

## Algorithm xxx: Fortran 90 subroutines to compute Mathieu functions for complex values of the parameter

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Software to compute angular and radial Mathieu functions is provided in the case that the parameter  $q$  is a complex variable and the independent variable  $x$  is real. After an introduction on the notation and the definitions of Mathieu functions and their related properties, Fortran 90 subroutines to compute them are described and validated with some comparisons. A sample application is also provided.

Categories and Subject Descriptors: G.4 [Mathematical Software]: documentation; J.2 [Physical Sciences and Engineering]: engineering, mathematics, physics

General Terms: algorithm, documentation, languages, verification

Additional Key Words and Phrases: Mathieu function, special function, computation, validation

### ACM Reference Format:

Erricolo, D., Carluccio G., 2011. Fortran 90 subroutines to compute Mathieu functions for complex values of the parameter. ACM Trans. Math. Softw. 38, 1, Article 1 (January 2011), 19 pages.  
DOI = 10.1145/0000000.0000000 <http://doi.acm.org/10.1145/0000000.0000000>

### 1. INTRODUCTION

Mathieu functions were introduced by Émile Léonard Mathieu in 1868 to study the vibration modes of an elliptical membrane [Mathieu 1868]. Since then, many other authors have investigated their properties and the most important results are included in [McLachlan 1964], [Meixner and Schäfke 1954], [Staff of the Computation Laboratory 1967]. Mathieu functions are the solutions of Mathieu's differential equation

$$\frac{d^2y}{dx^2} + (a - 2q \cos^2 x)y = 0, \quad (1)$$

which are also called angular Mathieu functions. Replacing  $x$  with  $ix$  in the previous equation yields the modified Mathieu differential equation

$$\frac{d^2y}{dx^2} - (a - 2q \cosh^2 x)y = 0, \quad (2)$$

whose solutions are called radial Mathieu functions.

Solutions of both differential equations depend upon the parameter  $q$ , which could take complex values.

The computation of Mathieu functions is not a trivial problem and software packages that compute them when  $q$  is real have been developed by, among others, Clemm

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This work was partially supported by the AFOSR under MURI grant FA9550-05-1-0443 and by the AFOSR grant FA9550-12-1-0174.

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DOI 10.1145/0000000.0000000 <http://doi.acm.org/10.1145/0000000.0000000>

[1969; 1970], Hodge [1972], Rengarajan [1980], Frisch [1972], Baker [1992], Shirts [1993a], [1993b], IMSL [1994], Zhang and Jin [1996], Alhargan [2001], Erricolo [2006] and [Wolfram 2003]. When  $q$  is complex, Mathematica® provides solutions, but only for the angular functions. The novelty of this article is to extend to the case of complex values of  $q$  the results presented in [Erricolo 2006].

## 2. ANGULAR AND RADIAL OF MATHIEU FUNCTIONS

In the following, we will use the notation of Blanch and Rhodes [Blanch and Rhodes 1955] (which is also available in [National Bureau of Standards 1951], [Staff of the Computation Laboratory 1967]) and Stratton [Stratton 1941].

Equation (1) is periodic and in many physical applications only periodic solutions are of interest. For a given  $q$  there exist two countable sets of values of  $a$  for which equation (1) admits periodic solutions. These values of  $a$  are called characteristic values and, depending upon the set, the period of the solution is either  $\pi$  or  $2\pi$ . There are four kinds of periodic solutions of (1) associated with the characteristic values  $a$ :

$$\text{Se}_{2r}(q, x) = \sum_{k=0}^{\infty} \text{De}_{2k}^{(2r)} \cos 2kx \quad (\text{of period } \pi) \quad (3)$$

$$\text{Se}_{2r+1}(q, x) = \sum_{k=0}^{\infty} \text{De}_{2k+1}^{(2r+1)} \cos(2k+1)x \quad (\text{of period } 2\pi) \quad (4)$$

$$\text{So}_{2r}(q, x) = \sum_{k=1}^{\infty} \text{Do}_{2k}^{(2r)} \sin 2kx \quad (\text{of period } \pi) \quad (5)$$

$$\text{So}_{2r+1}(q, x) = \sum_{k=0}^{\infty} \text{Do}_{2k+1}^{(2r+1)} \sin(2k+1)x \quad (\text{of period } 2\pi) \quad (6)$$

The functions  $\text{Se}_n(q, ix)$  and  $\text{So}_n(q, ix)$  clearly satisfy (2) for the same characteristic values  $a$ , but (3)-(6) converge slowly. Therefore the solutions of (2) are written in terms of rapidly converging series of products of Bessel functions associated with the same coefficients  $\text{De}_m, \text{Do}_m$  of the angular functions. Their expressions are:

$$\text{Re}_{2r}^{(1)}(q, x) = \frac{(-1)^r}{\text{De}_0^{(2r)}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^{\infty} (-1)^k \text{De}_{2k}^{(2r)} J_k(s) J_k(t), \quad (7)$$

$$\text{Re}_{2r+1}^{(1)}(q, x) = \frac{(-1)^r}{\text{De}_1^{(2r+1)}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^{\infty} (-1)^k \text{De}_{2k+1}^{(2r+1)} [J_{k+1}(s) J_k(t) + J_k(s) J_{k+1}(t)], \quad (8)$$

$$\text{Ro}_{2r}^{(1)}(q, x) = \frac{(-1)^r}{\text{Do}_2^{(2r)}} \sqrt{\frac{\pi}{2}} \sum_{k=1}^{\infty} (-1)^k \text{Do}_{2k}^{(2r)} [J_{k+1}(s) J_{k-1}(t) - J_{k+1}(t) J_{k-1}(s)], \quad (9)$$

$$\text{Ro}_{2r+1}^{(1)}(q, x) = \frac{(-1)^r}{\text{Do}_1^{(2r+1)}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^{\infty} (-1)^k \text{Do}_{2k+1}^{(2r+1)} [J_{k+1}(s) J_k(t) - J_k(s) J_{k+1}(t)], \quad (10)$$

where

$$s = \sqrt{q}e^x, \quad t = \sqrt{q}e^{-x}. \quad (11)$$

The radial functions of the first kind have parity either even,  $\text{Re}_n^{(1)}$ , or odd,  $\text{Ro}_n^{(1)}$ . A second set of solutions for the modified Mathieu's equation is obtained by replacing the Bessel functions  $J_m(s)$  in the previous equations with the Bessel functions  $Y_m(s)$ .

This substitution yields the modified functions of the second kind. In many physical applications they are referred to as radial functions of the second kind. They have parity either even,  $\text{Re}_n^{(2)}$ , or odd,  $\text{Ro}_n^{(2)}$ :

$$\text{Re}_{2r}^{(2)}(q, x) = \frac{(-1)^r}{\text{De}_0^{(2r)}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^{\infty} (-1)^k \text{De}_{2k}^{(2r)} Y_k(s) J_k(t), \quad (12)$$

$$\text{Re}_{2r+1}^{(2)}(q, x) = \frac{(-1)^r}{\text{De}_1^{(2r+1)}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^{\infty} (-1)^k \text{De}_{2k+1}^{(2r+1)} [Y_{k+1}(s) J_k(t) + Y_k(s) J_{k+1}(t)], \quad (13)$$

$$\text{Ro}_{2r}^{(2)}(q, x) = \frac{(-1)^r}{\text{Do}_2^{(2r)}} \sqrt{\frac{\pi}{2}} \sum_{k=1}^{\infty} (-1)^k \text{Do}_{2k}^{(2r)} [Y_{k+1}(s) J_{k-1}(t) - Y_{k-1}(s) J_{k+1}(t)], \quad (14)$$

$$\text{Ro}_{2r+1}^{(2)}(q, x) = \frac{(-1)^r}{\text{Do}_1^{(2r+1)}} \sqrt{\frac{\pi}{2}} \sum_{k=0}^{\infty} (-1)^k \text{Do}_{2k+1}^{(2r+1)} [Y_{k+1}(s) J_k(t) - Y_k(s) J_{k+1}(t)]. \quad (15)$$

Similar to Hankel functions, one defines modified functions of the third and fourth kinds. They are also referred to as radial functions and they have even parity

$$\text{Re}_n^{(3)} = \text{Re}_n^{(1)} + i\text{Re}_n^{(2)}, \quad (16)$$

$$\text{Re}_n^{(4)} = \text{Re}_n^{(1)} - i\text{Re}_n^{(2)}, \quad (17)$$

and odd parity

$$\text{Ro}_n^{(3)} = \text{Ro}_n^{(1)} + i\text{Ro}_n^{(2)}, \quad (18)$$

$$\text{Ro}_n^{(4)} = \text{Ro}_n^{(1)} - i\text{Ro}_n^{(2)}. \quad (19)$$

Let  $a_r(q)$  be the characteristic values associated with even solutions and  $b_r(q)$  those associated with odd solutions. Notice that  $r = 0, 1, 2, \dots$  and  $r = 0$  applies only to  $a_r$ . The following four properties hold for complex values of  $q$ .

