Asymptotic Analysis of Kinetic Models of Collective Behavior

by

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THESIS

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Dedicated to my parents and my sons.

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CONTRIBUTIONS OF AUTHORS

Chapter 1 and Chapter 2 represent the papers (27) and (28), respectively, that I jointly worked with my advisor, Professor Roman Shvydkoy.

Chapter 3 is from the preprint (26) that I am the sole author.

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SUMMARY

In this dissertation, we study the long-time behavior of the solutions of some kinetic equations arising from the studies of collective behavior. Propagation of chaos is a fundamental question in kinetic theory which enables the reduction of an N-particle description to a single partial differential equation. In Chapter 1, we prove the propagation of chaos for the classical Cucker-Smale system and its variant in which the system is additionally forced with Rayleigh-type friction and self-propulsion force. Moreover, the quantitative estimates of the rate of the convergence in Wasserstein-2 distance are shown. In Chapter 2, a continuous model of opinion dynamics is considered. The global well-posedness, the regularity, and asymptotic behavior of the solution are studied. In Chapter 3, we investigate the long-time behavior of the solution of a kinetic Fokker-Planck-type equation. The exponential relaxation of the solution to its equilibrium is proved here.

CHAPTER 1

PROPAGATION OF CHAOS UNDER HEAVY TAIL COMMUNICATION

(Previously published as V. Nguyen and R. Shvydkoy, Propagation of chaos for the Cucker-Smale systems under heavy tail communication, Communications in Partial Differential Equations, 47(9):1883–1906, 2022.)

1.1 Introduction and main results

One of the fundamental questions of the mathematical theory of large systems of particles is a derivation and formal justification of the corresponding kinetic models. Among the many systems describing collective phenomena this question has been successfully settled for the Cucker-Smale model describing the basic mechanism of alignment (9; 10):

$$\begin{cases} \dot{x}_i &= v_i, \qquad x_i(0) = x_i^0 \in \mathbb{R}^n, \\ \dot{v}_i &= \frac{1}{N}\sum_{j=1}^N \phi(x_i - x_j)(v_j - v_i), \quad v_i(0) = v_i^0 \in \mathbb{R}^n. \end{cases} \tag{1.1}$$

Here N is the number of particles and x_i, v_i denote the position and velocity of the i-th particle. ϕ is a non-negative non-increasing smooth communication kernel. The corresponding Vlasov-Alignment equation is given by

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (f F(f)) = 0, \quad f(0) = f_0 : \mathbb{R}^{2n} \to \mathbb{R}_+, \tag{1.2}$$

where

$$F(f)(x,v) = \int_{\mathbb{R}^{2n}} \phi(x-y)(w-v)f(y,w,t)\,dy\,dw.$$

A formal derivation of (Equation 1.2) via the BBGKY hierarchy was performed in Ha and Tadmor (19), and rigorously via the mean-field limit in Ha and Liu (18).

The hierarchy approach is based upon the classical idea of propagation of chaos, which postulates that the particles $(x_1, v_1, \ldots, x_N, v_N)$ whose joint probability distribution f^N is given by the solution to the Liouville transport equation

$$\partial_t f^N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f^N + \sum_{i=1}^{N} \nabla_{v_i} \cdot (f^N F_i^N) = 0, \tag{1.3}$$

would gradually decorrelate as $N \to \infty$ if initially so

$$f^N(0) = f_0^{\otimes N}, \qquad f_0 : \mathbb{R}^{2n} \to \mathbb{R}_+, \tag{1.4}$$

and their individual distributions would evolve according to (Equation 1.2). In other words,

$$\langle f^N, \varphi_1 \otimes \ldots \otimes \varphi_k \otimes 1 \otimes \cdots \otimes 1 \rangle \to \prod_{j=1}^{k} \langle f, \varphi_j \rangle, \qquad \varphi \in C_b(\mathbb{R}^{2nk}). \tag{1.5}$$

The mean-field limit on the other hand, is based on the weak convergence of a sequence of empirical measures built from solutions to (Equation 1.1),

$$\mu^N = \frac{1}{N}\sum_{j=1}^{N} \delta_{x_i(t)} \otimes \delta_{v_i(t)} \to f.$$

In fact, a more detailed analysis done in (17; 31) establishes Lipschitz continuity of measure-valued solutions to (Equation 1.2) with respect to the Wasserstein metric,

$$\mathcal{W}_p(\mu_t', \mu_t'') \leqslant C(t)\mathcal{W}_p(\mu_0', \mu_0'').$$

It is well-known, however, that propagation of chaos and the mean-field limit (in a somewhat more specific sense) are equivalent, see Sznitman (34). In fact, (Equation 1.5) holds if and only if for any $\varphi \in \mathrm{Lip}(\mathbb{R}^{2n})$ one has

$$E_\varphi(t) = \int_{\mathbb{R}^{2nN}} \left| \frac{1}{N}\sum_{j=1}^{N} \varphi(x_i(t), v_i(t)) - \langle f_t, \varphi\rangle \right|^2 f_0^{\otimes N}\, dX_0\, dV_0 \to 0, \tag{1.6}$$

where X_0, V_0 are the initial conditions for the characteristic flow $\{x_i(t), v_i(t)\}_{i=1}^N$. Note that initially $E_\varphi(0) \to 0$ by a direct verification. Technically, since not every initial ensemble X_0, V_0 in the support of $f_0^{\otimes N}$ forms an empirical measure weakly close to f_0, the limit (Equation 1.6) does not directly follow from (17; 18; 31). However, one can restore it using similar estimates on the deformation of the flow-map of (Equation 1.1) and coupling with the characteristics of (Equation 1.2).

In any case, Snitzman's general principle seems to provide little quantitative information on the rate of propagation in (Equation 1.5) as it avoids using any specificity of the system at hand. For stochastically forced systems, the work of Bolley, Cañizo and Carrillo (3) establishes such a quantitative estimate on the Wasserstein-2 distance:

$$\mathcal{W}_2(f_t^{(k)}, f_t^{\otimes k}) \leqslant C(T)\sqrt{\frac{k}{N^{e^{-Ct}}}}, \quad \forall t \leqslant T. \tag{1.7}$$

Recently, Natalini and Paul addressed the deterministic case in (24) and with additional chemotaxis forces in (25). For the forceless system, the estimate carries exponential dependence in time,

$$\mathcal{W}_2(f_t^{(k)}, f_t^{\otimes k}) \leqslant Ce^{\delta t}\sqrt{\frac{k}{N}}. \tag{1.8}$$

The estimates (Equation 1.7), (Equation 1.8) are finite-time bounds in spirit, in the sense that they do not take into account any flocking long-time behavior of the system. A natural question is: can one improve upon the time dependence in the deterministic case (Equation 1.8) when the system is known to flock exponentially fast? It is the result that goes back to Cucker and Smale (9) and improved and extended in (6; 18; 19) that the system (Equation 1.1) with a heavy tail radial communication,

$$\int_0^\infty \phi(r)\,dr = \infty \tag{1.9}$$

aligns with an exponential rate. Let us give a quantitative summary of this result for future reference, see also (31) for details.

Proposition 1.1.1. *Suppose* ϕ *satisfies* (Equation 1.9). *For any solution to* (Equation 1.1) *with initial data in* (X_0, V_0) *in a compact domain* $\Omega \subset \mathbb{R}^{2nN}$ *the following flocking estimates hold:*

$$\sup_{t>0} \max_{i,j=1,\dots,N} |x_i - x_j| = D < \infty, \qquad \max_{i,j=1,\dots,N} |v_i - v_j| \leqslant A_0 e^{-t\phi(D)}, \tag{1.10}$$

where A_0 *is the initial velocity fluctuation and* D *depends only on the initial diameter of the flock and* ϕ.

Similarly, for any solution f *to* (Equation 1.2) *with initial compact support one has*

$$\sup_{t>0} \operatorname{diam} \operatorname{supp} f_t = D < \infty, \qquad \max_{(x',v'),(x'',v'') \in \operatorname{supp} f_t} |v' - v''| \leqslant A_0 e^{-t\phi(D)}. \tag{1.11}$$

With the use of this additional flocking information we will improve the estimate (Equation 1.8) to being linear in time.

Theorem 1.1.2. *Suppose* ϕ *satisfies* (Equation 1.9), *and let* $f_0 \in C_0^1(\mathbb{R}^{2n})$ *be an initial distribution with a compact support. Let* f^N *be the solution to* (Equation 1.3)-(Equation 1.4), *while* f *be the solution to* (Equation 1.2). *Then there exists a constant* C *which depends only on* $\operatorname{diam}(\operatorname{supp} f_0)$ *and* ϕ *such that for all* $N \in \mathbb{N}$, $k < N$, *and* $t \geqslant 0$ *one has*

$$\mathcal{W}_2(f_t^{(k)}, f_t^{\otimes k}) \leqslant C\sqrt{k} \min\left\{1, \frac{t}{\sqrt{N}}\right\}. \tag{1.12}$$

Our general methodology relies on the same classical coupling method, which compares characteristic flow of the original system (Equation 1.1) to N copies of the flow-map of the kinetic transport (Equation 1.2), but it differs from (24) in two aspects. First, we run the entire argument from the Lagrangian point of view, which gives a direct access to characteristics and the flocking estimates. This is closer in spirit to the original mean-field approach of (18) or (3) in stochastic settings. Second, we rely on the flocking information of Proposition 1.1.1 to extract a crucial stabilizing exponential factor in the estimation of kinetic energy, see (Equation 1.23). The linear time dependence here comes primarily from the growth of the potential energy, and it seems not to be removable within the given framework.

Next, we consider the same problem in the context of systems forced with self-propulsion and Rayleigh-type friction force with variable characteristic parameters θ:

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \phi(x_i - x_j)(v_j - v_i) + \sigma(\theta_i - |v_i|^p)v_i, \\ \dot{\theta}_i = \dfrac{\kappa}{N}\displaystyle\sum_{j=1}^{N} \phi(x_i - x_j)(\theta_j - \theta_i), \end{cases} \qquad (x_i, v_i, \theta_i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+, \tag{1.13}$$

where $\kappa > 0$ is a coupling coefficient and $p > 0$. This model is relevant in the study of systems of agents with a tendency to adhere to their preferred characteristic speeds θ_i, see (16; 22). The recent study (22) introduced a general method of Grassmannian reduction that allows to prove flocking for solutions with velocities confined to a sector Σ of opening $< \pi$, so-called sectorial solutions, see Proposition 1.3.1 below. We give an extension of this method to the corresponding kinetic Vlasov equation in Proposition 1.3.3 and use it to prove propagation of chaos for the forced system (Equation 1.13). Specifically, we prove the following theorem:

Theorem 1.1.3. *Suppose the kernel* ϕ *satisfies* (Equation 1.25). *Let* $f_0 \in C_0^1(\Omega)$ *be a sectorial initial distribution, and* f^N, f *be the sectorial solutions to the system* (Equation 1.60) *and* (Equation 1.28)*, respectively. Then there exists a constant* C *which depends only on* $\mathrm{diam}(\mathrm{supp}\, f_0)$ *and* ϕ *such that for all* $N \in \mathbb{N}$*,* $k < N$*, and* $t \geqslant 0$ *one has*

$$W_2(f_t^{(k)}, f_t^{\otimes k}) \leqslant C\sqrt{k} \min\left\{1, \frac{t^2}{\sqrt{N}}\right\}. \tag{1.14}$$

To achieve this bound we employ monotonicity of the force to control the adverse self-propulsion component. The ultimate effect of its presence, however, is reflected in the quadratic dependence on time in (Equation 1.14).

1.2 Propagation of chaos for the forceless system

In this section, we focus on establishing propagation of chaos for the pure Cucker-Smale system (Equation 1.1), Theorem 1.1.2. To fix the notation let us consider a solution f^N to the full Liouville equation (Equation 1.3) with the chaotic initial condition (Equation 1.4) on the configuration space $(X, V) \in \mathbb{R}^{2nN}$. We can assume without loss of generality that f_0 is a probability distribution. The forces F_i^N's are given by the Cucker-Smale system

$$F_i^N(X, V) = \frac{1}{N}\sum_{j=1}^{N} \phi(x_i - x_j)(v_j - v_i).$$

Due to the symmetries of the forces, the solution will remain symmetric with respect to permutations of pairs (x_i, v_i) for all time.

We define the k-th marginal as usual by

$$f_t^{(k)}(x_1, v_1, \dots, x_k, v_k) = \int_{\mathbb{R}^{2n(N-k)}} f_t^N(x_1, v_1, \dots, x_N, v_N)\, dx_{k+1} \dots dv_N. \tag{1.15}$$

Let us introduce various characteristic maps that will be used in the proof. We denote by

$$\Phi_t^N = (x_1(t), v_1(t), \dots, x_N(t), v_N(t)) : \mathbb{R}^{2nN} \to \mathbb{R}^{2nN}$$

the flow-map of the Liouville equation (Equation 1.3), in other words these are solutions to the agent-based system

$$\begin{cases} \dot{x}_i &= v_i, \\ \dot{v}_i &= \frac{1}{N}\sum_{j=1}^N \phi(x_i - x_j)(v_j - v_i). \end{cases} \tag{1.16}$$

Then, f_t^N at any time $t > 0$ is a push-forward of the initial distribution by Φ_t^N,

$$f_t^N = \Phi_t^N \sharp f_0^{\otimes N}. \tag{1.17}$$

Now, denote by

$$\bar{\Phi}_t = (\bar{x}(t), \bar{v}(t)) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$$

the flow-map of the Vlasov equation (Equation 1.2), i.e.

$$\begin{cases} \dot{\bar{x}} & = \bar{v}, \\ \\ \dot{\bar{v}} & = \int_{\mathbb{R}^{2n}} \phi(\bar{x}-y)(w-\bar{v}) f(y,w,t)\, dy\, dw, \end{cases} \tag{1.18}$$

and by

$$\bar{\Phi}_t^{\otimes N} = (\bar{x}_1(t), \bar{v}_1(t), \dots, \bar{x}_N(t), \bar{v}_N(t)) : \mathbb{R}^{2nN} \to \mathbb{R}^{2nN}$$

the direct product of N copies of $\bar{\Phi}_t$'s. Thus,

$$f_t = \bar{\Phi}_t \sharp f_0, \qquad f_t^{\otimes N} = \bar{\Phi}_t^{\otimes N} \sharp f_0^{\otimes N}. \tag{1.19}$$

The proof of Theorem 1.1.2 can be reduced to establishing the following estimate

$$\int_{\mathbb{R}^{2nN}} |\Phi_t^N(X_0, V_0) - \bar{\Phi}_t^{\otimes N}(X_0, V_0)|^2 \, f_0^{\otimes N}(X_0, V_0)\, dX_0\, dV_0 \leqslant C \min\{N, t^2\}. \tag{1.20}$$

Indeed, let us recall that the Wasserstein-2 distance between two probability measures $\mu, \bar{\mu}$ on $\mathbb{R}^{2nk}$ can be defined in probabilistic sense as

$$\mathcal{W}_2^2(\mu, \bar{\mu}) = \inf \mathbb{E}[|Z - \bar{Z}|^2],$$

where the infimum is taken over $\mathbb{R}^{2nk}$-valued random variables $Z, \bar{Z}$ defined on any probability space with distributions given by μ and $\bar{\mu}$, respectively. To measure the distance between $f_t^{(k)}$

and $f_t^{\otimes k}$ we can pick the probability space $\mathbb{R}^{2nN}$ with measure $f_0^{\otimes N}(X_0, V_0)\, dX_0\, dV_0$, and random variables given by any selection of k coordinates of Φ_t^N and $\bar{\Phi}_t^{\otimes N}$, respectively, because their probability distributions relative to the chosen base space are exactly $f_t^{(k)}$ and $f_t^{\otimes k}$ according to (Equation 1.17) and (Equation 1.19).

So, let us denote by Σ_N^k is the set of all ordered subsets of $[1, \ldots, N]$ of size k. Clearly, its cardinality is $\binom{N}{k}$. Then, for any $\sigma \in \Sigma_N^k$,

$$\mathcal{W}_2^2(f_t^{(k)}, f_t^{\otimes k}) \leqslant \int_{\mathbb{R}^{2nN}} \sum_{i=1}^{k} |(x_{\sigma(i)}, v_{\sigma(i)}) - (\bar{x}_{\sigma(i)}, \bar{v}_{\sigma(i)})|^2 \, f_0^{\otimes N}(X_0, V_0)\, dX_0\, dV_0.$$

Summing up over all $\sigma \in \Sigma_N^k$, we obtain

$$\binom{N}{k} \mathcal{W}_2^2(f_t^{(k)}, f_t^{\otimes k}) \leqslant \int_{\mathbb{R}^{2nN}} \sum_{\sigma \in \Sigma_N^k} \sum_{i=1}^{k} |(x_{\sigma(i)}, v_{\sigma(i)}) - (\bar{x}_{\sigma(i)}, \bar{v}_{\sigma(i)})|^2 \, f_0^{\otimes N}(X_0, V_0)\, dX_0\, dV_0.$$

Observe that in the double sum inside the integral each coordinate will be repeated $\binom{N-1}{k-1}$ times. So,

$$\binom{N}{k} \mathcal{W}_2^2(f_t^{(k)}, f_t^{\otimes k}) \leqslant \binom{N-1}{k-1} \int_{\mathbb{R}^{2nN}} \sum_{i=1}^{N} |(x_i, v_i) - (\bar{x}_i, \bar{v}_i)|^2 \, f_0^{\otimes N}(X_0, V_0)\, dX_0\, dV_0.$$

Simplifying and using (Equation 1.20), we obtain

$$\mathcal{W}_2^2(f_t^{(k)}, f_t^{\otimes k}) \leqslant Ck \min\left\{1, \frac{t^2}{N}\right\},$$

as desired. Let us note that an alternative argument, relating a distance between k-th marginals to a particular realization (Equation 1.20) appeared in (15), where the authors use the original joint-distribution definition of $\mathcal{W}_2$.

To establish (Equation 1.20) let us break the expression under the integral into potential and kinetic part,

$$\mathcal{P} = \frac{1}{2}\int_{\mathbb{R}^{2nN}} |X_t - \bar{X}_t|^2 \, f_0^{\otimes N} \, dX_0 \, dV_0, \qquad \mathcal{K} = \frac{1}{2}\int_{\mathbb{R}^{2nN}} |V_t - \bar{V}_t|^2 \, f_0^{\otimes N} \, dX_0 \, dV_0. \tag{1.21}$$

Here, X_t, V_t and $\bar{X}_t, \bar{V}_t$ denote the corresponding components of Φ_t^N and $\bar{\Phi}_t^{\otimes N}$, respectively. By the Hölder inequality, we have

$$\frac{d}{dt}\mathcal{P} \leqslant 2\mathcal{P}^{1/2}\mathcal{K}^{1/2}. \tag{1.22}$$

Let us now write out the equation for the kinetic part,

$$\begin{aligned}
\frac{d}{dt}\mathcal{K} = \int_{\mathbb{R}^{2nN}} \sum_{i=1}^{N} (v_i - \bar{v}_i) \cdot \left(\frac{1}{N} \sum_{j=1}^{N} \phi(x_i - x_j)(v_j - v_i) - \int_{\mathbb{R}^{2n}} \phi(\bar{x}_i - y)(w - \bar{v}_i) f(y, w, t) \, dy \, dw \right) \\
\times f_0^{\otimes N} \, dX_0 \, dV_0 \\
= A + B + C,
\end{aligned}$$

where

$$A = \int_{\mathbb{R}^{2nN}} \sum_{i=1}^{N} (v_i - \bar{v}_i) \cdot \frac{1}{N} \sum_{j=1}^{N} [\phi(x_i - x_j) - \phi(\bar{x}_i - \bar{x}_j)](v_k - v_i)\, f_0^{\otimes N}\, dX_0\, dV_0,$$

$$B = \int_{\mathbb{R}^{2nN}} \sum_{i=1}^{N} (v_i - \bar{v}_i) \cdot \frac{1}{N} \sum_{j=1}^{N} \phi(\bar{x}_i - \bar{x}_j)[(v_j - \bar{v}_j) - (v_i - \bar{v}_i)]\, f_0^{\otimes N}\, dX_0\, dV_0,$$

$$C = \int_{\mathbb{R}^{2nN}} \sum_{i=1}^{N} (v_i - \bar{v}_i) \cdot \Big(\frac{1}{N} \sum_{j=1}^{N} \phi(\bar{x}_i - \bar{x}_j)(\bar{v}_j - \bar{v}_i) - \int_{\mathbb{R}^{2n}} \phi(\bar{x}_i - y)(w - \bar{v}_i) f(y, w, t)\, dy\, dw \Big) \times f_0^{\otimes N}\, dX_0\, dV_0.$$

Let us start with C. Apply the Hölder inequality first

$$\begin{aligned} C^2 \leqslant & \Big(\int_{\mathbb{R}^{2nN}} \sum_{i=1}^{N} |v_i - \bar{v}_i|^2 f_0^{\otimes N}\, dX_0\, dV_0 \Big) \\ & \times \Big(\int_{\mathbb{R}^{2nN}} \sum_{i=1}^{N} \Big| \frac{1}{N} \sum_{j=1}^{N} \phi(\bar{x}_i - \bar{x}_j)(\bar{v}_j - \bar{v}_i) - \int_{\mathbb{R}^{2n}} \phi(\bar{x}_i - y)(w - \bar{v}_i) f(y, w, t)\, dy\, dw \Big|^2 \\ & \qquad \times f_0^{\otimes N}\, dX_0\, dV_0 \Big) \\ & = 2\mathcal{K} \int_{\mathbb{R}^{2nN}} \sum_{i=1}^{N} \Big| \frac{1}{N} \sum_{j=1}^{N} \phi(\bar{x}_i - \bar{x}_j)(\bar{v}_j - \bar{v}_i) - \int_{\mathbb{R}^{2n}} \phi(\bar{x}_i - y)(w - \bar{v}_i) f(y, w, t)\, dy\, dw \Big|^2 \\ & \qquad \times f_0^{\otimes N}\, dX_0\, dV_0. \end{aligned}$$

Switching back to the Eulerian coordinates, whereby $\bar{x}_i, \bar{v}_i$ become dummy variables, we get

$$C^2 \leqslant 2\mathcal{K} \int_{\mathbb{R}^{2nN}} \sum_{i=1}^{N} \Big| \frac{1}{N} \sum_{j=1}^{N} \phi(\bar{x}_i - \bar{x}_j)(\bar{v}_j - \bar{v}_i) - \int_{\mathbb{R}^{2n}} \phi(\bar{x}_i - y)(w - \bar{v}_i) f(y, w, t)\, dy\, dw \Big|^2 \times f_t^{\otimes N} d\bar{X} d\bar{V}.$$

