phi f) / P(f)|$. Write $T = T_{P_{f}}(\tau)$, $K = \ker(\phi f)$. Then $K \trianglelefteq P(f)$ so

$$TK / K \cong T / T \cap K \cong T / U(f) / (T \cap K) / U(f) \cong P(f') / \ker(\phi f')$$

Moreover $P(f) / K \cong \phi f(P(f))$ and $P(f') / \ker(\phi f') \cong \phi f'(P(f'))$ and hence

$$|TK \setminus P(f)| = |P(f) / TK| = |P(f) / K / TK / K| = \left|\phi f(P(f)) / \phi f'(P(f'))\right|$$

Since $l_s < m$ we have nonzero $n-2l_s$, thus the image of $P(f)$ under $\phi f$ is $C_{q+1}$ so we have

$$|T_{P_{f}}(\tau) \ker(\phi f) \setminus P(f)| = \left|\phi f(P(f))\right| = \left|\phi f'(P(f'))\right| = \frac{q+1}{\left|\phi f'(P(f'))\right|}$$

Definition 7.3.3 Let $j_f$ be the smallest positive integer such that $j$ divides $j_f \cdot \frac{q+1}{\phi f'(P(f))}$. 
Remark: Observe that for $f$ of length 1, $j_e = j$ certainly holds. In general, for fixed $f \in F$, $|\phi_f(P(f))|$ is almost always equal to $q + 1$ and hence $j_f = j$. The image

$$\phi_f(P(f)) = \prod_{i=1}^{s}(C_{h_i})$$

where $C_{h_i}$ is the image of the $i$-the factor and so depends on $n_i(f)$. For $1 \leq i < s$, if $n_i(f) = 0$ then $h_i = 1$ whereas if $n_i(f) \neq 0$, $h_i = q + 1/\gcd(q + 1, i)$. In the $s$-th factor, if $n - 2l_s = 0$ then $h_s = q + 1/\gcd(q + 1, s)$ whereas if $n - 2l_s \neq 0$ then $h_s = q + 1$.

$$|\phi_f(P(f))| = C_L \text{ where } L = \text{lcm}(h_i)$$

If either of $n_1(f)$ or $n - 2l_s$ are nonzero, then $|\phi_f(P(f))| = q + 1$. If both are zero then $C$ cannot be covered and

1. $n = 2m$ and $l_s = m$
2. $C = \{l, 2l, \ldots, m\}$ so that $\partial C$ begins $(l, l, \ldots)$.

On the other hand, if $C$ can be covered and $f = (J, C, l)$ and $f' = (J, C \cup \{l_s + r\}, l)$ are in $F$ as in Equation 7.12 then

$$k_{d-d(f)}(P(f), \tau, \rho, \phi_f, j_f) = k_{d-d(f)}(P(f), \tau, \rho, \phi_f, j_f) = k_{d-d(f')}(P(f'), \rho, \phi_{f'}, j_{f'}).$$
We are interested in counting irreducible characters of $P(f)$ that correspond to irreducible characters of $U(f)$. The following is analogous to definitions 6.4.2, 6.4.4, and 6.5.5 in the discussion of unitary linear modules and central modules in the previous chapter.

**Definition 7.3.4**

1. Let $S^u(f)$ denote the subset of nonidentity characters in $\text{Irr}(U(f))$ which are trivial on $Z(f)$.

2. Let $S^{su}(f)$ denote the subset of characters in $\text{Irr}(U(f))$ which are trivial on $Z(f)$ and correspond to singular flags in the unitary space $V_2$.

3. Let $S^{nu}(f)$ denote the subset of characters in $\text{Irr}(U(f))$ which are trivial on $Z(f)$ and correspond to nonsingular flags in the unitary space $V_2$.

4. Let $S^z(f)$ denote the subset of characters in $\text{Irr}(U(f))$ which are not trivial on $Z(f)$.

Clearly $\text{Irr}(U(f)) = 1 \cup S^u(f) \cup S^z(f) = 1 \cup S^{su}(f) \cup S^{nu}(f) \cup S^z(f)$ holds.

For fixed $f$ with $l(f) \geq 2$

$$k_{d-d(f)}(P(f), \rho, \phi_f, j_f) = k^0_{d-d(f)}(P(f), U(f), \rho, \phi_f, j_f) + k^1_{d-d(f)}(P(f), U(f), \rho, \phi_f, j_f)$$

and

$$k^1_{d-d(f)}(P(f), U(f), \rho, \phi_f, j_f) = k_{d-d(f)}(P(f), S^{su}(f), \rho, \phi_f, j_f) + k_{d-d(f)}(P(f), S^{nu}(f), \rho, \phi_f, j_f) + k_{d-d(f)}(P(f), S^z(f), \rho, \phi_f, j_f).$$
Observe that if $X$ is the subset of $S^{su}(f)$ containing only characters of rank $r$, where $r \in J$, and kernels equal to complements of spaces stabilized by $P_r^{+\partial_{l_r}}$, then

$$k_{d-d(f)}(P(f), X, \rho, \phi_f, j_f) = \sum_{f'=(J,C',l)} k_{d-d(f')}(P(f'), \rho, \phi_{f'}, j_{f'}).$$

By the first reduction in Proposition 6.2.4, when summing over all $f$ of length 1 and $J \subseteq I$ has maximal element $l$, we need only sum over $\tau \in S^{su}(f)$ with $r \in J$ and $\ker(\tau) = w$, where $w$ is a complement to the space stabilized by $P_r^l$. By repeated application of this reasoning it follows that

**Proposition 7.3.5**

$$\sum_{f \in F} (-1)^{|f|} k_{d-d(f)}^1(P(f), U(f), \rho, \phi_f, j) =$$

$$\sum_{f \in F, l(f)=1} (-1)^{|f|} k_{d-d(f)}^0(P(f), U(f), \rho, \phi_f, j_f) +$$

$$\sum_{f \in F, l(f) \geq 2} (-1)^{|f|} k_{d-d(f)}^0(P(f), S^{su}(f), \rho, \phi_f, j_f) +$$

$$\sum_{f \in F, l(f) \geq 1} (-1)^{|e|} k_{d-d(e)}(P(f), S^Z(f), \rho, \phi_f, j_f). \quad (7.13)$$

Our next result involves the members of $E$ and $F$. This is a modification of Ku’s result. We have the additional parameters $\phi_e, \phi_f$ and $j_e, j_f$. This remarkable cancellation leads to the second reduction in Equation 4.6b.
Proposition 7.3.6

\[ \sum_{e \in E \atop l(e) \geq 2} (-1)^{|e|} k_0^0_{d-d(e)}(P(e), V(e), \rho, \phi_e, j_e) + \sum_{f \in F \atop l(f) \geq 2} (-1)^{|f|} k_0^0_{d-d(f)}(P(f), U(f), \rho, \phi_f, j_f) = 0 \]

**Proof:** We proceed by explicitly matching pairs in the sum with opposite parity. There are four cases.

1. Match \( f = (J, C, l) \) to \( e = (J \cup \{l_s\}, C, (l, l_s)) \). This is possible since \( s \geq 2 \).

\[ P(f) \cong P_{J_1(f)}^{+n_1(f)} \times \cdots \times P_{J_{s-1}(f)}^{+n_{s-1}(f)} \times \left( P_{J_s}^{+\partial l_s} \times U_{n-2l_s}(q) \right) \times U(f) \]

Then \( n_0(e) = n - 2l_s \) and \( J_0(e) = \emptyset, n_s(e) = l_s - l_{s-1} = \partial l_s \) so \( V(e) = 1 \). Moreover \( n_i(f) = n_i(e) \) for \( 1 \leq i < s \) and we have

\[ P(e) \cong U_{n-2l_s}(q) \times P_{J_1(f)}^{+n_1(f)} \times \cdots \times P_{J_{s-1}(f)}^{+n_{s-1}(f)} \times \left( P_{J_s}^{+\partial l_s} \times 1 \right) \times 1. \]

Thus

\[ P(f)/U(f) \cong P(e)/V(e) \quad \text{and} \quad |\phi_f(P(f))| = |\phi_e(P(e))| \]

since \( \phi_f \) and \( \phi_e \) agree on isomorphic factors of the quotient groups.

2. Match \( e = (J, C, (l, l')) \) to \( e' = (J \cup \{l_s\}, C, (l, l_s)) \), if \( l_s < l' \).

\[ P(e) \cong L_{J_0}^{n-2l'} \times P_{J_1(e)}^{+n_1(e)} \times \cdots \times \left( P_{J_s}^{+\partial l_s} \times \text{GL}_{l'-l_s}(q^2) \right) \times V(e). \]
Moreover since there are no elements in $J \cup \{l_s\}$ between $l_s$ and $l'$

$$L_{J_0(e')}^{n-2l_s} \cong L_{J_0(e)}^{n-2l'} \times \text{GL}_{l''-l_s}(q^2).$$

Thus

$$P(e)/V(e) \cong P(e')/V(e') \quad \text{and} \quad |\phi_e(P(e))| = |\phi_{e'}(P(e'))| = q + 1.$$

3. Match $e = (J, C, (l, l'))$ to $e' = (J \setminus \{l'\}, C, (l, l''))$, if $l_s = l' < \max J$, where $l, l', l''$ are consecutive elements in $J$.

$$P(e) \cong L_{J_0(e)}^{n-2l'} \times P_{J_1(e)}^{+n_1(e)} \times \cdots \times \left( P_{J_s}^{+\partial l_s} \times 1 \right) \times 1.$$

$$P(e') \cong L_{J_0(e')}^{n-2l''} \times P_{J_1(e)}^{+n_1(e)} \times \cdots \times \left( P_{J_s}^{+\partial l_s} \times \text{GL}_{l''-l_s}(q^2) \right) \times V(e').$$