*Property 1.* When  $q \rightarrow 0$ ,  $a_r(q) \rightarrow r^2$  and  $b_r(q) \rightarrow r^2$ .

*Property 2.*  $a_{2r}(-q) = a_{2r}(q)$ ,  $b_{2r}(-q) = b_{2r}(q)$ ,  $a_{2r+1}(-q) = b_{2r+1}(q)$ .

*Property 3.* When  $q$  is real and not zero, the characteristic values can be ordered as  $a_0 < b_1 < a_1 < b_2 < \dots < b_r < a_r < \dots$ ,  $q > 0$

$a_0 < a_1 < b_1 < b_2 < a_2 < a_3 < b_3 < \dots < a_{2r-1} < b_{2r-1} < b_{2r} < a_{2r} < \dots$ ,  $q < 0$

*Property 4.* When  $q$  is real, the solutions associated with  $a_r(q)$  and  $b_r(q)$  have  $r$  zeros in  $[0, \pi)$ .

Properties 1 and 3 establish the ordering principle for the solutions of Mathieu equation. Specifically, the order  $r$  of a Mathieu function is given by the position  $r$  of the characteristic value  $a_r$  (or  $b_r$ ) in the sequence of characteristic values. Mathieu functions of different order are associated with different expansion coefficients  $\text{De}^r$  ( $\text{Do}^r$ ), which depend on  $a_r$  ( $b_r$ ), as implied by the definitions (3)-(10), (12)-(19). When coupled with the continuity of  $a_r(q)$  and  $b_r(q)$ , Properties 1 and 3 are the only criterion available to order the characteristic values  $a_r(q)$  and  $b_r(q)$  in the complex plane [Blanch and Clemm 1969].

It is sufficient to know the solutions of Mathieu equation when  $q$  belongs to the first quadrant. In fact, all values of  $q$  may be reduced to the first quadrant because of Property 2 and also because if  $y(z)$  is a solution associated with  $a$  and  $q$  then  $y^*$  is a solution associated with  $q^*$  and  $a^*$ .

### 3. NORMALIZATION, PROPERTIES, ORTHOGONALITY, WRONSKIAN

There exist at least three different normalizations for the Mathieu functions. In this work, we use the normalization of Stratton [1941] and, accordingly, the Mathieu angular functions are normalized so that

$$\text{Se}_r(q, 0) = 1, \left. \frac{d\text{So}_r(q, x)}{dx} \right|_{x=0} = 1. \quad (20)$$

Therefore the expansion coefficients must satisfy

$$\sum_{k=0}^{\infty} \text{De}_{2k+p}^{(r)} = 1, \sum_{k=0}^{\infty} (2k+p) \text{Do}_{2k+p}^{(r)} = 1 \quad (21)$$

with  $p = 0, 1$  depending on the order  $r$ . Mathieu angular functions satisfy the following orthogonality relations

$$\int_0^{2\pi} \text{Se}_m(q, x) \text{Se}_n(q, x) dx = \begin{cases} 0 & m \neq n \\ N_m^e & m = n \end{cases} \quad (22)$$

$$\int_0^{2\pi} \text{So}_m(q, x) \text{So}_n(q, x) dx = \begin{cases} 0 & m \neq n \\ N_m^o & m = n \end{cases} \quad (23)$$

$$\int_0^{2\pi} \text{Se}_m(q_1, x) \text{So}_n(q_2, x) dx = 0, \forall m, n, q_1, q_2 \quad (24)$$

In the previous relations appear the normalization coefficients

$$N_{2r}^e = \pi \left[ 2 \left( \text{De}_0^{(2r)} \right)^2 + \left( \text{De}_2^{(2r)} \right)^2 + \left( \text{De}_4^{(2r)} \right)^2 + \dots \right], \quad (25)$$

$$N_{2r+1}^e = \pi \left[ \left( \text{De}_1^{(2r+1)} \right)^2 + \left( \text{De}_3^{(2r+1)} \right)^2 + \dots \right], \quad (26)$$

$$N_{2r}^o = \pi \left[ \left( \text{Do}_2^{(2r)} \right)^2 + \left( \text{Do}_4^{(2r)} \right)^2 + \dots \right], \quad (27)$$

$$N_{2r+1}^o = \pi \left[ \left( \text{Do}_1^{(2r+1)} \right)^2 + \left( \text{Do}_3^{(2r+1)} \right)^2 + \dots \right]. \quad (28)$$

#### 3.1. Relations between solutions for the parameter $q$ and $-q$

The following relations hold for the angular functions

$$\text{Se}_{2r}(-q, x) = \frac{\text{Se}_{2r}(q, \pi/2 - x)}{\text{Se}_{2r}(q, \pi/2)} \quad \text{De}_{2k}^{(2r)}(-q) = \frac{(-1)^k \text{De}_{2k}^{(2r)}(q)}{\text{Se}_{2r}(q, \pi/2)} \quad (29)$$

$$\text{Se}_{2r+1}(-q, x) = \frac{\text{So}_{2r+1}(q, \pi/2 - x)}{\text{So}_{2r+1}(q, \pi/2)} \quad \text{De}_{2k+1}^{(2r+1)}(-q) = \frac{(-1)^k \text{Do}_{2k+1}^{(2r+1)}(q)}{\text{So}_{2r+1}(q, \pi/2)} \quad (30)$$

$$\text{So}_{2r+1}(-q, x) = -\frac{\text{Se}_{2r+1}(q, \pi/2 - x)}{\text{Se}_{2r+1}'(q, \pi/2)} \quad \text{Do}_{2k+1}^{(2r+1)}(-q) = \frac{(-1)^{k+1} \text{De}_{2k+1}^{(2r+1)}(q)}{\text{Se}_{2r+1}'(q, \pi/2)} \quad (31)$$

$$\text{So}_{2r}(-q, x) = -\frac{\text{So}_{2r}(q, \pi/2 - x)}{\text{So}_{2r}'(q, \pi/2)} \quad \text{Do}_{2k}^{(2r)}(-q) = \frac{(-1)^k \text{Do}_{2k}^{(2r)}(q)}{\text{So}_{2r}'(q, \pi/2)} \quad (32)$$

For the radial functions of even order

$$\operatorname{Re}_{2r}^{(1)}(-q, x) = \operatorname{Re}_{2r}^{(1)}(q, x + i\pi/2) \quad (33)$$

$$\operatorname{Re}_{2r}^{(2)}(-q, x) = \operatorname{Re}_{2r}^{(2)}(q, x + i\pi/2) \quad (34)$$

$$\operatorname{Ro}_{2r}^{(1)}(-q, x) = \operatorname{Ro}_{2r}^{(1)}(q, x + i\pi/2) \quad (35)$$

$$\operatorname{Ro}_{2r}^{(2)}(-q, x) = \operatorname{Ro}_{2r}^{(2)}(q, x + i\pi/2) \quad (36)$$

and for the radial functions of odd order

$$\operatorname{Re}_{2r+1}^{(1)}(-q, x) = \operatorname{Ro}_{2r+1}^{(1)}(q, x + i\pi/2) \quad (37)$$

$$\operatorname{Ro}_{2r+1}^{(1)}(-q, x) = \operatorname{Re}_{2r+1}^{(1)}(q, x + i\pi/2) \quad (38)$$

$$\operatorname{Re}_{2r+1}^{(2)}(-q, x) = \operatorname{Ro}_{2r+1}^{(2)}(q, x + i\pi/2) \quad (39)$$

$$\operatorname{Ro}_{2r+1}^{(2)}(-q, x) = \operatorname{Re}_{2r+1}^{(2)}(q, x + i\pi/2) \quad (40)$$

