Complete Coordinate-free Characterization of Isolated Homogeneous Singularities and Derivations of Moduli

BY

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THESIS

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To my family:

my parents,

my wife Xue Luo and my son Eric Zuo.
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This work, as well as many things in my life, would not have been possible to complete by my own. Many people have been by my side all along, mathematically and otherwise. I would like to express my gratitude to my teacher and supervisor Stephen S.-T. Yau for his patience and encouragement.

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A special thanks to my family, they have always supported me.
This Ph.D. thesis is mainly based on several papers (91) (92), (93), (94), and (17). Three main topics are stated in this thesis. The first topic is about complete characterization of homogeneous isolated hypersurface singularities which will be considered in Chapter 3. The second topic is about Haperin conjecture in singularity theory which will be considered in Chapter 4. The third topic is about characterization of isolated complete intersection singularities with $\mathbb{C}^*$-action of dimension $n \geq 2$ by means of geometric genus and irregularity which will be considered in Chapter 5.
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Let \( X \) be a nonsingular projective variety in \( \mathbb{CP}^{n-1} \). Then the cone over \( X \) in \( \mathbb{C}^n \) is an affine variety \( V \) with an isolated singularity at the origin. It is a very natural and important question to ask when an affine variety with an isolated singularity at the origin is a cone over nonsingular projective variety.

This problem is very hard in general. In this thesis we shall treat the hypersurface case. Given a function \( f \) with a isolated singularity at the origin. We can ask whether \( f \) is a weighted homogeneous polynomial or a homogeneous polynomial after a biholomorphic change of variables. The former question was answered in a celebrated paper by Saito (52) in 1971. However, the latter question has remained open for 40 years until Xu-Yau solved it for \( f \) with three variables (79). In this thesis we first solved it for \( f \) with up to six variables based on the classification of weighed homogeneous singularities. This result is published in our paper (92). However the methods used in (92) is hard to be generalized. In this thesis, using different methods we can also solve the latter question for general \( n \) completely, i.e., we show that \( f \) is a homogeneous polynomial after a biholomorphic change of coordinates if and only if \( \mu = \tau = (\nu - 1)^n \), where \( \mu, \tau \) and \( \nu \) are Milnor number, Tjurina number and multiplicity of the singularity respectively. Thus we give a intrinsic complete characterization of homogenous isolated hypersuface singularities which is very important in the classification of isolated hypersurface singularities.

In this thesis we also deal with another fundamental problem in singularity theory, i.e., Halperin Conjecture. This thesis present a result about the finite-dimensional Lie algebras
SUMMARY (Continued)

associated with germs of isolated hypersurface singularities defined by polynomials with the number of monomials equal to the number of variables.

Let $R = \mathbb{C}\{x_1, x_2, \cdots, x_n\}/(f)$ where $f$ is a weighted homogeneous polynomial defining an isolated singularity at the origin. Then $R$ and $\text{Der}(R, R)$, i.e., the Lie algebra of derivations on $R$, are graded. It is well-known that $\text{Der}(R, R)$ has no negatively graded component (73). J. Wahl conjectured that the above fact is still true in higher codimensional case provided that $R = \mathbb{C}\{x_1, x_2, \cdots, x_n\}/(f_1, f_2, \cdots, f_m)$ is an isolated, normal and complete intersection singularity and $f_1, f_2, \cdots, f_m$ are weighted homogeneous polynomials with the same weight type $(w_1, w_2, \cdots, w_n)$. On the other hand Yau conjectured that the moduli algebra $A(V) = \mathbb{C}\{x_1, x_2, \cdots, x_n\}/(\partial f/\partial x_1, \cdots, \partial f/\partial x_n)$ has no negatively weighted derivation where $f$ is a weighted homogeneous polynomial defining an isolated singularity at the origin (77). By supposing this conjecture has a positive answer he proved a characterization of weighted homogeneous hypersurface singularities only using the Lie algebra $L(V)$ of derivations on $A(V)$.

The conjecture of Yau can be thought as an Artinian analogue of J. Wahl’s conjecture. For the dimension up to 3, the Yau conjecture 4.1.1 has a positive answer ((18), (15)). In this thesis we prove this conjecture for the fewnomial singularities with multiplicity at least 5.

It is well known that geometric genus $p_g$ and irregularity $q$ are two important invariants for isolated singularities. In this thesis we give a formula relating $p_g$ and $q$ for isolated singularities with $\mathbb{C}^*$-action in any dimension. We also give a simple characterization of the quasi-homogeneous isolated complete intersection singularities using $p_g$ and $q$. As a corollary,
we prove that \( q \) is a invariant topological type for two-dimensional weighted homogeneous hypersurface singularities.
CHAPTER 1

INTRODUCTION

1.1 Background and History

In singularity theory one has first studied hypersurface singularities, where a holomorphic map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is given. Especially simple is the example $f(z) = z_1^{a_1} + z_2^{a_2} + \cdots + z_n^{a_n}$ (Briskorn polynomial). A more general class is given by the weighted homogeneous polynomials: Let $d, w_1, \ldots, w_n$ be positive integers and $f \in \mathbb{C}[x_1, \cdots, x_n]$ be a polynomial.

**Definition 1.1.1.** $f$ is weighted homogeneous of degree $d$ with respect to the weights $w_1, \cdots, w_n$ if $f$ is a linear combination of monomials $x_1^{j_1} \cdots x_n^{j_n}, j_1 w_1 + \cdots + j_n w_n = d$.

The study of singularities lies on the cross-roads of many different areas of mathematics. Initially, during the nineteenth and early twentieth century, algebraic geometers worked on plane curve singularities. Since the late 1960s, new methods in singularity theory have been rapidly developed. One of the fundamental results is the fibration theorem of Milnor (46). It deals with hypersurface singularities related to functions of several complex variables. Milnor’s book (46) has been extremely influential and since then the development of the theory over the complex numbers $\mathbb{C}$ is ongoing. Besides, the interaction between the different methods makes the study of hypersurface singularities particularly fruitful.

A further central topic in singularity theory is the classification of hypersurface singularities. In the early 1970, Arnold introduced the notion of modality and developed the classification
over $\mathbb{C}$ with respect to right equivalence (4). First singularities of modality 0 are then classified. These are mostly known as simple or ADE-singularities. Also Arnold and especially Brieskorn (14) established the coincidence of this classification with that of simple Lie Groups. In subsequent papers Arnold classified singularities of modality 1 (5) and 2 (7). In (8), the reader is referred to a complete list of normal forms of simple, unimodular and bimodular singularities.

Types of singularities of modality 3 have been discussed by Wall in (76). In (55), unimodular plane curve singularities are classified for contact equivalence.

The goal of this dissertation is to give an intrinsic complete characterization of homogeneous isolated hypersurface singularities which provides the necessary tools, theoretically and computationally, for the purpose of classification of singularities.

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be the germ of a complex analytic function with an isolated critical point at the origin. Let $V = \{ z \in \mathbb{C}^n : f(z) = 0 \}$. Recall that the multiplicity of the singularity $V$ is defined to be the order of the lowest non-vanishing term in the power series expansion of $f$ at 0. The Milnor number $\mu$ and the Tjurina number $\tau$ of the singularity $(V, 0)$ are defined respectively by

$$\mu = \dim \mathbb{C}\{z_1, z_2, \ldots, z_n\}/(f_{z_1}, \ldots, f_{z_n}),$$

$$\tau = \dim \mathbb{C}\{z_1, z_2, \ldots, z_n\}/(f, f_{z_1}, \ldots, f_{z_n}).$$

They are numerical invariants of $(V, 0)$. In 1971, Saito proved the following theorem which gives a necessary and sufficient condition for $V$ to be defined by a weighted homogeneous polynomial.
Theorem 1.1.1. (52). Let \((V, 0) \subseteq (\mathbb{C}^n, 0)\) be the germ of an isolated hypersurface singularity defined by \(f \in \mathbb{C}\{x_1, \cdots, x_n\}\). The following conditions are equivalent:

(1): \((V, 0)\) is a weighted homogeneous isolated hypersurface singularities from different viewpoints.

(2): \(f \in \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right)\).

(3): \(\mu(f) = \tau(f)\).

(4): \((V, 0)\) admits a non-nilpotent vector field.

Sometimes we also use another definition of weighted homogeneous polynomial which is equivalent to definition 1.1.1. Let \(w = (w_1, \cdots, w_n)\) be an \(n\)-tuple of positive rational numbers. A polynomial \(f(z_1, \cdots, z_n)\) is said to be a weighted homogeneous polynomial with weights \(w\) if each monomial \(\alpha z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n}\) of \(f\) satisfies \(a_1/w_1 + \cdots + a_n/w_n = 1\). It has an isolated critical point at \(0 \in \mathbb{C}^n\) if \(\text{grad } f = (\partial f/\partial z_1, \cdots, \partial f/\partial z_n)\) is zero at \(0\) but \(\text{grad } f(z) \neq 0\) for all \(z\) in a neighborhood of \(0\).

Recall that a polynomial \(f(z_1, \cdots, z_n)\) is called quasi-homogeneous if \(f\) is in the Jacobian ideal of \(f\) i.e., \(f \in (\partial f/\partial z_1, \cdots, \partial f/\partial z_n)\).

By the above theorem of Saito, if \(f\) is quasi-homogeneous with an isolated critical point at \(0\), then after a biholomorphic change of coordinates, \(f\) becomes a weighted homogeneous polynomial. So there is no difference between quasi-homogeneous singularity and weighted homogeneous singularity after a biholomorphic change of coordinates.

Let \(f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)\) be the germ of a complex analytic function with an isolated critical point at the origin. Let \(V = \{z \in \mathbb{C}^n : f(z) = 0\}\). It is a natural question to ask when \(V\)
is defined by a weighted homogeneous polynomial up to biholomorphic change of coordinates. Saito solved this question. He gives a necessary and sufficient condition for $V$ to be defined by a weighted homogeneous polynomial (see the above Saito’s Theorem). It is a natural and important question to characterize homogeneous polynomial with isolated critical point at the origin. This question has remained open for 40 years. In fact it is the first important case of the following interesting problem. Let $X$ be a nonsingular projective variety in $\mathbb{CP}^{n-1}$. Then the cone over $X$ in $\mathbb{C}^n$ is the affine variety $V$ with an isolated singularity at the origin. It is a very natural and important question to ask when an affine variety with an isolated singularity at the origin is a cone over nonsingular projective variety.

For a two-dimensional isolated hypersurface singularity $V$, Xu and Yau ((78), (79)) found a coordinate free characterization for $V$ to be defined by a homogeneous polynomial. Recently Lin and Yau ((37), (35), (80)) and Chen, Lin, Yau and the author (17) gave necessary and sufficient conditions for a 3 and 4- dimensional isolated hypersurface singularities with $p_g \geq 0$ and $p_g > 0$ respectively, where $p_g$ is the geometric genus of the singularity. Based on classification of weighted homogeneous singularities Yau and the author solve the problem for $f$ with up to six variables (92). However, it is quite difficult to generalize their methods to give characterization of homogeneous polynomials for general $n$.

In 2005, Yau formulated the following conjecture

**Yau Conjecture 3.1.1.** (1) Let $\mu$ and $\nu$ be the Milnor number and multiplicity of $(V, 0)$ respectively. Then $\mu \geq (\nu - 1)^n$ and the equality holds if and only if $f$ is a semi-homogeneous function (see Definition 2.7.3).
(2) Moreover, if \( f \) is a quasi-homogeneous function, then \( \mu = (\nu - 1)^n \) if and only if \( f \) is a homogeneous polynomial after a biholomorphic change of coordinates.

In this thesis we verify **Yau Conjecture 3.1.1** affirmatively. As a result we have solved the characterization problem of homogeneous polynomial with an isolated critical point at the origin, i.e. we have shown that \( f \) is a homogeneous polynomial after a biholomorphic change of coordinates if and only if \( \mu = \tau = (\nu - 1)^n \).

Let \( \pi : (M, A) \to (V, 0) \) be a resolution of singularity of dimension \( n \) with exceptional set \( A = \pi^{-1}(0) \). The geometric genus \( p_g \) of the singularity \((V, 0)\) is the dimension of \( H^{n-1}(M, \mathcal{O}) \) and is independent of the resolution \( M \).

Using \( p_g, \mu \) and \( \nu \), Yau raised another conjecture which answer when a weighted homogeneous singularity is a homogeneous singularity.

**Yau Conjecture 3.1.2.** Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be a weighted homogeneous polynomial with an isolated singularity at the origin. Let \( \mu, p_g \) and \( \nu \) be the Milnor number, geometric genus and multiplicity of the singularity \( V = \{ z : f(z) = 0 \} \), then

\[
\mu - p(\nu) \geq n!p_g,
\]

where \( p(\nu) = (\nu - 1)^n - \nu(\nu - 1)\ldots(\nu - n + 1) \), and equality holds if and only if \( f \) is a homogeneous polynomial.

In fact, we proved in this thesis that if \( p_g = 0 \), then **Yau Conjecture 3.1.1** implies **Yau Conjecture 3.1.2.**
These conjectures are sharp estimate and have some important applications in geometry.

In the second part of this thesis, we present a result about the finite-dimensional Lie algebras associated with germs of isolated hypersurface singularities (IHS) defined by polynomials with the number of monomials equal to the number of variables. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function with an isolated critical point at the origin. Then $V = \{f = 0\}$ is an IHS germ. In (83), Yau was the first person who considers $L(V)$ which is the Lie algebra of derivations of the moduli algebra $A(V) = \mathbb{C}\{x_1, \ldots, x_n\}/(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$, i.e., $L(V) = \text{Der}_{\mathbb{C}}(A(V), A(V))$. According to (83), $L(V)$ is a finite-dimensional solvable Lie algebra which is often called the Lie algebra of singularity $V$. Following (29) and (95) we call it the Yau algebra of $V$ in order to distinguish from Lie algebras of other types appearing in singularity theory (10), (2), (11).

Recall that a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is called weighted homogeneous if there exist positive rational numbers $w_1, \ldots, w_n$ (called weights of indeterminates $x_i$) and $d$ such that, for each monomial $\prod x_i^{a_i}$ appearing in $f$ with nonzero coefficient, one has $\sum a_i w_i = d$. The number $d$ is called the weighted homogeneous degree ($w$-degree) of $f$ with respect to weights $w_j$.

Let $P = \mathbb{C}[x_1, \ldots, x_n]$ be the polynomial ring of $n$ weighted variables $x_1, \ldots, x_n$ with positive integer weights $w_1, w_2, \ldots, w_n$. For a monomial $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ in $P$ its weighted degree is defined to be $w_1 i_1 + \cdots + w_n i_n$. By this (weighted) degree the polynomial ring $P$ is graded. For a homogeneous ideal $I$ (i.e., an ideal generated by weighted homogeneous polynomials) $\subset P$ we have a graded quotient ring $R = P/I = \oplus_{i=0}^{\infty} R_i$ which corresponds to an affine variety with singularities at the origin. It is obvious that there is a $\mathbb{C}^*$-action on this variety with singularities.
When the Krull dimension of $R$ is zero we have a 0-dimensional scheme at the origin and $R$ is a graded Artinian algebra. Let $\text{Der}(R, R)$ be the $R$-module of derivations on $R$ (i.e., vector fields on the aforementioned variety with $\mathbb{C}^*$-action where the $\mathbb{C}^*$-action corresponds to the well-known Euler derivation $\Delta := w_1x_1\partial/\partial x_1 + w_2x_2\partial/\partial x_2 + \cdots + w_nx_n\partial/\partial x_n$), we have a natural grading on $\text{Der}(R, R) = \oplus_{k=-\infty}^{+\infty} \text{Der}(R, R)_k$ where $\text{Der}(R, R)_k = \{ D \in \text{Der}(R, R) : D(R_i) \subset R_{i+k} \text{ for any } i \}$. In this way, the Euler derivation $\Delta$ has weight 0. It is useful to know if $\text{Der}(R, R)$ has any derivation of negative weight, or other derivations of weight 0. Moreover, it is a natural and long-standing problem whether $\text{Der}(R, R)$ is non-negatively graded, i.e., $\text{Der}(R, R)_k = 0$ for any $k < 0$, nonexistence of negative weight derivation. Actually this problem has been motivated from both algebraic topology and singularities as follows.

A classic result of A. Borel ((13)) states that the Serre spectral sequence for rational cohomology of the universal bundle $G/H \to B_H \to B_G$ collapses if $G/H$ is a homogeneous space of equal rank pairs $(G, H)$ of compacted connected Lie groups. In 1976 S. Halperin made a very general conjecture which is one of the most important open problems in rational homotopy theory ((24), (42)).

**Halperin Conjecture.** Suppose that $F \to E \to B$ is a fibration with simply-connected base $B$ and the (rational) cohomology algebra of the fibre is an Artinian algebra of the following form

$$H^*(F, \mathbb{Q}) = \mathbb{Q}[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_n)$$

then the Serre spectral sequence for this fibration collapses.
It is clear that \( x_1, x_2, \ldots, x_n \) are weighted variables with weights of their dimensions and 
\( f_1, f_2, \ldots, f_n \) are weighted homogeneous polynomials from the topological background. Actually
the above conjecture is implied by the following conjecture about the nonexistence of negative
weight derivation ((24), (42)).

**Halperin Conjecture.** [equivalent form] Let \( x_1, x_2, \ldots, x_n \) be weighted variables and \( f_1, f_2, \ldots, f_n \)
be weighted homogeneous polynomials in \( P \). Suppose that \( R \) is an Artinian algebra of the form

\[
\mathbb{C}[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_n).
\]

Then there is no non-zero negative weight derivation on \( R \).

This conjecture has been proved when the fibre is homogeneous ((13), (42)) and \( n \leq 2 \) ((65),
(66) and (16)).

On the other hand S. S.-T. Yau has the following conjecture about the nonexistence of the
negative weight derivation from his work about *Lie* algebras of derivations on the moduli alge-
bras of isolated hypersurface singularities, and especially his micro-local characterization (only
use the *Lie* algebras of derivations on the moduli algebras) of quasi-homogeneous hypersurface
singularities ((82), (83), (41), (18)) and (77)).

**Yau Conjecture 4.1.1.** Let \( (V,0) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n: f(x_1, x_2, \ldots, x_n) = 0\} \)
be an isolated singularity defined by the weighted homogeneous polynomial \( f(x_1, x_2, \ldots, x_n) \).
Then there is no non-zero negative weight derivation on the moduli algebra (= Milnor al-
Algebra here) $A(V) = \mathbb{C}[x_1, x_2, \ldots, x_n]/(\partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n)$. i.e., the Yau algebra is non-negatively graded algebra.

In case $f$ is a homogeneous polynomial, then it was shown in (77) that $L(V)$ is a graded Lie algebra without negative weight. In fact they proved the following theorem.

**Theorem 1.1.1.** (77) Let $A = \oplus_{i=0}^t A_i$ be a commutative Artinian local algebra with $A_0 = \mathbb{C}$. Suppose that maximal ideal of $A$ is generated by $A_j$ for some $j > 0$. Then $L(A)$ is a nonnegatively graded Lie Algebra $\oplus_{k=0}^t L_k$.

This conjecture has also been proved in the low-dimensional case $n \leq 4$ ((18), (15)) by explicit calculations.

**Theorem 1.1.2.** (18) Let $f(x_1, x_2, x_3)$ be a weighted homogeneous polynomial of type $(w_1, w_2, w_3; d)$ with isolated singularity at the origin. Assume that $d \geq 2w_1 \geq 2w_2 \geq 2w_3$. Let $D$ be a derivation of the moduli algebra

$$\mathbb{C}[x_1, x_2, x_3]/(\partial f/\partial x_1, \partial f/\partial x_2, \partial f/\partial x_3).$$

Then $D \equiv 0$ if $D$ is negatively weighted.

**Theorem 1.1.3.** (15) Let $f(x_1, x_2, x_3, x_4)$ be a weighted homogeneous polynomial of type $(w_1, w_2, w_3, w_4; d)$ with isolated singularity at the origin. Assume that $d \geq 2w_1 \geq 2w_2 \geq 2w_3 \geq 2w_4$. Let $D$ be a derivation of the moduli algebra

$$\mathbb{C}[x_1, x_2, x_3, x_4]/(\partial f/\partial x_1, \partial f/\partial x_2, \partial f/\partial x_3, \partial f/\partial x_4).$$
Then $D \equiv 0$ if $D$ is negatively weighted.

However, Chen wrote in (15): “five variables case is very complicated than four variables and it seems that a general method (coordinate-free) is needed for the case of arbitrary dimension.” In fact the methods for proving Theorem 4.1.2 and Theorem 4.1.3 are very hard to be generalized to higher dimension.

Both conjectures of S. Halperin and S. S.-T. Yau are about 0-dimensional quasi-homogeneous singularities, i.e., graded Artinian algebras. The problem of nonexistence of negative weight derivation for positive-dimensional quasi-homogeneous singularities has also been considered from other motivations ((47), (73), (74), (75)). In (26) and (27) the nonexistence of negative weight derivation was proved for isolated quasi-homogeneous hypersurface singularities and quasi-homogeneous curve singularities. Kantor proved

(a) (26): If $A = \mathbb{C}[t^{n_1}, \cdots, t^{n_r}]$ is a non-regular monomial curve, then $A$ has no derivations of negative weight.

(b) (27): If $A = \mathbb{C}[X_1, \cdots, X_n]/(f)$ is a quasi-homogeneous hypersurface with isolated singularity and normalized grading, then $A$ has no derivations of negative weight, and the derivations of weight 0 can be read off the quadratic terms of $f$.

In (74), Wahl Proved that suppose $A$ is a graded normal surface singularity, and $A$ is not isomorphic to a cyclic quotient $\mathbb{C}[X, Y]^G (G \subset GL(2, \mathbb{C})$ cyclic). Then $\Delta$ is the only derivation of weight $\leq 0$.

In (74) Wahl proposed a very general conjecture about the nonexistence of negative weight derivation for positive-dimensional weighted-homogeneous singularities.
One special case of his conjecture for singular cones led him to give a beautiful cohomological characterization of complex projective space ((75)). Another special case of Wahl conjecture for complete intersections, which is the generalization of the well-known result in (27) was solved by A. G. Aleksandrov in (1). The answer is that: in the case of low-degree the negative weight derivations do exist; however except for those low-degree cases Wahl conjecture is true (also see (9, pp. 34–35)).

In this thesis, we deal with Yau conjecture 4.1.1 for fewnomial singularities. The concept of fewnomial was introduced firstly by A.Khovanski (28). Namely, we say that a polynomial $f$ in $n$ variables is a fewnomial if the number of of monomials appearing in $f$ does not exceed $n$. It is easy to show that, except for certain trivial cases, a fewnomial in $n$ variables can define an IHS only if it has exactly $n$ monomials, in which case we speak of fewnomial isolated singularity. In other words, fewnomial singularities are those which can be defined by $n$ monomials in $n$ indeterminates. Simple singularities are obviously fewnomial in this sense.

In this thesis we prove the following result which give positive answer to Halperin, Yau and Wahl conjectures for fewnomial singularities.

In the third part of this thesis, we shall present two results relate two important invariants of singularities, i.e., geometric genus $p_g$ and irregularity $q$.

Let $(V,0)$ be a Stein germ of an analytic space with an isolated singularity at 0. $(V,0)$ is a singularity with a (good) $\mathbb{C}^*$-action if the complete local ring of $V$ at 0 is the completion of a (positively) graded ring. $(V,0)$ is a quasi-homogeneous singularity if there exists an analytic isomorphism type of $(V,0)$ which is defined by weighted homogeneous polynomials.
Let \( f \in \mathbb{C}\{z_0, z_1, \cdots, z_n\} \) be a holomorphic function germ with an isolated singularity at the origin. It is well known that

\[
\mu = \dim \mathbb{C}\{z_0, z_1, \cdots, z_n\}/(\partial f/\partial z_0, \cdots, \partial f/\partial z_n)
\]

and

\[
\tau = \dim \mathbb{C}\{z_0, z_1, \cdots, z_n\}/(f, \partial f/\partial z_0, \cdots, \partial f/\partial z_n)
\]

are two very important invariants for hypersurface singularities. Clearly, \( \mu \geq \tau \), and the equality holds if and only if \( f \) is quasi-homogeneous singularity by a well known theorem of Saito (52). Both \( \mu \) and \( \tau \) can also be defined for \( n \)-dimensional isolated complete intersection singularity (ICIS) with \( n \geq 1 \) in the following manner:

\[
\mu = \text{rk} H_n(F)
\]

and

\[
\tau = \dim T^1_{V,0},
\]

where \( F \) is the Milnor fibre of a Milnor fibration of \((V, 0)\) (see (40)), and \( \tau \) is the dimension of the base space of a semi-universal deformation of \((V, 0)\). From the defining equations of \((V, 0)\), one can give formulae for \( \mu \) and \( \tau \) as dimensions of certain finite length modules, but it is no longer clear what is the relation between these invariants. This problem was first considered by Greuel (22), who conjectured \( \mu \geq \tau \), and proved the inequality in two cases: \( n = 1 \) or the link
of $V$ a rational homotopy sphere. Greuel also proved that (in every dimension) \( \mu = \tau \) if \((V,0)\) is quasi-homogeneous. Looijenga (39) proved that for ICIS of dimension \( n = 2 \), \( \mu \geq \tau + b \) where \( b = \) number of loops in the resolution dual graph of \((V,0)\). Then Looijenga and J. Steenbrink (40) generalized this result for all \( n \geq 2 \). In (71), Wahl proved that for two-dimensional isolated complete intersection singularity \((V,0)\), \( \mu \geq \tau + b \) and \( \mu = \tau + b \) if and only if \((V,0)\) is quasi-homogeneous \((for \ b = 0)\) or \((V,0)\) is cusp \((b = 1)\). More recently, Vosegaard (69) generalized this result for general \( n \). Let \((V,0)\) be an isolated complete intersection singularity of any dimension, he proved that \((V,0)\) is quasi-homogeneous if and only if \( \mu = \tau \).

Let \((V,0)\) be a normal surface singularity. Artin first defined an invariant geometric genus \( p_g \) for the singularity \((V,0)\). It turns out that this is an important invariant for the theory of normal surface singularities. In (88), Yau introduced another invariant called irregularity \( q \) of the singularity \((V,0)\). This invariant is interesting for the following reason. It is a long-term conjecture that normal surface singularities are not rigid, i.e., \( \dim T^1_V \geq 1 \) where \( T^1_V \) is the set of isomorphism classes of first order infinitesimal deformations of \( V \). In the case of Gorenstein surface singularities this irregularity actually gives a lower bound for \( \dim T^1_V \). In fact, both geometric genus and irregularity can also be defined for general \( n \)-dimensional isolated singularities. Let \((V,0)\) be a normal isolated singularity of dimension \( n(\geq 2) \) with \( 0 \) as its only isolated singularity. Let \( \pi : \tilde{V} \to V \) be a resolution of the singularity of \( V \) with exceptional set \( E = \pi^{-1}(0)_{\text{red}} \). Then \( p_g := \dim R^{n-1} \pi_* \mathcal{O}_{\tilde{V}} \), and \( q = \dim H^0(\Omega^n_{\tilde{V}} - E)/H^0(\Omega^n_{\tilde{V}}) \).
In (88), Yau gave a formulae for the irregularity in case \((V,0)\) is a hypersurface singularity or a two-dimensional singularity with \(\mathbb{C}^*\)-action. Moreover, for \(n\)-dimensional singularity with \(\mathbb{C}^*\)-action, Yau gave a lower estimate for irregularity in terms of geometric genus. He proved

\[ q \geq p_g - h^{n-1}(\mathcal{O}_E). \]

In this paper, one of our main results is to prove that the above inequality is actually an equality. Explicitly, we prove the following result: let \((V,0)\) be a normal isolated singularity of dimension \(n(\geq 2)\) with \(\mathbb{C}^*\)-action. Let \(\pi: \tilde{V} \to V\) be a good resolution of the singularity \((V,0)\) with \(E = f^{-1}(V)_{\text{red}}\). Then \(q = p_g - h^{n-1}(\mathcal{O}_E)\).

A natural question is how to use \(p_g\) and \(q\) to characterize quasi-homogeneous singularities. We prove that the converse of the above theorem is also correct for non-Du Bois isolated complete intersection singularities. Explicitly, we prove the following result: Let \((V,0)\) be a normal isolated complete intersection singularities of dimension \(n(\geq 2)\), and \(\pi: \tilde{V} \to V\) be a good resolution of the singularity \((V,0)\) with \(E = f^{-1}(0)_{\text{red}}\). If \(q = p_g - h^{n-1}(\mathcal{O}_E)\), then either \((V,0)\) has a \(\mathbb{C}^*\)-action or \((V,0)\) is a Du Bois singularity.

1.2 Main Results

Our main purpose in this thesis is to prove the following results and give some applications.
Theorem A. Let \( f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0) \) be a holomorphic germ defining an isolated hypersurface singularity \( V = \{ z : f(z) = 0 \} \) at the origin. Let \( \mu \) and \( \nu \) be the Milnor number and multiplicity of \( (V, 0) \) respectively. Then

\[
\mu \geq (\nu - 1)^n
\]

and the equality in (1.7) holds if and only if \( f \) is a semi-homogeneous function (i.e. \( f = f_\nu + g \) where \( f_\nu \) is a non-degenerate homogeneous polynomial of degree \( \nu \) and \( g \) consists of terms of degree at least \( \nu + 1 \)).

Theorem B. Let \( f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0) \) be a weighted homogeneous polynomial with an isolated singularity at the origin. Let \( \mu \) and \( \nu \) be the Milnor number and multiplicity of the singularity \( V = \{ z : f(z) = 0 \} \) respectively. Then

\[
\mu \geq (\nu - 1)^n
\]

and the equality in (1.8) holds if and only if \( f \) is a homogeneous polynomial after a biholomorphic change of coordinates.