All these terms, due to symmetry are independent of i. According to (24, Lemma 3.3), and our flocking estimate (Equation 1.11), each can be estimated by

$$\frac{4}{N} \sup_{(\bar{x}',\bar{v}'),(\bar{x}'',\bar{v}'')\in \operatorname{supp} f_t} |\phi(\bar{x}' - \bar{x}'')(\bar{v}' - \bar{v}'')|^2 \leqslant \frac{c}{N} e^{-\delta t}.$$

Thus,

$$C \leqslant c e^{-\delta t} \mathcal{K}^{1/2}.$$

Turning back to A, we use the smoothness of the kernel and exponential flocking estimates (Equation 1.10),

$$\begin{aligned}
|A| &\leqslant c e^{-\delta t} \sqrt{\mathcal{K}} \left(\int_{\mathbb{R}^{2nN}} \sum_{i=1}^{N} \left[\frac{1}{N} \sum_{j=1}^{N} (|x_i - \bar{x}_i| + |x_j - \bar{x}_j|) \right]^2 f_0^{\otimes N} \, dX_0 \, dV_0 \right)^{1/2} \\
&\leqslant c e^{-\delta t} \sqrt{\mathcal{K}} \left(\int_{\mathbb{R}^{2nN}} \sum_{i=1}^{N} \left[|x_i - \bar{x}_i|^2 + \frac{1}{N} \sum_{j=1}^{N} |x_j - \bar{x}_j|^2 \right] f_0^{\otimes N} \, dX_0 \, dV_0 \right)^{1/2} \\
&\leqslant c e^{-\delta t} \sqrt{\mathcal{K}} \left(2 \int_{\mathbb{R}^{2nN}} \left[\sum_{i=1}^{N} |x_i - \bar{x}_i|^2 \right] f_0^{\otimes N} \, dX_0 \, dV_0 \right)^{1/2} \\
&= c e^{-\delta t} \sqrt{\mathcal{K}} \sqrt{\mathcal{P}}.
\end{aligned}$$

Finally, one can see that B contributes a negative term,

$$\sum_{i=1}^{N} (v_i - \bar{v}_i) \cdot \frac{1}{N} \sum_{j=1}^{N} \phi(\bar{x}_i - \bar{x}_j)[(v_j - \bar{v}_j) - (v_i - \bar{v}_i)] = \frac{1}{N} \sum_{i,j=1}^{N} \phi(\bar{x}_i - \bar{x}_j)((v_i - \bar{v}_i) \cdot (v_j - \bar{v}_j) - |v_i - \bar{v}_i|^2)$$

and symmetrizing,

$$= \frac{1}{2}\frac{1}{N}\sum_{i,j=1}^{N} \phi(\bar{x}_i - \bar{x}_j)(-|v_j - \bar{v}_j|^2 + 2(v_i - \bar{v}_i)\cdot(v_j - \bar{v}_j) - |v_i - \bar{v}_i|^2)$$
$$= -\frac{1}{2}\frac{1}{N}\sum_{i,j=1}^{N} \phi(\bar{x}_i - \bar{x}_j)|(v_j - \bar{v}_j) - (v_i - \bar{v}_i)|^2 \leqslant 0.$$

Collecting all of the above we obtain

$$\frac{d}{dt}\mathcal{K} \leqslant ce^{-\delta t}(\mathcal{K}^{1/2} + \mathcal{K}^{1/2}\mathcal{P}^{1/2}). \tag{1.23}$$

Denoting $p = 1 + \mathcal{P}^{1/2}$, $k = \mathcal{K}^{1/2}$ we obtain the system

$$\dot{p} \leqslant k,\ p_0 = 1; \qquad \dot{k} \leqslant ce^{-\delta t}p,\ k_0 = 0. \tag{1.24}$$

Claim 1.2.1. *Any non-negative solution to* (Equation 1.24) *obeys an estimate* $p \leqslant 1 + Ct$, $k \leqslant C\min\{1, t\}$*, where* $C = C(c, \delta)$.

To see that let us fix an $\varepsilon > 0$ to be determined later and compute

$$\frac{d}{dt}(\varepsilon p^2 + k^2) \leqslant 2pk(\varepsilon + ce^{-\delta t}) \leqslant \sqrt{\varepsilon}(\varepsilon p^2 + k^2) + \frac{c}{\sqrt{\varepsilon}}e^{-\delta t}(\varepsilon p^2 + k^2).$$

Thus,

$$\varepsilon p^2 + k^2 \leqslant \varepsilon \exp\left\{\sqrt{\varepsilon}t + \frac{1}{\sqrt{\varepsilon}\delta}\right\}.$$

Setting $\varepsilon = \delta^2$, we can see that the growth rate of p does not exceed $\delta/2$, $p \lesssim e^{\delta t/2}$. Plugging this into k-equation we obtain $\dot{k} \lesssim e^{-\delta t/2}$. This proves the bound on k, and then solving for p, $p \leqslant 1 + Ct$.

Going back to the energies, we obtain

$$\mathcal{K} \leqslant C \min\{1, t^2\}, \qquad \mathcal{P} \leqslant Ct^2.$$

Due to the global bound on the support of the flock (Equation 1.10), (Equation 1.11), we also have $\mathcal{P} \leqslant CN$. Thus,

$$\mathcal{P} \leqslant C \min\{N, t^2\}.$$

Consequently, we obtain the required

$$\mathcal{K} + \mathcal{P} \leqslant C \min\{N, t^2\}.$$

1.3 Propagation of chaos for the forced system

In this section, we will prove Theorem 1.1.3. Using the basic energy estimates obtained in the previous section, we will now extend the result to the system with friction forces (Equation 1.13) and $\kappa > 0$. It is well-known that the flocking behavior of solutions to (Equation 1.13), even with constant $\theta_i = 1$ does not always hold even for global kernels $\phi \geqslant c_0 > 0$. The example exhibited in (16) shows misalignment dynamics when the initial configuration is symmetric $x_1 = -x_2$ and velocities are aimed in the opposite directions $v_1 = -v_2$. The work (22) proved that this is, in a sense, the only situation when no flocking occurs. As long as the initial condition is *sectorial*, meaning that all $v_i(0) \in \Sigma$, where Σ is an open conical sector of opening less than π, then the solutions align exponentially fast.

Proposition 1.3.1 ((22))**.** *Suppose that*

$$\phi(r) \geqslant \frac{\lambda}{(1+r^2)^{\beta/2}}, \quad \lambda > 0, \quad \beta \leqslant 1. \tag{1.25}$$

For any sectorial solution to (Equation 1.13) *there exists* $v_\infty \in \mathbb{R}^n$ *and* $\theta_\infty > 0$ *with* $|v_\infty|^p = \theta_\infty$, *such that one has*

$$\max_{i=1,\dots,N} (|v_i - v_\infty| + |\theta_i - \theta_\infty|) \leqslant Ce^{-\delta t}, \tag{1.26}$$

$$\sup_{t>0} \max_{i,j=1,\dots,N} |x_i - x_j| = D < \infty. \tag{1.27}$$

It is within the context of sectorial solutions that we will cast the propagation of chaos result. But first we establish a similar flocking estimates for solutions of the corresponding kinetic model.

1.3.1 Grassmannian reduction for Vlasov-alignment equation

Let us denote $\Omega = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$. The Vlasov equation corresponding to (Equation 1.13) is given by

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (fF(f)) + \nabla_v \cdot (fR) + \nabla_\theta \cdot (f\Theta(f)) = 0, \quad (x, v, \theta) \in \Omega, \quad t > 0, \tag{1.28}$$

subject to the initial condition

$$f(x, v, \theta, 0) = f_0(x, v, \theta), \tag{1.29}$$

where

$$F(f)(x, v, \theta) = \int_\Omega \phi(x-y)(w-v) f(y, w, \eta, t)\, dy\, dw\, d\eta,$$

$$R(x, v, \theta) = \sigma(\theta - |v|^p)v, \quad \sigma > 0, \quad p > 0,$$

$$\Theta(f)(x, v, \theta) = \kappa \int_\Omega \phi(x-y)(\eta - \theta) f(y, w, \eta, t)\, dy\, dw\, d\eta.$$

In this section, we will prove a similar flocking result for the sectorial solutions of (Equation 1.28). Let us define what they are in the kinetic context.

Definition 1.3.2. *A solution f to* (Equation 1.28) *is called* sectorial *if there exists a conical region Σ lying on one side of a hyperplane, i.e. with conical opening less than π such that $v \in \Sigma$ for any v in the velocity support of f, $(x, v, \theta) \in \operatorname{supp} f$ for some x, θ.*

Since the equation (Equation 1.28) is rotationally invariant, it will be convenient to assume that our solution belong the upper-half space: there exists $\varepsilon > 0$ such that

$$v_n \geqslant \varepsilon |v|, \quad \forall (x, v, \theta) \in \operatorname{supp} f, \tag{1.30}$$

By the weak maximum principle discussed below in Remark 1.3.6, it follows that if f is sectorial initially, then it will remain so for all time and the velocity support will lie in the same sector Σ.

Let us state our main result now.

Proposition 1.3.3. *Suppose the kernel satisfies* (Equation 1.25)*. For any sectorial solution* f *to* (Equation 1.28) *with initial compact support one has*

$$\sup_{t>0} \operatorname{diam} \operatorname{supp} f_t < \infty, \tag{1.31}$$

and there exist $v_\infty \in \mathbb{R}^n$*,* $\theta_\infty \in \mathbb{R}_+$*, with* $|v_\infty|^p = \theta_\infty$ *such that*

$$\max_{(x,v,\theta)\in \operatorname{supp} f_t} (|\theta - \theta_\infty| + |v - v_\infty|) \leqslant c e^{-\delta t}. \tag{1.32}$$

As in the discrete case the proof is based on examination of kinetic characteristics of the equation given by

$$\begin{cases} \dot{x} &= v, \qquad x(0) = x_0, \\ \dot{v} &= \int_\Omega \phi(x-y)(w-v) f(y,w,\eta,t)\, dy\, dw\, d\eta + \sigma(\theta - |v|^p) v, \quad v(0) = v_0, \\ \dot{\theta} &= \kappa \int_\Omega \phi(x-y)(\eta - \theta) f(y,w,\eta,t)\, dy\, dw\, d\eta, \quad \theta(0) = \theta_0. \end{cases} \tag{1.33}$$

Let us denote

$$\mathcal{D}(t) = \max_{(x,v,\theta),(x',v',\theta')\in \operatorname{supp} f_t} |x - x'|,$$

$$\mathcal{A}(t) = \max_{(x,v,\theta),(x',v',\theta')\in \operatorname{supp} f_t} |v - v'|,$$

$$\mathcal{Q}(t) = \max_{(x,v,\theta),(x',v',\theta')\in \operatorname{supp} f_t} |\theta - \theta'|,$$

$$M = \int_\Omega f(x,v,\theta,t)\,dx\,dv\,d\theta, \quad \theta_\infty = \frac{1}{M}\int_\Omega \theta f(x,v,\theta,t)\,dx\,dv\,d\theta,$$

$$\theta_+(t) = \max_{(x,v,\theta)\in \operatorname{supp} f_t} \theta, \qquad \theta_-(t) = \min_{(x,v,\theta)\in \operatorname{supp} f_t} \theta.$$

Then we have

$$\frac{d}{dt}\mathcal{D} \leqslant \mathcal{A}. \tag{1.34}$$

Indeed, at time t, let $\ell \in (\mathbb{R}^d)^*, |\ell| = 1, (x,v,\theta), (x',v',\theta') \in \operatorname{supp} f_t$ such that $\mathcal{D}(t) = \ell(x - x')$.

By Rademacher's Lemma and the first equation in the system (Equation 1.33) we have

$$\frac{d}{dt}\mathcal{D} = \ell(\dot{x} - \dot{x}') = \ell(v - v') \leqslant \mathcal{A}.$$

For $\mathcal{Q}$, we have

$$\frac{d}{dt}\mathcal{Q} \leqslant -\kappa\phi(\mathcal{D})\mathcal{Q}. \tag{1.35}$$

To prove that, at time t we choose $\ell \in \mathbb{R}^*, |\ell| = 1, (x, v, \theta), (x', v', \theta') \in \operatorname{supp} f_t$ which satisfy $\mathcal{Q}(t) = \ell(\theta - \theta')$. By Rademacher's Lemma and the third equation in the system (Equation 1.33) we get

$$\begin{aligned}\frac{d}{dt}\mathcal{Q} = \kappa \int_\Omega \phi(x-y)\ell(\eta-\theta)f(y,w,\eta,t)\,dy\,dw\,d\eta - \kappa \int_\Omega \phi(x'-y)\ell(\eta-\theta')f(y,w,\eta,t)\,dy\,dw\,d\eta \\ = \kappa \int_\Omega \phi(x-y)[\ell(\eta-\theta') - \ell(\theta-\theta')]f(y,w,\eta,t)\,dy\,dw\,d\eta \\ + \kappa \int_\Omega \phi(x'-y)[\ell(\theta-\eta) - \ell(\theta-\theta')]f(y,w,\eta,t)\,dy\,dw\,d\eta.\end{aligned}$$

Since $\ell(\eta-\theta') - \ell(\theta-\theta') \leqslant 0$ and $\ell(\theta-\eta) - \ell(\theta-\theta') \leqslant 0$, the right hand side of the above equality is nonpositive. Note that $\phi(x-y) \geqslant \phi(\mathcal{D})$ for all $x, y \in \operatorname{supp} f_t$. Therefore,

$$\frac{d}{dt}\mathcal{Q} \leqslant -\kappa\phi(\mathcal{D}) \int_\Omega \ell(\theta-\theta')f(y,w,\eta,t)\,dy\,dw\,d\eta \leqslant -\kappa\phi(\mathcal{D})\mathcal{Q}.$$

Similarly, using the third equation in (Equation 1.33) and Rademacher's Lemma, it is not hard to see that θ_+ is decreasing and θ_- is increasing. Thus,

$$\theta_+(t) \leqslant \theta^*, \quad \theta_-(t) \geqslant \theta_* \quad \forall t \geqslant 0, \tag{1.36}$$

where $\theta^* = \theta_+(0)$ and $\theta_* = \theta_-(0)$.

Before we proceed further let us discuss the boundedness of the velocity support of f and the weak maximum principle.

Lemma 1.3.4 (boundedness)**.** *There exists a constant* C *which depends on the initial data such that for any* $(x, v, \theta) \in \operatorname{supp} f_t$*, one has*

$$|v(t)| \leqslant C, \qquad \forall t > 0. \tag{1.37}$$

Proof. Let

$$|v_+|(t) = \max_{(x,v,\theta) \in \operatorname{supp} f_t} |v|.$$

At time t, let $\ell \in (\mathbb{R}^d)^*, |\ell| = 1$, $(x,\ v, \theta) \in \operatorname{supp} f_t$ such that $|v_+| = \ell(v)$. Then, by Rademacher's Lemma,

$$\begin{aligned}\frac{d}{dt}|v_+| &= \int_\Omega \phi(x-z)\,\ell(w-v) f(z,w,\eta,t)\ dz\,dw\,d\eta + \sigma\ell(v)(\theta - |v|^p)\\ &\leqslant \sigma|v_+|(\theta^* - |v_+|^p).\end{aligned}$$

Hence, if $\theta^* \leqslant |v_+|^p$ then

$$|v_+|(t) \leqslant |v_+|(0) \quad \forall t > 0.$$

Otherwise, we have

$$\frac{d}{dt}|v_+|^p \leqslant \sigma p |v_+|^p(\theta^* - |v_+|^p).$$

Solving the above ODI gives

$$|v_+|(t) \leqslant \frac{\sqrt[p]{\theta^*} e^{\sigma\theta^* t}}{(c + e^{\sigma p \theta^* t})^{1/p}} = \sqrt[p]{\theta^*} + \mathcal{O}(e^{-\sigma\theta^* t}), \tag{1.38}$$

where c is a positive constant depending on initial data. Thus, $|v_+|(t)$ is bounded for all $t > 0$. $\square$

Lemma 1.3.5 (weak maximum principle)**.** *If for a given functional* $\ell \in (\mathbb{R}^n)^*$*, all velocity vectors* v_0 *that lie in the support of the initial flock,* $(x_0, v_0, \theta_0) \in \operatorname{supp} f_0$*, satisfy*

$$\ell(v_0) \geqslant 0,$$

then at any positive time

$$\ell(v) \geqslant 0, \qquad \forall t > 0,\ \ (x, v, \theta) \in \operatorname{supp} f_t.$$

Proof. At time t, let

$$\ell(v) = \min_{(z,w,\eta) \in \operatorname{supp} f} \ell(w).$$

By Rademacher's Lemma,

$$\frac{d}{dt}\ell(v) = \int_\Omega \phi(x - z)\, \ell(w - v) f(z, w, \eta, t)\ dz\, dw\, d\eta + \sigma\ell(v)(\theta - |v|^p) \geqslant \sigma\ell(v)(\theta_* - |v|^p).$$

Then by Lemma 1.3.4 we get

$$\frac{d}{dt}\ell(v) \geqslant c\, \ell(v),$$

where c is constant. Solving this ODI we obtain the desired conclusion,

$$\ell(v) \geqslant \ell(v_0)e^{ct} \geqslant 0, \quad \forall t > 0.$$

$\square$

Remark 1.3.6. *By the weak maximum principle we note that if the support of* f_0 *in* v *lies in the convex sector defined by*

$$\Sigma_{\mathcal{F}} = \bigcap_{\ell \in \mathcal{F}} \{v \in \mathbb{R}^n : \ell(v) \geqslant 0\},$$

where $\mathcal{F}$ *is an arbitrary set of linear functionals on* $\mathbb{R}^n$*, then the velocity support of* f_t *will be confined to that sector for all time. Since the system* (Equation 1.33) *is invariant under rotations, without loss of generality we can assume that the support of* f_0 *in* v *lies above the hyperplane* $\Pi_n = \{v_n = 0\}$*, where* v_n *is the* n*-th coordinate of vector* v*.*

Lemma 1.3.7. *For any sectorial solution* f *to* (Equation 1.28) *there exists a positive constant* c_0 *depending on the initial data such that*

$$|v| \geqslant c_0, \quad \forall (x, v, \theta) \in \operatorname{supp} f_t. \tag{1.39}$$

Proof. At time t, let (x, v, θ) be a minimizer for $\min_{(x,v,\theta) \in \operatorname{supp} f_t} v_n$. Then

$$\frac{d}{dt} v_n = \int_{\Omega} \phi(x - z)(w_n - v_n) f(z, w, \eta, t) \, dz \, dw \, d\eta + \sigma v_n(\theta - |v|^p) \geqslant \sigma v_n(\theta_* - \varepsilon^{-p} v_n^p). \tag{1.40}$$

If $\theta_* \leqslant \varepsilon^{-p} v_n^p$ then

$$|v| \geqslant \varepsilon \sqrt[p]{\theta_*}.$$

Otherwise, solving (Equation 1.40) we get

$$v_n \geqslant \frac{\varepsilon \sqrt[p]{\theta_*} e^{\sigma\theta_* t}}{(c + e^{p\sigma\theta_* t})^{1/p}},$$

where c is a positive constant which depends on the initial data. Then the lemma follows. $\square$

Remark 1.3.8. *Lemma 1.3.7 tells us that for a sectorial solution* f, $\operatorname{supp} f(x, \cdot, \theta)$ *stays away from the origin. Then, by Lemma 1.3.4, it implies that* $\operatorname{supp} f(x, \cdot, \theta)$ *is contained in a sector. Lemma 1.3.7 also implies that for any sectorial solution* f *one has*

$$|v_-|(t) \geqslant c_0, \quad \forall t > 0, \tag{1.41}$$

where $|v_-|(t) = \min_{(x,v,\theta)\in \operatorname{supp} f} |v(t)|$.

Proof of Proposition 1.3.3. From now on we consider a sectorial solution f to the system (Equation 1.28). Denoting $\tilde{r} = \frac{r}{|r|}$ for any vector $r \in \mathbb{R}^n$. One has

$$\frac{d}{dt}\tilde{v} = \frac{1}{|v|}\left(\mathrm{Id} - \frac{v}{|v|} \otimes \frac{v}{|v|}\right)\dot{v} = \int_\Omega \frac{|w|}{|v|} \phi(x-z)(\mathrm{Id} - \tilde{v} \otimes \tilde{v})\tilde{w}\, f(z, w, \eta, t)\, dz\, dw\, d\eta. \tag{1.42}$$

Here, we used $(\mathrm{Id} - \tilde{v} \otimes \tilde{v})v = 0$.