The Mathieu radial functions of argument  $x + i\pi/2$  may be expressed in terms of the modified Bessel functions I and K using the following relations [Abramovitz and Stegun 1970]

$$J_m(iw) = i^m I_m(w), \quad Y_m(iw) = i^{m+1} \left[ I_m(w) + i \frac{2}{\pi} (-1)^m K_m(w) \right] \quad (41)$$

$$\cosh(w + i\pi/2) = i \sinh(w), \quad \sinh(w + i\pi/2) = i \cosh(w), \quad \tanh(w + i\pi/2) = \coth(w) \quad (42)$$

where  $w$  is a complex variable with  $-\pi < \arg w \leq \pi/2$ . In fact, using definition (11), when  $x \rightarrow x + i\pi/2$ , the argument of the Bessel functions become  $s \rightarrow is$  and  $t \rightarrow -it$ , so that the radial functions of the first kind become

$$\operatorname{Re}_{2r}^{(1)}(-q, x) = (-1)^r \sqrt{\frac{\pi}{2}} \operatorname{Ie}_{2r}(q, x) \quad (43)$$

$$\operatorname{Re}_{2r+1}^{(1)}(-q, x) = i(-1)^r \sqrt{\frac{\pi}{2}} \operatorname{Ie}_{2r+1}(q, x) \quad (44)$$

$$\operatorname{Ro}_{2r}^{(1)}(-q, x) = (-1)^r \sqrt{\frac{\pi}{2}} \operatorname{Io}_{2r}(q, x) \quad (45)$$

$$\operatorname{Ro}_{2r+1}^{(1)}(-q, x) = i(-1)^r \sqrt{\frac{\pi}{2}} \operatorname{Io}_{2r+1}(q, x) \quad (46)$$

where

$$\operatorname{Ie}_{2r}(q, x) = \sum_{k=0}^{\infty} (-1)^k \operatorname{De}_{2k}^{(2r)}(q) I_k(s) I_k(t) / \operatorname{De}_0^{(2r)}(q) \quad (47)$$

$$\operatorname{Ie}_{2r+1}(q, x) = \sum_{k=0}^{\infty} (-1)^k \operatorname{Do}_{2k+1}^{(2r+1)}(q) [I_{k+1}(s) I_k(t) + I_{k+1}(t) I_k(s)] / \operatorname{Do}_1^{(2r+1)}(q) \quad (48)$$

$$\operatorname{Io}_{2r}(q, x) = \sum_{k=1}^{\infty} (-1)^k \operatorname{Do}_{2k}^{(2r)}(q) [I_{k-1}(s) I_{k+1}(t) - I_{k+1}(s) I_{k-1}(t)] / \operatorname{Do}_2^{(2r)}(q) \quad (49)$$

$$\operatorname{Io}_{2r+1}(q, x) = \sum_{k=0}^{\infty} (-1)^k \operatorname{De}_{2k+1}^{(2r+1)}(q) [I_{k+1}(s) I_k(t) - I_k(t) I_{k+1}(s)] / \operatorname{De}_1^{(2r+1)}(q) \quad (50)$$

The radial functions of the second kind are complex for real value of  $x$ , yielding

$$\text{Re}_{2r}^{(2)}(-q, x) = (-1)^r \left[ i\sqrt{\pi/2} \text{Ie}_{2r}(q, x) - \sqrt{\frac{2}{\pi}} \text{Ke}_{2r}(q, x) \right] \quad (51)$$

$$\text{Re}_{2r+1}^{(2)}(-q, x) = (-1)^r \left[ -\sqrt{\pi/2} \text{Ie}_{2r+1}(q, x) + i\sqrt{\frac{2}{\pi}} \text{Ke}_{2r+1}(q, x) \right] \quad (52)$$

$$\text{Ro}_{2r}^{(2)}(-q, x) = (-1)^r \left[ i\sqrt{\pi/2} \text{Io}_{2r}(q, x) - \sqrt{\frac{2}{\pi}} \text{Ko}_{2r}(q, x) \right] \quad (53)$$

$$\text{Ro}_{2r+1}^{(2)}(-q, x) = (-1)^r \left[ -\sqrt{\pi/2} \text{Io}_{2r+1}(q, x) + i\sqrt{\frac{2}{\pi}} \text{Ko}_{2r+1}(q, x) \right] \quad (54)$$

where

$$\text{Ke}_{2r}(q, x) = \sum_{k=0}^{\infty} \text{De}_{2k}^{(2r)}(q) I_k(t) K_k(s) / \text{De}_0^{(2r)}(q) \quad (55)$$

$$\text{Ke}_{2r+1}(q, x) = \sum_{k=0}^{\infty} \text{Do}_{2k+1}^{(2r+1)}(q) [I_k(t) K_{k+1}(s) - I_{k+1}(t) K_k(s)] / \text{Do}_1^{(2r+1)}(q) \quad (56)$$

$$\text{Ko}_{2r}(q, x) = \sum_{k=0}^{\infty} \text{Do}_{2k}^{(2r)}(q) [I_{k-1}(t) K_{k+1}(s) - I_{k+1}(t) K_{k-1}(s)] / \text{Do}_2^{(2r)}(q) \quad (57)$$

$$\text{Ko}_{2r+1}(q, x) = \sum_{k=0}^{\infty} \text{De}_{2k+1}^{(2r)}(q) [I_{k+1}(t) K_k(s) + I_k(t) K_{k+1}(s)] / \text{De}_1^{(2r)}(q) \quad (58)$$

#### 4. COMPUTATION OF MATHIEU FUNCTIONS

Mathieu functions are computed according to the following three main steps: (1) computation of the characteristic values; (2) computation of the expansion coefficients  $\text{De}_m$  or  $\text{Do}_m$ ; and, (3) evaluation of the series expansions (3)-(10), (12)-(19).

##### 4.1. Computation of the characteristic values

The Mathieu characteristic values depend on the parameter  $q$ , the parity of the Mathieu function and its order  $n$ , and one approach to compute them is to solve an eigenvalue problem that leads to an infinite set of eigenvalues that approximate the characteristic values  $a_r$  ( $b_r$ ) associated with the parameter  $q$ .

When  $q$  is real, the characteristic values are real and it is possible to sort them, according to Property 3. This sorting provides the rule to determine the order  $n$  of the Mathieu function. Specifically, the solution of order  $n$  is the one generated using the characteristic value at position  $n$  in the sorted sequence. However, when  $q$  is not real, the characteristic values are complex and they cannot be sorted, which prevents a direct association between a specific characteristic value and its corresponding Mathieu function.

Fortunately, the continuity of the characteristic values  $a_r(q)$  and  $b_r(q)$  provides a method to determine the appropriate characteristic value to generate a requested Mathieu function of order  $n$ , which is explained next.

**4.1.1. Tracking algorithm.** Assume that  $q = q_{\text{real}} + iq_{\text{imag}}$  and that the parity and order of the Mathieu function are given. The following algorithm determines the appropri-

ate characteristic value,  $a_r(q_{\text{temp}})$  or  $b_r(q_{\text{temp}})$ , when  $q_{\text{temp}} = q_{\text{real}}$  and then tracks the characteristic value as  $q_{\text{temp}}$  is changed until it reaches  $q_{\text{temp}} = q$ .

*Step 1.* Let  $q_0 = q_{\text{real}}$  and compute the characteristic values associated with  $q_0$ . The characteristic values are the eigenvalues of an infinite matrix, whose entries depend on  $q$ ,  $n$ , and on the parity of the Mathieu function. Since  $q_0$  is real, the eigenvalues of the matrix are real and can be sorted. The infinite matrix is truncated to an order  $N > n$ , which is sufficiently large, ( $N = 10 + 2n$ ), so that the first  $n$  eigenvalues of the truncated matrix are a good approximation of the first  $n$  characteristic values. Each characteristic value  $a_r^0$  ( $b_r^0$ ) is associated with the eigenvalue at position  $r$  in the sequence and the one at the location  $r = n$  is selected. This is performed by `initial_eigenvalue_approximation`.

*Step 2.* Consider an increment  $\Delta q_{\text{imag}}$  so that  $\Delta q_{\text{imag}} \leq 0.5$  and set  $p = 0$ .

*Step 3.* Let  $p = p + 1$ ,  $q_{\text{temp}}^p = q_0 + ip\Delta q_{\text{imag}}$ . Then use the method described in [Blanch 1966] to refine the characteristic value  $a_n^p(q_{\text{temp}}^p)$  using as initial approximation  $a_n^{p-1}(q_{\text{temp}}^{p-1})$ .

*Step 4.* If  $q_{\text{temp}}^p \neq q$  then go back to Step 3, otherwise the last value found for  $a_n^p$  ( $b_n^p$ ) is the characteristic value of order  $n$  associated with  $q$ .

Steps 3 and 4 are both performed by `Mathieu_Eigenvalue_and_Coefficients`.

This software could be extended to allow for the simultaneous computation of all Mathieu functions of order below the maximum order  $n$ . This extension requires the simultaneous computation of all the characteristic values  $a_r(q)$  or  $b_r(q)$ ,  $r = 0, \dots, n$ , and could be accomplished with the following variation of the tracking algorithm. At Step 3, instead of using Blanch's algorithm, one could replace it with the following.