Moreover since there are no elements in $J \setminus \{l'\}$ between $l_s = l'$ and $l''$

$$L_{J_0(e')}^{n-2l''} \cong L_{J_0(e')}^{n-2l'} \times \text{GL}_{l''-l_s}(q^2).$$
and \( n - 2l' \neq 0 \). Thus

\[
P(e)/V(e) \cong P(e')/V(e') \quad \text{and} \quad |\phi_e(P(e))| = |\phi_{e'}(P(e'))| = q + 1.
\]

4. Match \( e = (J, C, (l, l')) \) to \( f = (J\setminus\{l'\}, C, l) \), if \( l_s = l' = \max J \).

\[
P(e) \cong \prod_{J_0(e)}^{n-2l'} \times P_{J_1(e)}^{+n_1(e)} \times \cdots \times \left( P_{J_s}^{+\partial l_s} \times 1 \right) \times 1
\]

where \( J_0(e) = \emptyset \). Then \( n_i(e) = n_i(f) \) for \( 1 \leq i < s \)

\[
P(f) \cong P_{J_1(f)}^{+n_1(f)} \times \cdots \times P_{J_{s-1}(f)}^{+n_{s-1}(f)} \times \left( P_{J_s}^{+\partial l_s} \times U_{n-2l'(q)} \right) \times U(f)
\]

Thus

\[
P(e)/V(e) \cong P(f)/U(f) \quad \text{and} \quad |\phi_e(P(e))| = |\phi_f(P(f))|
\]

since \( \phi_e \) and \( \phi_f \) agree on isomorphic factors of the quotient groups.

And we are done.
CHAPTER 8

COMPLETION OF THE VERIFICATION

Let \( \rho \) be an irreducible character of the center of \( U_n(q) \). Recall \( k_d^1(P_J, U_J, \rho, \det, j) \) is the number of irreducible characters \( \chi \in \text{Irr}(P_J) \) such that the unipotent radical \( U_J \) is not contained in \( \text{ker} \chi \), \( \chi \) lies over \( \rho \), and \( \chi \) restricted to \( \text{ker} \det \) is a sum of \( j' \) irreducible characters where \( j \) divides \( j' \).

In this chapter we prove Equation 4.6b which we restate here

\[
\sum_{J \subseteq I} (-1)^{|J|} k_d^1(P_J, U_J, \rho, \det, j) = - \sum_{\mu \vdash n \atop n'(\mu) = d \atop j \mid \gcd(q+1, \lambda(\mu))} \beta(\mu, a_\rho). \tag{8.1}
\]

This completes the proof of DOC for the finite special unitary groups.

We begin by reformulating the left hand side of Equation 8.1 via our parametrization using \( E \) and \( F \) from the previous chapter. After this reformulation we will refine the statement. Lastly we will need to make use of several layers of inductive arguments. To that end we introduce several propositions which are modifications of Ku’s results. The extra parameters \( \phi_f \) and \( j_f \) require even more involved combinatorial details.
1 The Reformulation

Recall, from the example on page 111 we have

\[ k_1^d(P_J, V(j_i, j_{i+1}), \rho, \det, j) = k_1^{d-d(e)}(P(e), V(e), \rho, \phi_e, j_e) \]

where \( e = (J, \{j_i\}, (j_i, j_{i+1})) \) and from the example on page 116 we have

\[ k_1^d(P_J, U_j, \rho, \det, j) = k_1^{d-d(f)}(P(f), U(f), \rho, \phi_f, j_f) \]

where \( f = (J, \{j_s\}, j_s) \). Hence for fixed \( J \) we have the decomposition

\[ k_1^d(P_J, U_J, \rho, \det, j) = \sum_{e = (J,C,t) \atop l(e) = 1} (-1)^{|e|} k_1^{d-d(e)}(P(e), V(e), \rho, \phi_e, j_e) + \sum_{f \in F \atop l(f) = 1} (-1)^{|f|} k_1^{d-d(f)}(P(f), U(f), \rho, \phi_f, j_f). \] (8.2)

Recall our definition of parity \(|e|\) and \(|f|\). Also note that for \( e \) and \( f \) of length 1, \( d(e) = 0, d(f) = 0, j_e = j, \) and \( j_f = j \). Thus the left hand side of Equation 8.1 can be written

\[ \sum_{J \subseteq I} (-1)^{|J|} k_1^d(P_J, U_J, \rho, \det, j) = \]

\[ \sum_{e \in E \atop l(e) = 1} (-1)^{|e|} k_1^{d-d(e)}(P(e), V(e), \rho, \phi_e, j) + \sum_{f \in F \atop l(f) = 1} (-1)^{|f|} k_1^{d-d(f)}(P(f), U(f), \rho, \phi_f, j) \] (8.3)
We have the further reduction by Proposition 7.3.2

\[ \sum_{e \in E, l(e) = 1} (-1)^{|e|} k_{d-d(e)}^1 (P(e), V(e), \rho, \phi_e, j_e) = \sum_{e \in E, l(e) \geq 2} (-1)^{|e|} k_{d-d(e)}^0 (P(e), V(e), \rho, \phi_e, j_e). \]

Moreover by Proposition 7.3.5,

\[ \sum_{f \in F, l(f) = 1} (-1)^{|e|} k_{d-d(f)}^1 (P(f), U(f), \rho, \phi_f, j_f) = \sum_{f \in F, l(f) \geq 2} (-1)^{|f|} k_{d-d(f)}^0 (P(f), U(f), \rho, \phi_f, j_f) + \sum_{f \in F, l(f) \geq 1} (-1)^{|e|} k_{d-d(e)}^1 (P(f), U(f), \rho, \phi_f, j_f) + \sum_{f \in F, l(f) \geq 1} (-1)^{|e|} k_{d-d(e)}^0 (P(f), S^{u}(f), \rho, \phi_f, j_f). \quad (8.4) \]

Recall the cancellation of Proposition 7.3.6, i.e.

\[ \sum_{e \in E, l(e) \geq 2} (-1)^{|e|} k_{d-d(e)}^0 (P(e), V(e), \rho, \phi_e, j_e) + \sum_{f \in F, l(f) \geq 2} (-1)^{|f|} k_{d-d(f)}^0 (P(f), U(f), \rho, \phi_f, j_f) = 0. \]

Thus, we can now omit all the terms in our sum where the characters correspond either to characters of so called general linear modules, or to the characters of so called unitary linear modules which themselves correspond to singular flags, i.e. all the characters of the \( P(e) \)'s together with the characters of the \( P(f) \)'s which correspond to characters of \( V(f) \) corresponding
to singular flags in a unitary vector space. We are left with characters of the $P(f)$’s that correspond to flags of nonsingular type for $V(f)$ and characters of $Z(f)$. The left hand side of Equation 8.1 may thus be written

$$
\sum_{J \subseteq I} (-1)^{|J|} k_d^1(P_J, U_J, \rho, \det, j) =
$$

$$\sum_{\substack{f \in F \ni l(f) \geq 1 \ni d(f) \leq s}} (-1)^{|f|} k_{d - d(f)}(P(f), S^{nu}(f), \rho, \phi_f, j_f) +
$$

$$\sum_{\substack{f \in F \ni l(f) \geq 1 \ni d(f) \leq s}} (-1)^{|f|} k_{d - d(f)}(P(f), S^z(f), \rho, \phi_f, j_f).
$$

(8.5)

2 The refinement

At this stage in order to proceed in our calculations we must separate the set of $P(f)$’s according to the length of $f$. Recall that $l(f) = |C|$ which keeps track of the number of times we have iterated the process of taking stabilizers. In this sense $C$ also keeps track of the structure of $P(f)$ as a product of block subgroups embedded in $U_n(q)$. Recalling definition 7.2.1 of $P(f)$, $C$ keeps track of the number of times each component of $P(f)$ appears in $U_n(q)$. By collecting all the $f$ of the same length $s$, we are gathering all the $P(f)$ which have the same structure as products of block subgroups embedded in $U_n(q)$.

**Proposition 8.2.1** For fixed $n$ and $s$, $1 \leq s \leq m$ the following hold:

$$
\sum_{\substack{f \in F \ni l(f) = s \ni d(f) \leq s}} (-1)^{|f|} k_{d - d(f)}(P(f), S^{nu}(f), \rho, \phi_f, j_f) = - \sum_{\substack{\mu \ni n \ni \mu' = d \ni l(\mu) = 2s + 1 \ni j \mid \gcd(\lambda(\mu), q + 1)}} \beta(\mu, a_\rho)
$$

(8.6)
\[
\sum_{f \in F} (-1)^{l(f)k_{d(f)}} (P(f), S^\zeta(f), \rho, \phi_f, j_f) = - \sum_{\mu^r \in \mathcal{D}(\lambda(\mu), q+1)} \beta(\mu, a_{\mu}) \quad (8.7)
\]

Summing over all \( s \), Proposition 8.2.1 certainly implies Equation 8.1. However, this is not enough. In order to prove this proposition we must further decompose the sets \( S^{nu}(f) \) and \( S^\zeta(f) \). As has been discussed, originally in chapter 6 and then in chapter 7, characters in \( S^{nu}(f) \) have non-singular rank by definition. Characters in \( S^\zeta(f) \) have non-singular rank if they do not correspond to singular flags. For clarity observe that in chapter 6 the roles of \( r \) and \( r' \) were reversed.

