Emergent spacetime in String Theory

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This thesis is dedicated to my family, without whom it would never have been accomplished.

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Contents

Summary

The attempt to achieve a quantum theory of gravity has to deal with the thorny issue of combining general relativity and quantum field theory. On the one hand general relativity's picture of the world presents gravity as intrinsically associated with the dynamics of spacetime itself, the latter being a fundamental dynamical and geometrical object in the theory. On the other hand, the canonical quantum field theory approach to gravity is an attempt to construe it as a force mediated by particles which propagate over a fixed spacetime. The latter is still a fundamental object in the theory, but plays the role that the classical Newtonian background plays for dynamics. In other words, a difficult theoretical marriage.

String theory plays an interesting role in facing this conceptual tension. The theory has the good property of predicting gravity. In fact, closed string theory describes a massless spin two particle that can be identified with the graviton. This hypothetical quantum is generated as one of the closed string's modes of oscillation. Moreover, it has been seen that the value of a type of excitation of the massless scalar field dilaton determines the value of the Newton gravitational constant through the string self-interaction constant. These theoretical findings - along with other theoretical discoveries (connected to dualities) showing the possible existence of a "minimal length" in space - seem to point out a dialectical synthesis between the two theories. Spacetime, not a fundamental object anymore, emerges from string theory's fundamental dynamical equations in a quantized shape.

String theory originally developed from studies of dual models of hadronic resonance at the end of sixties[1]. However, after this start, the project to consider string theory as a model for

[1] In 1968 Veneziano discovered that the Euler beta function if interpreted as scattering amplitude, later on named Veneziano amplitude, seems to explain physical properties as symmetry and duality of strongly interacting mesons. Veneziano's discovery is considered as what determined the birth of String Theory; see $http : //en.wikipedia.org/wiki/Veneziano_amplitude$. In order to read more about Veneziano's discovery see also [$Ven68$]

strong interactions was set aside for a while. Several facts caused this evolution: the great success achieved by the competitor theory of quantum chromodynamics, some conceptual and computational difficulties concerning internal aspects of the theory, for example extra dimensions required by string dynamics. A new interest in string theory started again about 1974 as soon as it became evident that, the theory could include gravitation without being affected by the divergences problems of the ordinary Quantum field theory[2].

The present work is an analysis of the emergent nature of spacetime in string theory. The notion of emergence is involved in many philosophical disputes and it branches off in a multitude of different subjects. In this work I shall not examine all its declinations. I shall be focused instead only on some particular uses of this notion which are central to the current debate about spacetime's role in string theory: spacetime emergence in the contest of theoretical dualities, in particular T-duality, and spacetime emergence in the context of time-space non-commutativity. The former requires making some preliminary remarks that set the stage.

How do the notions of spacetime emergence and of theoretical duality combine together? An exhaustive answer to this question requires unraveling the content of chapters two and three. However, I'll try to give a preliminary answer in this introduction.

The notion of emergence I am using here applies to a theoretical context. More precisely, a theory T^1 emerges from a theory T^2 if some of the physical entities or properties introduced by the former are emergent from those described by the latter. More precisely, an emergent physical entity or property of T^1 can be characterized as being described by T^1 as novel in relation to those described by T^2, i.e. it is not part of the description given by T^2 and in some sense it is an unexpected physical feature inside the latter. What I want to emphasize with this definition is that theoretical emergence is not a relation of physical equivalence between two theories. An illustrative example of emergence that clearly shows this feature is that defined by Butterfield and Isham in "Spacetime and the Philosophical Challenge of Quantum Gravity"[3]. In this paper they define a scenario in which the emergent theory T^1

[2] In 1974 some important works by Scherk, Schwarz and Yoneya, see [$JS74$],[$Sch75$],[$Yon74$], presented and interpreted some interesting findings: open strings' excitations always contains at least one photon, and in general Yang Mills fields, whereas closed strings' excitations always contains a graviton. The mutual interactions among gravitons, at distances much bigger than the string's dimension, are described by Young-Mills theory and by General relativity. Therefore, Scherk, Schwarz and Yoneya proposed re-interpreting String Theory as the basis for a quantum theory of gravity

[3] See [J.B99], page 71

is a limit theory approximating T^2,

> To "go beyond" such a structure, one strategy is to argue that it is emergent (in physics' jargon: "phenomenological"). "Emergence" is vague, and indeed contentious.[..] But here, we only need the general idea of one theory T^1 being emergent from another T^2 if in a certain part of T^2's domain of application (in physics' jargon, a "regime": usually specified by certain ranges of values of certain of T^2's quantities), the results of T^2 are well approximated by those of T^1 where "results" can include theoretical propositions as well as observational ones, and even "larger structures" such as derivations and explanations.

The lack of symmetry between physical contents of two theories related in this way introduces a distinction between two levels of description of reality, one more fundamental than the other. But then combining this notion with that of theoretical duality requires some more explanation, owing to the fact that the latter is by contrast characterized by that symmetry. In fact according to Seiberg[4] duality relation is a physical equivalence between theories,

> [...]This is the most natural description among all possible dual descriptions. However, two points should be stressed about this case. First, even though this description is the most natural one, there is nothing wrong with all other T-dual descriptions and they are equally valid[...].

Moreover, according to Rickles[5],

> Finally, dualities are of wider interest in philosophy of science since they point to a mechanism for generating new theories and results. In particular,they point to the possibility of "simulating" hard physics,in hard regimes, with simple physics. This is not simulation in the sense of approximation: the dualities (in string theory) are exact.

In string theory the nature of the conceptual relation between these two notions lies in the following fact. As we will see later on, dualities in string theory reveal physical equivalence between spacetime theories very different in appearance. Dual string theories can postulate

[4] See [Sei06],"Emergent Spacetime", page 4
[5] See [Ric10],"A philosopher looks at string dualities", page 56

geometrically inequivalent spacetimes and still produce the very same physics. Therefore dualities seem to point out to background independence of string physics. Although the latter is still a controversial open problem in string theory, it inevitably arises the issue of spacetime's emergence. For it crucially undermines the traditional idea that ordinary spacetime geometry, having a symmetric interaction with matter, plays a necessary role in producing fundamental physical dynamics. One of the most popular views about that is that of Witten. According to him dualities would show that in string theory we do not have ordinary spacetime but just the corresponding two-dimensional field theory[6]. The latter - along with the replacement of ordinary Feynman diagrams with stringy ones describing strings' propagation - is all we need. Still according to Witten, two dual theories postulating topologically inequivalent spacetimes can be read in terms of the same more fundamental field theory that introduces a deeper level of explanation of reality inside which its background independence seems to be revealed. But then this feature shows that inside that description, which is characterized by $\alpha^{'} \neq 0$, ordinary spacetime is an unexpected physical entity. The latter emerges in a derived, less fundamental, theory arising in the limit $\alpha^{'} \rightarrow 0$.

The present work is divided in the following way. Chapter 1 is basically an introduction to string theory through consideration of the bosonic string. That can provide us with the necessary mathematical and conceptual tools for studying T-duality. Moreover the chapter gives particular emphasis to a general feature of string theory, its conformal invariance. The conceptual consequences of this peculiarity are central to the issue of spacetime emergence. Chapter 2 is a mathematical and philosophical presentation of T-duality. Chapter 3 contains two parts. The first one is an attempt of interpretational proposal which unravels the role of moduli space in the theory. The aim is showing that the mathematical nature of this representational tool provides the theory with a notion of physical "background" intrinsically different from the traditional one. I will examine the main features of this notion and will show that the use of moduli spaces brings in a weaker notion of background independence. The second part of the chapter is instead a survey about the most popular views concerning implications of duality. Finally, chapter 4 approaches the issue of emergent spacetime inside the context of time-space non commutativity. The basic idea of the chapter can be succinctly described in the following way. Some interesting theoretical findings concerning causality in

[6] See [Wit96], page 29. Witten's view will be studied in detail both in chapter two and three.

space-time non commutative string theory, along with a view about the status of space-time uncertainty principle in the theory, are the main ingredients in my attempt to present a scenario of spacetime emergence alternative to that involved by background independence.

Chapter 1

Introduction to bosonic string theory, but not only

This chapter introduces string theory through consideration of the bosonic string. It is widely known that the fundamental objects of String theory are not point particles but extended objects called strings. Particles appear in the theory as different modes of strings vibration[1]. The bosonic case is an unrealistic theory, since it includes only bosons and therefore it is a theory without matter. However, it is an instructive case to start with because it is an easier way to approach string theory. Through the bosonic case it is possible to learn string theory's practical aspects inside a slightly simpler context. Many basic formal features arise in a simpler form. In particular, the bosonic case provides us with a concrete and simple case of T-duality, which will be calculated in the second chapter.

Before unraveling the bosonic case, I would like to mention the following fact. String theory can be approached basically in two ways, different from a conceptual point of view but mathematically equivalent. The nature of this basic distinction plays a crucial role inside the issue of spacetime emergence and amounts to a difference in viewing the $X^{\mu}(\sigma, \tau)s$ describing the string world-sheet. As we will see in this chapter strings dynamics are mathematically described by a function $X^{\mu}(\tau, \sigma)$ of two parameters living on the string's worldsheet, the

[1] The presentation contained in this section is a selection of topics taken from the textbook "String theory and M-Theory"; see [$KB07$], in particular chapters $1-5$. The selected parts will be presented through steps given mostly without proofs

"spatial' parameter σ and the "time' parameter τ.

If we take the values of this function to be spacetime coordinates, then we are studying physical dynamics of embedded strings. In fact the function's variation - depending on the variation of τ and σ - describes a two-dimensional string worldsheet embedded in an external spacetime. This surface is provided with an induced metric which determines distances on it. This metric is said to be induced because it is defined in function of the metric of the background spacetime.

If instead we take the $X^{\mu}(\sigma, \tau)s$ to be fields living over the string world-sheet, then we have a string field theory over a surface, as articulated e.g. by Edward Witten in "Reflections on the fate of spacetime", (see [$Wit96$]). The notion of an external spacetime, theater of strings dynamics, completely dissolves: away from the string's worldsheet there is no spacetime, no external environment in which the worldsheet is embedded. Then, according to this *weltanschauung* the idea of a metrical structure induced from outside the string is meaningless. So an internal structure is introduced by defining the "metric" of the worldsheet, i.e. the *auxiliary* "metric".

In what follows I will present basic facts about bosonic string by referring to both conceptual perspectives. A last introductory remark about this difference is that, despite physics textbooks usually privilege the embedded strings weltanschauung, there is nothing intrinsically special about it. There is no empirical or theoretical evidence supporting the idea of embedded strings in spacetime being a more truthful model of representation of physical string dynamics. The choice of one or the other only depends on the particular philosophical view about ontological commitment of string theory.

The chapter develops in the following way. I define the classical action of the bosonic string and derive from it the equations of motions. Then, I present a brief description of some quantization procedures of this action and I introduce its symmetries. One particular symmetry, the action's Weyl invariance is highlighted, since it constitutes a key ingredient in analyzing spacetime emergence, in particular presenting Witten's view in the last section of chapter 3.

Finally, I want to emphasize something just briefly touched earlier on, i.e. the fact that the two mentioned standpoints share the same mathematical formulation for the theory's action.

The auxiliary metric and the induced metric both show up in it. Each interpretation assigns to each metric different meanings though. For example, according to Witten's interpretation the "induced metric" is an internal product among fields over the string. What defines a distance over the worldsheet in this case is the auxiliary metric. By contrast, according to the first interpretation, the induced metric is responsible for the metrical structure of the worldsheet.

Let's start the introduction to bosonic string theory. The bosonic case includes both open and closed strings and it requires a spacetime's dimension D=26 for consistency reasons[2].

1.1 Classical action of the bosonic string

Strings are one-dimensional objects sweeping out two-dimensional world-sheets. As I said above, points of the string come labeled with σ and τ, respectively "spatial' and "time' parameters internal to the world-sheet. Open strings have two end-points, whereas closed ones are topologically like circles. If σ is periodic then the string is closed. If σ covers a finite interval, the string is open. The world-sheet is mathematically described by the map $(\tau, \sigma) \longrightarrow X^\mu(\tau, \sigma)$. As I said above the range of $X^\mu(\tau, \sigma)$ can be interpreted in two ways, either as spacetime coordinates describing an embedded surface or as fields over the world-sheet. Let's stick to the first one for now.

The following brief review about how action is defined for particles is useful to introduce string action. Let's consider a classical free particle of mass m moving in a gravitational field described by the metric tensor $g_{\mu\nu}(x)$, with $D-1$ positive eigenvalues and a negative one. The particle moves along its world-line which is embedded into the D-dimensional spacetime. The action describing its motion is

$$S = -m \int d\tau \sqrt{-\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} g_{\mu\nu}(x)} = -m \int ds, \tag{1.1}$$

i.e. it minimizes path length[3]. This action has two important properties. It is Poincaré invariant and it is invariant by re-parametrization of the world-line, i.e. re-parametrization

[2] See [$KB07$], chapter 2, page 46
[3] See [$KB07$], page 18

of the proper time τ[4].

However, this action has two disadvantages. First, the presence of a square root makes quantization difficult. Second, it obviously cannot be used to describe a massless particle. In order to overcome these difficulties we introduce the auxiliary temporal metric $\eta(\tau)$. That produces an action equivalent to the previous one, where $\dot{x}^\mu \equiv \frac{dx^\mu}{d\tau}$,

$$S' = \frac{1}{2}\int (\eta^{-1}\dot{x}^\mu \dot{x}_\mu - \eta m^2) d\tau. \tag{1.2}$$

The equation of motion relative to η,i.e.

$$\frac{\partial S'}{\partial \eta(\tau)} = 0, \tag{1.3}$$

produces a constraint equation which can be interpreted as mass-shell condition generalized to the case of propagation in a curved space,i.e.

$$\eta^2 = -\frac{\dot{x}^\mu \dot{x}_\mu}{m^2}. \tag{1.4}$$

The relativistic point particle sweeping out its world-line is described by an action which is proportional to this world-line's length. Finally, solving the equation for $\eta(\tau)$ and substituting back into S' will give S.

Let's extend these facts to a n-dimensional object[5]. In the case of a string the action ends up to be proportional to the world-sheet's area:

$$S_P = \frac{-T}{2}\int d\tau d\sigma \sqrt{-\gamma}\gamma^{ab}(\sigma) g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu, \tag{1.5}$$

where γ_{ab} is the auxiliary metric of the world-sheet with signature $(-+)$, $\gamma = det\gamma_{ab}$, $g_{\mu\nu}$ is the metric tensor of the D-dimensional spacetime in which the world-sheet is thought to be embedded, and finally T is the string's tension. The action above is called the Polyakov's action[6].

Defining $h_{a,b} = g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$ as the induced spacetime metric on the world-sheet, it is

[4] These two basic properties will be appropriately extended to the case of string's action

[5] What follows as far as the end of this subsection is based on $[KB07]$, pages $19 - 30$

[6] $S_P \sim \int ds^2$, i.e. to minimize area is the natural extension

possible to obtain the Nambu-Goto action:

$$S_{NG} = -T \int d\tau d\sigma \sqrt{-det(h_{ab})} \tag{1.6}$$

The two actions are equivalent. In order to show that it is sufficient to derive the equations of motion for the auxiliary metric γ_{ab},("metric" tensor of the world sheet), which will produce the constraint equation (1.4) for the coordinates X^μ.

So far we have been sticking to the idea that the world-sheet is a spacetime embedded manifold. According to that, S_P describes strings propagation in an arbitrarily dimensional spacetime. However, we will see shortly that quantizing the theory will impose a constraint on spacetime dimension, which for consistency reasons will have to be equal to 26. So, the $X^\mu s$ transform as vectors under 26-dimensional Poincaré transformations. Both actions are also invariant for a generic (τ, σ) coordinate transformation of the world-sheet.

Now, according to the non embedded strings approach, both actions S_P and S_{NG} describe a covariant $(1+1)$-dimensional field theory. As we said above, in such a theory the $X^\mu s$ play the role of scalar fields. So, they transform as scalars for world-sheet re-parametrizations.

1.2 Symmetries of string action

Starting with S_{NG}, the action is invariant with respect to the D-dimensional Poincaré group of transformations[7]

$$X'^{\mu}(\tau, \sigma) = \Lambda^\mu_\nu X^\nu(\tau, \sigma) + a^\mu, \tag{1.7}$$

where Λ^μ_ν is a Lorentz transformation and a^μ is a translation.

From the world-sheet internal point of view,(not embedded world-sheet perspective), these global transformations are internal symmetries of D free and massless fields, which propagate along the string.

Incidentally, that won't change after quantization of the X_μ fields. Then, since quantization of the theory is symmetric with respect to the Poincaré group, the Hilbert space will provide us with a unitary representation of such group. That means that particles' states will be characterized by mass and spin. Since we are dealing with strings free to oscillate

[7]What follows about symmetries of S_{NG} is based on $[KB07]$, pages $24 - 25$

and with an infinite number of harmonics, an infinite number of particles will appear in the spectrum.

Moreover S_{NG} is invariant respect to diffeomorphisms (or reparametrizations)

$$X'^{\mu}(\tau', \sigma') = X^{\mu}(\tau, \sigma), \tag{1.8}$$

where $\tau' = f(\tau)$ and $\sigma' = g(\sigma)$.

Let's see now the symmetries of the action S_P[8]. S_P is invariant respect to Poincaré D-dimensional transformations:

$$X'^{\mu}(\tau, \sigma) = \Lambda^{\mu}_{\nu} X^{\nu}(\tau, \sigma) + a^{\mu}, \tag{1.9}$$

$$\gamma'_{\alpha\beta}(\tau, \sigma) = \gamma_{\alpha\beta}(\tau, \sigma). \tag{1.10}$$

It is also invariant by the following diffeomorphisms, or reparametrizations

$$X'^{\mu}(\tau', \sigma') = X^{\mu}(\tau, \sigma), \tag{1.11}$$

$$\frac{\partial \sigma'^{\gamma}}{\partial \sigma^{\alpha}} \frac{\partial \sigma'^{\delta}}{\partial \sigma^{\beta}} \gamma'_{\gamma\delta}(\tau', \sigma') = \gamma_{\alpha\beta}(\tau, \sigma). \tag{1.12}$$

Moreover, S_P is invariant by Weyl transformation[9],

$$X'^{\mu}(\tau, \sigma) = X^{\mu}(\tau, \sigma), \tag{1.13}$$

$$\gamma'_{\alpha\beta}(\tau, \sigma) = e^{2\omega(\tau, \sigma)} \gamma_{\alpha\beta}(\tau, \sigma), \tag{1.14}$$

for any rescaling factor $\omega(\tau, \sigma)$.

I will come back on Weyl transformations later on to study in some depth their crucial role in string theory. Let's mention for now just few things.

Thinking inside the view of an embedded world-sheet, the Weyl-invariance of the action can be understood considering the equation (1.4) - the equations of motion relative to the auxiliary metric. In fact, (1.4) determines $\gamma_{\alpha\beta}$ only by a factor of local re-scaling. Therefore,

[8] What follows about symmetries of S_P is based on[$KB07$], pages $30 - 31$

[9] In theoretical physics the Weyl transformation is a local rescaling of the metric tensor $\gamma_{\alpha\beta}$ which produce another metric in the same conformal class.

metrics which are equivalent by Weyl local transformations, correspond to the same spacetime coordinate X^μ.

Moreover, as I said above, from the world-sheet internal point of view S_P describes a Klein-Gordon massless scalar field X_μ, coupled in a covariant way with the metric γ_{ab}. Therefore, in this perspective Poincaré invariance appears to be an internal symmetry, i.e. acting on fields with σ and τ both fixed.

Finally, the variation of the action respect to the metric defines the energy-momentum tensor, which in the case of S_P is

$$T^{ab} = -4\pi \frac{1}{\sqrt{-\gamma}} \frac{\partial S_P}{\partial \gamma_{ab}} = -2\pi T(\partial^a X^\mu \partial^b X_\mu - \frac{1}{2}\gamma^{ab}\partial_c X^\mu \partial^c X_\mu). \tag{1.15}$$

From the invariance with respect to diffeomorphisms the following conservation law follows:

$$\nabla_a T^{ab} = 0, \tag{1.16}$$

whereas the invariance of S_P respect to Weyl local transformations entails

$$\gamma_{ab}\frac{\delta S_P}{\delta \gamma_{ab}} = 0 \Rightarrow T^a_a = 0. \tag{1.17}$$

The vanishing of the trace of the energy momentum tensor is an important fact. It provides us with the constraint equation for fields in the classical perspective and with the constraint equation for physical states in the quantum perspective.

1.3 Equations of motion, but not only

In this section I will derive strings equation of motions. Since Weyl invariance is used here to find them in a particularly suitable form, I will also make an introductory remark about its crucial role for string theory. That will set the stage for further analysis developed in section 1.5.

The variation of S_P with respect to X^μ produces the following equations of motion[10]

$$\partial_a(\sqrt{-\gamma}\gamma^{ab}\partial_b X^\mu = \sqrt{-\gamma}\nabla^2 X^\mu = 0, \tag{1.18}$$

with the following boundary conditions (to ensure the vanishing of the surface term)[11]:
(i) Neumann boundary conditions for open strings

$$\partial_\sigma X^\mu(\tau, 0) = 0, \partial_\sigma X^\mu(\tau, \pi) = 0; \tag{1.19}$$

(ii) Dirichelet boundary conditions for closed strings (the fields in this case are periodic, the ending points of the string are joined to form a loop:

$$\partial_\sigma X^\mu(\tau, 0) = \partial_\sigma X^\mu(\tau, \pi) \tag{1.20}$$

$$X^\mu(\tau, 0) = X^\mu(\tau, \pi)$$

$$\gamma_{ab}(\tau, 0) = \gamma_{ab}(\tau, \pi).$$

As I said above, Weyl invariance is crucial for string theory. Here, we can start to see why. The internal metric tensor γ_{ab} has three independently variable components, being a symmetric 2×2 matrix. Weyl invariance contributes to fix them just choosing an appropriate gauge[12]. More precisely, the invariance of the action with respect to the diffeomorphisms internal to the world-sheet along with Weyl local invariance can fix the three variable parameters of the matrix. A convenient choice of gauge then is that of picking the *conformal gauge*[13],

$$\gamma_{ab} = \eta_{ab} e^\phi. \tag{1.21}$$

Here we have an internal, flat metric multiplied a positive function, which is known as conformal factor. The latter is a scaling factor, i.e. it preserves angles but not lengths. So, (1.21) is a *conformally flat* metric.