Theorem C. Let \( f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0) \) be a weighted homogeneous polynomial with an isolated singularity at the origin. Let \( \mu, p_g \) and \( \nu \) be the Milnor number, geometric genus and multiplicity of the singularity \( V = \{ z : f(z) = 0 \} \), if \( p_g = 0 \) then

\[
\mu - p(\nu) \geq n! p_g,
\]
where \( p(\nu) = (\nu - 1)^n - \nu(\nu - 1) \cdots (\nu - n + 1) \), and equality holds if and only if \( f \) is a homogeneous polynomial after a biholomorphic change of coordinates.

**Theorem D.** Let \( f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0) \) be a polynomial with an isolated singularity at the origin. Let \( \mu, \nu \) and \( \tau \) be the Milnor number, multiplicity and Tjurina number of the singularity \( V = \{ z : f(z) = 0 \} \) respectively. Then \( f \) is a homogeneous polynomial after a biholomorphic change of coordinates if and only if \( \mu = \tau = (\nu - 1)^n \).

In the second part of this thesis, we prove the following result which give positive answer to Halperin, Yau and Wahl conjectures for fewnomial singularities.

**Theorem E.** Let \( f \in \mathbb{C}\{x_1, \cdots, x_n\} \) be an isolated weighted homogeneous fewnomial singularity with positive weights \( w_1, w_2, \cdots, w_n \) and multiplicity at least 5. Let

\[
A(f) = \mathbb{C}\{x_1, \cdots, x_n\}/(f_{x_1}, f_{x_2}, \cdots, f_{x_n})
\]

be the Milnor algebra of \( f \). Then there is no non-zero negative weight derivation on \( A(f) \).

**Theorem F.** Let \( (V, 0) \) be a normal isolated singularity of dimension \( n(\geq 2) \) with \( \mathbb{C}^* \)-action. Let \( \pi : \tilde{V} \rightarrow V \) be a good resolution of the singularity \( (V, 0) \) with \( E = f^{-1}(V)_{\text{red}} \). Then \( q = p_g - h^{n-1}(\mathcal{O}_E) \).

**Theorem G.** Let \( (V, 0) \) be a normal isolated complete intersection singularity of dimension \( n(\geq 2) \), and \( \pi : \tilde{V} \rightarrow V \) be a good resolution of the singularity \( (V, 0) \) with \( E = f^{-1}(0)_{\text{red}} \). If \( q = p_g - h^{n-1}(\mathcal{O}_E) \), then either \( (V, 0) \) has a \( \mathbb{C}^* \)-action or \( (V, 0) \) is a Du Bois singularity.
1.2 Organization of the Material

In Chapter 2, we introduce some basic knowledge on isolated hypersurface singularities. In section 2.1 we give some notions which are needed throughout the thesis. In section 2.2 and 2.3, we introduce the hypersurface singularities and isolated hypersurface singularities. In section 2.4, we discuss the families of singularities. In section 2.5, two equivalent relations right and contact equivalence of hypersurface singularities are introduced. In section 2.6, we give a formula for Milnor number for Newton non-degenerate singularities. In section 2.7, we introduce two special classes singularities: quasihomogeneous and semi-quasihomogeneous singularities which are two main objects in our research. Chapter 3 is the first main part of this dissertation. In this chapter, we present some related results of isolated weighted homogeneous hypersurface singularities which are needed for proving the Yau conjecture 3.1.1. As a result we give a intrinsic completely characterization of isolated homogenous hypersurface singularities. Chapter 4 is devoted to proof of Halperin conjecture (or Yau Conjecture 4.1.1) of Yau algebra of isolated hypersurface fewnomial singularities. In section 4.1, we recall the background and history of Halperin conjecture. In section 4.2, duality for zero-dimensional singularities is introduced which is crucial ingredient in our proof of Halperin conjecture. In section 4.3 we review two useful theorems which are needed for our proof. In section 4.4 we formalize the notions of isolated weighted homogenous fewnomial singularities which are our main objects in this chapter. Section 4.5 is devoted to the proof of the Main Theorem. In Chapter 5, section 5.1-5.4, we study the irregularity and genus of singularities and present our two main results. We also give some application of our main theorem in section 5.5.
CHAPTER 2

PRELIMINARIES OF HYPERSURFACE SINGULARITIES

This chapter is an overview of the main knowledge of hypersurface singularities which will be
used in this dissertation. A good reference is (23). Overall, after some notation is fixed, we shall
define isolated hypersurface singularities and discuss related results. Afterwards, we overview
briefly right and contact equivalence and then we deal with the mostly relevant invariants for
this work.

2.1 Notations

(0) We are working over the field \( \mathbb{C} \) throughout this thesis.

(1) We write \( \mathbb{C}\{x\} := \mathbb{C}\{x_1, \ldots, x_n\} \) for the local ring of convergent power series over \( \mathbb{C} \),
having \( n \) variables and we denote by \( \mathfrak{m} \) its maximal ideal.

(2) For \( \alpha = (\alpha_1, \ldots, \alpha_n) \), we denote the monomial \( x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_n^{\alpha_n} \) by \( x^\alpha \). Moreover the
positive integer \( |\alpha| = \alpha_1 + \cdots + \alpha_n \) is called the total degree of \( x^\alpha \) and is denoted by \( \deg(x^\alpha) \).

(3) We denote by \( \mathbb{Z}_{>0} \) (\( \mathbb{Z}_{\geq0} \)) the set of strictly positive integers (nonnegative positive integers)
and \( \mathbb{N} \) the set of natural numbers. On the other hand, if \( \alpha = (\alpha_1, \cdots, \alpha_n) \), \( \beta = (\beta_1, \cdots, \beta_n) \) then we write
\[
< \alpha, \beta > = \sum_{i=1}^{n} \alpha_i \beta_i
\]
for the scalar product of \( \alpha \) and \( \beta \).
(4) Let 

\[ f = \sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha x^\alpha \in \mathbb{C}^n \{x\} \]

be a convergent power series. The support of \( f \) is the set

\[ \text{supp}(f) := \{ \alpha : a_\alpha \neq 0 \} \text{ or, sometimes, supp}(f) := \{ x^\alpha : a_\alpha \neq 0 \}. \]

Furthermore, the multiplicity of \( f \) is \( \text{mult}(f) := \inf \{|\alpha| : \alpha \in \text{supp}(f)\} \).

(5) For \( i = 1, \cdots, n \), we write \( f_{xi} := \frac{\partial f}{\partial x_i} \), and \( \partial_{xi} := \partial/\partial x_i \).

(6) We denote by \( \text{Der}_\mathbb{C}(\mathbb{C}\{x\}) \) the space of \( \mathbb{C} \)-derivations on \( \mathbb{C}\{x\} \). Furthermore, we observe that \( \text{Der}_\mathbb{C}(\mathbb{C}\{x\}) \) is isomorphic to the \( \mathbb{C}\{x\} \)-module \( \sum_i^n \mathbb{C}\{x\} \partial_{xi} \).

### 2.2 Isolated Hypersurface Singularities

**Definition 2.2.1.** Let \( U \subset \mathbb{C}^n \) be open. A complex valued function \( f : U \rightarrow \mathbb{C} \) is called **holomorphic**, if it is holomorphic at \( p \) for all \( p \in U \). That is for all \( p = (p_1, \cdots, p_n) \in U \) there is an open neighborhood \( W \subset U \) and a power series

\[ \sum_{|\alpha|=0}^\infty c_\alpha (x_1 - p_1)^{\alpha_1} \cdots (x_n - p_n)^{\alpha_n} \]

which converges in \( W \) to \( f \mid_W \). In particular, the coordinate functions \( x_1, \cdots, x_n \) of \( \mathbb{C}^n \), \( x_i : \mathbb{C}^n \rightarrow \mathbb{C}, p \mapsto p_i \), are holomorphic.
**Definition 2.2.2.** Let $U \subset \mathbb{C}^n$ be open, $f : U \to \mathbb{C}$ a holomorphic function, and $V = V(f) = f^{-1}(0)$ the hypersurface defined by $f$ in $U$. We call

$$\text{Crit}(f) := \text{Sing}(f) := \{x \in U \mid \partial f / \partial x_1(x) = \cdots = \partial f / \partial x_n(x) = 0\}$$

the set of **critical**, or **singular**, points of $f$ and

$$\text{Sing}(V) := \{x \in U \mid f(x) = \partial f / \partial x_1(x) = \cdots = \partial f / \partial x_n(x) = 0\}$$

the set of **singular points** of $V$.

A point $x \in U$ is called an isolated critical point of $f$, if there exists a neighborhood $W$ of $x$ such that $\text{Crit}(f) \cap W \setminus \{x\} = \emptyset$. It is called an isolated singular point of $V$ if $x \in V$ and $\text{Sing}(V) \cap W \setminus \{x\} = \emptyset$. Then we say also the germ $(V, x) \subset (\mathbb{C}^n, x)$ is an isolated hypersurface singularity.

**Definition 2.2.3.** Let $f \in m \setminus \{0\}$ be a convergent power series.

(1). The ideal

$$j(f) := \langle f_{x_1}, \cdots, f_{x_n} \rangle \subset \mathbb{C}\{x\}$$

is called the **Jacobian ideal**, and

$$tj(f) := \langle f \rangle + j(f) \subset \mathbb{C}\{x\}$$
is called the **Tjurina ideal** of \( f \).

(2). The \( \mathbb{C} \)-algebra

\[
M_f := \mathbb{C}\{x\}/j(f), \quad T_f := \mathbb{C}\{x\}/tj(f)
\]

are called the **Milnor** and **Tjurina algebra** (or **moduli algebra**) of \( f \).

(3). The numbers

\[
\mu := \dim_{\mathbb{C}}(M_f), \quad \tau := \dim_{\mathbb{C}}(T_f)
\]

are called the **Milnor** and **Tjurina numbers** of \( f \) respectively.

The Milnor and the Tjurina algebras and, in particular, their dimension play an important role in the study of isolated hypersurface singularities as we shall see later in this chapter.

**Remark 2.2.1.** Let \( f \in m \subset \mathbb{C}\{x\} \) be a non-zero element. It is straightforward from Definition 2.2.3 that, if \( \mu(f) \) is finite, then \( \tau(f) \) is also finite too. It is well known that the converse is also correct (cf. (23), Lemma 2.3).

An important property of isolated hypersurface singularities is as follows.

**Proposition 2.2.2.** (38) Let \( f : \mathbb{C}^n \to \mathbb{C} \) be a holomorphic function, and \( V = V(f) = f^{-1}(0) \) the hypersurface defined by \( f \) in \( \mathbb{C}^n \). Then the following statements are equivalent.

1. \((V,0) \subset (\mathbb{C}^n,0) \) is an isolated hypersurface singularity.
2. If there exists a \( k > 0 \) such that \( m^k \subset tj(f) \).
3. If there exists a \( k > 0 \) such that \( m^k \subset j(f) \).
4. \( \mu(f) < \infty \).
(5). $\tau(f) < \infty$.

In general the Milnor number and Tjurina number is hard to calculate. However, there is a compute program ”Singular” which can help us to calculate those two invariants easily.

**Example:** (1) ($E_7$-singularity)

\[
f = x_1(x_1^2 + x_2^2) + x_3^2 + \ldots + x_n^2, \ n \geq 2,
\]

\[
j(f) = \langle 3x_1^2 + x_2^3, x_1x_2^2, x_3, \ldots, x_n \rangle = t_j(f),
\]

since $x_1^3, x_1x_2^2 \in j(f)$, in particular, $f \in j(f)$. Hence $M_f = T_f \cong \mathbb{C}\{x_1, x_2\}/(3x_1^2 + x_2^3, x_1x_2^2)$.

(2) $f = x^5 + y^5 + x^2y^2$ has $j(f) = (5x^4 + 2xy^2, 5y^4 + 2x^2y)$.

Using ”Singular”, we compute a $\mathbb{C}$-basis of $T_f$ as $1, x, \cdots, x^4, xy, y, \cdots, y^4$ and a $\mathbb{C}$-basis of $M_f$, which has an additional monomial $y^5$. Hence, $10 = \tau(f) < \mu(f) = 11$.

”Singular” code:

```plaintext
ring r=0,(x,y),ds; // a ring with a local ordering
poly f=x^5+y^5+x^2y^2;
ideal j=jacob(f);
vdim(std(j)); // the Milnor number
// $\rightarrow 11$
ideal tj=f, j;
vdim(std(tj)); // the Tjurina number
// $\rightarrow 10$
```
kbase(std(tj));


The built-in commands:

LIB ”sing.lib”;

milnor (f);

// → 11

tjurina (f);

// → 10

**Topological meaning of μ**: The Milnor fiber \( X_t = f^{-1}(t) \), \( t \neq 0 \), is a bouquet of \( μ \) \((n-1)\)-dimensional spheres which shrink to a point, if \( t \) goes to 0. That is, \( μ \) is the number of vanishing cycles in the Milnor fiber.

### 2.3 Families of Singularities

**Definition 2.3.1.** \( F \in \mathbb{C}\{x, t\} = \mathbb{C}\{x_1, \cdots, x_n, t_1, \cdots, t_k\} \) is called an **unfolding** or **deformation** of \( f \in \mathbb{C}\{x_1, \cdots, x_n\} \) if \( F(x, 0) = f(x) \). We denote by

\[
F_t(x) = F(x, t), t \in \mathbb{C}^k,
\]

the family of power series \( F_t \in \mathbb{C}^k \).

**Definition 2.3.2.** A deformation is **semi-universal** if
(1) For any deformation $G(x, s), s \in \mathbb{C}^l$ of $f$ there exists a base change map $\phi : (\mathbb{C}^l, 0) \to (\mathbb{C}^k, 0)$ such that

$$V(G(x, s)) = V(F(x, \phi(s))).$$

(2) $k$ is minimal with respect to (1).

**Theorem 2.3.1.** (Tjurina) Let $f$ have an isolated singularity and let $g_1, \ldots, g_\tau \in \mathbb{C}\{x\}$ be a $\mathbb{C}$-vector space basis of the Tjurina algebra $T_f$. Then

$$F(x, t_1, \ldots, t_\tau) = f(x) + \sum_{i=1,\ldots,\tau} t_ig_i(x)$$

is the semi-universal deformation of $f$.

**Analytical meaning of $\tau$:** Hence $\tau(f)$ is the number of parameters in the semi-universal deformation of $f$. Note that in contrast to $\mu$, $\tau$ is not a topological invariant of $f$.

### 2.4 Right and Contact Equivalence of Hypersurface Singularities

In the following, we briefly review the notion of right and contact equivalence.

**Definition 2.4.1.** Let $f, g \in m \subset \mathbb{C}\{x\}$.

(1) $f$ is called right equivalent to $g$, $f \overset{r}{\sim} g$, if there exists an automorphism $\phi$ of $\mathbb{C}\{x\}$ such that $\phi(f) = g$.

(2) $f$ is called contact equivalent to $g$, $f \overset{c}{\sim} g$, if there exists an automorphism $\phi$ of $\mathbb{C}\{x\}$ and a unit $u \in \mathbb{C}\{x\}^*$ such that $f = u \cdot \phi(g)$. 
It is straightforward from the above definition that the right and the contact equivalence are equivalence relations on the set of convergent power series.

**Remark 2.4.1.** (1) It is clear, that \( f \sim g \) implies \( f \sim g \). However, it is well-known, that the converse does not hold even though the characteristic is zero.

(2) \( f \sim g \Longleftrightarrow f = g \circ \phi \) for some biholomorphic map germ \( \phi : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0) \).

(3) \( f \sim g \Longleftrightarrow \exists \phi : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0) \text{ s.t. } \phi(f^{-1}(0), 0) = (g^{-1}(0), 0) \Longleftrightarrow \mathbb{C}[x]/(f) \cong \mathbb{C}[x]/(g) \) as analytic \( \mathbb{C} \)-algebras.

**Lemma 2.4.2.** Let \( f, g \in \mathbb{C}\{x_1, \cdots, x_n\} \). Then

(i) \( f \sim g \Longrightarrow M_f \cong M_g \) and \( T_f \cong T_g \) as analytic algebras. In particular, \( \mu(f) = \mu(g) \) and \( \tau(f) = \tau(g) \).

(ii) \( f \sim g \Longrightarrow T_f \cong T_g \) and hence \( \tau(f) = \tau(g) \). Moreover \( \mu(f) = \mu(g) \) holds (difficult).

Even more difficult, \( \mu(f) \) is a topological invariant of \( (f^{-1}(0), 0) \) by a result of Milnor.

**Proof.** (i) If \( g = \varphi(f) = f \circ \phi \), then

\[
\left( \frac{\partial f \circ \phi}{\partial x_1}(x), \cdots, \frac{\partial f \circ \phi}{\partial x_n}(x) \right) = \left( \frac{\partial f}{\partial x_1}(\phi(x)), \cdots, \frac{\partial f}{\partial x_n}(\phi(x)) \right) \cdot D\phi(x)
\]

where \( D\phi(x) \) is the Jacobian matrix of \( \phi \), which is invertible in a neighborhood of \( x \). It follows that

\[
j(\varphi(f)) = \varphi(j(f)) \quad \text{and} \quad (\varphi(f), j(\varphi(f)) = \varphi((f, j(f))),
\]

which proves the claim.
(ii) By the product rule we have \((u \cdot f, j(u \cdot f)) = (f, j(f))\). Together with (i) this implies \(T_f = T_g\).

For isolated hypersurface singularities, the following celebrated theorem was obtained by Mather and Yau, see (41).

**Theorem 2.4.3.** Let \(f, g \in m \subset \mathbb{C}\{x\}\) be two isolated hypersurface singularities having isomorphic Tjurina (moduli) algebras \(T_f \cong T_g\). Then \(f \sim g\).

### 2.5 Finite Determinancy

**Definition 2.5.1.** The group

\[ \mathcal{R} := \text{Aut}(\mathbb{C}\{x\}) \]

of automorphisms of the algebra \(\mathbb{C}\{x\}\) is called the **right group**. The **contact group** is the semidirect product

\[ \mathcal{K} := (\mathbb{C}\{x\})^* \rtimes \mathcal{R} \]

of \(\mathcal{R}\) with the group of units of \(\mathbb{C}\{x\}\), where the product in \(\mathcal{K}\) is defined by

\[ (u', \phi')(u, \phi) = (u'\phi'(u), \phi' \phi) \]

Those groups act on \(\mathbb{C}\{x\}\) by

\[ \mathcal{R} \times \mathbb{C}\{x\} \to \mathbb{C}\{x\} : (\phi, f) \mapsto \phi(f). \]
\[ \mathcal{K} \times \mathbb{C}\{x\} \to \mathbb{C}\{x\} : ((u, \phi), f) \mapsto u \cdot \phi(f). \]

We have

\[ f \sim^R g \iff f \in R \cdot g, \quad f \sim^C g \iff f \in K \cdot g \]

where \( R \cdot g \) (respectively \( K \cdot g \)) denotes the **orbit** of \( g \) under \( R \) (respectively \( K \)).

Unfortunately \( R \) and \( K \) are not algebraic groups since they are infinite dimensional. Therefore we have to pass to the \( k \)-jets. We set

\[
J^{(k)} = \mathbb{C}\{x\}/(x)^{k+1},
\]

\[ \text{jet}(f, k) = \text{image of } f \text{ in } J^{(k)}, \]

\[ R^{(k)} := \{ \text{jet}(\phi, k) \mid \phi \in R \}, \]

\[ K^{(k)} := \{ (\text{jet}(u, k), \text{jet}(\phi, k)) \mid (u, \phi) \in K \}, \]

where \( \text{jet}(\phi, k) \) is the truncation of the power series of the component functions of \( \phi \).

\( R^{(k)} \) and \( K^{(k)} \) are affine algebraic groups acting algebraically on the jet space \( J^{(k)} \), which is a finite dimensional complex vector space. The action is give by

\[ \phi \cdot f = \text{jet}(\phi(f), k), \quad (u, \phi) \cdot f = \text{jet}(u \cdot \phi(f), k), \]

for \( \phi \in R^{(k)}, \quad (u, \phi) \in K^{(k)} \). Hence, we can apply the theory of algebraic groups to the action of \( R^{(k)} \) and \( K^{(k)} \) on \( J^{(k)} \).
Definition 2.5.2. (1) An (affine) algebraic group $G$ (over an algebraically closed field $K$) is a reduced (affine) algebraic variety over $K$, which is also a group such that the group operations are morphisms of varieties. That is, there exists an element $e \in G$ (the unit element) and morphisms of varieties over $K$

\[ G \times G \to G, \ (g, h) \mapsto g \cdot h \text{(the multiplication)}, \]

\[ G \to G, \ g \mapsto g^{-1} \text{(the inverse)} \]

satisfying the usual group axioms.

(2) A morphism of algebraic groups is a group homomorphism, which is also a morphism of algebraic varieties over $K$.

Definition 2.5.3. (1) An (algebraic) action of $G$ on an algebraic variety $V$ is given by a morphism of varieties

\[ G \times V \to V, \ (g, x) \mapsto g \cdot x, \]

satisfying $ex = x$ and $(gh)x = g(hx)$ for all $g, h \in G, x \in V$.

(2) The orbit of $x \in V$ under the action of $G$ on $V$ is the subset

\[ Gx := \{g \cdot x \in V \mid g \in G\} \subset V, \]

that is, the image of $G \times \{x\}$ in $V$ under the orbit map $G \times V \to V$. 
(3) The stabilizer of $x \in V$ is the subgroup

$$G_x := \{ g \in G \mid gx = x \}$$

of $G$, i.e. the preimage of $x$ under the induced map $G \times \{ x \} \to V$.

In this sense $R^{(k)}$ and $K^{(k)}$ act algebraically on $J^{(k)}$. Note that the somehow unexpected multiplication on $K^{(k)}$ as a semidirect product was introduced in order to guarantee $(gh)x = g(hx)$.

For the classification of singularities, the following properties of orbits is important.

**Theorem 2.5.1.** (23) Let $G$ be an affine algebraic group acting on an algebraic variety $V$, and $x \in V$ an arbitrary point. Then

1. $G$ is a smooth variety.
2. $Gx$ is open in its (Zariski-) closure $\overline{Gx}$.
3. $Gx$ is a smooth subvariety of $V$.
4. $\overline{Gx} \setminus Gx$ is a union of orbits of smaller dimension.
5. The stabilizer $G_x$ is a closed subvariety of $G$.
6. If $G$ is connected, then $\dim(Gx) = \dim(G) - \dim(G_x)$.

**Definition 2.5.4.** (1) $f \in \mathbb{C}\{x_1, \cdots, x_n\}$ is **right** (resp. **contact**) $k$-**determined** iff $\forall g \in \mathbb{C}\{x_1, \cdots, x_n\}$

$$f^{(k)} = g^{(k)} \Rightarrow f \sim^r g \ (\text{resp.} f \sim^c g).$$
(2) The minimal $k$ s.t. $f$ is $k$-determined is called the (right resp. contact) **determinacy** of $f$.

(3) $f$ is (right resp. contact) finitely determined iff there is a $k < \infty$ s.t. $f$ is (right resp. contact) $k$-determined

**Theorem 2.5.2.** (23) Let $f \in mC\{x\}$. Then

(1) $f$ is right $k$-determined if

$$m^{k+1} \subset m^2 \langle \partial f/\partial x_1, \cdots, \partial f/\partial x_n \rangle$$

(2) $f$ is contact $k$-determined if

$$m^{k+1} \subset m^2 \langle \partial f/\partial x_1, \cdots, \partial f/\partial x_n \rangle + m\langle f \rangle$$

**Corollary 2.5.3.** (23) Let $f \in \mathbb{C}\{x_1, \cdots, x_n\}$ have an isolated singularity. Then

(1) $f$ is $(\mu(f) + 1)$-right determined.

(2) $f$ is $(\tau(f) + 1)$-contact determined.

**Corollary 2.5.4.** (23) For $f \in \mathbb{C}\{x_1, \cdots, x_n\}$, $f(0) = 0$, the following are equivalent.

(1) $f$ has an isolated critical point.

(2) $f$ is right finitely-determined.

(3) $f$ is contact finitely-determined.

Example: Check the determinancy by different methods
LIB "classify.lib",
ring r = 0, (x, y), ds;
poly f = x^3 + xy^3 // E7-singularity
milnor (f);
// → 7
Hence f is (right) 8-determined by the Corollary. Now use the method of the theorem:
ideal j = maxideal (2) * jacob (f);
deg (highcorner (std(j)));
|→ 5 // hence f is 5-determined

2.6 Newton Non-Degenerate Singularities

Definition 2.6.1. Let $f = \sum_{\alpha \in \mathbb{N}^n} a_{\alpha} x^\alpha \in \mathbb{C}\{x\} = \mathbb{C}\{x_1, \cdots, x_n\}$, $a_0 = 0$. Then the convex hull of the support of $f$ in $\mathbb{R}^n$,

$$\Delta(f) := \text{conv}\{\alpha \in \mathbb{N}^n \mid a_{\alpha} \neq 0\},$$

is called the Newton polytope of $f$.

Set $K(f) := \text{conv}(\{0\} \cup \Delta(f))$, and $K_0(f) :=$ the closure of the set $K(f) \setminus \Delta(f)$. Then

$$\Gamma(f, 0) := K_0(f) \cap \Delta(f)$$
is called the **Newton diagram** of \( f \) at the origin. Moreover, We introduce for a face \( \sigma \subset \Gamma(f,0) \) the **truncation**

\[
 f^{\sigma} = \sum_{\alpha \in \sigma} c_\alpha x^\alpha.
\]

**Definition 2.6.2.** \( f \) is called **Newton non-degenerate (NND)** at 0 if, for all faces \( \sigma \subset \Gamma(f,0) \), the hypersurface \( \{f^{\sigma} = 0\} \) has no singular point in the torus \((\mathbb{C}^*)^n\). \( f \) is called **convenient** if \( \Gamma(f,0) \) meets all coordinate axes.

**Theorem 2.6.1.** (Kouchnirenko)(31) Let \( f \in \mathbb{C}\{x_1, \cdots, x_n\} \) be Newton non-degenerate and convenient. Then the Milnor number of \( f \) satisfies

\[
 \mu(f) = n! \text{Vol}_n K_0(f) + \sum_{i=1}^{n} (-1)^{n-i} \cdot (n-i)! \cdot \text{Vol}_{n-i}(K_0(f) \cap H_{n-i})
\]

where \( H_i \) denotes the union of all \( i \)-dimensional coordinate planes, and where \( \text{Vol}_i \) denotes the \( i \)-dimensional Euclidean volume.

### 2.7 Special Classes Singularities: Quasihomogeneous and Semi-quasihomogeneous Singularities

**Definition 2.7.1.** Let \( f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \in \mathbb{C}\{x_1, \cdots, x_n\} \) is called **weighted homogeneous** or **quasihomogeneous** of **type** \( (w; d) = (w_1, \cdots, w_n; d) \) if \( w_i, d \) are positive integers satisfying

\[
 w \cdot \text{deg}(x^\alpha) := w_1 \alpha_1 + \cdots + w_n \alpha_n = d
\]
for each \( \alpha \in \mathbb{N}^n \) with \( a_\alpha \neq 0 \). The numbers \( w_i \) are called the weights and \( d \) the weighted degree of \( f \) (This property is not invariant under coordinate changes).

**Remark 2.7.1.** A quasihomogeneous polynomial \( f \) of type \((w; d)\) obviously satisfies the relations

\[
d \cdot f = \sum_{i=1}^{n} w_i x_i \frac{\partial f}{\partial x_i} \in \mathbb{C}[x] \text{ (Euler relation)}
\]

\[
f(t^{w_1}x_1, \ldots, t^{w_n}x_n) = t^d \cdot f(x_1, \ldots, x_n) \text{ in } \mathbb{C}[x, t]
\]

The first relation implies that \( f \) is contained in \( j(f) \), hence \( \mu(f) = \tau(f) \).

The second relation implies that the hypersurface \( V(f) \subset \mathbb{C}^n \) is invariant under the \( \mathbb{C}^* \)-action \( \mathbb{C}^* \times \mathbb{C}^n \longrightarrow \mathbb{C}^n \)

\[
(\lambda, x) \longmapsto \lambda \circ x := (\lambda^{w_1}x_1, \ldots, \lambda^{w_n}x_n).
\]

In particular, the affine complex hypersurface \( V(f) \subset \mathbb{C}^n \) is contractible.

**Definition 2.7.2.** An isolated hypersurface singularity \((X, x) \subset (\mathbb{C}^n, x)\) is called quasihomogeneous \((QH)\) if there exists a quasihomogeneous polynomial \( f \in \mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n] \) such that \( \theta_{X,x} \cong \mathbb{C}\{x\}/(f) \).

**Lemma 2.7.2.** Let \( f \in \mathbb{C}[x] \) be quasihomogeneous and \( g \in \mathbb{C}\{x\} \) arbitrary. Then \( f \sim g \) if and only if \( f \sim g \).
Proof. One direction is obvious. Let $f$ be weighted homogeneous of type $(w_1, \cdots, w_n; d)$. If $f \sim g$ then there exists a unit $u \in \mathbb{C}\{x\}^*$ and an automorphism $\phi \in \text{Aut}\mathbb{C}\{x\}$ such that $u \cdot f = \phi(g)$. Choose a $d$-th root $u^{1/d} \in \mathbb{C}\{x\}$. The automorphism

$$
\varphi : \mathbb{C}\{x\} \longrightarrow \mathbb{C}\{x\}, \ x_i \longmapsto u^{w_i/d} \cdot x_i
$$

yields $\varphi(f(x)) = f(u^{w_1/d} \cdot x_1, \cdots, u^{w_n/d} \cdot x_n) = u \cdot f(x)$ by the Remark above. Hence $f = \varphi^{-1} \circ \phi(g)$. 

\[\square\]

**Theorem 2.7.3.** (K. Saito)(52) Let $(X, x) \subset (\mathbb{C}^n, x)$ be an isolated hypersurface singularity and let $f \in \mathbb{C}\{x_1, \cdots, x_n\}$ be any local equation for $(X, x)$, then

$$(X, x) \text{ is quasihomogeneous } \iff \mu(f) = \tau(f).$$

**Definition 2.7.3.** A power series $f \in \mathbb{C}\{x_1, \cdots, x_n\}$ is called **semi-quasihomogeneous (SQH)** at $0$ if there is a face $\sigma \subset \Gamma(f, 0)$ (Newton diagram) of dimension $n - 1$ such that the truncation $f^\sigma$ has no critical points in $\mathbb{C}^n \setminus \{0\}$. $f^\sigma$ is called the **main part**, or **principal part** of $f$.