Recall \( \partial C \) kept track of the rank of singular unitary characters for which we took stabilizers. At each iteration of the process the rank of the next singular character could not exceed the rank of the previous character. In fact \( \partial C \) is precisely the list of these ranks in order reading from left to right. We are done calculating stabilizers of singular unitary characters. However, we are not done calculating stabilizers for \( \tau \in S^{nu}(f) \) or \( S^\zeta(f) \). For fixed \( r \), if the minimal element of \( \partial C \) is greater than or equal to \( r \), then the last factor of \( P(f) \) is

\[
P^{nu}(f) \cong \left( P^{\partial l_s} \times U_{n-2l_s}(q) \right) \ltimes U(f)
\]

where \( \partial l_s \geq r \). Hence \( P(f) \) is big enough to contain \( \tau \in S^{nu}(f) \) or \( S^\zeta(f) \) of non-singular rank \( r \).
Now this is the key fact: Given $\tau$ of non-singular rank $r \in J$, the stabilizer of $\tau$ in the last factor of $P(f)$ contains a subgroup isomorphic to $P^r_{J(<r)}$. By gathering together the $f$ of the same length with $\partial C \geq r$ we are grouping all the $P(f)$ with the same block structure as embedded in $U_n(q)$ which have stabilizers containing copies of $P^r_{J(<r)}$. Hence we will be able to peel off this factor and make use of a layered inductive argument.

Recall from definition 7.3.4, $S^{nu}(f)$ is the set of irreducible characters of the unitary linear module $V(f)$ that correspond to nonsingular flags.

**Definition 8.2.2** We define the following subsets.

1. Let $S^r_{nu}(f)$ denote the elements of $S^{nu}(f)$ with non-singular rank $r$.

2. Let $S^r_{nu}(f)(r')$ denote the elements of $S^r_{nu}(f)$ with rank $r'$.

In keeping with the original notation of definition 6.4.4 from chapter 6, we have $S^r_{nu}(f)(r') = S^r_{nu}(V(f), J(< \partial l_s), r')$. Notice that

$$S^r_{nu}(f) = \bigcup_{r' = r}^{\min(\partial l_s, n - 2l_s)} S^r_{nu}(f)(r').$$

Recall from definition 7.3.4, $S^z(f)$ is the set of nontrivial irreducible characters of the central module $Z(f)$.

**Definition 8.2.3** We define the following subsets.

1. let $S^z_r(f)$ denote the elements of $S^z(f)$ with non-singular rank $r$. 
2. Let $S^r_z(f)(r')$ denote the elements of $S^r_z(f)$ with rank $r'$.

In keeping with the original notation of definition 6.5.7 from chapter 6, we have $S^r_z(f)(r') = S^r_z(Z(f), J(< \partial_s), r')$. Also recall that from this definition $S^r_z(f)(r)$ is the set of elements of $S^r(f)$ labeled by singular chains. Notice that

$$S^r_z(f) = \bigcup_{r'=r} S^r_z(f)(r').$$

Summing over all $r$, the following proposition certainly implies Proposition 8.2.1.

**Proposition 8.2.4** For fixed $n, s$, and $r$ with $1 \leq s, r \leq m$ the following hold:

$$\sum_{f \in F} \frac{(-1)^{|f|} k_{d_{d(f)}}(P(f), S^m_{r^*}(f), \rho, \phi_f, j_f)}{\min \partial C \geq r} = - \sum_{\mu \vdash n \atop n'(\mu)=d \atop l(\mu)=2s+1 \atop \min(\mu)=r \atop j | \gcd(\lambda(\mu), q+1)} \beta(\mu, a_p) \tag{8.8}$$

$$\sum_{f \in F} \frac{(-1)^{|f|} k_{d_{d(f)}}(P(f), S^r_z(f), \rho, \phi_f, j_f)}{\min \partial C \geq r} = - \sum_{\mu \vdash n \atop n'(\mu)=d \atop l(\mu)=2s \atop \min(\mu)=r \atop j | \gcd(\lambda(\mu), q+1)} \beta(\mu, a_p) \tag{8.9}$$

However, at this point the above proposition seems somewhat convoluted. In general inductive arguments can be less than transparent. Thus, before continuing we briefly remark on why odd partitions are involved in Equation 8.8 and even partitions are involved in Equation 8.9 in the above proposition.
**Remark:** Take \( \tau \in S_{nu}(f) \). Then \( T_{P(f)}(\tau) \) contains an isomorphic copy of \( P^r_{J(<r)} \) and the multiplicity of this factor as a block subgroup is \( 2s + 1 \). On the other hand, for \( \tau \in S_{z}(f) \) \( T_{P(f)}(\tau) \) contains an isomorphic copy of \( P^r_{J(<r)} \) and the multiplicity of this factor as a block subgroup is \( 2s \). At bottom, when for example the set \( J(<r) \) is empty so that \( P^r_{J(<r)} = U_r(q) \), we know from the remark following definition 3.2.4 in chapter 3 that irreducible characters of \( U_r(q) \) fall into classes of certain type given by partitions of \( r \). These partitions encode the degree of the characters and also the splitting upon restriction to \( SU_r(q) \). Since \( 2s + 1 \) or \( 2s \) copies of \( U_r(q) \) (according to whether \( \tau \in S_{nu}(f) \) or \( \tau \in S_{z}(f) \)) appear in \( U_n(q) \) it is not unreasonable to suppose that partitions of \( r \) will lead to partitions of \( n \). For fixed \( C \) and \( l \) the left hand sums in Equation 8.8 and Equation 8.9 involve all \( f \) with \( J = J' \cup J'' \) where \( J' \subseteq [r-1] \) varies and \( J'' \) with minimal member \( r \) is fixed. We will eventually be able to apply induction to the following sub-sums involved in Equation 8.8 and Equation 8.9

\[
\sum_{J \subseteq [r-1]} (-1)^{|J|} k_{d-d(f)-d'} (P^r_{J}, \rho, \det^{2s+1}, j') \quad \text{in Equation 8.8}
\]

\[
\sum_{J \subseteq [r-1]} (-1)^{|J|} k_{d-d(f)-d'} (P^r_{J}, \rho, \det^{2s}, j'') \quad \text{in Equation 8.9.}
\]

where \( j' = \frac{j}{\gcd(j, q + 1, 2s + 1)} \) and \( j'' = \frac{j}{\gcd(j, q + 1, 2s)} \).

3 **The rest**

Before proving Proposition 8.2.4, we state and then prove an important corollary necessary for the inductive step. Moreover we present two intermediate propositions which allow us to
rewrite the left hand sides of Equation 8.8 and Equation 8.9. We introduce two maps in order to keep track of the members of the involved alternating sums. These are extensions of maps defined by Ku ((15), Chapter 9). We use his notations: \( h, g \).

In order to streamline the very involved notation we define certain subsets of \( F^n \) and then introduce two \( \mathbb{Z} \)-valued maps \( h, g \) on those subsets.

**Definition 8.3.1** For fixed \( n \) and \( s \),

1. Let \( F^n(s, r) \) denote the \( f \) in \( F^n \) with \( l(f) = |C| = s \) and \( \min \partial C \geq r \).
2. Let \( F^n(\leq s) \) denote the \( f \in F^n \) of length \( l(f) \leq s \).

We now define our first map \( h \) which is involved in expressing the left hand side of Equation 8.8

**Definition 8.3.2** Fix \( n \) and \( 1 \leq r \leq m \). Define \( h_{n,d,\rho,s,r,j} : F^n-(2s+1)r(\leq s) \to \mathbb{Z} \) by

\[
h_{n,d,\rho,s,r,j}(f) = \begin{cases} 
  k_{d-d(f)}^0(P^n-(2s+1)r(f), U^n-(2s+1)r(f), \rho, \phi_f, j_f), & \text{if } 0 \leq l(f) < s; \\
  k_{d-d(f)}^0(P^n-(2s+1)r(f), Z^n-(2s+1)r(f), \rho, \phi_f, j_f), & \text{if } l(f) = s.
\end{cases}
\]

The following is a first crucial step which is involved in peeling off the factor \( P_{f(\leq r)}^r \) in the left hand side of Equation 8.8.
Proposition 8.3.3 Fix $n$ and $1 \leq r \leq m$. Then the alternating sum

$$
\sum_{\substack{f \in F \\
u(f) = s \ \text{min} C \geq r}} (-1)^{|f|} k_{d_1 - d(f)}(P(f), S^n_{tr}(f), \rho_f, j_f) = 
- \sum_{\rho_1, \rho_2} \sum_{d_1, d_2} \left( \sum_{J \subseteq \lfloor r/2 \rfloor} (-1)^{|J|} k_{d_1}(P_J, \rho_1, \det^{2s+1}, j_1) \sum_{f \in F^{n-(2s+1)r} \leq s} (-1)^{|f|} h_{n,d_2,\rho_2,s,r,j}(f) \right)
$$

where $j_1 = j / \gcd(j, q + 1, 2s + 1)$, $d_1 + d_2 = d - d(f)$, and $\rho_1 \rho_2 = \rho$.

Remark: If $n - (2s+1)r = 0$, then there is a unique $f = (\emptyset, 0, 0) \in F^{n-(2s+1)r} \leq s$. Moreover, we must set $j_f = 1$ regardless of the value of $j$ in order that our sum may accommodate this degenerate case. Thus

$$
h_{n,d,\rho,s,r,j}(f) = k_{d_1 - d(f)}^0(P^n-(2s+1)r(f), U^{n-(2s+1)r}(f), \rho, \phi_f, j_f)
= k_{d_1}^0(1, 1, \rho, \det, 1)
= 1 \text{ if and only if } \rho = 1 \text{ and } d = 0.
$$

Proof: Take $f \in F^n(s, r)$. We consider three cases.