So, using the gauge symmetry of the theory, S_P becomes the action of a free field and the

[10]The content of this section is a reformulation based on $[KB07]$, pages $31-34$
[11]Both types of boundary conditions are consistent with Poincaire D-dimensional invariance.
[12]For a more detailed analysis of this point see $[KB07]$, page 31, in particular the equation (2.23)
[13]See $[KB07]$, page 59, equation (3.2)

equations appear to be familiar. In fact they become the wave equation in dimension two

$$(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2})X^\mu(\tau,\sigma) = 0. \tag{1.22}$$

However, producing nicer and simpler equations is not the only interesting consequence of the theory's gauge symmetry. Before giving the general solution of this equation, I want to anticipate here something very important which will be developed in section 1.5. Weyl invariance in conjunction with diffeomorphism invariance means that the string is conformally invariant, i.e. string theory can be considered to be a conformal field theory. This fact has very important consequences for the formal articulation of the theory and for understanding a possible way of approaching the seemingly emergent character of spacetime in string theory. We will come back on this in section 1.5.

In conclusion, the general solution of the above equation breaks down in two parts

$$X^\mu(\tau,\sigma) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-), \tag{1.23}$$

where X^μ_R describes the right-moving modes of the string, X^μ_L the left-moving ones and finally the equations are given in *light-cone* coordinates, i.e. $\sigma^+ = \tau + \sigma$ and $\sigma^- = \tau - \sigma$.

The equations obtained from the action with fixed gauge must be joined to some constraint equations, which - as seen above - derive from the vanishing of the action variation respect to the metric or, in other words, deriving from the vanishing of the energy-momentum tensor's components as defined in (1.16). Let's reformulate these constraints in light-cone coordinates as well.

The metric in the new coordinates becomes:

$$ds^2 = -d\tau^2 + d\sigma^2 = -d\sigma^+ d\sigma^-$$

so

$$\eta_{-+} = \eta_{+-} = -\frac{1}{2} \tag{1.24}$$

$$\eta{-+} = \eta{+-} = -2$$

$$\eta_{++} = \eta_{--} = \eta{++} = \eta{--} = 0,$$

moreover

$$\partial_\tau = \partial_+ + \partial_-,$$

$$\partial_\sigma = \partial_+ - \partial_-.$$

Therefore, the constraints on the components of T become:

$$T_{++} = \frac{1}{2}(T_{\tau\tau} + T_{\tau\sigma}) = -\frac{1}{\alpha'}\partial_+ X^\mu \partial_+ X_\mu \equiv -\frac{1}{\alpha'}\dot{X}_L^2 = 0 \tag{1.25}$$

$$T_{--} = \frac{1}{2}(T_{\tau\tau} - T_{\tau\sigma}) = -\frac{1}{\alpha'}\partial_- X^\mu \partial_- X_\mu \equiv -\frac{1}{\alpha'}\dot{X}_R^2 = 0.$$

1.4 Mode expansion

For open strings the equations of motion above with Neumann boundary conditions have solutions[14]

$$X^\mu(\tau,\sigma) = x^\mu + 2\alpha' p^\mu \frac{\sigma^+ + \sigma^-}{2} + i\sqrt{(2\alpha')}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-i\frac{n}{2}\sigma^+} e^{-i\frac{n}{2}\sigma^-} cosn(\frac{\sigma^+ - \sigma^-}{2}); \tag{1.26}$$

whereas for closed strings with periodic boundary conditions:

$$X_R^\mu(\sigma^-) = \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-} \tag{1.27}$$

$$X_L^\mu(\sigma^+) = \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\widetilde{\alpha}_n^\mu e^{-in\sigma^+},$$

where in order to obtain real solutions we'll have to impose that

$$\alpha_{-n}^\mu = (\alpha_n^\mu)^*, \tag{1.28}$$

$$\widetilde{\alpha}_{-n}^\mu = (\widetilde{\alpha}_n^\mu)^*.$$

[14]Following [$KB07$], pages $34 - 37$

The mode expansions of $\partial_- X_R^\mu$ and $\partial_+ X_L^\mu$ can be computed in the same way,

$$\partial_- X_R^\mu = \sqrt{2\alpha'} \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{-in\sigma^-} \tag{1.29}$$

$$\partial_+ X_L^\mu = \sqrt{2\alpha'} \sum_{n=-\infty}^{\infty} \widetilde{\alpha}_n^\mu e^{-in\sigma^+}.$$

Therefore, we can clearly see that mode expansions for closed strings are like a couple of oscillators, traveling in opposite directions along the string, whereas modes expansion for open strings are standing waves which represent the fact that left-moving side and right-moving side are the reflections of one another, due to the Neumann boundary conditions. Moreover, in both cases x^μ and p^μ are position and momentum of the string's center of mass, and p^μ is also identified with the zero mode of expansion, $\alpha_0^\mu = \sqrt{2\alpha'}p^\mu$ for open strings and $\alpha_0^\mu = \widetilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$ for closed strings.

1.5 Some remarks on Conformal invariance.

As I said in section 1.3, Weyl invariance in conjunction with diffeomorphism invariance yield a theory which is conformally invariant. This fact is crucial. For it has deep implications for the notion of space and time over the string world-sheet. Moreover, Einstein's equations for the gravitational field can be mathematically derived from the theory's conformal invariance, supporting the idea of an emergent ordinary spacetime inside a more fundamental and potentially complete theory. These two issues, deeply connected to each other, will be analyzed in what follows. The first subsection will unravel the former by presenting some aspects of Witten's popular view, whereas the second subsection will develop the latter.

1.5.1 Theory's conformal invariance I: length and duration over the string lose physical significance

In "Time in Quantum Gravity", we describe Witten's idea about the ontology of string theory as a form of relationism: "since it reduces the space and time of experience to the spatiotemporal properties of material points, those of the string; and perhaps it is no more radical than other forms of relationism."[15]. Here I will not discuss our interpretation of Witten's view in terms of a "stringy" relationalism because that will be one of the topics presented in chapter 3. However, I shall use the way in which in that paper we extrapolated from Witten's claim about conformal invariance of the theory some crucial conceptual features of space and time over the string worldsheet.

A passage in Witten's "Reflections on the Fate of Spacetime" relevant to this aim is the following[16]:

> So we arrive at a quite beautiful paradigm. Whereas in ordinary physics one talks about spacetime and classical fields it may contain, in string theory one talks about an auxiliary two-dimensional field theory that encodes the information. The paradigm has a quite beautiful extension: a spacetime that obeys its classical field equations corresponds to a two-dimensional field theory that is conformally invariant. If one computes the conditions needed for conformal invariance of the quantum theory derived from the Lagrangian, assuming the fields to be slowly varying on the stringy scale, one gets generally covariant equations that are simply the Einstein equations plus corrections of order $\alpha^{'}$.

What Witten is discussing in this passage is the crucial fact that the two-dimensional quantum field theory describing fluctuations of the string world-sheet is conformally invariant. The latter feature imposes constraints on the fields used to construct the two-dimensional action that reduce to Einstein's equations. An ordinary spacetime is a geometrical and dynamical entity obeying Einstein's fields equations. So, in this sense it is a derived concept. A mathematical derivation of Einstein's equations from conformal invariance will be presented in the next subsection. Here I am concerned with a correlated conceptual issue entailed by Witten's passage about conformal invariance.

As I said in the introduction of this chapter, physics textbooks usually present strings as objects embedded in ordinary spacetime. But this way of looking at the $X^{\mu}(\tau, \sigma)$ should not be seen as a more truthful model of physical dynamics. In fact, according to Witten, talking about the ontological commitment of string theory we should accept things for what they appear. And what would appear is just that the $X^{\mu}(\tau, \sigma)$s are fields living on the

[15] See [N.Hng], page 11, 12

[16] [Wit96], page 28

strings. Interpreting the passage quoted above, the idea of an ordinary spacetime in which the string's worldsheet is embedded would play an unnecessary descriptive role in the theory. It is in fact an idea having a role only inside a derived description of reality arising from conformal invariance of a more fundamental field theory describing fluctuations of the string world-sheet. Moreover, somewhere else in the same paper Witten says[17]:

> So spacetime with its metric determines a two-dimensional field theory. And that two-dimensional field theory is *all* one needs to compute stringy Feynman diagrams.

Therefore, Witten suggests the need of shifting from a model postulating an unnecessary spacetime to a more fundamental model in which the X^μ is a field over the string, whose values depend on the stringy "spatial" and "temporal" parameters. The latter should be preferred to the former since it encodes just the necessary information. So, according to Witten, the emergent nature of ordinary spacetime in string theory should be conceptualized in terms of emergence from a conformally invariant formal structure.

But then a question arises: what should we say about τ and σ? Do these stringy parameters replace in the theory some space-like and time-like notions? They might seem to play the same role as ordinary spacetime after all, except for the fact that they are confined to the lower dimensional string's worldsheet. So, is there any possibility that their role in the theory undermines the idea of a strings' world completely lacking of ordinary metrical properties? Answering this question requires using the notion of Weyl invariance.

Let's recall the definition of Weyl transformation: it is as a local rescaling of the intrinsic string metric γ living on the worldsheet

$$\gamma \longrightarrow \exp^{\omega(\tau,\sigma)} \gamma$$

for any smooth function $\omega(\tau, \sigma)$.
Being Weyl invariant means being invariant under these local rescaling. Since now on, space and time over the string world-sheet will be called respectively σ-space and τ-time.

String worldsheets have causal structure since they can be divided into σ-spacelike, τ-timelike and lightlike paths. This causal structure is preserved by the Weyl invariance of the

[17] [Wit96], page 27

auxiliary metric γ because the quantity $\exp^{\omega}$ is strictly positive and hence it does not change the line element's sign. However, Weyl invariance of γ does not preserve the line element's lengths. This fact means that the length assigned by γ to any curve has no physical significance, since it can be rescaled to anything one chooses. Moreover, a profound implication for the notion of τ-time is that the"proper τ-time' of a τ-timelike curve, which we would expect to be an invariant quantity associated to the curve, is not preserved through Weyl transformations. So, the notion of proper tau-time is lost in the theory[18].

In conclusion, the Weyl invariance of the theory provides the stringy parameters τ and σ with features which are deeply incompatible with ordinary notions of space and time. The former lack of those basic metrical properties peculiar to the latter. This fact seems to undermine the idea that τ and σ could replace in the theory some space-like and time-like notions.

1.5.2 Theory's conformal invariance II: deriving Einstein's equation for the gravitational field

As I mentioned above, Einstein's gravitational field equation mathematically emerges from the conformal invariance of the theory. Getting ordinary spacetime out of the theory is part of the perturbative calculation of the string S-matrix in weakly curved spacetimes[19]. What follows describes the main aspects of this important derivation.

My starting point is the formula (25) in Rickle's paper, (see [Ric10], page 59),

$$P(X) = \sum_g \int_{M_g} \int D\Phi \exp^{iS[\Phi, G_{\mu\nu}]} . \tag{1.30}$$

How should we read this formula? The history of a physical system is represented by its worldsheet, hence it is represented by a Riemann surface, here denoted by Σ[20]. The space M_g over which we compute the external integral is the moduli space of Riemann surfaces. So that integral is a sum over all possible histories of the physical system.

[18] See [N.Hng], page 12.

[19] What follows is based on "String theory, Superstring theory and beyond",(vol I), by Joseph Polchinski, pages 108-120, see [Pol05]. See also Rickles in [Ric10], page 59.

[20] I will come back to Riemann surfaces later on in chapter 4. There some details about these surfaces will be useful to develop the issue of spacetime non commutativity. However, here for the purpose of the present section it is enough to say they represent strings worldsheets.

What are we summing over that space then? The quantity being summed is represented by the internal integral $\int D\Phi \exp^{iS[\Phi,G_{\mu\nu}]}$. Fixing a particular history Σ we have infinitely many ways of embedding that surface in a target spacetime X, so we have infinitely many maps like the following:

$$\Phi : \Sigma \longrightarrow X. \tag{1.31}$$

Choosing a local analytic coordinate z on the surface Σ and then embedding this surface in spacetime through one of the Φs above, we have that all Σ's points end up with having spacetime coordinates $\Phi^\mu(z)$. Now the internal integral is a sum over all possible way of embedding Σ in a target spacetime X, i.e it is a sum over all possible Φs. But then this integral is the partition function of the two-dimensional quantum field theory whose action is $S[\Phi, G_{\mu\nu}]$, where $G_{\mu\nu} = G_{\mu\nu}(\Phi)$ is the induced spacetime metric field over $\Phi(\Sigma)$, which depends on the embedding Φ. This action plays a central role in involving the background into the calculation:

$$S[\Phi, G_{\mu\nu}] = \frac{1}{4\pi\alpha'} \int_\Sigma d^2z \sqrt{g} g^{ab} G_{\mu\nu}(\Phi) \partial\Phi^\mu \partial\Phi^\nu, \tag{1.32}$$

where g^{ab} is the Euclidean worldsheet metric.

For detail about how to get its mathematical expression I refer the reader to Polchinski's chapter[21]. As Polchinski says, this action should be thought as describing a coherent state of gravitons."A curved spacetime is roughly speaking a coherent background of gravitons, and therefore in string theory is a coherent state of strings"[22]. Since we have a spacetime close to flat the curved metric has the form

$$G_{\mu\nu} = \eta_{\mu\nu} + \chi_{\mu\nu}, \tag{1.33}$$

with $\chi_{\mu\nu}$ being the very small contribution given by the string, formally represented by the vertex operator for the graviton state. I am not including here backgrounds of other massless string states.

[21] See [Pol05], page 108, formula (3.7.2).
[22] See [Pol05], page 108.

A field theory such as the action above is an interacting two-dimensional quantum field theory. Expanding the path integral around the classical solution ϕ_0 and replacing $\Phi(z) = \phi_0 + Y(z)$ in $G_{\mu\nu}(\Phi)$ we get the following expansion[23]

$$G_{\mu\nu}(\Phi) = G_{\mu\nu}(\phi_0^\mu + Y^\mu) = G_{\mu\nu}(\phi_0^\mu) + \partial_\omega G_{\mu\nu}(\phi_0^\mu)Y^\omega + \partial_\omega \partial^\rho G_{\mu\nu}(\phi_0^\mu)Y^\omega Y^\rho + ... \tag{1.34}$$

Then replacing in $G_{\mu\nu}(\Phi)\partial\Phi^\mu\partial\Phi^\nu$,

$$G_{\mu\nu}(\Phi)\partial\Phi^\mu\partial\Phi^\nu = [G_{\mu\nu}(\phi_0^\mu) + \partial_\omega G_{\mu\nu}(\phi_0^\mu)Y^\omega + \partial_\omega \partial^\rho G_{\mu\nu}(\phi_0^\mu)Y^\omega Y^\rho + ...]\partial Y^\mu \partial Y^\nu. \tag{1.35}$$

So the action can be rewritten as

$$S[\Phi] = \frac{1}{4\pi\alpha'} \int_\Sigma d^2z \sqrt{g} g^{ab} [G_{\mu\nu}(\phi_0^\mu) + \partial_\omega G_{\mu\nu}(\phi_0^\mu)Y^\omega + \partial_\omega \partial^\rho G_{\mu\nu}(\phi_0^\mu)Y^\omega Y^\rho + ...]\partial Y^\mu \partial Y^\nu. \tag{1.36}$$

We can see how the spacetime metric and its derivatives appear as infinitely many couplings in the two-dimensional field theory. Therefore, this is how the field theory encodes the spacetime background in which the strings move and interact.

Now, in order to compute a string S-matrix which is physically meaningful we need the partition function being well-defined on the moduli space of Riemann surfaces, which in other words means that such function should be invariant with respect to local rescaling by a local scale factor $\exp^{f(z)}$ in the two-dimensional string metric. This local scale invariance is equivalent to conformal invariance. The latter is characterized by the vanishing of the trace of the of the stress energy tensor over the string, i.e. $T_a^a = 0$ and the former means the vanishing of the renormalization group β-function. In fact the β functions govern the dependence of the physics on the world-sheet scale and they are expressed as the derivatives of the effective couplings of the field theory with respect to a change of the two-dimensional scale.

[23]The expansion above is obtained in the usual way, starting with an expansion up to the first order derivative

$$G_{\mu\nu}(\phi_0^\mu + Y^\mu) - G_{\mu\nu}(\phi_0^\mu) = \partial_\omega G_{\mu\nu}(\phi_0^\mu)Y^\omega$$

and then continuing the expansion in higher order derivatives.

Using Polchinski's formulas (3.7.12) and (3.7.13a)[24] along with reminding the reader we are here neglecting the presence of backgrounds of other massless string states, we have that the vanishing of the string metric tensor's trace is

$$0 = T^a_a = -\frac{1}{2\alpha^{'}}\beta^G_{\mu\nu} g^{ab}\partial_a\Phi^\mu\partial_b\Phi^\nu, \tag{1.37}$$

where the coefficients $\beta^G_{\mu\nu}$ are essentially the renormalization group β-functions. The variation of scale consist in taking the derivatives of the coupling showing up in our expansion (1.36). So considering the expansion up to linear order with respect to the component $\chi_{\mu\nu}$ of $G_{\mu\nu}$ and remembering we have scale invariance we have

$$0 = \beta^G_{\mu\nu} = -\frac{\alpha^{'}}{2}(\partial^2\chi_{\mu\nu} - \partial_\nu\partial^\omega\chi_{\mu\omega} - \partial_\mu\partial^\omega\chi_{\omega\nu} + \partial_\mu\partial_\nu\chi^\omega_\omega + ...). \tag{1.38}$$

A combination of the linear second derivative with high order contributions yields on the right hand side of the approximated equation the spacetime Ricci tensor[25]

$$0 = \beta^G_{\mu\nu} = \alpha^{'}\mathbf{R}_{\mu\nu} + O(\alpha^{'2}). \tag{1.39}$$

Therefore from the vanishing of the β-functions we obtain an equation that resembles that of Einstein for the gravitational field. So, the S-matrix is physically meaningful just in case the partition function of the two-dimensional field theory is scale invariant and hence just in case the two-dimensional field theory is conformally invariant. But this condition produces the fact that the spacetime background is a solution of the equation of motion.

1.6 Brief description of some quantization procedures

Butterfield and Isham[26] introduce an interesting distinction among different approaches to quantum gravity. Some strategies regards general relativity "just as another classical field theory" to be quantized in a standard way. Some others present general relativity as the low-energy limit of a quantization of a different classical theory, for example limit of quantized

[24]See [Pol05], page 111
[25]See [Pol05], page 111
[26]See Butterfield and Isham in [$J.B$99], section 3

string theories. Since the latter is the approach I am using here to study the notion of spacetime emergence, it seems worthy to have some grasp of how a quantized string theory looks like. To this aim I introduce in this section some basics of quantization procedures.

Repeating an important point, the Polyakov's action defined earlier on, i.e.

$$S_P = \frac{-T}{2} \int d\tau d\sigma \sqrt{-\gamma}\gamma^{ab}(\sigma) g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu, \tag{1.40}$$

can be considered either as the action describing the propagation of a string embedded in spacetime, or the action of a bi-dimensional conformal field theory. If we take X^μ to be spacetime coordinates then the process of quantization of the action is called first quantization, whereas if we take them to be fields it is called second quantization. The former is the procedure I will apply in this chapter.

There are several ways in which first quantization can be performed. One is the canonical approach[27], which uses equal time commutators and constraint equations to get the Hilbert space of the physical states. A second approach[28] uses instead the formalism of path integrals, more precisely the following path integral

$$Z = \int [dg(\sigma) dX(\sigma)] e^{-S[g,X]}, \tag{1.41}$$

where

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X_\mu. \tag{1.42}$$

In this section I shall describe the latter. S in the formula above is the same as the Polyakov action S_P seen above. In this action formula I assume that strings are propagating trough a flat spacetime and so I am replacing Minkowski metric $\gamma_{ab}(\tau, \sigma)$ with the euclidean metric $g_{ab}(\sigma^1, \sigma^2)$ of signature $(++)$. The integral is taken over all the possible $g_{ab}(\sigma^1, \sigma^2)$ and over all the possible $X^\mu(\sigma^1, \sigma^2)$. The advantage of using an euclidean path integral consists in having an integral which is well defined almost everywhere. In fact, the euclidean metric can be not singular even in cases of topologically non trivial surfaces. The Minkowski metric lacks this good property.

[27]See $[KB07]$,section 2.4

[28]The following schematic presentation of this procedure is based on the content of $http : //www.physics.thetangentbundle.net/wiki/Quantum_field_theory/gauge_theory/Faddeev-Popov_procedure$

This procedure is called Faddeev-Popov quantization. What follows is a schematic presentation of it.

The integral Z above should be rewritten more informatively as

$$Z = \int \frac{[dgdX]}{V_{diff \times weyl}} e^{-S}. \tag{1.43}$$

In fact, the first integral contains a redundancy due to the local symmetries of the action, symmetries that link up several metric and fields configurations: if (X, g) and (X', g') are connected by a diffeomorphism or by a Weyl transformation, then they represent the same physical configuration. That is why in (1.43) the measure is divided by the volume of this local symmetries group.

The computation of Z starts usually by fixing a gauge[29]. Then, we try to obtain the correct measure of integration over the portion of gauge volume[30].

Before specifying what the correct measure is, I'll briefly describe in words what is the main idea underlying the computaion of Z once we get the correct measure. In fact at that point we integrate over a portion that intersects each equivalence class. These classes are gauge orbits - geometrically speaking they are curves. Integrating all over the space of the fields and dividing it by that volume - which is what we do in (1.43) - is equivalent to integrating over a portion which intersects each orbit just one time, using the appropriate jacobian, which is the Faddeev Popov determinant.

Now, the correct measure of integration is the Faddeev-Popov measure $\triangle_{FP}$, defined in the following way

$$1 = \triangle_{FP}(g) \int [d\zeta] \delta(g - g^{\zeta}) \tag{1.44}$$

where $[d\zeta]$ is the gauge invariant measure over the group $G = Weyl \times diff$ and δ is the delta function.

$\triangle_{FP}$ is the jacobian - the Faddeev Popov determinant - of a suitable coordinates transformation ζ:$g \to g^{\zeta}$[31].