For a SQH $f$ we have

$$f = f_0 + g, \ \mu(f_0) < \infty$$
with $f_0 = f^\sigma$ a quasihomogeneous polynomial of type $(w = (w_1, \cdots, w_n); d)$, i.e., each monomial $x^{\alpha}$ appearing in $f_0$ satisfies $w\text{-deg}(x^{\alpha}) := \alpha_1 w_1 + \cdots + \alpha_n w_n = d$ and all monomials of $g$ being of $w\text{-deg}$ at least $d + 1$.

**Proposition 2.7.1.** (3) Let $f \in \mathbb{C}\{x\}$ be SQH with principal part $f_0$. Then $f$ has an isolated singularity at 0 and

$$\mu(f) = \mu(f_0) = \left(\frac{d}{w_1} - 1\right) \cdots \left(\frac{d}{w_n} - 1\right).$$
CHAPTER 3

COMPLETE CHARACTERIZATION OF ISOLATED HOMOGENEOUS HYPERSURFACE SINGULARITIES

Let $X$ be a nonsingular projective variety in $\mathbb{CP}^{n-1}$. Then the cone over $X$ in $\mathbb{C}^n$ is an affine variety $V$ with an isolated singularity at the origin. It is a very natural and important question to ask when an affine variety with an isolated singularity at the origin is a cone over nonsingular projective variety.

This problem is very hard in general. In this thesis we shall treat the hypersurface case. Given a function $f$ with a isolated singularity at the origin. We can ask whether $f$ is a weighted homogeneous polynomial or a homogeneous polynomial after a biholomorphic change of coordinates. The former question was answered in a celebrated paper by Saito (52) in 1971. However, the latter question has remained open for 40 years until Xu-Yau solved it for $f$ with three variables (79). Recently, Yau and the author solved it for $f$ with up to six variables (92). However the methods they used is hard to be generalized. In this thesis, we solve the latter question for general $n$ completely, i.e., we show that $f$ is a homogeneous polynomial after a biholomorphic change of coordinates if and only if $\mu = \tau = (\nu - 1)^n$, where $\mu$, $\tau$ and $\nu$ are Milnor number, Tjurina number and multiplicity of the singularity respectively.

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3.1 Introduction

Let \( w = (w_1, \cdots, w_n) \) be an \( n \)-tuple of positive rational numbers. A polynomial \( f(z_1, \cdots, z_n) \) is said to be a weighted homogeneous polynomial with weights \( w \) if each monomial \( \alpha z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n} \) of \( f \) satisfies \( a_1/w_1 + \cdots + a_n/w_n = 1 \). It has an isolated critical point at \( 0 \in \mathbb{C}^n \) if \( \nabla f = (\partial f/\partial z_1, \cdots, \partial f/\partial z_n) \) is zero at \( 0 \) but \( \nabla f \mid_{Z \neq 0} \neq 0 \) for all \( Z \) in a neighborhood of \( 0 \).

Recall that a polynomial \( f(z_1, \cdots, z_n) \) is called quasi-homogeneous if \( f \) is in the Jacobian ideal of \( f \) i.e., \( f \in (\partial f/\partial z_1, \cdots, \partial f/\partial z_n) \).

By a theorem of Saito (see Theorem 3.2.6), if \( f \) is quasi-homogeneous with an isolated critical point at \( 0 \), then after a biholomorphic change of coordinates, \( f \) becomes a weighted homogeneous polynomial.

Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be the germ of a complex analytic function with an isolated critical point at the origin. Let \( V = \{ z \in \mathbb{C}^n : f(z) = 0 \} \). It is a natural question to ask when \( V \) is defined by a weighted homogeneous polynomial up to biholomorphic change of coordinates. Saito solved this question in 1971 (52). He gives a necessary and sufficient condition for \( V \) to be defined by a weighted homogeneous polynomial. It is a natural and important question to characterize homogeneous polynomial with isolated critical point at the origin. This question has remained open for 40 years. In fact it is the first important case of the following interesting problem. Let \( X \) be a nonsingular projective variety in \( \mathbb{P}^{n-1} \). Then the cone over \( X \) in \( \mathbb{C}^n \) is the affine variety \( V \) with an isolated singularity at the origin. It is a very natural and important question to ask when an affine variety with an isolated singularity at the origin is a cone over nonsingular projective variety.
For a two-dimensional isolated hypersurface singularity \( V \), Xu and Yau ((78), (79)) found a coordinate free characterization for \( V \) to be defined by a homogeneous polynomial. Recently Lin and Yau ((37), (35), (80)) and Chen, Lin, Yau and the author (17) gave necessary and sufficient conditions for a 3 and 4- dimensional isolated hypersurface singularities with \( p_g \geq 0 \) and \( p_g > 0 \) respectively where \( p_g \) is the geometric genus of the singularity. Based on classification of weighted homogeneous singularities Yau and the author solve the problem for \( f \) with up to six variables (92). However, it is quite difficult to generalize their methods to give characterization of homogeneous polynomials for general \( n \). In 2005, Yau formulated the Yau Conjecture: (1) Let \( \mu \) and \( \nu \) be the Milnor number and multiplicity of \((V,0)\) respectively. Then \( \mu \geq (\nu - 1)^n \) and the equality holds if and only if \( f \) is a semi-homogeneous function. (2) Moreover, if \( f \) is a quasi-homogeneous function, then \( \mu = (\nu - 1)^n \) if and only if \( f \) is a homogeneous polynomial after a biholomorphic change of coordinates. In this thesis we verify Yau Conjecture affirmatively. As a result we have solved the characterization problem of homogeneous polynomial with an isolated critical point at the origin, i.e. we have shown that \( f \) is a homogeneous polynomial after a biholomorphic change of coordinates if and only if \( \mu = \tau = (\nu - 1)^n \).

Recall that the multiplicity of the singularity \( V \) is defined to be the order of the lowest non-vanishing term in the power series expansion of \( f \) at 0. The Milnor number \( \mu \) and the Tjurina number \( \tau \) of the singularity \((V,0)\) are defined respectively by

\[
\mu = \dim \mathbb{C}\{z_1, z_2, \ldots, z_n\}/(f_{z_1}, \ldots, f_{z_n}),
\]

\[
\tau = \dim \mathbb{C}\{z_1, z_2, \ldots, z_n\}/(f, f_{z_1}, \ldots, f_{z_n}).
\]
They are numerical invariants of $(V,0)$. In 1971, Saito proved the following theorem which gives a necessary and sufficient condition for $V$ to be defined by a weighted homogeneous polynomial.

**Theorem 3.1.1.** (52) $f$ is a weighted homogeneous polynomial after a biholomorphic change of coordinates if and only if $\mu = \tau$.

Let $\pi : (M, A) \to (V, 0)$ be a resolution of singularity of dimension $n$ with exceptional set $A = \pi^{-1}(0)$. The geometric genus $p_g$ of the singularity $(V, 0)$ is the dimension of $H^{n-1}(M, \mathcal{O})$ and is independent of the resolution $M$. In 1993, Xu and Yau [Xu-Ya 2] gave necessary and sufficient conditions for a 2-dimensional $V$ to be defined by a homogeneous polynomial.

**Theorem 3.1.2.** ((77) (79)) Let $(V, 0)$ be a 2-dimensional isolated hypersurface singularity defined by a holomorphic function $f(z_1, z_2, z_3) = 0$. Let $\mu$ be the Milnor number, $\tau$ be the Tyurina number, $p_g$ be the geometric genus, and $\nu$ be the multiplicity of the singularity. Then $f$ is a homogeneous polynomial after a biholomorphic change of variables if and only if $\mu = \tau$ and $\mu - \nu + 1 = 6p_g$.

Based on above theorem, a conjecture was made by Yau in 2005 as follows:

**Yau Conjecture 3.1.1.** (36) Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic germ defining an isolated hypersurface singularity $V = \{z: f(z) = 0\}$ at the origin. Let $\mu$ and $\nu$ be the Milnor number and multiplicity of $(V, 0)$ respectively. Then

$$\mu \geq (\nu - 1)^n \quad (1.1)$$
and the equality in (1.1) holds if and only if $f$ is a semi-homogeneous function (i.e. $f = f_\nu + g$ where $f_\nu$ is a non-degenerate homogeneous polynomial of degree $\nu$ and $g$ consists of terms of degree at least $\nu + 1$). Furthermore if $f$ is a quasi-homogeneous function, i.e. $f \in (\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n})$, then the equality in (1.1) holds if and only if $f$ is a homogeneous polynomial (after a biholomorphic change of coordinates).

Remark: a polynomial is called non-degenerate if it define a isolated singularities at the origin.

Using $p_g$, $\mu$ and $\nu$, Yau raised another conjecture which answer when a weighted homogeneous singularity is a homogeneous singularity.

**Yau Conjecture 3.1.2.** ((37), (17)) Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a weighted homogeneous polynomial with an isolated singularity at the origin. Let $\mu$, $p_g$ and $\nu$ be the Milnor number, geometric genus and multiplicity of the singularity $V = \{z : f(z) = 0\}$, then

$$\mu - p(\nu) \geq n! p_g,$$

where $p(\nu) = (\nu - 1)^n - \nu(\nu - 1) \cdots (\nu - n + 1)$, and equality holds if and only if $f$ is a homogeneous polynomial.

In fact, we proved in this thesis that if $p_g = 0$, then Yau Conjecture 3.1.1 implies Yau Conjecture 3.1.2.
These conjectures are sharp estimates and have some important applications in geometry. However, the Yau conjectures were proved only for lower dimensional singularities. For conjecture 3.1.1, Lin, Wu, Yau and Luk proved the following two theorems.

**Theorem 3.1.3.** (36) Let \( f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \) be a germ of holomorphic function defining an isolated plane curve singularity \( V = \{ z \in \mathbb{C}^2 : f(z) = 0 \} \) at the origin. Let \( \mu \) and \( \nu \) be the Milnor number and multiplicity of \( (V, 0) \) respectively. Then

\[
\mu \geq (\nu - 1)^2. \tag{1.2}
\]

Furthermore, if \( V \) has at most two irreducible branches at the origin, or if \( f \) is a quasihomogeneous function, then the equality in (1.2) holds if and only if \( f \) is a homogeneous polynomial (after a biholomorphic change of coordinates).

**Theorem 3.1.4.** (36) Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be a germ of holomorphic function defining an isolated hypersurface singularity \( V = \{ z \in \mathbb{C}^n : f(z) = 0 \} \) at the origin. Let \( \mu, \nu \) and \( \tau = \dim \mathbb{C}\{z_1, \ldots, z_n\}/(f, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}) \) be the Milnor number, multiplicity and Tjurina number of \( (V, 0) \) respectively. Suppose \( \mu = \tau \) and \( n \) is either 3 or 4. Then

\[
\mu \geq (\nu - 1)^n \tag{1.3}
\]

and the equality in (1.3) holds if and only if \( f \) is a homogeneous polynomial (after a biholomorphic change of coordinates).
For Conjecture 3.1.2, Lin, Tu and Yau have the following theorem.

**Theorem 3.1.5.** ((37) (35)) Let \((V, 0)\) be a 3-dimensional isolated hypersurface singularity defined by a weighted homogeneous polynomial \(f(x, y, z, w) = 0\). Let \(\mu\), \(\nu\), and \(p_g\) be the Milnor number, multiplicity, and geometric genus of the singularity respectively. Then

\[
\mu - (2\nu^3 - 5\nu^2 + 2\nu + 1) \geq 4!p_g
\]  

(1.4)

and equality in (1.4) holds if and only if \(f\) is a homogeneous polynomial.

Remark: The above theorem is proved in (37) with \(p_g > 0\). For \(p_g = 0\), the theorem is proved in (35).

A immediately corollary of Theorem 3.1.5 is the following.

**Corollary 3.1.6.** (35) Let \((V, 0)\) be a 3-dimensional isolated hypersurface singularity defined by a polynomial \(f(x, y, z, w) = 0\). Let \(\mu\), \(\nu\), \(p_g\) and \(\tau\) be the Milnor number, multiplicity, geometric genus and Tjurina number of the singularity respectively. Then \(f\) is a homogeneous polynomial after a biholomorphic change of coordinates if and only if \(\mu = \tau\) and \(\mu - (2\nu^3 - 5\nu^2 + 2\nu + 1) = 4!p_g\).

Chen, Lin Yau and the author (17) can generalize the above theorem to 4 dimension isolated hypersurface singularity with an additional assumption \(p_g > 0\).
Theorem 3.1.7. (17) Let \((V, 0)\) be a 4-dimensional isolated hypersurface singularity defined by a weighted homogeneous polynomial \(f(x, y, z, w, t) = 0\). Let \(\mu\), \(\nu\), and \(p_g\) be the Milnor number, multiplicity, and geometric genus of the singularity respectively. If \(p_g > 0\) then

\[
\mu - \left[ (\nu - 1)^5 + \nu(\nu - 1)(\nu - 2)(\nu - 3)(\nu - 4) \right] \geq 5! p_g
\]

and equality in (1.5) holds if and only if \(f\) is a homogeneous polynomial.

Corollary 3.1.8. (17) Let \((V, 0)\) be a 4-dimensional isolated hypersurface singularity defined by a polynomial \(f(x, y, z, w, t) = 0\). Let \(\mu\), \(\nu\), \(p_g\) and \(\tau\) be the Milnor number, multiplicity, geometric genus and Tjurina number of the singularity respectively. Moreover if \(p_g > 0\) then \(f\) is a homogeneous polynomial after a biholomorphic change of coordinates if and only if \(\mu = \tau\) and

\[
\mu - \left[ (\nu - 1)^5 + \nu(\nu - 1)(\nu - 2)(\nu - 3)(\nu - 4) \right] = 5! p_g.
\]

Yau and the author proved the following theorem in (92) basing on the classification of weighted homogeneous singularities, in their proof they need to use the following results.

The following theorem is about the relation of weight and multiplicity.

Theorem 3.1.9. (57) If \(f\) is a quasi-homogeneous isolated singularity of type \((\omega_1, \cdots, \omega_n)\), then \(\text{Mult}(f) = \min \{ m \in \mathbb{N} : m \geq \min \{ \omega_i : i = 1, \cdots, n \} \} \).

Lemma 3.1.1. Let \(f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)\) be a weighted homogeneous polynomial of weight type \((w_1, \cdots, w_n)\) with an isolated singularity at the origin. If \(w_i\) is not a integer, \(z_i^{a_i} z_j^{a_j} \in \text{Supp}(f)\) where \(a_i\) is a positive integer, \(j_i \neq i\) and \(|w_i| < \nu\), then \(\nu = a_i + 1\), \(\frac{a_i}{w_i} \neq 1\).
Proof. Since \( z_i^{a_i}z_j \in \text{Supp}(f) \), so \( \frac{a_i}{w_i} + \frac{1}{w_j} = 1 \). It follows from the fact that \( w_i \) is not an integer that \( \frac{w_i}{w_j} \neq 1 \). \( \frac{a_i}{w_i} + \frac{1}{w_j} = 1 \) implies that \( w_j > a_i \). Since \( \lfloor w_i \rfloor < \nu \), by Theorem 3.1.9, we have \( \nu = \lfloor w_i \rfloor + 1 \geq a_i + 1 \). By definition of multiplicity, we also have \( \nu \leq a_i + 1 \). Therefore \( \nu = a_i + 1 \).

Theorem 3.1.10. (92) Let \( f : (\mathbb{C}^k, 0) \to (\mathbb{C}, 0) \) where \( k \) is either 5 or 6, be a weighted homogeneous polynomial with an isolated singularity at the origin. Let \( \mu \) and \( \nu \) be the Milnor number and multiplicity of the singularity \( V = \{ z : f(z) = 0 \} \) respectively. Then

\[
\mu \geq (\nu - 1)^k
\]

(1.6)

and the equality in (1.6) holds if and only if \( f \) is a homogeneous polynomial after a biholomorphic change of coordinates.

Proof

We shall give a detail proof for \( k = 5 \).

Let \( f : (\mathbb{C}^5, 0) \to (\mathbb{C}, 0) \) be a weighted homogeneous polynomial with an isolated singularity at the origin. Let \( \mu \) and \( \nu \) be the Milnor number and multiplicity of the singularity \( V = \{ z : f(z) = 0 \} \) respectively. We want to show \( \mu \geq (\nu - 1)^5 \) and the equality holds if and only if \( f \) is a homogeneous polynomial. By Theorem A, it suffices to show that equality holds if and only if \( f \) is a homogeneous polynomial. Set \( w(z_i) = w_i, 1 \leq i \leq 5 \). Without loss of generality, we assume that \( 2 \leq w_1 \leq \min \{ w_2, \ldots, w_5 \} \) where \( w_i, i = 1, \cdots, 5 \) are positive rational numbers.
If \( \nu = 2 \), then the theorem is trivial by Milnor-Orlik formula (Theorem 3.2.2). In the following, we only consider \( \nu \geq 3 \), or equivalently \( w_1 > 2 \).

If \( w_1 \) is an integer, by Theorem 3.1.9, then \( \nu = w_1 \). Since \( \mu = (w_1 - 1) \cdots (w_5 - 1) \), so \( \mu = (\nu - 1)^5 \) if and only if \( w_1 = w_2 = \cdots = w_5 \). i.e. \( f \) is an homogeneous polynomail.

If \( w_1 \) is not an integer, by Theorem 3.1.9, \( \nu = \lceil w_1 \rceil + 1 \), where \( \lceil w_1 \rceil \) denotes the integer part of \( w_1 \). We want to show that \( \mu > (\nu - 1)^5 \). Since \( f \) is an isolated singularity, so for every \( i \in \{1, \cdots , 5\} \), either \( z_i^{a_i} \) or \( z_i^{a_i} z_j \) is in the support of \( f \), where \( a_i \) is an positive integer number.

By assumption, \( w_1 \) is not an integer, so \( z_1^{a_1} z_{j_1} \in \text{supp}(f) \). By Lemma 3.1.1, we have \( \nu = a_1 + 1 \).

We shall show that \( (\nu - 1)^2 < (w_1 - 1)(w_{j_1} - 1) \). Since \( \frac{w_1}{w_{j_1}} \cdot \frac{1}{w_{j_1}} = 1 \), so \( a_1 = w_1 - \frac{w_1}{w_{j_1}} \), \( \nu = w_1 - \frac{w_1}{w_{j_1}} + 1 \). Therefore the fraction part of \( w_1 \) is \( \frac{w_1}{w_{j_1}} \). In order to make the notation simple, we set \( x = \lceil w_1 \rceil \) where \( x \geq 2 \), \( y = \frac{w_1}{w_{j_1}} \) where \( 0 < y < 1 \), then \( x = \nu - 1 \), \( w_1 = x + y \), \( w_{j_1} = \frac{x+y}{y} \). By simple calculation, \( (\nu - 1)^2 < (w_1 - 1)(w_{j_1} - 1) \) is same as \( x^2 < (x+y-1)(\frac{x+y}{y} - 1) \) which is true for \( x \geq 2 \).

We use \( \{w_1, \cdots , w_5\} \setminus \{w_1, w_{j_1}\} \) denote the collection of the three remaining rational numbers \( \{\hat{w}_1, w_2, \cdots , \hat{w}_{j_1}, \cdots , w_5\} \) where \( \hat{w}_i, i = 1, j_1 \) means \( w_i \) is omitted. Without loss of generality, we assume that \( w_2 \in \{w_1, \cdots , w_5\} \setminus \{w_1, w_{j_1}\} \) which is not empty set is the minimal weight in this set.

If \( w_2 \) is a positive integer, then we have \( \nu \leq w_2 \), then \( \nu - 1 \leq w_2 - 1 \). Since \( w_2 \) is the minimal weight in the set \( \{w_1, \cdots , w_5\} \setminus \{w_1, w_{j_1}\} \), therefore \( \mu > (\nu - 1)^5 \).

If \( w_2 \) is not a positive integer and \( \lceil w_2 \rceil > \lceil w_1 \rceil \), then we have \( \nu - 1 < w_2 - 1 \). The same reason as before gives \( \mu > (\nu - 1)^5 \).
If \( w_2 \) is not a positive integer and \([w_2] = [w_1]\). Our goal is to prove \((\nu-1)^2 < (w_2-1)(w_{j_2}-1)\) where \( w_{j_2} \) depends on \( w_2 \). Since \( w_2 \) is not an integer, so \( \exists a_2 \), a positive integer number such that \( z_2^{a_2} z_{j_2} \in \text{supp} f \) where \( j_2 \neq 2 \). There will be 3 cases to be considered.

Case 1. If \( j_2 = 1 \), then \( z_2^{a_2} z_1 \in \text{supp} f \). \( \frac{a_2}{w_2} + \frac{1}{w_1} = 1 \) and \( w_2 \geq w_1 \) implies \( \frac{a_2+1}{w_2} \leq 1 \), \( w_2 \geq (a_2 + 1) \geq \nu \) which contradicts \( \nu = [w_1] + 1 = [w_2] + 1 \). This case cannot happen.

Case 2. If \( j_2 \in \{1, \cdots , 5\} \setminus \{1, 2, j_1\} \), then \( \frac{a_2}{w_2} + \frac{1}{w_{j_2}} = 1 \), since \( z_2^{a_2} z_{j_2} \in \text{supp} f \). We want to show that \((\nu-1)^2 < (w_2-1)(w_{j_2}-1)\). \( \nu \leq a_2 + 1 \), and \( \nu = a_1 + 1 \) implies \( a_2 \geq a_1 \).

\[ a_2 = w_2 - \frac{w_2}{w_{j_2}} \geq a_1 \geq \nu - 1. \]

Let \( x = \frac{w_2}{w_{j_2}} \), by Lemma 3.1.1 we have \( x \neq 1 \). Then \( 0 < x < 1 \).

\[ w_2 \geq \nu - 1 + x \text{ and } w_{j_2} \geq \frac{\nu-1+x}{x}. \]

It suffice to show that \((\nu-1)^2 < (\nu-1+x-1)(\frac{\nu-1+x}{x} - 1)\) which is true for \( \nu > 2 \) and \( 0 < x < 1 \).

Case 3. If \( j_2 = j_1 \), then \( z_2^{a_2} z_{j_1} \in \text{supp} f \). Since \( f \) has isolated singularity and both \( z_1^{a_1} z_{j_1} \in \text{supp} f \) and \( z_2^{a_2} z_{j_1} \in \text{supp} f \), then either \( z_1^{b_1} z_2^{b_2} \in \text{supp} f \), where \( b_i > 0, i = 1, 2 \) or \( z_1^{b_1} z_2^{b_2} z_{j_{12}} \in \text{supp} f \) where \( b_i \geq 0, \text{for } i = 1, 2 \). However in the latter case \( b_1 \) and \( b_2 \) can’t both 0 and \( j_{12} \in \{1, \cdots , 5\} \setminus \{1, 2, j_1\} \).

Subcase 1. \( z_1^{b_1} z_2^{b_2} \in \text{supp} f \) where \( b_i > 0, i = 1, 2 \). In this case, we have \( \frac{b_1}{w_1} + \frac{b_2}{w_2} = 1 \).

\[ \frac{b_1}{w_2} + \frac{b_2}{w_2} = \frac{b_1}{w_1} + \frac{b_2}{w_2} = 1 \text{ which implies } w_2 \geq b_1 + b_2 \geq \nu \text{ which contradicts to } \nu - 1 = [w_1] = [w_2]. \]

This case cannot happens.

Subcase 2. \( z_1^{b_1} z_2^{b_2} z_{j_{12}} \in \text{supp} f \) where \( b_i \geq 0, \text{for } i = 1, 2 \) and \( j_{12} \in \{1, \cdots , 5\} \setminus \{1, 2, j_1\} \). In this case we divide it into three subcases.

a). If \( b_1 = 0 \), then \( z_2^{b_2} z_{j_{12}} \in \text{supp} f \). This case is same as the previous case 2.
b). If \( b_2 = 0 \), then \( z_1^{b_1} z_{j12} \in suppf \). By Lemma 3.1.1, we have \( \nu = b_1 + 1 \). Therefore \( a_1 = b_1 \). Remember that we also have \( \frac{a_1}{w_1} + \frac{1}{w_{j1}} = 1 \), thus \( w_{j1} = w_{j12} \). Since we have proved \((\nu - 1)^2 < (w_1 - 1)(w_{j1} - 1)\), then we get \((\nu - 1)^2 < (w_2 - 1)(w_{j12} - 1)\).

c). If \( b_1 \neq 0 \) and \( b_2 \neq 0 \), then \( \frac{b_1}{w_1} + \frac{b_2}{w_2} + \frac{1}{w_{j12}} = 1 \) which implies \( \frac{b_1 + b_2}{w_2} + \frac{1}{w_{j12}} \leq 1 \). Since \( \nu \leq b_1 + b_2 + 1 \) and \( \nu = a_1 + 1 \), so \( a_1 \leq b_1 + b_2 \), then \( \frac{a_1}{w_2} + \frac{1}{w_{j12}} \leq 1 \) which implies \( a_1 \leq w_2 - \frac{w_2}{w_{j12}} \).

Since \( j_{12} \in \{1, \cdots, 5\} \setminus \{1, 2, j_1\} \), so \( w_{j12} \geq w_2 \). If \( w_{j12} = w_2 \), then \( w_2 \geq a_1 + \frac{w_2}{w_{j12}} = a_1 + 1 \) which contradicts to \([w_1] = [w_2] = \nu - 1\). Thus \( w_{j12} > w_2 \). Let \( x = \frac{w_2}{w_{j12}} \), then \( 0 < x < 1 \). Since \( w_2 \geq a_1 + x \), so \( w_{j12} \geq \frac{a_1 + x}{x} \). We want to show that \((\nu - 1)^2 < (w_2 - 1)(w_{j12} - 1)\). It suffice to show that \( a_1^2 < (a_1 + x - 1)(\frac{a_1 + x}{x} - 1) \) which follows from \( 0 < (a_1 - 1)(1 - x) \) where \( a_1 \geq 2 \) and \( 0 < x < 1 \).

After the above steps, either we finish the proof or after reordering the subindex, we have proved \((\nu - 1)^4 < (w_1 - 1)(w_2 - 1)(w_{j1} - 1)(w_{j2} - 1)\), where \( z_1, z_2, z_{j1}, z_{j2} \) are different variables. There is only one variable left. Without loss of generality, we use \( z_3 \) denotes the remaining variable. We know \( w_3 \geq w_2 \geq w_1 \), and \( w_1 \) and \( w_2 \) are not positive integers by the previous arguments.

If \( w_3 \) is a positive integer, or \( w_3 \) is not a positive integer and \([w_3] > [w_1]\), then we have \( \nu \leq w_3, \nu - 1 \leq w_3 - 1 \). Therefore \( \mu > (\nu - 1)^5 \) in this case. The proof ends.

Suppose that \( w_3 \) is not a positive integer and \([w_3] = [w_1]\). Since \( w_3 \geq w_1 \), so we have \( w_3 - 1 \geq w_1 - 1 \). We have already proved \((\nu - 1)^2 < (w_1 - 1)(w_{j1} - 1), (\nu - 1)^2 < (w_2 - 1)(w_{j2} - 1)\).

In order to prove \((\nu - 1)^5 < \mu\), it suffices to show that \((\nu - 1)^3 < (w_1 - 1)^2(w_{j1} - 1)\). In order to make the notation simple, we set \( x = [w_1] \) where \( x \geq 2 \), \( y = \frac{w_1}{w_{j1}} \) where \( 0 < y < 1 \). Then
\[
x = \nu - 1, \quad w_1 = x + y, \quad w_{j1} = \frac{x + y}{y}.
\]

By simple calculation, \((\nu - 1)^3 < (w_1 - 1)^2(w_{j1} - 1)\) is equivalent to
\[
x^3 \leq (x + y - 1)^2\left(\frac{x + y}{y} - 1\right), \text{ i.e. } x(x - 2)(1 - y) + (y - 1)^2 > 0 \text{ which follows from }
x \geq 2 \text{ and } 0 < y < 1.
\]

In summary, we have proved \((\nu - 1)^5 = \mu\) if and only if the \(f\) is homogeneous polynomial.

For \(k = 6\).

Following the same argument as \(k = 5\), we can obtain \((\nu - 1)^2 < (w_1 - 1)(w_{j1} - 1), (\nu - 1)^3 < (w_2 - 1)(w_{j2} - 1)\). Without loss of generality we assume \(w_3\) and \(w_4\) are the remaining two weights.

Then the same argument as above we can show that
\[
(\nu - 1)^3 < (w_1 - 1)(w_{j1} - 1)(w_3 - 1) \quad \text{and} \quad (\nu - 1)^3 < (w_2 - 1)(w_{j2} - 1)(w_4 - 1).
\]

Thus we have \((\nu - 1)^6 < (w_1 - 1)(w_2 - 1)\cdots(w_6 - 1) = \mu\). which is what we want for proving \((\nu - 1)^6 = \mu\) if and only if the \(f\) is homogeneous polynomial. Q.E.D.

**Theorem 3.1.11.** (91) Let \(f : (\mathbb{C}^k, 0) \to (\mathbb{C}, 0)\) where \(k\) is either 5 or 6, be a polynomial with an isolated singularity at the origin. Let \(\mu, \nu\) and \(\tau\) be the Milnor number, multiplicity and Tjurina number of the singularity \(V = \{z : f(z) = 0\}\) respectively. Then \(f\) is a homogeneous polynomial after a biholomorphic change of coordinates if and only if \(\mu = \tau = (\nu - 1)^k\).

In this chapter, our main purpose is to prove the following main results.