1. Let $\partial C = \{r\}$. Then $C = (r, 2r, \ldots, sr)$ and

$$
P(f) = \left( P_{\partial \mathbb{C}^{-2r}}^{+} \times U_{n-2sr} \right) \times U(f).
$$
$S_{r}^{\text{nu}}(f)$ is non-empty if and only if $n - 2sr \neq 0$ and $J(< r) \subset \{1, 2, \ldots, \lfloor \frac{n-2sr}{2} \rfloor \}$. If $S_{r}^{\text{nu}}(f) \neq \emptyset$ then it has one orbit. Take $\tau \in S_{r}^{\text{nu}}(f)$, then

$$T = T_{P_{f}}(\tau) = \left( P_{J(<r)}^{r} \times U_{n-2sr-r}(q) \right) \times U(f).$$

Thus $k_{d-d(f)}(P(f), S_{r}^{\text{nu}}(f), \rho, \phi_{f}, j_{f}) = k_{d-d(f)}(T/U(f), \rho, \phi_{f}, j')$ where $j = j_{f}$, $j'$ is the least positive integer such that $j$ divides $j' \cdot \frac{q + 1}{|\phi_{f}(T)|}$ and $d'$ is $q$-height in the index of $T$ in $P(f)$. We have $U_{n-2sr-r}(q) = P^{n-(2s+1)r}(f')$ for $f' = (\emptyset, \emptyset, 0) \in F^{n-(2s+1)r}(\leq s)$ and

$$k_{d-d(f)}(P(f), S_{r}^{\text{nu}}(f), \rho, \phi_{f}, j_{f})$$

$$= \sum_{\rho_{1},\rho_{2}} \sum_{d_{1},d_{2}} k_{d_{1}}(P_{J(<r)}^{r},\rho_{1},(\det)^{2s+1},j_{1})k_{d_{2}}(P^{n-(2s+1)r}(f'),\rho_{2},\phi_{f'},j_{f'})$$

$$= \sum_{\rho_{1},\rho_{2}} \sum_{d_{1},d_{2}} k_{d_{1}}(P_{J(<r)}^{r},\rho_{1},(\det)^{2s+1},j_{1})h_{n,d_{3},\rho_{2},s,r,j}(f')$$

where $j_{1} = j / \gcd(j, q + 1, 2s + 1)$, $d_{1} + d_{2} = d - d(f)$, and $\rho_{1}\rho_{2} = \rho$.

2. Let $\partial C \supseteq \{r\}$. Then $\partial C = (l_{1}, \ldots, r)$ where $l_{1} > r$.

$$P(f) = P_{J_{1}(f)}^{n_{1}(f)} \times \cdots \times P_{J_{s-1}(f)}^{n_{s-1}(f)} \times P_{J_{s}(f)}^{n_{s}(f)}$$

where $n_{i}(f)$ is nonzero for some $1 \leq i < s$ and the last factor decomposes

$$P_{J_{s}(f)}^{n_{s}(f)} = \left( P_{J_{1}(<r)}^{r} \times U_{n-2l_{s}}(g) \right) \times U(f).$$
$S^u_r(f)$ is non-empty if and only if $n - 2l_s \neq 0$ and $J(< r) \subset \{1, 2, \ldots, \lceil \frac{n-2l_s}{2} \rceil \}$. If $S^u_r(f) \neq \emptyset$ then it has one orbit. Take $\tau \in S^u_r(f)$, then

$$T = T_{P(f)}(\tau) = P^+_{J_1(f)} \times \cdots \times P^+_{J_{s-1}(f)} \times \left( P^r_{J_1(\tau) \cup U_{n-2l_s-r}(q)} \right) \times U(f).$$

Thus $k_{d-d(f)}(P(f), S^u_r(f), \rho, \phi_f, j_f) = k_{d-d(f)-d'}(T/U(f), \rho, \phi_f, j')$ where $j = j_f$, $j'$ is the least positive integer such that $j$ divides $j' \cdot \frac{q+1}{|\phi_f(T)|}$ and $d'$ is $q$-height in the index of $T$ in $P(f)$. Let $J' = \{ j-r | j \in J(> r) \}$ and $C' = \{ l_i - ir | l_i \in C \} \{ 0 \}$. For $f' = (J', C', l_1 - r) \in F^{n-(2s+1)r}(\leq s)$ we have

$$P^{n-(2s+1)r}(f') = P^+_{J_1(f')} \times \cdots \times P^+_{J_{s-2}(f')} \times \left( P^+_{J_{s-1}(f')} \times U_{n-2l_s-r}(q) \right) \times U^{n-(2s+1)r}(f').$$

Thus

$$k_{d-d(f)}(P(f), S^u_r(f), \rho, \phi_f, j_f)
= \sum_{\rho_1, \rho_2, d_1, d_2} k_{d_1}(P^r_{J_1(\tau)}, \rho_1, (\det)^{2s+1}, j_1) k_{d_2}^{\emptyset}(P^{n-(2s+1)r}(f'), U^{n-(2s+1)r}(f'), \rho_2, \phi_{f'}, j_{f'})
= \sum_{\rho_1, \rho_2, d_1, d_2} k_{d_1}(P^r_{J_1(\tau)}, \rho_1, (\det)^{2s+1}, j_1) h_{n, d_2, \rho_2, s, r, j}(f')$$

where $j_1 = j/\gcd(j, q + 1, 2s + 1)$, $d_1 + d_2 = d - d(f)$, and $\rho_1 \rho_2 = \rho$.

3. Let $\min(\partial C) > r$. Then $\partial l_s > r$. We have

$$P(f) = P^+_{J_1(f)} \times \cdots \times P^+_{J_{s-1}(f)} \times P^u_{J_s(f)}.$$
The last factor decomposes

\[ P_{J_s(f)}^{n_s} = \left( P_{J(<r)}^{+\partial l_s} \times U_{n-2l_s}(g) \right) \times U(f). \]

Assume \( S_{lu}^n(f) \) is non-empty, so \( n - 2l_s \neq 0 \) and \( J(<r) \subset \{ 1, 2, \ldots, \left\lceil \frac{n-2l_s}{2} \right\rceil \} \). Take \( \tau \in S_{lu}^n(f) \) of rank \( r' \). Then \( \tau \) corresponds to a pair of chains \((c_1, c_2)\) where \( c_1 \) is a singular chain in a unitary space of dimension \( r \) and \( c_2 \) is a flag in a unitary space of dimension \( n - 2l_s - r \). Let \( \tau' \) correspond to \( c_2 \), then

\[ T = T_{P(f)}(\tau) = P_{J_1(f)}^{+n_1} \times \cdots \times P_{J_{s-1}(f)}^{+n_{s-1}}(f) \times T_{P_{J_s(f)}}(\tau) \]

where

\[ T_{P_{J_s(f)}}(\tau) = \left( P_{J_1}^{+(\partial l_s - r')} \times P_{J(<r)}^r \times T_{U_{n-2l_s-2r'}(q)(\tau')} \right) \times U(f). \]

Here \( J_1 = \{ j - r' \mid j \in J_1(f)(>r') \} \).

And we are done.

We now define our second map \( g \) which is involved in expressing the left hand side of Equation 8.9.
Definition 8.3.4 Fix $n$ and $1 \leq r \leq m$. Define $g_{n,d,\rho,s,r,j}: F^{n-2sr} \rightarrow \mathbb{Z}$ by

$$g_{n,d,\rho,s,r,j}(f) = \begin{cases} 
k_{d-d(f)}^{-1}((P^{n-2sr}(f), U^{n-2sr}(f), \rho, \phi_f, j_f), & \text{if } 0 \leq l(f) < s; \\k_{d-d(f)}(P^{n-2sr}(f), \rho, \phi_f, j_f), & \text{if } l(f) = s. \end{cases}$$

We have a corresponding first crucial step which is involved in peeling off the factor $P^r_{f(<r)}$ in the left hand side of Equation 8.9.

Proposition 8.3.5 For fixed $n$ and $1 \leq r \leq m$, the alternating sum

$$\sum_{l(f)=s \atop \text{min} \partial C \geq r} (-1)^{|f|} k_{d-d(f)}((P(f), S_r^z(f), \rho, \phi_f, j_f)) =$$

$$- \sum_{\rho_1, \rho_2} \sum_{d_1, d_2} \left( \sum_{J \subseteq [r/2]} (-1)^{|J|} k_{d_1}(P_J, \rho_1, \det^{2s}, j_1) \sum_{f \in F^{n-2sr} \leq s} (-1)^{|f|} g_{n,d_2,\rho_2,s,r,j}(f) \right)$$

where $j_1 = j / \gcd(j, q + 1, 2s)$, $d_1 + d_2 = d - d(f)$, and $\rho_1 \rho_2 = \rho$.

Remark: If $n - 2sr = 0$, then there is a unique $f = (0, 0, 0) \in F^{n-2sr} \leq s$. Moreover, we must set $j_f = 1$ regardless of the value of $j$ in order that our sum may accommodate this degenerate case. Thus

$$g_{n,d,\rho,s,r,j}(f) = k_0^{-1}((P^{n-2sr}(f), U^{n-2sr}(f), \rho, \phi_f, j_f))$$

$$= k_0^{-1}(1, 1, \rho, \det, 1)$$

$$= 1 \text{ if and only if } \rho = 1 \text{ and } d = 0.$$
**Proof:** The proof is entirely analogous the proof of Proposition 8.3.3 and consists of considering the same three cases for $f \in F^n(s, r)$:

1. $\partial C = \{r\}$.
2. $\partial C \supseteq \{r\}$, and
3. $\min(\partial C) > r$.

We omit the proof, but remark on the following difference. For $\tau \in \text{Irr}(V(f))$, $\tau$ is linear and hence extendible to $T_P(f)(\tau)$. However this does not hold for $\tau \in \text{Irr}(Z(f))$. Rather non-linear $\phi \in \text{Irr}(U(f))$ is extendible to $T_P(f)(\phi)$ as $U(f)/(\ker(\phi))$ is an extra special $p$-group.

We now state the important corollary to Proposition 8.2.4 which as mentioned will be needed for the inductive case in the proof of Proposition 8.2.4.