[29] $g_{ab} = \eta_{ab} e^{\phi}$

[30] The correct measure will be defined here without explaining how it can be derived. For more detail about how this derivation works, see $http://www.physics.thetangentbundle.net/wiki/Quantum_field_theory/gauge_theory/Faddeev-Popov_procedure$

[31] More detail here is not needed.

Replacing the left hand side of (1.44) in (1.43) we obtain

$$Z[g] = \int \frac{[d\zeta dg dX]}{V_{diff \times weyl}} \Delta_{FP} \delta(g - g^{\zeta}) e^{-S[X,g]}. \tag{1.45}$$

The action is invariant under transformations ζ, i.e. $S[X,g] = S[X,g^{\zeta}]$. Integrating (1.45) with respect to g and renaming the variable X with X^{ζ} we have:

$$Z[g] = \int \frac{[d\zeta dX^{\zeta}]}{V_{diff \times weyl}} \Delta_{FP}(g^{\zeta}) e^{-S[X^{\zeta},g^{\zeta}]}. \tag{1.46}$$

Using the invariance of S, Δ_{FP} and $[dX]$ we have

$$Z[g] = \int \frac{[d\zeta dX]}{V_{diff \times weyl}} \Delta_{FP}(g) e^{-S[X,g]}. \tag{1.47}$$

Let's notice at this point that nothing inside the integral (1.47) depends on ζ, which is one of the variable of integration. Therefore, the integration with respect of ζ will produce a multiplicative factor, which is exactly the volume $V_{Weyl \times diff}$ of the group $G = Weyl \times diff$. This multiplicative factor will cancel the denominator in (1.43).
So, we obtain the following expression:

$$Z[g] = \int [dX] \Delta_{FP}(g) e^{-S[X,g]}. \tag{1.48}$$

The computation of the Faddeev-Popov determinant $\Delta_{FP}(g)$ would require the introduction of new fields, called *gosths* and *antigosths*, but this goes beyond the scope of this section. Therefore, it is sufficient to say here that what we have in (1.48) is the Polyakov path integral.

Chapter 2

T-duality. Philosophical and mathematical features

This chapter starts with the presentation of an old and familiar argument, Poincaré's under-determination problem[1].

The problem of under-determination inevitably comes up in the analysis of dual string theories. Two string theories postulating two geometrically inequivalent backgrounds for string propagation are in principle experimentally distinguishable. In fact, physical properties of vibrating strings, like their masses and the force charges they carry, are largely determined by the postulated geometrical properties of the extra-dimensions. However, it turns out that if they are two *dual* theories, they are experimentally indistinguishable. In other words, same physical properties and same physical dynamics of vibrating strings can be determined by geometrically inequivalent backgrounds. Hence, Poincaré's under-determination argument seems to apply to this case because empirical data concerning physical properties and physical dynamics of strings are not sufficient to determine which one between the two postulated geometries is the correct geometry of the world. But the implications of duality can be worst than that. It's not just that there are no facts about the correct geometry of the fundamental world, but it is also that there are no facts about being there some ultimate geometrical structure.

[1] I don't attempt a historical reading of Poincaré, but I follow the account offered by L. Sklar in "Space, time and spacetime", (see [$L.S77$], section F, pages $89 - 103$).

The second part of the chapter presents further philosophical remarks on the notion of dual theories, in particular T-dual ones, along with an introduction to the mathematical features of T-duality.

2.1 Poincaré and the under-determination problem

Before starting I would like to say that the bibliographic source which guided my understanding of Poincaré's position is the book of L.Sklar, "Space, time and spacetime"[2]. Sklar develops a conventionalist position, which he attributes to Poincaré. Here I will not be concerned with the issue of correctness of his interpretation, but on the view that Sklar presents, since that will illuminate the important philosophical points. Therefore, let's say that here to speak of Poincaré is short-hand for "Sklar's Poincaré".

This section is a kind of warm-up for what will follows about the conceptual features of T-duality and also dualities in general. Poincaré's argument in favor of conventionalism in geometry contains a step which is called "the under-determination problem".This problem was formulated for the first time by Duhem, a French physicist who lived at the turn of the 20^{th} Century[3]. Duhem argued that the problem of scientific under-determination posed serious challenges to our efforts to confirm theories in physics. As Quine[4] suggested later on such challenges applied not only to the confirmation of physics theories, but to all knowledge claims.

It is widely known that Poincaré formulated the under-determination problem inside the conceptual framework of geometry. Moreover, thereafter many philosophers and scientists applied the structure of his argument to several different contests of epistemological discussions concerning scientific theories. Here I will be concerned with analyzing how his under-determination argument comes up in string theory relatively to the problem of dualities.

[2]See [L.S77], section F, pages 89-103

[3]For a general discussion on Duhem's presentation of the problem see http://plato.stanford.edu/entries/scientific-underdetermination/, whereas for a more deep analysis of his work see his "The Aim and Structure of Physical Theory", trans. P.Wiener, Princenton Science Library, 1991. Here I will not be concerned with an analysis of the content of Duhem's position on this issue

[4]For a general discussion on Quine's position see http://plato.stanford.edu/entries/scientific-underdetermination/, whereas for a detailed analysis of his position see his "On Empirically Equivalent Systems of the World", Erkenntnis, 9 : 313328, 1975. Here I will not be concerned with a study of the philosophical view of Quine on this issue

Poincaré[5] was able to develop a deep and clever examination of the status of geometric knowledge right before the revolution of general relativity. He started by presenting a consistency proof for non Euclidean geometries, confuting in this way all those claims about the logical inconsistency of these geometries. He then criticized the Kantian view of Euclidean geometry as the correct geometrical structure of the world.

Let's briefly present Kant's view of Euclidean geometry. That might provide us with a deeper understanding of Poincaré's critique of Kant. What did Kant really mean in viewing Euclidean geometry as the correct geometrical structure of the world? It is widely known that one of the main goals that Kant pursued in the First Critique was that of unearthing the a priori foundations of Newtonian physics, which describes the structure of the world in terms of Euclidean geometry. How did he achieve that? [6]. Kant maintained that our understanding of the physical world had its foundations not merely in experience, but in both experience and "a priori" concepts. Here I will not analyze in detail his transcendental arguments. I will just mention that he argues that the possibility of sensory experience depends on certain necessary conditions which he calls "a priori" forms and that these conditions structure and hold true of the world of experience. As he maintains in the "Transcendental Aesthetic", Space and Time are not derived from experience but rather are its preconditions[7]. Experience provides those things which we sense. It is our mind, though, that processes this information about the world and gives it order, allowing us to experience it. Our mind supplies the conditions of space and time to experience objects. Thus "space" for Kant is not something existing - as it was for Newton. Space is an "a priori" form that structures our perception of objects in conformity to the principles of the Euclidean geometry. In this sense, then, the latter is the correct geometrical structure of the world. It is necessarily correct because it is part of the "a priori" principles of organization of our experience[8].

This claim is exactly what Poincaré criticized about Kant's view of geometry. Poincaré[9] did not agree with Kant's view of space as precondition of experience. He thought that our knowledge of the physical space is the result of inferences made out of our direct perceptions.

[5] See Sklar, [$L.S77$], section$F$,pages $89 - 90$

[6] The answer to this question is grounded in my interpretation of some passages of the First Critique: see [Kan98], A 23/B 38, A 39/B 56

[7] See [Kan98], A 23/B 38

[8] See [$Kan98$],$A39/B56$

[9] See Sklar, [$L.S77$], section$F$,pages 90

This knowledge is a theoretical construct, i.e, we infer the existence and nature of the physical space as an explanatory hypothesis which provides us with an account for the regularity we experience in our direct perceptions. But this hypothesis does not possess the necessity of an "a priori" principle that structures what we directly perceive. Although Poincaré does not endorse an empiricist account, he seems to think, though, that an empiricist view of geometry is more adequate than Kantian conception. In fact, the idea that only a large number of observations inquiring the geometry of physical world can establish which geometrical structure is the correct one, is considered by him as more plausible. But, this empiricist approach is not going to work as well. In fact Poincaré does not endorse an empiricist view of geometry. The outcome of his considerations about a comparison between the empiricist and Kantian accounts of geometry is well described by Sklar[10]: "Nevertheless", Sklar says, "the empiricist account is wrong. For, given any collections of empirical observations a multitude of geometries, all incompatible with one another, will be equally compatible with the experimental results".

This is the problem of under-determination of hypotheses about the geometrical structure of physical space by experimental evidence. The under-determination is not due to our ability to collect experimental facts. No matter how rich and sophisticated are our experimental procedures for accumulating empirical results, these results will be never enough compelling to support just one of the hypotheses about the geometry of physical space - ruling out the competitors once for all. Actually, it is even worse than that: empirical results seem not to give us any reason at all to think one of the other hypothesis correct. Poincaré thought that this problem was grist to the mill of the conventionalist approach to geometry. The adoption of a geometry for physical space is a matter of making a conventional choice.

A brief description of Poincaré disk model might unravel a bit more the issue that is coming up here[11]. The short story about this imaginary world shows that an empiricist account of geometry fails to be adequate. In fact, Poincaré describes a scenario in which Euclidean and hyperbolic geometrical descriptions of that physical space end up being equally consistent with the same collection of empirical data. However, what this story tells us can be generalized to any other scenario, including ours, in which a scientific inquiry concerning

[10] See [$L.S77$], page 89

[11] See [$L.S77$], pages $92, 93$

the intrinsic geometry of the world is performed.

The imaginary world described in Poincaré's example is an Euclidean two dimensional disk heated to a constant temperature at the center, whereas, along the radius R, it is heated in a way that produces a temperature's variation described by $R^2 - r^2$. Therefore, the edge of the disk is uniformly cooled to 0^0.

A group of scientists living on the disk are interested in knowing what the intrinsic geometry of their world is. As Sklar says, the equipment available to them consists in rods uniformly dilating with increasing temperatures, i.e. at each point of the space they all change their lengths in a way which is directly proportional to temperature's value at that point. However, the scientists are not aware of this peculiar temperature distortion of their rods. So, without anybody knowing, every time a measurement is performed, rods shrank or dilated, depending if they are close to the edge or to the center. After repeated measurements all over the disk, they have a list of empirical data that seems to support strongly the idea that their world is a Lobachevskian plane. So, this view becomes the official one. However, a different data's interpretation is presented by a member of the community who, striking a discordant note, claims that those empirical data can be taken to indicate that the world is in fact an Euclidean disk, but equipped with fields shrinking or dilating lengths.

Although the two geometrical theories about the structure of the physical space are competitors, the empirical results collected by the scientists support both of them. According to our external three-dimensional Euclidean perspective we know their bi-dimensional world is Euclidean and so we know that only the innovator's interpretation is the correct one. Using our standpoint the problem of under-determination would seem indeed a problem of epistemic access due to the particular experimental repertoire of the inhabitants. After all expanding this repertoire and increasing the amount of empirical data can overcome the problem. But according to Poincaré that would completely miss the point. Moving from our "superior"[12] perspective to their one would collocate us exactly in the same situation as they are, i.e. in the impossibility to decide which geometry is the correct one. But more importantly, Poincaré seems to say that any arbitrarily large amount of empirical data cannot refute a geometric hypothesis[13]. In fact, a scientific theory about space is divided in two branches, a

[12] See [$L.S77$], page 93.
[13] See [$L.S77$], page 97

geometric one and a “physical” one. These two parts are deeply related. It would be possible to save from experimental refutation any geometric hypothesis about space, suitably changing some features of the physical branch of the theory[14]. According to Sklar, this fact forces Poincaré to the conclusion that the choice of one hypothesis among several competitors is purely conventional.

The problem of under-determination comes up in the analysis of dual string theories. As I mentioned in the introduction two string theories postulating two geometrically inequivalent backgrounds, if dual, can produce the same experimental results: same expectation values, same scattering amplitude, and so on. Therefore, similarly to Poincaré's short story, empirical data relative to physical properties and physical dynamics of strings are not sufficient to determine which one between the two different geometries postulated for the background is the right one, or if there is any more fundamental geometry at all influencing physical dynamics.

In the next sections I will attempt to present a more detailed analysis of these issues. We'll be mainly focused on T-duality.

2.2 T-duality. Mathematical features and conceptual implications

The notion of T-duality arises from combining together certain compact dimensions and strings. Intuitively speaking, in string theory “the action” of compactifying one or more extended dimensions of a manifold means basically wrapping each of them in a circle. In this way they become periodic variables with period 2π times the circle's radius. The mathematical procedure involved in this process will be here presented in the case of bosonic string theory. Unraveling technical details about bosonic T-duality and bosonic theory compactification requires introducing some preliminary remarks about duality and compactification procedures.

Before unraveling technical details let's introduce a broader conceptual framework.

[14]See Sklar in [$L.S77$], page 97.

2.2.1 Some preliminary remarks on dual theories. What does it mean "same physical content"?

The notion of duality is different from that of physical symmetry and gauge symmetries[15]. In chapter one we saw what physical symmetries of bosonic string theory are. Physical symmetries map distinct physical states to one another. An orbit under their group action over the Hilbert space of physical states $\mathcal{H}$ is made of points, each of them representing different physical situations. Gauge symmetries map physical states to one another as well, but both the transformed and untransformed states are the very same physical situation. This fact is the reason for gauge symmetry being labeled as un-physical, a kind of redundancy among different ways of describing just one physical situation. This descriptive redundancy can be swept away taking the quotient of $\mathcal{H}$ respect to the action of the gauge group. If, like Rickles[16], we re-define the notion of a physical state in terms of being a gauge orbit, then the difference between physical symmetries and gauge symmetries can be restated in the following: physical symmetries map gauge orbits to one another, whereas gauge symmetries are transformations inside gauge orbits.

Duality is a type of symmetry that involves a space whose elements are not physical states. Both the transformed and untransformed objects are physical theories. Two distinct physical theories are dual of one another if they produce the same physics, or, in other words, if they have the same physical content. Let's see more closely this notion of same physical content. After that, I shall present a more rigorous definition of duality.

Rickles in his *A philosopher looks at string dualities*[17]exemplifies the standard philosophical view about this notion. Two theories have the same physical content when they make precisely the same predictions about all observable phenomena, or in other words, the expected values of any observable in any state are the same in both theories. So two dual theories are physically equivalent in this sense.

At first glance this definition of same physical content seems to introduce in the debate the old verificationist idea of meaning of a physical theory. According to this view, the physical content of a theory must match perfectly with the complete set of its observables. But I think

[15]See Rickles in [$Ric10$], pages $54 - 56$

[16]Ibid page 55

[17]see [Ric10],pages 63-64

that reading Rickles in this way would be incorrect. Nowhere in the paper Rickles commits himself to a such definition of physical content. What seems to me more plausible instead is that Rickles is saying that same physical content means producing same expectation values for each observable *that can be "observed"*. He is not assuming that all the observables of the theory correspond to some observable phenomena. These debate about dualities do not rule out that a physical theory has also a "representational part", a subset of observables which are not "observable". So, I think that when Rickles and others use the notion of physical content they actually refer to that part of a theory that becomes connected to its ontological view of the world. In other words, physical content of a physical theory outrun its observables.

So back to dual theories, they can appear to be completely different, which means, all observable phenomena they equally predict (same physical content) are studied inside two completely different physical world's views or inside very different "ontological commitments". More precisely, a dual couple can contain two theories having a different number of dimensions, two theories such that one is quantum and one is not, one is strongly coupled and its dual is weakly coupled. Now, one of the things that is worthy of attention is that in the context of duality this notion of equivalent physical content reveals to be a powerful tool for computational purposes. Why is that? Let's use a scenario similar to that presented by Brian Greene in *The Elegant Universe*[18].

Let's imagine we are trying to calculate some physical properties, like force charges or particle masses and so on, inside the conceptual framework of some theory. Let's assume there are some technological obstacles that presently make quite difficult matching these computational results with experiments. Therefore, we can just rely on our mathematical tools and on theoretical predictions about which possible physical results we are more likely to obtain. However, at a certain point, we reach a stage in which these mathematical computations are too difficult. The problem is becoming intractable then. However, later on someone informs us that we are lucky after all, since the theory in which we are computing these physical properties has a dual partner, i.e. a different theory in which it is possible to calculate the same physical properties using a different computational machinery. The lucky part of the story is that the new computational techniques are much easier than our previ-

[18]See [Gree00], pagexx

ous ones. This fact may enable us to at least perform a complete mathematical calculation concerning the original problem. However, it may turn out of course that inside the dual theoretical framework the available computational tools are even more difficult. The happy end is not part of the story about duality, but what instead is an important part is something that also Rickles[19] seems to say, i.e. that the property of having the same physical content is a powerful conceptual and computational tool, because it makes the correspondence between theories an exact one, a correspondence that allows one to translate results from one side to another without using approximations between theories. Finally a correspondence which has nothing to do with redundancy, unless it is an instance of self-duality. Let's now move on some more rigorous analysis.

Vafa in his *Geometric Physics* gives a definition of duality which I introduce below, (definition also unravelled in chapter three)[20]. The definition relies on an interesting idea of what a physical theory is. The latter appears to have ubiquitous features at the interface between pure mathematics and physics. More precisely, physical concepts characterizing system's dynamics are combined with abstract objects imported from pure mathematics, like for example moduli spaces. This feature reveals to be very useful particularly when it applies to the string theory case whose domain of application is a complex area where findings go both ways, from mathematics to physics, from physics to mathematics. I am being intentionally vague since a detailed analysis of moduli space's nature and role in string theory will be developed in chapter three. Therefore I will give here just a basic introduction.

According to Vafa the description of a physical system mainly depends on a number of parameters λ. All these parameters glue together to form a space, also called the moduli space M of the theory. How we should think about these gluing conditions will be one of the main issues unraveled in chapter three. Let's just say here that this "space" associated to the theory encodes multi-faceted information. Its topological structure along with a specific fiber bundle over it reflect data about physical backgrounds' geometry and data concerning system's physical dynamics[21].

For now, let's stick to the basic introduction given by Vafa in his paper. Let $\{O_{\alpha_i}\}$ be the complete set of observables of a physical theory T describing the behavior of a physical

[19] $[Ric10]$, page 56
[20] See Vafa in $[Vaf98]$, pages $539-540$
[21] What I am anticipating in this paragraph will be extensively analyzed in chapter three.

system Q. The correlation functions associated to these observables are defined over the moduli space in the following way: for every $\lambda \in M$ and for every n, i.e. every set of observable $\{O_{\alpha_1}, O_{\alpha_2}, ..., O_{\alpha_n}\}$ of the theory, we have

$$< O_{\alpha_1}...O_{\alpha_n} >= f_{\alpha_1...\alpha_n}(\lambda). \tag{2.1}$$

That is a general introductory representation of how parameters λ encode through correlation functions dynamical information[22]. However, this suffices to present Vafa's definition of duality. Let's denote a physical theory T describing the dynamics of a physical system Q with $T = Q[M, O_{\alpha_i}]$.

Definition 2.2.1. Two distinct physical theories T=$Q[M, O_\alpha]$ and T'=$Q'[M', O_\beta]$ are considered to be dual to one another if (1) The two parameters moduli spaces M and M' are isomorphic, (2) there is an isomorphism between O_α and O_β compatible with all the correlation functions.

In other words, T' is dual to T if there exist an isomorphism h

$$M \leftrightarrow^h M'$$

$$\lambda \leftrightarrow \lambda'$$

such that for all $\lambda \in M \;\; \exists! \; \lambda' \in M'$ and for all $\alpha \;\; \exists! \; \beta$ such that

$$f_\alpha(\lambda) = f'_\beta(\lambda').$$

As Rickles points out features of the above isomorphism between the two moduli spaces reveal information about which type of duality connects the two corresponding theories[23]. Basically, if the isomorphism between two moduli spaces is trivial then the duality relation is self-duality. This is the case in which two apparently distinct dual theories reveal to be manifestations of a same underlying theory. Then, the map that originally related one moduli space to the other becomes an automorphism of the moduli space associated to the underlying

[22] More on this in chapter 3.
[23] See also Rickles, [Ric10], page 57

theory. This automorphism maps the space in itself, relating internal regions associated with one form of the underlying theory to those internal regions associated with the other form of the same. Differently, if the isomorphism between two moduli spaces is not trivial then the corresponding duality relation is not trivial as well and in general it will relate very different theories.

As we will see later on, T-duality in bosonic string theory is a case of self-duality. In chapter 3 we will explicitly see how its moduli space looks like. Its automorphisms are underlain by trivially homeomorphic backgrounds. T-duality in general requires homeomorphic backgrounds, whereas mirror symmetries correspond to non trivial moduli spaces isomorphisms underlain by much weaker forms of topological similarities between physical backgrounds, in some cases even by topologically inequivalent backgrounds. In fact, as generalization of T-dualities, mirror symmetries extend the background independence implied by T-duality.

2.2.2 Some preliminary remarks on compactification

As I said above, T-duality applies to string theories over compact dimensions. Why do we need to introduce compact dimensions in String Theory? Let's say intuitively that compactification can be one way to provide the theory with some topological features necessary to produce realistic physics. An example of this kind of strategy is given by quantum superstring theory. It is widely known that in this context compactification plays a key role in making the theory Lorentz invariant. In fact this invariance is guaranteed only if spacetime has ten dimensions. However, assuming the existence of these six extra dimensions increases the level of abstraction so much that the resulting theory risks loosing contact with concrete physics. Therefore, given the fact that we want to assume their existence and that we want to maintain a realistic theory, we should represent them like having a such tiny volume to be not detectable because smaller than the smallest length scales we can probe. Heuristically speaking, something small and also circular seems to be appropriate. We may assume then that at any point of the extended dimensions, small curled up extra dimensions live without meeting our eyes.

However, I would like to say something more precise about compactification of dimensions in string theory. To this aim, let's start with introducing two equivalent mathematical definitions of compactness[24]

Definition 2.2.2. A topological space (or topological set) S is compact if and only if every open cover of S has a finite subcover.

or, in case a topological space is also a metric space

Definition 2.2.3. A topological space (or topological set) S is compact if and only if it is closed and bounded,

where"closed" means a set containing all its limit points and "bounded" means, in a certain sense, being of finite size.