**Theorem A.** Let \(f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)\) be a holomorphic germ defining an isolated hypersurface singularity \(V = \{z : f(z) = 0\}\) at the origin. Let \(\mu\) and \(\nu\) be the Milnor number and multiplicity of \((V, 0)\) respectively. Then
\[
\mu \geq (\nu - 1)^n \quad \text{(1.7)}
\]
and the equality in (1.7) holds if and only if \( f \) is a semi-homogeneous function (i.e. \( f = f_\nu + g \) where \( f_\nu \) is a non-degenerate homogeneous polynomial of degree \( \nu \) and \( g \) consists of terms of degree at least \( \nu + 1 \)).

**Theorem B.** Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be a weighted homogeneous polynomial with an isolated singularity at the origin. Let \( \mu \) and \( \nu \) be the Milnor number and multiplicity of the singularity \( V = \{ z : f(z) = 0 \} \) respectively. Then

\[
\mu \geq (\nu - 1)^n
\]  

(1.8)

and the equality in (1.8) holds if and only if \( f \) is a homogeneous polynomial after a biholomorphic change of coordinates.

**Theorem C.** Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be a weighted homogeneous polynomial with an isolated singularity at the origin. Let \( \mu, p_g \) and \( \nu \) be the Milnor number, geometric genus and multiplicity of the singularity \( V = \{ z : f(z) = 0 \} \), if \( p_g = 0 \) then

\[
\mu - p(\nu) \geq n! p_g,
\]

where \( p(\nu) = (\nu-1)^n - \nu(\nu-1) \cdots (\nu-n+1) \), and equality holds if and only if \( f \) is a homogeneous polynomial after a biholomorphic change of coordinates.

**Theorem D.** Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be a polynomial with an isolated singularity at the origin. Let \( \mu, \nu \) and \( \tau \) be the Milnor number, multiplicity and Tjurina number of the singularity
\( V = \{ z : f(z) = 0 \} \) respectively. Then \( f \) is a homogeneous polynomial after a biholomorphic change of coordinates if and only if \( \mu = \tau = (\nu - 1)^n \).

In §2, we recall the necessary materials which are needed to prove the Main Theorems. In §3, we prove the Main Theorems.

### 3.2 Preliminary

In this section, we recall some known results which are needed to prove the Main Theorem. Let \( f(z_1, \ldots, z_n) \) be a germ of an analytic function at the origin such that \( f(0) = 0 \). Suppose \( f \) has an isolated critical point at the origin. \( f \) can be developed in a convergent Taylor series

\[
f(z_1, \ldots, z_n) = \sum a_\lambda z^\lambda \quad \text{where} \quad z^\lambda = z_1^{\lambda_1} \cdots z_n^{\lambda_n}.
\]

Recall that Newton boundary \( \Gamma(f) \) is the union of compact faces of \( \Gamma_+(f) \) where \( \Gamma_+(f) \) is the convex hull of the union of subsets \( \{ \lambda + \mathbb{R}_+^n \} \) for \( \lambda \) such that \( a_\lambda \neq 0 \). Finally, let \( \Gamma_-(f) \), the Newton polyhedron of \( f \), be the cone over \( \Gamma(f) \) with cone point at 0. For any closed face \( \Delta \) of \( \Gamma(f) \), we associate the polynomial

\[
f_\Delta(z) = \sum_{\lambda \in \Delta} a_\lambda z^\lambda.
\]

We say that \( f \) is non-degenerate if \( f_\Delta \) has no critical point in \( (\mathbb{C}^*)^n \) for any \( \Delta \in \Gamma(f) \) where \( \mathbb{C}^* = \mathbb{C} - \{ 0 \} \). We say that a point \( p \) of the integral lattice \( \mathbb{Z}^n \) in \( \mathbb{R}^n \) is positive if all coordinates of \( p \) are positive. The following beautiful theorem is due to Merle-Teissier [Me-Te].

**Theorem 3.2.1.** (Merle-Teissier)(43). Let \( (V, 0) \) be an isolated hypersurface singularity defined by a non-degenerate holomorphic function \( f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0) \). Then the geometric genus

\[
p_g = \# \{ p \in \mathbb{Z}^n \cap \Gamma_- : p \text{ is positive} \}.
\]

Recall that a polynomial \( f(z_1, \ldots, z_n) \) is weighted homogeneous of type \( (w_1, \ldots, w_n) \), where \( w_1, \ldots, w_n \) are fixed positive rational numbers, if it can be expressed as a linear combination
of monomials $z_1^{i_1} \cdots z_n^{i_n}$ for which $i_1/w_1 + \cdots + i_n/w_n = 1$. As a consequence of the theorem of Merle-Teissier, for isolated singularity defined by a weighted homogeneous polynomial, computing the geometric genus is equivalent to counting the number of positive integral points in the tetrahedron defined by $x_1/w_1 + \cdots + x_n/w_n \leq 1, x_1 \geq 0, \ldots, x_n \geq 0$. We also need the following result.

**Theorem 3.2.2.** (Milnor-Orlik) (45) Let $f(z_1, \ldots, z_n)$ be a weighted homogeneous polynomial of type $(w_1, \ldots, w_n)$ with isolated singularity at the origin. Then the Milnor number $\mu = (w_1 - 1) \cdots (w_n - 1)$.

In (81), Yau give a lower bound for $p_g$ of hypersurface singularity.

**Theorem 3.2.3.** (81) Let $f(z_1, \cdots, z_{n-1}, z_n) = z_n^m + a_1(z_1, \cdots, z_{n-1})z_n^{m-1} + \cdots + a_m(z_1, \cdots, z_{n-1})$ be holomorphic near $(0, \cdots, 0)$. Let $d_i$ be the order of the zero of $a_i(z_1, \cdots, z_{n-1})$ at $(0, \cdots, 0)$, $d_i \geq i$. Let $d = \min_{1 \leq i \leq m}(\frac{d_i}{i})$. Suppose that

$$V = \{(z_1, \cdots, z_n) : f(z_1, \cdots, z_n) = 0\}$$

defined in a suitably small polydisc, has $p = (0, \cdots, 0)$ as its only singularity. Let $\pi : M \to V$ be resolution of $V$. Then $\dim H^{n-2}(M, \mathcal{O}) > (m - 1)d - (n - 1)$.

Remark: Here, the singularity is $(n - 1)$-dimensional, so $\dim H^{n-2}(M, \mathcal{O}) = p_g$.

Recall that we have the following definition.
Definition 3.2.1. Let \( f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be germs of holomorphic functions defining an isolated hypersurface singularities \( V_f = \{ z : f(z) = 0 \} \) and \( V_g = \{ z : g(z) = 0 \} \) respectively. Let \( \phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0) \) be a germ of biholomorphic map:

1. \( f \) is contact equivalent to \( g \), if \( \phi(V_f) = V_g \).
2. \( f \) is right equivalent to \( g \), if \( g = f \circ \phi \).

The Milnor number is invariant under right equivalence and the Tjurina number is invariant under contact equivalence. It is a non-trivial theorem that the Milnor number is indeed an invariant of contact equivalence (see (21)).

Theorem 3.2.4. (21) Let \( \mathbb{K} \) be an algebraically closed field of characteristic zero and \( f, g \in \mathbb{K}\{z_1, z_2, \ldots, z_n\} \). If \( f \) is contact equivalent to \( g \), then \( \mu(f) = \mu(g) \).

Theorem 3.2.5. ((56),(84)) If \( f \) and \( g \) be germs of isolated weighted homogeneous singularities at origin in \( \mathbb{C}^n \), then \( f \) and \( g \) are right equivalent if and only if \( f \) and \( g \) are contact equivalent.

Theorem 3.2.6. (52) Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be the germ of a complex analytic function with an isolated critical point at the origin.

(a) \( f \) is right equivalent to a weighted homogeneous polynomial if and only if

\[
f \in J_f := \left( \frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n} \right)
\]

(b) If \( f \) is weighted homogeneous with normalized weight system \((w_1, \cdots, w_n, 1)\) with \( 0 < w_1 \leq \cdots \leq w_n < 1 \) and if \( f \in m^{3}_{\mathbb{C}^n,0} \), then the weight system is unique and \( 0 < w_1 \leq \cdots \leq w_n < \frac{1}{2} \).
(c) If \( f \in J_f \) then \( f \) is right equivalent to a weighted homogeneous polynomial \( g(z_1, \ldots, z_k) + z_{k+1}^2 + \cdots + z_n^2 \) with \( g \in \mathfrak{m}^3_\mathbb{C}_{n,0}. \) Especially, its normalized weight system satisfies \( 0 < w_1 \leq \cdots \leq w_k < w_{k+1} = \cdots = w_n = \frac{1}{2}. \)

(d) If \( f \) and \( \bar{f} \in O_{\mathbb{C}^n,0} \) are right equivalent and weighted homogeneous with normalized weight systems \( (w_1, \ldots, w_n, 1) \) and \( (\bar{w}_1, \ldots, \bar{w}_n, 1) \) with \( w_1 \leq \cdots \leq w_n \leq \frac{1}{2} \) and \( \bar{w}_1 \leq \cdots \leq \bar{w}_n \leq \frac{1}{2} \) then \( w_i = \bar{w}_i. \)

3.3 Multiplicity of Filtered Rings

Let \( (A, m) \) be a local Noetherian ring and \( M \) a finitely generated \( A \)-module with dim \( M =: d > 0. \) For an \( m \)-primary ideal \( q \subset A \) denote by

\[
P_M(n) := P_M^q(n) := l(M/q^{n+1}M)
\]

the Hilbert-Samuel function, where \( l = l_A \) is the length function over \( A. \) The function \( P_M(n) \) is for \( n \gg 0 \) a polynomial in \( n \) of degree \( d \) and so can be written as

\[
P_M := e(q, M) \frac{n^d}{d!} + \text{terms of lower degree}
\]

where \( e(q, M) \) is the multiplicity of \( M. \) It is well-known that \( e(q, M) \) is an integer \( > 0. \) We set \( e(q, M) = 0 \) if \( M = 0. \)

Let \( (A, m) \) be a \( d \)-dimensional local ring. A filtration \( F = \{F^k\}_{k \geq 0} \) is a decreasing sequence of ideals of \( A \) that satisfies \( F^0 = A, F^1 = m, F^k \supset F^{k+1}, \) and \( F^k \cdot F^j \subset F^{k+j}. \) We always assume that \( R = \bigoplus_{k \geq 0} F^k \cdot T^k \subset A[T] \) is a finitely generated \( A \)-algebra. Here \( T \) is an indeterminate.
There is a positive integer \( N \) such that \( F^{kN} = (F^N)^k \) holds for \( k \geq 0 \), and we assume that \( F^N \) is \( m \)-primary. One can find general information on the theory of filtered rings and filtered blowing-up in (68). Let \( y_1, \ldots, y_d \in m \) be a system of parameters of \( A \), and assume that the relation \( y_i \in F^{a_i} \) holds for each \( i \), where \( a_i \in \mathbb{Z}_{\geq 1} \). Let \( G = \oplus_{k \geq 0} F^k/F^{k+1} \) be the associated graded ring, and let \( P(G, \lambda) = \sum l(G_k) \lambda^k \in \mathbb{Z}[[\lambda]] \). Then the arguments of the proof of Theorem A of (67) yield the following result on the multiplicity \( e((y_1, y_2, \ldots, y_d), A) \).

**Theorem 3.3.1.** (19) (1) We have the inequality

\[
e((y_1, y_2, \ldots, y_d), A) \geq (\prod_{i=1}^d a_i) \lim_{\lambda \to 1} (1 - \lambda)^d P(G, \lambda).
\]

(2) The equality holds in the above if and only if the following two conditions hold: First, \( y_i \in F^{a_i} - F^{a_i+1} \), for \( i = 1, \ldots, d \), and Second, \( \bar{y}_i \in G_{a_i} \), for \( i = 1, \ldots, d \), defines a systems of parameters of \( G \).

In case of (2), we obtain the equality \( e((\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_d), A) = e((y_1, y_2, \ldots, y_d), A) \).

### 3.4 Characterization of Semi-quasihomogeneous Hypersurface Singularities

In the following theorem, it is convenient for us using another definition of weight type.

Let \( f \in \mathbb{C}\{z_1, \ldots, z_n\} \) define an isolated singularities at the origin. Let \( w = (w_1, \cdots, w_n) \) be a weight on the coordinates \( (z_1, \cdots, z_n) \) by positive integer numbers \( w_i, i = 1, \cdots, n \). We have the weighted Taylor expansion \( f = f_\rho + f_{\rho+1} + \cdots \) with respect to \( w \) and \( f_\rho \neq 0 \), where \( f_k \) is a weighted homogeneous of type \( (w_1, \cdots, w_n; k) \), for \( k \geq \rho \). i.e. \( f_k \) is linear combination of monomials \( z_1^{i_1} \cdots z_n^{i_n} \) for which \( i_1 w_1 + \cdots + i_n w_n = k \). We only use this definition of weight for
the following Theorem 3.4.1 as well as in the proof of Theorem A. For any other place we use the previous definition before Theorem 3.2.2 for weight type.

**Theorem 3.4.1.** (19). Let $f \in \mathbb{C}\{z_1, \cdots, z_n\}$ define an isolated singularity at the origin. With the above situation, then: (1) The following inequality holds:

$$\mu(f) \geq (\frac{\rho}{w_1} - 1) \cdots (\frac{\rho}{w_n} - 1).$$

(2) The equality holds in the above if and only if $f_{\rho}$ defines an isolated singularity at the origin.

Here we recall that $f$ is called a semi-quasihomogeneous function if the initial term $f_{\rho}$ defines an isolated singularity at the origin.

**Proof.** Let $F^k$ be the filtration on $\mathbb{C}\{x_1, \cdots, x_d\}$ by the weight $w$ on the coordinate $x_1, \cdots, x_d$. We have the relation $f \in F^\rho - F^{\rho+1}$. Since $f$ defines an isolated singularity at the origin, $\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_d}$ define a system of parameters of $\mathbb{C}\{x_1, \cdots, x_d\}$.

We have the weighted Taylor expansion $f = \sum_{k \geq \rho} f_k$ where $f_k$ is a weighted homogeneous polynomial of the type $(w_1, \cdots, w_d; k)$. Furthermore, we have $\frac{\partial f}{\partial x_1} = \sum_{k \geq \rho} \frac{\partial f_k}{\partial x_1}$. Here $\frac{\partial f_k}{\partial x_i}$ is zero or a weighted homogeneous polynomial of type $(w_1, \cdots, w_d; k - w_i)$. Here $\frac{\partial f}{\partial x_i} \in F^{\rho - w_i}$ for $i = 1, \cdots, d$. By (1) of Theorem 3.3.1, we obtain the relation

$$e(\langle \frac{\partial f_k}{\partial x_1}, \cdots, \frac{\partial f_k}{\partial x_d} \rangle, \mathbb{C}\{x_1, \cdots, x_d\}) \geq \prod_{i=1}^{d} (\rho - w_i) \lim_{\lambda \to 1} (1 - \lambda)^d P(G, \lambda).$$

Since $\mathbb{C}\{x_1, \cdots, x_d\}$ is Cohen-Macaulay, it is well known that
This coincides with \( \mu(f) \).

By our definition of the filtration \( F, G = \mathbb{C}\{x_1, \cdots, x_d\} \) and we can easily see the relation

\[
P(G, \lambda) = \frac{1}{(1 - \lambda^{w_1} \cdots 1 - \lambda^{w_d})}.
\]

This completes the proof of (1) of Theorem 3.4.1.

(2) Let us employ the notation of (1). First, we show the necessity part of our assertion.

Suppose the equality holds in (1) of Theorem 3.4.1. Then by (2) of Theorem 3.3.1, we obtain

the relations

\[
\frac{\partial f}{\partial x_i} \in F_{\rho - w_i} - F_{\rho - w_i + 1}
\]

for \( i = 1, \cdots, d \). Hence \( \frac{\partial f}{\partial x_i} \neq 0 \) and we may write \( \frac{\partial f}{\partial x_i} \in G \) as \( \frac{\partial f}{\partial x_i} \in \mathbb{C}\{x_1, \cdots, x_d\} = G \). By

Theorem 3.3.1, \( \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_d} \) define a system of parameters at the origin. Hence \( f_\rho \) defines a

system of parameters at the origin.

Next we show the sufficiency part. Assume that \( f_\rho \) defines an isolated singularity. Then

\( \frac{\partial f}{\partial x_i} \neq 0 \) for \( i = 1, \cdots, d \), and \( \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_d} \) define a system of parameters at the origin. Since

we may write \( \frac{\partial f}{\partial x_i} \in G \) as \( \frac{\partial f}{\partial x_i} \in \mathbb{C}\{x_1, \cdots, x_d\} = G \). Theorem 3.3.1 implies the relation

\[
e((\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_d}), \mathbb{C}\{x_1, \cdots, x_d\}) = \prod_{i=1}^{d} (\rho - w_i) \lim_{\lambda \to 1} (1 - \lambda)^d P(G, \lambda).
\]
3.5 Characterization of Isolated Homogeneous Hypersurface Singularities

Proof of the Theorem A

Let \( f(z_1, \cdots, z_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be a holomorphic function with an isolated singularity at the origin. Let \( \mu \) and \( \nu \) be the Milnor number and multiplicity of the singularity \( V = \{ z : f(z) = 0 \} \). By an analytic change of coordinates, one can assume that the \( z_n \)-axis is not contained in the tangent cones of \( V \), so that \( f(0, \cdots, 0, z_n) \neq 0 \). By the Weierstrass preparation theorem, near 0, the germ \( f \) can be represented as a product \( f(z_1, \cdots, z_n) = u(z_1, \cdots, z_n)g(z_1, \cdots, z_n) \), where \( u(0, \cdots, 0) \neq 0 \) and \( g(z_1, \cdots, z_n) = z_n^\nu + a_1(z_1, \cdots, z_n)z_n^{\nu-1} + \cdots + a_\nu(z_1, \cdots, z_n) \) where \( \nu \) is the multiplicity of \( f(z_1, \cdots, z_n) \) and \( a_i \in (x_1, \cdots, x_{n-1})^i \), for \( i = 1, \cdots, \nu \). Therefore \( f(z_1, \cdots, z_n) \) is contact equivalent to \( g(z_1, \cdots, z_n) \).

Let \( d_i \) be the order of the zero of \( a_i(z_1, \cdots, z_{n-1}) \) at \( (0, \cdots, 0) \), \( d_i \geq i \). Let \( d = \min_{1 \leq i \leq \nu} \left[ \frac{d_i}{i} \right] \), then \( d \geq 1 \). We define a new weight \( w \) on the coordinate systems, \( w(z_n) = d \), \( w(z_i) = 1 \), for \( 1 \leq i \leq n-1 \). Here the definition of weight type is the same as Theorem 3.4.1. With respect to the new weights, \( z_n^\nu \) has degree \( d\nu \), and \( a_i(z_1, \cdots, z_{n-1})z_n^{\nu-i} \) has degree at least \( d(\nu - i) + d_i \geq d\nu - di + di = d\nu \). Thus the initial term of \( f(z_1, \cdots, z_n) \) has the degree \( \rho = d\nu \).

By Theorem 3.2.4, Milnor number is a invariant under the contact equivalence. By Theorem 3.4.1 (1) we have \( \mu = \mu(g) \geq (\frac{d\nu}{\nu} - 1)(\frac{d\nu}{\nu} - 1) \cdots (\frac{d\nu}{\nu} - 1) = (\nu - 1)(d\nu - 1)^{n-1} \geq (\nu - 1)^n \).

Suppose \( f \) is a semi-homogeneous polynomial. Since the Milnor number of \( f \) is the same as the Milnor number of its initial part (see (3)), so the fact that \( \mu = (\nu - 1)^n \) is obvious.

If \( \mu = (\nu - 1)^n \), then by \( \mu \geq (\nu - 1)(d\nu - 1)^{n-1} \geq (\nu - 1)^n \), we have \( d = 1 \) and by Theorem 3.4.1 (2) \( g_{d\nu}(z_1, \cdots, z_n) = g_\nu(z_1, \cdots, z_n) \) is homogeneous polynomial of degree \( \nu \) defining an iso-
lated singularity. Hence $f(z_1, \cdots, z_n)$ is contact equivalent to a semi-homogeneous singularity.

Q.E.D.

**Proof of the Theorem B**

We only need to show that if $f$ is a weighted homogeneous singularity, then $\mu = (\nu - 1)^n$ if and only if $f$ is equivalent to a homogeneous singularity.

The ‘if’ part is trivial. We only need to consider the ‘only if’ part. By Saito’s theorem (see Theorem 3.2.6(c)) we can choose a normalized weights for $f$, that means these weights satisfy $0 < w_i \leq \frac{1}{2}, 1 \leq i \leq n$. By the proof of Theorem A, we know there exists a $g(z_1, \cdots, z_n)$ so that $f$ is contact equivalent to $g$, moreover the initial part of $g$ is $g_{\nu}$ which is a homogeneous polynomial with degree $\nu$ and $g_{\nu}$ also define an isolated singularity at the origin. We can rewrite $f$ and $g$ as following:

\[
f(z_1, \cdots, z_n) = f_{\nu}(z_1, \cdots, z_n) + f_{\nu+1}(z_1, \cdots, z_n) + \cdots,
\]

\[
g(z_1, \cdots, z_n) = g_{\nu}(z_1, \cdots, z_n) + g_{\nu+1}(z_1, \cdots, z_n) + \cdots,
\]

where $f_i$ and $g_i$, $i \geq \nu$ are homogeneous part of $f$ and $g$ respectively and $g_{\nu}$ define a isolated singularity at origin. Since for weighted homogenous singularity the contact equivalence is same as the right equivalence (See Theorem 3.2.5), so there exists a biholomorphism at the origin.

\[
\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)
\]

\[
(z_1, \cdots, z_n) \to (\phi_1(z_1, \cdots, z_n), \cdots, \phi_n(z_1, \cdots, z_n)).
\]
such that \( f(z_1, \cdots, z_n) = g(\phi_1(z_1, \cdots, z_n), \cdots, \phi_n(z_1, \cdots, z_n)) \), and

\[
\begin{align*}
\phi_1(z_1, \cdots, z_n) &= a_{11}z_1 + \cdots + a_{1n}z_n + H_1^2 + H_1^3 + \\
&\vdots \\
\phi_n(z_1, \cdots, z_n) &= a_{n1}z_1 + \cdots + a_{nn}z_n + H_n^2 + H_n^3 + 
\end{align*}
\]

where \( H_i^j = \sum_{\alpha_1 + \cdots + \alpha_n = j} c_i(\alpha_1, \cdots, \alpha_n)z_1^{\alpha_1} \cdots z_n^{\alpha_n} \). Since \( \phi \) is biholomorphism at the origin, we have \(|(a_{ij})| = \det (a_{ij}) \neq 0 \). It follows from \( f(z_1, \cdots, z_n) = g(\phi_1(z_1, \cdots, z_n), \cdots, \phi_n(z_1, \cdots, z_n)) \)
that \( g_\nu(\phi_1(z_1, \cdots, z_n), \cdots, \phi_n(z_1, \cdots, z_n)) + g_\nu+1(\phi_1(z_1, \cdots, z_n), \cdots, \phi_n(z_1, \cdots, z_n)) + \cdots 
= f_\nu(z_1, \cdots, z_n) + f_{\nu+1}(z_1, \cdots, z_n) + \cdots. \)
Comparing the degree of both sides, we have
\( g_\nu(\bar{\phi}_1(z_1, \cdots, z_n), \cdots, \bar{\phi}_n(z_1, \cdots, z_n)) = f_\nu(z_1, \cdots, z_n) \), where \( \bar{\phi}_i = a_{i1}z_1 + \cdots + a_{in}z_n, 1 \leq i \leq n. \)
Since \( \det (a_{ij}) \neq 0 \), so \( f_\nu \) is right equivalent to \( g_\nu \). Therefore \( f_\nu \) also define a isolated singularity.

Now we have two normalized weights for \( f_\nu \), one is \((w_1, \cdots, w_n)\) and another is \((\frac{1}{\nu}, \cdots, \frac{1}{\nu})\). By Theorem 3.2.6, we have \( w_1 = w_2 = \cdots = w_n = \frac{1}{\nu}. \) Therefore \( f(z_1, \cdots, z_n) = f_\nu(z_1, \cdots, z_n) \) is a homogeneous polynomail. Q.E.D.

**Proof of the Theorem C**

Since \( p_g = 0 \), then by Theorem 3.2.3, we have \( 0 > (\nu - 1)d - (n - 1) \) where \( d = \min_{1 \leq i \leq \nu} (\frac{d_i}{\nu}) \),
and \( d_i \) is the order of the zero of \( a_i(x_1, \cdots, x_n) \) at \((0, \cdots, 0)\) with \( d_i \geq i \). Then \( \nu < \frac{n-1}{d} + 1. \) Since \( d \geq 1, \nu \) is an integer at least 2 for isolated hypersurface singularities, we have \( 2 \leq \nu \leq n - 1. \)
Therefore \( p(\nu) = (\nu - 1)^n - \nu(\nu - 1) \ldots (\nu - n + 1) = (\nu - 1)^n \). The theorem is reduced to prove that

\[
\mu \geq (\nu - 1)^n,
\]

where equality holds if and only if \( f \) is a homogeneous polynomial after coordinate change. The proof follows from Theorem B obviously. Q.E.D.

**Proof of the Theorem D**

It follows from the Theorem B and Theorem 3.1.1. Q.E.D.
CHAPTER 4

NONEXISTENCE OF NEGATIVE WEIGHT DERIVATIONS ON WEIGHTED HOMOGENEOUS ISOLATED FEWNOMIAL SINGULARITIES

This chapter presents a result about the finite-dimensional Lie algebras associated with germs of isolated hypersurface singularities defined by polynomials with the number of monomials equal to the number of variables.

Let \( R = \mathbb{C}\{x_1, x_2, \ldots, z_n\}/(f) \) where \( f \) is a weighted homogeneous polynomial defining an isolated singularity at the origin. Then \( R \) and \( \text{Der}(R, R) \), i.e., the Lie algebra of derivations on \( R \), are graded. It is well-known that \( \text{Der}(R, R) \) has no negatively graded component (73). J. Wahl conjectured that the above fact is still true in higher codimensional case provided that \( R = \mathbb{C}\{x_1, x_2, \ldots, x_n\}/(f_1, f_2, \ldots, f_m) \) is an isolated, normal and complete intersection singularity and \( f_1, f_2, \ldots, f_m \) are weighted homogeneous polynomials with the same weight type \((w_1, w_2, \ldots, w_n)\). On the other hand the first author Yau conjectured that the moduli algebra \( A(V) = \mathbb{C}\{x_1, x_2, \ldots, x_n\}/(\partial f/\partial x_1, \ldots, \partial f/\partial x_n) \) has no negatively weighted derivation where \( f \) is a weighted homogeneous polynomial defining an isolated singularity at the origin (77). By supposing this conjecture has a positive answer he proved a characterization of weighted homogeneous hypersurface singularities only using the Lie algebra \( L(V) \) of derivations on \( A(V) \).

The conjecture of Yau can be thought as an Artinian analogue of J. Wahl’s conjecture. For the
dimension up to 3, the Yau conjecture 4.1.1 has a positive answer ((18), (15)). In this paper we prove this conjecture for the fewnomial singularities with multiplicity at least 5.

4.1 Introduction

The aim of this chapter is to present a result about the finite-dimensional Lie algebras associated with germs of isolated hypersurface singularities (IHS) defined by polynomials with the number of monomials equal to the number of variables. Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be a holomorphic function with an isolated critical point at the origin. Then \( V = \{ f = 0 \} \) is an IHS germ. In (83), Yau was the first person who considers \( L(V) \) which is the Lie algebra of derivations of the moduli algebra \( A(V) = \mathbb{C}\{x_1, \ldots, x_n\}/(\partial f/\partial x_1, \ldots, \partial f/\partial x_n) \), i.e., \( L(V) = \text{Der}_\mathbb{C}(A(V), A(V)) \). According to (83), \( L(V) \) is a finite-dimensional solvable Lie algebra which is often called the Lie algebra of singularity \( V \). Following (29) and (95) we call it the Yau algebra of \( V \) in order to distinguish from Lie algebras of other types appearing in singularity theory (10), (2), (11).

Recall that a polynomial \( f \in \mathbb{C}[x_1, \ldots, x_n] \) is called weighted homogeneous if there exist positive rational numbers \( w_1, \ldots, w_n \) (called weights of indeterminates \( x_i \)) and \( d \) such that, for each monomial \( \prod x_i^{a_i} \) appearing in \( f \) with nonzero coefficient, one has \( \sum a_i w_i = d \). The number \( d \) is called the weighted homogeneous degree (\( w \)-degree) of \( f \) with respect to weights \( w_j \).

Let \( P = \mathbb{C}[x_1, \ldots, x_n] \) be the polynomial ring of \( n \) weighted variables \( x_1, \ldots, x_n \) with positive integer weights \( w_1, w_2, \ldots, w_n \). For a monomial \( x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \) in \( P \) its weighted degree is defined to be \( w_1 i_1 + \cdots + w_n i_n \). By this (weighted) degree the polynomial ring \( P \) is graded. For a homogeneous ideal \( I \) (i.e., an ideal generated by weighted homogeneous polynomials) \( \subset P \) we
have a graded quotient ring $R = P/I = \oplus_{i=0}^{\infty} R_i$ which corresponds to an affine variety with
singularities at the origin. It is obvious that there is a $\mathbb{C}^*$-action on this variety with singularities.

When the Krull dimension of $R$ is zero we have a 0-dimensional scheme at the origin and $R$
is a graded Artinian algebra. Let $\text{Der}(R, R)$ be the $R$-module of derivations on $R$ (i.e., vector
fields on the aforementioned variety with $\mathbb{C}^*$-action where the $\mathbb{C}^*$-action corresponds to the well-
known Euler derivation $\Delta := w_1 x_1 \partial / \partial x_1 + w_2 x_2 \partial / \partial x_2 + \cdots + w_n x_n \partial / \partial x_n$), we have a natural
grading on $\text{Der}(R, R) = \oplus_{k=-\infty}^{+\infty} \text{Der}(R, R)_k$ where $\text{Der}(R, R)_k = \{D \in \text{Der}(R, R) : D(R_i) \subset R_{i+k} \text{ for any } i\}$. In this way, the Euler derivation $\Delta$ has weight 0. It is useful to know if
$\text{Der}(R, R)$ has any derivation of negative weight, or other derivations of weight 0. Moreover,
it is a natural and long-standing problem whether $\text{Der}(R, R)$ is non-negatively graded, i.e.,
$\text{Der}(R, R)_k = 0$ for any $k < 0$, nonexistence of negative weight derivation. Actually this
problem has been motivated from both algebraic topology and singularities as follows.