**Corollary 8.3.6** Assume that Proposition 8.2.4 holds. Let $r = 0$ in the definition of $h_{n,d,p,s,0,j}$ and $g_{n,d,p,s,0,j}$. For each $1 \leq s \leq m$ we have the following:

$$\sum_{f \in F^n(s)} (-1)^{|f|} h_{n,d,p,s,0,j}(f) = \sum_{\mu \vdash n \atop \delta'(\mu) = d \atop \ell(\mu) \leq 2s \atop j \mid \gcd(\lambda(\mu), q+1)} \beta(\mu, a_p) \quad (8.10)$$
\[
\sum_{f \in F^n(\leq s)} (-1)^{|f|} g_{n,d,\rho,s,0,j}(f) = \sum_{\mu \vdash n, n' = \mu, l(\mu) = d, \frac{1}{2}l(\mu) \leq 2s-1, j | \gcd(l(\mu),q+1)} \beta(\mu, a_\rho)
\] (8.11)

**Proof:** The assumption that Proposition 8.2.4 holds implies a number of results. In order of implication these are Proposition 8.2.1, Equation 8.1 (via Equation 8.5), and thus Theorem 4.1.1, the main theorem of this paper. We proceed by induction on \(s\).

**The small case:** Let \(s = 1\) so that \(f \in F^n(\leq 1)\) and \(f\) has length 0 or 1.

If \(l(f) = 0\) then \(f = (\emptyset, \emptyset, 0)\) so \(P(f) = U_n(q), U(f) = 1,\) and \(d(f) = 0\). The contribution to the left hand side of Equation 8.11 is \(g_{n,d,\rho,s,0,j}(f) = k_d(U_n(q), \rho, \det, j)\). If \(l(f) = 1\) then \(f = (J, \{l\}, l)\) so \(P(f) = P_J, U(f) = U_l,\) and \(d(f) = 0\). The contribution is \((-1)^{|f|} g_{n,d,\rho,s,0,j}(f) = (-1)^{|J|} k_d(P_J, \rho, \det, j)\).

Hence the left hand side of Equation 8.11 is

\[
k_d(U_n(q), \rho, \det, j) + \sum_{0 \neq J \subset I} (-1)^{|J|} k_d(P_J, \rho, \det, j)
\]

\[
= \begin{cases} 
\beta((n), a_\rho), & \text{if } d = \frac{n}{2} \text{ and } j = 1; \\
0, & \text{otherwise}
\end{cases}
\] (8.12)

by Equation 8.1.
The only partition $\mu$ of $n$ with $l(\mu) \leq 1$ is $\mu = (n)$. Moreover $n'(n) = \binom{n}{2}$ and the only $j$ dividing $\lambda((n))$ is $j = 1$. Hence the right hand side of Equation 8.11 is equal to the left. Thus Equation 8.11 holds for $s = 1$.

As for Equation 8.10 if $l(f) = 0$ then $f = (\emptyset, \emptyset, 0)$ so $P(f) = U_n(q)$, $U(f) = 1$, and $d(f) = 0$. The contribution to the left hand side of Equation 8.10 is $h_{n,d,s,0,j}(f) = k_d(U_n(q), \rho, \det, j)$. If $l(f) = 1$ then $f = (J, \{l\}, l)$ so $P(f) = P_J$, $U(f) = U_l$, and $d(f) = 0$. The contribution is

$$(-1)^{|J|}k_d^0(P_J, Z_I, \rho, \det, j) = (-1)^{|J|}(k_d(P_J, \rho, \det, j) - k_d^1(P_J, Z_I, \rho, \det, j)).$$

Hence the left hand side of Equation 8.10 is given by

$$k_d(U_n(q), \rho, \det, j) + \sum_{\emptyset \neq J \subset I} (-1)^{|J|}k_d(P_J, \rho, \det, j) - \sum_{l(f) = 1} (-1)^{|J|}k_d^1(P(f), Z(f), \rho, \phi_f, j_f)$$

$$= \sum_{f \in P_n(\leq 1)} (-1)^{|J|}g_{n,d,s,0,j}(f) + \sum_{\mu \vdash n} \beta(\mu, a_\rho)$$

$$= \sum_{\mu \vdash n} \beta(\mu, a_\rho)$$

(8.13)
by Equation 8.11 and Proposition 8.2.4. Hence Equation 8.11 holds for $s = 1$ and the small case is proved.

**The inductive case:** Let us assume that Corollary 8.3.6 holds for all $s' < s$. Our first observation is that

$$g_{n,d,p,s,r,j}(f) = g_{n-2sr,d,p,s,0,j}(f) \quad \text{and} \quad h_{n,d,p,s,r,j}(f) = h_{n-(2s+1)r,d,p,s,0,j}(f).$$

Now fixing $n$ and letting $r = 0$ we further observe that for $f \in F^n$ with $l(f) \leq s - 2$

$$g_{n,d,p,s,0,j}(f) = h_{n,d,p,s-1,0,j}(f) = k_0^d(P^n(f), U^n(f), \rho, \phi_f, j_f).$$

For the rest of the proof we will drop the superscript $n$ and let $P(f)$ denote $P^n(f)$, $U(f)$ denote $U^n(f)$, $Z(f)$ denote $Z^n(f)$, and $V(f)$ denote $V^n(f)$.

Writing $P(f) = Q(f) \times H(f)$ where $Q(f) = \prod_{i=1}^{s-1} P_{j_i(f)}^{n_i}(f)$ and $H(f) = P_{j_s(f)}^{n_s}(f)$. Let $L(f)$ be a complement to $U(f)$ in $H(f)$. The quotient $P(f)/Z(f) \cong Q(f) \times (L(f) \times V(f))$ and hence

$$k_{d-d(f)}^0(P(f), Z(f), \rho, \phi_f, j_f) = k_{d-d(f)}^0(P(f), U(f), \rho, \phi_f, j_f) + k_{d-d(f)}^1(P(f), V(f), \rho, \phi_f, j_f).$$

For brevity write $g_{d,p,s} = g_{n,d,p,s,0,j}$ and $h_{d,p,s-1} = h_{n,d,p,s-1,0,j}$. Then
\[
\sum_{f \in F^{n}(\leq s)} (-1)^{|f|} g_{d,\rho,s}(f)
= \sum_{f \in F^{n} \atop l(f)=s} (-1)^{|f|} k_{d-d(f)}(P(f), \rho, \phi_{f}, jf) + \sum_{f \in F^{n} \atop l(f)=s-1} (-1)^{|f|} k_{d-d(f)}^{0}(P(f), \rho, \phi_{f}, jf)
+ \sum_{f \in F^{n}(\leq s-1)} (-1)^{|f|} h_{d,\rho,s-1}(f) - \sum_{f \in F^{n} \atop l(f)=s-1} (-1)^{|f|} k_{d-d(f)}^{0}(P(f), Z(f), \rho, \phi_{f}, jf)
= \sum_{f \in F^{n} \atop l(f)=s} (-1)^{|f|} k_{d-d(f)}(P(f), \rho, \phi_{f}, jf) - \sum_{f \in F^{n} \atop l(f)=s-1} (-1)^{|f|} k_{d-d(f)}^{1}(P(f), V(f), \rho, \phi_{f}, jf)
+ \sum_{f \in F^{n}(\leq s-1)} (-1)^{|f|} h_{d,\rho,s-1}(f)
=- \sum_{f \in F^{n} \atop l(f)=s-1} (-1)^{|f|} k_{d-d(f)}(P(f), \rho, \phi_{f}, jf) + \sum_{f \in F^{n}(\leq s-1)} (-1)^{|f|} h_{d,\rho,s-1}(f)
= + \sum_{\substack{n' \vdash n \atop l'(\mu)=d \atop j \gcd(q+1, \lambda(\mu))}} \beta(\mu, a_{\rho}) + \sum_{\substack{n' \vdash n \atop l'(\mu)=d \atop j \gcd(q+1, \lambda(\mu))}} \beta(\mu, a_{\rho})
\]

Hence Equation 8.11 holds at \( s \).

As to Equation 8.10 notice for \( f \in F^{n} \) \( h_{d,\rho,s}(f) = g_{d,\rho,s}(f) \) if \( l(f) \leq s \). Thus
\[
\sum_{f \in F^n(\leq s)} (-1)^{|f|} h_{d,\rho,s}(f) = \sum_{f \in F^n(\leq s)} (-1)^{|f|} g_{d,\rho,s}(f) - \sum_{f \in F^n} (-1)^{|f|} k_{d-d(f)}(P(f), \rho, \phi_f, j_f) + \sum_{f \in F^n} (-1)^{|f|} k_{d-d(f)}^0(P(f), Z(f), \rho, \phi_f, j_f)
\]
\[
= \sum_{f \in F^n(\leq s)} (-1)^{|f|} g_{d,\rho,s}(f) - \sum_{f \in F^n} (-1)^{|f|} k_{d-d(f)}^1(P(f), Z(f), \rho, \phi_f, j_f)
\]
\[
= \sum_{\mu\vdash n'} \beta(\mu, a_\rho) + \sum_{\mu\vdash n'_{\lambda}\mu=d \quad l(\mu)=2s \quad j|\gcd(q+1,\lambda(\mu))} \beta(\mu, a_\rho)
\]
\[
= \sum_{\mu\vdash n'_{\lambda}\mu=d \quad l(\mu)=2s \quad j|\gcd(q+1,\lambda(\mu))} \beta(\mu, a_\rho).
\]

Hence Equation 8.10 holds at \(s\) and Corollary 8.3.6 is proved.

**Proof of Proposition 8.2.4:** We proceed by induction on \(m\) where \(n = 2m\) or \(n = 2m+1\).

The small cases: Let \(m = 0\) so that \(n = 1\). Then both sums are empty and hence Proposition 8.2.4 is vacuously true.

Now let \(m = 1\) so that \(n = 2\) or \(n = 3\). In either case we have \(s = r = 1\). For \(f\) with \(l(f) = 1\) we have \(f = (J, \{l\}, l)\) where \(\max J = l\). Thus \(f = (\{1\}, \{1\}, 1)\) is the only \(f \in F^n\) of
nonzero length and $P(f) = P_{11}$ the Borel subgroup of $U_n(q)$ with $U(f)$ the unipotent radical of $P_{11}$. Let $L(f)$ be a complement in $P(f)$ to $U(f)$. Notice that $d(f) = 0$ by definition.

