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Norm inflation for generalized magneto-hydrodynamic system

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Abstract

We consider the incompressible magneto-hydrodynamic system with fractional powers of the Laplacian in the three-dimensional case. We discover a wide range of spaces where the norm inflation occurs and hence small initial data results are out of reach. The norm inflation occurs not only in scaling invariant (critical) spaces, but also in supercritical and, surprisingly, subcritical ones.

Keywords: norm inflation, generalized magneto-hydrodynamic system, Besov spaces, interactions of plane waves
Mathematics Subject Classification: 76D03, 35Q35

1. Introduction

In this paper we study the three dimensional incompressible magneto-hydrodynamic (MHD) system with fractional powers of the Laplacian:

$$\begin{aligned} &u_t + (u\cdot\nabla)u - (b\cdot\nabla)b + \nabla p = -\nu(-\Delta)^{\alpha_1}u, \\ &b_t + (u\cdot\nabla)b - (b\cdot\nabla)u = -\mu(-\Delta)^{\alpha_2}b, \\ &u(x,0) = u_0, \quad b(x,0) = b_0, \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^3$, $t \geqslant 0$, u is the fluid velocity, p is the pressure of the fluid and b is the magnetic field. The parameter ν denotes the kinematic viscosity coefficient of the fluid and μ denotes the reciprocal of the magnetic Reynolds number. The initial data u_0 and b_0 are divergence free. The power $\alpha_1 = \alpha_2 = 1$ corresponds to the classical MHD system. A vast amount of literature has been devoted to these equations, for background we refer the readers to [3, 7, 11–14].

Solutions to the generalized MHD system (1.1) are scaling invariant when $\alpha_1 = \alpha_2 = \alpha > 0$. In this case, if $(u(x,t), p(x,t), b(x,t))$ solves system (1.1) with the initial data $(u_0(x), b_0(x))$, then

$$u_\lambda(x,t) = \lambda^{2\alpha-1}u(\lambda x, \lambda^{2\alpha}t), \quad p_\lambda(x,t) = \lambda^{2(2\alpha-1)}p(\lambda x, \lambda^{2\alpha}t)$$
$$b_\lambda(x,t) = \lambda^{2\alpha-1}b(\lambda x, \lambda^{2\alpha}t)$$

solves system (1.1) with the initial data

$$u_{0\lambda} = \lambda^{2\alpha-1}u_0(\lambda x), \; b_{0\lambda} = \lambda^{2\alpha-1}b_0(\lambda x).$$

A space that is invariant under the above scaling is called a critical space. The largest critical space for the generalized MHD system (1.1) when $\alpha_1 = \alpha_2 = \alpha$ is the Besov space $\dot{B}^{1-2\alpha}_{\infty,\infty}$ (see [2]).

In the case $b = 0$, (1.1) reduces to the generalized Navier–Stokes system that has been studied extensively. Since the global regularity is only known for $\alpha_1 \geqslant 5/4$, the question of global well-posedness in various critical spaces is of great interest. In the classical case $\alpha_1 = 1$ the best small initial data result is due to Koch and Tataru. In [8], they established the global well-posedness of the Navier–Stokes equations with small initial data in the space BMO^{-1}. In the case $\alpha_1 \in (1/2, 1)$ the situation is better as Yu and Zhai [16] showed global well-posedness for small initial data in the largest critical space $\dot{B}^{1-2\alpha_1}_{\infty,\infty}$, which is out of reach for $\alpha_1 \in [1, 5/4)$. Obstacles here are illustrated by some Ill-posedness results: Bourgain and Pavlović [1] showed the norm inflation for the classical Navier–Stokes equations in $\dot{B}^{-1}_{\infty,\infty}$ in case $\alpha_1 = 1$, Cheskidov and Shvydkoy [5, 6] proved the existence of discontinuous Leray–Hopf solutions of the Navier–Stokes equations in $\dot{B}^{1-2\alpha_1}_{\infty,\infty}$ with arbitrarily small initial data for $\alpha_1 \in [1, 5/4)$, and Yoneda [15] showed the norm inflation for the classical Navier–Stokes equation in a generalized Besov space near BMO^{-1}.