Both definitions do not contain any explicit reference to the property of "being circular". However, in string theory "being compact" means "being curled up or being circular". That is also correct and almost equivalent to the mathematical definitions. Thinking of compactness in terms of being curled up is just a more heuristic way to emphasize some sort of finiteness of the space's size. Moreover, the mathematical procedure of getting compactness involves the "action" of curling up something flat, by curving up the edges. As Zweibach[25]says, we can think of the circle as the open line along with an identification. That is, we decide that points with coordinates that differ by $2nR$ are the same point. More precisely, two points P and Q are declared to be the same point if their coordinates differ by an integer number of $2\pi R$:

$$P \sim Q \Leftrightarrow x(P) = x(Q) + 2n\pi R.$$

The property of having a small volume is not entailed by the property of being compact. The extra-dimensions' property of being small is just added to their compactness property. Actually, compactness can be also a property of very large and extended dimensions. For example, as Greene[26]says, the three observable spatial dimensions of our universe have a visible extension of about 15 billion light-years. No astronomical observations can currently tell us what happens beyond that distance. They could either continue to extend indefinitely or curl up in the shape of a huge circle that cannot be seen with our current telescopes. Therefore, our familiar extended dimensions might be compact as well.

[24]For both definitions see $http://en.wikipedia.org/wiki/Compact_space$

[25]See $[Zwi04]$, pages $31 - 32$

[26]See B. Green, [Gree99], p.248

Assuming the existence of such compact extra-dimensions produces a type of background for strings which mathematically can be represented as a product space of the form $S \times K$, where S is usually an ordinary four-dimensional spacetime and K is a n-dimensional compact complex manifold, $n > 0$, whose background fields are gauge fields[27]. As we will see in chapter three, $S \times K$ should be thought as a "prima facie" background of the theory since the notion of background actually shows feature more abstract than those owned by the former. The compact part K comes into the story in a more sophisticated way. What we really have in a string theory is a space of deformations of this compact complex manifold, producing a family of compact parts sometimes metrically different, but always characterized by topological invariants. In that chapter I will unravel my interpretational proposal about what in the theory produces this notion of background and which kind of implications it has for the controversial issue of background independence. Here it suffices to say, roughly speaking, that topological invariants of K are chosen in a way that enable us to get out physics that match with phenomena. In other words, physics comes out from the topological properties of these compact dimensions.

As we'll see in detail in the next sections, an example of string theory on one compact dimension is that of a bosonic theory over a circle of radius R. We will also see that bosonic T-duality is a physical equivalence between a bosonic theory like that and a bosonic theory over a circle of radius $\frac{1}{R}$. But they are actually the same theory. Bosonic T-duality is a case of self-duality. Deforming the radius does not end up with producing a background outside the family parameterized by the moduli space of the theory. This deformation yields an automorphism of the moduli space in itself. Much more on this in chapter three, for now all I want to say is that bosonic T-duality shows the insensitivity of strings' physics to the difference between two inversely proportional radii.Next step is an attempt to give a mathematical presentation of T-duality through bosonic strings' case study.

[27] See also Rickles in [Ric10], page 60. Anyway, as we will see in the bosonic string case, compactification is not required for the all set of extra dimensions

2.2.3 Brief presentation of compactification in field theory: Kaluza-Klein method

It is generally known that string theory requires that we subject our conceptions of space and time to a radical revision. The foundations of modern and contemporary physics are shaken by string theory to the point that even the general accepted number of dimensions in our universe is challenged. However, the challenge did not originate in the context of string theory. In fact a polish mathematician named Kaluza about 1920, much earlier than the development of string theory, proposed that the spatial structure of the universe might posses more than the three dimensions of ordinary experience[28]. By adding one compact spatial extra dimension to the ordinary four ones, he provided us with a compelling conceptual framework for unifying Einstein's General Relativity and Maxwell electromagnetic theory. Thus, Kaluza along with the Swedish mathematician Klein - who refined Kaluza initial proposal - were the initiators of the idea that the spatial fabric of our universe may have both extended and curled up dimensions. The mathematical techniques they used inside the context of field theory are the same as those used later on in string theory - even if in the latter the procedure has been modified by some additional features due to the string's peculiarities. A detailed analysis of Kaluza-Klein reduction is not part of the topic of this paper. Nevertheless, mentioning here the basic features of their work will turn out to be useful for a comprehension of the string version of compactification[29].

Let's consider a five-dimensional space, with metric G_{MN}, M,N $=0, ..., 4$, with one compact direction. A possible compactified spacetime in this case has the topological structure of $\mathbb{R}^4 \times S^1$. More clearly, we have the usual four coordinates on $\mathbb{R}^4$, x^μ,$\mu = 0, ...3$, plus one periodic coordinate, $x^4 = x^4 + 2\pi R$, with R being the radius of the circle. This five-dimensional theory is invariant under the coordinates transformation (in the fifth dimension):

$$x^M \longrightarrow x'^M = x^M + \epsilon^M(x).$$

[28] See, B. Green in [Gre99], pages $186 - 192$

[29] The mathematical presentation I'm giving here is based on *D-branes*, by Clifford Johnson, see [Joh03], pages? along with some material that can be found at http: $//staff.science.uva.nl/\ jpschaar/report/node12.html$

Under this transformation the metric transforms as:

$$G_{MN} \longrightarrow G'_{MN} = G_{MN} - \partial_M \epsilon_N - \partial_N \epsilon_M. \tag{2.2}$$

In particular,

$$G^5_{\mu 4} \longrightarrow G'^5_{\mu 4} = G^5_{\mu 4} - \partial_\mu \epsilon_4(x). \tag{2.3}$$

The transformation above is just a gauge transformation $U(1)$: $A_\mu \longrightarrow A_\mu - \partial_\mu \Lambda(x)$, where A_μ is a vector proportional to $G^5_{\mu 4}$. In this way the symmetry $U(1)$ of the Electromagnetism can be thought as the result of compactification of gravity, being the gauge field a component internal to the metric. Fixing a radius R small enough - over some much larger scale of distances - the world would appear four dimensional, therefore physical quantities would turn out to be independent from x^4, the fifth dimension.

Let's consider a scalar field and let's see briefly how the independence of the theory from the coordinate x^4 comes up. The moment along the periodic direction becomes quantized:

$$p^4 = \frac{n}{R}. \tag{2.4}$$

Every scalar in dimension five, which satisfies the motion equation $\partial_M \partial^M \phi = 0$, has the expansion

$$\phi(x^M) = \sum_{n \in \mathbb{Z}} \phi_n(x^\mu) e^{\frac{inx^4}{R}}. \tag{2.5}$$

This expansion, if inserted in the motion equation provide us with:

$$\partial^\mu \partial_\mu \phi_n(x^\mu) - \frac{n^2}{R^2} \phi_n(x^\mu) = 0. \tag{2.6}$$

In this way it is possible to see that the modes ϕ_n of a five-dimensional field look in dimension four like an infinite family of scalars with four-dimensional mass

$$-p^\mu p_\mu = \frac{n^2}{R^2}, \tag{2.7}$$

which is not zero for all those fields having moment p^4 different from zero. We get a tower of states that become heavier as soon as R becomes smaller. In other words the more R is small, the less these states become excitable and hence eventually only those fields independent from x^4 are visible. The theory becomes 4-dimensional. If the energy levels become higher than $\frac{1}{R}$, then it will be possible to see the tower of states of Kaluza-Klein.

2.2.4 Compactification in the closed strings case

We are now in a twenty six dimensional space, $D = 0, ..., 25$.

Let's rewrite the expansion modes of a closed string[30]:

$$X^\mu(z, \overline{z}) = X^\mu_L(z) + X^\mu_R(\overline{z}) = \tag{2.8}$$

$$\frac{1}{2}x^\mu - i\sqrt{\frac{\alpha'}{2}}\alpha^\mu_0 \log(z) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n z^{-n} + \frac{1}{2}x^\mu - i\sqrt{\frac{\alpha'}{2}}\widetilde{\alpha}^\mu_0 \log(\overline{z}) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\widetilde{\alpha}^\mu_n \overline{z}^{-n} =$$

$$\frac{x^\mu}{2} + \frac{\widetilde{x}^\mu}{2} - i\sqrt{\frac{\alpha'}{2}}(\alpha^\mu_0 + \widetilde{\alpha}^\mu_0)\tau + \sqrt{\frac{\alpha'}{2}}(\alpha^\mu_0 - \widetilde{\alpha}^\mu_0)\sigma + ...oscillators$$

The space-time total momentum of the string, coinciding with the mass center's momentum, is given by

$$p^\mu = T\int_0^{2\pi} d\sigma \frac{dX^\mu}{d\tau}(\sigma) = \frac{1}{\sqrt{2\alpha'}}(\alpha^\mu_0 + \widetilde{\alpha}^\mu_0). \tag{2.9}$$

Before starting the compactification of one dimension, let's move around the closed string, which means

$$\sigma \longrightarrow \sigma + 2\pi, \tag{2.10}$$

and since the oscillation terms are periodic in σ, we have

$$X^\mu(z, \overline{z}) \longrightarrow X^\mu(z, \overline{z}) + 2\pi\sqrt{\frac{\alpha'}{2}}(\alpha^\mu_0 - \widetilde{\alpha}^\mu_0). \tag{2.11}$$

Since X^μ must be a one-value function, we have

$$\alpha^\mu_0 - \widetilde{\alpha}^\mu_0 = 0 \Rightarrow \alpha^\mu_0 = \widetilde{\alpha}^\mu_0 \tag{2.12}$$

[30] In this section and in the next one, I'll make an attempt to unpack the dense presentation given by BBS in [$KB07$], section 6.1, pages $188 - 192$

Replacing this equality in (52), we have

$$p^{\mu} = \sqrt{\frac{2}{\alpha'}}\alpha_0^{\mu} = \sqrt{\frac{2}{\alpha'}}\widetilde{\alpha}_0^{\mu}. \tag{2.13}$$

We did not introduce any compact dimension yet. We are just moving along the closed string. In fact, p^{μ} has a continuum spectrum of values. This continuity means that the direction of X^{μ} is not periodic and provide us with the usual p^{μ} relation already seen above.

Let's see at this point what the consequences are if we make a compactification of one direction, for example $X^{25} \cong X^{25} + 2\pi R$. The momentum along the compact direction turns out to be quantized and therefore it can have only discrete values:

$$p^{\mu} = \frac{n}{R}. \tag{2.14}$$

This situation is the same as in field theory or, I should say, almost the same. In fact, if we now move along the closed string, X^{25} is not a one-value function anymore, but it varies periodically:

$$X^{25}(\tau, \sigma + 2\pi) = X^{25}(\tau, \sigma) + 2\pi\omega R, \tag{2.15}$$

where $\omega \in \mathbb{Z}$ is called *winding number*. In other words, a closed string can wind around the compact dimension. This kind of string's behavior is not something that can be found in field theory. Point particles do not wind around anything, therefore they do not have winding numbers.

In this new situation we have two equations:

$$p^{\mu} = \frac{n}{R}, \tag{2.16}$$

$$\alpha^{25} - \widetilde{\alpha}^{25} = \sqrt{\frac{2}{\alpha'}}\omega R, \tag{2.17}$$

that provide us with the zero modes

$$\alpha_0^{25} = (\frac{n}{R} + \frac{\omega R}{\alpha'})\sqrt{\frac{\alpha'}{2}} = \sqrt{\frac{\alpha'}{2}}p_L^{25} \tag{2.18}$$

$$\widetilde{\alpha}_0^{25} = (\frac{n}{R} - \frac{\omega R}{\alpha'})\sqrt{\frac{\alpha'}{2}} = \sqrt{\frac{\alpha'}{2}} p_R^{25}.$$

We can use these relations to compute the mass spectrum of the remaining dimensions not compactified:

$$M^2 = -p^\mu p_\mu = (p_L^{25})^2 + \frac{4}{\alpha'}(N-1) = \tag{2.19}$$

$$(p_L^{25})^2 + \frac{4}{\alpha'}(\widetilde{N}-1) =$$

$$\frac{n^2}{R^2} + \frac{\omega^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \widetilde{N} - 2),$$

where $\mu = 0, ...24$, N and $\widetilde{N}$ are excitations levels of the left and right moving oscillators. Moreover the condition $L_0 = \widetilde{L}_0 = 1$ has been used[31]. If we use the fact that summation of these operators is equal to the hamiltonian and to the level-matching we get the further condition $n\omega + N - \widetilde{N} = 0$.

Let's notice at this point that the spectrum has been modified not just by the tower of momentum 's states ($n \in \mathbb{N}$), as in field theory, but also by the tower of winding states ($\omega \in \mathbb{Z}$). In the mass formula the first term carries the contribution of the tower of momentum's states of Kaluza-Klein for the string (the compact moment), the second term represents the potential energy of the string which is winding, i.e the term deriving from the tower of the winding states, the third one represents the usual excitation levels of the oscillator.

Here a couple of preparatory remarks. Let's notice on one hand that the vibrational excitations of a string have energies that are inversely proportional to the radius of the circular dimension. A heuristic appeal to the "uncertainty principle" may explain that: the more you confine a string inside a small radius the more its energy increases. On the other hand, the winding mode energies are directly proportional to the radius. This is due to the fact that the minimum length of wound strings (hence their minimum winding energy) is proportional to the radius.

These two remarks prepare the ground for the following consideration. Let's see in some details the behavior of the spectrum when R varies. The states of zero mass for a generic value

[31]Closed string have the following property. While $L_0 + \overline{L}_0$ generates temporal translations on the world-sheet, $L_0 - \overline{L}_0$ generates translations in σ. Within the closed string's perspective the notion of being at some point has not any physical meaning. That translates into a condition of invariance under translations in σ. That is equivalent to the condition $L_0 - \overline{L}_0 = 0$, which at the quantum level will become the condition of *level-matching* $N = \overline{N}$ - i.e. equality between the number of right-moving excited oscillators and the number of left-moving ones. N is the counting operator $N = \sum_1^\infty \alpha_{-n} \cdot \alpha_n = 0, 1, 2, ...$

of R are characterized by quantic numbers $n = \omega = 0$, and by excitation levels $N = \widetilde{N} = 1$. These are the states already seen in the non compact theory, below presented separating the internal direction 25 from the space-time directions μ:

$$\alpha_{-1}^{\mu}\widetilde{\alpha}_{-1}^{\nu}|0;k>, (\alpha_{-1}^{\mu}\widetilde{\alpha}_{-1}^{25} + \alpha_{-1}^{25}\widetilde{\alpha}_{-1}^{\mu})|0;k>$$

$$(\alpha_{-1}^{\mu}\widetilde{\alpha}_{-1}^{25} - \alpha_{-1}^{25}\widetilde{\alpha}_{-1}^{\mu})|0;k>, \alpha_{-1}^{25}\widetilde{\alpha}_{-1}^{25}|0;k> .$$

If $R \longrightarrow \infty$, then the mass formula shows that the winding states ($\omega \neq 0$) tend to disappear, being infinitely massive. Large radius entails large winding energy. Whereas the momentum states become lighter and therefore preferable from the energetic point of view. Therefore, the states characterized by $\omega = 0$ and $n \neq 0$ produce a continuum spectrum, describing in this way a physical configuration without compact dimensions.
If $R \longrightarrow 0$, then momentum states tend to disappear. In ordinary field theory the consequences of this fact would be that the remaining fields are just independent of the compact coordinate. But in the string context things are different. Small values of the radius entails small winding energy. So, the pure winding states survive ($\omega \neq 0$, $n = 0$) and they will produce a continuum spectrum.

At this point, the key fact about bosonic T-duality. Let's consider a string configuration. First, physical properties are sensitive to the total energy of this configuration, but not to the way in which the total energy splits into a vibration part and a winding part. Second, let's consider a large circular radius background for string propagation. Based on what I said above on spectrum's behavior associated to the variation of R, I can say that there exist a corresponding small circular radius background such that its vibration energies are equal to the winding energies in the large radius background and such that its winding energies are equal to the vibration energies in the same large one. But since the total energy is the same in both cases, there is no physical distinction between these two backgrounds. Two bosonic theories referring to these two backgrounds are empirically indistinguishable. More precisely, they are the same theory. In fact each background of the pair can be thought as result of smoothly deforming the radius of the other one[32]. Shifting from one to the other

[32] More on this in chapter 3.

means mapping a point over the theory's moduli space onto another point of the same moduli space. As we said bosonic T-duality is self-duality.

2.2.5 T-duality for closed strings

Let's give a more formal shape to what we just saw in the last part of the previous section[33]. The spectrum above is invariant under the following transformation:

$$T : n \leftrightarrow \omega, R \leftrightarrow R' \equiv \frac{\alpha'}{R} \tag{2.20}$$

This symmetry is called T-duality. The compactified string theory over a circle of radius R', obtained switching winding number with moment number, produces the same physics as the theory of radius R. Given T-duality:

$$\alpha_0^{25} = (\frac{n}{R} + \frac{\omega R}{\alpha'})\sqrt{\frac{\alpha'}{2}} \rightarrow (\frac{\omega}{R'} + \frac{nR'}{\alpha'})\sqrt{\frac{\alpha'}{2}} = (\frac{\omega R}{\alpha'} + \frac{n}{R})\sqrt{\frac{\alpha'}{2}} \tag{2.21}$$

$$\widetilde{\alpha}_0^{25} = (\frac{n}{R} - \frac{\omega R}{\alpha'})\sqrt{\frac{\alpha'}{2}} \rightarrow (\frac{\omega}{R'} - \frac{nR'}{\alpha'})\sqrt{\frac{\alpha'}{2}} = (\frac{\omega R}{\alpha'} - \frac{n}{R})\sqrt{\frac{\alpha'}{2}},$$

or in other words

$$p_R^{25} \longrightarrow -p_R^{25}, p_L^{25} \longrightarrow p_L^{25}; \tag{2.22}$$

from the technical point of view, we just rewrite the theory of radius R switching

$$X^{25}(z, \overline{z}) = X_R^{25}(\overline{z}) + X_L^{25}(z) \rightarrow X'^{25}(z, \overline{z}) = X_L'^{25}(z) - X_R'^{25}(\overline{z}). \tag{2.23}$$

The T-duality transformation also maps all the remaining modes as

$$\alpha_\mu^{25} \longrightarrow \alpha_\mu^{25}, \tag{2.24}$$

$$\widetilde{\alpha}_\mu^{25} \longrightarrow -\widetilde{\alpha}_\mu^{25}.$$

Therefore, the field X'^{25} keeps sharing with X^{25} the same energy-momentum tensor. Quantities like correlation functions will be invariant. Hence, the theory of radius R and the theory

[33] As I said in the previous footnote,see [$KB07$], section 6.1, pages $188 - 192$ for bibliographic references on this part

of radius $\frac{\alpha'}{R}$ are physically identical.

Let's briefly point out that in the formalism above we can see how the extended nature of strings plays, through the parameter α', a crucial role in determining T-duality. In the theory the parameter α' fixes the minimum distance. In fact, it is related to the string tension,

$$\alpha' = \frac{1}{2\pi T}, \tag{2.25}$$

and the minimum distance scale we can see in string theory is $x \sim \sqrt{\alpha'}$. Looking at the map above we see that when $R = \sqrt{\alpha'}$ then $R' = R$. It looks like $R = \sqrt{\alpha'}$ is the minimum radius. If you try to shrink R below that value you will get a theory for a large radius.

2.2.6 T-duality for open strings

Let's still consider[34] a spacetime with one compact direction $X^{25} \cong X^{25} + 2\pi R$. As I already said the compactification of X^{25} transforms the momentum in that direction in a quantized momentum, $p^{25} = \frac{n}{R}$, with $n \in \mathbb{Z}$. However, open strings cannot wrap around the periodic dimension. Therefore they don't have a winding number around that dimension. Since $\omega = 0$, we have that the formula $n\omega + N - \widetilde{N} = 0$ becomes $N = \widetilde{N}$ and so the mass formula is

$$M^2 = \frac{n^2}{R^2} + \frac{4}{\alpha'}(N-1). \tag{2.26}$$

Let's briefly study the spectrum behavior within the two limits $R \to 0$ and $R \to \infty$ in this open string case as well. In the first limit, states with not zero internal momentum get an infinite mass and so they disappear. However, differently from the case of closed strings, any continuum spectrum of states deriving from the winding tower won't be found. Therefore the remaining fields will be independent of the compact dimension and we'll have a theory with one less dimension than the initial theory. The compact dimension is lost. This fact may be disturbing. In fact interacting open string theories must contain closed strings, but taking the limit $R \to 0$ we can see that closed strings live in a D-dimensional spacetime, whereas open strings in a $(D-1)$-dimensional one. How can we explain this inconsistency ? Actually, the consequence of the limit $R \to 0$ can be thought as condition on the end points of the

[34] In this section, I'll attempt to unpack the BBS presentation of this topic contained in [$KB07$], section 6.1, pages $192 - 195$

open string, since, after all, the internal part of the open string cannot be distinguished from the internal part of the closed one. It will just keep vibrating in D dimensions. The part that distinguishes the two kinds of strings are just the extremities. These extremities, in the case of an open string, live inside a hyper-plane of dimension $D-1$. Let's write the mode expansions:

$$X^\mu(z,\overline{z}) = X^\mu(z) + X^\mu(\overline{z}) = \tag{2.27}$$

$$= \frac{x^\mu}{2} + \frac{x'^\mu}{2} - i\alpha' p^\mu log(z) + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu z^{-n} +$$

$$+\frac{x^\mu}{2} - \frac{x'^\mu}{2} - i\alpha' p^\mu log(\overline{z}) + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu \overline{z}^{-n},$$

where x'^μ is an arbitrary quantity that disappears when working with the usual open string coordinates. Let's focus in particular on the compact coordinate X^{25}. From the T-dual transformation above on $X^{25}(z,\overline{z})$, we have

$$X^{25}(z) \longrightarrow X^{25}(z); X^{25}(\overline{z}) \longrightarrow -X^{25}(\overline{z}) \tag{2.28}$$

therefore

$$X^{25}(z,\overline{z}) = X^{25}(z) + X^{25}(\overline{z}) \longrightarrow X'^{25}(z,\overline{z}) = X^{25}(z) - X^{25}(\overline{z}), \tag{2.29}$$

$$X'^{25}(z,\overline{z}) = X^{25}(z) - X^{25}(\overline{z}) =$$

$$= x'^{25} - i\alpha' p^{25} \log(\frac{z}{\overline{z}}) + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n}\alpha_n^{25} e^{-in\tau} sin(n\sigma) =$$

$$= x'^{25} + 2\alpha' p^{25}\sigma + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n}\alpha_n^{25} e^{-in\tau} sin(n\sigma) =$$

$$= x'^{25} + 2\alpha' \frac{n}{R}\sigma + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n}\alpha_n^{25} e^{-in\tau} sin(n\sigma).$$

In the sector of 0 modes there is no dependence on the world-sheet's coordinate τ. Since the mass center's momentum arises from this element, we have that the momentum is zero. In fact, the oscillators' quantities vanishes along the extremities of the open string,i.e $\sigma = 0$ and $\sigma = \pi$. These extremities don't move along the direction X'^{25}. The T-dual string turns out to be fixed, i.e.

$$p'^{25} = T \int_0^\pi d\sigma \partial_\tau X'^{25} = 0, \tag{2.30}$$

hence the Neumann boundary conditions, $\partial_n X \equiv \partial_\sigma X = 0$, have been replaced with the Dirichlet boundary conditions, $\partial_t X \equiv i\partial_\tau X = 0$, where n is the direction normal to the boundary and t the tangent one. More precisely, the Dirichlet conditions that constrain the two extremities of an open string to one point are:

$$\partial_\sigma X^{25}(\sigma = \pi) = \partial_\sigma X^{25}(\sigma = 0) = 0, \tag{2.31}$$

and from the point of view of the T-dual string become:

$$X'^{25}(\tau, \pi) - X'^{25}(\tau, 0) = \int_0^\pi d\sigma \frac{\partial X'^{25}}{\partial \sigma} = 2\pi\alpha' \frac{n}{R} = 2\pi n R'. \tag{2.32}$$

Therefore, Neumann conditions on the original coordinates become Dirichlet conditions on the dual coordinates.