Assume $n = 2$ so that we are working in $U_2(q)$. Then $P(f) \cong \text{GL}_1(q^2) \rtimes F_q$. Under this isomorphism the map $\phi_f$ is defined $\phi_f(A) = (\det A)^{1-q}$ where $A \in \text{GL}_1(q^2)$. The group $L(f) \cong \text{GL}_1(q^2)$ and $U(f) = Z(f)$ is elementary abelian of order $q$. We have $V(f) = U(f)/Z(f) = 1$ so that $S_n^{\mu}(f)$ is empty. Thus the left hand side of Equation 8.8 is

$$-k_{d-d(f)}(P(f), S_1^{\mu}(f), \rho, \phi_f, j_f) = 0.$$ 

This is equal to the right hand side of Equation 8.8 since there are no partitions $\mu \vdash 2$ of length 3.

As for Equation 8.9, note the set $S_1^\tau(f)$ is the collection of nontrivial irreducible characters of $F_q$ on which $\text{GL}_1(q^2) \rtimes F_q$ acts transitively. Indeed let us recall $g \in L(f)$ acts on $\tau \in S_1^\tau(f)$ via

$$\begin{pmatrix} a & 0 \\ 0 & a^{-q} \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a^q \end{pmatrix} = \begin{pmatrix} 1 & acaq \\ 0 & 1 \end{pmatrix}.$$ 

Take any nontrivial $\tau \in \text{Irr}(F_q)$ then $T_{\text{GL}_1(q^2)}(\tau) = \mathbb{C}_{q+1}$. Moreover the map $\phi_f$ restricted to $T_{\text{GL}_1(q^2)}(\tau)$ is defined $\phi_f(A) = (\det A)^2$ for $A \in \mathbb{C}_{q+1}$. We have

$$k_{d}(\text{GL}_1(q^2) \ltimes F_q, \tau, \rho, \phi_f, j) = k_{d}(T_{\text{GL}_1(q^2)}(\tau), \rho, \det^2, j')$$
where $j'$ is the least positive integer such that $j$ divides $j' \cdot \gcd(2, q + 1)$. Hence assuming that $j$ is a divisor of 2

$$j' = j / \gcd(j, q + 1, 2) = \begin{cases} 
1, & \text{if } j = 2; \\
1, & \text{if } j = 1.
\end{cases}$$

Then for $j = 1$ or $j = 2$ the left hand side of Equation 8.9 is given by

$$-k_d(P(f), S_1^*(f), \rho, \phi_f, j) = -k_d(\GL_1(q^2) \ltimes F_q, \tau, \rho, \phi_f, j)$$

$$= -k_d(\mathbb{C}_{q+1}, \rho, \det^{2}, 1) = \begin{cases} 
-1, & \text{if } d = 0; \\
0, & \text{otherwise}.
\end{cases}$$

This is certainly equal to the right hand side of Equation 8.9 since the only partition $\mu \vdash 2$ with $l(\mu) = 2$ is $\mu = (1^2)$ with minimal element 1, $n'((1^2)) = 0$, and $\lambda((1^2)) = 2$ which is divisible by $j = 1$ or $j = 2$.

Assume now $n = 3$ so that we are working in $U_3(q)$. The group

$$P(f) = P_{\{1\}} \cong (\GL_1(q^2) \times U_1(q)) \ltimes U_{\{1\}}.$$ 

Under this isomorphism the map $\phi_f$ is defined $\phi_f(A, B) = (\det A)^{1-q} \det B$ where $A \in \GL_1(q^2)$ and $B \in U_1(q)$. In this case $U(f) = U_{\{1\}}$ is no longer equal to its center. Let $L(f)$ be a complement to $U(f)$ in $P(f)$. 
The quotient $V(f) \cong M_{1,1}(F_q^2)$ and $S_{1}^{nu}(f)$ has one orbit under $P_{\{1\}}$. Take $\tau \in S_{1}^{nu}(f)$ then $T_{L(f)}(\tau) = U_1(q) \cong \mathbb{C}_{q+1}$. Moreover the map $\phi_f$ restricted to $T_{L(f)}(\tau)$ is defined $\phi_f(B) = (\det B)^3$ for $B \in U_1(q)$.

Then for $j = 1$ or $j = 3$ the left hand side of Equation 8.8 is given by

$$-k_d(P(f), S_{1}^{nu}(f), \rho, \phi_f, j) = -k_d(P(f), \tau, \rho, \phi_f, j)$$

$$= -k_d(T_{L(f)}(\tau), \rho, \phi_f, j / \gcd(j, 3, q + 1))$$

$$= -k_d(\mathbb{C}_{q+1}, \rho, \det^3, 1)$$

$$= \begin{cases} 
-1, & \text{if } d = 0; \\
0, & \text{otherwise.} 
\end{cases}$$

This is certainly equal to the right hand side of Equation 8.8 since the only partition $\mu \vdash 3$ with $l(\mu) = 2(1) + 1 = 3$ is $\mu = (1^3)$ with minimal element 1, $n'(1^3) = 0$, and $\lambda((1^3)) = 3$ which is divisible by $j = 1$ or $j = 3$.

As for Equation 8.9, $S_{1}^z(f)$ is the set of non-trivial characters of $Z(f)$ on which $P(f)$ acts transitively. Take $\tau \in S_{1}^z(f)$. Then there exists a unique $\psi \in \Irr(U(f))$ lying over $\tau$ with $\psi(1) = q$. We have

$$T_P(\tau) = T_P(\psi) = (U_1(q) \times U_1(q)) \ltimes U_{\{1\}}$$
and $q$ does not divide $|P|/|T_P(\psi)|$. The map $\phi_f$ restricted to $T_P(\psi)$ is defined $\phi_f(A, B) = (\det A)^2(\det B)$ for $A$ in the first factor $U_1(q)$ and $B$ in the second factor $U_1(q)$. Also notice that

$$|P(f)/T_P(\psi)| = 1.$$  

Hence the left hand side of Equation 8.8 is given by

$$-k_d(P(f), S_1(f), \rho, \phi_f, j) = -k_d(P(f), \psi, \rho, \phi_f, j)$$

$$= -k_d(P(f), \psi, \rho, \phi_f, j)$$

$$= -k_{d-1}(T_{P(f)}(\psi)/U_{\{1\}}, \rho, \phi_f, j)$$

$$= -k_{d-1}(C_{q+1} \times C_{q+1}, \rho, \det^2 \cdot \det, j)$$

$$= -\sum_{\rho_1, \rho_2 = \rho} \sum_{d_1 + d_2 = d - 1} k_{d_1}(C_{q+1}, \rho_1, \det^2, j/gcd(j, 2, q + 1))k_{d_2}(C_{q+1}, \rho_2, \det, j)$$

$$= \begin{cases} -(q + 1), & \text{if } d = 1 \text{ and } j = 1; \\ 0, & \text{otherwise}. \end{cases}$$

This is certainly equal to the right hand side of Equation 8.9 since the only partition $\mu \vdash 3$ of length 2 is $\mu = (2, 1)$ with $n'(\mu) = 1, \min(\mu) = 1, \text{ and } \lambda((2, 1)) = 1$. Indeed $-\beta((2, 1), a_\rho) = -(q + 1)$.

**The inductive case:** We assume Proposition 8.2.4 holds for all $m' < m$ where $m \geq 2$ and $n = 2m$ or $n = 2m + 1$. Recall that this assumption implies that for all such $m'$, not only does Corollary 8.3.6 hold at $m'$, but also Proposition 8.2.1, Equation 8.1, and hence Theorem 4.1.1,
the main theorem of this paper, hold at $m'$.

Recall our observation in the proof of 8.3.6 that

$$h_{n,d,p,s,r,j}(f) = h_{n-(2s+1)r,d,p,s,0,j}(f).$$

Moreover since $1 \leq r < n$ by induction assumption we have

$$\sum_{J \subseteq \lfloor r/2 \rfloor} (-1)^{|J|} k_{d_1}(P_J, \rho_1, \det^{2s+1}, j_1) = \begin{cases} \beta((r), a_\rho), & \text{if } d_1 = (\rho) \text{ and } j_1 = 1; \\ 0, & \text{otherwise.} \end{cases}$$
Notice that if \( j_1 = 1 \) we have \( j = \gcd(j, 2s + 1, q + 1) \) in the statement of Proposition 8.3.3.