Denoting by $\widehat{(v,u)}$ the angle between two vectors v and u, then $\cos\widehat{(v,u)} = \tilde{v}\cdot\tilde{u}$. Thus, if $(x,v,\theta),(y,u,\zeta)$ are the solutions to (Equation 1.33) with respect to the initial conditions $(x_0,v_0,\theta_0),(y_0,u_0,\zeta_0)$, respectively, then

$$\begin{aligned}\frac{d}{dt}\cos\widehat{(v,u)} &= \int_\Omega \frac{|w|}{|v|}\phi(x-z)[\cos\widehat{(u,w)} - \cos\widehat{(v,u)}\cos\widehat{(v,w)}]f(z,w,\eta,t)\,dz\,dw\,d\eta \\ &+ \int_\Omega \frac{|w|}{|u|}\phi(y-z)[\cos\widehat{(v,w)} - \cos\widehat{(v,u)}\cos\widehat{(u,w)}]f(z,w,\eta,t)\,dz\,dw\,d\eta.\end{aligned} \tag{1.43}$$

Note that if v, u, and w are three vectors lying in the same two dimenstional plane and

$$\widehat{(v,u)} = \widehat{(v,w)} + \widehat{(w,u)} < \pi - \delta \quad \text{for some } \delta > 0, \tag{1.44}$$

then the followings hold:

$$\begin{aligned}\cos\widehat{(u,w)} - \cos\widehat{(v,u)}\cos\widehat{(v,w)} &= \cos\left(\widehat{(v,u)} - \widehat{(v,w)}\right) - \cos\widehat{(v,u)}\cos\widehat{(v,w)} \\ &= \sin\widehat{(v,u)}\sin\widehat{(v,w)} \geqslant 0,\end{aligned}$$

$$\cos\widehat{(v,w)}\cos\widehat{(v,u)}\cos\widehat{(u,w)} \geqslant 0,$$

$$\cos\widehat{(u,w)} + \cos\widehat{(v,w)} = \cos\frac{\widehat{(v,u)}}{2}\cos\frac{\widehat{(u,w)} - \widehat{(v,w)}}{2} \geqslant \left(\cos\frac{\pi-\delta}{2}\right)^2.$$

Therefore, if the support of f in v is on a two dimensional plane and (Equation 1.44) is satisfied, then by Lemma 1.3.4 , Lemma 1.3.7 and (Equation 1.43), one has

$$\begin{aligned}\frac{d}{dt}\cos\widehat{(v,u)} &\geqslant c\phi(\mathcal{D})\int_{\Omega}\left(\cos\widehat{(u,w)}+\cos\widehat{(v,w)}\right)\left(1-\cos\widehat{(v,u)}\right)f(z,w,\eta,t)\,dz\,dw\,d\eta\\ &\geqslant c\phi(\mathcal{D})\left(1-\cos\widehat{(v,u)}\right).\end{aligned}$$

Equivalently,

$$\frac{d}{dt}\left(1-\cos\widehat{(v,u)}\right)\leqslant -c\phi(\mathcal{D})\left(1-\cos\widehat{(v,u)}\right). \tag{1.45}$$

Now let Π be a fixed two dimensional plane which contains the v_n-axis. Denoting by v^{Π} the projection of any $v\in \operatorname{supp} f$ onto Π. Projecting the second equation in (Equation 1.33) onto Π we have the following equation:

$$\dot{v}^{\Pi}=\int_{\Omega}\phi(x-z)(w^{\Pi}-v^{\Pi})f(z,w,\eta,t)\,dz\,dw\,d\eta+\sigma v^{\Pi}(\theta-|v|^{p}) \tag{1.46}$$

Therefore, we can write the equation for $\cos\widehat{(v^{\Pi},u^{\Pi})}$ as follows:

$$\begin{aligned}\frac{d}{dt}\cos\widehat{(v^{\Pi},u^{\Pi})}&=\int_{\Omega}\frac{|w^{\Pi}|}{|v^{\Pi}|}\phi(x-z)[\cos\widehat{(u^{\Pi},w^{\Pi})}-\cos\widehat{(v^{\Pi},u^{\Pi})}\cos\widehat{(v^{\Pi},w^{\Pi})}]f(z,w,\eta,t)\,dz\,dw\,d\eta\\ &+\int_{\Omega}\frac{|w^{\Pi}|}{|u^{\Pi}|}\phi(y-z)[\cos\widehat{(v^{\Pi},w^{\Pi})}-\cos\widehat{(v^{\Pi},u^{\Pi})}\cos\widehat{(u^{\Pi},w^{\Pi})}]f(z,w,\eta,t)\,dz\,dw\,d\eta.\end{aligned} \tag{1.47}$$

Let us denote $\mathcal{G}(1, n-1)$ the space of all two dimensional subspaces of $\mathbb{R}^n$ which contain v_n-axis. Since $\mathcal{G}(1, n-1)$ can be identified with 1-Grassmannian manifold of $\mathbb{R}^{n-1}$ which is compact, we can define

$$\gamma^{2D} = \max_{\substack{\Pi \in \mathcal{G}(1,n-1) \\ (x,v,\theta),(y,u,\zeta) \in \operatorname{supp} f}} (\widehat{v^{\Pi}, u^{\Pi}}). \tag{1.48}$$

We note that

$$\gamma^{2D} \leqslant \pi - \delta \quad \text{for some } \delta > 0.$$

Since the n-th coordinate of any $v \in \operatorname{supp} f$ does not change when it is projected onto Π, $|v^{\Pi}|$ is still bounded above and below by positive constants. Therefore, choosing a maximizing triple Π, u, v for $(\widehat{v^{\Pi}, u^{\Pi}})$, from (Equation 1.47) we deduce that

$$\frac{d}{dt}(1 - \cos \gamma^{2D}) \leqslant -c\phi(\mathcal{D})((1 - \cos \gamma^{2D}). \tag{1.49}$$

Denoting

$$\gamma = \max_{(x,v,\theta),(y,u,\zeta) \in \operatorname{supp} f} \widehat{(u, v)}.$$

Claim 1.3.9. *We have* $\gamma \leqslant \gamma^{2D}$.

Proof of Claim 1.3.9. For any $(x, v, \theta), (y, u, \zeta) \in \operatorname{supp} f$, consider the two dimensional subspace $\Pi = \operatorname{span}\{e_n, \tilde{u} - \tilde{v}\}$ where $e_n = (0, \dots, 0, 1)$. We have $\Pi \in \mathcal{G}(1, n-1)$ and $\tilde{u} - \tilde{v} = \tilde{u}^{\Pi} - \tilde{v}^{\Pi}$. By the law of cosines, we get

$$\begin{aligned} 2(1 - \cos\widehat{(u, v)}) = |\tilde{u} - \tilde{v}|^2 = |\tilde{u}^{\Pi} - \tilde{v}^{\Pi}|^2 &= 2|\tilde{u}^{\Pi}|^2(1 - \cos\widehat{(u^{\Pi}, v^{\Pi})}) \\ &\leqslant 2(1 - \cos\widehat{(u^{\Pi}, v^{\Pi})}). \end{aligned}$$

It implies that for any $(x, v, \theta), (y, u, \zeta) \in \operatorname{supp} f$ there exists $\Pi \in \mathcal{G}(1, n-1)$ such that $\widehat{(u, v)} \leqslant \widehat{(u^{\Pi}, v^{\Pi})}$. Therefore, the claim is followed. $\square$

Remark 1.3.10. *Claim 1.3.9 and the inequality* (Equation 1.49) *imply that if* $\mathcal{D}(t) \leqslant D < \infty$ *then*

$$1 - \cos\gamma \leqslant 1 - \cos\gamma^{2D} \lesssim e^{-c\phi(D)t}.$$

Now we set

$$\mathcal{R} = \max_{(x,v,\theta),(y,u,\zeta) \in \operatorname{supp} f} \frac{|v|^2}{|u|^2}.$$

Suppose that $(x, v, \theta), (y, u, \zeta)$ maximize $\mathcal{R}$ at time t, we have

$$
\begin{aligned}
\frac{d}{dt}\mathcal{R} &= \frac{2}{|u|^2}\left[\int_{\Omega} \phi(x-z)(v\cdot w - |v|^2) f(z,w,\eta,t)\,dz\,dw\,d\eta + \sigma|v|^2(\theta - |v|^p)\right] \\
&\quad - \frac{2|v|^2}{|u|^4}\left[\int_{\Omega} \phi(y-z)(u\cdot w - |u|^2) f(z,w,\eta,t)\,dz\,dw\,d\eta + \sigma|u|^2(\zeta - |u|^p)\right] \\
&= \frac{2}{|u|^2}\int_{\Omega} \phi(x-z)(v\cdot w - |v|^2) f(z,w,\eta,t)\,dz\,dw\,d\eta \qquad (1.50) \\
&\quad + \frac{2|v|^2}{|u|^4}\int_{\mathbb{R}^{2d}} \phi(y-z)(|u|^2 - u\cdot w) f(z,w,\eta,t)\,dz\,dw\,d\eta + 2\sigma\mathcal{R}(\theta - \zeta + |u|^p - |v|^p).
\end{aligned}
$$

Since u, v maximize $\mathcal{R}$, we have $v\cdot w - |v|^2 \leqslant |v|(|w| - |v|) \leqslant 0$ for all $w \in \operatorname{supp} f$. Hence, the first term on the right hand side of (Equation 1.50) is nonpositve. For the second term, we have

$$|u|^2 - u\cdot w = |u|^2 - |u||w|\cos\widehat{(u,w)} \lesssim 1 - \cos\gamma.$$

Note that $\mathcal{R}$ is bounded from above and below, hence,

$$2\sigma\mathcal{R}(\theta - \zeta + |u|^p - |v|^p) = 2\sigma\mathcal{R}(\theta - \zeta) + \frac{2\sigma\mathcal{R}}{|u|^p}(1 - \mathcal{R}^{p/2}) \lesssim \mathcal{Q} + (1 - \mathcal{R}).$$

Therefore, there exist positive constants c_1, c_2, c_3 such that

$$\frac{d}{dt}(\mathcal{R} - 1) \leqslant -c_1(\mathcal{R} - 1) + c_2(1 - \cos\gamma) + c_3\mathcal{Q}. \qquad (1.51)$$

Firstly, we see that the flock diameter grows at most linearly in time,

$$\mathcal{D}(t) \lesssim t \tag{1.52}$$

since

$$\frac{d}{dt}\mathcal{D}(t) \leqslant \mathcal{A}(t) \tag{1.53}$$

and $|v|$ is bounded for all $(x, v, \theta) \in \operatorname{supp} f$. It is not hard to see the relation

$$\mathcal{A}^2 \lesssim (\mathcal{R} - 1) + (1 - \cos\gamma). \tag{1.54}$$

Thus, to prove an exponential alignment it suffices to show that both $(\mathcal{R} - 1)$ and $(1 - \cos\gamma)$ decay exponentially fast.

We now consider two cases for β:

Case I: $\beta < 1$. Our assumption on the kernel and (Equation 1.52) imply that

$$\phi(\mathcal{D}) \gtrsim \frac{1}{(1 + t^2)^{\beta/2}}. \tag{1.55}$$

Plugging it into (Equation 1.49) and applying the Grönwall's Lemma we get

$$1 - \cos\gamma \leqslant 1 - \cos\gamma^{2D} \lesssim e^{-c\langle t\rangle^{1-\beta}}. \tag{1.56}$$

Plugging (Equation 1.55) into (Equation 1.35) and solving for $\mathcal{Q}$ we also have

$$\mathcal{Q} \lesssim e^{-c\langle t\rangle^{1-\beta}}. \tag{1.57}$$

Combining these inequalities with (Equation 1.51) and solving for $\mathcal{R} - 1$ we obtain

$$\mathcal{R} - 1 \lesssim e^{-c\langle t\rangle^{1-\beta}}. \tag{1.58}$$

From (Equation 1.53), (Equation 1.54), (Equation 1.56) and (Equation 1.58), we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{D} \lesssim e^{-c\langle t\rangle^{(1-\beta)/2}}.$$

Solving this ODI gives

$$\mathcal{D}(t) \leqslant D < \infty. \tag{1.59}$$

Thus, (Equation 1.35) implies that

$$\mathcal{Q}(t) \leqslant \mathcal{Q}(0)e^{-t\phi(D)}.$$

Hence, $\theta(t)$ aligns to θ_∞ exponentially fast for all $(x, v, \theta) \in \operatorname{supp} f$. Due to finite flock diameter (Equation 1.59) and Remark 1.3.10, we have

$$1 - \cos\gamma \lesssim e^{-c\phi(D)t}.$$

Putting the estimates for $\mathcal{Q}$ and $(1-\cos\gamma)$ into (Equation 1.51) and solving for $\mathcal{R}-1$ where we use the Grönwall's Lemma, we obtain the exponential decay for $\mathcal{R}-1$ as well. Therefore, we arrive at an alignment with an exponential rate.

Denoting by E any quantity which decays exponentially fast. So far we have $|\theta-\theta_\infty| = E(t)$, $|v-u| = E(t)$ for any $\theta, v, u \in \operatorname{supp} f$. By (Equation 1.41) and Lemma 1.3.4, $|v_\pm|(t)$ are bounded, hence, the following equations hold for $|v_\pm|^p(t)-\theta_\infty$:

$$\frac{d}{dt}(|v_\pm|^p-\theta_\infty) = (\sigma p|v_\pm|^p(\theta_\infty-|v_\pm|^p)+E) \quad \sim \quad (-(|v_\pm|^p-\theta_\infty)+E).$$

It follows that $|v_\pm|^p(t)$ converges to θ_∞ exponentially fast. Therefore, from the characteristic equation for $v \in \operatorname{supp} f$ in (Equation 1.33) we deduce that

$$\frac{d}{dt}v = E, \quad \forall v_0 \in \operatorname{supp} f_0.$$

The existence of v_∞ is followed then.

Case II: $\beta = 1$. In this case, we have $\phi(\mathcal{D}) \gtrsim \dfrac{1}{\sqrt{1+t^2}}$, hence,

$$1-\cos\gamma \leqslant 1-\cos\gamma^{2D} \lesssim \langle t\rangle^{-\alpha}, \qquad \text{and}$$

$$\mathcal{Q} \lesssim \langle t\rangle^{-\alpha}, \text{ for some } \alpha > 0.$$

Therefore,

$$\frac{d}{dt}(\mathcal{R}-1) \lesssim -(\mathcal{R}-1)+\langle t\rangle^{-\alpha}.$$

Solving this ODI we yield

$$\mathcal{R} - 1 \lesssim \langle t \rangle^{-\alpha}.$$

Here we used the fact that $e^{-ct} * \langle t \rangle^{-\alpha} \sim \langle t \rangle^{-\alpha}$. It implies that

$$\mathcal{A} \lesssim \langle t \rangle^{-\alpha/2},$$

and hence,

$$\mathcal{D} \lesssim \langle t \rangle^{1-\alpha/2}.$$

Thus,

$$\phi(\mathcal{D}) \gtrsim \phi(\langle t \rangle^{1-\alpha/2}) \gtrsim \frac{1}{(1+t^2)^{\tilde{\beta}/2}} \text{ for some } \tilde{\beta} < 1.$$

Now we can argue exactly as in the case $\beta < 1$ replacing β by $\tilde{\beta}$ to reach the conclusions of the theorem. □

1.3.2 Proof of Theorem 1.1.3

Using Proposition 1.3.3 as a key ingredient we now prove our main result for the Rayleigh-forced system, Theorem 1.1.3. So, let us we consider the full Liouville equation for a probability density f^N on Ω^N:

$$\partial_t f^N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f^N + \sum_{i=1}^{N} \nabla_{v_i} \cdot (f^N F_i^N) + \sum_{i=1}^{N} \nabla_{v_i} \cdot (f^N R_i^N) + \sum_{i=1}^{N} \nabla_{\theta_i} \cdot (f^N \Theta_i^N) = 0, \quad (1.60)$$

subject to the initial condition

$$f^N(0) = f_0^{\otimes N}, \tag{1.61}$$

where $f_0 : \Omega \to \mathbb{R}_+$ and for $(X, V, \Theta) = (x_1, \ldots, x_N, v_1, \ldots, v_N, \theta_1, \ldots, \theta_N)$,

$$\begin{aligned}
F_i^N(X, V, \Theta) &= \frac{1}{N} \sum_{k=1}^{N} \phi(x_i - x_k)(v_k - v_i), \\
\Theta_i^N(X, V, \Theta) &= \frac{1}{N} \sum_{k=1}^{N} \phi(x_i - x_k)(\theta_k - \theta_i), \\
R_i^N(X, V, \Theta) &= \sigma v_i(\theta_i - |v_i|^p).
\end{aligned}$$

We introduce a similar notation for the flow-maps. Denote by

$$\Phi_t^N = (x_1(t), v_1(t), \theta_1(t) \ldots, x_N(t), v_N(t), \theta_N(t)) : \Omega^N \to \Omega^N$$

the flow-map of the discrete system (Equation 1.13) which is also the characteristic flow of (Equation 1.60). Then, as before, f^N is the push forward of $f_0^{\otimes N}$ under Φ_t^N,

$$f^N = \Phi_t^N \sharp f_0^{\otimes N}.$$

Let also

$$\bar{\Phi}_t = (\bar{x}(t), \bar{v}(t), \bar{\theta}(t)) : \Omega \to \Omega$$

be the characteristic map of (Equation 1.28), which consists of solutions to (Equation 1.33). The direct product of N copies will be denoted $\bar{\Phi}_t^{\otimes N}$. Then we have

$$f = \bar{\Phi}_t \sharp f_0, \quad f^{\otimes N} = \bar{\Phi}_t^{\otimes N} \sharp f_0^{\otimes N}. \tag{1.62}$$

By the same logic as before the theorem reduces to establishing the bound

$$\int_{\mathbb{R}^{2nN}} |\Phi_t^N(X_0, V_0) - \bar{\Phi}_t^{\otimes N}(X_0, V_0)|^2 \, f_0^{\otimes N}(X_0, V_0) \, dX_0 \, dV_0 \leqslant C \min\{N, t^4\}. \tag{1.63}$$

We split the integrand into three components:

$$\begin{aligned} \mathcal{P} &= \frac{1}{2} \int_{\Omega^N} |X_t(X_0, V_0, \Theta_0) - \bar{X}_t(X_0, V_0, \Theta_0)|^2 \, f_0^{\otimes N}(X_0, V_0, \Theta_0) \, dX_0 \, dV_0 \, d\Theta_0, \\ \mathcal{K} &= \frac{1}{2} \int_{\Omega^N} |V_t(X_0, V_0, \Theta_0) - \bar{V}_t(X_0, V_0, \Theta_0)|^2 \, f_0^{\otimes N}(X_0, V_0, \Theta_0) \, dX_0 \, dV_0 \, d\Theta_0, \\ \mathcal{C} &= \frac{1}{2} \int_{\Omega^N} |\Theta_t(X_0, V_0, \Theta_0) - \bar{\mathrm{Th}}_t(X_0, V_0, \Theta_0)|^2 \, f_0^{\otimes N}(X_0, V_0, \Theta_0) \, dX_0 \, dV_0 \, d\Theta_0. \end{aligned} \tag{1.64}$$

For the potential energy we will use the same inequality as before, (Equation 1.22). For $\mathcal{K}$, we obtain

$$\frac{d}{dt}\mathcal{K} = \mathcal{S}_1 + \mathcal{S}_2,$$

where $\mathcal{S}_1$ is the exact same alignment term that we handled before, but now with the use of Proposition 1.3.1 and Proposition 1.3.3,

$$\mathcal{S}_1 \leqslant c e^{-\delta t} \mathcal{K}^{1/2} (1 + \mathcal{P}^{1/2}). \tag{1.65}$$

And $\mathcal{S}_2$ is given by

$$\mathcal{S}_2 = \int_{\Omega^N} \sum_{i=1}^N (v_i - \bar{v}_i) \cdot \big(\sigma v_i(\theta_i - |v_i|^p) - \sigma \bar{v}_i(\bar{\theta}_i - |\bar{v}_i|^p)\big)\, f_0^{\otimes N}(X_0, V_0, \Theta_0)\, dX_0\, dV_0\, d\Theta_0.$$

Let us write $\mathcal{S}_2$ as follows

$$\begin{aligned} \mathcal{S}_2 &= \sigma \int_{\Omega^N} \sum_{i=1}^N (v_i - \bar{v}_i) \cdot \big(\theta_i v_i - \bar{\theta}_i \bar{v}_i\big)\, f_0^{\otimes N}(X_0, V_0, \Theta_0)\, dX_0\, dV_0\, d\Theta_0 \\ &\quad - \sigma \int_{\Omega^N} \sum_{i=1}^N (v_i - \bar{v}_i) \cdot (v_i |v_i|^p) - \bar{v}_i |\bar{v}_i|^p)\, f_0^{\otimes N}(X_0, V_0, \Theta_0)\, dX_0\, dV_0\, d\Theta_0 \\ &:= J_1 - J_2. \end{aligned}$$

Since

$$(v_i - \bar{v}_i) \cdot \big(\theta_i v_i - \bar{\theta}_i \bar{v}_i\big) = \frac{1}{2}(\theta_i + \bar{\theta}_i)|v_i - \bar{v}_i|^2 + \frac{1}{2}(v_i - \bar{v}_i) \cdot [(\theta_i - \bar{\theta}_i)(v_i + \bar{v}_i)],$$

one has

$$J_1 = \frac{\sigma}{2} \int_{\Omega^N} \sum_{i=1}^N \Big((\theta_i + \bar{\theta}_i)|v_i - \bar{v}_i|^2 + (v_i - \bar{v}_i) \cdot [(\theta_i - \bar{\theta}_i)(v_i + \bar{v}_i)]\Big)\, f_0^{\otimes N}(X_0, V_0, \Theta_0)\, dX_0\, dV_0\, d\Theta_0.$$

For J_2, since

$$(v_i - \bar{v}_i) \cdot (v_i |v_i|^p - \bar{v}_i |\bar{v}_i|^p) = \frac{1}{2}(|v_i|^p + |\bar{v}_i|^p)|v_i - \bar{v}_i|^2 + \frac{1}{2}(|v_i|^2 - |\bar{v}_i|^2)(|v_i|^p - |\bar{v}_i|^p),$$

and

$$\frac{1}{2}(|v_i|^2 - |\bar{v}_i|^2)(|v_i|^p - |\bar{v}_i|^p) \geqslant 0,$$

we get

$$-J_2 \leqslant -\frac{\sigma}{2}\int_{\Omega^N}\sum_{i=1}^{N}(|\bar{v}_i|^p + |v_i|^p)|v_i - \bar{v}_i|^2 f_0^{\otimes N}(X_0, V_0, \Theta_0)\,dX_0\,dV_0\,d\Theta_0.$$

Therefore,

$$\begin{aligned}\mathcal{S}_2 = J_1 - J_2 \leqslant \frac{\sigma}{2}\int_{\Omega^N}\sum_{i=1}^{N}(\theta_i - |v_i|^p + \bar{\theta}_i - |\bar{v}_i|^p)|v_i - \bar{v}_i|^2 f_0^{\otimes N}(X_0, V_0, \Theta_0)\,dX_0\,dV_0\,d\Theta_0 \\ + \frac{\sigma}{2}\int_{\Omega^N}\sum_{i=1}^{N}(v_i - \bar{v}_i)\cdot(\theta_i - \bar{\theta}_i)(\bar{v}_i + v_i) f_0^{\otimes N}(X_0, V_0, \Theta_0)\,dX_0\,dV_0\,d\Theta_0. \quad (1.66)\end{aligned}$$

Because $|\theta_i - |v_i|^p| \leqslant ce^{-\delta t}$ and $|\bar{\theta}_i - |\bar{v}_i|^p| \leqslant ce^{-\delta t}$, the first integral on the right hand side of (Equation 1.66) is less than or equal to $ce^{-\delta t}\mathcal{K}$. Then, we apply the Hölder inequality and the boundedness of $|\bar{v}_i|$ and $|v_i|$ to the second integral to obtain

$$\mathcal{S}_2 \leqslant c(e^{-\delta t}\mathcal{K} + \mathcal{K}^{1/2}\mathcal{H}^{1/2}). \quad (1.67)$$