*Step 3.* Let  $p = p + 1$ ,  $q_{\text{temp}}^p = q_0 + ip\Delta q_{\text{imag}}$ . Create a new truncated matrix, similar to Step 1, for the parameter  $q_{\text{temp}}^p$  and compute its eigenvalues  $e_\ell$ ,  $\ell = 1, \dots, N$ . Let  $d_{\ell s}$  be the distance between the eigenvalue  $e_\ell$  and the previous characteristic value approximation  $a_s^{p-1}(q_{\text{temp}}^{p-1})$  or  $b_s^{p-1}(q_{\text{temp}}^{p-1})$ . For each  $s = 1, \dots, n$ , compute all distances  $d_{\ell s}$ . The characteristic value  $a_s^p(q_{\text{temp}}^p)$  or  $b_s^p(q_{\text{temp}}^p)$  is the eigenvalue  $e_\ell$  for which the distance  $d_{\ell s}$  is minimal.

#### 4.2. Computation of the expansion coefficients

The expansion coefficients  $\text{De}_m$  ( $\text{Do}_m$ ) are computed using four recurrence relations obtained by substituting the series expansion for the angular functions (3)-(6) into Mathieu differential equation (1).

Blanch [1966] showed that each relation may be written as both a forward or a backward recurrence for any characteristic value  $a_r$  ( $b_r$ ). In particular, she found the property that when the value of  $a_r$  ( $b_r$ ) is the characteristic value then both the forward and the backward recurrence relation produce the same expansion coefficients. This property is also used at Steps 3 and 4 of the previous section to refine the value computed for  $a_r$  ( $b_r$ ), by computing appropriate derivatives of  $a_r$  ( $b_r$ ) and using them into a Newton scheme to improve the initial value of  $a_r$  ( $b_r$ ). Once the value of  $a_r$  ( $b_r$ ) has been determined, the expansion coefficients are computed using both the forward and the backward recurrence relation by the subroutine `Mathieu_Eigenvalue_and_Coefficients`.

In the computation of the expansion coefficients, it is sufficient to consider values of the parameter  $q$  such that  $\text{real}(q) \geq 0$  because of the relations between Mathieu functions for the parameter  $q$  and  $-q$  presented in Section 3.1.

### 4.3. Evaluation of the series expansions

**4.3.1. Angular functions.** Mathieu angular functions are computed using the series expansions (3)-(6) by the function `MathieuAngular` for real values of the variable  $x$  and complex values of the parameter  $q$ . When  $x$  takes complex values, the appropriate function call is `MathieuAngular_cmplx`.

**4.3.2. Radial functions.** Mathieu radial functions are computed using the series expansions (7)-(10), (12)-(19) by the function `MathieuRadial`, which accepts real values of the variable  $x$  and complex values of the parameter  $q$ .

Specifically the function `MathieuRadial` operates in the following way. When  $\text{real}(q) \geq 0$ , it applies the tracking algorithm, it determines the expansion coefficients and computes the appropriate series expansions using the definitions (7)-(10), (12)-(19). When  $\text{real}(q) \leq 0$ , it introduces a temporary variable  $q1 = -q$  and performs the same steps as for the previous case, where  $q1$  replaces  $q$ . Then, the radial functions are obtained by applying the relations given in Section 3.1.

## 5. VALIDATION

These subroutines have been validated by making comparisons as described in the following.

- When  $|q|$  is sufficiently small, comparisons can be made with the McLaurin expansions of the Mathieu functions described in [Larsen et al. 2008].
- Comparison with Mathematica®.
- Another check is obtained by controlling the wronskian value.

In the following,  $u$  is the argument of the angular Mathieu function and  $v$  is the argument of the radial Mathieu function because of the physical meaning of  $u$  and  $v$ , which is related to the transformation from elliptic cylinder to rectangular Cartesian coordinates given in eq. (62) as part of the sample application given in Section 6.

### 5.1. Comparison with power series

We compare our results to the values obtained by using the small argument series expansion approximation [Larsen et al. 2008] of the solutions calculated by a subroutine programmed in MATLAB®. Fig. 1 shows the results of the comparison in the case of the even angular Mathieu function  $Se_m(q, v)$  for  $m = 15$  and  $q = -0.9 + i0.9$ . Fig. 2

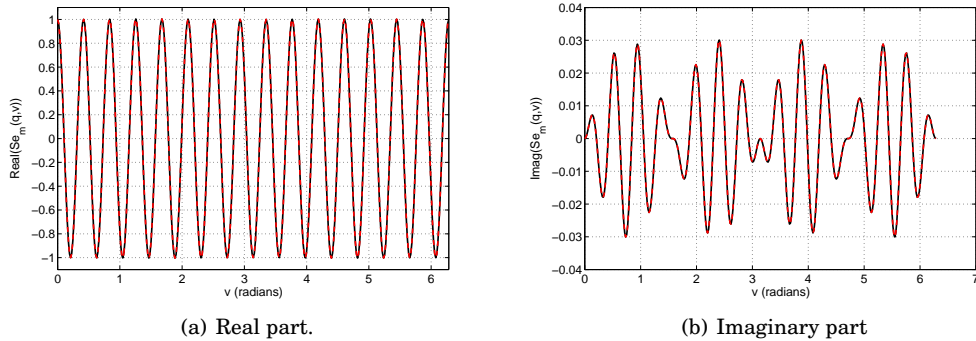


Fig. 1. Plot of  $Se_m(q, v)$  for  $m = 15$  and  $q = -0.9 + i0.9$ . Fortran results are indicated with a solid black line, while power series MATLAB® results are indicated with a dashed red line. The two curves overlap.



shows the results of the comparison in the case of the odd angular Mathieu function  $So_m(q, v)$  for  $m = 10$  and  $q = -0.9 + i0.9$ .

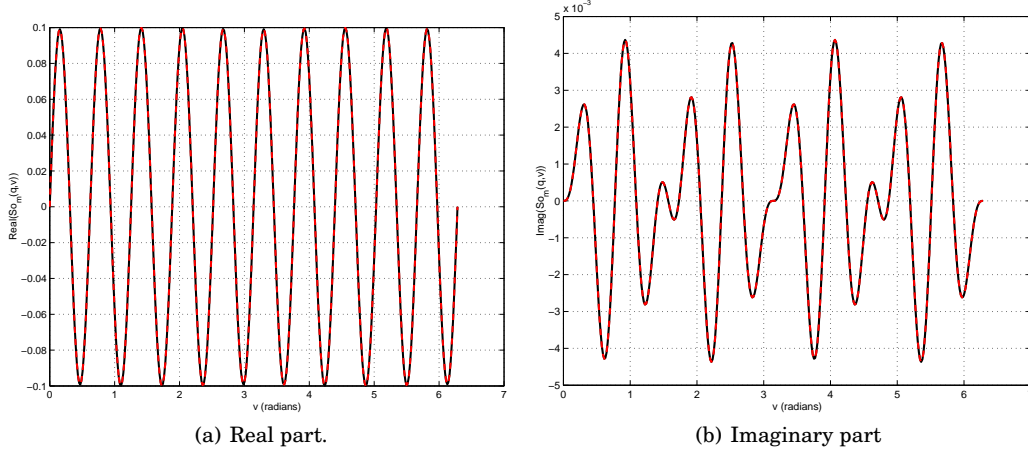


Fig. 2. Plot of  $So_m(q, v)$  for  $m = 10$  and  $q = -0.9 + i0.9$ . Fortran results are indicated with a solid black line, while power series MATLAB® results are indicated with a dashed red line. The two curves overlap.

Fig. 3 shows the results of the comparison in the case even radial Mathieu functions of the first kind  $Re_m^{(1)}(q, u)$  for  $m = 5$  and  $q = -0.8 + i0.7$ . Fig. 4 shows the results

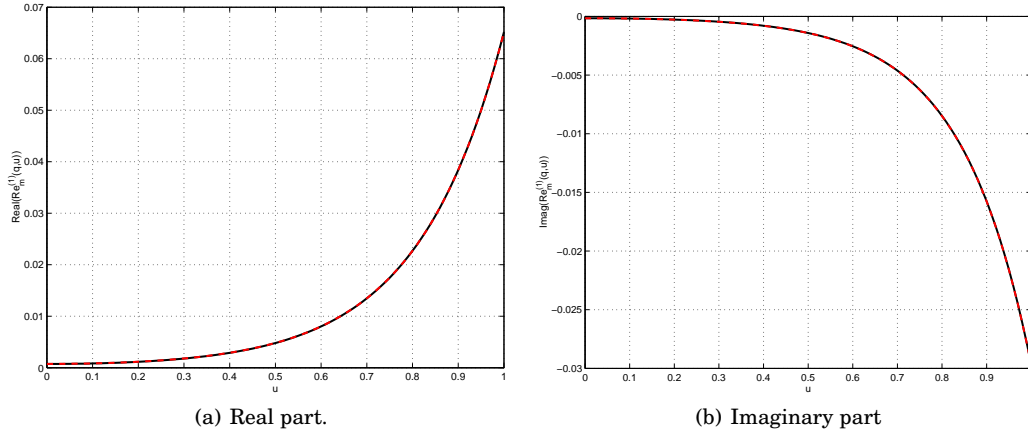


Fig. 3. Plot of  $Re_m^{(1)}$  for  $m = 5$  and  $q = -0.8 + i0.7$ . Fortran results are indicated with a solid black line, while power series MATLAB® results are indicated with a dashed red line. The two curves overlap.