The two equations show that the extremities are compelled to live on the same hyper-plane. In particular

$$X'^{25}(\tau, \pi) = X'^{25}(\tau, 0) + n2\pi R', \tag{2.33}$$

shows that extremities lie on the same hyper-plane in the periodic T-dual space.

That is consistent with the idea that in general T-duality switches the definitions of normal derivative and tangent derivative:

$$\partial_n X^{25}(z, \overline{z}) = \frac{\partial X^{25}}{\partial z}(z) + \frac{\partial X^{25}}{\partial \overline{z}}(\overline{z}) = \partial_t X'^{25}(z, \overline{z}), \tag{2.34}$$

$$\partial_t X^{25}(z, \overline{z}) = \frac{\partial X^{25}}{\partial z}(z) - \frac{\partial X^{25}}{\partial \overline{z}}(\overline{z}) = \partial_n X'^{25}(z, \overline{z}).$$

The T-duality transformation we considered acted along the periodic direction X^{25}. So, the end points of an open string who are fixed along the T-dual direction by

$$X'^{25}(\tau, \pi) - X'^{25}(\tau, 0) = n2\pi R'$$

are free to move along the remaining 24 spatial dimensions on which T-duality did not act. In other words each of the two extremities are free to move on an hyper-plane, which are called *D-branes*. In general, if the periodic coordinates are more than one, the T-duality transform the Neumann conditions on each of the coordinates X^μ into Dirichlet conditions on the T-dual coordinates X'^μ. In this way the end points of open strings are confined to move on N hyper-plane of dimension $p+1$, where p is equal to D minus the number of coordinates on which T-duality acted. These hyper-planes are called D_p-branes. They are very important in string theory. It is believed that quantum fields described by Yang-Mills theories (for example electromagnetism) involve strings that are attached by D-branes. This idea seems to have good explanatory power, because the quantum of gravity (gravitons) are not attached to D-branes. They can travel through a D-brane and that would explain why we cannot see them. So, the universe in this picture has a three-dimensional brane embedded in a higher dimensional spacetime called the bulk. The interactions in our world are mediated by particles that are really strings stuck to the brane. Gravity is mediated by strings that can leave the brane and travel away from it into the bulk. That would explain also why gravity is actually a much weaker force than electromagnetic force.

Chapter 3

Taking a closer look at string theory: some thoughts on background independence.

Background independence is a key notion in the debate on spacetime emergence. As I said earlier on the notion is usually presented as one of the conceptual implications of theoretical dualities. In the first part of this chapter I will consider background independence in a slightly different perspective. In fact I will attempt an interpretational proposal. More precisely, we saw that the set of data necessary to define a string theory include the notion of moduli space[1]. I shall first investigate what exactly can be recovered from the moduli space and how the latter precisely encodes the theory. Then I will show that the outcome of this analysis reveals that moduli space on its own - without involving non trivial dualities - might point out to a weaker form of background independence. Then, using the mathematical language introduced, I will formulate a way of representing string theory which I think highlights its main algebraic aspects. Such formulation unpacks Vafa's notion introduced in the previous chapter. Finally, I shall read again duality inside this alternative formulation.

The second part of the chapter analyzes some important views about the implications of dualities in string theory for the ordinary notion of spacetime. The indifference showed

[1] See Rickles in [$Ric10$], page 60 and also Vafa in [$Vaf98$], page 540

sometimes by physical content toward geometrical structure of the background for strings propagation undermines the classical idea of a fundamental spacetime metric influencing physical dynamics. Whether or not a more fundamental non spacetime "metric" produces such influence is the main issue on which the views I present will be compared.

3.1 Unravelling the role of moduli spaces: deformation spaces as more fundamental than ordinary spaces.

Vafa and Rickles' accounts of theoretical duality in string theory[2] well exemplify the general conceptual debate on this matter. As I said in the previous chapter, this complex debate relies on a notion of physical theory presenting ubiquitous features at the interface between physics and pure mathematics. This section is an attempt to unpack such notion by limiting my analysis only to the string theory case and by unraveling the role played by moduli space inside its mathematical framework. The goal is that of showing that moduli space on its own might point to a weaker form background independence. In fact we will see that moduli space encodes geometrical and dynamical properties of an ordinary background by encoding its space of deformations, hence replacing in the the theory the former with the latter. Therefore, what we have is a family of different backgrounds taken as data for constructing the theory. We will see that this fact can be taken as a starting point for formulating a weaker notion of background independence to which the theory's moduli space equipped with some extra structure might point out. As we know background independence means that string physics does not detect differences between topologically inequivalent backgrounds. That explains why what we have here is a weaker form of independence. In fact in this case the insensitivity of the physics would be just toward different but diffeomorphic members of the same family, along with the fact that this blindness does not arise for any arbitrarily chosen pairs of topologically different backgrounds inside the family.

Moreover, if deformations of a given background for string dynamics introduce a notion of mathematical possibility then the theory delivers an ontological view taking into account these possibilities as ontologically fundamental.

[2]See Vafa in [Vaf98] and Rickles in [Ric10]

Let's start thinking to the theory as a collection of mathematical objects and of maps linking the former in a way that fits their mathematical structure. The way in which a physical theory was denoted in the previous chapter, $T = Q[M, O_\alpha]$, partially reveals which are the spaces involved, but let's see more closely.

Theories prima facie describe strings dynamics along a background spacetime of propagation. The latter is a metrical space. As I said in the previous chapter, assuming the existence of compact extra-dimensions in string theory produces a type of spacetime that can be mathematically represented as a product space $S \times K_0$, where S is a four-dimensional ordinary spacetime and K_0 is a n-dimensional complex compact manifold whose topological and complex structure yields the low energy physics in S.

Moreover, as I mentioned in the previous chapter, descriptions of strings dynamics depend on certain parameters λ that glue together to form the moduli space M of the theory. Once some gluing conditions are satisfied by these parameters, the space M, along with a fiber bundle over it, ends up encoding all the crucial information about the geometry and dynamics of the physical system described by the theory. These gluing conditions can be understood by getting an idea of what a local structure of the moduli space M should look like.

Let's think for now just to the moduli space M without the structure of fiber bundle over it. As we will see the latter has to do more with observables of the system and at this stage of my analysis I want to be focused just on how the moduli space's local structure encodes information about background spacetime's geometry. This will clarify the notion of "encoding different backgrounds". However, the reader should keep in mind that this separation between topological and dynamical information encoded by a moduli space is introduced just for the sake of simplicity. In fact they usually overlap, for example in all those cases in which moduli encoding couplings do depend on the moduli relative to topological properties of the compact part.

The notion of moduli space was already introduced in the previous part of this work. However, in order to unravel the latter issue, I need to say something more about it. What follows is not a detailed and exhaustive presentation of moduli space's properties, but just an attempt to describe those local features which are central to establish its representational role.

A first simple case of moduli space can be found describing points in a plane by using cartesian coordinates (x, y). Assuming that each pair (x, y) is a couple of real numbers, we have that the space of coordinates $\mathbb{R}^2$ can be seen as a picture of a specific set of mathematical objects, i.e. points in a plane. Therefore $\mathbb{R}^2$ is the moduli space for the set of points in a plane. In simple words moduli spaces are pictures of sets of mathematical objects, encoding in some way their mathematical properties.

A moduli space for a set of geometric objects is a "geometric" object itself, though not necessarily of the same kind. The necessary requirements a "space" M should satisfy for being the moduli space for a set N of geometric objects are the following. First, the correspondence between points of M and objects of N has to be one to one. Second, if two objects are close to each other in N, the corresponding points over M must end up being close as well. However, I want to emphasize that the notion of being close in N is not in principle the same as that of being close in the moduli space M.

Back to our original question, how does the moduli space M of the theory encode the geometry of the spacetime $S \times K_0$? As I said above K_0 is a complex compact manifold[3].

To answer this question I need to introduce some new important notions concerning compact complex manifolds[4]. What I'm introducing here are the notions of manifolds' family and of manifold's deformations. Informally speaking, we use the idea of a complex manifold K_λ depending on a parameter λ as a "function" of λ, but unlike a function there is no space containing its range of variations. However, we can consider the set

$$K = \bigcup_{\lambda \in B} K_\lambda,$$

where B is a complex domain that for simplicity we will assume to be one-dimensional[5]. Working with a real parameter λ will allow us to use the idea of K_λ not just as a "function" of λ, but also as a C^∞-"function" of the same. That will make possible to refer to the notion

[3]An intuitive definition of complex compact manifold is the following: a complex manifold is a manifold that can be entirely covered by an atlas of charts which are open unit disks in $\mathbb{C}^n$ along with holomorphic transition maps. A compact complex manifold is a complex manifold covered by a finite number of these charts.

[4]What follows about deformation is based on the content of "Complex manifolds and deformation of complex structures" by Kunihiko Kodaira, see [Kod86] pages $182 - 208$. Also pages $265 - 266$ of "Algebraic geometry" by Robin Hartshorne, see [Har77], along with "Lectures on deformations of complex manifolds" by Marco Manetti, see [Man04].

[5]I am following here Kodaira's approach, see [kod86], pages $182 - 192$.

of differentiable family of compact complex manifolds. By using the concept of differentiable family we will get a clearer image of what a deformation of a compact complex manifold is[6].

Definition 3.1.1. Suppose given a domain B in $\mathbb{C}$ and a set $\{K_\lambda|\lambda \in B\}$ of complex manifolds K_λ depending on λ. We can say that K_λ will have a C^∞ dependence on λ and that $\{K_\lambda|\lambda \in B\}$ is a *differentiable family* of compact complex manifolds if there are a differentiable manifold K as above and a C^∞ map ϕ of K onto B satisfying the following conditions[7]:

(1)The rank of the Jacobian matrix of ϕ is equal to the dimension of B, or in other words, the differential of ϕ, $\phi_\star$: $T_pK \longrightarrow T_{\phi(p)}B$ is surjective at every point $p \in K$, where T_pK and $T_{\phi(p)}B$ are respectively the tangent space to K at the point p and the tangent space to B at the point $\phi(p)$.

(2) K is a non empty complex compact manifold and for each $\lambda \in B$, $\phi^{-1}(\lambda) = K_\lambda$ is a compact differentiable submanifold of K.

(3) There are locally finite open covering $\{V_j|j = 1, 2, ...\}$ of K and complex-valued C^∞ functions $z_j^1(p), ..., z_j^n(p)$, $j = 1, 2, ...$, defined[8] on V_j such that for each λ

$$\{p \to (z_j^1(p), ..., z_j^n(p))|V_j \bigcap \phi^{-1}(\lambda) \neq \emptyset\}$$

form a system of local complex coordinates of K_λ.

Let's call $\lambda \in B$ the parameter of the differentiable family $\{K_\lambda|\lambda \in B\}$ and B its parameter space or base space.

Coming back to the system of local coordinates, since the coordinate transformation

$$z_k(p) \to z_j(p)$$

on $V_j \bigcap V_k \neq \emptyset$ does not affect λ, we can rewrite it as

$$(z_k^1(p), ..., z_k^n(p), \lambda) \to (z_j^1(p), ..., z_j^n(p), \lambda)$$

[6]See Kodaira in [kod86], pages 182 − 192. In particular see definition 4.1, page 184.

[7]An intuitive definition of differentiable manifold is the following: it is a type of manifold that is locally similar enough to a linear space to allow one to do calculus.

[8]$n = dim_{\mathcal{C}}(K_\lambda)$.

$$= (f^1_{jk}(z_k(p, \lambda), \lambda), ..., f^n_{jk}(z_k(p, \lambda), \lambda), \lambda),$$

where $f_{jk}(z_k, \lambda)$ on $V_j \bigcap V_k \bigcap \phi^{-1}(\lambda)$ are C^∞ functions of λ and also holomorphic in z_k. These functions are very important since they tell us how the open neighborhoods V_j and V_k joins together for every λ. Expressing their dependence on the parameter λ introduces their central role in what follows about deformation of a complex compact manifold.

Now having introduced the notion of a differentiable family (K, B, ϕ) should produce a clearer image of the deformation of a compact complex manifold. What we want here is a specific type of differentiable family of complex compact manifolds, i.e. a family of *deformations* of one of its members, here denoted by K_0. Namely, given a compact complex manifold K_0, if there is a differentiable family (K, B, ϕ) with the features described in definition 3.1.1 along with the fact that $\phi^{-1}(0) = K_0$, then each $K_\lambda = \phi^{-1}(\lambda)$ is called a deformation of K_0. Moreover, K is also called the total space of deformations of K_0.

So we can think to a deformation of the compact manifold K_0 as a change in the way of glueing together V_1,..,V_j, i.e. as a change of the coordinate transformations $f_{jk}(z_k, \lambda)$ depending on λ's variation. Deforming the gluing functions $f_{jk}(z_k, \lambda)$ of a compact complex manifold means deforming its complex structure. So, despite the K_λs of the family are all diffeomorphic to K_0 they typically carry different complex structures. Generally a complex structure over a manifold (whether or not compact) comes along with a hermitian metric compatible with it, which in principle is not unique[9]. Such metric consists of smoothly varying, positive definite inner product on the tangent bundle, which is hermitian with respect to the complex structure on the tangent space at each point. There is no a priori relation between metric and complex structure, but there are different compatibility conditions between them, depending on which hermitian metric we are choosing, that amount to mathematical correspondences involving holomorphic functions in the manifold's atlas of charts. We will not enter in detail concerning these mathematical relations, which are different from case to case. However, we will assume in what follows that given a compatibility condition between complex structure and one of the induced metrics, it is possible to deform the complex structure also involving changes to the metrical one. This fact will yield a family of different backgrounds, different in both complex and geometrical structures.

[9]In fact such metrics always exist in abundance on any complex manifold.

Inside the family of deformations we will be focused on a specific type, i.e. first-order infinitesimal deformations. Introducing the latter, along with the Kodaira-Spencer map presented below, completes this description of how the local structure of the moduli space reflects geometrical properties of the physical background.

Let's consider the point $0 \in B$ such that $\phi^{-1}(0) = K_0$, the central fiber of the deformation family. Let's take the first order neighborhood B_ϵ of 0, i.e.

$$B_\epsilon = \{\lambda \in B | \lambda \in (-\epsilon, \epsilon)\}.$$

If we have a differentiable function over this interval by differentiating it in λ we will know how this function infinitesimally varies around 0. As we saw above, a differentiable family differs in many aspects from a differentiable function. Nevertheless, we can perform in a similar way the differentiation in $\lambda \in (-\epsilon, \epsilon)$ of

$$f_{ik}(z_k, \lambda) = f_{ij}(f_{jk}(z_k, \lambda), \lambda), \tag{3.1}$$

over $V_i \bigcap V_j \bigcap V_k \neq \emptyset$.

I will not introduce here the more detailed description in coordinates $f_{ik}^{\alpha}(z_k, \lambda)$ with $\alpha = 1, ...n$, but I will just present the general idea that leads interesting consequences for us.

Taking the derivative in λ on both sides of (3.1) we have

$$\frac{\partial f_{ik}}{\partial \lambda} = \frac{\partial f_{ij}}{\partial f_{jk}} \frac{\partial f_{jk}}{\partial \lambda}. \tag{3.2}$$

Let's now focus on the infinitesimal neighborhood B_ϵ of 0. Left hand side of the identity above can be used to define a holomorphic field at $\lambda = 0$

$$\theta_{ik}(0) = \frac{\partial f_{ik}(z_k, \lambda)}{\partial \lambda}|_{\lambda=0}. \tag{3.3}$$

Applying differentiation rules for composite functions we get this important equality:

$$\theta_{ik}(0) = \theta_{ij}(0) + \theta_{jk}(0). \tag{3.4}$$

The above equality holds true on $V_i \bigcap V_j \bigcap V_k \neq \emptyset$, over K_0, and it can also be written as

$$\theta_{jk}(0) - \theta_{ik}(0) + \theta_{ij}(0) = 0. \tag{3.5}$$

Replacing $i = k$ we have $\theta_{kk}(0) = 0$ and hence (3.5) becomes

$$\theta_{kj}(0) = -\theta_{jk}(0). \tag{3.6}$$

Identities (3.4), (3.5) and (3.6) hold true for every $\lambda \in B$. However, we now focus on the central fiber of the family $K_0 = \phi^{-1}(0)$ and its first-order infinitesimal neighborhood

$$\bigcup_{\lambda \in (-\epsilon, \epsilon)} K_\lambda = \phi^{-1}(B_\epsilon),$$

where $B_\epsilon = (-\epsilon, \epsilon)$ is the first order infinitesimal neighborhood of 0 mentioned above.

From (3.5) and (3.6) we can say that $\theta_{jk}(0)$ is a 1-cocycle of the sheaf T_{K_0} of holomorphic vector fields over K_0. So with $\theta(0)$ I denote the element of the cohomology group $H^1(K_0, T_{K_0})$, i.e. the cohomology class of $\{\theta_{jk}(0)\}$[10]. Informally speaking, as I said above, $\theta(0)$ represents the "derivative" at $\lambda = 0$ of the complex structure of K_λ obtained differentiating in $\lambda \in (-\epsilon, \epsilon)$. In other words the vector field $\theta(0)$ bends infinitesimally the manifold K_0 along a direction and in this sense $\theta(0)$ is an *infinitesimal first-order deformation* of K_0 along a certain direction

$$\frac{dK_\lambda}{d\lambda}|_{\lambda=0} = \theta(0). \tag{3.7}$$

Therefore the cohomology group $H^1(K_0, T_{K_0})$, generated as vector space by a basis of non isomorphic θs, represents all the infinitesimal deformations of K_0. Now, the map onto $H^1(K_0, T_{K_0})$ I am introducing below is going to complete the picture, finally clarifying how the theory's moduli space encodes the geometry of the background $S \times K_0$ and in which sense it replaces the latter with its space of deformations, becoming vehicle of information not just on a given background but also on possible different ones.

The tenor of what follows will be quite informal, since grasping the main point will not require a rigorous description.

[10] For an exhaustive and introductory presentation of the notions of sheaves, cohomology groups and 1-cocycle I refer the reader to Kodaira in [kod86], pages 109-133.

Recall we have a first-order family of deformations $\bigcup_\epsilon K_\epsilon \longrightarrow^\phi B_\epsilon$, and so we have a pull-back map $\phi^\star : Fun(B_\epsilon) \longrightarrow Fun(\bigcup_\epsilon K_\epsilon)$[11]. Fixed $0 \in B$, the first order neighborhood B_ϵ as topological space is just the point $\{0\}$, but its sets of functions differs from the set over $\{0\}$, since the former contains functions not vanishing at 0 and whose first-order derivatives do not vanish at 0[12]. So, each of these functions corresponds to a tangent vector to B at 0. Then $Fun(B_\epsilon)$ is the tangent space to B at 0, i.e. $T_{B,0}$.

Now, a tangent vector in $T_{B,0}$ is pulled-back by the map $\phi^\star$ to a function over the infinitesimal family. The latter will be a function not vanishing on K_0 and whose first derivative does not vanish on K_0[13]. So these pull-backs describe directions in K which are normal to K_0 and along which functions vary. Therefore, they can be thought as first order deformations of K_0 and so they can be represented as elements of $H^1(K_0, T_{K_0})$.

So we are now ready to introduce the following map:

$$\rho : T_{B,0} \longrightarrow H^1(K_0, T_{K_0}). \tag{3.8}$$

This is the Kodaira-Spencer map at 0 of the family of deformations (K, B, ϕ). Through this map a tangent vector to the parameter space B at 0 is sent to a first order infinitesimal deformation K_λ, $\lambda \in (-\epsilon, \epsilon)$, of K_0 along the direction of that vector. Kodaira-Spencer map is definable at each point λ of the parameter space B.

Now, why does this map complete the picture? The way in which the local structure of the moduli space encodes the geometry of the background $S \times K_0$ is described by the way in which the base space B parameterizes the family of deformations of K_0. The base space B is the local structure of the global moduli space M of the theory, whereas $B_\epsilon \subset B$ as its first order infinitesimal structure around the point 0 corresponding to K_0. What the moduli space's local structure actually parameterizes are the first order deformations K_λ of K_0's complex and geometrical structures obtained by deforming the way of glueing together $V_1, ..., V_j$. So the moduli space encodes K_0's geometry and its possible variations.

At this point two relevant facts follow.

[11] Strictly speaking I should say the sheaf of holomorphic functions over the two topological spaces.

[12] In fact the sheaf of functions over B_ϵ is equal to the quotient $\frac{O_{B,0}}{m^2_{B,0}}$, where $m^2_{B,0}$ consists of functions vanishing at 0.