Combining (Equation 1.65) and (Equation 1.67) we get

$$\frac{d}{dt}\mathcal{K} \leqslant ce^{-\delta t}\mathcal{K}^{1/2}(\mathcal{K}^{1/2} + 1 + \mathcal{P}^{1/2}) + \mathcal{K}^{1/2}\mathcal{H}^{1/2}. \quad (1.68)$$

Let us now turn to the characteristic parameters term $\mathcal{C}$:

$$\frac{d}{dt}\mathcal{C} = \int_{\Omega^N} \sum_{i=1}^{N} (\theta_i - \bar{\theta}_i) \cdot \Big(\frac{1}{N} \sum_{k=1}^{N} \phi(x_i - x_k)(\theta_k - \theta_i) \\ - \int_{\Omega} \phi(\bar{x}_i - y)(\eta - \bar{\theta}_i) f(y, w, \eta, t)\, dy\, dw\, d\eta \Big) f_0^{\otimes N}\, dX_0\, dV_0\, d\Theta_0 \\ := I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{\Omega^N} \sum_{i=1}^{N} (\theta_i - \bar{\theta}_i) \cdot \frac{1}{N} \sum_{k=1}^{N} \big[\phi(x_i - x_k) - \phi(\bar{x}_i - \bar{x}_k)\big] (\theta_k - \theta_i)\, f_0^{\otimes N}\, dX_0\, dV_0\, d\Theta_0,$$

$$I_2 = \int_{\Omega^N} \sum_{i=1}^{N} (\theta_i - \bar{\theta}_i) \cdot \frac{1}{N} \sum_{k=1}^{N} \phi(\bar{x}_i - \bar{x}_k) \big[(\theta_k - \bar{\theta}_k) - (\theta_i - \bar{\theta}_i)\big]\, f_0^{\otimes N}\, dX_0\, dV_0\, d\Theta_0,$$

$$I_3 = \int_{\Omega^N} \sum_{i=1}^{N} (\theta_i - \bar{\theta}_i) \cdot \Big(\frac{1}{N} \sum_{k=1}^{N} \phi(\bar{x}_i - \bar{x}_k)(\bar{\theta}_k - \bar{\theta}_i) \\ - \int_{\Omega} \phi(\bar{x}_i - y)(\eta - \bar{\theta}_i) f(y, w, t)\, dy\, dw\, d\eta \Big) f_0^{\otimes N}\, dX_0\, dV_0\, d\Theta_0.$$

We have $I_2 \leqslant 0$ because

$$I_2 = \int_{\Omega^N} \frac{1}{N} \sum_{i,k=1}^{N} \phi(\bar{x}_i - \bar{x}_k)[(\theta_i - \bar{\theta}_i) \cdot (\theta_k - \bar{\theta}_k) - |\theta_i - \bar{\theta}_i|^2]\, f_0^{\otimes N}\, dX_0\, dV_0\, d\Theta_0 \\ = -\int_{\Omega^N} \frac{1}{2N} \sum_{i,k=1}^{N} \phi(\bar{x}_i - \bar{x}_k) |(\theta_i - \bar{\theta}_i) - (\theta_k - \bar{\theta}_k)|^2\, f_0^{\otimes N}\, dX_0\, dV_0\, d\Theta_0.$$

For I_1, we obtain, using Proposition 1.3.1,

$$\begin{aligned}
|I_1|^2 &\leqslant 2\mathcal{C}\int_{\Omega^N}\sum_{i=1}^N\left|\frac{1}{N}\sum_{k=1}^N[\phi(x_i-x_k)-\phi(\bar{x}_i-\bar{x}_k)](\theta_k-\theta_i)\right|^2 f_0^{\otimes N}\,dX_0\,dV_0\,d\Theta_0\\
&\leqslant 2|\nabla\phi|_\infty^2\mathcal{C}\int_{\Omega^N}\sum_{i=1}^N\left(\frac{1}{N}\sum_{k=1}^N|(x_i-x_k)-(\bar{x}_i-\bar{x}_k)||\theta_k-\theta_i|\right)^2 f_0^{\otimes N}\,dX_0\,dV_0\,d\Theta_0\\
&\leqslant ce^{-2\delta t}\mathcal{C}\int_{\Omega^N}\sum_{i=1}^N\left(\frac{1}{N}\sum_{k=1}^N(|x_i-\bar{x}_i|+|x_k-\bar{x}_k|)\right)^2 f_0^{\otimes N}\,dX_0\,dV_0\,d\Theta_0\\
&\leqslant ce^{-2\delta t}\mathcal{C}\int_{\Omega^N}\sum_{i=1}^N\left(|x_i-\bar{x}_i|^2+\frac{1}{N}\sum_{k=1}^N|x_k-\bar{x}_k|^2\right) f_0^{\otimes N}\,dX_0\,dV_0\,d\Theta_0\\
&= ce^{-2\delta t}\mathcal{C}\mathcal{P}.
\end{aligned}$$

Thus,

$$|I_1|\leqslant ce^{-\delta t}\mathcal{C}^{1/2}\mathcal{P}^{1/2}. \tag{1.69}$$

For I_3, we have

$$
\begin{aligned}
|I_3|^2 \leqslant 2\mathcal{C} \int_{\Omega^N} \sum_{i=1}^N \left| \frac{1}{N} \sum_{k=1}^N \phi(\bar{x}_i - \bar{x}_k)(\bar{\theta}_k - \bar{\theta}_i) - \int_\Omega \phi(\bar{x}_i - y)(\eta - \bar{\theta}_i) f(y, w, \eta, t)\, dy\, dw\, d\eta \right|^2 \\
\times f_0^{\otimes N}(X_0, V_0, \Theta_0)\, dX_0\, dV_0\, d\Theta_0 \\
= 2\mathcal{C} \int_{\Omega^N} \sum_{i=1}^N \left| \frac{1}{N} \sum_{k=1}^N \phi(\bar{x}_i - \bar{x}_k)(\bar{\theta}_k - \bar{\theta}_i) - \int_\Omega \phi(\bar{x}_i - y)(\eta - \bar{\theta}_i) f(y, w, \eta, t)\, dy\, dw\, d\eta \right|^2 \\
\times f^{\otimes N}(\bar{X}, \bar{V}, \bar{\Theta}, t) d\bar{X} d\bar{V} d\bar{\Theta} \\
= 2\mathcal{C}N \int_{\Omega^N} \left| \frac{1}{N} \sum_{k=1}^N \phi(\bar{x}_1 - \bar{x}_k)(\bar{\theta}_k - \bar{\theta}_1) - \int_\Omega \phi(\bar{x}_1 - y)(\eta - \bar{\theta}_1) f(y, w, \eta, t)\, dy\, dw\, d\eta \right|^2 \\
\times f^{\otimes N}(\bar{X}, \bar{V}, \bar{\Theta}, t) d\bar{X} d\bar{V} d\bar{\Theta} \\
\leqslant 2\mathcal{C}N \frac{4}{N} \sup_{(\bar{x}, \bar{v}, \bar{\theta}), (\bar{x}', \bar{v}', \bar{\theta}') \in \operatorname{supp} f_t} |\phi(\bar{x} - \bar{x}')(\bar{\theta} - \bar{\theta}')|^2 \leqslant c\mathcal{C} e^{-2\delta t}.
\end{aligned}
$$

Here in the penultimate step we used again (24, Lemma 3.3). Therefore,

$$
|I_3| \leqslant c e^{-\delta t} \mathcal{C}^{1/2}. \tag{1.70}
$$

Combining the three estimates for I_1, I_2, I_3, we obtain

$$
\frac{d}{dt}\mathcal{C} \leqslant c e^{-\delta t} (1 + \mathcal{P}^{1/2}) \mathcal{C}^{1/2}. \tag{1.71}
$$

Setting $p = 1+\mathcal{P}^{1/2}$, $k = \mathcal{K}^{1/2}$, $q = \mathcal{C}^{1/2}$. By (Equation 1.22), (Equation 1.68) and (Equation 1.71) we obtain the system of ODIs:

$$\begin{cases} \dot{p} & \leqslant k, \quad p_0 = 1, \\ \dot{k} & \leqslant ce^{-\delta t}(p+k) + cq, \quad k_0 = 0, \\ \dot{q} & \leqslant ce^{-\delta t}p, \quad q_0 = 0. \end{cases} \tag{1.72}$$

Claim 1.3.11. *For any nonnegative solution* (p, k, q) *to* (Equation 1.72)*, there exists a constant* C *depending on* c, δ *such that*

$$p \leqslant 1 + Ct^2, \quad k \leqslant Ct, \quad z \leqslant C\min\{1, t\}. \tag{1.73}$$

Proof of the Claim 1.3.11. Fix $\varepsilon, \tau > 0$ to be chosen later. We have

$$\begin{cases} \frac{d}{dt}(\varepsilon p^2) \leqslant 2\varepsilon pky \leqslant \sqrt{\varepsilon}(\varepsilon p^2 + k^2), \\ \frac{d}{dt}k^2 \leqslant ce^{-\delta t}(2pk + 2k^2) + 2ckq \leqslant ce^{-\delta t}\left[\frac{1}{\sqrt{\varepsilon}}(\varepsilon p^2 + k^2) + 2k^2\right] + \frac{c}{\sqrt{\tau}}(p^2 + \tau q^2), \\ \frac{d}{dt}(\tau q^2) \leqslant 2\tau ce^{-\delta t}pq \leqslant \frac{c\sqrt{\tau}e^{-\delta t}}{\sqrt{\varepsilon}}(\varepsilon p^2 + \tau q^2). \end{cases}$$

It implies that

$$\frac{d}{dt}(\varepsilon p^2 + k^2 + \tau q^2) \leqslant c(\tau, \varepsilon)e^{-\delta t}(\varepsilon p^2 + k^2 + \tau q^2) + \left(\sqrt{\varepsilon} + \frac{c}{\sqrt{\tau}}\right)(\varepsilon p^2 + k^2 + \tau q^2).$$

Applying Grönwall's lemma we get

$$\varepsilon p^2+k^2+\tau q^2 \leqslant \varepsilon \exp\left(\left(\sqrt{\varepsilon}+\frac{c}{\sqrt{\tau}}\right)t+\frac{c(\varepsilon,\tau)}{\delta}(1-e^{-\delta t})\right) \leqslant \varepsilon \exp\left(\left(\sqrt{\varepsilon}+\frac{c}{\sqrt{\tau}}\right)t+\frac{c(\varepsilon,\tau)}{\delta}\right).$$

Now choosing $\varepsilon = \delta^2/4, \tau = 4c^2/\delta^2$, we obtain

$$p \lesssim e^{\delta t/2}.$$

Plugging it into the third equation in (Equation 1.72) and solving for q we have

$$q \leqslant c\int_0^t e^{-\delta s/2}\,ds \leqslant C\min\{1,t\}.$$

Substituting p, q into the second equation in (Equation 1.72) we have

$$\frac{d}{dt}k \leqslant ce^{-\delta t}k + ce^{-\delta t/2} + C\min(1,t).$$

It implies that

$$k \leqslant Ct.$$

Hence, by the first equation in (Equation 1.72) we get

$$p \leqslant 1 + Ct^2.$$

The Claim 1.3.11 follows that

$$\mathcal{P} \leqslant \mathsf{C}\mathsf{t}^4, \quad \mathcal{K} \leqslant \mathsf{C}\mathsf{t}^2, \quad \mathcal{C} \leqslant \mathsf{C}\min\{1, \mathsf{t}^2\}.$$

On the other hand, in view of the global estimates on the support of the flock, $\mathcal{P} \leqslant \mathsf{C}\mathsf{N}$. Due to the alignment we also have $\mathcal{K} \leqslant \mathsf{C}\mathsf{N}$. Therefore,

$$\mathcal{P} + \mathcal{K} + \mathcal{C} \leqslant \mathsf{C}\min\{\mathsf{N}, \mathsf{t}^4\},$$

as desired. $\square$

CHAPTER 2

CONTINUOUS MODEL OF OPINION DYNAMICS WITH CONVICTIONS

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2.1 Introduction

In this chapter, we study regularity and long time behavior of solutions to the following transport equation

$$\partial_t \mu + \partial_y (u(\mu)\mu) = 0, \tag{2.1}$$

where $\mu = \mu(t, y, \theta)$ is a measure on $\Omega = \mathbb{R}_+ \times \mathbb{R}_+$ for each $t \geqslant 0$, and

$$u(\mu) = \partial_y (W * \mu + \sigma V), \tag{2.2}$$

$$W(y) = -\frac{1}{2} y^2, \quad V(y, \theta) = \frac{1}{2}\theta y^2 - \frac{1}{p+2} y^{p+2}. \tag{2.3}$$

Here, σ and p are positive parameters. The variable θ can be thought of as a parameter as well, however, note that the convolution $W * \mu$ couples all the measures together across the family.

The motivation for this particular model is twofold. First, it represents the kinetic counterpart of the corresponding discrete dynamical system:

$$\dot{y}_i = \frac{1}{N}\sum_{k=1}^{N}(y_k - y_i) + \sigma(\theta_i - y_i^p)y_i, \tag{2.4}$$

where θ_i's are constant parameters. In fact, the empirical distributions

$$\mu^N = \frac{1}{N}\sum_{i=1}^{N}\delta_{\theta_i}\otimes\delta_{y_i(t)} \tag{2.5}$$

solve (Equation 2.1) in the weak sense if and only if y_i's solve (Equation 2.4), and formally the mean-field limit $\mu^N \to \mu$ yields a solution to (Equation 2.1). The discrete system (Equation 2.4) was derived in (22) as the effective limiting dynamics of the speeds $y_i = |v_i|$ of agents governed by the corresponding alignment model with all-to-all communication and Rayleigh friction/self-propulsion force

$$\dot{x}_i = v_i, \qquad \dot{v}_i = \frac{1}{N}\sum_{k=1}^{N}(v_k - v_i) + \sigma(\theta_i - |v_i|^p)v_i. \tag{2.6}$$

When all velocities v_i belong to a sector of opening less than π, the vectors v_i will dynamically align themselves along one direction $v_i \sim y_i\hat{v}$, where $y_i = |v_i|$, and the evolution of y_i is governed by (Equation 2.4) up to an exponentially decaying force.

The system (Equation 2.6) is a very important example of a collective behavior model of Cucker-Smale type that was introduced in (9; 10) and studied under this particular forcing in the earlier works (16; 22; 8; 27). Kinetic limits in the context of forced systems including potential interaction and friction/self-propulsion were established in (6; 5; 3; 7). The first order conservation models of type (Equation 2.1) appeared in the context of aggregation models in the works of Topaz et al (35; 36). All these works correspond to the non-parametric case, i.e. $\theta = \text{const}$, where friction force appears. The variable θ case, beyond the work (22), was considered more recently in (27) where propagation of chaos with quantified rate was established for sectorial solutions, as described above, to the full Cucker-Smale system.

Our second motivation for this study comes from interpretation of the equation (Equation 2.1) as a continuous model of opinion dynamics. To put it in prospective of a vast existing literature let us compare it to several related models. The classical Hegselmann-Krause model (20) focuses on exchange of opinions only under local environmental averaging protocol – one that is based on interactions of agents with close views. A more elaborate protocol of opinion updates based on randomization of interaction schemes between groups were studied in works of Galam, see (13) and references there in. Equations (Equation 2.1), (Equation 2.4) belong to a class of models that incorporate 'conviction' parameter θ whose role is to pull the opinion of an agent to its value while remaining unchanged. As far as we can trace such models, also called models with 'stubborn' agents, appeared first in the work of Friedkin and Johnsen (12) and later became a staple in many studies on opinion dynamics, see for example (2; 14) and literature therein. In those works, however, the conviction pull is defined by a linear force, which in our notation would correspond to a constant multiple of $\theta_i - y_i$. The model proposed here uses the most basic all-to-all communication rule, but it incorporates the nonlinear conviction force. Phenomenologically it describes the effect of strengthening the pull towards conviction as the latter becomes more extreme. Such a model is necessarily not Galilean invariant and is fully non-linear, which makes the analysis of an 'agreement' or even its existence a challenging problem.

For the discrete variant (Equation 2.4) the problem was addressed in (22) where the model was interpreted as a non-cooperative game in the sense of Nash (23). The limiting state of opinions is characterized as a Nash equilibrium – an agreement deviation from which is of no benefit to any player, although may not necessarily be the most optimal value to anyone. Clearly, such an agreement is not expected to be a perfect consensus due to adherance to convictions. The existence, uniqueness and stability of the equilibrium was proved in (22) using the Brouwer topological degree theory.

Theorem 2.1.1. *For any positive set of parameters* $(\theta_1, \dots, \theta_N, \sigma) \in \mathbb{R}_+^N \times \mathbb{R}_+$ *there exists a unique stable Nash equilibrium* $y^* = (y_1^*, \dots, y_N^*) \in \mathbb{R}_+^N$ *of system* (Equation 2.4) *relative to payoffs*

$$p_i(y) = \sigma\left(\frac{1}{2}\theta_i y_i^2 - \frac{1}{p+2} y_i^{p+2}\right) - \frac{1}{2}(\bar{y} - y_i)^2, \quad \bar{y} = \frac{1}{N}\sum_j y_j. \tag{2.7}$$

Any solutions with positive initial data will remain positive and converge to y^* *as* $t \to \infty$. *Moreover, if* $\theta_i = \theta_j$ *then* $y_i = y_j$.

The main difficulty in establishing the result is that the natural gradient structure of (Equation 2.4)

$$\dot{y} = -\nabla\Phi(y)$$

involves energy $\Phi(y) = \sum_{i=1}^N p_i(y)$ that is not globally convex.

The purpose of this present study is to recreate a similar result for the kinetic model (Equation 2.1). First, we justify it as the mean-field model of (Equation 2.4) by establishing the limit $\mu^N \to \mu$. Such analysis is rather standard for first-order models, which is done by proving a general weak-Lipschitzness of the solution map $\mu_0 \to \mu_t$ with respect to the Wasserstein-1 metric, (1),

$$\mathcal{W}_1(\mu_t, \nu_t) \leqslant Ce^{ct}\mathcal{W}_1(\mu_0, \nu_0), \quad t > 0,$$

see Section 2.2. However, the details include a quantitative maximum principle of Lemma 2.2.2 that will be used later in the paper. So, we present the argument in full.

Our primary focus will be on the analysis of the Nash equilibrium of the continuous model (Equation 2.1). To state the main result let us fix some notation. Let us observe that the θ-marginal given by

$$d\pi(\theta, t) = \int_{y \in \mathbb{R}_+} d\mu(y, \theta, t), \tag{2.8}$$

is conserved $\frac{d}{dt}\pi = 0$. This is a reflection of the principle that convictions do not change. By the disintegration theorem, see (1), for π-a.e. $\theta \in \mathbb{R}_+$ there is a unique family of probability 'slicing' measures $\{\mu^\theta\}_{\theta \in \mathbb{R}_+}$ such that $\mu = \mu^\theta \otimes d\pi(\theta)$, that is,

$$\int_\Omega \varphi(y, \theta) d\mu(y, \theta) = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \varphi(y, \theta) d\mu^\theta(y) d\pi(\theta), \quad \forall \varphi \in C_0(\Omega). \tag{2.9}$$

Each measure μ^θ represents distribution of opinions of agents that share the same conviction θ.

Our main result states that each of these slicing measures approaches a mono-opinion state, i.e. a Dirac measure at a fixed point $g(\theta)$ for some smooth strictly increasing function g. In other words,

$$\mu_t \to \delta_{g(\theta)} \otimes d\pi(\theta), \quad t \to \infty.$$

To put it formally we assume that our initial measure is located within a box compactly inside Ω:

$$\operatorname{supp} \mu_0 \subset R_0 := [y_{\min}, y_{\max}] \times [\theta_{\min}, \theta_{\max}], \qquad y_{\min}, \theta_{\min} > 0. \tag{2.10}$$

Theorem 2.1.2. *Let* μ *be the measure-valued solution to* (Equation 2.1) *with initial data satisfying* (Equation 2.10). *Then there exists a function* $g \in C^\infty([\theta_{\min}, \theta_{\max}])$ *strictly increasing such that*

$$\sup_{\theta \in \operatorname{supp} \pi} \mathcal{W}_1(\mu_t^\theta, \delta_{g(\theta)}) \leqslant Ce^{-ct}, \quad t > 0, \tag{2.11}$$

where $C, c > 0$ *depend only on* μ_0 *and the parameters of the model. Moreover, under the assumption*

$$\sigma\theta_{\min} > \frac{p+1}{p} \quad \textit{or} \quad \frac{\theta_{\max}}{\theta_{\min}} < p+1. \tag{2.12}$$

the map $\pi \to g$ *is Lipschitz,*

$$\sup_{\theta \in [\theta_{\min}, \theta_{\max}]} |g(\theta) - \tilde{g}(\theta)| \leqslant C\mathcal{W}_1(\pi, \tilde{\pi}). \tag{2.13}$$

In particular g *is unique for each* π.

Structurally, the equation (Equation 2.1) can be considered as a fibered gradient system in the sense of (29) where the fibers are parametrized by convictions θ and the free energy is given by

$$\mathcal{E}(\mu) = \frac{1}{2}\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} W(x-y)d\mu(y,\theta)d\mu(x,\eta) - \sigma\int_{\mathbb{R}^2_+} V(y,\theta)d\mu(y,\theta).$$

The equation can be written as a gradient dynamics

$$\partial_t\mu = -\partial\mathcal{E}(\mu),$$

where ∂ is understood as a fibered variant of the Fréchet subdifferential relative to a properly defined fibered Wasserstein distance. Without getting further into details one can obtain directly the following energy dissipation law

$$\frac{d}{dt}\mathcal{E} = -\int_{\mathbb{R}_+\times\mathbb{R}_+} |u(\mu)|^2 d\mu(y,\theta).$$

The law demonstrates perpetual descent of the solution down the energy surface and suggests convergence to a local minimum. The general results of this nature were established in (29) under a properly formulated convexity condition on the energy. However, just as in the discrete case, such convexity is not always true in our settings. Therefore, the statement of Theorem 2.1.2 does not directly follow from the theory developed in (29). Our method is based on the Lagrangian approach, which involves detailed analysis of asymptotic behavior of characteristics of (Equation 2.1). Let us note that in the discrete case the uniqueness of the limiting state is unconditional. Removing assumptions (Equation 2.12) for the kinetic model remains an open issue.