of the comparison in the case of even radial Mathieu functions of the second kind  $Re_m^{(2)}(q, u)$  for  $m = 5$  and  $q = -0.8 + i0.7$ . Fig. 5 shows the results of the comparison in the case of odd radial Mathieu functions of the first kind  $Ro_m^{(1)}(q, u)$  for  $m = 3$  and  $q = 0.2025 + i0.5$ . Fig. 6 shows the results of the comparison in the case of even radial Mathieu functions of the second kind  $Ro_m^{(2)}(q, u)$  for  $m = 3$  and  $q = 0.2025 + i0.5$ . Next we show the comparisons with the small argument power series as a function of  $q$ . In Fig.7, the angular Mathieu function  $Se_m(q, v)$  is evaluated for  $m = 1$ ,  $v = 1$  and

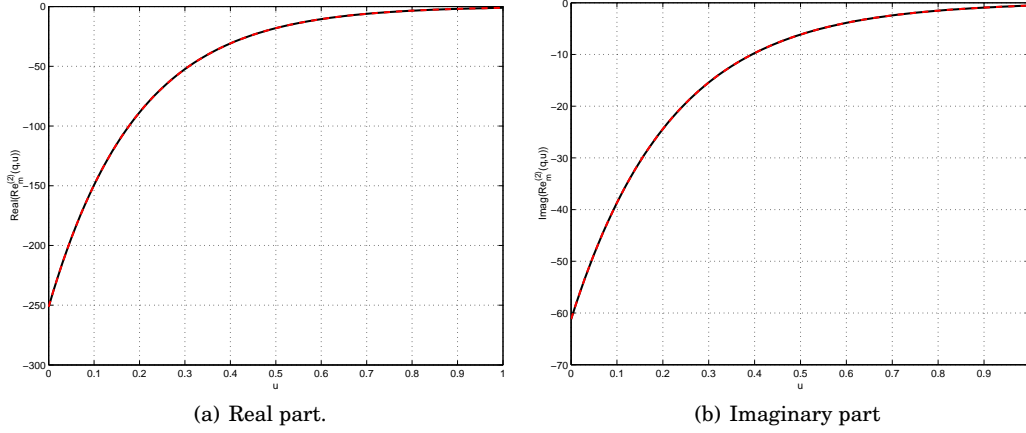


Fig. 4. Plot of  $\text{Re}_m^{(2)}(q, u)$  for  $m = 5$  and  $q = -0.8 + i0.7$ . Fortran results are indicated with a solid black line, while power series MATLAB® results are indicated with a dashed red line. The two curves overlap.

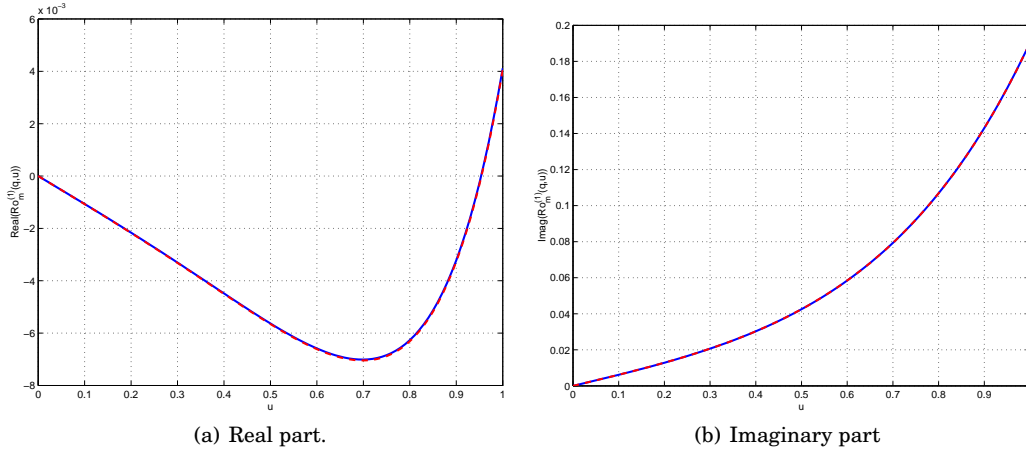


Fig. 5. Plot of  $\text{Ro}_m^{(1)}(q, u)$  for  $m = 3$  and  $q = 0.2025 + i0.5$ . Fortran results are indicated with a solid black line, while power series MATLAB® results are indicated with a dashed red line. The two curves overlap.

$q = q_r + iq_i$ , with  $q_i = 0.5$  while  $-0.5 < q_r < 0.5$ . In Fig. 8, the even radial Mathieu function of the first kind  $\text{Re}_m^1(q, u)$  is evaluated for  $m = 1$ ,  $u = 1$  and  $q = q_r + iq_i$ , with  $q_i = 0.5$  while  $-0.5 < q_r < 0.5$ . In both cases, there is a good agreement between the Fortran results and the small argument power expansion.

## 5.2. Comparison with Mathematica®

For larger value of  $|q|$ , Mathieu angular functions are evaluated using Mathematica® and compared to the results obtained by our subroutines. The comparison with Mathematica® is limited to the angular Mathieu functions, since there are no Mathematica® functions that compute radial Mathieu functions.

Angular Mathieu functions are evaluated in three steps: (1) given a small parameter  $q$  and order  $n$ , the Mathieu characteristic values  $a_r$  and  $b_r$  are computed using the Mathematica® functions `MathieuCharacteristicA[m,q]` and `MathieuCharacteristicB[m,q]`; (2) even angular Mathieu functions are computed us-

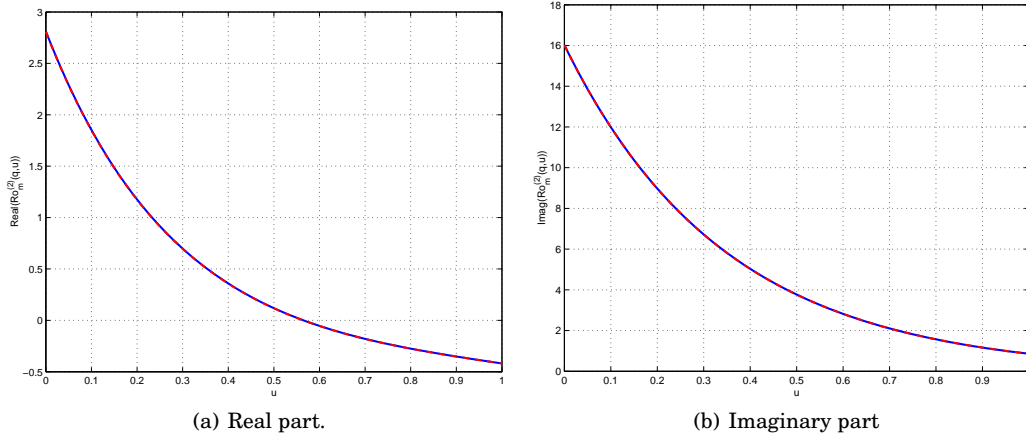


Fig. 6. Plot of  $Ro_m^{(2)}(q, u)$  for  $m = 3$  and  $q = 0.025 + i0.5$ . Fortran results are indicated with a solid black line, while power series MATLAB® results are indicated with a dashed red line. The two curves overlap.

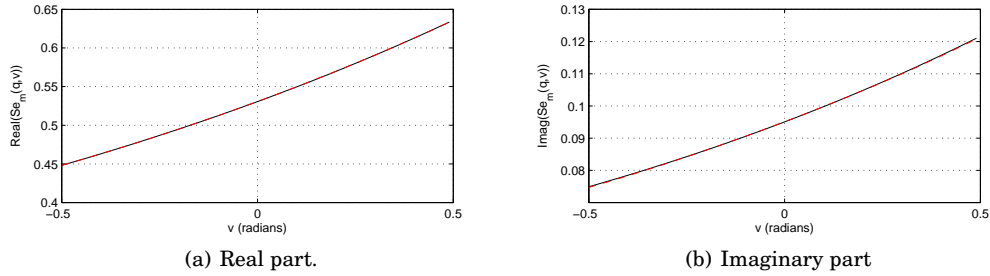


Fig. 7. Plot of  $Se_m(q, v)$  for  $m = 1$ ,  $v = 1$  while the real part of  $q = q_r + iq_i$  is varied with  $q_i = 0.5$ . Fortran results are indicated with a solid black line, while power series MATLAB® results are indicated with a dashed red line. The two curves overlap.