A classic result of A. Borel ((13)) states that the Serre spectral sequence for rational coho-
mology of the universal bundle $G/H \to B_H \to B_G$ collapses if $G/H$ is a homogeneous space of
equal rank pairs $(G, H)$ of compacted connected Lie groups. In 1976 S. Halperin made a very
general conjecture which is one of the most important open problems in rational homotopy
theory ((24), (42)).

**Halperin Conjecture.** Suppose that $F \to E \to B$ is a fibration with simply-connected base $B$
and the (rational) cohomology algebra of the fibre is an Artinian algebra of the following form

$$H^*(F, \mathbb{Q}) = \mathbb{Q}[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_n)$$
then the Serre spectral sequence for this fibration collapses.

It is clear that $x_1, x_2, \ldots, x_n$ are weighted variables with weights of their dimensions and $f_1, f_2, \ldots, f_n$ are weighted homogeneous polynomials from the topological background. Actually the above conjecture is implied by the following conjecture about the nonexistence of negative weight derivation ((24), (42)).

**Halperin Conjecture** (equivalent form). Let $x_1, x_2, \ldots, x_n$ be weighted variables and $f_1, f_2, \ldots, f_n$ be weighted homogeneous polynomials in $P$. Suppose that $R$ is an Artinian algebra of the form

$$\mathbb{C}[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_n).$$

Then there is no non-zero negative weight derivation on $R$.

This conjecture has been proved when the fibre is homogeneous ((13), (42)) and $n \leq 2$ ((65), (66) and (16)).

On the other hand S. S.-T. Yau has the following conjecture about the nonexistence of the negative weight derivation from his work about Lie algebras of derivations on the moduli algebras of isolated hypersurface singularities, and especially his micro-local characterization (only use the Lie algebras of derivations on the moduli algebras) of quasi-homogeneous hypersurface singularities ((82), (83), (41), (18)) and (77)):

**Yau Conjecture 4.1.1.** Let $(V, 0) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n : f(x_1, x_2, \ldots, x_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial $f(x_1, x_2, \ldots, x_n)$. Then there is no non-zero negative weight derivation on the moduli algebra (= Milnor algebra here) $A(V) =$
\[ \mathbb{C}[x_1, x_2, \ldots, x_n]/(\partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n). \] i.e., the Yau algebra is non-negatively graded algebra.

In case \( f \) is a homogeneous polynomial, then it was shown in (77) that \( L(V) \) is a graded Lie algebra without negative weight. In fact they proved the following theorem.

**Theorem 4.1.1.** (77) Let \( A = \bigoplus_{i=0}^t A_i \) be a commutative Artinian local algebra with \( A_0 = \mathbb{C} \). Suppose that maximal ideal of \( A \) is generated by \( A_j \) for some \( j > 0 \). Then \( L(A) \) is a non-negatively graded Lie Algebra \( \bigoplus_{k=0}^t L_k \).

This conjecture has also been proved in the low-dimensional case \( n \leq 4 \) ((18), (15)) by explicit calculations.

**Theorem 4.1.2.** (18) Let \( f(x_1, x_2, x_3) \) be a weighted homogeneous polynomial of type \((w_1, w_2, w_3; d)\) with isolated singularity at the origin. Assume that \( d \geq 2w_1 \geq 2w_2 \geq 2w_3 \). Let \( D \) be a derivation of the moduli algebra

\[ \mathbb{C}[x_1, x_2, x_3]/(\partial f/\partial x_1, \partial f/\partial x_2, \partial f/\partial x_3). \]

Then \( D \equiv 0 \) if \( D \) is negatively weighted.

**Theorem 4.1.3.** (15) Let \( f(x_1, x_2, x_3, x_4) \) be a weighted homogeneous polynomial of type \((w_1, w_2, w_3, w_4; d)\) with isolated singularity at the origin. Assume that \( d \geq 2w_1 \geq 2w_2 \geq 2w_3 \geq 2w_4 \). Let \( D \) be a derivation of the moduli algebra

\[ \mathbb{C}[x_1, x_2, x_3, x_4]/(\partial f/\partial x_1, \partial f/\partial x_2, \partial f/\partial x_3, \partial f/\partial x_4). \]
Then $D \equiv 0$ if $D$ is negatively weighted.

However, Chen wrote in (15): “five variables case is very complicated than four variables and it seems that a general method (coordinate-free) is needed for the case of arbitrary dimension”. In fact the methods for proving Theorem 4.1.2 and Theorem 4.1.3 are very hard to be generalized to higher dimension.

Both conjectures of S. Halperin and S. S.-T. Yau are about 0-dimensional quasi-homogeneous singularities, i.e., graded Artinian algebras. The problem of nonexistence of negative weight derivation for positive-dimensional quasi-homogeneous singularities has also been considered from other motivations ((47), (73), (74), (75)). In (26) and (27) the nonexistence of negative weight derivation was proved for isolated quasi-homogeneous hypersurface singularities and quasi-homogeneous curve singularities. Kantor proved

(a) (26): If $A = \mathbb{C}[t^{m_1}, \cdots, t^{m_r}]$ is a non-regular monomial curve, then $A$ has no derivations of negative weight.

(b) (27): If $A = \mathbb{C}[X_1, \cdots, X_n]/(f)$ is a quasi-homogeneous hypersurface with isolated singularity and normalized grading, then $A$ has no derivations of negative weight, and the derivations of weight 0 can be read off the quadratic terms of $f$.

In (74), Wahl Proved that suppose $A$ is a graded normal surface singularity, and $A$ is not isomorphic to a cyclic quotient $\mathbb{C}[X, Y]^G (G \subset GL(2, \mathbb{C})$ cyclic). Then $\Delta$ is the only derivation of weight $\leq 0$.

In (74), Wahl proposed a very general conjecture about the nonexistence of negative weight derivation for positive-dimensional weighted-homogeneous singularities.
One special case of his conjecture for singular cones led him to give a beautiful cohomological characterization of complex projective space (75). Another special case of Wahl conjecture for complete intersections, which is the generalization of the well-known result in (27) was solved by A. G. Aleksandrov in (1). The answer is that: in the case of low-degree the negative weight derivations do exist; however except for those low-degree cases Wahl conjecture is true (also see pp. 34–35 (9)).

In this thesis, we deal with Yau conjecture 4.1.1 for fewnomial singularities. The concept of fewnomial was introduced by A. Khovanski (28). Namely, we say that a polynomial \( f \) in \( n \) variables is a fewnomial if the number of monomials appearing in \( f \) does not exceed \( n \). It is easy to show that, except for certain trivial cases, a fewnomial in \( n \) variables can define an IHS only if it has exactly \( n \) monomials, in which case we speak of fewnomial isolated singularity. In other words, fewnomial singularities are those which can be defined by \( n \) monomials in \( n \) indeterminates. Simple singularities are obviously fewnomial in this sense.

In this paper we prove the following result which give positive answer to Halperin, Yau and Wahl conjectures for fewnomial singularities.

**Theorem E.** Let \( f \in \mathbb{C}\{x_1, \cdots, x_n\} \) be a weighted homogeneous fewnomial isolated singularity with positive weights \( w_1, w_2, \cdots, w_n \) and multiplicity at least 5. Let

\[
A(f) = \mathbb{C}\{x_1, \cdots, x_n\}/(f_{x_1}, f_{x_2}, \cdots, f_{x_n})
\]

be the Milnor algebra of \( f \). Then there is no non-zero negative weight derivation on \( A(f) \).
4.2 Duality for Zero-dimensional Singularities

Let $k$ be a field of characteristic zero and let $A$ be a local analytic $k$-algebra, i.e., a factor algebra of the convergent power series ring $H = k[x_1, \cdots, x_n]$ in $n$ variables over $k$. We shall denote the maximal ideal of $A$ by $m_A$ and the module of regular holomorphic differential 1-forms of $H$ by $\Omega^1_{H/k} \cong H\{dx_1, \cdots, dx_n\}$.

Let $I$ be an ideal of $H$ and let $A = H/I$ be a factor algebra of $H$. One can define the module $\Omega^1_{A/k}$ of Kähler differentials of $A$ over $k$ by the following standard exact sequence of $A$-modules

$$I/I^2 \xrightarrow{d} \Omega^1_{H/k} \otimes H A \xrightarrow{\Omega^1_{A/k}} 0,$$

(2.1)

where $d$ is given by $d(\bar{f}) = d_{H/k}(\bar{f}) \otimes 1$ for $\bar{f} \in I/I^2$ and $d_{H/k} : H \to \Omega^1_{H/k}$ is the universal differential.

Let us consider the functor $\text{Der}_k(A, -)$ of $k$-derivations (from $A$-modules to $A$-modules). Then there is a fundamental functorial isomorphism $\text{Der}_k(A, -) \cong \text{Hom}_A(\Omega^1_{A/k}, -)$. We put $\text{Der}(A) = \text{Der}_k(A, A)$ so that $\text{Der}(A)$ is the $A$-module of $k$-derivations of $A$.

Remark: Suppose that the ideal $I$ is generated by the sequence of functions $f_1, \cdots, f_k \in H$. Then

$$\Omega^1_A \cong \Omega^1_H / (\sum_{j=1}^k f_j \cdot \Omega^1_H + H \cdot df_j).$$

Let $k$ be the field $\mathbb{C}$ of complex numbers. Then the local $\mathbb{C}$-algebra $A = H/I$ corresponds to the germ $X \subset (\mathbb{C}^n, 0)$ with the dual analytic $\mathbb{C}$ algebra $\mathcal{O}_X \cong A$. The modules $\Omega^1_A$ and $\text{Der}(A)$ are usually called the module of regular holomorphic differential 1-forms and the module of
holomorphic vector fields on the germ $X$ respectively. They are also denoted by $\Omega^1_\chi$ and $\text{Der}(X)$.

We will also consider the tangent and cotangent functors of analytic algebras (see [Pa]) denoted by $T_i$ and $T^i$ respectively, $i \geq 0$. By definition, for any $A$-module $M$ there exist the following isomorphisms

$$T_0(A/k, M) \cong \Omega^1_{A/k} \otimes_A M, \quad T^0(A/k, M) \cong \text{Der}_k(A, M)$$

and exact sequences of $A$-modules

$$0 \to T^1(A/k, M) \to I/I^2 \otimes_A M \xrightarrow{d \otimes 1_m} \Omega^1_{H/k} \otimes_H M \to T_0(A/k, M) \to 0$$

The first sequence contains (2.1) tensored with $M$ over $A$. Applying the functor $\text{Hom}_A(-, M)$ to (2.1), we get the second sequence.

For brevity we shall denote the tangent and cotangent modules $T_i(A/k, A)$ and $T^i(A/k, A)$ by $T_i(A)$ and $T^i(A)$ respectively.

Hence

$$T_0(A) \cong \Omega^1_{\chi}, \quad T^0(A) \cong \text{Der}(A), \quad T_1(A) \cong \text{Ker}(d).$$
Moreover, if \( A = A_{\text{red}} \) is reduced and an \( A \) module \( M \) has no embedded associative primes then there is an isomorphism (see (30), (1.4.3))

\[
T^1(A/k, M) \cong \text{Ext}_A^1(\Omega^1_{A/k}, M).
\]

This is also true if instead of the above condition on \( A \) we shall assume that the analytic \( \mathbb{C} \)-algebra \( A \) correspond to the germ \( X \subset (\mathbb{C}^m, 0) \) has positive depth along its singular locus.

For convenience we also recall the following assertion (see (2), (2.2)).

**Proposition 4.2.1.** (2). Let \( A = H/I \) be an Artinian complete intersection, that is, \( I \) is generated by a regular \( H \) sequence \( f_1, \cdots, f_n \in H \). Then the tangent and cotangent module \( T_i(A) \) and \( T^j(A) \) have the same dimension for \( i, j = 0, 1 \), i.e.,

\[
\dim_k \Omega^1_{A/k} = \dim_k T^1(A/k, A) = \dim_k \text{Der}(A, A) = \dim_k T^1(A/k, A).
\]

**Definition.** Let \( f \in H \) be an analytic function with isolated critical point at the origin. The ideal \( I \) generated by partial derivatives \( f_{x_i} = \partial f / \partial x_i, i = 1, \cdots, n \) is called the gradient ideal. In this case the \( H \)-sequence \( f_{x_1}, \cdots, f_{x_n} \) is regular so that \( A = H/I \) is an Artinian complete intersection.

**Theorem 4.2.1.** (1) Let \( I \) be the gradient ideal defined by an analytic function \( f \in H \) with isolated critical point at the origin. Let \( A = H/I \). Then there exist two canonical non-degenerate pairings

\[
T^0(A) \times T^1(A) \to k,
\]
\[ T_0(A) \times T_1(A) \rightarrow k. \]

**Remark.** Using elementary properties of tangent cohomology one may easily calculate an explicit representation of the pairings from the above theorem:

\[
\text{Der}(A) \times A^n / \text{Hess}(f) \cdot A^n \rightarrow A \rightarrow k
\]

(\[ \nu_1, \cdots, \nu_n \times (a_1, \cdots, a_n) \mapsto \sum_i \nu_i a_i \mapsto k \]

where

\[
\text{Hess}(f) = \begin{pmatrix}
\frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\
\frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n}
\end{pmatrix}
\]

and the last map is the projection on the one dimensional component of the algebra \( A \) which is called socle that corresponds to the class of \( \text{Det}(\text{Hess}(f)) = df_{x_1} \wedge \cdots \wedge df_{x_n} / dx_1 \wedge \cdots \wedge dx_n \).

### 4.3 Some Related Useful Results

The following Theorem which we want to use later is proved by K. Saito in ((53)).

**Theorem 4.3.1.** (53) Let \( f \in \mathcal{O}_{\mathbb{C}^n,0} \) be a germ of a holomorphic function which defines a hypersurface with isolated singularity at 0. Then

(1) \[ \text{Det}(\partial^2 f / \partial x_i \partial x_j)_{i,j=1,\ldots,n} \notin (f_{x_1}, \cdots, f_{x_n}) \mathcal{O}_{\mathbb{C}^n,0} \]

and

\[ m\text{Det}(\partial^2 f / \partial x_i \partial x_j)_{i,j=1,\ldots,n} \in (f_{x_1}, \cdots, f_{x_n}) \mathcal{O}_{\mathbb{C}^n,0} \]
where $m$ is the maximal ideal of $\mathcal{O}_{\mathbb{C}^n,0}$.

(2) For each $g \in \mathcal{O}_{\mathbb{C}^n,0}$ with

$$g \notin (f_{x_1}, \cdots, f_{x_n})\mathcal{O}_{\mathbb{C}^n,0}$$

there is a $h \in \mathcal{O}_{\mathbb{C}^n,0}$ such that

$$hg - \text{Det}(\partial^2 f/\partial x_i \partial x_j)_{i,j=1,\cdots,n} \in (f_{x_1}, \cdots, f_{x_n})\mathcal{O}_{\mathbb{C}^n,0}$$

(3) If $f$ is weighted homogeneous polynomial with weight $(w_1, \cdots, w_n; d)$, then

$$\text{Det}(\partial^2 f/\partial x_i \partial x_j)_{i,j=1,\cdots,n}$$

is also a weighted homogeneous polynomial with the same weight type $(w_1, \cdots, w_n)$ as $f$ and of degree $nd - 2 \sum_{i=1}^n w_i$.

The following concepts and results enable one to compute the Yau algebras of many concrete singularities we are going to consider. Let $A, B$ be associative algebras over a field $k$ of characteristic zero which in the sequel will be $\mathbb{C}$. Recall that the multiplication algebra $M(A)$ of $A$ is defined as the subalgebra of endomorphisms of $A$ generated by the identity element and left and right multiplications by elements of $A$. The centroid $C(A)$ is the set of endomorphisms of $A$ which commute with all elements of $M(A)$. Clearly, $C(A)$ is a unital subalgebra of $\text{End}(A)$. 
The following statement is a particular case of a general result from Proposition 1.2. of (12). Let $S = A \otimes B$ be a tensor product of finite dimensional associative algebras with units. Then

$$\text{Der}S \cong (\text{Der}A) \otimes C(B) + C(A) \otimes (\text{Der}B).$$

We will only use this result for commutative associative algebras with unit, in which case the centroid coincides with the algebra itself. Thus for commutative associative algebras $A, B$ one has:

**Theorem 4.3.2.** For commutative associative algebras $A, B$,

$$\text{Der}S \cong (\text{Der}A) \otimes B + A \otimes (\text{Der}B) \quad (3.3)$$

This formula will be used in the sequel.

**Definition 4.3.1.** For an ideal $J$ in an analytic algebra $S$, denote by $\text{Der}_J S \subseteq \text{Der}_S S$ the Lie subalgebra of all $\sigma \in \text{Der}_S S$ for which $\sigma(J) \subseteq J$.

We shall use the following well-known result to compute the derivations.

**Theorem 4.3.3.** Let $J$ be an ideal in $R = \mathbb{C}\{x_1, \cdots, x_n\}$. Then there is a natural isomorphism of Lie algebras

$$(\text{Der}_J R)/(J \cdot \text{Der}_\mathbb{C} R) \cong \text{Der}_\mathbb{C}(R/J).$$

**Proof.** By definition, there is a map $\varphi : \text{Der}_J R \to \text{Der}_\mathbb{C}(R/J)$ whose kernel contains $J \cdot \text{Der}_\mathbb{C} R$. Note that $\text{Der}_\mathbb{C} R$ is a free $R$-module with basis $\partial/\partial x_1, \cdots, \partial/\partial x_n$ and that the coefficient of
\(\partial/\partial x_i\) in \(\sigma \in \text{Der}_C R\) is \(\sigma(x_i)\). So if \(\sigma \in \text{Ker}\phi\), then \(\sigma(x_i) \in J\) and hence \(\sigma \in J \cdot \text{Der}_C R\). This proves injectivity. By a result of Scheja and Wiebe (54), any \(\bar{\sigma} \in \text{Der}_C(R/J)\) lifts to a \(\sigma \in \text{Der}_C R\) which is then necessarily in \(\text{Der}_J R\). This proves surjectivity and the claim follows.

\[\Box\]

4.4 **Weighted Homogenous Fewnomial Isolated Singularities**

We first recall the setting of the so-called fewnomials introduced in (28). Let us first establish precise terminology which will be different from the setting of (28) where the term fewnomial was introduced. Let \(P\) be a polynomial in \(n\) variables. We shall say that \(P\) is a fewnomial if the number of monomials entering in \(P\) does not exceed \(n\). Obviously, the number of monomials in \(P\) may depend on the system of coordinates. In order to obtain a rigorous concept we shall only admit linear changes of coordinates and say that \(P\) (or rather its germ at the origin) is a \(k\)-nomial if \(k\) is the smallest natural number such that \(P\) becomes a polynomial consisting of \(k\) monomials after (possibly) a linear change of coordinates. For linguistic flexibility it is convenient to say in such case that the nomiality \(\text{nom}(P)\) of \(P\) is equal to \(k\). Nomiality may be considered as a sort of elementary complexity measure of polynomials which appears relevant in some problems of enumerative algebraic geometry (28). An isolated hypersurface singularity \((V,0)\) is called \(k\)-nomial if there exists an IHS \(Y\) analytically isomorphic to \(V\) which can be defined by a \(k\)-nomial and \(k\) is the smallest such number. It turns out then, except for some non-interesting cases, a singularity defined by a fewnomial \(P\) can be isolated only if \(\text{nom}(P) = n\), i.e., if \(P\) is a \(n\)-nomial in \(n\) variables. We shall formulate this result separately for further reference.
Lemma 4.4.1. (52) Fix an \( i \in \{1, \cdots, n\} \). For an isolated singularity \( f \), at least one of the monomials of the form \( x_i^a x_j, a \geq 1, j = 1, \cdots, n \) appears in the series \( f \) with a nonzero coefficient.

Lemma 4.4.2. (29) A \( k \)-nomial \( P \) in \( n \) variables which does not contain monomials of order less than three, cannot have an isolated critical point at the origin if \( k < n \).

Proof. Fix a number \( n \), by lemma 4.4.1 for each \( i \) there exists a monomial \( x_i^a x_j \) entering in \( P \). For each \( i \), fix a monomial of such form with the minimal \( j = j(i) \). Since there are no monomials of degree two, two monomials of such type chosen for two different numbers \( i_1 \neq i_2 \) cannot coincide. This obviously implies that the number of monomials in \( P \) cannot be less than the number of coordinates \( n \), which gives the result.

Remark: Using terminology of (10), the requirement that there are no quadratic terms can be expressed by saying that \( P \) is of (maximal) corank \( n \) at the origin. The reason why we have to exclude quadratic terms, is that otherwise the formulation given above would not be correct. Indeed, a stabilization of \( A_1 \) singularity can be defined by a polynomial in \( 2k \) variables of the form \( x_1 x_2 + \cdots + x_{2k-1} x_{2k} \) which contains only \( k \) monomials. Notice that polynomials of the form \( x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n} \) give evident examples of \( n \)-nomials with isolated singularity at the origin of \( \mathbb{C}^n \).

We introduce some terminology.
**Definition 4.4.1.** We say that an IHS in $\mathbb{C}^n$ is fewnomial if it can be defined by a $n$-nomial in $n$ variables and we say that it is a weighted homogenous fewnomial isolated singularity if it can be defined by a weighted homogenous fewnomial.

Notice that a direct sum of weighted homogenous fewnomial isolated singularity is also a weighted homogenous fewnomial isolated singularity. Moreover, according to (3.3) derivation algebras of direct sums can be easily computed. For this reason our strategy will be to prove the Theorem E for certain series of weighted homogeneous fewnomial isolated singularities and then extend it to direct sums of singularities from those series.

**Theorem 4.4.1.** Let $f$ be a weighted homogeneous fewnomial isolated singularity with $\text{mult}(f) \geq 3$. then $f$ analytically equivalent to a linear combination of the following three series.

- **Type A.** $x_1^{a_1} + x_2^{a_2} + \cdots + x_{n-1}^{a_{n-1}} + x_n^{a_n}, n \geq 1$
- **Type B.** $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1, n \geq 2$
- **Type C.** $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n}, n \geq 2$

**Proof.** Let’s first introduce a Lemma which is crucial part of the proof of the theorem.

**Lemma 4.4.3.** Let $f(x_1, \cdots, x_n)$ be a weighted homogeneous fewnomial which define an isolated singularity at origin with multiplicity at least 3. Then $x_1^{a_1} x_j(i_1), x_2^{a_2} x_j(i_2)$, where $j(i_1) \neq i_1$, $j(i_2) \neq i_2$, $i_1 \neq i_2$ and $j(i_1) = j(i_2)$, cannot both appear in the support of $f$. 
Proof. It is enough to prove that $x_1^{a_1}x_3$ and $x_2^{a_2}x_3$ cannot both appear in the support of $f$. Since $f$ is weighted homogeneous fewnomial with isolated critical point at the origin, in view of Lemma 4.4.1, $f$ can be written as

$$f = c_1x_1^{a_1}x_3 + c_2x_2^{a_2}x_3 + \sum_{k=3}^{n} c_k x_k^{a_k} x_{j(k)}, \ c_i \neq 0, \text{ for } 1 \leq i \leq n.$$ 

Then

$$\frac{\partial f}{\partial x_1} = c_1 a_1 x_1^{a_1 - 1} x_3 + g_1(x_3, \cdots, x_n)$$

$$\frac{\partial f}{\partial x_2} = c_2 a_2 x_2^{a_2 - 1} x_3 + g_2(x_3, \cdots, x_n)$$

$$\frac{\partial f}{\partial x_3} = c_1 x_1^{a_1} + c_2 x_2^{a_2} + c_3 a_3 x_3^{a_3 - 1} x_{j(3)} + g_3(x_4, \cdots, x_n)$$

$$\frac{\partial f}{\partial x_k} = c_k a_k x_k^{a_k - 1} x_{j(k)} + g_k(x_3, \cdots, x_n), \text{ for } k \geq 4,$$

where multiplicity of $g_i$ is at least one for $1 \leq i \leq n$. Clearly the singular set of $f$ is given by

$$\{(x_1, x_2, 0 \cdots, 0) : c_1 x_1^{a_1} + c_2 x_2^{a_2} = 0\}$$

which is not an isolated singularity. Hence $x_1^{a_1}x_3$ and $x_2^{a_2}x_3$ cannot both appear in the support of $f$. 

Then the theorem 4.4.1 is an immediately corollary of Lemma 4.4.1, Lemma 4.4.2 and Lemma 4.4.3 up to nonzero coefficients. After rescaling, we can make all the coefficients of the monomials in $f$ to be 1.
4.5 Proof of the Theorem E

**Proposition 4.5.1.** Let \( f = x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n} \) be a weighted homogeneous fewnomial isolated singularity of type A. Then there is no non-zero negative weight derivation on \( A(f) = \mathbb{C}\{x_1, \cdots, x_n\}/(f_{x_1}, f_{x_2}, \cdots, f_{x_n}) \) is the moduli algebra of \( f \).

*Proof.* Since

\[
A(f) := \mathbb{C}\{x_1, \cdots, x_n\}/(f_{x_1}, f_{x_2}, \cdots, f_{x_n}) = \mathbb{C}\{x_1, \cdots, x_n\}/(a_1 x_1^{a_1-1}, a_2 x_2^{a_2-1}, \cdots, a_n x_n^{a_n-1})
\]

\[
\cong \mathbb{C}\{x_1\}/(x_1^{a_1-1}) \otimes \mathbb{C}\{x_2\}/(x_2^{a_2-1}) \otimes \cdots \otimes \mathbb{C}\{x_n\}/(x_n^{a_n-1})
\]

By (3.3), it suffices to show that \( \mathbb{C}\{x\}/(x^{a_k-1}) \) has no negative weight derivations. By Theorem 4.3.3, It is easy to compute the \( \text{Der}(\mathbb{C}\{x\}/(x^{a_k-1})) \) which is spanned by \( x^i \partial x, 1 \leq i \leq a_k - 2, \)

\( 1 \leq k \leq n \). Each of those generators has nonnegative weight. Thus there is no non-zero negative weight derivation on \( A(f) \).

\[ \square \]

**Proposition 4.5.2.** Let \( f = x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1 \) \( n \geq 2 \) be a weighted homogeneous fewnomial isolated singularity of type B and \( \text{mult}(f) \geq 4 \). Let

\[
A(f) = \mathbb{C}\{x_1, \cdots, x_n\}/(f_{x_1}, f_{x_2}, \cdots, f_{x_n})
\]

be the moduli algebra of \( f \). Then there is no non-zero negative weight derivation on \( A(f) \).
In order to prove Proposition 4.5.2, we first introduce some lemmas.

In the moduli algebra $A(f) = \mathbb{C}\{x_1, x_2, \ldots, x_n\}/(f_{x_1}, f_{x_2}, \ldots, f_{x_n})$ we have the following relations:

\begin{align*}
x_1^{a_1} &= -a_2 x_2^{a_2-1} x_3 \\
x_2^{a_2} &= -a_3 x_3^{a_3-1} x_4 \\
\vdots \\
x_i^{a_i} &= -a_{i+1} x_{i+1}^{a_{i+1}-1} x_{i+2} \\
\vdots \\
x_{n-1}^{a_{n-1}} &= -a_n x_n^{a_n-1} x_1 \\
x_n^{a_n} &= -a_1 x_1^{a_1-1} x_2.
\end{align*} 

(5.4)

It follows that

\begin{align*}
x_1^{a_1} x_2 &= 0 \\
x_2^{a_2} x_3 &= 0 \\
\vdots \\
x_{n-1}^{a_{n-1}} x_n &= 0 \\
x_n^{a_n} x_1 &= 0.
\end{align*} 

(5.5)
It is well-known that $A(f) = \mathbb{C}\{x_1, x_2, \cdots, x_n\}/(f_{x_1}, f_{x_2}, \cdots, f_{x_n})$ is a vector space spanned by monomial bases $\{x_1^{k_1}x_2^{k_2} \cdots x_{n-1}^{k_{n-1}}x_n^{k_n}\}$, where $(k_1, k_2, \cdots, k_n) \in \{0 \leq k_i \leq a_i - 1, 1 \leq i \leq n\}$ (see (32)).

**Lemma 4.5.1.** Let $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1$ be a weighted homogeneous fewnomial isolated singularity of type B and mult$(f) \geq 4$, with positive weights $w_1, \cdots, w_n$. Then $w_i < \frac{d}{3}$ for $1 \leq i \leq n$.

**Proof.** Since mult$(f) \geq 4$, so $a_i \geq 3$ for $1 \leq i \leq n$. It follows from $aw_i + w_{i+1} = d, a_i \geq 3$ that $w_i < \frac{d}{3}$ for $1 \leq i \leq n$. \qed

The main ideal of the proof of Proposition 4.5.2 is using duality Theorem (see Theorem 4.2.1 and its remark). Since the pairing map in Theorem 4.2.1 is non-degenerate, it suffices to prove that for any negative derivation in Der$(A)$, when pairing with all elements in $A(f)^n$/Hess$(f)A(f)^n$ map to 0. We can simplify $A(f)^n$/Hess$(f)A(f)^n$ firstly.