2.2 Well-posedness and mean-field limit

In this section, we will prove the existence of measure-valued solutions to the equation (Equation 2.1). First of all, let us introduce some notations and definitions. Let $\Omega = \mathbb{R}^2_+$ and denote $\mathcal{P}_0(\Omega)$ the set of probability measures on Ω which have compact support in the interior of Ω.

Definition 2.2.1. *Given* $0 \leqslant T < \infty$*, a map* $\mu : [0,T) \to \mathcal{P}_0(\Omega)$, $t \mapsto \mu_t$*, is called a measure-valued solution to* (Equation 2.1) *with initial data* μ_0 *if it satisfies the following conditions:*

i) μ *is weakly* continuous,*

ii) For any $\varphi \in C_0^\infty([0,T) \times \Omega)$ *and* $0 < t < T$,

$$\int_\Omega \varphi(t,y,\theta)\, d\mu_t(y,\theta) = \int_\Omega \varphi(0,y,\theta)\, d\mu_0(y,\theta) + \int_0^t \int_\Omega [\partial_s \varphi + u\, \partial_y \varphi]\, d\mu_s(y,\theta)\, ds.$$

Let us note that we do not make any specific assumptions about the class of measures we consider as solutions. In particular, μ is purely atomic, see (Equation 2.5) then it is easy to check that the definition of a solution is equivalent to the ODE (Equation 2.4).

To make further notation simpler let us observe that by making the change of variables

$$y \to \sigma^{\frac{1}{p}} y, \qquad \theta \to \sigma\theta, \qquad \mu \to \sigma^{1+\frac{1}{p}} \mu, \tag{2.14}$$

we can scale out the parameter σ from the equation altogether. So, from now on we can assume that $\sigma = 1$, and be mindful that all the constants that appear later eventually depend on the original parameter σ.

If $\mu : [0, T) \to \mathcal{P}_0(\Omega)$ is a measure-valued solution to (Equation 2.1) with initial data μ_0, by the classical transport theory, μ is a push-forward of μ_0 along characteristics (Y, Θ):

$$\frac{d}{dt} Y(t, y, \theta) = \int_\Omega (Y' - Y) d\mu_0(y', \theta') + Y(\Theta - Y^p), \quad Y(0, y, \theta) = y, \tag{2.15}$$

$$\frac{d}{dt} \Theta(t, y, \theta) = 0, \quad \Theta(0, y, \theta) = \theta. \tag{2.16}$$

Note that Θ is not changing in time, so in the equation (Equation 2.15) we can replace Θ by its initial θ and view θ as a parameter.

The local well-posedness of the system (Equation 2.15) - (Equation 2.16) follows from the standard fixed point argument for integro-differential equations and local Lipschitzness relative to continuous maps (Y, Θ) of the right hand side. Global well-posedness will follow as soon as we establish a priori bounds on the support of Y.

Our standing assumption on the initial support of μ_0 will always be (Equation 2.10). Let us denote

$$Y_{\max}(t) = \max_{R_0} Y(t, \cdot), \quad Y_{\min}(t) = \min_{R_0} Y(t, \cdot).$$

Note that $y_{\max} = Y_{\max}(0)$ and $y_{\min} = Y_{\min}(0)$.

Lemma 2.2.2. *For any solution* Y *to* (Equation 2.15) *on a time interval* $[0, T)$, *we have for all* $t < T$,

$$Y_{\max}^p \leqslant \frac{\theta_{\max} y_{\max}^p e^{p\theta_{\max} t}}{\theta_{\max} + y_{\max}^p (e^{p\theta_{\max} t} - 1)}, \tag{2.17}$$

$$Y_{\min}^p \geqslant \frac{\theta_{\min} y_{\min}^p e^{p\theta_{\min} t}}{\theta_{\min} + y_{\min}^p (e^{p\theta_{\min} t} - 1)}. \tag{2.18}$$

Proof. Evaluating (Equation 2.15) at a point of maximum on R_0, using Rademacher's lemma (see (31)), we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} Y_{\max}^p &= pY_{\max}^{p-1} \underbrace{\int_\Omega (Y' - Y_{\max}) d\mu_0(y', \theta')}_{\leqslant 0} + Y_{\max}^p (\theta - Y_{\max}^p) \\ &\leqslant pY_{\max}^p (\theta_{\max} - Y_{\max}^p). \end{aligned}$$

The right hand side of (Equation 2.17) solves the above equation exactly. So, by the classical comparison principle, we obtain (Equation 2.17).

Similarly,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} Y_{\min}^p &= pY_{\min}^{p-1} \underbrace{\int_\Omega (Y' - Y_{\min}) d\mu_0(y', \theta')}_{\geqslant 0} + Y_{\min}^p (\theta - Y_{\min}^p) \\ &\geqslant pY_{\min}^p (\theta_{\min} - Y_{\min}^p). \end{aligned}$$

The comparison principle implies (Equation 2.18). $\square$

The lemma shows that on any finite time interval the characteristics will not leave Ω and in fact the image $Y(t, \operatorname{supp} \mu_0)$ will be compactly embedded in Ω and remain uniformly bounded a priori. Consequently, by extension, the system (Equation 2.15) - (Equation 2.16) is globally well-posed. By the push-forward transport, there is a global measure-valued solution to (Equation 2.1).

Theorem 2.2.3. *Given any measure* $\mu_0 \in \mathcal{P}_0(\Omega)$ *with* (Equation 2.10) *there exists a unique measure-valued solution to* (Equation 2.1) *with initial condition* μ_0 *and such that* $\operatorname{supp} \mu_t \subset \Omega$ *remains bounded and bounded away from* $\partial\Omega$ *uniformly for all times.*

Let us now show continuity of the map $\mu_0 \to \mu_t$ in weak topology, which is the basis for justification of the mean-field limit.

Lemma 2.2.4. *Let* μ *and* ν *be two measure-valued solutions to* (Equation 2.1) *with* μ_0, ν_0 *satisfying* (Equation 2.10)*. Then for any* $t > 0$ *one has*

$$\mathcal{W}_1(\mu_t, \nu_t) \leqslant C e^{ct} \mathcal{W}_1(\mu_0, \nu_0),$$

where $C, c > 0$ *depend on the initial condition and the parameters of the model.*

Proof. Denote $L^\infty := L^\infty(R_0)$. Let us also denote by Y the characteristics of μ and by Z the characteristics of ν.

In what follows, C and c are constants which are varying line by line. By the definition of the Wasserstein distance, we have

$$\begin{aligned}
\mathcal{W}_1(\mu_t, \nu_t) &= \sup_{\|\varphi\|_{\mathrm{Lip}} \leqslant 1} \left| \int_\Omega \varphi(y,\theta) d\mu_t(y,\theta) - \int_\Omega \varphi(y,\theta) d\nu_t(y,\theta) \right| \\
&= \sup_{\|\varphi\|_{\mathrm{Lip}} \leqslant 1} \left| \int_\Omega \varphi(Y,\theta) d\mu_0(y,\theta) - \int_\Omega \varphi(Z,\theta) d\nu_0(y,\theta) \right| \\
&= \sup_{\|\varphi\|_{\mathrm{Lip}} \leqslant 1} \left| \int_\Omega \varphi(Y,\theta) d\mu_0(y,\theta) - \int_\Omega \varphi(Y,\theta) d\nu_0(y,\theta) + \int_\Omega [\varphi(Y,\theta) - \varphi(Z,\theta)] d\nu_0(y,\theta) \right| \\
&\leqslant (1 + \|\nabla Y\|_\infty) \mathcal{W}_1(\mu_0, \nu_0) + \int_\Omega |Y - Z| d\nu_0(y,\theta) \\
&\leqslant (1 + \|\nabla Y\|_\infty) \mathcal{W}_1(\mu_0, \nu_0) + \|Y - Z\|_\infty.
\end{aligned} \tag{2.19}$$

The proof reduces to the estimation of $\|\nabla Y\|_\infty$ and $\|Y - Z\|_\infty$.

Taking the gradient

$$\nabla Y = (\partial_y Y, \partial_\theta Y)$$

of (Equation 2.15) we obtain

$$\frac{d}{dt} \nabla Y = -\nabla Y + \theta \nabla Y + (0, Y) - (p+1) Y^p \nabla Y.$$

Evaluating at a point where $\|\nabla Y\|_\infty$ is achieved, by Rademacher's lemma, we have

$$\frac{d}{dt} \|\nabla Y\|_\infty \leqslant -(1-\theta) \|\nabla Y\|_\infty - (p+1) Y^p \|\nabla Y\|_\infty + \|Y\|_\infty. \tag{2.20}$$

By (Equation 2.17),

$$\frac{d}{dt}\|\nabla Y\|_\infty \leqslant C\|\nabla Y\|_\infty + C,$$

and hence,

$$\|\nabla Y\|_{L^\infty} \leqslant Ce^{ct}. \tag{2.21}$$

Now let us compute the derivative of $\|Y - Z\|_\infty$. We have

$$\begin{aligned}
\frac{d}{dt}(Y - Z) = &\int_\Omega (Y' - Y) d\mu_0(y', \theta') - \int_\Omega (Z' - Z) d\nu_0(y', \theta') \\
&+ (\theta - Y^p)Y - (\theta - Z^p)Z \\
= &\int_\Omega Y' d\mu_0(y', \theta') - \int_\Omega Y' d\nu_0(y', \theta') + \int_\Omega Y' d\nu_0(y', \theta') - \int_\Omega Z' d\nu_0(y', \theta') \\
&+ (\theta - 1)(Y - Z) - (Y^{p+1} - Z^{p+1}).
\end{aligned}$$

Evaluating at a point of maximum and noting that $Y^{p+1} - Z^{p+1} = (p+1)\tilde{Y}^p(Y - Z)$ for some $\tilde{Y}$ between Y and Z we obtain

$$\begin{aligned}
\frac{d}{dt}\|Y - Z\|_\infty &\leqslant \|\nabla Y\|_\infty \mathcal{W}_1(\mu_0, \nu_0) + (|\theta - 1| + 1)\|Y - Z\|_\infty - (p+1)\tilde{Y}^p\|Y - Z\|_\infty \\
&\leqslant \|\nabla Y\|_\infty \mathcal{W}_1(\mu_0, \nu_0) + C\|Y - Z\|_\infty.
\end{aligned}$$

Combining with (Equation 2.21) and by Grönwall's lemma, it implies that

$$\|Y - Z\|_\infty \leqslant Ce^{ct}\mathcal{W}_1(\mu_0, \nu_0). \tag{2.22}$$

where c is a constant depending on σ and the supports of μ_0, ν_0 with respect to θ. Therefore, plugging (Equation 2.22) and (Equation 2.21) into (Equation 2.19) we obtain

$$\mathcal{W}_1(\mu_t, \nu_t) \leqslant Ce^{ct}\mathcal{W}_1(\mu_0, \nu_0)$$

which concludes the lemma. □

For any $N \in \mathbb{N}$, if $\{(y_i, \theta_i)\}_{i=1,\dots,N}$ is a solution to the system (Equation 2.4) with the initial conditions $y_i(0) = y_i^0, \theta_i(0) = \theta_i$, then

$$\mu_t^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{y_i(t)} \otimes \delta_{\theta_i},$$

is a measure-valued solution to (Equation 2.1) with the initial condition

$$\mu_0^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{y_i^0} \otimes \delta_{\theta_i}.$$

So, if $\mu_0^N \to \mu_0$ weakly, then by Lemma 2.2.4, $\mu_t^N \to \mu_t$, for any $t > 0$. Which justifies the weak approximation by empirical measures.

This method can be used to give an alternative proof of global existence for (Equation 2.1) without the use of general characteristics Y and simply based on the fact that the discrete system (Equation 2.4) is globally well-posed.

Another proof of Theorem 2.2.3. Let us pick any weak*-approximation of μ_0 by empirical measures

$$\mu_0^N = \sum_{k=1}^{N} m_k \delta_{y_k^0} \otimes \delta_{\theta_k} \to \mu_0.$$

Let

$$\mu_t^N := \sum_{k=1}^{N} m_k \delta_{y_k(t)} \otimes \delta_{\theta_k}.$$

Since μ^N is a measure-valued solution to (Equation 2.1) with the initial data μ_0^N we apply Lemma 2.2.4 to get

$$\mathcal{W}_1(\mu_t^N, \mu_t^M) \leqslant Ce^T \mathcal{W}_1(\mu_0^N, \mu_0^M), \quad \text{for } N, M > 0,\ t \leqslant T.$$

Hence $\{\mu_t^N\}_N$ is weakly*-Cauchy in the complete metric space $(\mathcal{P}_+(\Omega), \mathcal{W}_1)$, and consequently there is a limit $\mu_t^N \to \mu_t \in \mathcal{P}_+(\Omega)$, and moreover

$$\mathcal{W}_1(\mu_t^N, \mu_t) \leqslant C_T \mathcal{W}_1(\mu_0^N, \mu_0), \quad \text{for } N > 0,\ t \leqslant T. \tag{2.23}$$

Now we prove the weak*-continuity of the map $t \to \mu_t$. Note that for $\psi \in C_0^\infty(\Omega)$ the sequence $\{\int_\Omega \psi(y,\theta)d\mu_t^N(y,\theta)\}_N$ is uniformly Lipschitz continuous on $[0,T]$. Indeed, for $t \in [0,T)$ and $\Delta t > 0$ with $t+\Delta t \in [0,T]$ we have

$$\begin{aligned}\left|\int_\Omega \psi(y,\theta)d\mu_{t+\Delta t}^N(y,\theta) - \int_\Omega \psi(y,\theta)d\mu_t^N(y,\theta)\right| \leqslant & \int_\Omega \left|\psi(Y^N(t+\Delta t),\theta) - \psi(Y^N(t),\theta)\right| d\mu_0^N(y,\theta) \\ \leqslant & |\nabla\psi|_\infty \int_\Omega \left|Y^N(t+\Delta t) - Y^N(t)\right| d\mu_0^N(y,\theta) \\ \leqslant & C\Delta t,\end{aligned}$$

where Y^N denotes the characteristics of μ^N. For the last inequality we used the uniform Lipschitzness of $\{Y^N\}_N$ on $[0,T]$. Letting $N \to +\infty$, we have

$$\left|\int_\Omega \psi(y,\theta)d\mu_{t+\Delta t}(y,\theta) - \int_\Omega \psi(y,\theta)d\mu_t(y,\theta)\right| \leqslant C\Delta t,$$

which implies the weak*-continuity of the map $t \to \mu_t$.

We will show that this μ is a measure-valued solution to (Equation 2.1) with the given initial μ_0.

Because μ^N is a measure-valued solution, for any test function $\varphi \in C_0^\infty([0,T) \times \Omega)$,

$$\int_\Omega \varphi(t,y,\theta)d\mu_t^N(y,\theta) = \int_\Omega \varphi(0,y,\theta)d\mu_0^N(y,\theta) + \int_0^t \int_\Omega [\partial_s\varphi + u_s^N \partial_y \varphi]d\mu_s^N(y,\theta)\, ds, \quad (2.24)$$

where

$$u_s^N = \int_\Omega y' d\mu_s^N(y',\theta') - y + (\theta - y^p)y := P^N(s) + F(y,\theta).$$

All linear terms weakly converge to the natural limits. Since F is a fixed continuous function we also have

$$\int_0^t \int_\Omega F\partial_y \varphi d\mu_s^N(y,\theta)\,ds \longrightarrow \int_0^t \int_\Omega F\partial_y \varphi d\mu_s(y,\theta)\,ds \qquad \text{as } N \to \infty.$$

Note that the moments $P^N(s)$ is just a sequence of numbers for which we have, by (Equation 2.23),

$$|P^N(s) - P(s)| = \left| \int_\Omega y'(d\mu_s^N(y',\theta') - d\mu_s(y',\theta')) \right| \leqslant \mathcal{W}_1(\mu_s^N,\mu_s) \leqslant C_T \mathcal{W}_1(\mu_0^N,\mu_0) \to 0.$$

So, $P^N \to P$ uniformly on $[0,T)$. Consequently,

$$\int_0^t \int_\Omega P^N(s)\partial_y \varphi d\mu_s^N(y,\theta)\,ds \to \int_0^t \int_\Omega P(s)\partial_y \varphi d\mu_s(y,\theta)\,ds.$$

It follows that μ satisfies (ii). □

2.3 Existence and uniqueness of the mono-opinion state

Let μ be a measure-valued solution to (Equation 2.1) with the initial μ_0. Let π be its time-independent conviction marginal (Equation 2.8).

Let us derive the equation for μ^θ. By Definition 2.2.1 and (Equation 2.9), for any $\varphi \in C_0^\infty([0,T) \times \Omega)$ and $0 < t < T$ one has

$$\begin{aligned}\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \varphi(t,y,\theta) d\mu_t^\theta(y) d\pi(\theta) = &\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \varphi(0,y,\theta) d\mu_0^\theta(y) d\pi(\theta) \\ &+ \int_0^t \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} [\partial_s \varphi + u_s \partial_y \varphi] d\mu_s^\theta(y) d\pi(\theta)\,ds.\end{aligned}$$

It implies that for π-almost every θ, the probability measure μ^θ is a measure-valued solution with the initial μ_0^θ to the equation

$$\partial_t \mu^\theta + \partial_y \left[u \mu^\theta \right] = 0, \tag{2.25}$$

where

$$u(t, y, \theta) = \int_\Omega (z - y) d\mu_t^\eta(z) d\pi(\eta) + (\theta - y^p) y.$$

Note that the family of equations are all coupled through the velocity u, but otherwise represent transport of each individual slicing measure μ^θ. The characteristics that transport μ^θ, denoted Y_θ are nothing but $Y_\theta(t, y) = Y(t, y, \theta)$ as defined by (Equation 2.15). We will view them, however, as individual trajectories satisfying the coupled system

$$\frac{d}{dt} Y_\theta = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} (Y'_{\theta'} - Y_\theta) d\mu_0^{\theta'}(y') d\pi(\theta') + (\theta - Y_\theta^p) Y_\theta. \tag{2.26}$$

In particular we will derive an individual comparison bound from below as an alternative to global (Equation 2.18).

Lemma 2.3.1. *For any* $\theta \in [\theta_{\min}, \theta_{\max}]$ *such that* $\theta > 1$ *one has*

$$Y_\theta^p(t, y) \geqslant \frac{y^p (\theta - 1) e^{p(\theta-1)t}}{(\theta - 1) + y^p (e^{p(\theta-1)t} - 1)}, \quad \forall t \geqslant 0, \ \forall y > 0. \tag{2.27}$$

Proof. To achieve (Equation 2.27) we decouple the system (Equation 2.26) by ignoring the entire coupling term

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} Y'_{\theta'} d\mu_0^{\theta'}(y') d\pi(\theta') \geqslant 0.$$

So,

$$\frac{d}{dt} Y_\theta^p \geqslant p\left(\theta - 1 - Y_\theta^p\right) Y_\theta^p. \tag{2.28}$$

The lemma follows from the comparison principle. □

Let us note that in principle the statement of the lemma holds for any $\theta - 1$, but it is most meaningful when the parameter is positive in view of the universal support from below for all characteristics (Equation 2.18).

2.3.1 Mono-opinion state

In the next step we will show that for each $\theta \in \operatorname{supp}\pi$, the slicing measure μ^θ will converge to a Dirac measure in Wasserstein distance with different rates depending on θ.

Lemma 2.3.2. *Let* μ *be the measure-valued solution to* (Equation 2.1) *satisfying* (Equation 2.10) *and* π *being the conviction marginal* (Equation 2.8). *Then there exists a function* $g \in \mathrm{Lip}[\theta_{\min}, \theta_{\max}]$ *such that*

$$\sup_{\theta \in \operatorname{supp}\pi} \mathcal{W}_1(\mu_t^\theta, \delta_{g(\theta)}) \leqslant Ce^{-ct}, \quad t > 0, \tag{2.29}$$

where $C, c > 0$ *depend only on* μ_0 *and the parameters of the model.*

Proof. Differentiating the characteristic equation (Equation 2.26) we obtain

$$\partial_t \partial_y Y_\theta = (\theta - 1)\partial_y Y_\theta - (p+1)Y_\theta^p \partial_y Y_\theta. \tag{2.30}$$

In what follows we denote $L^\infty = L^\infty(R_0)$. By Rademacher's lemma, at a point of maximum y such that $(y, \theta) \in R_0$, we get

$$\frac{d}{dt}\|\partial_y Y_\theta\|_\infty = (\theta - 1)\|\partial_y Y_\theta\|_\infty - (p+1)Y_\theta^p \|\partial_y Y_\theta\|_\infty. \tag{2.31}$$

Let us first consider the stable case when $\theta - 1 \leqslant \varepsilon_0$, with $\varepsilon_0 > 0$ to be determined later. Using (Equation 2.18) we find that $Y_\theta^p \geqslant c_0$, which is determined only by the initial condition and the parameters of the model. Plugging in (Equation 2.31), we obtain

$$\frac{d}{dt}\|\partial_y Y_\theta\|_\infty \leqslant \varepsilon_0\|\partial_y Y_\theta\|_\infty - (p+1)c_0\|\partial_y Y_\theta\|_\infty \leqslant -\varepsilon_0\|\partial_y Y_\theta\|_\infty \tag{2.32}$$

by setting $\varepsilon_0 = \frac{(p+1)c_0}{2}$.