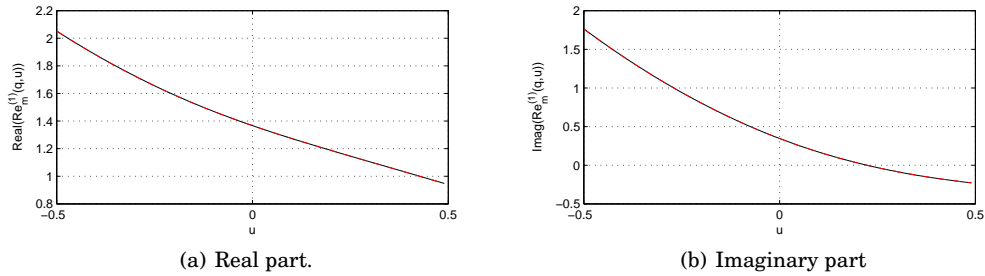


Fig. 8. Plot of  $Re_m(q, u)$  for  $m = 1$ ,  $u = 1$  while the real part of  $q = q_r + iq_i$  is varied with  $q_i = 0.5$ . Fortran results are indicated with a solid black line, while power series MATLAB® results are indicated with a dashed red line. The two curves overlap.

ing the function `MathieuC[ar, q, v]` and odd angular Mathieu functions are computed using `MathieuS[br, q, v]`; (3) the results are normalized according to Stratton, *i.e.*

$$Se_m(q, v) = \frac{ce_m(q, v)}{ce_m(q, 0)}, So_m(q, v) = \frac{se_m(q, v)}{se'_m(q, 0)} \quad (59)$$

where  $se'_m(q, 0)$  is computed using the function `MathieuSPrime[br, q, v]` [Erricolo 2003].

Even angular functions  $Se_m(q, v)$  are examined for  $m = 15$ ,  $q = 3 + 2j$  in Fig. 9(b) and Fig. 9(b) by comparing the real and imaginary parts, respectively.

Odd angular functions  $So_m(q, v)$  are examined for  $m = 10$ ,  $q = 4 + 2j$  in Fig. 10(a) and Fig. 10(b) by comparing the real and imaginary parts, respectively.

It is clear from the plots that the Fortran subroutines are very accurate in calculating angular functions for small values of the parameter.

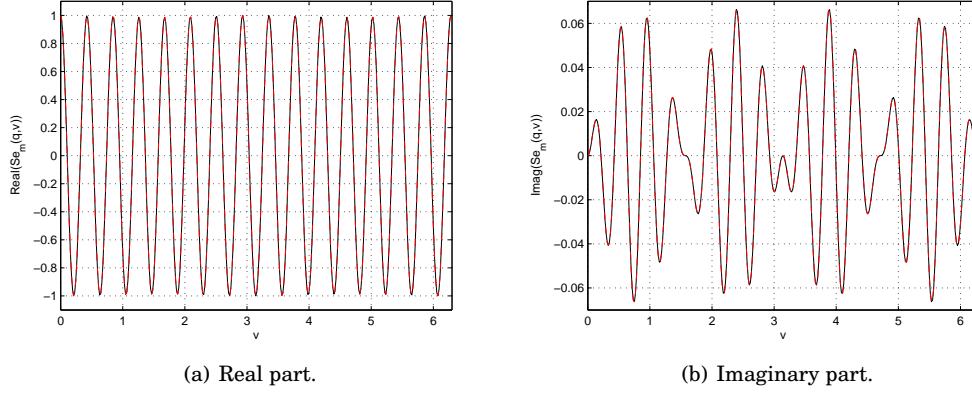


Fig. 9. Comparisons between Fortran subroutines and Mathematica® in the case of the even angular function  $Se_m(q, v)$  when  $m = 15$  and  $q = 3 + 2i$ . Fortran results are indicated with a solid black line, while Mathematica® results are indicated with a dashed red line. The two curves overlap.

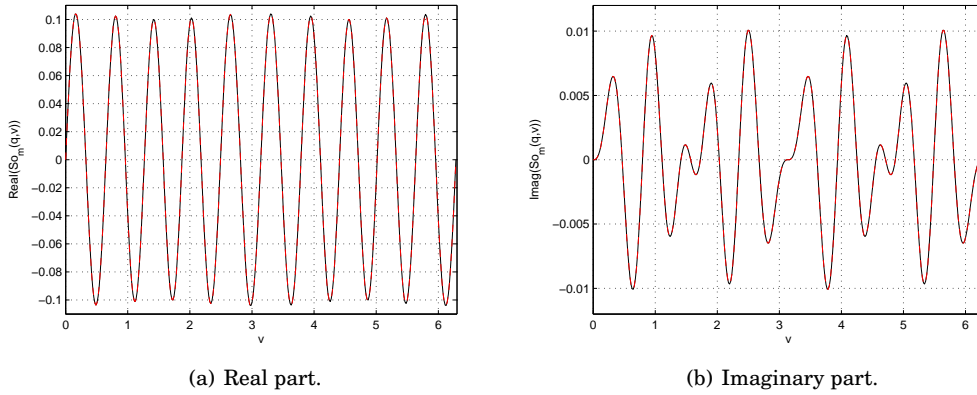


Fig. 10. Comparisons between Fortran subroutines and Mathematica® in the case of the odd angular function  $So_m(q, v)$  when  $m = 10$  and  $q = 4 + 2j$ . Fortran results are indicated with a solid black line, while Mathematica® results are indicated with a dashed red line. The two curves overlap.

Table I. Wronskian test for  $q = -0.9 + j0.9$  and  $u = 2$ .

| n  | real( $W_e - 1$ ) | imag( $W_e$ ) | real( $W_o - 1$ ) | imag( $W_o$ ) |
|----|-------------------|---------------|-------------------|---------------|
| 0  | -0.497676E-07     | -0.104774E-07 |                   |               |
| 1  | 0.582077E-10      | 0.000000E+00  | -0.145519E-10     | 0.000000E+00  |
| 2  | -0.203727E-09     | 0.232831E-09  | 0.000000E+00      | 0.000000E+00  |
| 3  | 0.000000E+00      | 0.000000E+00  | -0.291038E-10     | 0.000000E+00  |
| 4  | 0.756700E-09      | -0.631553E-08 | 0.151049E-07      | 0.321306E-07  |
| 5  | 0.538421E-09      | -0.258660E-08 | 0.698492E-09      | -0.217915E-08 |
| 6  | -0.625732E-09     | 0.894943E-09  | 0.298314E-09      | -0.107138E-08 |
| 7  | -0.483851E-09     | 0.605269E-09  | -0.471118E-09     | 0.537057E-09  |
| 8  | 0.647105E-09      | -0.112777E-09 | -0.369027E-09     | 0.383125E-09  |
| 9  | 0.696215E-09      | -0.852083E-10 | 0.655563E-09      | -0.596287E-10 |
| 10 | -0.106102E-08     | -0.669896E-09 | 0.688974E-09      | -0.345644E-10 |
| 11 | -0.138795E-08     | -0.100460E-08 | -0.129301E-08     | -0.978877E-09 |
| 12 | 0.119574E-08      | 0.515956E-08  | -0.162720E-08     | -0.136690E-08 |
| 13 | 0.147028E-08      | 0.851756E-08  | 0.125216E-08      | 0.811601E-08  |
| 14 | 0.224559E-07      | -0.356762E-07 | 0.123325E-08      | 0.124891E-07  |
| 15 | 0.441156E-07      | -0.586881E-07 | 0.426029E-07      | -0.535508E-07 |
| 16 | -0.530413E-06     | 0.677142E-07  | 0.859084E-07      | -0.964482E-07 |
| 17 | -0.919311E-06     | 0.312372E-06  | -0.878474E-06     | 0.138578E-06  |
| 18 | 0.875018E-05      | 0.841740E-05  | -0.245003E-05     | 0.366472E-07  |
| 19 | 0.275663E-04      | 0.256637E-04  | 0.356364E-04      | 0.243828E-04  |
| 20 | -0.324346E-03     | -0.298998E-05 | 0.566832E-04      | -0.444547E-04 |

### 5.3. Wronskian check

Another way to check the accuracy of the computation of radial functions is to examine their wronskian. This test is based on the wronskian property

$$W_e = \text{Re}_m^{(1)}(q, u) \text{Re}_m^{(2')}(q, u) - \text{Re}_m^{(2)}(q, u) \text{Re}_m^{(1')}(q, u) = 1 \quad (60)$$

$$W_o = \text{Ro}_m^{(1)}(q, u) \text{Ro}_m^{(2')}(q, u) - \text{Ro}_m^{(2)}(q, u) \text{Ro}_m^{(1')}(q, u) = 1 \quad (61)$$

where the prime symbol refers to the derivative with respect to argument  $u$ . Table I shows the results of this test when  $q = -0.9 + j0.9$  and  $u = 2$ . Table II shows the results of this test when  $q = 2.5 + j1.8$  and  $u = 2$ .