[13] As before, the family $\{K_\epsilon\}$ as topological space is equal to K_0, but its sheaf of holomorphic functions differs from that of K_0 because it contains the pull-back of such functions.

(1) First, in this theoretical context the background of dynamics is not an ordinary manifold anymore. In fact inside a first order infinitesimal neighborhood of the central fiber $K_0 = \phi^{-1}(0)$ we just found a family of diffeomorphic manifolds which are geometrically different. The following diagram shows this point :
$K_0 = \phi^{-1}(0)$, $0 \in B_\epsilon$, and $\forall\ \lambda \in B_\epsilon$,

$$S \times K_0 \quad \longrightarrow^{id \times i} \quad S \times Def_\lambda(K_0) \quad \longrightarrow^{id \times \phi} \quad S \times B_\epsilon, \quad \longrightarrow^{p_2} \quad B_\epsilon \subset M,$$

such that $(p_2 \circ (id \times \phi) \circ (id \times i))(K_0) = 0$.

We can see that introducing the notion of moduli space in the definition of the theory produces a notion of "spacetime background" of string dynamics quite different from that of an ordinary manifold. The notion of "background" actually in charge in string theory seems to be represented by the space of deformations $S \times Def_\lambda(K_0)$ parameterized by the moduli space. Studying this space of deformations means probing the local structure of the moduli space. As we will see in the next section, defining over this local structure some additional mathematical tools might serve the purpose of revealing the insensitivity of the string physics to certain deformations. In the ideal situation in which the family of backgrounds can be taken as the data for constructing a string theory without any prejudice to choice of a particular member, we would have a full background independence restricted to that family. But what we have here is something weaker since only certain deformations seems to be not detected by physics. As an instance of that let's consider the bosonic string case of self-duality we saw in chapter two. In that case string physics does not detect the difference between the compact part S_r and $S_{\frac{1}{r}}$, but this does not hold true if we deform the radius of S_r in a different way.
(2) Second, the existence of a Kodaira-Spencer map at $\lambda = 0$, (see formula (3.8)), tells us that each tangent vector at $\lambda = 0$ to the moduli space identifies a possible deformation of the manifold K_0. If the Kodaira-Spencer map is surjective then every first order deformation of K_0 is represented by a tangent vector to the moduli space. So under the latter condition a local linear approximation of moduli space around the chosen point represents an infinitesimal neighborhood of K_0 in K. This neighborhood can be thought as a space representing mathematical possibilities. In fact deforming the complex and metrical structure of K_0 amounts

to looking for possible alternatives to K_0 which are compatible with the same physics. But if the local structure of moduli space is what plays the role of "background", then the theory delivers an ontological view in which spaces of mathematical possibilities replace ordinary spacetime backgrounds inside a more fundamental description of reality.

The condition under which the Kodaira-Spencer map is surjective are beyond the aim of this chapter. I will refer the reader to "Lectures on deformations of complex manifolds" by Marco Manetti, (see [$Man04$], page 12). Let's briefly say here that the surjectivity of the map is equivalent to the statement that the family K is *complete*,i.e. it contains all the first-order deformations of K_0.

An example of one-dimensional moduli space is that parameterizing the bosonic string. In chapter two, studying T-duality, we saw the case in which $S \times K_0$ is a 25-dimensional minkowski spacetime S with one compact dimension which is a circle of radius r_0, i.e. $S \times S^{r_0}$. In this case the compact part K_0 does not have an underlying complex structure since it is a real compact manifold. Still this example can be useful to understand more general cases. In order to find the local structure of the theory's moduli space I will consider the space of infinitesimal deformations of the compact part $K_0 = S^{r_0}$. The parameter of deformation is the radius r of the circle, so the space of deformation is $K = \bigcup_r S^r$. The moduli space M parameterizing these deformation is $\mathbb{R}^+$, the axis of positive real numbers. The way in which it parameterizes all possible values circles' radii can assume is the following:

$$S \times S^{r_0} \quad \longrightarrow^{id \times i} \quad S \times K \quad \longrightarrow^{id \times \phi} \quad S \times \mathbb{R}^+_r \quad \longrightarrow^{p_2} \quad \mathbb{R}^+_r.$$

I would like to emphasize again here that bosonic T-duality studied in chapter two, i.e. the empirical equivalence between two bosonic string theories, one with radius r and the other with radius $\frac{1}{r}$ is actually a case of self-duality. In fact the map sending S^r to $S^{\frac{1}{r}}$ is just a deformation of the compact part. Both backgrounds belong to the same differentiable family of deformations of a central fiber S^0.

3.2 An attempt of algebraic representation of the theory: weaker background independence

In this section I will complete the picture of moduli space's local structure. We will see in basic mathematical terms how moduli space conveys information about the observables of a physical system described by the theory. The outcome of this analysis is an algebraic representation of the theory that will reveal in which sense the moduli space might point to a weaker form of background independence.

Let's consider physical dynamics of a quantum system of strings, here denoted by Q, along a fixed background of propagation. Typically a such system has many observables, which we can measure by computing their expectation values. Observables of a system are linear self-adjoint operators acting on the Hilbert space of the system's states. Let's denote with ψ a quantum state of Q, with H the Hilbert space of Q's quantum states, with O the vector space generated by the observables O_α of the system. We can think of the vector space O as a group of linear transformations O_α acting on H by mapping quantum states of the system onto different quantum states of the same system. Moreover, it is possible to define multi-linear maps over this space O that, depending on their ranks, map a certain number of observables onto their expectation values. Such multi-linear maps generate a vector space, here denoted by $Func(O)$, defined in the following way:

$$Func(O) = \{h_n : O \longrightarrow \mathbb{C}; \forall n \in \mathbb{N}\},$$

such that

$$\forall \alpha, n = 1, h_1(O_\alpha) =< O_\alpha >,$$

$$\forall n > 1, h_n(O_{\alpha_1} O_{\alpha_2} ... O_{\alpha_n}) =< O_{\alpha_1} O_{\alpha_2} ... O_{\alpha_n} > .$$

Introducing observables of a system Q inside our picture requires the use of a specific type of algebraic objects, namely fiber bundles over the moduli space M[14]. Before describing how these fiber bundles look like let's briefly define what a fiber bundle is.

Definition 3.2.1. A fiber bundle with fiber F is a map $p : E \longrightarrow D$ where E is called

[14]See also Vafa in [Vaf98],page 540, footnote 2.

the total space of the fiber bundle and D the base space of the fiber bundle. The main condition for the map to be a fiber bundle is that every point in the base space $d \in D$ has a neighborhood U such that $p^{-1}(U)$ is homeomorphic to $U \times F$ in a special way. More precisely, if q is the homeomorphism, i.e. q is defined by

$$q : p^{-1}(U) \longrightarrow U \times F$$

then $proj_U \circ q = p_{|p^{-1}(U)}$, where the map $proj_U$ is the projection onto the U component of the cartesian product. The homeomorphisms q which "commute with projection" are called local trivializations for the fiber bundle p. In other words, E, at least locally, looks trivial, i.e. like the product $D \times F$, except that the fibers $p^{-1}(d)$, for $d \in D$ may be a bit "twisted"[15].

Applying the above definition to our case, let's consider a string system Q along with the Hilbert space of its quantum states and with the space of its observables. The moduli space M of the theory describing the system is the base space and the fiber bundle over that base is $\overline{H}$, i.e. we have

$$\begin{array}{c} \overline{H} \\ \downarrow^{p} \\ M \end{array}$$

The fiber bundle $\overline{H}$ is defined as $\coprod_{\lambda \in M}(H_\lambda \times \mathbb{C})$. Each disjunct $H_\lambda \times \mathbb{C}$ is a fiber over a point λ of the base space M. H_λ is the Hilbert space of Q's states whose dynamics are considered along the background parameterized by λ and the vector space of complex numbers $\mathbb{C}$ represent all the numerical values assumed over λ by the transition functions $f_{\alpha_1\alpha_2\ldots\alpha_n}$ of the system. In other words,

$$f_{\alpha_1\alpha_2\ldots\alpha_n}(\lambda) = < O_{\alpha_1} O_{\alpha_2} ... O_{\alpha_n} >,$$

where for any given λ, $f_{\alpha_1\alpha_2\ldots\alpha_n}(\lambda)$ assume the values of the functions $h_n(O_{\alpha_1}, ..., O_{\alpha_n})$ introduced above considering the system over a fixed background.

So, $\overline{H}$ is the fiber bundle resulting from the disjunct union of all the possible Hilbert

[15] For more detail see http://mathworld.wolfram.com/FiberBundle.html

spaces H_λ of the system, each of them equipped with some extra structure. Such union is taken by varying over the family of backgrounds parameterized by λ.

Let's now define a section of the bundle. Let's consider a set of observables $\{O_{\alpha_1}, O_{\alpha_2}, ...O_{\alpha_n}\}$ of the system Q. Recall that fixing a λ means fixing a background of propagation. By definition we have that $\forall\ \lambda \in M$, $s(\lambda) \in (H_\lambda \times \mathbb{C})$. More precisely, the section of the fiber bundle map a point λ of the base space M over a vector in the fiber in the following way,

$$s(\lambda) = (\psi_\lambda, f_{\alpha_1 \ldots \alpha_n}(\lambda)),$$

where ψ_λ is a quantum state of the system along the background λ and $f_{\alpha_1 \ldots \alpha_n}(\lambda)$ is the correlation function value of the chosen set of observables, relative to that quantum state and that background. Let's also denote the vector $(\psi_\lambda, f_{\alpha_1 \ldots \alpha_n}(\lambda))$ with v_λ.

At this point we have all the necessary tools to formulate the issue of background independence in this context. What we would like to have in order to say that the structure of moduli space on its own points to backround independence is that given a set of observables like above and an arbitrarily picked member λ_0 of the family, the value $f_{\alpha_1 \ldots \alpha_n}(\lambda_0) =< O_{\alpha_1} O_{\alpha_2} ... O_{\alpha_n} >$ remains constant over all backgrounds of the family,
i.e. $\forall\ \lambda \in M$

$$f_{\alpha_1 \ldots \alpha_n}(\lambda_0) = f_{\alpha_1 \ldots \alpha_n}(\lambda).$$

As I said before this goal cannot be reached over all members of the family but just over subsets of it. Let's unravel this point.

In order to study in which cases the transition functions' values remain constants, we need a way to compare vectors lying in different fibers of the vector bundle over two different points λ_1 and λ_2. So we need to pick a *flat connection* and some path over M from λ_1 to λ_2 to drag a vector in the fiber over λ_1 along that path, ending up with a vector in the fiber over λ_2. In other words we need to make a "parallel transport" of information that allows us to identify the fibers[16]. Let's make these remarks more precise[17].

Let $[0,T] \longrightarrow M$, $t \in [0,T]$, be a smooth path $\lambda(t)$ over the moduli space M from λ_1 to λ_2, i.e. $\lambda_1 = \lambda(0)$ and $\lambda_2 = \lambda(T)$. Let's denote with $v_{\lambda(t)}$ the vector in the fiber of $\overline{H}$ over $\lambda(t)$. We need to say that $v_{\lambda(t)}$ is "parallel transported" along $\lambda(t)$. That can be done by finding an equation involving the covariant derivative of $v_{\lambda(t)}$ in the direction $\lambda(t)$ is going, i.e. $\lambda^{'}(t)$. Since we need to use a flat connection our equation comes from the vanishing of the covariant derivative[18],

$$D_{\lambda^{'}(t)} v_{\lambda(t)} = 0. \tag{3.9}$$

So, we can say that $v_{\lambda(t)}$ is parallel transported along $\lambda(t)$ if and only if the above covariant derivative vanishes $\forall\ t \in [0,T]$.

The differential equation above can always be solved and if we give an initial condition over λ_1 the solution is unique an it is the vector with which we wind up in the fiber over λ_2. At this point we should recall two things.

First, $\forall \lambda$, v_λ in our case is $(\psi_\lambda, f_{\alpha_1 \ldots \alpha_n}(\lambda)) \in H_\lambda \times \mathbb{C}$ and $\{O_{\alpha_1}, O_{\alpha_1}, ..., O_{\alpha_1}\}$ is a set of observables of the system.

Second, fixing an initial condition $v_\lambda(0) = v_{\lambda_1} = (\psi_{\lambda_1}, f_{\alpha_1 \ldots \alpha_n}(\lambda_1))$, we would like that the solution v_{λ_2} is a quantum state of the system over the background λ_2 preserving all the expectation values assumed over λ_1 by the set of observables mentioned above. This requirement, if satisfied for any arbitrarily fixed initial condition λ and for any observable of the system would point out to background independence.

Now, if on one side the existence of a solution is guaranteed, on the other side it is not true in general that the solution has the above requirement. We can always have a flat connection over a fiber bundle which represents a canonical way to drag a vector in the fiber

[16] What follows about connection completely relies on "Gauge fields, knots and gravity", by John Baez and Javier Muniain, see [JB08], pages $223-242$. For an introduction to the preliminary notions on which my presentation relies, I refer the reader to this book. Finally, I was inspired to use flat connections over the fiber bundle as mathematical tool of investigation about the issue of background independence by a paper of Witten, "Quantum background independence in string theory", see [Wit93]. In this paper flat connections are used in a different context concerning mirror symmetries. Also I do not use his results, still I found Witten's use of this mathematics very inspiring.

[17] See Baez and Javier in [JB08], page $233-235$. Here Baez present the general case of a fiber bundle over a manifold. In what follows I will apply the same pattern of reasoning to my specific case.

[18] For a definition of covariant derivative see Baez, [JB08], pages $223-229$. Here such derivative is defined for a section s of the fiber bundle. However, from page 233 to page 234, Baez using the idea of a section s as vector-space-valued function, re-define by analogy the covariant derivative of a vector in a fiber along the path over the manifold,i.e. the derivative I introduced above.

over a point along a path, ending up with a vector in the fiber over another point. But it is not guaranteed that the latter satisfies the condition above, either it might or it might not. However, there are cases in which the requirement of preserving expectation values is met by certain subsets of the backgrounds family. For example, as I said, the bosonic string case. Here we have a string system Q whose physical properties are insensitive to a change of the background radius from r to $\frac{1}{r}$. In particular the observables $\{O_{\alpha_1}, O_{\alpha_2}, ..., O_{\alpha_n}\}$ sensitive to the total energy of the string configuration do not detect the difference and so we have

$$f_{\alpha_1 ... \alpha_n}(\lambda_r) = f_{\alpha_1 ... \alpha_n}(\lambda_{\frac{1}{r}}) =< O_{\alpha_1} O_{\alpha_2} ... O_{\alpha_n} > .$$

I consider this fact as being an instance of weaker background independence.

Finally, we have an overall representation of the theory's algebraic structure exemplified by the following diagram, which I will call the theory-diagram:

$$\begin{array}{lllllll} \overline{H} & & & & & & \\ \downarrow^{p} & & & & & & \\ M & \longleftarrow^{p_2} & S \times M & \longleftarrow^{id \times \phi} & S \times Def(K_0) & \longleftarrow^{i} & S \times K_0 \end{array}$$

$$p^{-1}(\lambda) = (\psi_\lambda, f_{\alpha_1 ... \alpha_n}(\lambda)) \in H_\lambda \times \mathbb{C}.$$

3.3 Reading again duality

Let's consider again Vafa's definition of duality: two distinct string theories are dual just in case (1) they have isomorphic moduli spaces and (2) there is an isomorphism between observables compatible with all the correlation functions,(see pages 25, 26). In this section I want to unravel this definition by using the diagram introduced above. Let's then consider two dual string theories,$T[M, S \times Def(K_0), O_\alpha]$,$T^{'}[M^{'}, S^{'} \times Def(K_0^{'}), O_\beta]$. Using the theory-diagram above I want to build another diagram exemplifying the fact that T and $T^{'}$ are dual, which I will call duality-diagram.

As we saw earlier on in string theory the topological and complex structure of the compact manifold determines the low energy physics in the real non compact dimensions. Two different compact manifolds, if respectively connected to two dual string theories, produce the same low energy physics. Inquiring about which kind of differences are allowed we found out that duality does not require too many constraints. T-duality only holds for homeomorphic manifolds, but in general duality can involve topologically inequivalent compact manifolds, like in the case of mirror symmetry. The latter involves theories having Calabi-Yau manifolds as complex compact manifolds. What it is required by string theory in order to maintain consistency with the observed particle physics is a Calabi-Yau manifold with Euler characteristic[19]of ± 6. An interesting fact is that two mirror symmetric theories can have Calabi Yau manifolds with opposite Euler number.

Let's consider in what follows this case of topologically inequivalent backgrounds. They are in correspondence through the mirror mapping ψ_0 that changes the sign of the euler number,

$$S \times K_0 \longrightarrow^{\psi_0} S^{'} \times K_0^{'}.$$

Each background is parameterized by the moduli space of the corresponding theory according to the description I made in the previous section:

$$\begin{array}{lcr}
\overline{H} & & \overline{H'} \\
\downarrow^{p} & & \downarrow^{q} \\
M & & M^{'} \\
\uparrow^{\phi} & & \uparrow^{\varphi} \\
S \times Def_{\lambda}(K_0) & \longrightarrow^{\psi} & S^{'} \times Def_{\lambda'}(K_0^{'}) \\
\uparrow^{i} & & \uparrow^{i} \\
S \times K_0 & \longrightarrow^{\psi_0} & S^{'} \times K_0^{'}.
\end{array}$$

Heuristically speaking, smoothness of the deformations of the central fiber K_0 preserves its euler number inside the family and allows the map ψ_0 to lift to the map ψ which applies to every fiber of the family maintaining the same property of being a mirror mapping.

Now, despite the two "backgrounds" $S \times Def_{\lambda}(K_0)$ and $S^{'} \times Def_{\lambda'}(K_0^{'})$ look very different

[19]For a surface of genus g, the euler characteristic is defined like $\chi(g) = 2 - 2g$.

topologically, from the point of view of their moduli spaces the difference vanishes. In fact they are two isomorphic spaces. But this is understandable if we think that the moduli space point of view is that of the string theory "living" on that background. And this perspective seems to be nothing else than the point of view of the two-dimensional conformal field theory with couplings encoding the backgrounds'topological properties. As we saw earlier on the conformal field theory is insensitive to the mirror mapping and in this sense it shows a certain degree of background independence. So the map η below is the corresponding isomorphism between the two moduli spaces equipped with structures of fiber bundle given by the theories'obervables:

$$\begin{array}{ccc}
\overline{H} & \longrightarrow^{\eta} & \overline{H'} \\
\downarrow^{p} & & \downarrow^{q} \\
M & \longrightarrow^{\psi^{\star}} & M^{'} \\
\uparrow^{\phi} & & \uparrow^{\varphi} \\
S \times Def_{\lambda}(K_0) & \longrightarrow^{\psi} & S^{'} \times Def_{\lambda^{'}}(K_0^{'}) \\
\uparrow^{i} & & \uparrow^{i} \\
S \times K_0 & \longrightarrow^{\psi_0} & S^{'} \times K_0^{'}.
\end{array}$$

η is an isomorphism between the two fiber bundles $\overline{H}$ and $\overline{H'}$ such that $\forall(\psi_{\lambda}, f_{\alpha_1 \ldots \alpha_n}(\lambda)) \in \overline{H}$,

$$\exists! \eta(\psi_{\lambda}, f_{\alpha_1 \ldots \alpha_n}(\lambda)) = (\phi_{\psi^{\star}(\lambda)}, f_{\beta_1 \ldots \beta_n}(\psi^{\star}(\lambda)) \in \overline{H'},$$

such that the following compatibility condition is respected: $\forall\ \lambda \in M$ and $\forall\ n \in \mathbb{N}$

$$f_{\alpha_1 \ldots \alpha_n}(\lambda) = f_{\beta_1 \ldots \beta_n}(\psi^{\star}(\lambda)).$$

The fact that the fiber bundle map η is a fiber bundle isomorphism satisfying certain compatibility conditions makes the above diagram a duality-diagram, i.e a diagram exemplifying the duality relation between the theories T and $T^{'}$.

Finally, the duality-diagram would exemplify a self-duality relation if the map ψ_0 represents a trivial diffeomorphism between backgrounds. In fact in this case the map ψ would not be a map between two different families, but just a deformation of one central fiber of the same differentiable family.

3.4 Conceptual implications of dualities for ordinary spacetime

In this section I'll present some popular views about the conceptual implications of dualities for an ordinary notion of spacetime. Dualities reveal indifference of the string physics toward geometrical differences of physical backgrounds. Despite background independence in string theory is still a huge open question, this fact not surprisingly downplays the traditional idea of a fundamental ordinary spacetime metric governing physical dynamics. Ordinary metrical quantities of spacetime seem in a certain sense to be not "real" for string theory, because they appear to be irrelevant inside the theory's description of the fundamental reality.

Whether or not a more fundamental non-spacetime "metric" replaces the ordinary one restoring a fundamental role is the main issue on which the views I'm presenting will be compared. If duality makes the ordinary notion of spacetime not fundamental, we have the problem to figure out how we should go beyond it. An interesting difference between two strategies is presented by Butterfield and Isham[20]. The first one is to argue that ordinary spacetime is an emergent notion - in the phenomenological sense I specified in the introduction. This strategy does not quantize directly ordinary spacetime, but it recovers this notion as the low-energy limit of a quantized string theory. The second strategy instead consists in trying to quantize directly classical spacetime and then to recover it as classical limit of the ensuing quantum theory.

The authors I'm presenting here seem to follow the first approach, although with several individual differences. The main characteristic of this approach is that of refusing the idea that ordinary spacetime can be fundamental object of the theory.

Brian Greene and Nicholas Huggett seem to develop their ideas within the general view that quantum string theory might postulate some stringy "metric" from which the ordinary one would emerge through a low-energy limit. In the chapter on T-duality I mentioned how the extended nature of strings, along with some geometrical properties of the dimension

[20] See [$J.B$99], page $71 - 79$.

around which strings wrap, are responsible for this geometrical ambiguity. This ambiguity is interpreted by these authors as a symptom of being there two interconnected definitions of distance. This feature is peculiar only to string theory because in point particles theory, where the notion of winding is meaningless, only one definition of distance shows up.

Greene in "The elegant universe"[21] develops some interesting ideas about that. In the chapter on mathematical features of T-duality we saw that momentum and winding modes both contribute to the total energy spectrum of a string. We also saw how both modes can became heavier or lighter depending on the length's variation of R. Moreover, I also pointed out that heavy states are less preferable from an energetic point of view owing to the fact that they are less excitable and hence requiring too much energy to prepare a localized light signal from them. That is why they are not visible.