For the unstable case $\theta - 1 \geqslant \varepsilon_0$, the inequality (Equation 2.27) implies that

$$\begin{aligned} Y_\theta^p &\geqslant \frac{y^p(\theta-1)e^{p(\theta-1)t}}{(\theta-1) + y^p e^{p(\theta-1)t}} = \theta - 1 - \frac{(\theta-1)^2}{(\theta-1)+y^p e^{p(\theta-1)t}} \\ &\geqslant \theta - 1 - (\theta-1)^2 y^{-p} e^{-p(\theta-1)t}. \end{aligned}$$

Therefore, in this case we have

$$Y_\theta^p \geqslant \theta - 1 - c_1 e^{-c_2 t}, \tag{2.33}$$

where $c_1, c_2 > 0$ depend only on the initial condition and parameters of the model. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_y Y_\theta\|_\infty \leqslant (\theta-1)\|\partial_y Y_\theta\|_\infty - (p+1)(\theta-1-c_1e^{-c_2t})\|\partial_y Y_\theta\|_\infty \leqslant \left(-p\varepsilon_0 + c_1e^{-c_2t}\right)\|\partial_y Y_\theta\|_\infty.$$

In either case we obtain, by Grönwall's lemma,

$$\|\partial_y Y_\theta\|_{L^\infty} \leqslant c_3 e^{-c_4 t}. \tag{2.34}$$

Consequently,

$$|Y_\theta(y,t) - Y_\theta(y',t)| \leqslant c_5 e^{-c_4 t}, \qquad \text{for any } (y,\theta),(y',\theta) \in R_0. \tag{2.35}$$

We can see that the characteristics are squeezing as t approaches infinity. Since the trajectories are also precompact, for each $\theta \in [\theta_{\min}, \theta_{\max}]$ there exists $g(\theta)$ such that

$$\sup_{y\in[y_{\min},y_{\max}]} |Y_\theta(y,t) - g(\theta)| \leqslant c_5 e^{-c_4 t}.$$

We compute

$$\begin{aligned}
\mathcal{W}_1(\mu_t^\theta, \delta_{g(\theta)}) &= \sup_{\|\varphi\|_{\mathrm{Lip}} \leqslant 1} \left| \int_{\mathbb{R}_+} \varphi(y) d\mu_t^\theta(y) - \int_{\mathbb{R}_+} \varphi(y) \delta_{g(\theta)}(y) \right| \\
&= \sup_{\|\varphi\|_{\mathrm{Lip}} \leqslant 1} \left| \int_{\mathbb{R}_+} \varphi(Y_\theta) d\mu_0^\theta(y) - \varphi(g(\theta)) \right| \\
&= \sup_{\|\varphi\|_{\mathrm{Lip}} \leqslant 1} \left| \int_{\mathbb{R}_+} \big(\varphi(Y_\theta) - \varphi(g(\theta))\big) d\mu_0^\theta(y) \right| \\
&\leqslant \|Y_\theta - g(\theta)\|_\infty.
\end{aligned}$$

The statement (Equation 2.29) follows.

It remains to show that g is a Lipschitz function on $[\theta_{\min}, \theta_{\max}]$. Indeed, computing the evolution of $\partial_\theta Y_\theta$ we obtain

$$\partial_t \partial_\theta Y_\theta = Y_\theta + \big(\theta - 1 - (p+1) Y_\theta^p\big) \partial_\theta Y_\theta.$$

Note that Y_θ remains bounded on R_0 by Lemma 2.2.2, and the remainder of the equation has the same structure as in (Equation 2.30). So,

$$\frac{d}{dt} \|\partial_\theta Y_\theta\|_\infty \leqslant c_1 + (-c_2 + c_3 e^{-c_4 t}) \|\partial_\theta Y_\theta\|_\infty.$$

We obtain

$$\|\partial_\theta Y_\theta\|_\infty < C.$$

Consequently,

$$|Y(y,\theta,t) - Y(y,\theta',t)| \leqslant C|\theta - \theta'|.$$

Letting $t \to \infty$ we obtain

$$|g(\theta) - g(\theta')| \leqslant C|\theta - \theta'|.$$

This finishes the proof. □

2.3.2 Uniqueness and stability

The uniqueness of the limiting state follows from the lemma below and holds under either of the two conditions on parameters

$$\theta_{\min} > \frac{p+1}{p} \quad \text{or} \quad \frac{\theta_{\max}}{\theta_{\min}} < p+1. \tag{2.36}$$

Note that under the change (Equation 2.14) this translates into condition (Equation 2.12).

Lemma 2.3.3. *Let μ and $\tilde{\mu}$ be two solutions to* (Equation 2.1) *starting in a box R_0 and sharing the same conviction measure π. And suppose either of the assumptions* (Equation 2.36) *hold. Then for any $t \in [0,T)$ one has*

$$\sup_{\theta \in \operatorname{supp} \pi} \mathcal{W}_1(\mu_t^\theta, \tilde{\mu}_t^\theta) \leqslant c_1 e^{-c_2 t} \sup_{\theta \in \operatorname{supp} \pi} \mathcal{W}_1(\mu_0^\theta, \tilde{\mu}_0^\theta),$$

where $c_1, c_2 > 0$ depend on the initial data and parameters of the model.

Proof. In what follows $L^\infty := L^\infty([y_{\min}, y_{\max}])$. Denoting $\tilde{Y}_\theta, \tilde{Y}'_\theta$ the characteristics of $\tilde{\mu}^\theta$ starting from y, y' respectively. For fixed $\theta \in \operatorname{supp} \pi$,

$$\begin{aligned}
\mathcal{W}_1(\mu_t^\theta, \tilde{\mu}_t^\theta) &= \sup_{\|\varphi\|_{\mathrm{Lip}} \leqslant 1} \left| \int_{\mathbb{R}_+} \varphi(y) d\mu_t^\theta(y) - \int_{\mathbb{R}_+} \varphi(y) d\tilde{\mu}_t^\theta(y) \right| \\
&= \sup_{\|\varphi\|_{\mathrm{Lip}} \leqslant 1} \left| \int_{\mathbb{R}_+} \varphi(Y_\theta) d\mu_0^\theta(y) - \int_{\mathbb{R}_+} \varphi(\tilde{Y}_\theta) d\tilde{\mu}_0^\theta(y) \right| \\
&= \sup_{\|\varphi\|_{\mathrm{Lip}} \leqslant 1} \left| \int_{\mathbb{R}_+} \varphi(Y_\theta) d\mu_0^\theta(y) - \int_{\mathbb{R}_+} \varphi(Y_\theta) d\tilde{\mu}_0^\theta(y) + \int_{\mathbb{R}_+} [\varphi(Y_\theta) - \varphi(\tilde{Y}_\theta)] d\tilde{\mu}_0^\theta(y) \right| \\
&\leqslant \quad \|\partial_y Y_\theta\|_{L^\infty} \mathcal{W}_1(\mu_0^\theta, \tilde{\mu}_0^\theta) + \|Y_\theta - \tilde{Y}_\theta\|_{L^\infty}.
\end{aligned}$$

We proved the uniform exponential contraction for $\|\partial_y Y_\theta\|_{L^\infty}$ in (Equation 2.34).

Let us now focus on $\|Y_\theta - \tilde{Y}_\theta\|_{L^\infty}$. We have

$$\begin{aligned}
\frac{d}{dt}(Y_\theta - \tilde{Y}_\theta) &= \int_{\mathbb{R}_+} \left[\int_{\mathbb{R}_+} Y'_{\theta'} d\mu_0^{\theta'}(y') - \int_{\mathbb{R}_+} \tilde{Y}'_{\theta'} d\tilde{\mu}_0^{\theta'}(y') \right] d\pi(\theta') \\
&\quad + (\theta - 1)(Y_\theta - \tilde{Y}_\theta) - (Y_\theta^{p+1} - \tilde{Y}_\theta^{p+1}) \\
&= \int_{\mathbb{R}_+} \left[\int_{\mathbb{R}_+} Y'_{\theta'} (d\mu_0^{\theta'}(y') - d\tilde{\mu}_0^{\theta'}(y')) + \int_{\mathbb{R}_+} (Y'_{\theta'} - \tilde{Y}'_{\theta'}) d\tilde{\mu}_0^{\theta'}(y') \right] d\pi(\theta') \\
&\quad + (\theta - 1)(Y_\theta - \tilde{Y}_\theta) - (p+1)\hat{Y}_\theta^p (Y_\theta - \tilde{Y}_\theta),
\end{aligned}$$

where $\hat{Y}_\theta$ is between Y_θ and $\tilde{Y}_\theta$. Denote

$$\mathcal{D}(t) = \sup_{\theta \in [\theta_{\min}, \theta_{\max}]} \|Y_\theta - \tilde{Y}_\theta\|_{L^\infty}.$$

At a point of maximum we obtain using (Equation 2.34),

$$\frac{d}{dt}\mathcal{D} \leqslant c_3 e^{-c_4 t} \sup_{\theta \in \operatorname{supp} \pi} \mathcal{W}_1(\mu_0^\theta, \tilde{\mu}_0^\theta) + \theta\mathcal{D} - (p+1)\min\{Y_\theta^p, \tilde{Y}_\theta^p\}\mathcal{D}.$$

Using (Equation 2.33),

$$\begin{aligned}\frac{d}{dt}\mathcal{D} &\leqslant c_3 e^{-c_4 t} \sup_{\theta \in \operatorname{supp} \pi} \mathcal{W}_1(\mu_0^\theta, \tilde{\mu}_0^\theta) + \theta\mathcal{D} - (p+1)[\theta - 1 - c_1 e^{-c_2 t}]\mathcal{D} \\ &= c_3 e^{-c_4 t} \sup_{\theta \in \operatorname{supp} \pi} \mathcal{W}_1(\mu_0^\theta, \tilde{\mu}_0^\theta) + [p + 1 - p\theta + c_1 e^{-c_2 t}]\mathcal{D}\end{aligned}$$

The result follows provided $\theta_{\min} > \frac{p+1}{p}$. Alternatively, using the lower bound (Equation 2.18),

$$\frac{d}{dt}\mathcal{D} \leqslant c_3 e^{-c_4 t} \sup_{\theta \in \operatorname{supp} \pi} \mathcal{W}_1(\mu_0^\theta, \tilde{\mu}_0^\theta) + [\theta_{\max} - (p+1)\theta_{\min} + c_1 e^{-c_2 t}]\mathcal{D}$$

and the result follows provided $\frac{\theta_{\max}}{\theta_{\min}} < p + 1$. □

Under the stability assumption (Equation 2.36) the limiting states are also stable with respect to perturbation of convictions. So, a small change even in the weak topology of conviction marginal π results in a small change in the limiting mono-opinion state. This can be proved via a minor modification of the argument above.

First, since we will be comparing slicing measures that are technically defined not on the same set let us adopt a convention that if $\theta \notin \operatorname{supp} \pi$, then $\mu^\theta = 0$.

Lemma 2.3.4. *Let* μ *and* $\tilde{\mu}$ *be two measure-valued solutions to* (Equation 2.1) *with the conviction marginals* π *and* $\tilde{\pi}$*, respectively, and parameters satisfying* (Equation 2.36)*. Then for any* $t \in [0, T)$ *one has*

$$\sup_{\theta\in[\theta_{\min},\theta_{\max}]} \mathcal{W}_1(\mu_t^\theta, \tilde{\mu}_t^\theta) \leqslant c_1 e^{-c_2 t} \sup_{\theta\in[\theta_{\min},\theta_{\max}]} \mathcal{W}_1(\mu_0^\theta, \tilde{\mu}_0^\theta) + c_3 e^{-c_4 t} + c_5 \mathcal{W}_1(\pi, \tilde{\pi}),$$

where $c_i > 0$ *depend only on the initial condition and parameters of the model.*

By sending $t \to \infty$ and using that fact that

$$\sup_{\theta\in[\theta_{\min},\theta_{\max}]} |g(\theta) - \tilde{g}(\theta)| = \sup_{\theta\in[\theta_{\min},\theta_{\max}]} \mathcal{W}_1(\delta_{g(\theta)}, \delta_{\tilde{g}(\theta)}),$$

we obtain the statement (Equation 2.13) of Theorem 2.1.2.

Proof. We only need to focus on estimation of $\mathcal{D}(t)$. We have

$$\begin{aligned}
\frac{d}{dt}(Y_\theta - \tilde{Y}_\theta) &= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} Y'_{\theta'} d\mu_0^{\theta'}(y') d\pi(\theta') - \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \tilde{Y}'_{\theta'} d\tilde{\mu}_0^{\theta'}(y') d\tilde{\pi}(\theta') \\
&\quad + (\theta - 1)(Y_\theta - \tilde{Y}_\theta) - (Y_\theta^{p+1} - \tilde{Y}_\theta^{p+1}) \\
&= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} Y'_{\theta'} d\mu_0^{\theta'}(y') d\pi(\theta') - \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \tilde{Y}'_{\theta'} d\mu_0^{\theta'}(y') d\pi(\theta') \\
&\quad + \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \tilde{Y}'_{\theta'} d\mu_0^{\theta'}(y') d\pi(\theta') - \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \tilde{Y}'_{\theta'} d\tilde{\mu}_0^{\theta'}(y') d\pi(\theta') \\
&\quad + \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \tilde{Y}'_{\theta'} d\tilde{\mu}_0^{\theta'}(y') d\pi(\theta') - \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \tilde{Y}'_{\theta'} d\tilde{\mu}_0^{\theta'}(y') d\tilde{\pi}(\theta') \\
&\quad + (\theta - 1)(Y_\theta - \tilde{Y}_\theta) - (p+1)\hat{Y}_\theta^p (Y_\theta - \tilde{Y}_\theta).
\end{aligned}$$

Hence,

$$\frac{d}{dt}\mathcal{D} \leqslant c_3 e^{-c_4 t} \sup_{\theta\in[\theta_{\min},\theta_{\max}]} \mathcal{W}_1(\mu_0^\theta, \tilde{\mu}_0^\theta) + \int_{\mathbb{R}^+} G(\theta')[d\pi - d\tilde{\pi}] + \theta\mathcal{D} - (p+1)\min\{Y_\theta^p, \tilde{Y}_\theta^p\}\mathcal{D},$$

where

$$G(\theta) := \int_{\mathbb{R}_+} \tilde{Y}_\theta(y) d\tilde{\mu}_0^\theta(y) = \int_{\mathbb{R}_+} (\tilde{Y}_\theta(y) - \tilde{g}(\theta)) d\tilde{\mu}_0^\theta(y) + \tilde{g}(\theta).$$

Since the first term is bounded exponentially, and $\tilde{g} \in \mathrm{Lip}$, we have

$$\int_{\mathbb{R}^+} G(\theta')[d\pi - d\tilde{\pi}] \leqslant c_1 e^{-c_2 t} + \|\tilde{g}\|_{\mathrm{Lip}} \mathcal{W}_1(\pi, \tilde{\pi}).$$

Coming back to the $\mathcal{D}$-equation and estimating the rest of the right hand side as previously we obtain

$$\frac{d}{dt}\mathcal{D} \leqslant c_3 e^{-c_4 t} \sup_{\theta\in[\theta_{\min},\theta_{\max}]} \mathcal{W}_1(\mu_0^\theta, \tilde{\mu}_0^\theta) + c_1 e^{-c_2 t} + c_5 \|\tilde{g}\|_{\mathrm{Lip}} \mathcal{W}_1(\pi, \tilde{\pi}) - c_6 \mathcal{D}.$$

The result follows. □

2.4 Properties of mono-opinion states

The results of the previous sections establish that for each conviction measure there is at least one (and in some cases only one) limiting distributions of opinions $g \in \mathrm{Lip}[\theta_{\min}, \theta_{\max}]$. Technically it makes material sense to only consider values of g on the $\operatorname{supp}\pi$, but to study analytic properties of g it will be convenience to make full use of its existence on the closed interval $[\theta_{\min}, \theta_{\max}]$.

We have the following equation for g:

$$\int_{\mathbb{R}_+} g(\eta) d\pi(\eta) + (\theta - 1) g(\theta) - g^{p+1}(\theta) = 0, \qquad \forall \theta \in [\theta_{\min}, \theta_{\max}]. \tag{2.37}$$

Although it is difficult to find the function g explicitly, solutions to (Equation 2.37) exhibit certain universal features.

Remark 2.4.1. *One instance where g is computable is when $p = 1$. Indeed, let*

$$\alpha := \int_{\mathbb{R}_+} g(\eta)\, d\pi(\eta),$$

then by (Equation 2.37) *we have*

$$g^2 + (1 - \theta) g - \alpha = 0.$$

This second order equation always has a positive solution

$$g = \frac{1}{2} \left(\theta - 1 + \sqrt{(1 - \theta)^2 + 4\alpha} \right),$$

for any parameter $\alpha > 0$. Note that this expression is still implicit as α depends on g. But whatever α is we can see in particular that g is strictly increasing and convex.

Let us discuss these properties more systematically.

First, let us consider the extreme values

$$g_{\max} = \max_{[\theta_{\min},\theta_{\max}]} g(\theta), \qquad g_{\min} = \min_{[\theta_{\min},\theta_{\max}]} g(\theta).$$

We claim that

$$\theta_{\min} \leqslant g_{\min}^{p}, \quad g_{\max}^{p} \leqslant \theta_{\max}. \tag{2.38}$$

Indeed, the equation (Equation 2.37) can be rewritten as

$$\int_{\mathbb{R}_+} [g(\eta) - g(\theta)] d\pi(\eta) + \theta g(\theta) - g^{p+1}(\theta) = 0, \qquad \forall \theta \in [\theta_{\min}, \theta_{\max}]. \tag{2.39}$$

Let $\bar{\theta}$ be the point such that $g_{\min} = g(\bar{\theta})$. Since

$$\int_{\mathbb{R}_+} [g(\eta) - g_{\min}] d\pi(\eta) \geqslant 0,$$

by the equation (Equation 2.39), we have

$$\bar{\theta} g_{\min} - g_{\min}^{p+1} \leqslant 0.$$

Therefore,

$$\theta_{\min} \leqslant \bar{\theta} \leqslant g_{\min}^{p}.$$

Similarly, we have

$$g^p_{\max} \leqslant \theta_{\max}.$$

By (Equation 2.37), we also have that

$$(\theta - 1)g(\theta) - g^{p+1}(\theta) \leqslant 0, \qquad \forall \theta \in [\theta_{\min}, \theta_{\max}].$$

Thus, for each $\theta \in [\theta_{\min}, \theta_{\max}]$ the following estimate holds true

$$g^p(\theta) \geqslant \theta - 1. \tag{2.40}$$

A more refined estimate will be obtained next.

Lemma 2.4.2. *Let* g *be a solution to the equation* (Equation 2.37)*. Then* $g \in C^\infty([\theta_{\min}, \theta_{\max}])$*,* g *is strictly increasing on* $[\theta_{\min}, \theta_{\max}]$*, and for each* $\theta \in [\theta_{\min}, \theta_{\max}]$*,*

$$g^p(\theta) \geqslant \theta + \pi([\theta, \infty)) - 1. \tag{2.41}$$

Proof. Since g is Lipschitz we can conclude monotonicity from the sign of the derivative,

$$g' = \frac{g}{1 - \theta + (p+1)g^p}. \tag{2.42}$$

If $1 \geqslant \theta$, then using (Equation 2.38), it is clear that the denominator is positive, and so $g' > 0$. If $1 < \theta$ we have by the rough bound (Equation 2.40)

$$1 - \theta + (p+1)g^p \geqslant p(\theta - 1) > 0.$$

This establishes monotonicity. Also, since the denominator of (Equation 2.42) is always positive, by bootstrapping this implies $g \in C^\infty([\theta_{\min}, \theta_{\max}])$.

Combining monotonicity with the equation (Equation 2.37) we obtain

$$\int_{\{\eta \geqslant \theta\}} g(\theta) d\pi(\eta) - g(\theta) + [\theta - g^p(\theta)]g(\theta) \leqslant 0.$$

Since $g(\theta) \geqslant 0$ for all $\theta \in [\theta_{\min}, \theta_{\max}]$ we must have

$$\int_{\{\eta \geqslant \theta\}} d\pi(\eta) - 1 + \theta - g^p(\theta) \leqslant 0.$$

The estimate (Equation 2.41) follows. □

Let us discuss convexity. The second derivative of $g(\theta)$ is given by

$$g'' = \frac{g'[1 - \theta + (p+1)g^p] - g[-1 + p(p+1)g^{p-1}g']}{[1 - \theta + (p+1)g^p]^2}$$

and using (Equation 2.42) to replace g' we obtain

$$g'' = \frac{2(1-\theta)g + (2+p-p^2)g^{p+1}}{[1-\theta+(p+1)g^p]^3}. \tag{2.43}$$

The denominator is always positive, and we note that in view of (Equation 2.40) the numerator is also positive regardless of the range of θ provided $p \leqslant 1$. So, g is globally convex in this case.

In other cases, the convexity may change. In fact for $p = 2$ we have

$$g'' = \frac{2(1-\theta)g}{[1-\theta+3g^2]^3}.$$

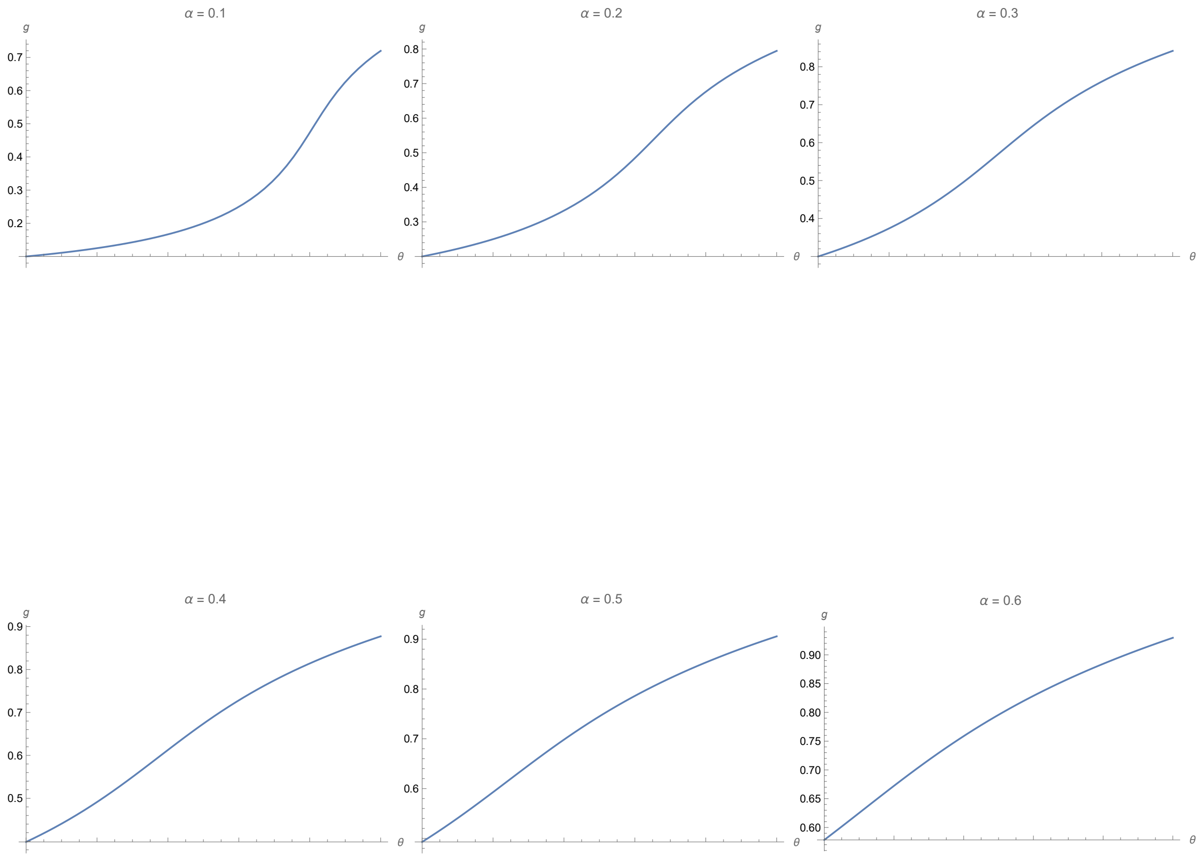
So, $\theta = 1$ is an inflection point.