We can see that the Hessian matrix Hess$(f)$ of $f$ is the following.

$$
\begin{pmatrix}
  a_1(a_1-1)x_1^{a_1-2}x_2 & a_1x_1^{a_1-1} & 0 & \cdots & 0 & a_nx_n^{a_n-1} \\
  a_2x_1^{a_1-1} & a_2(a_2-1)x_2^{a_2-2}x_3 & a_2x_2^{a_2-1} & \cdots & 0 & 0 \\
  0 & a_2x_2^{a_2-1} & a_3(a_3-1)x_3^{a_3-2}x_4 & \cdots & 0 & 0 \\
  0 & 0 & a_3x_3^{a_3-1} & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & a_{n-1}(a_{n-1}-1)x_{n-1}^{a_{n-1}-2}x_n & a_{n-1}x_{n-1}^{a_{n-1}-1} \\
  a_nx_n^{a_n-1} & 0 & 0 & \cdots & a_{n-1}x_{n-1}^{a_{n-1}-1} & a_{n-1}x_{n-1}^{a_{n-1}-2}x_1 
\end{pmatrix}
$$

By Theorem 4.3.1(3), the maximal weight of the monomial base of $A(f)$ is the class of Det(Hess$(f)$) which is called socle and has weight $nd - 2(w_1 + w_2 + \cdots + w_n)$. 
Then we can use relations (5.4) and (5.5) to simplify the $A(f)^n / \text{Hess}(f) A(f)^n$, we get

$$A(f)^n / \text{Hess}(f) A(f)^n \subseteq A(f)/I_1 \oplus A(f)/I_2 \oplus \cdots \oplus A(f)/I_n$$

where $I_i = \langle x_i^{a_i-1} x_{i+1} x_{i-1}, x_i^{a_i-1} x_{i+1} x_{i+2}, x_{i-2} x_{i-1} x_i^{a_i-1} x_{i+1}, 1 \leq i \leq n \text{ and } x_0 = x_n, x_{n-1} = x_n-1, x_{n+1} = x_1, x_{n+2} = x_2. \rangle$

We want to give a detail description how we get

$$I_1 = \langle x_1^{a_1-1} x_2 x_n, x_1^{a_1-1} x_2 x_3, x_{n-1} x_n x_1^{a_1-1} x_2, \rangle$$

For other $I_i$, $2 \leq i \leq n$, the argument is similar. In Hess($f$), there are only 3 columns with nonzero first coordinate, i.e., the first, second and the last column. We consider the first column which is $(a_1(a_1-1)x_1^{a_1-2} x_2, a_1 x_1^{a_1-1}, 0, \cdots, 0, a_n x_n^{a_n-1})^T$. By relation (5.5), in $A(f)$, $a_1 x_1^{a_1-1}$ is killed by $x_1 x_2$ and $a_n x_n^{a_n-1}$ is killed by $x_n x_1$. So if we multiply lcm($x_1 x_2, x_1 x_n$)=$x_1 x_2 x_n$ to this column, we get $(a_1(a_1-1)x_1^{a_1-1} x_2^2 x_3, 0, \cdots, 0)^T$. Similarly, from the second column, we can get $(a_1 x_1^{a_1-1} x_2^2 x_3, 0, \cdots, 0)^T$ and from the last column we can get $(a_n x_n-1 x_n^{a_n+1}, 0, \cdots, 0)^T$ which is $(-a_1 a_n x_1^{a_1-1} x_2 x_{n-1} x_n, 0, \cdots, 0)^T$ by relation (5.4). Thus we can take

$$I_1 = \langle x_1^{a_1-1} x_2 x_n, x_1^{a_1-1} x_2 x_3, x_{n-1} x_n x_1^{a_1-1} x_2, \rangle$$
Proposition 4.5.3. Let \( f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1, \) \( n \geq 2 \) be a weighted homogeneous fewnomial isolated singularity of type B and \( \text{mult}(f) \geq 4 \), with positive weights \( w_1, \cdots, w_n \). Then \( w_i + \text{maximal weight of monomial bases of } \mathcal{A}(f)/I_i \leq nd - 2(w_1 + w_2 + \cdots + w_n), \ 1 \leq i \leq n. \)

Proof. We give a detail proof for \( i = 1 \). For other \( 2 \leq i \leq n \), the argument is similar.

\[ \mathcal{A}(f)/I_1 = \mathcal{A}(f)/(x_1^{a_1-1}x_2^2x_n, x_1^{a_1-1}x_2^2x_3, x_{n-1}x_n^{a_1-1}x_2) \] has a grading. Since we already know the monomial bases of \( \mathcal{A}(f) \), so it is easy to obtain the candidates for maximal weight of graded bases of \( \mathcal{A}(f)/I_1 \) are:

- case (B) 1.1. \( x_1^{a_1-2}x_2^{a_2-1}\cdots x_n^{a_n-1} \)
- case (B) 1.2. \( x_1^{a_1-1}x_2x_3^{a_3-1}\cdots x_n^{a_n-1} \)
- case (B) 1.3. \( x_1^{a_1-1}x_2x_3^{a_3-1}\cdots x_{n-2}^{a_{n-2}}x_n^{a_n-1} \)
- case (B) 1.4. \( x_1^{a_1-1}x_2^{-1}x_4^{a_4-1}\cdots x_{n-1}^{a_{n-1}} \)
- case (B) 1.5. \( x_1^{a_1-1}x_3^{a_3-1}\cdots x_{n-1}^{a_{n-1}}x_n^{a_n-1} \)

Lemma 4.5.2. For case (B) 1.1-case (B) 1.5 above, the weight of the monomial bases is less than or equal to \( nd - 2(w_1 + \cdots + w_n) - \text{wt}(x_1) \).

Proof. For case (B) 1.1. It is obviously that \( \text{wt}(x_1^{a_1-2}x_2^{a_2-1}\cdots x_n^{a_n-1}) + \text{wt}(x_1) = nd - 2(w_1 + \cdots + w_n) \). So the lemma is true in this case.
For case (B) 1.2. We want to show that

\[ wt(x_1^{a_1-1} x_2^{a_2-1} \cdots x_n^{a_n-1}) + wt(x_1) \leq nd - 2(w_1 + \cdots + w_n) \]

i.e., \((a_1 - 1)w_1 + w_2 + (a_3 - 1)w_3 + \cdots + (a_{n-1} - 1)w_{n-1} + w_1 \leq nd - 2(w_1 + \cdots + w_n)\). The above inequality is equivalent to \(2w_1 + 2w_2 + w_3 + w_n \leq 2d\) in view of \(a_i w_i + w_{i+1} = d\), for \(1 \leq i \leq n\).

We need to show \(2w_1 + 2w_2 + w_3 + w_n \leq 2d\) which is true by Lemma 4.5.1.

For case (B) 1.3. We want to show that \(wt(x_1^{a_1-1} x_2^{a_2-1} \cdots x_{n-2}^{a_{n-2}-1} x_n^{a_n-1}) + wt(x_1) \leq nd - 2(w_1 + \cdots + w_n)\), which is equivalent to \(w_1 + 2w_2 + w_3 + w_{n-1} + w_n \leq 2d\). The result follows from Lemma 4.5.1.

For case (B) 1.4. We want to show that \(wt(x_1^{a_1-1} x_3^{a_3-1} x_4^{a_4-1} \cdots x_{n-1}^{a_{n-1}-1}) + wt(x_1) \leq nd - 2(w_1 + \cdots + w_n)\), which is equivalent to \(2w_1 + w_3 + w_4 + w_n \leq 2d\). The result follows from Lemma 4.5.1.

For case (B) 1.5. We want to show that \(wt(x_1^{a_1-1} x_3^{a_3-1} \cdots x_{n-1}^{a_{n-1}-1} x_n^{a_n-1}) + wt(x_1) \leq nd - 2(w_1 + \cdots + w_n)\), which is equivalent to \(w_1 + w_2 + w_3 \leq d\). The result follows from Lemma 4.5.1.

For \(2 \leq i \leq n\), the argument is similar as above. The candidates for maximal weight of graded bases of \(A(f)/I_i = A(f)/(x_{i-1} x_i^{a_i-1} x_{i+1}^{a_{i+1}}, x_i^{a_i-1} x_{i+1}^{a_{i+1}} x_{i+2}, x_{i-2} x_{i-1} x_i^{a_i-1} x_{i+1})\), \(2 \leq i \leq n\) are:
Remark: Here $a_i = a_j$ and $x_i = x_j$ for $i \equiv j \mod(n)$.

**Lemma 4.5.3.** For case(B) i.1-case(B) i.5 above, the weight of the monomial bases is less than or equal to $nd - 2(w_1 + \cdots + w_n) - wt(x_i)$.

**Proof.** The proof is the same as those of Lemma 4.5.2. □

The Proposition 4.5.3 follows from Lemma 4.5.3 immediately. □

Now we are ready to give a proof of Proposition 4.5.2.

**Proof of Proposition 4.5.2.** Let $D \in \text{Der}(A(f))$ be a negative weight derivation, we can write $D = \sum_{i=1}^{n} g_i \partial / \partial x_i$, where $g_i \in A(f)$ and $wt(g_i) < w_i$. By the Proposition 4.5.3, we have $wt(g_i) + \text{maximal weight of monomial bases of } A(f)/I_i < nd - 2(w_1 + w_2 + \cdots + w_n), 1 \leq i \leq n$. Since by Theorem 4.3.1(3), the weight of socle is $nd - 2(w_1 + w_2 + \cdots + w_n)$, thus $wt(g_i) + \text{maximal weight of monomial bases of } A(f)/I_i < wt(\text{socle})$. This means that the projection map (2.2)
will send the pairing of \( D \) and any element of \( A(f)^n/\text{Hess}(f) \cdot A(f)^n \) to zero. By Theorem 4.2.1 we conclude that \( D = 0 \). Thus \( \text{Der}(A(f)) \) has no negative weight derivations. Q.E.D.

**Proposition 4.5.4.** Let \( f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}, \ n \geq 2 \) be a weighted homogeneous fewnomial isolated singularity of type C, \( \text{mult}(f) \geq 5 \), and \( n \) is even. Let \( A(f) = \mathbb{C}\{x_1, \cdots, x_n\}/(f_{x_1}, f_{x_2}, \cdots, f_{x_n}) \) be the moduli algebra of \( f \). Then there is no non-zero negative weight derivation on \( A(f) \).

Remark: By Theorem 4.1.3, our main Theorem E is true for \( 2 \leq n \leq 4 \). So we can assume \( n \geq 5 \) in the proof of Proposition 4.5.4 and Proposition 4.5.5. However, for \( n \leq 4 \) the proof is almost the same.

In the moduli algebra \( A(f) = \mathbb{C}\{x_1, x_2, \cdots, x_n\}/(f_{x_1}, f_{x_2}, \cdots, f_{x_n}) \), we have the following relations:

\[
\begin{align*}
  x_1^{a_1} &= -a_2 x_2^{a_2-1} x_3 \\
  x_2^{a_2} &= -a_3 x_3^{a_3-1} x_4 \\
  &\vdots \\
  x_i^{a_i} &= -a_{i+1} x_{i+1}^{a_{i+1}-1} x_{i+2} \\
  &\vdots \\
  x_{n-2}^{a_{n-2}} &= -a_{n-1} x_{n-1}^{a_{n-1}-1} x_n \\
  x_{n-1}^{a_{n-1}} &= -a_n x_n^{a_n-1}.
\end{align*}
\]
and
\[ x_1^{a_1-1} x_2 = 0 \]
\[ x_2^{a_2} x_3 = 0 \]
\[ \vdots \]
\[ x_{n-1}^{a_{n-1}} x_n = 0 \]
\[ x_n^{a_n} = 0. \]  

It is also well-know that \( A(f) = \mathbb{C}\{x_1, x_2, \ldots, x_n\}/(f_{x_1}, f_{x_2}, \ldots, f_{x_n}) \) is a vector space spanned by monomial bases \( \{x_1^{k_1} x_2^{k_2} \cdots x_{n-1}^{k_{n-1}} x_n^{k_n}\} \), where \((k_1, k_2, \ldots, k_n)\) satisfy \( \{0 \leq k_1 \leq a_1 - 2, 0 \leq k_j \leq a_j - 1 \text{ for } 2 \leq j \leq n\} \)
\[ \cup \{ k_1 = a_1 - 1, k_2 = 0, 0 \leq k_3 \leq a_3 - 2, 0 \leq k_j \leq a_j - 1 \text{ for } 4 \leq j \leq n\} \]
\[ \cup \cdots \cup \{ k_{2j-1} = a_{2j-1} - 1, k_{2j} = 0, \text{ for } 1 \leq j \leq \frac{n}{2} \}, k_{2i+1} = a_{2i+1} - 1, 0 \leq k_{2j+2} \leq a_{2j+2} - 1, \text{ for } 1 \leq i \leq j \leq \frac{n}{2} \} \cup \cdots \cup \{ k_{2j-1} = a_{2j-1} - 1 \text{ and } k_{2j} = 0 \text{ for } 1 \leq j \leq \frac{n}{2} \} \) (see (32)).

Write down the Hess(\(f\)) =

\[
\begin{pmatrix}
 a_1(a_1-1)x_1^{a_1-2}x_2 & a_1x_1^{a_1-1} & 0 & \ldots & 0 & 0 \\
 a_1x_1^{a_1-1} & a_2(a_2-1)x_2^{a_2-2}x_3 & a_2x_2^{a_2-1} & \ldots & 0 & 0 \\
 0 & a_2x_2^{a_2-1} & a_3(a_3-1)x_3^{a_3-2}x_4 & \ldots & 0 & 0 \\
 0 & 0 & a_3x_3^{a_3-1} & \ldots & 0 & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & 0 & \ldots & a_{n-2}x_{n-2}^{a_{n-2}-1} & 0 \\
 0 & 0 & 0 & \ldots & a_{n-1}(a_{n-1}-1)x_n^{a_{n-1}-2}x_n & a_{n-1}x_{n-1}^{a_{n-1}-1} \\
 0 & 0 & 0 & \ldots & a_{n-1}x_{n-1}^{a_{n-1}-1} & a_n(a_n-1)x_n^{a_n-2}
\end{pmatrix}
\]
Lemma 4.5.4. Let \( f = x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} \), \( n \geq 2 \) be a weighted homogeneous fewnomial isolated singularity of type C and \( \text{mult}(f) \geq 3 \), with positive weights \( w_1, \ldots, w_n \), and \( n \) is even, then

\[
\begin{align*}
&\text{wt}(x_1^{a_1-2} x_2^{a_2-1} x_3^{a_3-1} x_4^{a_4-1} x_5^{a_5-1} x_6^{a_6-1} \cdots x_{n-1}^{a_{n-1}-1} x_n^{a_n-1}) \\
&\text{wt}(x_1^{a_1-1} x_2^{a_2-1} x_3^{a_3-1} x_4^{a_4-1} x_5^{a_5-1} x_6^{a_6-1} \cdots x_{n-1}^{a_{n-1}-1} x_n^{a_n-1}) \\
&\text{wt}(x_1^{a_1-1} x_2^{a_2-1} x_3^{a_3-1} x_4^{a_4-1} x_5^{a_5-1} x_6^{a_6-1} \cdots x_{n-1}^{a_{n-1}-1} x_n^{a_n-1}) \\
&\quad \vdots \\
&\text{wt}(x_1^{a_1-1} x_2^{a_2-1} x_3^{a_3-1} x_4^{a_4-1} x_5^{a_5-1} x_6^{a_6-1} \cdots x_{n-1}^{a_{n-1}-1} x_n^{a_n-1}) \\
&\text{wt}(x_1^{a_1-1} x_2^{a_2-1} x_3^{a_3-1} x_4^{a_4-1} x_5^{a_5-1} x_6^{a_6-1} \cdots x_{n-1}^{a_{n-1}-1} x_n^{a_n-1})
\end{align*}
\]

Remark: all monomials above are monomial bases in \( A(f) \).

Proof. We just check the first inequality, the other inequality are similar.

\[
\begin{align*}
&\text{wt}(x_1^{a_1-2} x_2^{a_2-1} x_3^{a_3-1} x_4^{a_4-1} x_5^{a_5-1} x_6^{a_6-1} \cdots x_{n-1}^{a_{n-1}-1} x_n^{a_n-1}) - \text{wt}(x_1^{a_1-1} x_2^{a_2-1} x_3^{a_3-1} x_4^{a_4-1} x_5^{a_5-1} x_6^{a_6-1} \cdots x_{n-1}^{a_{n-1}-1} x_n^{a_n-1}) = \\
&-w_1 + (a_2 - 1) w_2 + w_3 = d - w_1 - w_2. \quad \text{Since } \text{mult}(f) \geq 3, \ w_i < \frac{d}{2}, 1 \leq i \leq n, \text{ thus } d - w_1 - w_2 \geq 0 \n\end{align*}
\]
and the first inequality is proved. \( \square \)
Lemma 4.5.5. Let $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_n^{a_{n-1}}x_n + x_n^{a_n}$, $n \geq 2$ be a weighted homogeneous fewnomial isolated singularity of type C and $\text{mult}(f) \geq 4$, with positive weights $w_1, \ldots, w_n$, and $n$ is even. Then for any $1 \leq i \leq n$,

$$w_i + wt(x_1^{a_1-1}x_2^{a_2-1}x_3^{a_3-1}x_4^{a_4-1}x_5^{a_5-1}x_6^{a_6-1} \cdots x_n^{a_{n-1}-1}x_n^{a_n-1}) \leq nd - 2(w_1 + \cdots + 2w_n)$$

Proof. Notice that $wt(x_1^{a_1-2}x_2^{a_2-1}x_3^{a_3-1}x_4^{a_4-1}x_5^{a_5-1}x_6^{a_6-1} \cdots x_n^{a_{n-1}-1}x_n^{a_n-1}) = nd - 2(w_1 + \cdots + 2w_n)$. By Lemma 4.5.4, it suffices to show that, for any $1 \leq i \leq n$,

$$w_i + wt(x_1^{a_1-1}x_2^{a_2-1}x_3^{a_3-1}x_4^{a_4-1}x_5^{a_5-1}x_6^{a_6-1} \cdots x_n^{a_{n-1}-1}x_n^{a_n-1}) \leq$$

$$wt(x_1^{a_1-2}x_2^{a_2-1}x_3^{a_3-1}x_4^{a_4-1} \cdots x_n^{a_{n-1}-1}x_n^{a_n-1}) = nd - 2(w_1 + \cdots + 2w_n)$$  \hspace{1cm} (5.8)

which is equivalent to $d - (w_i + w_1 + w_2) \leq 0$. Since $\text{mult}(f) \geq 4$, so $a_i \geq 3$, then $w_i < \frac{d}{4}$, hence $d - (w_i + w_1 + w_2) \leq 0.$ \hfill \square

Remark: This Lemma 4.5.5 will help us to prove Proposition 4.5.5 below. Explicitly they are used to determine the candidates of maximal graded pieces in $A(f)^n/\text{Hess}(f) \cdot A(f)^n$. 

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Lemma 4.5.6. Let \( f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}, \ n \geq 2 \) be a weighted homogeneous fewnomial isolated singularity of type C, \( \text{mult}(f) \geq 5 \), with positive weights \( w_1, \ldots, w_n \). Let \( A(f) = \mathbb{C}\{x_1, \ldots, x_n\}/(f_1, f_2, \ldots, f_n) \) be the moduli algebra of \( f \). Then \( w_i < \frac{d}{4} \) for all \( 1 \leq i \leq n-1 \) and \( w_n \leq \frac{d}{5} \).

Proof. Similar argument as Lemma 4.5.1 will give the proof. \( \square \)

Proposition 4.5.5. Let \( f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}, \ n \geq 2 \) be a weighted homogeneous fewnomial isolated singularity of type C with positive weights \( w_1, \ldots, w_n \). Suppose that \( \text{mult}(f) \geq 5 \), and \( n \) is even. Let \( A(f) = \mathbb{C}\{x_1, \ldots, x_n\}/(f_1, f_2, \ldots, f_n) \) be the moduli algebra of \( f \). Then for any \( 1 \leq i \leq n \), maximal weight of monomial bases of \( A(f)/I_i \leq nd - 2(w_1 + w_2 + \cdots + w_n) \).

Before giving the proof of Proposition 4.5.5, we first prove several lemmas.

Similar as before, we can use the above relations (5.6) and (5.7) to simplify \( A(f)/\text{Hess}(f) \cdot A(f)^n \). We obtain (notice that we assume \( n \geq 5 \)):

\[
A(f)^n/\text{Hess}(f)A(f)^n \subseteq A(f)/I_1 \oplus A(f)/I_2 \oplus \cdots \oplus A(f)/I_n
\]

where \( I_1 = \langle x_1^{a_1-2}x_2^2 >, I_2 = \langle x_2^{a_2-1}x_3 >, I_3 = \langle x_2x_3^{a_3-1}x_4, x_3^{a_3-1}x_4^2 >, I_i = \langle x_{i-1}x_i^{a_i-1}x_{i+1}, x_i x_{i+1}x_{i+2} > \rangle \) where \( 4 \leq i \leq n-2 \). \( I_{n-1} = \langle x_n^{a_n-1}x_n^2, x_{n-3}x_n^{a_n-1}x_n >, I_n = \langle x_n^{a_n} > \).

For \( I_1 = \langle x_1^{a_1-2}x_2^2 > \). By Lemma 4.5.4 and Lemma 4.5.5, the candidates for maximal weight of bases of \( A(f)/I_1 \) are:
Lemma 4.5.7. The assumption is the same as in Proposition 4.5.5. Then for case(C) 1.1-
case(C) 1.3 above, the weight of the monomial bases is less or equal to
\[ nd - 2(w_1 + w_2 + \cdots + w_n) - wt(x_1). \]

Proof. Case(C) 1.1. It is obviously that
\[ wt(x_1^{a_1-3}x_2^{a_2-1} \cdots x_n^{a_n-1}) + wt(x_1) = nd - 2(w_1 + \cdots + w_n). \]
So the lemma is true in this case.

Case(C) 1.2. Notice that in this case, we want to show that
\[ wt(x_1^{a_1-2}x_2x_3^{a_3-1} \cdots x_n^{a_n-1}) + wt(x_1) \leq nd - 2(w_1 + \cdots + w_n). \]
This is equivalent to show \( w_1 + 2w_2 + w_3 \leq d \) in view of \( a_iw_i + w_{i+1} = d \), for \( 1 \leq i \leq n - 1 \) and
\( a_nw_n = d \). On the other hand \( w_1 + 2w_2 + w_3 \leq d \) is correct from Lemma 4.5.6.

Case(C) 1.3. We want to show that \( wt(x_1^{a_1-1}x_3^{a_3-2}x_4^{a_4-1} \cdots x_n^{a_n-1}) + wt(x_1) \leq nd - 2(w_1 + \cdots + w_n) \). This is equivalent to show \( 2w_1 + w_2 \leq d \). The last inequality is true obviously.

For \( I_2 = \langle x_2^{a_2-1}x_3 \rangle \). By Lemma 4.5.4 and Lemma 4.5.5, the candidates for maximal
weight of basis of \( A(f)/I_2 \) are:
Lemma 4.5.8. The assumption is the same as in Proposition 4.5.5. Then for case (C) 2.1-case (C) 2.3 above, the weight of the monomial bases is less or equal to \(nd - 2(w_1 + w_2 + \cdots + w_n)\).

Proof. Case (C) 2.1: It is obviously that \(wt(x_1^{a_1-2}x_2^{a_2-2}x_3^{a_3-1} \cdots x_{n-1}^{a_{n-1}}) + wt(x_2) = nd - 2(w_1 + \cdots + w_n)\). So the lemma is true in this case.

Case (C) 2.2: Notice that in this case, we want to show that

\[
wt(x_1^{a_1-2}x_2^{a_2-1}x_3^{a_3-1} \cdots x_{n-1}^{a_{n-1}}) + wt(x_2) \leq nd - 2(w_1 + \cdots + w_n),
\]

which is correct by Lemma 4.5.6.

Case (C) 2.3: We want to show that \(wt(x_1^{a_1-1}x_3^{a_3-2}x_4^{a_4-1} \cdots x_{n-1}^{a_{n-1}}) + wt(x_2) \leq nd - 2(w_1 + \cdots + w_n)\), This is equivalent to show \(w_1 + 2w_2 \leq d\) which is correct by Lemma 4.5.6.

For \(I_3 = \langle x_2x_3^{a_3-1}x_4, x_3^{a_3-1}x_4^2x_5 \rangle\). The candidates for maximal weight of basis of \(A(f)/I_3\) are:
Lemma 4.5.9. The assumption is the same as in Proposition 4.5.5. Then for case(C) 3.1-case(C) 3.4 above, the weight of the monomial bases is less or equal to $nd - 2(w_1 + w_2 + \cdots + w_n) - wt(x_3)$.

Proof. Case(C) 3.1: It is obviously that $wt(x_1^{a_1-2}x_2^{a_2-1}x_3^{a_3-1}x_4^{a_4-1} \cdots x_n^{a_n-1}) + wt(x_3) = nd - 2(w_1 + \cdots + w_n)$. So the lemma is true in this case.

Case(C) 3.2: Note that in this case, we want to show that

$$wt(x_1^{a_1-2}x_2^{a_2-1}x_3^{a_3-1}x_4^{a_4-1} \cdots x_n^{a_n-1}) + wt(x_3) \leq nd - 2(w_1 + \cdots + w_n)$$

This is equivalent to show $w_3 + w_4 + w_5 \leq d$ which is correct by Lemma 4.5.6.

Case(C) 3.3: We want to show that $wt(x_1^{a_1-2}x_2^{a_2-1}x_3^{a_3-1}x_4^{a_4-1}x_5^{a_5-1}x_6^{a_6-1}x_7^{a_7-1} \cdots x_n^{a_n-1}) + wt(x_3) \leq nd - 2(w_1 + \cdots + w_n)$. This is equivalent to show $w_2 + 2w_3 + w_5 + w_6 \leq 2d$ which is correct by Lemma 4.5.6.
Case(C) 3.4: We want to show that \( wt(x_1^{a_1-1}x_3^{a_3-2}x_4^{a_4-1}x_5^{a_5-1}x_6^{a_6-1}\cdots x_n^{a_n-1}) + wt(x_3) \leq nd - 2(w_1 + \cdots + w_n) \). This is correct by Lemma 4.5.5.

For \( I_4 = < x_3^{a_4-1}x_5^2, x_2x_3^{a_4-1}x_5, x_4^{a_4-1}x_5^2x_6 > \). The candidates for maximal weight of basis of \( A(f)/I_4 \) are:

\[
\begin{align*}
\text{case(C) 4.1: } & x_1^{a_1-2}x_2^{a_2-1}x_3^{a_3-1}x_4^{a_4-2}x_5^{a_5-1}\cdots x_n^{a_n-1} \\
\text{case(C) 4.2: } & x_1^{a_1-2}x_3^{a_3-1}x_4^{a_4-1}x_5x_6^{a_6-1}\cdots x_n^{a_n-1} \\
\text{case(C) 4.3: } & x_1^{a_1-2}x_2^{a_2-1}x_4^{a_4-1}x_5x_6^{a_6-1}\cdots x_n^{a_n-1} \\
\text{case(C) 4.4: } & x_1^{a_1-2}x_2^{a_2-1}x_3^{a_3-1}x_4^{a_4-1}x_6^{a_6-1}\cdots x_n^{a_n-1} \\
\text{case(C) 4.5: } & x_1^{a_1-2}x_2^{a_2-1}x_4^{a_4-1}x_5^{a_5-1}x_7^{a_7-1}\cdots x_n^{a_n-1}
\end{align*}
\]

Lemma 4.5.10. The assumption is the same as in Proposition 4.5.5. Then for case(C) 4.1-
case(C) 4.6 above, the weight of the monomial bases is less or equal to \( nd - 2(w_1 + w_2 + \cdots + w_n) - wt(x_4) \).

Proof. Case(C) 4.1: It is obviously that \( wt(x_1^{a_1-2}x_2^{a_2-1}x_3^{a_3-1}x_4^{a_4-2}x_5^{a_5-1}\cdots x_n^{a_n-1}) + wt(x_4) = nd - 2(w_1 + \cdots + w_n) \). So the lemma is true in this case.

Case(C) 4.2: Note that in this case, we want to show that

\[
wt(x_1^{a_1-2}x_3^{a_3-1}x_4^{a_4-1}x_5x_6^{a_6-1}\cdots x_n^{a_n-1}) + wt(x_4) \leq nd - 2(w_1 + \cdots + w_n).
\]
This is equivalent to show $w_2 + w_3 + w_4 + 2w_5 + w_6 \leq 2d$ which is correct by Lemma 4.5.6.

Case(C) 4.3: We want to show that

$$wt(x_1^{a_1-2}x_2^{a_2-1}x_4^{a_4-1}x_5x_6^{a_6-1}\cdots x_n^{a_n-1}) + wt(x_4) \leq nd - 2(w_1 + \cdots + w_n),$$

This is equivalent to show $w_3 + 2w_4 + 2w_5 + w_6 \leq 2d$ which is correct by Lemma 4.5.6.

Case(C) 4.4: We want to show that $wt(x_1^{a_1-2}x_2^{a_2-1}x_4^{a_4-1}x_5^{a_5-1}x_7^{a_7-1}\cdots x_n^{a_n-1}) + wt(x_4) \leq nd - 2(w_1 + \cdots + w_n)$. This is equivalent to show $w_4 + w_5 + w_6 \leq d$ which is correct by Lemma 4.5.6.

Case(C) 4.5: We want to show that $wt(x_1^{a_1-2}x_2^{a_2-1}x_4^{a_4-1}x_5^{a_5-1}x_7^{a_7-1}\cdots x_n^{a_n-1}) + wt(x_4) \leq nd - 2(w_1 + \cdots + w_n)$. This is equivalent to $w_3 + 2w_4 + w_6 + w_7 \leq 2d$ which is correct by Lemma 4.5.6.