Greene explains how defining distances mainly consists in giving experimental procedures for measuring them[22]. What makes two measurement procedures different is the kind of probe we use. He introduces two types of definitions, "the first definition uses strings that are not wound around a circular dimension, whereas the second definition uses strings that are wound". Unwound strings can move without obstacles along the entire circumference of the dimension, therefore they are sensitive to distances that are proportional to R, whereas wound strings having minimal energy proportional to R are sensitive to distances that are proportional to $\frac{1}{R}$. The results obtainable by using both types of strings are hence inversely related to one another.

However, as Greene points out, we always carry out just one operational definition, despite being there an alternative one. How can we explain this fact? According to Greene whenever the dimension R differs greatly from the value $\sqrt{\alpha'}$, the measurement procedure involves highly massive probe and therefore it becomes extremely difficult to perform. Producing the heavy string configurations is presently beyond our technological abilities. However, the two definitions of distance, despite deeply different, are both valid concepts. Due to technological limitations we are familiar with just the light one.

As the radius R - the quantity measured by the unwound strings - shrink to the value $\sqrt{\alpha'}$ and continue to get smaller, we have that $\frac{1}{R}$ - the quantity measured by wound strings

[21] See [$Gree00$],chapter 10

[22] See [$Gree00$], pages $249 - 252$.

- grows to the value $\sqrt{\alpha'}$ and gets larger. So, the winding modes are now lighter than the unwound modes and if we keep using the light string modes - the traditional notion - then we encounter the minimal value $\sqrt{\alpha'}$ below which R cannot go.

What is the meaning of this minimal length? According to Greene, it tells us about our present inability to extract information below this value. It is a limit to our epistemic access. However, this fact does not undermine the possibility of using a notion of "distance" for $R < \sqrt{\alpha'}$, despite the "metric" involved is still unknown. Such "metric" should be thought as some kind of algebraic structure with a completely different meaning from the metric associated to the traditional notion of distance.

A similar idea about being there two incomparably different notions of distance is that developed by Huggett, (see [$Hugg07$],pages 2,5, $6-8$). According to him spacetime's destiny in string theory is completely different from the destiny that the newtonian space met in general relativity. What we have here is a physical theory that presents spatial quantities as derivable from an underlying "manifold" in two different ways. The first one produces a notion of metric in the "momentum sense", which is the "g_{ij}" metric assigned to our universe by general relativity. The second one yields a more fundamental notion of "metric" in the "winding sense", which is that assigned by string theory. So, ordinary spacetime is an emergent entity because it arises from an underlying structure. It is also completely different from the stringy spacetime. So far Huggett's position seems very similar to that of Green. Both of them seem to point out the emergent role of ordinary spacetime and the fact that there is a more fundamental theory, still maintaining some kind of "metric" notion which is deeply phenomenally different from the ordinary "g_{ij}" metric. But while Green seems to point to the string " metric" as the one that plays this role, Huggett does not seem to share this idea. He speaks about an "underlying manifold"[23] from which both metric notions arise. So, although he thinks that the stringy quantum world's description is more fundamental than the ordinary one, he allows the possibility of being there some theory more fundamental than string theory. In this new theory it is still meaningful to speak about " distances" and " metrical quantities", these notions will be encoded in the kind of observable operators that characterize the theory.

[23]See [$Hugg07$], page7

Another popular view is that of Edward Witten. In "Time in Quantum Gravity"[24] we present Witten's idea concerning the ontology of string theory as a form of relationism. A passage from "Reflections on the Fate of Spacetime" seems to support this interpretation[25]:

> Thus, once one replaces ordinary Feynman diagrams with stringy ones, one does not really need spacetime any more; one just needs a two-dimensional field theory describing the propagation of strings. And perhaps more fatefully still, one does not have spacetime any more, except to the extent that one can extract it from a two-dimensional field theory.

Let's unpack this passage. As I anticipated in chapter 1, in particular in section (1.5), what we have here is an approach that consider the string worldsheet all we can find in the ontology of the theory. There is no bigger spacetime embedding string worldsheet, but somehow the bigger spacetime emerges from the fundamental material fields over the string. The equation introduced by Witten for the string worldsheet lagrangian can briefly clarify the basic features of his relationism[26]:

$$I = \frac{1}{2\alpha'} \int d^2\sigma \sum_{ij\alpha} g_{ij}(X) \frac{dX^i}{d\sigma^\alpha} \frac{dX^j}{d\sigma^\alpha}, \tag{3.10}$$

where the integral is carried out with respect to τ and σ. The formula above is a lagrangian of a two-dimensional quantum field theory over the string worldsheet. Inside I an arbitrary ordinary spacetime metric shows up. Depending on which g_{ij} we replace, the corresponding field theory represented by I encodes a spacetime theory with some properties. If g_{ij} is replaced with the flat Minkowski metric, the spacetime theory emerging from I has the property of unbroken Poincaré invariance. Therefore, spacetime can be extracted from a two-dimensional field theory describing strings configurations. In this sense Witten's view is a form of stringy relationism, since it reduces ordinary spacetime to a set of possible spatiotemporal properties that material fields instantiate over the string worldsheet[27].

Then, the ambiguity introduced by dualities in string theory should be read as a relation between two spacetimes that are different in ordinary physics but both emerging from two physically equivalent field theories over the string worldsheet.

[24] See [N.Hng], pages 11,12
[25] [Wit96], page 28
[26] See [Wit96], pages 27,28
[27] See [N.Hng]page 11.

So again, ordinary spacetime as non fundamental entity emerging from some deeper "metrical" structure. How should we think about this underlying structure according to Witten? He claims that among several theoretical dualities, T-duality is the one that more clearly points out to"quantum geometry", revealing that "there is a smallest circle in string theory"[28]. In fact if you shrink a theory of radius R to a theory of radius equal to $\sqrt{\alpha'}$ and then try to shrink things down to a radius shorter than that, "space will re-expand in another direction peculiar to string theory", which is the winding direction[29]. So far we find the same description of the theory's peculiarity as that presented by Greene. But what makes Witten's approach slightly different from that of Greene is the kind of consequences he derives from that peculiarity. For such impossibility to create a compact dimension characterized by a radius shorter than $\sqrt{\alpha'}$ tells us that " smallest distances just are not there"[30]. This seems to be an ontological claim and not just an epistemological one about about our ability to extract information below a certain space scale. The limit seems to be on space itself. To support my ontological reading of Witten's position I will mention his reformulation of uncertainty principle in a stringy version[31]:

$$\triangle x \geq \frac{\hbar}{\triangle p} + \alpha' \frac{\triangle p}{\hbar}. \tag{3.11}$$

The new term describes a new kind of uncertainty due to string theoretical context. Witten seems to make an ontological claim about this principle: Heisenberg microscope does not work in string theory if the energy is too high. Accelerating beyond the string scale we don't end up with probing short distances, but with watching large strings'propagation. According to Witten, this fact does not imply that at those short distances there are facts about distances inaccessible to us. Smaller distances are just not there.

[28]See [$Wit96$], page 28
[29]See [$Wit96$], page 29
[30]See [$Wit96$], page 29
[31]See [$Wit96$], page 29,30

Chapter 4

Emergence of ordinary spacetime via space-time non commutativity

A potential feature of quantized space and time is that of failing to commute. Under certain conditions that happens in string theory. In the first section I will study the implications of space-time non commutativity for the causal structure of the theory. The question is whether causality is still fundamental. Some interesting results will be presented. But why is that relevant to our discussion on spacetime emergence? Knowing these implications might contribute to unravel the idea of an emergent ordinary spacetime in string theory via a perspective which is different from that produced by dualities and background independence arguments. This claim will be developed in the third section. For now I want to remind that there are basically two main approaches to the issue of emergence. On the one hand spacetime is considered to be a phenomenological entity emerging from an underlying structure devoid of metrical properties. On the other hand, the idea of spacetime emergence is developed inside a theoretical framework in which there isn't any deeper underlying structure postulated by the ontology of the theory. I will show that the theoretical findings about space-time non commutativity might support the first approach.

Then, in the second section, I shall analyze the nature of the conditions under which space-time non commutativity arises in string theory. Does this behavior appear just in perturbative regimes or it is instead an intrinsic property of the theory? Two different approaches on this issue will be presented.

Finally, in the previous chapter I presented my interpretational proposal about which notion of "background" is actually in charge in string theory. The idea of a deformation space playing this role introduces an abstract notion of background which ends up with supporting a weaker form of background independence. Although deformation spaces will not be used in this chapter, still we will deal here with an abstract notion of background, a non commutative one, which I consider to be the generalization of an ordinary commutative one. This is the conceptual framework within which I present here the notion of an emergent ordinary spacetime in string theory, emergent because it can be obtained as particular case of a more general non space-time commutative structure by imposing a constraint on the latter. I will show that this constraint amounts to a mathematical limit. But then something needs to be clarified, i.e. whether we still can apply the notion of emergence to this limit. My answer is that the applicability conditions are still there. Two reasons for that. First, as I defined in the introduction, an emergent entity of a theory T is mainly characterized by being a novel or unexpected feature inside the theory S from which T arises. As we will see later on, the notions of length and of point space are completely missing in the non-commutative general structure, whereas they appear in the particular commutative limit. Second, I will show that this mathematical limit has a physical counterpart in the canonical low energy limit - mentioned in previous chapters - which formally describes spacetime emergence.

4.1 Is causality still fundamental in the theory?

Does space-time non commutativity affect the causal structure of the theory[1]? The section starts considering the issue of causality in the perspective of non-commutative field theories. In fact, some of these field theories arise as low-energy limits of non-commutative string theory. Some others cannot arise in this way. In both cases they provide us with crucial

[1]This section unravels and develops the ideas presented in the fourth section of "Time in Quantum Gravity", see [N.Hng]. Although this is a joint paper, I wrote the original draft of the section "Space-time non commutativity", whereas Nick Huggett wrote the original draft of the section"Time in string theory".

information concerning the problem of whether causality is preserved in non-commutative string theory.

Non-commutative field theories are field theories over a spacetime equipped with a non-commutative geometry. The latter can be heuristically presented in the following way. The ordinary notion of space is well described by the Euclidean metric. For simplicity let's chose a space of dimension $D = 2$. A two-dimensional Euclidean space is a plane equipped with Euclidean metric, i.e. a metric induced by ordinary "$\cdot$" multiplication[2]. In this space the computation of the area A of a rectangle of sides x^1 and x^2 does not depend on the order of the factors in multiplication, i.e. $x^1 \cdot x^2 = x^2 \cdot x^1 = A$.
An algebraic generalization of the ordinary multiplication is the $\star$-product defined by the following:

$$x^1 \star x^2 - x^2 \star x^1 = \theta^{12},$$

where θ^{12} is a non zero antisymmetric parameter. So, $\star$ fails to be commutative. In this case the order of multiplication matters since $x^1 \star x^2 = x^2 \star x^1 + \theta^{12}$. Then, which kind of metrical structure, if any, will be induced over the space by the $\star$-product? The interesting point is that in a space like that computing areas would become an impossible task. Notions like length, area and volume are devoid of meaning. So, metric disappears and space is not a point space anymore. The $\star$-product introduces a more abstract notion of space, which is that of an *algebraic space*. This 'structural' notion, based on algebraic relations only, is not in competition with the ordinary one introduced above, but instead it is its generalization. Non commutative geometry is the mathematics that allow us to work with non geometric spaces. Mutatis mutandis a non-commutative spacetime of dimension D works exactly like the two dimensional case. In a D-dimensional spacetime non-commutativity can be either purely spatial or space-time non commutativity.

Some interesting comparative studies concerning space-time non commutativity in field theory and string theory are presented by two papers of Seiberg, Susskind and Toumbas along with a paper of Gomis and Mehen[3]. In these works two classes of non commutative

[2]That is just an intuitive way of characterizing the Euclidean metric. This is not the place for a detail consideration of the mathematics involved, but still it is worthy to say something a bit more precisely. The Euclidean metric over the point space is induced by the Euclidean scalar product defined over the underlying vector space along with the "$\cdot$" multiplication defined over the number field associated to the vector space.

[3]See Seiberg, Susskind and Toumbas in [$N.S00$],[$NST00$]. Also, Gomis and Mehen in [$J.G00$]. Other bibliographic references can be found in what follows

field theories are formally derived from a commutative field theory by replacing the ordinary fields product with the $\star$-product defined above. The antisymmetric parameter $\theta^{\mu\nu}$ is more precisely an antisymmetric matrix and it is defined by the non commutation relation among coordinates in the following[4]

$$[x^\mu, x^\nu] = x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}, \tag{4.1}$$

μ,$\nu = 0, ..., D-1$, with D being the spacetime dimension[5]. The first class of non commutative field theories is characterized by just space-space non commutativity. In this case the antisymmetric parameter is such that $\theta^{0i} = 0$. The second class is instead characterized by space-time non commutativity,i.e. $\theta^{0i} \neq 0$. This first group of findings is compared by the authors with a second group of results concerning string dynamics along non commutative backgrounds.

Before analyzing these studies I want to briefly say something about my use of the above mentioned papers. Here I will use the findings described in those works without sharing the authors' view - or I should say what their view looks to me - concerning the conceptual relation between ordinary space-time commutative theories and space-time non commutative theories. In fact, I don't think they would agree with the conceptual framework I am using here, in which emergence of an effective commutative spacetime theory is seen as result of specialization of a more general, underlying, non commutative one.

For in their case things seem to go in the other way around: a space-time non commutative theory arises from a commutative one as a particular case obtainable by perturbating the latter's background with an electric field. However, they do not seem to be committed to the idea that ordinary spacetime is a fundamental entity. Still it is an effective entity emerging from deeper underlying structures but apparently, according to them, space-time non commutativity is not the right way of characterizing these more fundamental structures, since space and time exhibit non commutative behavior only perturbing in a particular way a commutative spacetime. As we will see in section 4.2, this fact has to do with the conceptual

[4] $[J.G00]$, page 1,$[N.S00]$,page 1

[5] Then, the $\star$-product between two fields is

$$\phi_1(x) \star \phi_2(x) = \exp^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial\alpha^\mu}\frac{\partial}{\partial\beta^\nu}} \phi_1(x+\alpha)\phi_2(x+\beta)|\alpha = \beta = 0. \tag{4.2}$$

status they assign to the space-time uncertainty principle.

But the problem with this view, as it will appear later on, is that one of their theoretical findings does not seem to fit properly inside this picture. In fact inside the second group of results concerning string dynamics along a space-time non commutative background, something shows that space-time non commutativity appears also not in presence of perturbing fields. This fact seems to support another view that will be object of study in section 4.2, that of Yoneya, according to which space-time non commutativity in string theory, originating from a more general conformally invariant principle of the string world-sheet, is an intrinsic feature of the theory. I take Yoneya's view as being compatible with my attempt to define emergence via specialization of a more general space-time non commutative structure. This point will be developed in section 4.2.

Let's unravel the content of the above mentioned papers. Firstly I shall describe briefly the case of space-space non commutative field theory and what it tells us about space-space non commutative string theory. Secondly, following the same pattern, I will analyze more extensively the case of space-time non commutativity.

In case of space-space non commutativity, the field theory is not local in space, but still local in time. This kind of non-locality destroys the Lorentz invariance of the theory. In fact, the Lorentz transformations acting on this type of non commutative space end up being applied also to the new background field represented by $\theta^{\mu\nu}$, since this parameter has lorentz indices[6]. As Carroll et al.(2001)claim[7], there are two different types of Lorentz transformation. The first one is the rotation of the "observer inertial frame" which do not change the physics because the field operators in the lagrangian and the $\theta^{\mu\nu}$ are invariant under them. The second one is the "rotation of a particle"[8] inside a "fixed observer frame". The field $\theta^{\mu\nu}$ is not invariant under their action and this produce a different physics. So, more precisely, any space-space noncommutative field theory violates a particular kind of Lorentz symmetry. However, breaking this symmetry does not entail a lack of unitarity of the theory and so it does not raise the specter of indeterminism[9].

The presence of $\theta^{\mu\nu}$ does not change the rules and the usual framework of quantum

[6] [$Lic05$], page 1

[7] [$SC01$], pages $2-3$

[8] Also rotation of a "localized field configuration"

[9] An explanation of what unitarity means and how it relates to determinism will be shortly presented in the more important case of space-time non commutativity.

mechanics. In fact, locality in time of the action allows the construction of a Hamiltonian that yields a unitary time evolution[10]. The unitary structure of these field theories provides us with crucial information about string theory because they arise as low energy limits of string theory in the presence of a background magnetic field[11]. The low-energy limit is formally implemented by the limit $\alpha^{'} \to 0$ - where $\alpha^{'}$ is the string parameter - because the only dimensionless parameter in the theory is $(\alpha^{'}E^2)$[12]. In this low energy limit the dynamics are described by the non commutative field theory of the massless open strings, (the massless modes are all we can observe in that regime because the massive ones are too heavy to be seen). So, the fact that this limit can be performed tells us that space-space non commutativity preserves the deterministic structure of string theory as well. In other words, the effective field theory comes out from a "consistent truncation of the full unitary string theory"[13].

Things are different for space-time non commutative field theories, ($\theta^{0i} \neq 0$). The non commutative behavior of time in these field theories causes two important anomalies. The first one is the presence of acausal behavior at the first perturbation level, (tree level) - see Seiberg and Toumbas in [$NST00$]. The second one is the arising of a non unitary time evolution of the fields, i.e. a failure of determinism, at the second perturbation level (1-loop level) - see Gomis and Mehen in [$J.G00$] , Seiberg and Toumbas in [$NST00$].

The physical process I heuristically present here is taken from Seiberg et al.(2000). This is the scattering process of localized scalar fields or, in other words, the scattering of the wave packets of high energy particles after a collision[14]. I will use the same reduced number of dimensions (one spatial and one temporal) to make the description easier. Also, writing just θ is a shorthand for $\theta^{\mu\nu}$. Before the collision we have two incoming particles having momenta respectively k_1 and k_2, after the collision we have two outgoing scattering particles having momenta respectively p_1 and p_2. Using perturbation theory it is possible to calculate the S-matrix relative to this interaction[15]. Now, an important fact to emphasize about the

[10] [$J.G00$], page 1

[11] [$J.G00$], section 2,[$NS00$], pages 1,2

[12] See [$BBS07$], page 301

[13] [$J.G00$], section 2,[$NS00$], pages 1,2

[14] For an exhaustive mathematical presentation of this problem see [$NST00$], pages 5-9

[15] S matrices are unitary matrices connecting asymptotic particles states, i.e. particles'states between time equal to minus infinity and time equal to plus infinity; the entries of the matrix are scattering amplitudes which are produced by the interaction terms of the Lagrangian. These entries look like $< p_1, p_2|S|k_1, k_2 >$.

process described here is that the collision of the two incoming particles is the cause of the scattering, since the particles only interact at that time; before and after the collision the system is free. This remark will be relevant for what follows about acausal behavior that involves the first perturbation level.

In a commutative theory the S-matrix entries contain an invariant amplitude Ω, which at the first level of perturbation, is equal to '$-ig$', where g is a coupling constant[16] appearing in the interaction term of the lagrangian.
Switching to the non commutative case, the ordinary product among fields is replaced by the $\star$-product - introduced above - which is defined in function of the non-commutativity parameter θ. This new product brings several changes in the interaction term of the system's lagrangian. In fact, the interaction term will appear like[17]

$$g\phi \star \phi \star \phi \star \phi. \tag{4.3}$$

Therefore, this $\star$-product introduces an additional new phase in the interaction, called Moyal phase, which depends on the energies p^2 of the outgoing particles and on the θ parameter. This can be also seen from the entries of the new S-matrix which contains the new amplitude $\Omega = g(f(p^2\theta)$, where $f(p^2\theta)$ is a periodic function. The outgoing wave function, hence, will contain this new phase[18]:

$$\Phi_{out}(x) \sim \int dp(f(p^2\theta))\phi_{in}, \tag{4.4}$$

where ϕ_{in} is the incoming wave-packet.

Integrating over the momenta dp, we will get an outgoing wave-packet - after the collision - that splits in three parts, but only two are relevant to us. The first one appears time-delayed with respect to the collision of the incoming packet. It is called the *retarded* wave packet. The second outgoing term is instead an *"advanced"* wave-packet since it t appears before the collision[19]. The former represents a physical process compatible with causality, whereas the latter represents an acausal physical process, also considered by the authors as reason to reject a space-time non commutative field theory as pathological.

[16]This amplitude is computed using Feynman diagrams. For more detail see [Vel94], pages 46-62
[17]See [NST00], page 5, (2.26)
[18]See [NST00], page 6
[19]The use of quotation marks - also used by the authors - will be clarified shortly.

But is this acausal behavior an instance of backward causation? If that is the case it should not be seen as a violation of the physical laws governing the process. Physical laws are time-symmetric after all. The fact that the cause always precedes the effect is not a lawful fact.It is something that should be explained independently from the laws. An example from classical electrodynamics can help to explain more clearly this point. Learning to compute a solution in this context reveals to us that a wave equation is formally associated with both the retarded G^+ and advanced G^- Green's functions. However, any actual field configuration (solution to the wave equation) can be constructed selecting just one of the two Green functions. Physicists usually choose the retarded Green's function. This function is usually thought as the "causal" one. However, selecting the advanced Green function would be an equivalent choice. There is nothing special about the retarded Green function that makes it consistent with causality. That choice just depends on the particular Cauchy problem we are dealing with. The notion of cause in classical electrodynamics does not include the requirement that the cause must precede the effect. A cause is a physical event - usually an interaction - which is formally described by a Green's function that can be chosen either advanced or retarded.

Therefore,*mutatis mutandis*, we could just apply the same pattern of interpretation to the appearance of our advanced wave-packet, hence concluding that it does not violate the physical laws governing the scattering process. But in this case that appearance cannot be read as an instance of backward causation, and that is why it is labeled as "advanced". Why is that?