For $p > 2$, the solution has no more than one inflection point. This can be seen by solving for $g'' = 0$ in (Equation 2.43). We have

$$2(1-\theta) = (p^2 - p - 2)g^p.$$

The left hand side is a decreasing function and the right hand side is increasing for $p > 2$. So, the two can meet at most at one point.

The exact value of α depends on g and since the solution is in general not possible to compute explicitly we present in the figure below solutions to (Equation 2.37) with several 'passive' choices of α for illustration.


α = 0.1
g
0.7
0.6
0.5
0.4
0.3
0.2
θ
α = 0.2
g
0.8
0.7
0.6
0.5
0.4
0.3
θ
α = 0.3
g
0.8
0.7
0.6
0.5
0.4
θ
α = 0.4
g
0.9
0.8
0.7
0.6
0.5
θ
α = 0.5
g
0.9
0.8
0.7
0.6
θ
α = 0.6
g
0.90
0.85
0.80
0.75
0.70
0.65
0.60
θ

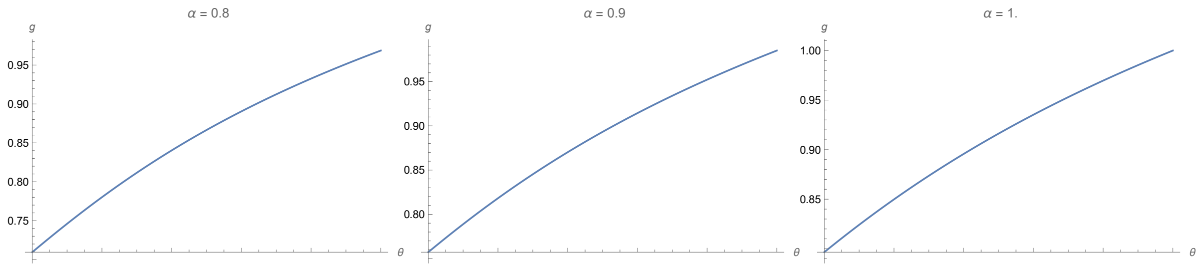


Figure 1: The behavior of $g(\theta)$ for the case $p = 6$. Here $\theta \in (0, 1]$ and α change in $(0, 1]$ at discrete steps of 0.1.

CHAPTER 3

EXPONENTIAL RELAXATION TO EQUILIBRIUM FOR A KINETIC FOKKER-PLANCK-TYPE EQUATION

3.1 Introduction

In this chapter we consider a kinetic Fokker-Plank-alignment equation which is derived from general environmental averaging models. More specifically, let $\Omega \subset \mathbb{R}^n$ be a periodic domain. An agent is featured by its position $x \in \Omega$ and its velocity $v \in \mathbb{R}^n$. The density of agents who has position x and velocity v at time $t \geqslant 0$, denoted by $f = f(x, v, t)$, is governed by the following equation:

$$\partial_t f + v \cdot \nabla_x f = s_\rho \big[\Delta_v f + \nabla_v \cdot \big(\big[(v - [u]_\rho) + F(v)\big] f\big)\big], \tag{3.1}$$

subject to the initial condition

$$f(x, v, 0) = f_0(x, v).$$

Here ρ and u are macroscopic density and macroscopic velocity defined by

$$\rho(x) = \int_{\mathbb{R}^n} f(x, v)\, dv, \quad u\rho(x) = \int_{\mathbb{R}^n} v f(x, v)\, dv. \tag{3.2}$$

The family of pairs $(\kappa_\rho, [\cdot]_\rho)$ with $d\kappa_\rho := s_\rho d\rho$ satisfies the conditions for a material environmental averaging model introduced in (33). The Rayleigh-type friction and self-propulsion force F is given by

$$F(v) = \frac{\sigma(|v|^p - 1)v}{\eta(|v|)}, \tag{3.3}$$

where $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth, positive and increasing function satisfying

$$\eta(z) = 1 \text{ if } z \leqslant R \text{ for some } R > 0; \text{ and } \eta(z) \sim z^q \text{ for some } q > p \text{ as } z \to \infty. \tag{3.4}$$

Our goal is to show that the solution of (Equation 3.1) relaxes exponentially fast toward its equilibrium. We utilize the Desvillettes-Villani's method (see (11; 38)) for collisional models to modify the entropy and establish a global hypocoercivity. Without additional force, Shvydkoy gave the first result on global hypocoercivity for this type of model in (32). In that paper, the averaging operator is given by

$$[u]_\rho := \phi * \Big(\frac{\phi * (u\rho)}{\phi * \rho}\Big),$$

where ϕ is a radial non-negative non-increasing function satisfying

$$\int_\Omega \phi(x)\, dx = 1, \quad \phi(x) \geqslant c_0 \mathbb{1}_{\{|x| < r_0\}}.$$

Then the result was extended to a class of kinetic equations in (33). In this work, we show that if an extra force is added then we still have a global hypocoercivity and hence, an exponential relaxation to equilibrium provided that the force is small in the sense of assumption (iv) below.

Before stating our result, let us give some motivation for studying the equation (Equation 3.1). The study of collective behavior has attracted a lot of attention from the scientific community because it has diverse applications ranging from biology, physics, computer science, social science etc., see e.g. (4; 31; 37; 39) and the references therein.

For microscopic descriptions, many models of collective behavior can be described as follows:

$$\begin{cases} \dot{x}_i = v_i, \qquad (x_i, v_i) \in \Omega \times \mathbb{R}^n, \\ \\ \dot{v}_i = s_i([v]_i - v_i) + F_i, \quad i = 1, \dots, N, \end{cases} \tag{3.5}$$

where s_i, F_i are respectively the communication strength and the force corresponding to the i-th agent; $v = (v_1, \dots, v_N)\mathbb{R}^{nN}$ and $[v]_i$ denotes the averaging operator acts on the i-th agent. The celebrated Cucker-Smale system (9; 10) can be written in form (Equation 3.5) with

$$s_i = \sum_{j=1}^{N} m_j \phi(|x_i - x_j|), \qquad [v]_i = \frac{\sum_{j=1}^{N} m_j \phi(|x_i - x_j|) v_j}{\sum_{j=1}^{N} m_j \phi(|x_i - x_j|)}, \tag{3.6}$$

where ϕ is a smooth radial non-increasing function, m_i is the communication weight of the i—th agent. In this model $F_i = 0$. For examples with nontrivial force F_i, the readers can see (22; 30; 31). If we take F_i in (Equation 3.5) to be the combination of a deterministic force and a noise of the form

$$F_i = \frac{\sigma(1 - |v_i|^p) v_i}{\eta(|v_i|)} + \sqrt{2 s_i(x)} \dot{B}_i, \quad 0 < \sigma < 1 \text{ and } p > 0, \tag{3.7}$$

here η is given by (Equation 3.4) and $B_i's$ are independent Brownian motions in $\mathbb{R}^n$, then the stochastic mean-field limit of (Equation 3.5) formally leads to the kinetic equation (Equation 3.1).

In this chapter, we will merely focus on the long-time behavior of the solution of (Equation 3.1) provided it exists. For a rigorous derivation of (Equation 3.1) via stochastic mean-field limit one can consult the scheme from (3; 33). For the existence of solution, we refer to (3; 21; 33). We assume the solution f to (Equation 3.1) belongs to some weighted Sobolev space

$$H_l^k(\Omega \times \mathbb{R}^n) := \left\{ f : \sum_{k' \leqslant k} \sum_{|\alpha| = k'} \int_{\Omega \times \mathbb{R}^n} \langle v \rangle^{l + 2(k-k')} \, |\partial_{x,v}^\alpha f|^2 \, dx \, dv < \infty \right\},$$

where $\langle v \rangle = \sqrt{1 + |v|^2}$ and α denotes a multiindex.

Next let us introduce some more notations. Letting $G : \mathbb{R}_+ \to \mathbb{R}$ be the function defined by

$$G(z) := \int_0^z \frac{\sigma(y^{p+1} - y)}{\eta(y)} \, dy,$$

and letting

$$V(v) = \frac{|v|^2}{2} + G(|v|). \tag{3.8}$$

Then the gradient and Hessian matrix of V can be computed explicitly,

$$\nabla V = v + F(v), \tag{3.9}$$

$$\nabla^2 V = \left(1 + \frac{\sigma(|v|^p - 1)}{\eta(|v|)}\right) \mathbb{I} + \frac{\sigma |v|^p}{\eta(|v|)} \frac{v}{|v|} \otimes \frac{v}{|v|} - \frac{\sigma(|v|^p - 1)|v| \eta'(|v|)}{\eta^2(|v|)} \frac{v}{|v|} \otimes \frac{v}{|v|}, \tag{3.10}$$

where $\mathbb{I}$ is the identity matrix.

Remark 3.1.1. *By the assumption* (Equation 3.4) *and the identity* (Equation 3.10) *we see that the Hessian matrix of* V *is bounded. Thus, there exists a positive constant* Λ *such that*

$$|(\nabla^2 V)(y)| \leqslant \Lambda |y|, \quad \forall y \in \mathbb{R}^n. \tag{3.11}$$

We also note that for $y \in \mathbb{R}^n$*,*

$$y^T (\nabla^2 V) y \geqslant \left(1 - \frac{\sigma}{\eta(|\nu|)} - \frac{\sigma |\nu|^{p+1} \eta'(|\nu|)}{\eta^2(|\nu|)}\right) |y|^2 \geqslant \lambda |y|^2, \tag{3.12}$$

where $\lambda > 0$ *is a constant depending on* σ*.*

We expect that the solution to (Equation 3.1) converges to

$$f_\infty := \frac{1}{Z} e^{-V(\nu)} \quad \text{with } Z = \int_{\Omega \times \mathbb{R}^n} e^{-V(\nu)} \, d\nu \, dx. \tag{3.13}$$

The macroscopic field u_F is defined by

$$\rho u_F(x) = \int_{\mathbb{R}^n} F(\nu) f(x, \nu) \, d\nu.$$

Denote $L^2(\kappa_\rho) := L^2(d\kappa_\rho)$. The inner product in $L^2(\kappa_\rho)$ is denoted by $\langle \cdot, \cdot \rangle_{\kappa_\rho}$. Our main result is the following:

Theorem 3.1.2. *Suppose that* $f \in H^k_l(\Omega \times \mathbb{R}^n)$ *is a solution to* (Equation 3.1) *such that* $\rho(t)$ *satisfies the following assumptions for all* $t \geqslant 0$*:*

(i) $c_0 \leqslant s_\rho \leqslant c_1$ *and* $\|\nabla s_\rho\|_\infty \leqslant c_2$*, where* c_0, c_1, c_2 *are positive constants,*

(ii) $\nabla_x(s_\rho[\cdot]_\rho) : L^2(\rho) \to L^2(\rho)$ *is uniformly bounded,*

(iii) there exists a constant $0 < \varepsilon_0 < 1$ *such that*

$$\sup \left\{ \langle w, [w]_\rho \rangle_{\kappa_\rho} \mid w \in L^2(\kappa_\rho), \|w\|_{L^2(\kappa_\rho)} = 1 \right\} \leqslant 1 - \varepsilon_0,$$

(iv) there exists a constant $0 < \varepsilon_1 < 1$ *such that*

$$\|u_F\|_{L^2(\kappa_\rho)} \leqslant \varepsilon_1 \|u\|_{L^2(\kappa_\rho)}.$$

Then f *converges to* f_∞ *exponentially fast:*

$$\|f(t) - f_\infty\|_{L^1(\Omega \times \mathbb{R}^n)} \leqslant Ce^{-\delta t},$$

where $C > 0$ *is a constant depending on initial data* f_0 *and given parameters;* $\delta > 0$ *is a constant depending only on given parameters.*

Remark 3.1.3. *Observe that in the case of Cucker-Smale model, since* $s_\rho = \phi * \rho$ *and* $s_\rho[u]_\rho = \phi * (u\rho)$*, condition (ii) holds automatically and condition (i) holds if* $\phi * \rho \geqslant \underline{\rho}$ *for some* $\underline{\rho} > 0$.

3.2 Proof of main result

In this section, we will prove Theorem 3.1.2. Firstly, let us introduce some notations and definitions.

3.2.1 Notations and preliminaries

The relative entropy is defined by

$$\mathcal{H}(f|f_\infty) = \int_{\Omega\times\mathbb{R}^n} f \log \frac{f}{f_\infty}\, dv\, dx.$$

For our convenient computation, we will derive an equation for h satisfying $f = hf_\infty$. Plugging this f into equation (Equation 3.1), we have the following equation for h:

$$\partial_t h + v\cdot\nabla_x h = s_\rho\big(\Delta_v h - \nabla V\cdot\nabla_v h + h[u]_\rho\cdot\nabla V - [u]_\rho\cdot\nabla_v h\big). \tag{3.14}$$

Letting

$$A = \nabla_v, \qquad B = v\cdot\nabla_x,$$

and A^* be the adjoint of A with respect to the inner product in the weighted space $L^2(\mu)$:

$$\langle\varphi_1,\varphi_2\rangle = \int_{\Omega\times\mathbb{R}^n}\varphi_1\varphi_2 d\mu, \quad d\mu = f_\infty\, dv\, dx.$$

We can calculate A^* explicitly,

$$A^* = (\nabla V - \nabla_v)\cdot.$$

Then we can write (Equation 3.14) in the abstract form:

$$h_t = -s_\rho A^* A h - B h + s_\rho A^*(h[u]_\rho). \tag{3.15}$$

Following the notations from the paper (33), let us define the partial Fisher information functionals as follows:

$$\mathcal{I}_{vv}(h) = \int_{\Omega\times\mathbb{R}^n} \frac{|\nabla_v h|^2}{h} d\mu, \quad \mathcal{I}_{xv}(h) = \int_{\Omega\times\mathbb{R}^n} \frac{\nabla_x h \cdot \nabla_v h}{h} d\mu, \quad \mathcal{I}_{xx}(h) = \int_{\Omega\times\mathbb{R}^n} \frac{|\nabla_x h|^2}{h} d\mu.$$

The full Fisher information is defined by

$$\mathcal{I} = \mathcal{I}_{vv} + \mathcal{I}_{xx}.$$

For our convenience we use the notation

$$(\varphi)_\mu := \int_{\Omega\times\mathbb{R}^n} \varphi d\mu.$$

Denote $\bar{h} = \log h$ and

$$\mathcal{D}_{vv} = (s_\rho h |\nabla_v^2 \bar{h}|^2)_\mu, \quad \mathcal{D}_{xv} = (s_\rho h |\nabla_v \nabla_x \bar{h}|^2)_\mu,$$

where $\nabla_v^2 \bar{h}$ is the Hessian matrix with respect to v of $\bar{h}$. We will use the notations J_A, J_B, J_u to refer to the terms related to the operators A, B and related to u respectively. They are different in the proof of each lemma in the sequel. We denote by C, c positive constants which may vary from line to line.

3.2.2 Proof of Theorem 3.1.2

By the Csiszár-Kullback inequality,

$$\|f - f_\infty\|^2_{L^1(\Omega\times\mathbb{R}^n)} \leqslant c\mathcal{H}. \tag{3.16}$$

Therefore, it suffices to show that the entropy function $\mathcal{H}$ decays exponentially fast in time. Using (Equation 3.1) and integration by parts, we have

$$\frac{d}{dt}\mathcal{H} = -\int_{\Omega\times\mathbb{R}^n} s_\rho \frac{|\nabla_v f + \nabla V f|^2}{f}\, dv\, dx + \langle u_V, [u]_\rho \rangle_{\kappa_\rho}, \tag{3.17}$$

where

$$u_V = u + u_F. \tag{3.18}$$

Define the partial Fisher information functional $\mathcal{I}_{vv}$ by

$$\mathcal{I}_{vv} = \int_{\Omega\times\mathbb{R}^n} s_\rho \frac{|\nabla_v f + \nabla V f|^2}{f}\, dv\, dx.$$

By the assumption (i) we have

$$\frac{d}{dt}\mathcal{H} \leqslant -c_0\mathcal{I}_{vv} + \langle u_V, [u]_\rho \rangle_{\kappa_\rho}. \tag{3.19}$$

We can also rewrite (Equation 3.17) in the dissipative form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} = -\int_{\Omega\times\mathbb{R}^n} s_\rho \frac{|\nabla_v f + (\nabla V - u_V)f|^2}{f}\,\mathrm{d}v\,\mathrm{d}x - \|u_V\|^2_{L^2(\kappa_\rho)} + \langle u_V, [u]_\rho\rangle_{\kappa_\rho}. \tag{3.20}$$

By the triangle inequality and assumption (iv) we have

$$\|u\|_{L^2(\kappa_\rho)} \leqslant \frac{1}{1-\varepsilon_1}\|u_V\|_{L^2(\kappa_\rho)}. \tag{3.21}$$

Then by the Cauchy-Schwarz inequality, assumptions (iii) and (iv) we have

$$\begin{aligned}\langle u_V, [u]_\rho\rangle_{\kappa_\rho} &= \langle u_V, [u_V]_\rho\rangle_{\kappa_\rho} - \langle u_V, [u_F]_\rho\rangle_{\kappa_\rho}\\ &\leqslant (1-\varepsilon_0)\|u_V\|^2_{L^2(\kappa_\rho)} + \frac{\varepsilon_1}{1-\varepsilon_1}\|u_V\|^2_{L^2(\kappa_\rho)}\\ &\leqslant (1-c_3)\|u_V\|^2_{L^2(\kappa_\rho)},\end{aligned} \tag{3.22}$$

where $c_3 > 0$ depending on $\varepsilon_0, \varepsilon_1$. Plugging this inequality into (Equation 3.20) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} \leqslant -c_3\|u_V\|^2_{L^2(\kappa_\rho)}. \tag{3.23}$$

Combining (Equation 3.19), (Equation 3.23) and (Equation 3.22) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} \leqslant -\frac{c_0c_3}{1+c_3}\mathcal{I}_{vv} - \frac{c_3^2}{1+c_3}\|u_V\|^2_{L^2(\kappa_\rho)} \leqslant -c\mathcal{I}_{vv} - c\|u_V\|^2_{L^2(\kappa_\rho)}, \tag{3.24}$$

where $c > 0$ depending on $\varepsilon_0, \varepsilon_1, c_0$.

By (Equation 3.12), f_∞ satisfies a logarithmic Sobolev inequality, see (38). Thus, we have

$$\mathcal{H} \leqslant c\mathcal{I}. \tag{3.25}$$

We have the following three estimates on the time derivative of partial Fisher information functionals. Their proofs will be presented in the next subsection.

Lemma 3.2.1. *We have*

$$\frac{d}{dt}\mathcal{I}_{vv}(h) \leqslant -2\mathcal{D}_{vv} - \lambda c_0 \mathcal{I}_{vv} - 2\mathcal{I}_{xv} + c\|u\|^2_{L^2(\kappa_\rho)}, \tag{3.26}$$

where c *is a positive constant depending on* $c_0, c_1, \lambda, \Lambda$.

Lemma 3.2.2. *We have*

$$\frac{d}{dt}\mathcal{I}_{xv} \leqslant c\mathcal{I}_{vv} - \frac{1}{2}\mathcal{I}_{xx} + 2\mathcal{D}_{vv} + \mathcal{D}_{xv} + c\|u\|^2_{L^2(\kappa_\rho)}, \tag{3.27}$$

where c *is dependent on* $c_0, c_1, c_2, \lambda, \Lambda$.

Lemma 3.2.3. *We have*

$$\frac{d}{dt}\mathcal{I}_{xx}(h) \leqslant c\mathcal{I}_{vv} - \mathcal{D}_{xv} + c\|u\|^2_{L^2(\kappa_\rho)},$$

where c *is a constant depending on* λ, Λ *and the parameters in the assumption (i), (ii).*

Choosing $\varepsilon > 0$ small so that if we define

$$\tilde{\mathcal{I}} = \mathcal{I}_{vv} + \varepsilon \mathcal{I}_{xv} + \frac{\lambda c_0}{c} \mathcal{I}_{xx}, \tag{3.28}$$

then $\mathcal{I} \sim \tilde{\mathcal{I}}$. Combining three lemmas above and the assumption (iv) we have

$$\frac{d}{dt}\tilde{\mathcal{I}} \leqslant -\lambda c_0 \mathcal{I}_{vv} - \frac{\varepsilon}{2}\mathcal{I}_{xx} + C\|u\|^2_{L^2(\kappa_\rho)} \leqslant -\lambda c_0 \mathcal{I}_{vv} - \frac{\varepsilon}{2}\mathcal{I}_{xx} + C\|u_V\|^2_{L^2(\kappa_\rho)}. \tag{3.29}$$

From (Equation 3.24), (Equation 3.29) and (Equation 3.25) we can choose a constant γ such that

$$\frac{d}{dt}(\tilde{\mathcal{I}} + \gamma \mathcal{H}) \lesssim -\mathcal{I} \leqslant -\delta(\tilde{\mathcal{I}} + \gamma \mathcal{H}). \tag{3.30}$$

Thus, by Grönwall's inequality we obtain

$$\tilde{\mathcal{I}} + \gamma \mathcal{H} \leqslant (\tilde{\mathcal{I}}_0 + \gamma \mathcal{H}_0) e^{-\delta t} \leqslant c \mathcal{I}_0 e^{-\delta t}. \tag{3.31}$$

Then we can conclude the theorem.

3.2.3 Proof of three technical lemmas

In this subsection, we will give the proofs of three lemmas mentioned previously.