Tables I and II are computed using the version of our subroutines that employs the IMSL numerical subroutines to evaluate Bessel functions and obtain the initial characteristic values  $a_r^0$  ( $b_r^0$ ). Since IMSL numerical subroutines may not be available to everyone, the software associated with this article contains other Fortran libraries to compute the Bessel functions [Amos 1986] and the eigenvalues. We would like to point out that higher accuracy is achieved when IMSL subroutines are used.

## 6. SAMPLE APPLICATION

Mathieu functions are frequently applied in problems involving elliptical cylinder coordinates [Caorsi et al. 1997; Caorsi et al. 1998; 1999a; 1999b; Erricolo et al. 2005b; 2005a; Berardi et al. 2004; Erricolo and Uslenghi 2005b; 2004; Erricolo et al. 2006; Erricolo and Uslenghi 2005a; Valentino and Erricolo 2006; 2007; Canta and D. Erricolo 2008; Olshansky and Turkel 2008; Erricolo et al. 2008; Akgol et al. 2011b; 2011a; Akgol et al. 2011; Gupta et al. 2008; Baktur et al. 2010; Bakker and Kuhlman 2011; Gil et al. 2011]. As a sample application of the Mathieu functions, we consider the two-dimensional problem of the computation of the electromagnetic field distribution inside an infinitely long lossy elliptic cylinder made of three layers, shown in Fig. 11(a), when the structure is illuminated by an incident time-harmonic plane wave. The time factor dependence  $e^{i\omega t}$  is assumed and neglected throughout.

Table II. Wronskian test for  $q = 2.5 + j1.8$  and  $u = 2$ .

| n  | real( $W_e - 1$ ) | imag( $W_e$ ) | real( $W_o - 1$ ) | imag( $W_o$ ) |
|----|-------------------|---------------|-------------------|---------------|
| 0  | 0.153153E-08      | 0.375110E-09  |                   |               |
| 1  | 0.122702E-08      | -0.992827E-09 | 0.483169E-11      | 0.964064E-10  |
| 2  | -0.105049E-07     | -0.793534E-09 | -0.142109E-11     | 0.583213E-10  |
| 3  | -0.110762E-07     | 0.100954E-09  | 0.852651E-13      | 0.227374E-12  |
| 4  | 0.833893E-10      | -0.295699E-09 | -0.238742E-11     | -0.454747E-12 |
| 5  | 0.170530E-12      | 0.248690E-12  | 0.113687E-12      | 0.568434E-13  |
| 6  | -0.476831E-09     | -0.367226E-08 | 0.290066E-08      | 0.727772E-08  |
| 7  | -0.598845E-10     | -0.171605E-08 | 0.107889E-09      | -0.176777E-08 |
| 8  | -0.641023E-09     | -0.612374E-09 | 0.253621E-09      | -0.876405E-09 |
| 9  | -0.464766E-09     | -0.531671E-09 | -0.446349E-09     | -0.581082E-09 |
| 10 | -0.378250E-09     | 0.373390E-10  | -0.281403E-09     | -0.498641E-09 |
| 11 | -0.435564E-09     | -0.457980E-10 | -0.441648E-09     | -0.713598E-10 |
| 12 | -0.236587E-09     | 0.336847E-09  | -0.453785E-09     | -0.181238E-09 |
| 13 | -0.478929E-09     | 0.334858E-09  | -0.496685E-09     | 0.315323E-09  |
| 14 | -0.282224E-10     | 0.923046E-09  | -0.796111E-09     | 0.188488E-09  |
| 15 | -0.672772E-09     | 0.141019E-08  | -0.722514E-09     | 0.138122E-08  |
| 16 | 0.130262E-08      | 0.345546E-08  | -0.202385E-08     | 0.173100E-08  |
| 17 | -0.465085E-09     | 0.750809E-08  | -0.679129E-09     | 0.742430E-08  |
| 18 | 0.160675E-07      | 0.199750E-07  | -0.673906E-08     | 0.137299E-07  |
| 19 | 0.165958E-07      | 0.583738E-07  | 0.157591E-07      | 0.580741E-07  |
| 20 | 0.226897E-06      | 0.157335E-06  | -0.941438E-08     | 0.142480E-06  |

Without any loss of generality, we select a coordinate system where the  $z$  axis is the axis of the cylinder and the  $x$  and  $y$  axes correspond to the axes of symmetry of the elliptic cross section. For this geometry, it is possible to express the solution in terms of series expansions of Mathieu functions. Accordingly, we select an elliptic cylinder coordinate system [Stratton 1941; Moon and Spencer 1961],  $(u, v, z)$  with focal distance  $d$ , having two foci located along the  $x$  axis at  $\pm d/2$ . The elliptic cylinder coordinates  $(u, v, z)$  and the Cartesian coordinates  $(x, y, z)$  are related by

$$\begin{aligned}
 x &= \frac{d}{2} \cosh u \cos v \\
 y &= \frac{d}{2} \sinh u \sin v \\
 z &= z
 \end{aligned} \tag{62}$$

From the previous definition, surfaces having  $u = \text{constant}$  represent families of confocal elliptic cylinders, while surfaces having  $v = \text{constant}$  represent families of confocal hyperbolic cylinders. There are four regions in Fig. 11(a): region 1, located inside the surface  $u = u_1$ ; region 2, limited between the surface  $u = u_2$  and the surface  $u = u_1$ ; region 3, limited between the surface  $u = u_3$  and the surface  $u = u_2$ ; and region 0, which is external to the surface  $u = u_3$ .

The source of the electromagnetic field is a plane wave, whose direction of propagation lies in the  $xy$  plane and makes the angle  $\phi_0$  with the negative  $x$  axis, and polarized with the electric field along the  $z$  direction, i.e.

$$E_z^i = E_0 e^{ik_0(x \cos \phi_0 + y \sin \phi_0)}. \tag{63}$$

The main steps that lead to the exact electromagnetic solution of this problem are briefly summarized in the following.

We first express the  $z$  component of the electric field in each region as a series expansion containing Mathieu functions [Stratton 1941; McLachlan 1964]. In doing so, we assume that each region is filled with a medium that is linear, uniform, and isotropic.

The wavevector in region  $\ell$  is  $k_\ell$ . The  $z$  component of the electric field in regions 1, 2 and 3 are given by [Bowman et al. 1987]

$$E_{1,z} = \sum_{m=0}^{+\infty} e_{1,m} \text{Re}_m^{(1)}(q_1, u) \text{Se}_m(q_1, v) + \sum_{m=1}^{+\infty} f_{1,m} \text{Ro}_m^{(1)}(q_1, u) \text{So}_m(q_1, v), \quad (64)$$

$$E_{2,z} = \sum_{m=0}^{+\infty} \left[ e_{2,m} \text{Re}_m^{(1)}(q_2, u) \text{Se}_m(q_2, v) + f_{2,m} \text{Re}_m^{(2)}(q_2, u) \text{Se}_m(q_2, v) \right] \\ + \sum_{m=1}^{+\infty} \left[ g_{2,m} \text{Ro}_m^{(1)}(q_2, u) \text{So}_m(q_2, v) + h_{2,m} \text{Ro}_m^{(2)}(q_2, u) \text{So}_m(q_2, v) \right], \quad (65)$$

$$E_{3,z} = \sum_{m=0}^{+\infty} \left[ e_{3,m} \text{Re}_m^{(1)}(q_3, u) \text{Se}_m(q_3, v) + f_{3,m} \text{Re}_m^{(2)}(q_3, u) \text{Se}_m(q_3, v) \right] \\ + \sum_{m=1}^{+\infty} \left[ g_{3,m} \text{Ro}_m^{(1)}(q_3, u) \text{So}_m(q_3, v) + h_{3,m} \text{Ro}_m^{(2)}(q_3, u) \text{So}_m(q_3, v) \right], \quad (66)$$

respectively. In region 0, external to the elliptic cylinder, the electric field is given by the sum of the incident field  $E_z^i$  (63) which can be expanded in terms of Mathieu functions

$$E_z^i = \sqrt{8\pi} \left[ \sum_{m=0}^{+\infty} \frac{i^m}{N_m^e} \text{Re}_m^{(1)}(q_0, u) \text{Se}_m(q_0, v) \text{Se}_m(q_0, \phi_0) \right. \\ \left. + \sum_{m=1}^{+\infty} \frac{i^m}{N_m^o} \text{Ro}_m^{(1)}(q_0, u) \text{So}_m(q_0, v) \text{So}_m(q_0, \phi_0) \right], \quad (67)$$

and the scattered field

$$E_{0,z} = \sum_{m=0}^{+\infty} e_{0,m} \text{Re}_m^{(4)}(q_0, u) \text{Se}_m(q_0, v) + \sum_{m=1}^{+\infty} f_{0,m} \text{Ro}_m^{(4)}(q_0, u) \text{So}_m(q_0, v), \quad (68)$$

where radial Mathieu functions of the fourth kind are used to satisfy the Sommerfeld radiation condition and  $q_\ell = (k_\ell d)^2/16$ . For a lossy cylinder, its wavenumber  $k_\ell$  is complex, and consequently,  $q_\ell$  is complex too. This motivates the use of the special functions presented in this work to compute the fields distribution. The magnetic field is obtained from the electric field using the relation

$$\mathbf{H}(u, v) = H_u(u, v) \hat{\mathbf{u}} + H_v(u, v) \hat{\mathbf{v}} = \frac{i}{\omega \mu d \sqrt{\cosh u^2 \cos v^2}} \left[ \hat{\mathbf{u}} \frac{\partial}{\partial v} - \hat{\mathbf{v}} \frac{\partial}{\partial u} \right] E_z(u, v). \quad (69)$$