For $I_i = < x_{i-1}x_1^{a_1-1}x_{i+1}^{a_{i+1}-1}, x_{i-2}x_{i-1}x_i^{a_i-1}x_{i+1}, x_i^{a_i-1}x_{i+1}x_{i+2} >$ where $4 \leq i \leq n - 2$. Similar as $i = 4$ case, the candidates for maximal weight of bases of $A(f)/I_i$ are:

case(C) 4.1. $x_1^{a_1-2}x_2^{a_2-1}\cdots x_{i-1}^{a_{i-1}-1}x_i^{a_i-2}x_{i+1}^{a_{i+1}-1}\cdots x_n^{a_n-1}$

\[case(C) 4.2. x_1^{a_1-2}x_2^{a_2-1}\cdots x_{i-2}^{a_{i-2}-1}x_i^{a_i-1}x_{i+1}x_{i+2}^{a_{i+2}-1}\cdots x_n^{a_n-1}\]

case(C) 4.3. $x_1^{a_1-2}x_2^{a_2-1}\cdots x_{i-2}^{a_{i-2}-1}x_i^{a_i-1}x_{i+1}x_{i+2}^{a_{i+2}-1}\cdots x_n^{a_n-1}$

\[case(C) 4.4. x_1^{a_1-2}x_2^{a_2-1}\cdots x_{i-1}^{a_{i-1}-1}x_i^{a_i-1}x_{i+2}^{a_{i+2}-1}\cdots x_n^{a_n-1}\]

\[case(C) 4.5. x_1^{a_1-2}x_2^{a_2-1}\cdots x_{i-2}^{a_{i-2}-1}x_i^{a_i-1}x_{i+3}^{a_{i+3}-1}\cdots x_n^{a_n-1}\]
Lemma 4.5.11. The assumption is the same as in Proposition 4.5.5. Then for case(C) i.1-case(C) i.5 above, the weight of the monomial bases is less or equal to $nd - 2(w_1 + w_2 + \cdots + w_n) - wt(x_i)$ where $4 \leq i \leq n - 2$.

Proof. The proof is the same as $I_4$. □

For $I_{n-1} = \langle x_n^{a_{n-1} - 1}, x_n^2, x_n^{-3}x_n^{-2}x_n^{a_{n-1} - 1}, x_n \rangle$. The candidates for maximal weight of basis of $A(f)/I_{n-1}$ are:

- case(C) $(n - 1).1$: $x_1^{a_1 - 2}x_2^{a_2 - 1} \cdots x_{n-2}^{a_{n-2} - 1}x_{n-1}^{a_{n-1} - 2}x_n^{a_n - 1}$
- case(C) $(n - 1).2$: $x_1^{a_1 - 2}x_2^{a_2 - 1} \cdots x_{n-3}^{a_{n-3} - 1}x_{n-1}^{a_{n-1} - 1}x_n$
- case(C) $(n - 1).3$: $x_1^{a_1 - 2}x_2^{a_2 - 1} \cdots x_{n-4}^{a_{n-4} - 1}x_{n-3}^{a_{n-3} - 1}x_{n-1}^{a_{n-1} - 1}x_n$
- case(C) $(n - 1).4$: $x_1^{a_1 - 2}x_2^{a_2 - 1} \cdots x_{n-3}^{a_{n-3} - 1}x_{n-2}^{a_{n-2} - 1}x_{n-1}^{a_{n-1} - 1}$

Lemma 4.5.12. The assumption is the same as in Proposition 4.5.5. Then for case(C) $(n - 1).1$-case(C) $(n - 1).4$ above, the weight of the monomial bases is less or equal to $nd - 2(w_1 + w_2 + \cdots + w_n) - wt(x_{n-1})$.

Proof. Case(C) $(n - 1).1$ : It is obviously that $wt(x_1^{a_1 - 2}x_2^{a_2 - 1} \cdots x_{n-2}^{a_{n-2} - 1}x_{n-1}^{a_{n-1} - 2}x_n^{a_n - 1}) + wt(x_{n-1}) = nd - 2(w_1 + \cdots + w_n)$. So the lemma is true in this case.

Case(C) $(n - 1).2$ : Note that in this case, we want to show that

$$wt(x_1^{a_1 - 2}x_2^{a_2 - 1} \cdots x_{n-3}^{a_{n-3} - 1}x_{n-1}^{a_{n-1} - 1}x_n) + wt(x_{n-1}) \leq nd - 2(w_1 + \cdots + w_n).$$
This is equivalent to $w_{n-2} + 2w_{n-1} + 2w_n \leq 2d$ which is correct by Lemma 4.5.6.

Case (C) $(n - 1).3$ : We want to show that

$$wt(x_1^{a_1-2} x_2^{a_2-1} \cdots x_{n-4}^{a_{n-4}-1} x_{n-2}^{a_{n-2}-1} x_{n-1}^{a_{n-1}-1} x_n) + wt(x_{n-1}) \leq nd - 2(w_1 + \cdots + w_n).$$

This is equivalent to $w_{n-3} + w_{n-2} + w_{n-1} + 2w_n \leq 2d$ which is correct by Lemma 4.5.6.

Case (C) $(n - 1).4$ : We want to show that $wt(x_1^{a_1-2} x_2^{a_2-1} \cdots x_{n-2}^{a_{n-2}-1} x_{n-1}^{a_{n-1}-1}) + wt(x_{n-1}) \leq nd - 2(w_1 + \cdots + w_n)$. This is equivalent to we get $w_{n-1} + w_n \leq d$ which is correct by Lemma 4.5.6.

For $I_n = < x_{n-1}^{a_{n-1}} >$. The candidates for maximal weight of bases of $A(f)/I_n$ are:

- **case (C) n.1**: $x_1^{a_1-2} x_2^{a_2-1} \cdots x_{n-2}^{a_{n-2}-1} x_{n-1}^{a_{n-1}-1} x_n^{a_n-2}$
- **case (C) n.2**: $x_1^{a_1-2} x_2^{a_2-1} \cdots x_{n-2}^{a_{n-2}-1} x_{n-1}^{a_{n-1}-1} x_n^{a_n-1}$

**Lemma 4.5.13.** The assumption is the same as in Proposition 4.5.5. Then for case (C) n.1-case (C) n.2 above, the weight of the monomial bases is less or equal to $nd - 2(w_1 + w_2 + \cdots + w_n)$ - $wt(x_n)$

**Proof.** Case (C) n.1: It is obviously that $wt(x_1^{a_1-2} x_2^{a_2-1} \cdots x_{n-1}^{a_{n-1}-1} x_n^{a_n-2}) + wt(x_n) = nd - 2(w_1 + \cdots + w_n)$. So the lemma is true in this case.
Case(C) n.2: Note that in this case, we want to show that

\[ \text{wt}(x_1^{a_1-2}x_2^{a_2-1} \cdots x_{n-2}^{a_{n-2}-1}x_n^{a_n-1}) + \text{wt}(x_n) \leq nd - 2(w_1 + \cdots + w_n). \]

This is equivalent to \( w_{n-1} + 2w_n \leq 2d \) which is correct by Lemma 4.5.6.

Now we can give the proof of Proposition 4.5.5 easily.

**Proof.** The Proposition 4.5.5 follows from Lemma 4.5.7, Lemma 4.5.8, Lemma 4.5.9, Lemma 4.5.11, Lemma 4.5.12 and Lemma 4.5.13 immediately.

Now we can give the proof of Proposition 4.5.4.

**Proof.** Let \( D \in \text{Der}(A(f)) \) be a negative weight derivation. We can write \( D = \sum_{i=1}^{n} g_i \partial / \partial x_i \), where \( g_i \in A(f) \) and \( \text{wt}(g_i) < w_i \). By the Proposition 4.5.5, we have \( \text{wt}(g_i) + \text{maximal weight of monomial bases of } A(f)/I_i < nd - 2(w_1 + w_2 + \cdots + w_n) \), \( 1 \leq i \leq n \). Since by Theorem 4.3.1(3), the weight of socle is \( nd - 2(w_1 + w_2 + \cdots + w_n) \), thus \( \text{wt}(g_i) + \text{maximal weight of monomial bases of } A(f)/I_i < \text{wt(socle)} \). This means that the projection map (2.2) will sent the pairing of \( D \) and any element of \( A(f)^n/\text{Hess}(f)A(f)^n \) to zero. By Theorem 4.2.1 we conclude that \( D = 0 \). Thus \( \text{Der}(A(f)) \) has no negative weight derivations.

**Proposition 4.5.6.** Let \( f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}, \ n \geq 2 \) be a weighted homogeneous fewnomial isolated singularity of type C, \( \text{mult}(f) \geq 5 \), and \( n \) is odd. Let \( A(f) = \overline{A(f)} \). Then \( \text{Der}(A(f)) \) has no negative weight derivations.
Let $\mathbb{C}\{x_1, \cdots, x_n\}/(f_{x_1}, f_{x_2}, \cdots, f_{x_n})$ be the moduli algebra of $f$. Then there is no non-zero negative weight derivation on $A(f)$.

**Proof.** The only difference of Proposition 4.5.4 and Proposition 4.5.6 is about whether $n$ is odd or even which will affect the monomial bases of $A(f)$. In case Proposition 4.5.4, $A(f) = \mathbb{C}\{x_1, x_2, \cdots, x_n\}/(f_{x_1}, f_{x_2}, \cdots, f_{x_n})$ is a vector space spanned by monomial basis

$$\left\{x_1^{k_1} x_2^{k_2} \cdots x_{n-1}^{k_{n-1}} x_n^{k_n}\right\}.$$ where $(k_1, k_2, \cdots, k_n) \in \{0 \leq k_1 \leq a_1 - 2, 0 \leq k_j \leq a_j - 1 \text{ for } 2 \leq j \leq n\} \cup \{k_1 = a_1 - 1, k_2 = 0, 0 \leq k_3 \leq a_3 - 2, 0 \leq k_j \leq a_j - 1 \text{ for } 4 \leq j \leq n\} \cup \cdots \cup \{k_{2j-1} = a_{2j-1} - 1, k_{2j} = 0, \text{ for } 1 \leq j \leq \frac{n-2}{2}\} \cup \cdots \cup \{k_{2j-1} = a_{2j-1} - 1 \text{ and } k_{2j} = 0 \text{ for } 1 \leq j \leq \frac{n-2}{2} \text{ and } 0 \leq k_n \leq a_n - 2\}$ (see page 4 (32)). However, the difference doesn’t affect the proof. Similar argument as Proposition 4.5.4 will give the proof of Proposition 4.5.6.

The following Proposition will be used in the proof of main Theorem.

**Proposition 4.5.7.** Let $f(x_1, \cdots, x_n)$ and $g(x_{n+1}, \cdots, x_m)$ be holomorphic functions with isolated singularity at origin in $\mathbb{C}^n$ and $\mathbb{C}^{m-n}$. Let $A(f)$, $A(g)$ and $A(f+g)$ be the moduli algebra of $f$, $g$, and $f+g$ respectively. If $f(x_1, \cdots, x_n)$ is a weighted-homogeneous holomorphic function with an isolated singularity at origin. Then $A(f+g) \cong A(f) \otimes A(g)$. 
Proof.

\[ A(f + g) := \mathbb{C}\{x_1, \ldots, x_m\}/(f + g, fx_1, \ldots, fx_n, gx_{n+1}, \ldots, gx_m) \]
\[ = \mathbb{C}\{x_1, \ldots, x_m\}/(f, f_{x_1}, \ldots, f_{x_n}, g_{x_{n+1}}, \ldots, g_{x_m}) \]
\[ \cong \mathbb{C}\{x_1, \ldots, x_n\}/(f_{x_1}, \ldots, f_{x_n}) \otimes \mathbb{C}\{x_{n+1}, \ldots, x_m\}/(g, g_{x_{n+1}}, \ldots, g_{x_m}) \]
\[ = A(f) \otimes A(g) \]

The second and last equalities come from the fact that \( f \) is weighted homogeneous singularities.

\[ \square \]

**Theorem E.** Let \( f \in \mathbb{C}\{x_1, \ldots, x_n\} \) be a weighted homogeneous fewnomial isolated singularity with positive weights \( w_1, w_2, \ldots, w_n \) and multiplicity at least 5. Let

\[ A(f) = \mathbb{C}\{x_1, \ldots, x_n\}/(fx_1, fx_2, \ldots, fx_n) \]

be the Milnor algebra of \( f \). Then there is no non-zero negative weight derivation on \( A(f) \).

**Proof.** Since \( f \in \mathbb{C}\{x_1, \ldots, x_n\} \) is a weighted homogeneous fewnomial isolated singularity, \( f \) is summation of the following three types.

Type A. \( x_1^{a_1} + x_2^{a_2} + \cdots + x_{n-1}^{a_{n-1}} + x_n^{a_n}, n \geq 1 \)

Type B. \( x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1, n \geq 2 \)

Type C. \( x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}, n \geq 2 \)
By Proposition 4.5.7, the moduli algebra $A(f)$ is a tensor product of those moduli algebras of above three types. By Theorem 4.3.2, it suffices to prove that the moduli algebra of above three types have no non-zero negative weight derivations. That is exactly what we have proved in Proposition 4.5.1, Proposition 4.5.2, Proposition 4.5.4 and Proposition 4.5.6. Therefore there are no non-zero negative weight derivations on $A(f)$. □
CHAPTER 5

CHARACTERIZATION OF ISOLATED COMPLETE INTERSECTION SINGULARITIES WITH $\mathbb{C}^*$-ACTION OF DIMENSION $N \geq 2$ BY MEANS OF GEOMETRIC GENUS AND IRREGULARITY

It is well known that geometric genus $p_g$ and irregularity $q$ are two important invariants for isolated singularities. In this chapter we give a formula relating $p_g$ and $q$ for isolated singularities with $\mathbb{C}^*$-action in any dimension. We also give a simple characterization of the quasi-homogeneous isolated complete intersection singularities using $p_g$ and $q$. As a corollary, we prove that $q$ is an invariant topological type for two-dimensional weighted homogeneous hypersurface singularities.

5.1 Introduction

Let $(V,0)$ be a Stein germ of an analytic space with an isolated singularity at $0$. $(V,0)$ is a singularity with a (good) $\mathbb{C}^*$-action if the complete local ring of $V$ at $0$ is the completion of a (positively) graded ring. $(V,0)$ is a quasi-homogeneous singularity if there exists an analytic isomorphism type of $(V,0)$ which is defined by weighted homogeneous polynomials.

Let $f \in \mathbb{C}\{z_0, z_1, \cdots, z_n\}$ be a holomorphic function germ with an isolated singularity at the origin. It is well know that

$$\mu = \dim \mathbb{C}\{z_0, z_1, \cdots, z_n\}/(\partial f/\partial z_0, \cdots, \partial f/\partial z_n)$$
and

$$\tau = \dim \mathbb{C}\{z_0, z_1, \ldots, z_n\}/(f, \partial f/\partial z_0, \ldots, \partial f/\partial z_n)$$

are two very important invariants for hypersurface singularities. Clearly, $\mu \geq \tau$, and the equality holds if and only if $f$ is quasi-homogeneous singularity by a well know theorem of Saito (52).

Both $\mu$ and $\tau$ can also be defined for $n$-dimensional isolated complete intersection singularity (ICIS) with $n \geq 1$ in the following manner:

$$\mu = \text{rk} H_n(F)$$

and

$$\tau = \dim T_{V,0}^1,$$

where $F$ is the Milnor fibre of a Milnor fibration of $(V, 0)$ (see (40)), and $\tau$ is the dimension of the base space of a semi-universal deformation of $(V, 0)$. From the defining equations of $(V, 0)$, one can give formulae for $\mu$ and $\tau$ as dimensions of certain finite length modules, but it is no longer clear what is the relation between these invariants. This problem was first considered by Greuel (22), who conjectured $\mu \geq \tau$, and proved the inequality in two cases: $n = 1$ or the link of $V$ a rational homotopy sphere. Greuel also proved that (in every dimension) $\mu = \tau$ if $(V, 0)$ is quasi-homogeneous. Looijenga (39) proved that for ICIS of dimension $n = 2$, $\mu \geq \tau + b$ where $b =$ number of loops in the resolution dual graph of $(V, 0)$. Then Looijenga and Steenbrink (40) generalized this result for all $n \geq 2$. In (71), Wahl proved that for two-dimensional isolated
complete intersection singularity \((V,0)\), \(\mu \geq \tau + b\) and \(\mu = \tau + b\) if and only if \((V,0)\) is quasi-homogeneous (for \(b = 0\)) or \((V,0)\) is cusp (\(b = 1\)). More recently, Vossgaard (69) generalized this result for general \(n\). Let \((V,0)\) be an isolated complete intersection singularity of any dimension, he proved that \((V,0)\) is quasi-homogeneous if and only if \(\mu = \tau\).

Let \((V,0)\) be a normal surface singularity. Artin first defined an invariant geometric genus \(p_g\) for the singularity \((V,0)\). It turns out that this is an important invariant for the theory of normal surface singularities. In (88), Yau introduced another invariant called irregularity \(q\) of the singularity \((V,0)\). This invariant is interesting for the following reason. It is a long-term conjecture that normal surface singularities are not rigid, i.e., \(\dim T^1_V \geq 1\) where \(T^1_V\) is the set of isomorphism classes of first order infinitesimal deformations of \(V\). In the case of Gorenstein surface singularities this irregularity actually gives a lower bound for \(\dim T^1_V\). In fact, both geometric genus and irregularity can also be defined for general \(n\)-dimensional isolated singularities. Let \((V,0)\) be a normal isolated singularity of dimension \(n(\geq 2)\) with 0 as its only isolated singularity. Let \(\pi : \tilde{V} \to V\) be a resolution of the singularity of \(V\) with exceptional set \(E = \pi^{-1}(0)_{\text{red}}\). Then \(p_g := \dim R^{n-1}_*\pi_*\mathcal{O}_{\tilde{V}},\) and \(q = \dim H^0(\Omega_{\tilde{V}-E}^{n-1})/H^0(\Omega_{\tilde{V}}^{n-1})\).

In (88), Yau gave a formulae for the irregularity in case \((V,0)\) is a hypersurface singularity or a two-dimensional singularity with \(\mathbb{C}^*\)-action. Moreover, for \(n\)-dimensional singularity with \(\mathbb{C}^*\)-action, Yau gave a lower estimate for irregularity in terms of geometric genus. He proved

\[ q \geq p_g - h^{n-1}(\mathcal{O}_E). \]
In this paper, one of our main results is to prove that the above inequality is actually an equality.

**Theorem F.** Let \((V, 0)\) be a normal isolated singularity of dimension \(n(\geq 2)\) with \(\mathbb{C}^*\)-action. Let \(\pi : \tilde{V} \to V\) be a good resolution of the singularity \((V, 0)\) with \(E = f^{-1}(V)_{\text{red}}\). Then
\[ q = p_g - h^{n-1}(\mathcal{O}_E). \]

A natural question is how to use \(p_g\) and \(q\) to characterize quasi-homogeneous singularities. We prove that the converse of the above theorem is also correct for non-Du Bois isolated complete intersection singularities.

**Remark 5.1.1.** For two-dimensional isolated Gorenstein singularity \((V, 0)\), then \((V, 0)\) is a Du Bois singularity if and only if \((V, 0)\) is either rational, simple elliptic or cusp (see (25)).

**Theorem G.** Let \((V, 0)\) be a normal isolated complete intersection singularities of dimension \(n(\geq 2)\), and \(\pi : \tilde{V} \to V\) be a good resolution of the singularity \((V, 0)\) with \(E = f^{-1}(0)_{\text{red}}\). If \(q = p_g - h^{n-1}(\mathcal{O}_E)\), then either \((V, 0)\) has a \(\mathbb{C}^*\)-action or \((V, 0)\) is a Du Bois singularity.

Theorem G is a generalization of Wahl’s well-known theorem in two-dimensional case (Theorem 1.9 (71)). Let \((V, 0)\) be a normal surface singularity, \(\pi : \tilde{V} \to V\) a good resolution, and \(E \subset \tilde{V}\) the (reduced) exceptional fibre. \(E\) is a union of smooth curves \(E_i, i = 1, \cdots, k\). Let \(g_i\) be genus of \(E_i\), \(g = \sum_{i=1}^{k} g_i\). Also define \(b = \) first Betti number of the dual graph of \(E\) (= number of loops). Then \(h^1(\mathcal{O}_E) = g + b, \dim H^1(E, \mathbb{C}) = 2g + b\). Denote the geometric genus by \(p_g = h^1(\mathcal{O}_V)\) and irregularity by \(q = \dim H^0(\Omega^1_{\tilde{V} - E})/H^0(\Omega^1_V)\).

Wahl introduced three other invariants \(\alpha, \beta, \gamma\) which are non-negative integers (see (1.8) (71)).
Theorem 5.1.1. (Wahl) Let $E \subset \hat{V} \to V$ be a good resolution of a normal surface singularity, with $p_g, g, b$ and $\alpha, \beta, \gamma \geq 0$ as above. Then the irregularity $q$ is given by

$$q = p_g - g - b - \alpha - \beta - \gamma.$$ 

Theorem 5.1.2. (Wahl) Let $(V,0)$ be a two-dimensional Gorenstein surface singularity. Then $\alpha = \beta = \gamma = 0$ iff either $(V,0)$ is quasi-homogeneous (so $b = 0$), or $(V,0)$ is a cusp (so $b = 1$).

Our Theorem G is a generalization of Theorem 5.1.2 to ICIS in arbitrary dimension.

In Section 5.2, we recall the basic propositions of geometric genus and irregularity. Theorem F and Theorem G are proved in Section 5.3 and Section 5.4 respectively. In Section 5.5, we give a formula of irregularity in terms of weights for weighted homogeneous polynomial in 3 variables. In particular, the irregularity is invariant of topological type in this case.

5.2 Geometric Genus and Irregularity

We first recall the concept of the geometric genus of normal isolated singularities; for more details the reader is referred to Laufer (33) and Yau (85), (86), (88).

Let $(V,0)$ be a normal isolated singularity of dimension $n(\geq 2)$ with 0 as its only isolated singularity. Let $\pi : \hat{V} \to V$ be a resolution of the singularity of $V$ with exceptional set $E = \pi^{-1}(0)_{\text{red}}$. We define $s^{(i)}, 1 \leq i \leq n$, of singularity $(V,0)$ to be $\dim \Gamma(\hat{V} - E, \Omega^{i}_{\hat{V}})/\Gamma(\hat{V}, \Omega^{i}_{\hat{V}})$ where $\Omega^{i}_{\hat{V}}$ is the sheaf of germs of holomorphic $i$-forms on $\hat{V}$.

Let $\bar{\Omega}_{\hat{V}}$ be the 0th direct image sheaf $\pi_{\ast}\Omega^{i}_{\hat{V}}$ of $\Omega^{i}_{\hat{V}}$. By Grauert’s direct image theorem((20), P.207), $\bar{\Omega}_{\hat{V}}$ is a coherent sheaf. Let $\iota : V - \{0\} \to V$ be the inclusion map. Then the 0-th direct
image sheaf $\tilde{\Omega}_V^i := \iota_* \Omega^i_{V - \{0\}}$ is coherent (59). Hence the quotient sheaf $\tilde{\Omega}_V^i / \tilde{\Omega}_V^i$ is coherent and supported on 0. $s^{(i)}$ is exactly $\dim \tilde{\Omega}_V^i / \tilde{\Omega}_V^i$. Therefore the invariants $s^{(i)}, 1 \leq i \leq n$, are indeed invariants of isolated singularities. Artin defined geometric genus $p_g$ of the singularity to be $\dim R^i \pi_* \mathcal{O}_V$. It is proved in (33), (34), (85) that $p_g = s^{(n)}$. Irregularity $q$ of singularity $(V, 0)$ is defined to be $s^{(n-1)}$.

The geometric genus of the singularity $(V, 0)$ can also be calculated using square integrable forms.

**Definition 5.2.1.** $\omega \in \Gamma(V - \{0\}, \mathcal{O}(K))$ is called square integrable if

$$\int_{W - \{0\}} \omega \wedge \bar{\omega} < \infty$$

where $W$ is any sufficiently small relatively compact neighborhood of 0 in $V$.

Let $L^2(V - \{0\}, \Omega^n)$ be the subspace of $\Gamma(V - \{0\}, \mathcal{O}(K)$ consisting of holomorphic $n$-form on $V - \{0\}$, which are square integrable near the origin. Then $\Gamma(V - \{0\}, \mathcal{O}(K))/L^2(V - \{0\}, \Omega^n)$ is a finite dimensional vector space. It can be shown that this integer is independent of the choice of the Stein neighborhoods (34).

**Theorem 5.2.1.** (Laufer(34), Yau(85), (86)) The geometric genus of a normal isolated singularity $(V, 0)$ is

$$p_g = \dim \Gamma(V - \{0\}, \mathcal{O}(K))/L^2(V - \{0\}, \Omega^n).$$
5.3 Proof of Theorem F

Let $E$ be a reduced divisor on a smooth manifold $\tilde{V}$, and let $\Omega^j_{\tilde{V}}(E)$ stand for the sheaf of differential $j$-forms on $\tilde{V}$ with at most simple poles along $E$.

**Definition 5.3.1.** A differential $j$-form with logarithmic poles along $E$ on an open subset $U \subset \tilde{V}$ is a meromorphic $j$-form $\omega$ regular on $\tilde{V} \setminus E$ and such that both $\omega$ and $d\omega$ have at most simple poles along $E$.

Differential $j$-forms with logarithmic poles along $E$ form a sheaf denoted by $\Omega^j_{\tilde{V}}(\log E)$. For any open subset $U \subset \tilde{V}$ we have

$$\Gamma(U, \Omega^j_{\tilde{V}}(\log E)) = \{ \omega \in \Gamma(U, \Omega^j_{\tilde{V}}(E)) : d\omega \in \Gamma(U, \Omega^{j+1}_{\tilde{V}}(E)) \}.$$

A normal crossing divisor $E$ in $\tilde{V}$, is a reduced divisor which is locally defined by an equation of the form $f = x_1 \cdots x_p$, where $x_1, \ldots, x_n$ are local coordinates for $\tilde{V}$, $p \leq n$.

If $E$ is a normal crossing divisor then $\Omega^j_{\tilde{V}}(\log E)$ is locally free sheaf. In this case a form $\omega \in \Omega^j_{\tilde{V}}(\log E)$ can be written locally in the following way

$$\omega = \sum_{1 \leq k_1 < \cdots < k_j \leq n} a_{k_1 \cdots k_j} \delta_{k_1} \wedge \cdots \wedge \delta_{k_j}$$

where

$$\delta_i = \begin{cases} \frac{dx_i}{x_i}, & \text{if } i \leq p \\ dx_i, & \text{if } i > p. \end{cases}$$
In particular, we have
\[ \Omega^j_V(\log E) = \wedge^j \Omega^1_V(\log E). \]

**Definition 5.3.2.** Let \( \tilde{V} \to V \) be a resolution of \((V,0)\) is called a good resolution if \( \tilde{V} \) is non-singular, \( E = \pi^{-1}(0)_{\text{red}} \) is a divisor with normal crossings, and \( \pi \) is a surjective and proper holomorphic map that restricts to an isomorphism \( \pi : \tilde{V} \setminus E \to V \setminus \{0\} \).

By Hironaka’s big theorem, there exists a good resolution for any singularity \((V,0)\).

**Theorem 5.3.1.** (Stenbrink(61)) Let \((V,0)\) be a normal isolated singularity of dimension \( n \), where \( V \) is a contractible Stein space. Let \( \pi : \tilde{V} \to V \) be a good resolution of the singularity \((V,0)\) with \( E = \pi^{-1}(0)_{\text{red}} \). Then
\[ H^q(\tilde{V}, E(\Omega^p_V(\log E))) = 0, \text{ for } p + q > n, \]

where \( E \) is the ideal sheaf of the divisor \( E \).

**Proof.** See Theorem 2b (61).

Since \( H^i(\tilde{V}, \mathcal{F})^* \) is dual to \( H^{n-i}_E(\tilde{V}, \mathcal{F}^* \otimes \omega_{\tilde{V}}) \) for any \( \mathcal{F} \) locally free sheaf on \( \tilde{V} \) and \((\mathcal{I}_E \Omega^p_V(\log E))^* \otimes \omega_{\tilde{V}} \cong \Omega^{n-p}_{\tilde{V}}(\log E) \) (see pp.99 (63)), by duality we have
\[ H^q_E(\tilde{V}, \Omega^p_V(\log E)) = 0, \text{ for } p + q < n. \]

**Theorem 5.3.2.** (Straten-Steenbrink (63)) Let \((V,0)\) be a normal isolated singularity of dimension \( n \), where \( V \) is a contractible Stein space. Let \( U = V \setminus \{0\} \). Let \( \pi : \tilde{V} \to V \) be a reso-
olution of the singularity \((V,0)\) with \(E = f^{-1}(0)_{\text{red}}\). Then the map \(d : H^0(\Omega_U^{n-1})/H^0(\Omega_{\tilde{V}}^{n-1}) \rightarrow H^0(\Omega_U^n)/H^0(\Omega_{\tilde{V}}^n(\log E))\) induced by differentiation is injective.

Proof. Since \(H^0(\Omega_{\tilde{V}}^{n-1}) = H^0(\Omega_{\tilde{V}}^{n-1}(\log E))\) (see Theorem 1.3 (63)), hence

\[
H^0(\Omega_U^{n-1})/H^0(\Omega_{\tilde{V}}^{n-1}) = H^0(\Omega_U^{n-1})/H^0(\Omega_{\tilde{V}}^{n-1}(\log E))
\]

and it is proved in (40) that the differential map

\[
H^1_E(\tilde{V}, \Omega_{\tilde{V}}^{n-1}(\log E)) \xrightarrow{\beta} H^1_E(V, \Omega_{\tilde{V}}^{n}(\log E))
\]

is injective. In the commutative diagram

\[
\begin{array}{ccc}
H^0(\Omega_U^{n-1})/H^0(\Omega_{\tilde{V}}^{n-1}) & \xrightarrow{d} & H^0(\Omega_U^n)/H^0(\Omega_{\tilde{V}}^{n}(\log E)) \\
\alpha \downarrow & & \downarrow \\
H^1_E(\tilde{V}, \Omega_{\tilde{V}}^{n-1}(\log E)) & \xrightarrow{\beta} & H^1_E(V, \Omega_{\tilde{V}}^{n}(\log E))
\end{array}
\]

the maps \(\alpha\) and \(\beta\) are injective, hence \(d\) is injective too.