Then

\[
\sum_{f \in F \atop l(f) = s \atop \min \delta C \geq r} (-1)^{|f|} |k_{d - d(f)}| (P(f), S_r^m(f), \rho, \phi_f, jf) = \\
- \sum_{\rho_1, \rho_2 \atop d_1, d_2} \left( \sum_{J \subseteq [r/2]} (-1)^{|J|} k_{d_1} (P_J, \rho_1, \det f^{2s+1}, j_1) \sum_{f \in F_n - (2s+1)r \atop l(f) \leq s} (-1)^{|f|} |h_{n, d_2, \rho_2, s, r, j} (f)\right) \\
- \sum_{\rho_1, \rho_2 \atop d_1, d_2} \left( \sum_{J \subseteq [r/2]} (-1)^{|J|} k_{d_1} (P_J, \rho_1, \det f^{2s+1}, j_1) \sum_{f \in F_n - (2s+1)r \atop l(f) \leq s} (-1)^{|f|} |h_{n - (2s+1)r, d_2, \rho_2, s, 0, j} (f)\right) \\
= - \sum_{\mu \atop r^n (\mu) = d_1} \left( \beta((r), a_{\rho_1}) \sum_{\mu^n (n - (2s+1)r) \atop n'(\mu) = d_2 \atop l(\mu) \leq 2s} \beta(\mu, a_{\rho_2}) \right) \\
= - \sum_{\tilde{\mu} \atop n'(\tilde{\mu}) = d_2 \atop \min(\tilde{\mu}) = r \atop l(\tilde{\mu}) \leq 2s+1 \atop j | \gcd(q+1, \lambda(\tilde{\mu}))} \beta(\tilde{\mu}, a_{\rho_2})
\]

where for fixed \( \mu = (a_1^{l_1}, a_2^{l_2}, \ldots, a_{\delta(\mu)}^{l_{\delta(\mu)}}) \vdash (n - (2s + 1)r) \) the corresponding partition of \( n \) is given by

\[
\tilde{\mu} = (r^{2s+1}) + \mu = ((a_1 + r)^{l_1}, (a_2 + r)^{l_2}, \ldots, (a_{\delta(\mu)} + r)^{l_{\delta(\mu)}}, r^e) \vdash n.
\]

and \( \sum_{i=1}^{e} l_i + e = 2s + 1 \) so \( j \) divides \( e \) i.e. \( j \) divides \( \lambda(\tilde{\mu}) \).
Recall our observation in the proof of 8.3.6 that

\[ g_{n,d,ρ,s,r,j}(f) = g_{n-2sr,d,ρ,s,0,j}(f). \]

Moreover since \( 1 \leq r < n \) by induction assumption we have

\[
\sum_{J \subseteq \left\{ \frac{r}{2} \right\}} (-1)^{|J|} k_{d_1}(P_J, ρ_1, \det^{2s}, j_1) = \begin{cases} 
β((r), a_ρ), & \text{if } d_1 = \left( \frac{r}{2} \right) \text{ and } j_1 = 1; \\
0, & \text{otherwise.}
\end{cases}
\]

Notice that if \( j_1 = 1 \) we have \( j = \gcd(j, 2s, q + 1) \) in the statement of Proposition 8.3.5. Then

\[
\sum_{\substack{f \in F \\text{ \text{s.t.} } l(f) = s \\text{ min} \partial C \geq r}} (-1)^{|f|} k_{d-f}(P(f), S^*_f(f), ρ, φ_f, j_f) = \\
- \sum_{ρ_1, ρ_2} \sum_{d_1, d_2} \left( \sum_{J \subseteq \left\{ \frac{r}{2} \right\}} (-1)^{|J|} k_{d_1}(P_J, ρ_1, \det^{2s}, j_1) \sum_{f \in F^{n-2sr}(\leq s)} (-1)^{|f|} g_{n, d_2, ρ_2, s, r, j}(f) \right) \\
- \sum_{ρ_1, ρ_2} \sum_{d_1, d_2} \left( \sum_{J \subseteq \left\{ \frac{r}{2} \right\}} (-1)^{|J|} k_{d_1}(P_J, ρ_1, \det^{2s}, j_1) \sum_{f \in F^{n-2sr}(\leq s)} (-1)^{|f|} g_{n-2sr, d_2, s, 0, j}(f) \right) \\
= - \sum_{ρ_1, ρ_2} \left( \beta((r), a_ρ) \sum_{\substack{μ = (n-2sr) \\
\mu' = μ = d_2 \\
l(μ) = 2s-1 \\
j | \gcd(2s, q+1, λ(μ))}} \beta(μ, a_ρ) \right) \\
= - \sum_{\substack{μ = (n) \\
\mu' = μ = d_2 \\
l(μ) = r \\
j | \gcd(2s+1, λ(μ))}} \beta(μ, a_ρ).
where for fixed \( \mu = (a_1^{l_1}, a_2^{l_2}, \ldots, a_{\delta(\mu)}^{l_{\delta(\mu)}}) \vdash (n-2sr) \) the corresponding partition of \( n \) is given by

\[
\tilde{\mu} = (r^{2s}) + \mu = ((a_1 + r)^{l_1}, (a_2 + r)^{l_2}, \ldots, (a_{\delta(\mu)} + r)^{l_{\delta(\mu)}}, r^e) \vdash n.
\]

and \( \sum_{i=1}^{\delta(\mu)} l_i + e = 2s \) so \( j \) divides \( e \) i.e. \( j \) divides \( \lambda(\tilde{\mu}) \).

And we are done, Proposition 8.2.4 is proved. Thus Proposition 8.2.1 also holds, so too does Equation 8.1 (via Equation 8.5), and thus the main result of this thesis Theorem 4.1.1 is proved.
CHAPTER 9

AN EXAMPLE: DIMENSION 4

Let $K = \overline{F}_q$. Let $\tilde{G} = \text{GL}_4(K)$. Under the Frobenius $F(a_{i,j}) = M(a_{j,i}^q)M^{-1}$ we have $\tilde{G}^F = U_4(q)$ which we denote by $G$. The Weyl group, $\tilde{W}$, of $\tilde{G}$ is of type $A_3$, i.e. $\tilde{W} = S_3$, the symmetric group on three elements. Let \{1,2,3\} be an index set for the distinguished generators of $\tilde{W}$. The Weyl group, $W$, of $G$ is of type $B_2$. The $F$-orbits on the reflections is given by \{\{1,3\}, \{2\}\}. Let $I = \{1,2\}$ index this set. Let $\tilde{B}$ be the Borel subgroup of upper triangular matrices in $\tilde{G}$. Observe that $\tilde{B}$ is $F$-stable. Let $B = \tilde{B}^F$. Then $B$ is upper triangular.

In keeping with the notation of this thesis we have the following parabolic subgroups,

$$P_\emptyset = G, \quad P_{\{1\}}, \quad P_{\{2\}}, \quad P_{\{1,2\}} = B$$
Where each $P_J$ has Levi decomposition $L_JU_J$. Writing the Levi subgroups as block matrices, we have the following $L_J$:

- $L_\emptyset = G = \{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \}$
- $L_{\{1\}} = \{ \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \} \cong \text{GL}_1(q^2) \times \text{U}_2(q)$
- $L_{\{2\}} = \{ \begin{pmatrix} * & * & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \} \cong \text{GL}_2(q^2)$
- $L_{\{1,2\}} = \{ \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \} \cong \text{GL}_1(q^2) \times \text{GL}_1(q^2)$.
Writing the unipotent radicals of the $P_J$ as block matrices, we have the following $U_J$ together with their respective normal series:

\[
U_\emptyset = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} = 1,
\]

\[
U_{\{1\}} = \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} > \left\{ \begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} > 1,
\]

\[
U_{\{2\}} = \left\{ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} = Z_{\{2\}} > 1,
\]

\[
U_{\{1,2\}} = \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} > \left\{ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} = Z_{\{2\}} > 1.
\]

For nonempty $J$, we enumerate the quotient modules for the $P_J$ and orbit representatives.
\[ J = \{1\}: \]

\[
U_{\{1\}} / Z_{\{1\}} \cong \left\{ \begin{pmatrix}
1 & a & b & 0 \\
0 & 1 & 0 & -a^q \\
0 & 1 & 0 & -b^q \\
0 & 0 & 0 & 1
\end{pmatrix} \mid a, b \in F_{q^2} \right\} \cong M_{1,2}(q^2) \text{is a unitary module for } L_{\{1\}}.
\]

Let \( \tau_s \) correspond to a singular chain of rank 1 in unitary space of dimension 2.

Let \( \tau_n \) correspond to a non-singular chain of rank 1 in unitary space of dimension 2.

\[
Z_{\{1\}} \cong \left\{ \begin{pmatrix}
1 & 0 & 0 & c \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \mid c + c^q = 0 \right\} \cong M_{1,1}(q) \text{is a central module for } L_{\{1\}}.
\]

Let \( x_1 = (\epsilon) \) be the unique non-trivial orbit representative.

\[ J = \{2\}: \]

\[
Z_{\{2\}} \cong \left\{ \begin{pmatrix}
1 & 0 & a & d_1 \\
0 & 1 & d_2 & -a^q \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \mid a, d_i \in F_{q^2} \text{ and } d_1 + d_1^q = 0 \right\} \cong M_{2,2}(q)
\]
is a central module for $L_{\{2\}}$. Let $x_2 = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$ be an orbit representative for elements of rank 2.

$J = \{1, 2\}$:

$U_{\{1,2\}}/U_{\{2\}} \cong \left\{ \begin{array}{cccc} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -a^q \\ 0 & 0 & 0 & 1 \end{array} \right\} | a \in F_{q^2} \right\} \cong M_{1,1}(q^2)$ is a general linear module for $L_{\{1,2\}}$.