The expansion coefficients  $e_{\ell,m}$ ,  $f_{\ell,m}$ ,  $g_{\ell,m}$ ,  $h_{\ell,m}$  are obtained by imposing the electromagnetic boundary conditions on the continuity of the tangential components of the total electric and magnetic fields across the interfaces between different materials.

This approach is similar to, for example, the one of the multilayer circular cylinder. However, in this case it is not possible to write explicitly in closed form the expansion coefficients of the scattered field in terms of the expansion coefficients of the incident field because the angular Mathieu functions depend on the material properties through the parameter  $q$ , which does not happen in the circular cylinder case. It is also worth noting that the orthogonality relations (22)-(23) are not valid across an interface when the material properties change, however they may be used as done, for example in [Caorsi et al. 1997], [Caorsi et al. 1998], [Caorsi et al. 1999b]. Therefore,

the point-matching method, which enforces the boundary conditions at predetermined points [Balanis 1989], is used in this work. Let  $m_{\max}$  be the maximum order of the expansion coefficients, then  $N_t = 6m_{\max} + 3$  is the total number of uniformly distributed points where the boundary conditions are enforced, as shown in Fig. 11(b).

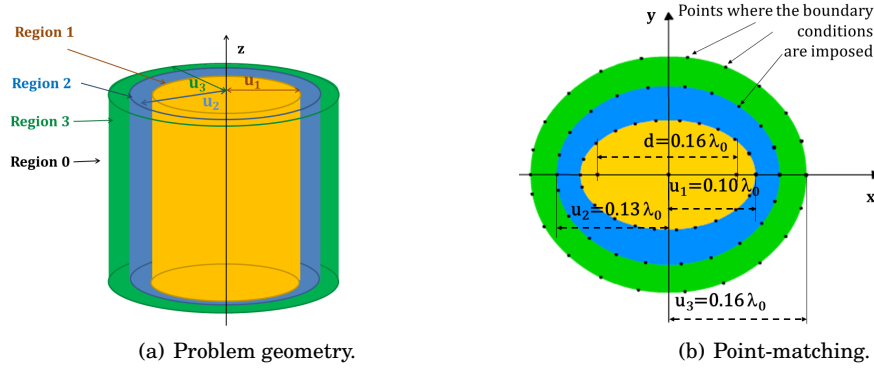


Fig. 11. Sample problem.

The application of the boundary conditions at each point leads to a system of  $12m_{\max} + 6$  equations that is solved by a matrix inversion.

Numerical results are shown in Fig. 12 for the electric field computed with the methods presented in [Caorsi et al. 1998] and with the method presented in this work. The geometry is the same in all cases, where the semimajor axes of the three layers of the elliptic cylinder have length  $u_1 = 0.10\lambda_0$ ,  $u_2 = 0.13\lambda_0$  and  $u_3 = 0.16\lambda_0$  and  $d = 0.16\lambda_0$ . The complex relative dielectric permittivities  $\epsilon_i = \epsilon'_i - i\epsilon''_i$  of the three layers are  $\epsilon'_1 = 1.0$ ,  $\epsilon'_2 = 4.0$ ,  $\epsilon'_3 = 2.5$  and  $\epsilon''_1 = \epsilon''_2 = \epsilon''_3 = \epsilon''$ . The solution has been computed for  $\epsilon'' = 0$  (pure dielectric cylinder),  $\epsilon'' = 0.06$  (weakly lossy cylinder),  $\epsilon'' = 0.6$  (lossy cylinder) and  $\epsilon'' = 1.89$  (very lossy cylinder).

All results are computed by considering  $m_{\max} = 9$ . For all values of  $\epsilon''$  there is good agreement between the methods presented in [Caorsi et al. 1998] and the method presented here. The differences between the two results may be attributed to how the boundary conditions are applied and to the computation of the Mathieu functions. Specifically, in [Caorsi et al. 1998] the boundary conditions are enforced using the Galerkin method, which is different from the point matching method used in this article. In addition, in [Caorsi et al. 1998] the Mathieu functions for complex values of the parameter  $q$  are computed using a Taylor series expansion.

## 7. CONCLUSIONS

This article describes the computation of Mathieu functions for complex values of the parameter  $q$ . The associated Fortran 90 software was validate in different ways and a sample problem the requires the use of such functions was provided.

## ACKNOWLEDGMENTS

The authors would like to thank Prof. Matteo Pastorino and Prof. Mirco Raffetto for providing the data to plot the comparisons, shown in Fig. 12, between our results and theirs, which were originally published in



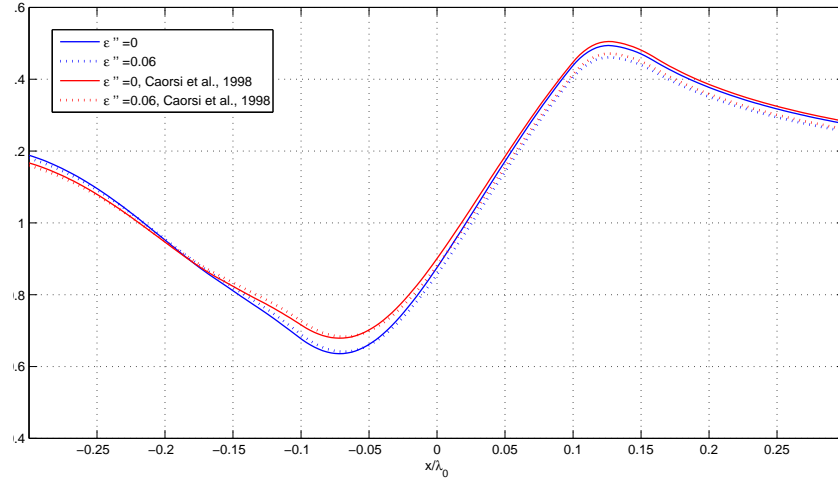
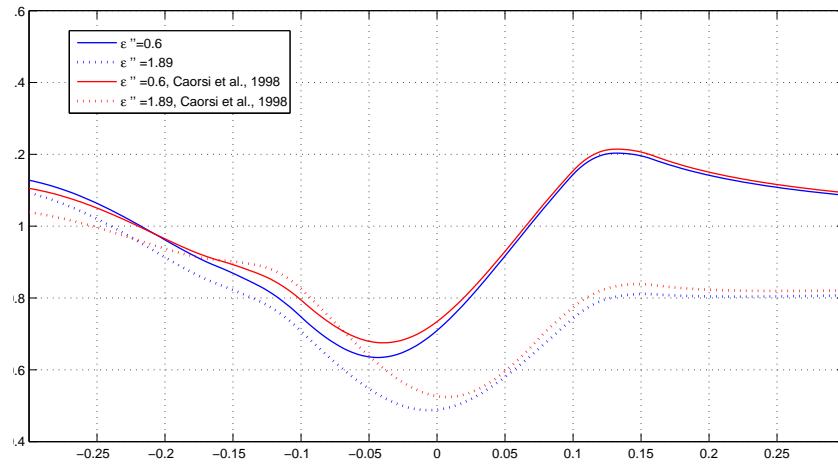
(a) Results for  $\epsilon'' = 0$  and  $\epsilon'' = 0.06$ .(b) Results for  $\epsilon'' = 0.6$  and  $\epsilon'' = 1.89$ .

Fig. 12. Magnitudes  $|E_z|$  in the four regions of the geometry: comparison between the computation method presented in [Caorsi et al. 1998] and the method presented here.

[Caorsi et al. 1998]. The authors would like to thank also Mr. Ramon Acosta for providing the comparisons with Mathematica® and the anonymous Reviewers for their comments to improve this manuscript.

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