**Proposition 5.3.1.** Let \((V,0)\) be a normal isolated singularity of dimension \(n\) where \(V\) is a contractible Stein space. Let \(U = V - \{0\}\), and \(\pi : \tilde{V} \rightarrow V\) be a good resolution of the singularity \((V,0)\) with \(E = f^{-1}(0)_{\text{red}}\). Then \(q \leq p_g - h^{n-1}(\mathcal{O}_E)\).
Proof. Since $p_g = \dim H^0(U, \Omega^n_U)/H^0(\tilde{V}, \Omega^n_{\tilde{V}})$, so we have

$$\dim H^0(\Omega^n_U)/H^0(\Omega^n_{\tilde{V}}(\log E)) = \dim H^0(\Omega^n_U)/H^0(\Omega^n_{\tilde{V}}) - \dim H^0(\Omega^n_U(\log E)/H^0(\Omega^n_{\tilde{V}}) = p_g - \dim H^0(\Omega^n_U(\log E)/H^0(\Omega^n_{\tilde{V}})).$$

The exact sequence of sheaves

$$0 \to \mathcal{I}_E \otimes \Omega^n_{\tilde{V}}(\log E) \to \Omega^n_{\tilde{V}}(\log E) \to \Omega^n_{\tilde{V}}(\log E) \otimes \mathcal{O}_E \to 0$$

gives the long exact sequence

$$0 \to H^0(\mathcal{I}_E \otimes \Omega^n_{\tilde{V}}(\log E)) \to H^0(\Omega^n_{\tilde{V}}(\log E)) \to H^0(\Omega^n_{\tilde{V}}(\log E) \otimes \mathcal{O}_E) \to H^1(\mathcal{I}_E \otimes \Omega^n_{\tilde{V}}(\log E)) \to \cdots$$

By Theorem 5.3.1, $H^1(\mathcal{I}_E \otimes \Omega^n_{\tilde{V}}(\log E)) = 0$. Since $h^0(\mathcal{I}_E \otimes \Omega^n_{\tilde{V}}(\log E)) = h^0(\Omega^n_{\tilde{V}})$ and recall that the dualing sheaf $\omega_E$ satisfies

$$\omega_E \cong \Omega^n_{\tilde{V}}(\log E) \otimes \mathcal{O}_E$$

((60) page 523), hence $H^0(\Omega^n_{\tilde{V}}(\log E) \otimes \mathcal{O}_E) \cong H^{n-1}(\mathcal{O}_E)$ by Serre duality. Thus we have

$$\dim H^0(\Omega^n_{\tilde{V}}(\log E))/H^0(\Omega^n_{\tilde{V}}) = h^{n-1}(\mathcal{O}_E).$$

By Theorem 5.3.2, we get $q \leq p_g - h^{n-1}(\mathcal{O}_E)$. 


Theorem 5.3.3. (Yau (87)) Let \((V,0)\) be a \(n(\geq 2)\)-dimensional normal isolated singularity with \(\mathbb{C}^*\)-action. Let \(\pi: \tilde{V} \to V\) be a good resolution of the singularity \((V,0)\) with \(E = \pi^{-1}(0)_{\text{red}}\). Then \(q \geq p_g - h^{n-1}(\mathcal{O}_E)\).

Proof. See (87).

Proof of Theorem F. It is a immediately corollary of Proposition 5.3.1 and Theorem 5.3.3.

5.4 Proof of Theorem G

We first recall the definition of Du Bois singularity.

Definition 5.4.1. A normal isolated singularity \((V,0)\) is called a Du Bois singularity if the canonical map \((R^i\pi_*\mathcal{O}_{\tilde{V}})_0 \to H^i(E, \mathcal{O}_E)\) is an isomorphism for each \(i > 0\), where \(\pi: \tilde{V} \to V\) is a good resolution and \(E = \pi^{-1}(0)_{\text{red}}\).

If \((V,0)\) is an isolated complete intersection, then we have more simple equivalent definition as follows.

Definition 5.4.2. Let \((V,0)\) be an isolated complete intersection singularity (ICIS) of dimension \(n \geq 2\). We can assume that \(V\) is contractible. Put \(U = V \setminus 0\). Let \(\pi: \tilde{V} \to V\) be a good resolution of \((V,0)\), and \(E = \pi^{-1}(0)_{\text{red}}\). The ICIS singularity \((V,0)\) is said to be

1. rational if \(R^{n-1}\pi_*\mathcal{O}_{\tilde{V}} = 0\);
2. Du Bois if \(R^{n-1}\pi_*\mathcal{O}_{\tilde{V}}(-E) = 0\);
3. purely elliptic if it is Du Bois and not rational.
Remark 5.4.1. These definitions 5.4.2 do not apply to more general situations in the given formulation. For instance, the definition 5.4.1 amounts to saying that an arbitrary normal singularity is Du Bois if the natural map $\pi^*: \mathfrak{m}_{V,0} \to \pi_*(\mathcal{O}_V(-E))$ is an isomorphism and $R^j\pi_*(\mathcal{O}_V(-E))_0 = 0$ for $j > 0$. In the setup above for ICIS, this is equivalent to (2) (see 4.2 (69)).

Looijenga and Steenbrink (40) gave a formula

$$\mu - \tau = \sum_{j=0}^{n-2} h^{0,j} + a_1 + a_2 + a_3$$

(4.1)

expressing the difference between the Milnor number and the Tjurina number of $(V,0)$ as a sum of non-negative integers.

In the formula, $h^{i,j}$ denotes the $(i,j)$-th Hodge number of the mixed Hodge structure on $(n-1)$-cohomology of the link of $(V,0)$.

The term $a_1$ is the dimension of the vector space

$$A_1 = \text{Coker}(H^0(\Omega_U^{n-1}) \xrightarrow{d} H^0(\Omega_U^n)) / H^0(\Omega_U^n(\log E))$$

We will not define $a_2$ and $a_3$ explicitly here. According to Lemma 2.7 (70) we have the following equality:

$$a_2 + a_3 = c_{2,0}^{n,0} + b_2^{(1)}$$
where $e_2^{n,0}$ and $b_2^{(1)}$ are the dimensions of the vector spaces

$$E_2^{n,0} = \text{Coker}(H^0(\Omega_{\tilde{V}}^{n-1}(\log E)(-E)) \xrightarrow{d} H^0(\Omega_{\tilde{V}}^n)),$$

and

$$B_2^{(1)} = \text{Coker}(H^0(\Omega_{\tilde{V}}^{n-1}(\log E)) \to H^0(\Omega_{\tilde{V}}^{n-1}(\log E) \otimes \mathcal{O}_E))$$

respectively.

Set $\chi = \sum_{j=0}^{n-2} h^{0,j} + b_2^{(1)}$. By equation (4.1), we have $\mu - \tau = e_2^{n,0} + \chi + a_1$.

The following theorem which is crucial ingredient in our proof is due to Vosegaard.

**Theorem 5.4.1.** (Vosegaard (69)) Let $(V,0)$ be an ICIS of dimension $n \geq 2$. Then $(V,0)$ is quasi-homogeneous if and only if

1. $e_2^{n,0} = 0$, in case $(V,0)$ is rational;
2. $\chi = 0$ in case $(V,0)$ is purely elliptic;
3. $a_1 = 0$, in case $(V,0)$ is non-Du Bois.

In particular, $(V,0)$ is quasi-homogeneous if and only if $\mu = \tau$.

**Proof of Theorem G.** Let $\pi : \tilde{V} \to V$ be a good resolution of $(V,0)$, and $E = \pi^{-1}(0)_{\text{red}}$. We claim that $a_1 = p_y - h^{n-1}(\mathcal{O}_E) - q$. Since $p_y = \dim H^0(U, \Omega_U^n)/H^0(\tilde{V}, \Omega_{\tilde{V}}^n)$ and by Theorem 5.3.2, the map

$$d : H^0(\Omega_{\tilde{V}}^{n-1})/H^0(\Omega_{\tilde{V}}^n) \to H^0(\Omega_U^n)/H^0(\Omega_V^n(\log E))$$
induced by differentiation is injective. Hence,

\[ a_1 = \dim \text{Coker} \left( \frac{H^0(\Omega^n_U)}{H^0(\Omega^n_{\tilde{V}}(\log E))} \right) \]

By proposition 5.3.1 \( \dim H^0(\Omega^n_U)/H^0(\Omega^n_{\tilde{V}}(\log E)) = p_g - h^{n-1}(\mathcal{O}_E) \). It follows that \( a_1 = p_g - h^{n-1}(\mathcal{O}_E) - q \). By assumption we have \( a_1 = 0 \) and if \((V, 0)\) is non-Du Bois, then \((V, 0)\) is quasi-homogeneous by Theorem 5.4.1 (3).

5.5 Applications

In this section we shall give two ways of applications of Theorem F.

5.5.1 Singularities With \( q = 0 \)

**Proposition 5.5.1.** Let \((V, 0)\) be a \( n \)-dimensional normal isolated Du Bois singularity with \( \mathbb{C}^* \)-action. Let \( \pi : \tilde{V} \rightarrow V \) be a good resolution of the singularity \((V, 0)\) with \( E = \pi^{-1}(0)_{\text{red}} \). Then \( q = 0 \)

Proof. Since a singularity is Du Bois, then by definition 5.4.1 the canonical map \( R^i \pi_* \mathcal{O}_{\tilde{V}} \rightarrow H^i(E, \mathcal{O}_E) \) is an isomorphism for each \( i \), in particular \( p_g = \dim (R^{n-1} \pi_* \mathcal{O}_{\tilde{V}})_0 = h^{n-1}(\mathcal{O}_E) \). By Theorem F, we have \( q = 0 \).

The converse of above proposition is also correct.

We recall the following lemma.
Lemma 5.5.1. (Steenbrink (60)) Let \((V,0)\) be a normal isolated singularity of dimension \(n\), where \(V\) is a contractible Stein space, with \(\mathbb{C}^*\)-action. Let \(\pi : \tilde{V} \to V\) be a good resolution of the singularity \((V,0)\) with \(E = \pi^{-1}(0)_{\text{red}}\). Then for all \(i \geq 0\) the natural map

\[ H^i(\tilde{V}, \mathcal{O}_{\tilde{V}}) \to H^i(E, \mathcal{O}_E) \]

is surjective.

Proposition 5.5.2. Let \((V,0)\) be a normal isolated Cohen-Macaulay singularity of dimension \(n \geq 2\), with \(\mathbb{C}^*\)-action. Let \(\pi : \tilde{V} \to V\) be a good resolution of the singularity \((V,0)\) with \(E = \pi^{-1}(0)_{\text{red}}\). Then \((V,0)\) is a Du Bois singularity if and only if \(q = 0\).

Proof. Let \(h^{p,q} = \dim H^q(\tilde{V}, \Omega_p^0(\log E)(-E))\). Since \((V,0)\) is Cohen-Macaulay, i.e. has depth \(n\), then by Proposition 1(62) the only possible non-zero \(h^{0,q}\) is \(h^{0,n-1}\). Therefore \((V,0)\) is Du Bois singularity if and only if \(h^{0,n-1} = 0\) (see (62)), i.e. \(H^{n-1}(\tilde{V}, \mathcal{O}_{\tilde{V}}(-E)) = 0\). We have the short exact sequence

\[ 0 \to \mathcal{O}_{\tilde{V}}(-E) \to \mathcal{O}_{\tilde{V}} \to \mathcal{O}_E \to 0 \]

which gives a long exact sequence

\[ \cdots \to H^{n-2}(\tilde{V}, \mathcal{O}_{\tilde{V}}) \to H^{n-2}(E, \mathcal{O}_E) \to H^{n-1}(\mathcal{O}_{\tilde{V}}(-E)) \]

\[ \to H^{n-1}(\mathcal{O}_{\tilde{V}}) \to H^{n-1}(\mathcal{O}_E) \to H^n(\mathcal{O}_{\tilde{V}}(-E)) \to \cdots . \]
Since by Lemma 5.5.1, \( H^i(\tilde{V}, \mathcal{O}_{\tilde{V}}) \to H^i(E, \mathcal{O}_E) \) for \( i \geq 0 \) are surjective, in particular, \( H^{n-2}(\tilde{V}, \mathcal{O}_{\tilde{V}}) \to H^{n-2}(E, \mathcal{O}_E) \) is surjective. Hence we get a long exact sequence

\[
0 \to H^{n-1}(\mathcal{O}_{\tilde{V}}(-E)) \to H^{n-1}(\mathcal{O}_{\tilde{V}}) \to H^{n-1}(\mathcal{O}_E) \to H^n(\mathcal{O}_{\tilde{V}}(-E)) \to \cdots
\]

By Siu’s theorem (58), we have \( H^n(\mathcal{O}_{\tilde{V}}(-E)) = 0 \), thus \( H^{n-1}(\mathcal{O}_{\tilde{V}}(-E)) = 0 \) if and only if \( h^{n-1}(\mathcal{O}_{\tilde{V}}) = h^{n-1}(\mathcal{O}_E) \), i.e. \( p_g = h^{n-1}(\mathcal{O}_E) \) which is equivalent to \( q = 0 \) by Theorem F.

### 5.5.2 Topological Invariants of Singularities

In this section we will give a formula of \( q \) in terms of weights for isolated weighted homogeneous hypersurface singularities of dimension 2. As a byproduct we prove \( q \) is an invariant of topological type for those singularities.

A polynomial \( f(z_0, z_1, \cdots, z_n) \) is a weighted homogenous of type \((w_0, w_1, \cdots, w_n)\), where \((w_0, w_1, \cdots, w_n)\) are fixed positive rational numbers, if it can be expressed as a linear combination of monomials \( z_0^{i_0} z_1^{i_1} \cdots z_n^{i_n} \) for which \( \frac{i_0}{w_0} + \frac{i_1}{w_1} + \cdots + \frac{i_n}{w_n} = 1 \). \((w_0, w_1, \cdots, w_n)\) are called weights of \( f \). In (89) the first author and Xu proved the following theorem.

**Theorem 5.5.1.** (89) Let \((V, 0)\) be an isolated quasi-homogeneous surface singularity defined by a weighted homogenous polynomial in \( \mathbb{C}^3 \) with weights \((w_0, w_1, w_2)\). Then the topological type of \((V, 0)\) determines and is determined by its weights \((w_0, w_1, w_2)\).
Let \( f(z_0, z_1, z_2) \) be weighted homogeneous with weights \( w_i = \frac{u_i}{v_i}, i = 0, 1, 2 \) in reduced form. For integers \( a_1, a_2, a_3 \), let \((a_1, a_2, a_3)\) denote their greatest common divisor. Define

\[
c = (u_0, u_1, u_2); \quad c_0 = (u_1, u_2)/c; \quad c_1 = (u_0, u_2)/c; \quad c_2 = (u_0, u_2)/c.
\]

Then for some positive integers \( \gamma_0, \gamma_1, \gamma_2 \), we have

\[
u_0 = cc_1c_2\gamma_0; \quad u_1 = cc_0c_2\gamma_1; \quad u_2 = cc_0c_1\gamma_2.
\]

Note that \( c_0, c_1, c_2 \) are pairwise relatively prime, \( \gamma_0, \gamma_1, \gamma_2 \) likewise and \((c_i, \gamma_i) = 1\) for \( i = 0, 1, 2\).

Thus we have

\[
d = \langle w_0, w_1, w_2 \rangle = cc_0c_1c_2\gamma_0\gamma_1\gamma_2
\]

and

\[
q_0 = v_0c_0\gamma_1\gamma_2; \quad q_1 = v_1c_1\gamma_0\gamma_2; \quad q_2 = v_2c_2\gamma_0\gamma_1.
\]

The link \( K_f = f^{-1}(0) \cap S^5 \), where \( S^5 \) is a sphere with center at origin, is a Seifert fibered 3-manifold. Orlik and Wagreich (49) have calculated the Seifert invariants of \( K_f \),

\[
\{-b; g; n_1(\alpha_1, \beta_1), n_2(\alpha_2, \beta_2), n_3(\alpha_3, \beta_3), n_4(\alpha_4, \beta_4)\},
\]
where $g$ is the genus of the central curve in the minimal good resolution of the singularity $(V,0)$ which is defined by $f$ and $b$ is the self intersection number of the central curve. Since here we are only interested in $g$, hence we don’t give the definitions of $n_i, \alpha_i$ and $\beta_i$ which can be found in (49). Explicitly, $g$ is given by the following formula.

**Proposition 5.5.3.** (49) *With the notation above*

$$g = \frac{2c_0c_1c_2 - c(c_0v_0 + c_1v_1 + c_2v_2) + v_0v_1 + v_1v_2 + v_2v_0 - v_0v_1v_2}{2v_0v_1v_2}$$

where $c, c_i, v_i, 0 \leq i \leq 2$ depend only on $w_0, w_1, w_2$.

In fact $p_g$ can also be calculate with respect to weights for non-degenerate weighted homogeneous hypersurface singularities. Let $(V,0)$ be an isolated singularity defined by a non-degenerate weighted homogeneous polynomial $f(x_0, x_1, \cdots, x_n)$ of type $(w_0, w_1, \cdots, w_n)$, where $w_0, w_1, \cdots, w_n$ are positive. As a consequence of the Theorem 3.2.1, we know that in case of isolated singularity defined by a weighted homogeneous polynomial, computing the geometric genus is equivalent to counting the number of positive integral points in a tetrahedron. i.e. we have

$$p_g = \#\{(x_0, x_1, \cdots, x_n) \in \mathbb{Z}_+^n : \frac{x_0}{\omega_0} + \frac{x_1}{\omega_1} + \cdots + \frac{x_n}{\omega_n} \leq 1\},$$

where $\mathbb{Z}_+$ is the set of positive integers.
Theorem 5.5.2. With the notation above. Let \((V, 0)\) be an isolated singularity defined by a weighted homogeneous polynomial \(f(x_0, x_1, x_2)\) of type \((w_0, w_1, w_2)\). Then

\[
q = \# \left\{ (x_0, x_1, x_2) \in \mathbb{Z}_+^3 : \frac{x_0}{w_0} + \frac{x_1}{w_1} + \frac{x_2}{w_2} \leq 1 \right\} - \frac{c^2 c_0 c_1 c_2 - c(c_0 v_0 + c_1 v_1 + c_2 v_2) + v_0 v_1 + v_1 v_2 + v_2 v_0 - v_0 v_1 v_2}{2v_0 v_1 v_2}
\]

Proof. By Theorem F, \(q = p_g - h^1(\mathcal{O}_E)\). Since for weighted homogeneous singularities the dual resolution graph is star-shaped dual graph, and except the central curve of dual graph of resolution the other components are rational. Thus \(h^1(\mathcal{O}_E) = g\) where \(g\) is the genus of the curve correspond to the central curve of the dual graph of resolution. By Proposition 5.5.3 we get

\[
h^1(\mathcal{O}_E) = \frac{c^2 c_0 c_1 c_2 - c(c_0 v_0 + c_1 v_1 + c_2 v_2) + v_0 v_1 + v_1 v_2 + v_2 v_0 - v_0 v_1 v_2}{2v_0 v_1 v_2},
\]

By Theorem 3.2.1 we have

\[
p_g = \# \left\{ (x_0, x_1, x_2) \in \mathbb{Z}_+^3 : \frac{x_0}{w_0} + \frac{x_1}{w_1} + \frac{x_2}{w_2} \leq 1 \right\}.
\]

Therefore,

\[
q = \# \left\{ (x_0, x_1, x_2) \in \mathbb{Z}_+^3 : \frac{x_0}{w_0} + \frac{x_1}{w_1} + \frac{x_2}{w_2} \leq 1 \right\} - \frac{c^2 c_0 c_1 c_2 - c(c_0 v_0 + c_1 v_1 + c_2 v_2) + v_0 v_1 + v_1 v_2 + v_2 v_0 - v_0 v_1 v_2}{2v_0 v_1 v_2}
\]
**Corollary 1.** Let \((V, 0)\) be an isolated singularity defined by a weighted homogeneous polynomial \(f(x_0, x_1, x_2)\) of type \((w_0, w_1, w_2)\). Then the irregularity \(q\) is a topological invariant.

**Proof.** By Theorem 5.5.2 the formula for \(q\) only dependent on weights of the weighted homogeneous polynomial and by Theorem 5.5.1 weights are topological invariant of weighted homogeneous surface singularities. We obtain the irregularity \(q\) of weighted homogeneous isolated surface singularity \(f\) is also a topological invariant. This means that if \(f, f'\) are two weighted homogeneous isolated surface singularities and \((\mathbb{C}^3, V(f), 0)\) is homeomorphic to \((\mathbb{C}^3, V(f'), 0)\), then \(q = q'\) where \(q\) and \(q'\) are irregularities of \(f\) and \(f'\) respectively.

Recall that \(p_g\) can be bounded by Milnor number for weighted homogenous singularities.

**Theorem 5.5.3.** (Yau-Zhang (90)) Let \((V, 0)\) be an isolated singularity defined by a weighted homogeneous polynomial \(f(z_0, z_1, \cdots, z_n)\). Then \((n + 1)!p_g \leq \mu\) and equality holds if and only if \(\mu = 0\).

In general, it is difficult to compute \(q\). However there is an upper bound for \(q\) using the weights for weighted homogenous surface singularities.

**Corollary 2.** Let \((V, 0)\) be an isolated singularity defined by a weighted homogeneous polynomial \(f(x_0, x_1, x_2)\) of type \((w_0, w_1, w_2)\). Then

\[
q \leq \frac{1}{6} \prod_{i=0}^{2} (w_i - 1) - \frac{c_0^2 c_1 c_2 - c(c_0 v_0 + c_1 v_1 + c_2 v_2) + v_0 v_1 + v_1 v_2 + v_2 v_0 - v_0 v_1 v_2}{2 v_0 v_1 v_2}.
\]

**Proof.** Milnor and Orlik (see (45)) have proven that \(\mu = \prod_{i=0}^{2} (w_i - 1)\). Then the above follows inequality follows from 5.5.3 and 3.2.1.
Example 1  Consider $f = z_0^{105} + z_1^{9} + z_1 z_2^{14}$ with weights $(105, 9, 63/4)$. Thus $c = 3, c_0 = 3, c_1 = 7, c_2 = 1, v_0 = v_1 = 1, v_2 = 4$, and by Proposition (5.5.3) we get $g = 19$. Since $\mu = (105 - 1)(9 - 1)(63/4 - 1) = 12272$, therefore by Corollary 2 we get $q \leq 2026$.

Example 2  Consider $f = z_0^{7} + z_1^{3} + z_2^{2}$ with weights $(7, 3, 2)$. Obviously $f$ is non-degenerate. By Proposition 5.5.3, we have $g = 0$, and it is easy to see only one positive point $(1, 1, 1)$ lying in the Newton polyhedron of $f$. Thus $p_g = 1$. Since the dual graph of the resolution of singularity defined by $f$ is star shape, then there is no loop. By Theorem F we get $q = 1$.

We have $q = p_g$ for this example. In fact the singularity is very special, we know the dual resolution graph is

\[\begin{array}{c}
-7 \\
-1 \\
-2 \\
-3
\end{array}\]

Note that a surface singularity has rational homology sphere (QHS) link if and only if the dual minimal resolution (not necessarily good) graph is tree and each irreducible component is rational. It is easy to check the singularity in Example 2 has QHS link. In fact, it has integral homology sphere link (i.e., QHS and $\det(E_i \cdot E_j) = \pm 1$). For those special surface singularities with QHS links, we have the following theorem.
**Theorem 5.5.4.** Let $(V,0)$ be an isolated surface singularity with $\mathbb{C}^*$-action and QHS link, then $q = p_g$ and both are topological invariant.

Proof. Since $(V,0)$ has QHS link, the dual graph of resolution is a tree and has rational components, therefore $h^1(\mathcal{O}_E) = 0$, hence by Theorem F we have $q = p_g$. Observe that weighted homogenous singularities with QHS are splice quotient singularities (see (48)), and it is well-known that $p_g$ can be calculated from the dual graph of resolution for splice quotient singularities (50). Therefore both $p_g$ and $q$ are topological invariant.

Let $(V,0)$ be defined by $f(z_0,z_1,z_2)$ which is a weighted homogeneous singularity with weights $(w_0,w_1,w_2)$. Let $w_i = \frac{u_i}{v_i}$, $i = 0, 1, 2$ in reduced form. Let $d = \langle u_0, u_1, u_2 \rangle$ denote their least common multiple. Define $q_i = d/w_i$ and $l = d - (q_0 + q_1 + q_2)$.

**Proposition 5.5.4.** (Wahl (71)) With the notation as above, then $l < 0$ iff $(V,0)$ is an RDP, $l = 0$ iff $(V,0)$ is simple elliptic. The Milnor algebra

$$J = \mathbb{C}[z_0, z_1, z_2]/\langle \frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2} \rangle$$

is graded with $J = \oplus_0^N J_l$, where $N = 2l + d$. $J_N$ has dimension 1, and is spanned by the Hessian of $f$. Moreover,

$$p_g = \dim \oplus_{l \leq 1} J_i$$

$$g = \dim J_l.$$  

Proof. See Lemma 4.3 in (71).
**Remark 5.5.2.** The above beautiful results is true not only for hypersurface surface singularities, but for Gorenstein surface singularities with $\mathbb{C}^*$-action (see (72)).

We get the following useful corollary.

**Corollary 3.** With the same assumption as Proposition 5.5.4. We have

$$q = \dim \bigoplus_{i < l} J_i.$$ 

Proof. It is immediately corollary by Theorem F and Proposition 5.5.4.

An immediately application of above Corollary 3 is that we can reprove Theorem 5.5.5 of Yau (i.e., Theorem A in (87)) easily.

**Theorem 5.5.5.** (Yau (87)) Let $(V,0)$ be a Gorenstein surface singularity with $\mathbb{C}^*$-action. Then $q = 0$ if and only if $(V,0)$ is either a rational double point or a simple elliptic singularity.

Proof. Case 1. If $(V,0)$ is a hypersurface singularity, then by Corollary 3, $q = 0$ if and only if $l \leq 0$. It follows from Proposition 5.5.4 that $(V,0)$ is either a rational double point or a simple elliptic singularity.

Case 2. If $(V,0)$ is not hypersurface singularities, by Remark 5.5.2, the result can be derived using the same arguments given in Case 1.
APPENDICES


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Publications


2. Classification of gradient space of dimension 8 with a reducible $sl(2,C)$ action (with Yung Yu and Stephen S.T. Yau) Science in China, Series A Mathematics, (2009), 52 (12), 2792-2828.


11. A sharp estimate of Dickman-De Bruijn function and a sharp polynomial estimate of positive integral points in 4-dimensional tetrahedron, (with Xue luo and Stephen S.T. Yau), 20 pp. in ms., submitted for publication.

12. Non-constant CR morphisms between compact strongly pseudo-convex CR manifolds and etale covering between resolutions of isolated singularities (with Yu-Chao Yu and Stephen S.T. Yau), 18 pp. in ms., submitted for publication.

13. Characterization of isolated complete intersection singularities with $\mathbb{C}^*$-action of dimension $n \geq 2$ by means of geometric genus and irregularity, (with Stephen S.T. Yau) 15 pp. in ms., submitted for publication.


Presentations and Talks

1. Connectedness of Moduli Spaces of Certain Types of Hyperplane Arrangements, in Birational Geometry and Moduli Spaces workshop, MRC 2010, Snowbird, Utah June 16, 2010

2. Application of Bogomolov’s instability criterion: Reider’s theorem on linear system on algebraic surfaces, in Graduate Algebraic Geometry Seminar, October 31 2007

3. An brief introduction to Leray-Serre Spectral sequence, in Algebraic Topology Course Fall 2008
4. The proof of Gesieker’s potential stability theorem, in UIC Graduate Algebraic Geometry Seminar, Fall 2008

5. Bend and Break Lemmas Review, in UIC Graduate Algebraic Geometry Seminar, August 31, 2009

6. Existence of Flips and Extensions of the Minimal Model Program, in UIC Graduate Algebraic Geometry Seminar, November 16, 2009

7. Examples of Contractions of Extremal Rays: small contraction, in UIC Graduate Algebraic Geometry Seminar, November 30, 2009

8. Filtrations of Vector Bundles on Smooth Algebraic Curves, in UIC Graduate Algebraic Geometry Seminar, Spring 2010

9. Invariant Geometry - An Introduction, in UIC Graduate Algebraic Geometry Seminar, Spring 2010

10. The Theorem of Cube I, in UIC Graduate Algebraic Geometry Seminar, Fall 2010

Other Academic Activities

1. 5-6 October 2007, AMS Fall Central Section Meeting, DePaul University.


4. April 11-13, 2008, Birthday conference for Robin Hartshorne, UIC,
   http://math.uic.edu/~coskun/hartshorne.html

5. October 18-19, 2008, UM-OSU-UIC Algebraic Geometry Workshop, University of Michigan,

   Lazarsfeld and Mircea Mustata, The Ohio State University, http://www.mri.osu.edu/special

7. March 6-8, 2009, 2nd Bluegrass Algebra Conference, University of Kentucky Lexington,
   http://www.ms.uky.edu/~algebra/ky09/

8. November 14-15, 2009 The OSU/UIC/UM Weekend Algebraic Geometry Workshop,
   University of Michigan, Ann Arbor,
   http://www.math.lsa.umich.edu/~mmustata/OSUUICMworkshops.html

    State University, Columbus, http://www.math.ohio-state.edu/node/46342

10. April 23-25, 2010, Midwest Algebra, Geometry and their Interactions Conference - MAGIC’10,
    University of Notre Dame, Notre Dame http://www.nd.edu/~magic/MAGIC’10/

11. May 10-14, 2010, Algebraic Geometry On Varieties And Manifolds, Fudan University,
    Shanghai, China http://homepage.fudan.edu.cn/~mchen/2010Workshop/main.html

12. June 1-11, 2010, School on birational geometry and moduli spaces, University of Utah,
    Salt Lake City, Utah http://www.math.utah.edu/vigre/minicourses/birational/index.html
   http://sites.google.com/site/birational/


Teaching

I have taught the following courses:

- Math 181 Calculus II, Spring 2012
- Math 180 Calculus I, Fall 2011
- Math 181 Calculus II Summer 2010
- Math 180 Calculus I, Spring 2011
- Math 121 Precalculus, Fall 2010
- Math 210 Calculus III, Fall 2009

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