$Z_{\{2\}} \cong \left\{ \begin{array}{cccc} 1 & 0 & a & d_1 \\ 0 & 1 & d_2 & -a^q \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right\} | a, d_i \in F_{q^2} \text{ and } d_i + d_i^q = 0 \right\} \cong M_{2,2}(q)$

is a central module for

$B/Z_{\{2\}} \cong \left\{ \begin{array}{ccc} * & * & 0 \\ * & 0 & 0 \\ 0 & * & * \\ 0 & 0 & * \end{array} \right\} \cong P_{\{1\}}^{+2}$. 
We enumerate the members of $E$ and $F$:

$$E = \{(\emptyset, \emptyset, 0), $$
$$\{(1, 2), \{1\}, (12)\}$$
$$\{(1, 2), \{1, 2\}, (12)\}\}, \text{ and}$$

$$F = \{(\emptyset, \emptyset, 0), $$
$$\{(1), \{1\}, 1, (2), (2), (1, 2), (2), \}$$
$$\{(1), \{1, 2\}, 1\}\}. $$

First observation: Take $e_1 = (\{1, 2\}, \{1\}, (12))$ so that $P(e_1) = P_{\{1,2\}}/U_{\{2\}} \cong P_{\{1\}}^{+2}$ with $V(e_1) = V(1, 2) \cong M_{1,1}(q^2)$. Take nontrivial $\tau_g \in \text{Irr}(V(1, 2))$ Then

$$T_{P(e_1)}(\tau) = GL_1(q^2) \ltimes V(e_1) \cong P(e_2) \ltimes V(e_1).$$

where $e_2 = (\{1, 2\}, \{1, 2\}, (12))$ since $P(e_2) \cong GL_1(q^2)$ by definition. Also note $V(e_2) = 1$ by definition. Let $g \in T_B(\tau_g)/U_{\{1,2\}}$. As a block matrix $g$ can be written:

$$g = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a^{-q} & 0 \\ 0 & 0 & 0 & a^{-q} \end{pmatrix}$$
Second observation: Take \( f_1 = (\{1\}, \{1\}, 1) \) so that \( P(f_1) \cong P_{\{1\}} \) with \( U(f_1) = U_{\{1\}} \) Then

\[
T_{P_{\{1\}}} (\tau_s) / U_{\{1\}} \cong T_{L_{\{1\}}} (\tau_s) = P^2_{\{1\}} \cong GL_1(q^2) \ltimes M_{1,1}(q) \cong P(f_2) \ltimes U(f_2)
\]

where \( f_2 = (\{1\}, \{1, 2\}, 1) \). Let \( g \in T_{L_{\{1\}}} (\tau_s) \). As a block matrix \( g \) can be written:

\[
g = \begin{pmatrix}
a & 0 & 0 & 0 \\
0 & a & b & 0 \\
0 & 0 & a^{-q} & 0 \\
0 & 0 & 0 & a^{-q}
\end{pmatrix}, \text{ where } b + b^t = 0.
\]

The elements \( e_2 \) and \( f_2 \) have opposite parity and hence lead to cancellation in the alternating sum in the statement of DOC, i.e. for \( j' = j / \gcd(j, q + 1, 2) \)

\[
k_d(B, \tau_g, \rho, \det, j) = k_{d-d'}(T_B(\tau_g) / V(1, 2), \rho, \det, j') = k^0_{d-d(e_2)}(P(e_2), V(e_2), \rho, \phi_{e_2}, j_{e_2})
\]

\[
k_d(P_{\{1\}}, \tau_s, \rho, \det, j) = k_{d-d''}(T_{P_{\{1\}}} (\tau_s) / U_{\{1\}}, \rho, \det, j')
\]

\[
= k^0_{d-d(f_2)}(P(f_2), V(f_2), \rho, \phi_{f_2}, j_{f_2}) + k^1_{d-d(f_2)}(P(f_2), U(f_2), \rho, \phi_{f_2}, j_{f_2})
\]

where \( d' \) is the power of \( q \) in \( |T_B(\tau_g) \backslash B| \) and \( d'' \) is the power of \( q \) in \( |T_{P_{\{1\}}} (\tau_s) \backslash P_{\{1\}}| \), i.e. \( d' = d'' = 0 \) Then

\[
k^0_{d-d(e_2)}(P(e_2), V(e_2), \rho, \phi_{e_2}, j_{e_2}) + k^0_{d-d(f_2)}(P(f_2), U(f_2), \rho, \phi_{f_2}, j_{f_2}) = 0.
\]
With regard to the splitting of characters upon restriction to the kernel of the determinant map in general, observe that 4 is divisible by 1, 2, and 4. Moreover, assuming that \( q + 1 \) is divisible by 4, we may consider \( j = 1, 2, \) or 4.

Let \( j = 4 \). Then \( k_d(L_J, \rho, \det, 4) = 0 \) for nonempty subsets \( J \) in \( \{1, 2\} \) so

\[
\sum_{J \subseteq I} (-1)^{|J|} k_d^0(P_J, U_J, \rho, \det, 4) = k_d(U_4(q), \rho, \det, 4)
\]

\[
= \begin{cases} 
\beta((1^4, a_\rho) = 1, & \text{if } d = 0; \\
0, & \text{otherwise}.
\end{cases}
\]

We have \( k_d^1(P_J, U_J, \rho, \det, 4) = 0 \) except for \( J = \{1\} \). Take \( \tau_s \), the orbit representative discussed above, and consider

\[
T_{P_{\{1\}}}(\tau_s)/U_{\{1\}} \cong P_{\{1\}}^2 \cong \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a^{-q} & 0 \\ 0 & 0 & 0 & a^{-q} \end{pmatrix} \mid a, b \in F_q^2, b + b^q = 0 \right\}.
\]

The determinant map restricted to \( P(f_2) = P_{\{1\}}^2 \) is \( \phi_{f_2} = \det^2 \). Moreover, \( 4_{f_2} \) is the least positive integer such that 4 divides

\[
4 \text{ divides } 4_{f_2} \cdot \frac{q + 1}{(q + 1)/\gcd(q + 1, 2)}.
\]
so \(4f_2 = 2\). By definition \(d(f_2) = 0\) and indeed the exponent of \(q\) in \(|TP_{(1)}(\tau_s) \setminus P_{(1)}|\) is zero as already mentioned.

Take non trivial \(x \in \text{Irr}(Z^2_{\{1\}})\). Then

\[
TP_{(1)}(x) = U_1(q) \ltimes Z^2_{\{1\}} \cong \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, b \in F_q, a^{1+q} = 1, \text{ and } b + b^q = 0 \right\}.
\]

The determinant map restricted to \(TP_{(1)}(x) = \det^4\). Let \(j'\) be the least positive integer such that

\[
2 \text{ divides } j' \cdot \frac{|P^2_{(1)}|}{|TP_{P^2_{(1)}}(x) \cdot \ker(\det^4)|},
\]

then \(j' = 1\). Hence

\[
\sum_{J \subseteq I} (-1)^{|J|}k^1_d(P_J, U_J, \rho, \det, 4) = -k^1_{d-d(f_2)}(P(f_2), U(f_2), \rho, \phi_{f_2}, j_{f_2})
\]

\[
= -k^1_d(P^2_{(1)}, Z^2_{\{1\}}, \rho, \det^2, 2)
\]

\[
= -k_d(U_1(q), \rho, \det^4, 1)
\]

\[
= \begin{cases} 
\beta((1^4), a) = 1, & \text{if } d = 0; \\
0, & \text{otherwise.}
\end{cases}
\]
Let \( j = 2 \), then \( k_d(L_J, \rho, \det, 2) = 0 \) for subsets \( J = \{1\} \) and \( \{1, 2\} \). The calculations are repetitive so we will not present them for all for \( d = 0, 1, 2, \) and 3, but rather present some of the more interesting calculations. Consider \( d = 0 \). Our first observation is that this case includes the above, since 2 divides 4.

\[
\sum_{J \subseteq I} (-1)^{|J|} k_0^d(P_J, U_J, \rho, \det, 4) = k_0(U_4(q), \rho, \det, 2) - k_0(GL_2(q^2), \rho, \det^{1-q}, 2)
\]

\[
= q\beta((1^4), a_\rho) - \beta((1^2), a_\rho)
\]

\[
= q - (q - 1)
\]

\[
= (q - 1) + 1 - (q - 1)
\]

\[
= 1.
\]

Notice that the remaining character splits into 4 irreducibles upon restriction to the kernel of the determinant map. We have \( k_1^0(P_J, U_J, \rho, \det, 2) = 0 \) except for \( J = \{1\} \), which is worked out above.

Continue to assume that \( j = 2 \) and consider \( d = 1 \). Then \( k_1(GL_2(q^2), \rho, \det^{1-q}, 2) = 0 \) since 0 and 2 are the only possible \( q \)-heights for \( \chi \in \text{Irr}(GL_2(q^2)) \). Moreover \( k_1(U_4(q), \rho, \det, 2) = 0 \) since 2 does not divide \( \lambda(\mu) \) for \( \mu = (2, 1^2) \). Hence,

\[
\sum_{J \subseteq I} (-1)^{|J|} k_1^0(P_J, U_J, \rho, \det, 4) = 0
\]
which doesn’t seem like an interesting calculation. However, examining the other side of the alternating sum is somewhat more interesting since we see cancellation. Take \(x_2 \in S^2(f_2)\) and \(S^2(f_3)\) as above, then

\[
T_{P(2)}(x_2)/U(2) \cong U_2(q) \text{ and } T_{P(1,2)}(x_2)/U(2) \cong P_{\{1\}}^2.
\]

The exponent of \(q\) in

\[
\left|U_2(q) \backslash \text{GL}_2(q^2)\right| = \left|P_{\{1\}}^2 \backslash P_{\{1\}}^{+2}\right| \text{ is } 1.
\]

If \(K\) is the kernel of the determinant map restricted to the stabilizers of \(x_2\), then

\[
2 \text{ divides } \left|T_{P(2)}(x_2)K \backslash P_{\{2\}}\right| = \left|T_{P(1,2)}(x_2)K \backslash P_{\{1,2\}}\right|.
\]

Hence,

\[
k_d(P_{\{2\}}, x_2, \rho, \text{det}^{1-q}, 2) = k_{d-1}(U_2(q), \rho, \text{det}^2, 1) \text{ and } k_d(P_{\{1,2\}}, x_2, \rho, \text{det}^{1-q}, 2) = k_{d-1}(P_{\{1\}}^2, \rho, \text{det}^2, 1).
\]

Thus

\[
\sum_{J \subseteq I} (-1)^{|J|} k_1^1(P_J, U_J, \rho, \text{det}, 2) = -k_0(U_2(q), \rho, \text{det}^2, 1) + k_0(P_{\{1\}}^2, \rho, \text{det}^2, 1) = -0
\]

since \(\binom{3}{2} = 1 > 0\).
Let $j = 1$. Since 1 divides every integer, this case is trivial. Indeed,

$$\sum_{J \subseteq I} (-1)^{|J|} k_d(P_J, U_J, \rho, \det, 1) = \sum_{J \subseteq I} (-1)^{|J|} k_d(P_J, U_J, \rho)$$

which has already been shown by Ku ((15)).
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