Recently, Cheskidov and Dai [4] showed the norm inflation in subcritical spaces $\dot{B}^{-s}_{\infty,\infty}$ for all $s \geqslant \alpha_1, \alpha_1 \geqslant 1$. This provides a wide range of spaces where a small initial data result is not expected. Note that the natural space for the norm inflation $B^{-\alpha_1}_{\infty,\infty}$ is only scaling invariant in the classical case $\alpha_1 = 1$, and is subcritical for $\alpha_1 > 1$. This explains why small initial data results are only available for $\alpha_1 < 1$.

The goal of this paper is to find natural norm inflation spaces for the generalized MHD system (1.1) and show that in general they are not scaling invariant, even in the classical case. Namely, we prove that

Theorem 1.1. *Let $\alpha_1, \alpha_2 \geqslant 1$. Assume $\theta_1 + \theta_2 = 2\alpha_2$ for $\theta_1, \theta_2 > 0$. For any $\delta > 0$ there exists a smooth space-periodic solution $(u(t), b(t))$ of (1.1) with period 2π and the initial data*

$$\|u(0)\|_{\dot{B}^{-\theta_1}_{\infty,\infty}} + \|b(0)\|_{\dot{B}^{-\theta_2}_{\infty,\infty}} \lesssim \delta$$

that satisfies, for some $0 < T < \delta$ and all $s > 0$,

$$\|b(T)\|_{\dot{B}^{-s}_{\infty,\infty}} \gtrsim \frac{1}{\delta}.$$

We refer the readers to the beginning of section of Preliminaries for the definition of the symbol $\lesssim$. Note that the homogeneous and non-homogeneous Besov norms are equivalent for periodic functions. Therefore, for the space-periodic solution in theorem 1.1 we have

$$\|b(0)\|_{\dot{B}^{-s}_{\infty,\infty}} \lesssim \|b(0)\|_{B^{-s}_{\infty,\infty}} \lesssim \|b(0)\|_{\dot{B}^{-\theta_2}_{\infty,\infty}} \lesssim \delta \qquad \text{for all} \quad s \geqslant \theta_2,$$

which means that the norm inflation for b occurs in all the spaces $\dot{B}^{-s}_{\infty,\infty}$, $s > 0$. More precisely, we have the following.

Corollary 1.2. *Let $\alpha_1, \alpha_2 \geqslant 1$. For any $s > 0$, $\delta > 0$, and positive θ such that $\theta \geqslant 2\alpha_2 - s$ there exists a smooth space-periodic solution $(u(t), b(t))$ of (1.1) with the initial data*

$$\|u(0)\|_{\dot{B}^{-\theta}_{\infty,\infty}} + \|b(0)\|_{\dot{B}^{-s}_{\infty,\infty}} \lesssim \delta$$

that satisfies, for some $0 < T < \delta$,

$$\|b(T)\|_{\dot{B}^{-s}_{\infty,\infty}} \gtrsim \frac{1}{\delta}.$$

In a previous work of Dai *et al* [7], the authors obtained norm inflations for the classical MHD system, i.e. $\alpha_1 = \alpha_2 = 1$ in (1.1). More precisely, the main result reads that initial data can be constructed such that

$$\|u(0)\|_{\dot{B}^{-1}_{\infty,\infty}} + \|b(0)\|_{\dot{B}^{-1}_{\infty,\infty}} \lesssim \delta,$$

while

$$\|b(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \gtrsim \frac{1}{\delta}, \qquad \text{for some } 0 < T < \delta.$$

One can see the above result is a particular case of corollary 1.2 corresponding to $\alpha_1 = \alpha_2 = \theta = s = 1$. In fact, taking $\alpha_1 = \alpha_2 = 1$, corollary 1.2 guarantees that the magnetic field b develops norm inflation in all the spaces $\dot{B}^{-s}_{\infty,\infty}$, $s > 0$, not only the critical one $\dot{B}^{1-2\alpha_1}_{\infty,\infty}$.

We point out another difference with [7]. For the classical MHD system, the proof of the norm inflation is based on the continuity of the bilinear operator (see (2.7) in [7]) associated with the heat kernel on certain Koch–Tataru space to control the nonlinear interactions in the error terms. For the bilinear operator corresponding to the fractional heat kernel, as discussed in [4], the continuity property on a modified Koch–Tataru adapted space is not available. Instead, the bilinear operator is estimated in L^∞ in order to control the error terms in section 4.3.

The rest of the paper is organized as follows. In section 2 we introduce some notations that shall be used throughout the paper and some auxiliary results; in section 3 we describe how the diffusions of plane waves interact in the fractional MHD system; in section 4 we devote to proving theorem 1.1.