Let' think for a moment to how an ordinary scattering process - i.e. a process following the ordinary temporal sequence - would appear. In this case we would see an incoming wave-packet corresponding to two increasingly close incoming particles moving from $t = -\infty$ to $t = 0$, the collision time. Then, for $t > 0$, we would see the effect of the collision, i.e. an outgoing wave packet corresponding to two increasingly distant outgoing particles moving toward $t = \infty$. For simplicity reasons, imagine a watch traveling along with the center of mass of the particles system with its hands moving clock-wise.

Now, if such system takes part in a backward causation process what we would see is the watch's hands starting to move counter-clockwise, since the time sense of the system would appear reversed. So we would see an advanced wave-packet - the advanced effect - now corresponding to two increasingly near incoming particles, traveling backward from $+\infty$ toward $t = 0$. Then, for $t < 0$ we see now two increasing distant outgoing particles, traveling backward toward $-\infty$. In this case we can say that the effect appears as advanced with respect to the cause - the collision - and the entire process appears reversed.

But this is not what happens in our scattering process. As Seiberg and Toumbas[20] crucially point out the advanced part of the outgoing wave packet (the advanced effect) propagates toward $+\infty$. So the "advanced" effect here consist in two outgoing increasingly distant particles - both moving toward $+\infty$ - appearing before the collision. That means that a watch moving along with them would keep rotating its hands clockwise. But if the time sense of the system is preserved we cannot apply the notion of backward causation.

This violation of causality seems to have a different and problematic status. The authors[21] try to explain this acausal behavior in terms of a similarity with an apparent advanced process in the dynamics of a rigid rod thrown against a wall from an initial distance. The rigid rod has a center of mass moving along with it. One of its ends will eventually hit the wall before the center of mass does the same. So, the center of mass will appear to bounce back before it hits the wall. As they claim, the rigidity implies that the effect has a space-like cause. But then this process is impossible because it violates relativity. That would label the theory as pathological. However, their reading of the "advanced" scattered wavepacket as the space-like effect of the collision is in tension with the fact that earlier on in the paper they seem to say that the "advanced" packet appears in the past cone of the collision.

In order to introduce the further problem arising at the 1-loop level in the scattering case I refer back to what I just said about Green functions and actual field configurations in electrodynamics. The reader should keep in mind that in what follows about 1-loop level in the scattering case the notions of advanced and retarded will be used in the sense of classical electrodynamics.

Although computing a solution involve an arbitrary choice of one of the two Green functions, the actual configuration of an electric field at any fixed time should not contain both functions (advanced and retarded with respect to the fixed time in which we look at the

[20] [NST00],page 7
[21] [NST00],page 8

actual configuration). Equivalently, the actual configuration of the electric field at a present time should not have a value depending both on the sources in space at a past time and on the same sources in space at some future time. That would be a simultaneous dependence of the present field configuration on two different initial conditions. But two different initial conditions cannot generate two solutions intersecting each other somewhere. That would violate determinism. Here, I am identifying the notion of deterministic evolution of a physical system with the mathematical property that an operator describing that evolution must have, i.e. being a one-to-one map among states.

Referring to the scattering field case we have first to translate what we just said about violation of determinism in the language of quantum phenomena. In quantum physics the notion of deterministic evolution of a system is often given via the definition of unitarity. Unitarity is a restriction on the allowed time evolutions that a quantum system can possibly have. Time evolution has to be mathematically described by a unitary operator, as a result of which probability is conserved. Unitary operators are automorphisms of Hilbert spaces, i.e., they preserve the linear space structure and the inner product of the space on which they act. In particular, since they are automorphisms, they have the property of being one-to-one maps between states. So the mathematical property of describing deterministic evolutions of the system is built inside the mathematical definition of being a unitary operator. So in this sense violation of determinism via violation of the mathematical requirement of being a one-to-one operator is a violation of unitarity. An example of that would be the simultaneous dependence of the present state's value of a system on both future and past states' values.

What we just said should clarify in which sense Gomes and Mehen speak about violation of unitarity at the second (1-loop) level of perturbation. In their "Space-Time Noncommutative Field Theories And Unitarity",(see [J.G00]) they analyze the same field scattering case studied by Seiberg et al.([NST00]). They show that, since time does not commute,($\theta^{0i} \neq 0$), the lagrangian contains non local time derivatives, which makes the theory non local in time[22]. The non locality of the action produces at the 1-loop level an actual configuration of the field in which its value at a present time depends simultaneously on both past and future times[23]. Therefore, at the 1-loop level the advanced and the retarded wave-packets

[22] [J.G00], page 2

[23] See also [NST00], page 1

appear together in the same outgoing wave-function. Formally speaking, this simultaneous presence seems to be due to the appearance of the extra phase containing the antisymmetric parameter θ mentioned above, (the Moyal phase). Therefore space-time non commutativity in field theory causes a failure of unitarity in the sense specified above.

Now let's consider what happens if we have space-time non-commutativity in a case of scattering strings.

Why is this result about non commutative space-time field theories relevant to the string case? The interesting fact is the following: although space-time non commutativity, ($\theta^{0i} \neq 0$), can be obtained in string theory[24], there is no way to obtain non commutative space-time field theories as low-energy limits of string theory[25]. Why is that? Because space-time non commutativity in string theory has implications for causality which are different from those found in field theory. I shall briefly describe the scattering process in non commutative open string theory presented by Seiberg, Susskind and Toumbas[26].

Before considering the scattering case studied by the three authors, let's mention some mathematical aspects involved in the presence of an electric field along a background of string's propagation. Let G be the metric of the open strings, $\theta^{0i} = \theta$, let E be the background electric field and let E_c be the critical upper bound value of the electric field. The critical upper bound is the value of the electric field beyond which the string perturbative regime breaks down[27]. All these parameters are related to each other by the following relation[28]

$$\alpha' G^{-1} = \frac{1}{2\pi} \frac{E}{E_c} \theta. \tag{4.5}$$

Intuitively speaking, we can see in this relation that $\alpha' \sim \theta$, for finite values of G. A low energy-limit of an open string theory is a limit in which $\alpha' \to 0$. This limit then implies also the vanishing of the non commutative parameter. Therefore, what would seem to arise in this case is an ordinary commutative field theory. From the formula above we can see that since $\alpha' \sim \theta$ and $E \sim \frac{1}{\theta}$, the energy scale that allows space-time noncommutativity to appear is the scale at which $\theta >> 0$. But then in this regime the massive open string states can not

[24] In the next subsection I shall analyze two different views about the nature of conditions under which space-time non commutativity arises.
[25] [NST00],page 9, [N.S00], pages 4-8,[J.G00], pages 9-12
[26] [NST00], pages 9-14
[27] See [J.G00], pages 10-11.
[28] See [J.G00], page 11, (3.3)

be neglected. However, this basic mathematical explanation of why we cannot perform the above mentioned low-energy limit should be pulled alongside of a physical explanation. To this aim let's move to the authors' presentation of the string scattering case.

We just saw that in non commutative field theory the $\star$-product among fields changes the interaction term of the lagrangian in such a way that a new phase depending on θ appears. This new phase is responsible for a field scattering process that violates causality. In the case of open string scattering, a new feature due to the oscillation of the strings changes the non commutative parameter θ in the following way[29]:

$$\theta^{'} = 2\pi(n \pm \frac{E}{E_c}). \tag{4.6}$$

This fact has important consequences. Although the amplitude acquires the Moyal phase as well, the modification of θ covers those features of the phase which in field theory are responsible for pathological acausal behavior at the "tree level", and failure of determinism at the "1-loop" level. In fact, in this case, the actual configuration of the physical system given by the outgoing wave function contains only time delay terms. The advanced terms are not there. Broadly speaking, the acausal phase gets multiplied by a function generated by the oscillation effects of the string. What it is left over by this multiplication is a formula in which the advanced terms do not appear because canceled. Therefore, the acausal behavior, generated by the non commutative parameter θ at the two levels of perturbation, is removed. Therefore, space-time non commutativity does not compromise the deterministic structure of string theory and does not introduce pathological behaviors. The lack of pathological temporal features of strings physically explains why space-time non commutative field theories cannot arise from low-energy limits of string theory.

This fact has profound implications for the ordinary notion of spacetime in string theory. My claim is that it supports the view assigning to it an emergent character in the theory. I shall come back to this point in the chapter's conclusion.

Finally, the appearance of a string's feature changing the non commutative parameter as in (4.6) has a further important consequence. In fact the formula clearly illustrates that

[29] I won't present here the mathematical steps relative to this point. For more detail see [NST00], page 11,12, formula (3.9)

despite removing the source of perturbation by turning off the electric field E, the non commutative parameter does not vanish. Based on (4.6) time and space can exhibit non commutative behavior also not in presence of a perturbation field. This fact does not seem to fit into the general picture presented by Seiberg, Susskind and Toumbas, in which, as we saw, space-time non commutativity holds just in that particular perturbative regime. The issue introduces us to the content of the next section.

4.2 Space-time uncertainty principle in string theory

Under what conditions space-time non commutativity arises in string theory? Are we in the presence of some intrinsic feature of the theory or instead of an extrinsic feature exhibited just in perturbative regimes? Answering these question I shall consider two different approaches.

The first view presented by Seiberg, Toumbas and Susskind[30] claims that String Theory shows space-time non commutativity only in the presence of a background electric field. The latter would produce a non-commutative perturbation[31] of the theory in which a stringy spacetime uncertainty principle can be consequently formulated. Its formal expression is in function of the $\alpha^{'}$ string parameter[32]

$$\triangle t \triangle x \geq \alpha^{'}. \tag{4.7}$$

So, according to them, space-time uncertainty principle is an extrinsic principle derivable only in perturbative regimes of the theory.

A second view is that presented by Yoneya. In his essay "String Theory and the Space-Time Uncertainty Principle",(see [Yon]), he claims that space-time non commutativity is an intrinsic feature of the theory, which is not necessarily related to the presence of perturbative fields. More precisely, it arises from an intrinsic property of strings dynamics. Therefore it appears also in a nonperturbative conceptual framework of the theory[33]. The world-sheet intrinsic property from which it arises is connected in a special way with space-time

[30] See [NST00], section 3

[31] [NST00], page 15

[32] The space-time uncertainty principle tells us that when we try to probe short distances along a time-like direction the amount of uncertainty relative to probing distances along a space-like direction increases, and viceversa.

[33] [Yon], pages 33-39

uncertainty principle at the string length scale. Let's see in detail.

On the one side, according to Seiberg, Toumbas and Susskind, the non commutative relation $[t, x] = i\theta$ - valid only perturbatively - entails inside the same perturbative regime the spacetime uncertainty principle. On the other side, according to Yoneya, things are in the other way around: space-time uncertainty principle entails space-time non commutativity. More clearly, the principle is thought to be one of those peculiar properties of the theory that can be derived from conformally invariant features of the string's world-sheet dynamics, (we will see below more precisely how this derivation works). This principle is therefore "universally" valid[34]. More precisely, the uncertainty relation between space-like direction and time-like direction is just a particular expression derivable from a more general conformally invariant principle defined over the world-sheet. The latter, if read with respect to the Minkowski metric, produces the former.

The method of derivation of the uncertainty principle from world-sheet's properties is schematically presented in what follows[35]. The idea behind the derivation relies on the use of Riemann surfaces[36]. Let's introduce some preliminary notions[37].

An arc γ on a Riemann surface has a length $L(\gamma, \rho)$ with respect to a metric $ds = \rho(z, \overline{z})|dz|$. In general $L(\gamma, \rho_1) \neq L(\gamma, \rho_2)$ if $\rho_1 \neq \rho_2$. However, it is possible to define "distances" on a Riemann surface which are conformally invariant, i.e very peculiar "distances" that do not change if dilated or contracted. Let Ω be a region on the surface, let Γ be a set of arcs in that region, we can define the *extremal length* of the collection Γ of curves as a conformal invariant of Γ, which means that, given a conformal mappings $f : \Omega \to \Omega'$, the extremal length of Γ is equal to the extremal length of the image of Γ under f. The extremal length of Γ is defined as[38]

$$\lambda_\Omega(\Gamma) = sup \frac{L(\Gamma, \rho)^2}{A(\Gamma, \rho)}, \tag{4.8}$$

[34] [Yon], page 7

[35] For more detail see [Yon], section 2.3

[36] In mathematics, a Riemann surface is a complex manifold. It is usually represented as a deformation of the complex plane, which locally (near every point) presents the same topological properties of such a plane, but globally appears to be topologically very different. A sphere is a good example of Riemann surface

[37] See [Yon] , pages 11-13

[38] See [Yon], pages 11-13, (2.7)

where $L(\Gamma, \rho) = inf_{\gamma \in \Gamma}\ L(\gamma, \rho)$ and

$$A(\Omega, \rho) = \int_\Omega \rho^2 dz d\bar{z} = \int_\Omega (\rho dz d\bar{z})^2 = \int_\Omega ds^2. \tag{4.9}$$

In particular, let "Ω be a quadrilateral segment"[39] (being β, β' the first couple of opposite edges and ξ, ξ' the second couple of opposite edges) and let "Γ be the set of all connected sets of arcs"[40] joining ξ and ξ'. Finally let Γ^* be the set of all connected sets of arcs joining β and β'.

What it is really important about the relation between these two sets of arcs is their *reciprocity relation*[41],

$$\lambda_\Omega(\Gamma)\lambda_\Omega(\Gamma^*) = 1 \tag{4.10}$$

The conformal invariance of this relation comes from the conformal invariance of the extremal length $\lambda_\Omega(\Gamma)$.

At this point is important to recall what we saw in chapter one about theory's partition function and path integrals. A particular history of a string physical system, i.e. a string world-sheet, is represented by a Riemann surface - as Jeffrey Olson shows in his "Worldsheets, Riemann Surfaces, and Moduli"[42]. So, in particular string world-sheets are conformally invariant surfaces on which we can define extremal lengths and their reciprocity relation. As we saw in 1.5.2, path integrals are identified with maps from string worldsheets to a target spacetime[43].

Now, the crucial thing to understand about derivation of space-time uncertainty principle is that the latter defined over the target spacetime of the path integral arises from the reciprocity relation between extremal lengths over the string world-sheet. This derivation can be understood looking at how a string's amplitude can be written inside the path integral[44]. In this amplitude two extremal lengths $\lambda_\Omega(\Gamma)$ and $\lambda_\Omega(\Gamma^*)$ show up as the measure involved in probing spacetime's structure[45]:

[39] See [Yon], page 12

[40] See [Yon], page 12

[41] For an explanation of how this relation can be obtained see [Yon], pages 11-12, (2.8)

[42] See [*Ols*], sections 2 and 3

[43] See also [Yon], page 11

[44] For more detail on how Yoneya derives this expression see [Yon], pages $11-14$

[45] see [Yon], page 13

$$exp^{-\frac{1}{l_s^2}(\frac{A^2}{\lambda(\Gamma)}+\frac{B^2}{\lambda(\Gamma^*)})}. \tag{4.11}$$

Now,

$$\Delta A \sim \sqrt{\lambda(\Gamma)} l_s, \tag{4.12}$$

$$\Delta B \sim \sqrt{\lambda(\Gamma^*)} l_s, \tag{4.13}$$

hence the length l_s, the length probed by strings amplitudes in spacetime, can be expressed like

$$l_s \sim \frac{\Delta A}{\sqrt{\lambda(\Gamma)}} \sim \frac{\Delta B}{\sqrt{\lambda(\Gamma^*)}}. \tag{4.14}$$

Moreover,the extremal lengths $\lambda(\Gamma)$ and $\lambda(\Gamma^*)$ show up respectively in ΔA and ΔB. Therefore, if we are examining both directions at the same time, the reciprocity principle involving those extremal lengths works as a constraint on how much information you can get at short distances. In fact, combining (4.10) with (4.12) and (4.13), we get

$$\frac{\Delta A}{l_s^2} \cdot \frac{\Delta B}{l_s^2} = 1, \tag{4.15}$$

which can be re-written like

$$\Delta A \cdot \Delta B \geq l_s^2. \tag{4.16}$$

Interpreting what Yoneya says[46], we can now read the above relation with respect to the Minkowski metric. If we do that we will get a relation between a space-like direction and a time-like direction, which is exactly what Yoneya is trying to derive, i.e. space-time uncertainty principle at the length scale probed by strings.

Therefore, space-time uncertainty principle can be derived from the more general conformally invariant duality relation over the string's worldsheet. Then, according to this approach, the principle is an intrinsic property of string theory which in its turn entails space-time non commutativity. As Yoneya says[47], "Thus the noncommutativity of space and time is indeed there in a hidden form.[...] What is in mind here is a different representation of string theory with manifest non commutativity that is, however, equivalent at the level of

[46] [Yon], page 13
[47] [Yon], page 33

the on-shell S-matrix, to the usual formulation" (where the equivalence at the level of the on-shell S-matrix means equivalence with respect to dynamics that obey the same equation of motion).

4.3 Conclusion

In the previous section I considered two different ways of describing the conditions under which space and time fail to commute in string theory. According to Seiberg, Toumbas and Sussukind space-time non commutativity seems to be an extrinsic feature of string theory since space and time fail to commute only in some specific cases. According to Yoneya, despite it is unquestionable that such perturbations yield space-time non commutativity, they are not necessary condition for the latter to happen. In fact he shows that space-time non commutativity can be alternatively derived from a conformally invariant principle defined over the string world-sheet. This fact qualifies space-time non commutativity as an intrinsic principle of the theory, hence holding true in a non perturbative formulation.

However, both views share the same idea that space-time non commutativity does not raise the specter of indeterminism and acausal behavior. This fact has profound implications for the ordinary notion of space and time. The question is how the unbroken causality would support the emergent role of ordinary spacetime in string theory. As I said in the introduction of this chapter, one way of thinking about emergence is that of considering ordinary spacetime as an entity emergent from an underlying structure postulated by the ontology of a more fundamental theory. How should we think about this underlying structure and about this more fundamental theory?

Let's first characterize a bit more precisely the notion of an underlying structure. The latter should be thought as an abstract algebraic space, conceptually very far from the ordinary notion of point space. It should be devoid of any metrical property, characterized by some sort of discreteness or quantized shape and by some kind of non locality. Now, the non commutative "spacetime" presented by the authors above mentioned seems to satisfy these requirements. In fact, space-time non commutativity is a potential feature of a quantized "manifold" and it also introduces some sort of non locality. So, non commutative "spacetime" seems to be the kind of algebraic space that a fundamental theory would postulate.

But let's say something more precise about non commutative"spacetime".
Non commutative"spacetime" is an algebraic object arising from the extension of a commutative algebra of functions over an ordinary spacetime. Algebraic properties of a commutative algebra of functions perfectly encode the metrical properties of the manifold over which the functions are defined. For example, integrating over spacetime corresponds to computing the trace of some operator belonging to the algebra. So, in the commutative case, we have a perfect correspondence between the category of algebraic objects (algebra of functions over spacetime) and the category of geometrical objects (spacetime).

However, when we extend the commutative algebra of functions over spacetime to a more general non commutative algebra of operators - which includes the former as its particular case - we lose the correspondence between algebraic and metrical properties, because we end up with lacking a geometrical manifold over which the non commutative algebra of operators is defined. So, non commutative "spacetime" is identified with the non commutative algebra of operators and in this sense it is an algebraic space.

But then in which sense ordinary spacetime would emerge from non commutative "spacetime"? As we saw in the short presentation of non commutative geometry, the latter contains the commutative case as its particular case obtainable by imposing a vanishing condition on the anti commutative parameter. So, ordinary commutative geometry would emerge from non commutative geometry via the mathematical limit of $\theta \to 0$.

But in order to speak about the emergence of ordinary spacetime from non commutative "spacetime" we need to figure out some kind of physical counterpart involved in that mathematical limit. Looking at (4.5) we can see that the non commutative parameter θ and the physical parameter $\alpha^{'}$ are directly proportional. So, the mathematical limit $\theta \to 0$ is formally connected to the physical low-energy limit $\alpha^{'} \to 0$. As we saw earlier on the limit $\alpha^{'} \to 0$ implements the physical low-energy limit since in the theory the only dimensionless parameter is $(\alpha^{'} E^2)$[48]. Then, the low-energy limit is what can characterize in a full physical sense this idea of an emergent spacetime from an underlying non commutative "spacetime".

At this point we reached some kind of account of how this emergence of ordinary spacetime would work and of how this underlying structure postulated by the ontology of a fundamental theory looks like. What can we say about this fundamental theory? We need a theoretical

[48] See [BBS07], page 301

framework in which the “quantized spacetime” described above does not produce any undesirable consequences for the theory, compromising its physical significance. The theoretical findings we examined in the previous section showed that introducing a non commutative spacetime in a concrete case of string scattering does not break causality and determinism. So, in string theory postulating some underlying “quantized spacetime” does not bring about any pathological consequences that would label the theory as unphysical. Space-time non commutativity in the theory do not compromise the basic temporal features of strings’ dynamics. That can be read as an important step toward the accomplishment of some kind of “spacetime quantization” inside the conceptual framework of string theory. But then the fate of our ordinary spacetime seems to be inevitably that of non necessary component for string theory, since it emerges as derived concept from the deeper underlying non commutative structure that the theory postulates as fundamental.

The approach I am presenting here is diametrically opposed to the view that Seiberg, Susskind and Toumbas seems to endorse. In their paper they do not explicitly deal with the issue of emergence. Still, extrapolating from their case study, they appear to assume that the non commutative string theory emerges as particular case inside the perturbative formulation of the commutative one.

A completely different relation holds instead between the view I am endorsing here and that of Yoneya. The latter presents space-time non commutativity - at the length scale probed by strings - as an intrinsic property “emerging” from an underlying conformally invariant feature of the string world-sheet, i.e. the reciprocity principle. What I would like to point out is that my way of reading his view can be consistently combined with the view I am endorsing here.

A possible broader conceptual framework that might serve the purpose consist in thinking that despite non commutative spacetime is a more fundamental entity postulated by string theory, from which the ordinary one emerges in the way described above, still it is not the most fundamental one, since it appears to arise from a deeper underlying world-sheet structure.

So we have a chain of emergent theoretical facts starting from the string world-sheet, where the reciprocity principle expresses a duality relation between ”distances” peculiar to the strings world. Then, by discovering that space-time uncertainty relation is actually hidden in that principle, we can see that, still at the spacetime length scale probed by strings, a space-time non commutative structure arises from that world-sheet by imposing particular constrains on the reciprocity principle. Finally, leaving the string length scale by a low energy limit taken on the non commutative spacetime string theory, an ordinary spacetime emerges.

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