Proof of Lemma 3.2.1. Let us rewrite $\mathcal{I}_{\nu\nu}$ in the form

$$\mathcal{I}_{\nu\nu} = (\nabla_\nu h \cdot \nabla_\nu \bar{h})_\mu.$$

By chain rule and equation (Equation 3.15) we get

$$\frac{d}{dt}\mathcal{I}_{\nu\nu} = 2(\nabla_\nu h_t \cdot \nabla_\nu \bar{h})_\mu - (|\nabla_\nu \bar{h}|^2 h_t)_\mu := J_A + J_B + J_u,$$

where

$$\begin{aligned}
J_A &= -2(s_\rho \nabla_\nu A^* A h \cdot \nabla_\nu \bar{h})_\mu + (s_\rho |\nabla_\nu \bar{h}|^2 A^* A h)_\mu, \\
J_B &= -2(\nabla_\nu B h \cdot \nabla_\nu \bar{h})_\mu + (|\nabla_\nu \bar{h}|^2 B h)_\mu, \\
J_u &= 2(s_\rho \nabla_\nu A^*([u]_\rho h) \cdot \nabla_\nu \bar{h})_\mu - (s_\rho |\nabla_\nu \bar{h}|^2 A^*([u]_\rho h))_\mu.
\end{aligned}$$

For notational convenience we will use the Einstein summation convention in the sequel.

We firstly consider the term J_A. Using the identity

$$\partial_{\nu_i}(A^* A h) = A^* A h_{\nu_i} + \nabla V_{\nu_i} \cdot \nabla_\nu h,$$

J_A equals to

$$-2(s_\rho A^* A h_{\nu_i} \bar{h}_{\nu_i})_\mu - 2(s_\rho (\nabla V_{\nu_i} \cdot \nabla_\nu h) \bar{h}_{\nu_i})_\mu + (s_\rho |\nabla_\nu \bar{h}|^2 A^* A h)_\mu =: J_A^1 + J_A^2 + J_A^3.$$

By (Equation 3.12) we have

$$J_A^2 = -2(s_\rho h^{-1}(\nabla_v h)^T \nabla^2 V \nabla_v h)_\mu \leqslant -2\lambda(s_\rho h^{-1}\nabla_v h \cdot \nabla_v h)_\mu.$$

Then the assumption (i) in Theorem 3.1.2 implies that

$$J_A^2 \leqslant -2\lambda c_0 \mathcal{I}_{vv}.$$

By switching A^* in J_A^1, J_A^3 we can write

$$\begin{aligned} J_A^1 + J_A^3 =& -2(s_\rho A h_{v_i} \cdot A\bar{h}_{v_i})_\mu + (s_\rho A(|\nabla_v \bar{h}|^2) \cdot Ah)_\mu \\ =& -2(s_\rho h A\bar{h}_{v_i} \cdot A\bar{h}_{v_i})_\mu - 2(s_\rho \bar{h}_{v_i} Ah \cdot A\bar{h}_{v_i})_\mu + 2(s_\rho \bar{h}_{v_i} A\bar{h}_{v_i} \cdot Ah)_\mu \\ =& -2(s_\rho h A\bar{h}_{v_i} \cdot A\bar{h}_{v_i})_\mu = -2\mathcal{D}_{vv}. \end{aligned}$$

Combining the above estimates we obtain

$$J_A \leqslant -2\mathcal{D}_{vv} - 2\lambda c_0 \mathcal{I}_{vv}. \tag{3.32}$$

For the term J_B, plugging $B = v \cdot \nabla_x$ into J_B we have

$$J_B = -2(\nabla_x h \cdot \nabla_v \bar{h})_\mu - 2((v \cdot \nabla_x h_{v_i})\bar{h}_{v_i})_\mu + (|\nabla_v \bar{h}|^2 v \cdot \nabla_x h)_\mu.$$

Using the identity $\bar{h}_{v_i} = h_{v_i} h^{-1}$ and integration by parts, we get

$$2((v \cdot \nabla_x h_{v_i})\bar{h}_{v_i})_\mu = \left(v \cdot \nabla_x |h_{v_i}|^2 h^{-1}\right)_\mu = \left(|h_{v_i}|^2 h^{-2} v \cdot \nabla_x h\right)_\mu = (|\nabla_v \bar{h}|^2 v \cdot \nabla_x h)_\mu.$$

Substituting this into J_B we yield

$$J_B = -2\mathcal{I}_{xv}. \tag{3.33}$$

For the last term J_u, we have

$$\begin{aligned}
J_u =& 2(s_\rho \nabla_v A^*([u]_\rho h) \cdot \nabla_v \bar{h})_\mu - (s_\rho |\nabla_v \bar{h}|^2 A^*([u]_\rho h))_\mu \\
=& 2(s_\rho \nabla_v (\nabla V \cdot [u]_\rho h - [u]_\rho \cdot \nabla_v h) \cdot \nabla_v \bar{h})_\mu - (s_\rho \nabla_v |\nabla_v \bar{h}|^2 \cdot [u]_\rho h)_\mu \\
=& 2(s_\rho \nabla^2 V([u]_\rho h) \cdot \nabla_v \bar{h})_\mu + 2(s_\rho (\nabla V \cdot [u]_\rho)\nabla_v h \cdot \nabla_v \bar{h})_\mu - 2(s_\rho \nabla_v^2 h([u]_\rho) \cdot \nabla_v \bar{h})_\mu \\
&- 2(s_\rho \nabla_v^2 \bar{h}(\nabla_v \bar{h}) \cdot [u]_\rho h)_\mu \\
=:& J_u^1 + J_u^2 + J_u^3 + J_u^4.
\end{aligned}$$

Plugging

$$\bar{h}_{v_i v_j} = h^{-1} h_{v_i v_j} - h^{-2} h_{v_i} h_{v_j}$$

into J_u^4 we get

$$\begin{aligned}
J_u^4 =& - 2(s_\rho h^{-1} h_{v_i v_j} \bar{h}_{v_j} [u_i]_\rho h)_\mu + 2(s_\rho h^{-2} h_{v_i} h_{v_j} \bar{h}_{v_j} [u_i]_\rho h)_\mu \\
=& - 2(s_\rho \nabla_v^2 h([u]_\rho) \cdot \nabla_v \bar{h})_\mu + 2(s_\rho |\nabla_v \bar{h}|^2 \nabla_v h \cdot [u]_\rho)_\mu \\
=& J_u^3 + 2(s_\rho |\nabla_v \bar{h}|^2 \nabla_v h \cdot [u]_\rho)_\mu.
\end{aligned}$$

Therefore,

$$\begin{aligned}
J_u^2 + J_u^3 + J_u^4 &= 2(s_\rho(\nabla V \cdot [u]_\rho h)\nabla_v \bar{h} \cdot \nabla_v \bar{h})_\mu - 2(s_\rho|\nabla_v \bar{h}|^2 \nabla_v h \cdot [u]_\rho)_\mu + 2J_u^4 \\
&= 2(s_\rho A^*([u]_\rho h)|\nabla_v \bar{h}|^2)_\mu + 2J_u^4 \\
&= 2(s_\rho h[u]_\rho \cdot A(|\nabla_v \bar{h}|^2))_\mu + 2J_u^4 \\
&= 4(s_\rho h[u]_\rho \cdot \nabla_v^2 \bar{h}(\nabla_v \bar{h}))_\mu + 2J_u^4 = 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
J_u &= 2(s_\rho \nabla^2 V([u]_\rho h) \cdot \nabla_v \bar{h})_\mu = 2(s_\rho \nabla^2 V([u]_\rho) \cdot \nabla_v h)_\mu \\
&\leqslant 2\Lambda c_1 \|[u]_\rho\|_{L^2(\kappa_\rho)} \sqrt{\mathcal{I}_{vv}} \qquad \text{(by (i), (Equation 3.11) and Hölder inequality)} \\
&\leqslant c\|u\|_{L^2(\kappa_\rho)} \sqrt{\mathcal{I}_{vv}} \\
&\leqslant c\|u\|_{L^2(\kappa_\rho)}^2 + \lambda c_0 \mathcal{I}_{vv} \quad \text{(by Young's inequality)}. \qquad (3.34)
\end{aligned}$$

Here the last constant c depends on $c_0, c_1, \lambda, \Lambda$.

Combining (Equation 3.32), (Equation 3.33) and (Equation 3.34) we have the conclusion of this lemma. □

Proof of Lemma 3.2.2. Computing the derivative of $\mathcal{I}_{xv}$ with respect to t we get

$$\frac{d}{dt}\mathcal{I}_{xv}(h) = (\nabla_x h_t \cdot \nabla_v \bar{h})_\mu + (\nabla_x \bar{h} \cdot \nabla_v h_t)_\mu - (h_t \nabla_v \bar{h} \cdot \nabla_x \bar{h})_\mu =: J_A + J_B + J_u,$$

where

$$J_A = -(\nabla_x(s_\rho A^*Ah)\cdot\nabla_v\bar{h})_\mu - (\nabla_x\bar{h}\cdot\nabla_v(s_\rho A^*Ah))_\mu + (s_\rho A^*Ah\nabla_v\bar{h}\cdot\nabla_x\bar{h})_\mu =: J_A^1 + J_A^2 + J_A^3,$$

$$J_B = -(\nabla_x(v\cdot\nabla_x h)\cdot\nabla_v\bar{h})_\mu - (\nabla_x\bar{h}\cdot\nabla_v(v\cdot\nabla_x h))_\mu + ((v\cdot\nabla_x h)\nabla_v\bar{h}\cdot\nabla_x\bar{h})_\mu := J_B^1 + J_B^2 + J_B^3,$$

$$J_u = (\nabla_x(s_\rho A^*([u]_\rho h))\cdot\nabla_v\bar{h})_\mu + (\nabla_x\bar{h}\cdot\nabla_v(s_\rho A^*([u]_\rho h)))_\mu - (s_\rho A^*([u]_\rho h)\nabla_v\bar{h}\cdot\nabla_x\bar{h})_\mu.$$

Let us firstly estimate J_A. Switching A^* and using the identity

$$\nabla_v h_{x_i} = \bar{h}_{x_i}\nabla_v h + h\nabla_v\bar{h}_{x_i}$$

we have

$$\begin{aligned} J_A^1 &= -(s_\rho A^*Ah_{x_i}\bar{h}_{v_i})_\mu - ((s_\rho)_{x_i}A^*Ah\bar{h}_{v_i})_\mu = -(s_\rho\nabla_v h_{x_i}\cdot\nabla_v\bar{h}_{v_i})_\mu - ((s_\rho)_{x_i}\nabla_v h\cdot\nabla_v\bar{h}_{v_i})_\mu \\ &= -(s_\rho h\nabla_v\bar{h}_{x_i}\cdot\nabla_v\bar{h}_{v_i})_\mu - (s_\rho\bar{h}_{x_i}\nabla_v h\cdot\nabla_v\bar{h}_{v_i})_\mu - \left(\frac{(s_\rho)_{x_i}}{s_\rho^{1/2}}\frac{\nabla_v h}{h^{1/2}}\cdot s_\rho^{1/2}h^{1/2}\nabla_v\bar{h}_{v_i}\right)_\mu. \end{aligned}$$

In view of assumption (i) in Theorem 3.1.2,

$$J_A^1 \leqslant -(s_\rho h\nabla_v\bar{h}_{x_i}\cdot\nabla_v\bar{h}_{v_i})_\mu - (s_\rho\bar{h}_{x_i}\nabla_v h\cdot\nabla_v\bar{h}_{v_i})_\mu + c\sqrt{\mathcal{I}_{vv}\mathcal{D}_{vv}},$$

where $c > 0$ is a constant depending on c_0, c_2. Next let us consider J_A^2. Since

$$\partial_{v_i}(A^*Ah) = A^*Ah_{v_i} + \nabla V_{v_i}\cdot\nabla_v h \text{ and } \nabla_v h_{v_i} = h\nabla_v\bar{h}_{v_i} + \bar{h}_{v_i}\nabla_v h,$$

we have

$$
\begin{aligned}
J_A^2 =& -(s_\rho \bar{h}_{x_i} A^* A h_{v_i})_\mu - (s_\rho \bar{h}_{x_i} \nabla V_{v_i} \cdot \nabla_v h)_\mu \\
=& -(s_\rho \nabla_v \bar{h}_{x_i} \cdot \nabla_v h_{v_i})_\mu - (s_\rho \bar{h}_{x_i} \nabla V_{v_i} \cdot \nabla_v h)_\mu \\
=& -(s_\rho h \nabla_v \bar{h}_{x_i} \cdot \nabla_v \bar{h}_{v_i})_\mu - (s_\rho \bar{h}_{v_i} \nabla_v \bar{h}_{x_i} \cdot \nabla_v h)_\mu - (s_\rho \nabla_x \bar{h} \cdot (\nabla^2 V)(\nabla_v h))_\mu.
\end{aligned}
$$

Then

$$
\begin{aligned}
J_A^1 + J_A^2 \leqslant& -(s_\rho \nabla_x \bar{h} \cdot (\nabla^2 V)(\nabla_v h))_\mu - 2(s_\rho h \nabla_v \bar{h}_{x_i} \cdot \nabla_v \bar{h}_{v_i})_\mu - (s_\rho A h \cdot A(\nabla_v \bar{h} \cdot \nabla_x \bar{h}))_\mu \\
&+ c\sqrt{\mathcal{I}_{vv}\mathcal{D}_{vv}} \\
\leqslant& -(s_\rho \nabla_x \bar{h} \cdot (\nabla^2 V)(\nabla_v h))_\mu + 2\sqrt{\mathcal{D}_{vv}\mathcal{D}_{xv}} + c\sqrt{\mathcal{I}_{vv}\mathcal{D}_{vv}} - J_A^3 \\
\leqslant& \, c_1 \Lambda \sqrt{\mathcal{I}_{vv}\mathcal{I}_{xx}} + 2\sqrt{\mathcal{D}_{vv}\mathcal{D}_{xv}} + c\sqrt{\mathcal{I}_{vv}\mathcal{D}_{vv}} - J_A^3.
\end{aligned}
$$

Thus, combining all the terms of J_A and applying Young's inequality we yield

$$
\begin{aligned}
J_A \leqslant& \, c_1 \Lambda \sqrt{\mathcal{I}_{vv}\mathcal{I}_{xx}} + 2\sqrt{\mathcal{D}_{vv}\mathcal{D}_{xv}} + c\sqrt{\mathcal{I}_{vv}\mathcal{D}_{vv}}, \\
\leqslant& \, c\mathcal{I}_{vv} + \frac{1}{4}\mathcal{I}_{xx} + \frac{3}{2}\mathcal{D}_{vv} + \mathcal{D}_{xv}.
\end{aligned}
\tag{3.35}
$$

Now we consider J_B. We have

$$
\begin{aligned}
J_B^2 =& -(\nabla_x \bar{h} \cdot \nabla_x h)_\mu - (\bar{h}_{x_i} v_j h_{x_j v_i})_\mu \\
=& -\mathcal{I}_{xx} + (\bar{h}_{x_i x_j} v_j h_{v_i})_\mu \\
=& -\mathcal{I}_{xx} + (h_{x_i x_j} v_j \bar{h}_{v_i})_\mu - (\bar{h}_{x_i} \bar{h}_{x_j} v_j h_{v_i})_\mu = -\mathcal{I}_{xx} - J_B^1 - J_B^3.
\end{aligned}
$$

In the last row we used the identity

$$\bar{h}_{x_i x_j} = h^{-1} h_{x_i x_j} - \bar{h}_{x_i} \bar{h}_{x_j}.$$

It follows that

$$J_B = -\mathcal{I}_{xx}. \tag{3.36}$$

Lastly let us examine J_u. We have

$$\begin{aligned} J_u &= ((s_\rho)_{x_i} A^*([u]_\rho h)\bar{h}_{v_i})_\mu + (s_\rho A^*(([u]_\rho)_{x_i} h)\bar{h}_{v_i})_\mu + (s_\rho A^*([u]_\rho h_{x_i})\bar{h}_{v_i})_\mu \\ &\quad + (s_\rho \bar{h}_{x_i} A^*([u]_\rho h_{v_i}))_\mu + (s_\rho \nabla_x \bar{h} \cdot (\nabla^2 V)([u]_\rho h))_\mu - (s_\rho h[u]_\rho \cdot \nabla_v(\nabla_v \bar{h} \cdot \nabla_x \bar{h}))_\mu \\ &= (h(s_\rho [u]_\rho)_{x_i} \cdot \nabla_v \bar{h}_{v_i})_\mu + (s_\rho h[u]_\rho \bar{h}_{x_i} \cdot \nabla_v \bar{h}_{v_i})_\mu + (s_\rho h \nabla_v \bar{h}_{x_i} \cdot [u]_\rho \bar{h}_{v_i})_\mu \\ &\quad + (s_\rho \nabla_x \bar{h} \cdot (\nabla^2 V)([u]_\rho h))_\mu - (s_\rho h[u]_\rho \cdot \nabla_v(\nabla_v \bar{h} \cdot \nabla_x \bar{h}))_\mu \\ &=: J_u^1 + J_u^2 + J_u^3 + J_u^4 + J_u^5. \end{aligned}$$

Since

$$J_u^2 + J_u^3 = (s_\rho h[u]_\rho \cdot \nabla_v(\nabla_x \bar{h} \cdot \nabla_v \bar{h}))_\mu = -J_u^5,$$

we get

$$J_u = J_u^1 + J_u^4.$$

By the assumption (ii) in Theorem 3.1.2,

$$J_u^1 = (h(s_\rho [u]_\rho)_{x_i} \cdot \nabla_v \bar{h}_{v_i})_\mu \leqslant c\|u\|_{L^2(\kappa_\rho)} \sqrt{\mathcal{D}_{vv}}.$$

For J_u^4 we use the assumption (i) and (Equation 3.11) to get

$$\begin{aligned} J_u^4 &= (s_\rho \nabla_x \bar{h} \cdot (\nabla^2 V)([u]_\rho h))_\mu \\ &\leqslant c\|u\|_{L^2(\kappa_\rho)} \sqrt{\mathcal{I}_{xx}}, \end{aligned}$$

where c is a constant depending on c_1, Λ. Hence, by Young's inequality we obtain

$$J_u \leqslant \frac{1}{4}\mathcal{I}_{xx} + \frac{1}{2}\mathcal{D}_{vv} + c\|u\|^2_{L^2(\kappa_\rho)}. \tag{3.37}$$

Combining three estimates (Equation 3.35), (Equation 3.36) and (Equation 3.37) it implies the conclusion of this lemma. □

Proof of Lemma 3.2.3. Computing the derivative of $\mathcal{I}_{xx}(h)$ with respect to t we get

$$\frac{d}{dt}\mathcal{I}_{xx}(h) = 2(\nabla_x h_t \cdot \nabla_x \bar{h})_\mu - (|\nabla_x \bar{h}|^2 h_t)_\mu =: J_A + J_B + J_u,$$

where

$$\begin{aligned} J_A &= -2(\nabla_x(s_\rho A^* A h) \cdot \nabla_x \bar{h})_\mu + (s_\rho |\nabla_x \bar{h}|^2 A^* A h)_\mu, \\ J_B &= -2(\nabla_x(v \cdot \nabla_x h) \cdot \nabla_x \bar{h})_\mu + (|\nabla_x \bar{h}|^2 v \cdot \nabla_x h)_\mu, \\ J_u &= 2(\nabla_x(s_\rho A^*([u]_\rho h)) \cdot \nabla_x \bar{h})_\mu - (s_\rho |\nabla_x \bar{h}|^2 A^*([u]_\rho h))_\mu. \end{aligned}$$

For J_A we have

$$J_A = -2((s_\rho)_{x_i} Ah \cdot A\bar{h}_{x_i})_\mu - 2(s_\rho Ah_{x_i} \cdot A\bar{h}_{x_i})_\mu + (s_\rho A|\nabla_x \bar{h}|^2 \cdot Ah)_\mu =: J_A^1 + J_A^2 + J_A^3.$$

By the assumption (i) in Theorem 3.1.2,

$$J_A^1 = -2\Big(\frac{(s_\rho)_{x_i}}{s_\rho^{1/2}} \frac{\nabla_v h}{h^{1/2}} \cdot s_\rho^{1/2} h^{1/2} \nabla_v \bar{h}_{x_i}\Big)_\mu \leqslant c\sqrt{\mathcal{I}_{vv}\mathcal{D}_{xv}}.$$

Using the identity $\nabla_v h_{x_i} = h\nabla_v \bar{h}_{x_i} + \bar{h}_{x_i}\nabla_v h$, we have

$$J_A^2 = -2(s_\rho h \nabla_v \bar{h}_{x_i} \cdot \nabla_v \bar{h}_{x_i})_\mu - 2(s_\rho \bar{h}_{x_i} \nabla_v h \cdot \nabla_v \bar{h}_{x_i})_\mu = -2\mathcal{D}_{xv} - J_A^3.$$

Therefore,

$$J_A \leqslant c\sqrt{\mathcal{I}_{vv}\mathcal{D}_{xv}} - 2\mathcal{D}_{xv}. \tag{3.38}$$

We have $J_B = 0$ because

$$-2(\nabla_x(v \cdot \nabla_x h) \cdot \nabla_x \bar{h})_\mu = -2((v \cdot \nabla_x h_{x_i})h_{x_i} h^{-1})_\mu = -((v \cdot \nabla_x |\nabla_x h|^2 h^{-1})_\mu = -(|\nabla_x \bar{h}|^2 v \cdot \nabla_x h)_\mu.$$

For J_u, we have

$$\begin{aligned}
J_u &= 2(h(s_\rho[u]_\rho)_{x_i} \cdot \nabla_v \bar{h}_{x_i})_\mu + 2(s_\rho h \bar{h}_{x_i}[u]_\rho \cdot \nabla_v \bar{h}_{x_i})_\mu - (s_\rho \nabla_v(|\nabla_x \bar{h}|^2) \cdot [u]_\rho h)_\mu \\
&= 2(h(s_\rho[u]_\rho)_{x_i} \cdot \nabla_v \bar{h}_{x_i})_\mu \\
&\leqslant c\|u\|_{L^2(\kappa_\rho)} \sqrt{\mathcal{D}_{xv}} \qquad \text{(by the assumption (ii) in Theorem 3.1.2).}
\end{aligned}$$

Combining all the estimates for J_A, J_B and J_u we get

$$\frac{d}{dt}\mathcal{I}_{xx}(h) \leqslant c\sqrt{\mathcal{D}_{xv}\mathcal{I}_{vv}} - 2\mathcal{D}_{xv} + c\|u\|_{L^2(\kappa_\rho)}\sqrt{\mathcal{D}_{xv}}.$$

Then by Young's inequality, the lemma is derived. □

APPENDIX

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