2. Preliminaries

2.1. Notation

We denote by $A \lesssim B$ an estimate of the form $A \leqslant CB$ with some absolute constant C, and by $A \sim B$ an estimate of the form $C_1 B \leqslant A \leqslant C_2 B$ with some absolute constants C_1, C_2. For simplification of the notation, we denote $\|\cdot\|_p = \|\cdot\|_{L^p}$.

2.2. Norm of Besov spaces

We recall the definitions of norms for the homogeneous and non-homogeneous Besov spaces $\dot{B}^{-s}_{\infty,\infty}$ and $B^{-s}_{\infty,\infty}$ (see [9])

$$\|f\|_{\dot{B}^{-s}_{\infty,\infty}} = \sup_{t>0} t^{\frac{s}{2\alpha}} \|\mathrm{e}^{-t(-\Delta)^\alpha} f\|_{L^\infty},$$

$$\|f\|_{B^{-s}_{\infty,\infty}} = \sup_{0<t<1} t^{\frac{s}{2\alpha}} \|\mathrm{e}^{-t(-\Delta)^\alpha} f\|_{L^\infty}.$$

Note that for the periodic functions the homogeneous and non-homogeneous norms are equivalent (see [10]). It is then easy to observe that

$$\|f\|_{\dot{B}^{-s}_{\infty,\infty}(\mathbb{T}^3)} \leqslant \|f\|_{L^\infty(\mathbb{T}^3)}, \tag{2.2}$$

since $\|\mathrm{e}^{-t(-\Delta)^\alpha} f\|_{L^\infty} \leqslant \|f\|_{L^\infty}$.

2.3. Bilinear operator

Let $\mathbb{P}$ denote the projection on divergence-free vector fields, which acts on a function ϕ as

$$\mathbb{P}(\phi) = \phi + \nabla \cdot (-\triangle)^{-1}\mathrm{div}\phi.$$

Define the bilinear operator for $\alpha > 1$

$$\mathcal{B}_\alpha(u, v) = \int_0^t \mathrm{e}^{-(t-\tau)(-\Delta)^\alpha}\mathbb{P}\nabla \cdot (u \otimes v)\,\mathrm{d}\tau. \tag{2.3}$$

As shown in [4] the following estimate for the bilinear operator $\mathcal{B}_\alpha$ holds.

Lemma 2.1. *Let $u, v \in L^1(0, T; L^\infty)$ be such that $u \otimes v \in L^1(0, T; L^\infty)$. Then for all $\alpha > 0$, the bilinear operator satisfies*

$$\|\mathcal{B}_\alpha(u, v)\|_\infty \lesssim \int_0^t \frac{1}{(t-\tau)^{1/2\alpha}}\|u(\tau)\|_\infty\|v(\tau)\|_\infty\,\mathrm{d}\tau. \tag{2.4}$$

3. Interactions of plane waves

3.1. The first iteration approximation of a mild solution

Let (u, b) be a solution to (1.1). We write it in the form

$$u = \mathrm{e}^{-t(-\Delta)^{\alpha_1}}u_0 - u_1 + y, \tag{3.5}$$

$$b = \mathrm{e}^{-t(-\Delta)^{\alpha_2}}b_0 - b_1 + z, \tag{3.6}$$

where

$$\begin{aligned} u_1(x,t) = {} & \mathcal{B}_{\alpha_1}(\mathrm{e}^{-t(-\Delta)^{\alpha_1}}u_0(x), \mathrm{e}^{-t(-\Delta)^{\alpha_1}}u_0(x)) \\ & -\mathcal{B}_{\alpha_1}(\mathrm{e}^{-t(-\Delta)^{\alpha_1}}b_0(x), \mathrm{e}^{-t(-\Delta)^{\alpha_1}}b_0(x)), \end{aligned} \tag{3.7}$$

$$\begin{aligned} b_1(x,t) = {} & \mathcal{B}_{\alpha_2}(\mathrm{e}^{-t(-\Delta)^{\alpha_2}}u_0(x), \mathrm{e}^{-t(-\Delta)^{\alpha_2}}b_0(x)) \\ & -\mathcal{B}_{\alpha_2}(\mathrm{e}^{-t(-\Delta)^{\alpha_2}}b_0(x), \mathrm{e}^{-t(-\Delta)^{\alpha_2}}u_0(x)). \end{aligned} \tag{3.8}$$

A simple calculation shows that

$$y(t) = -\int_0^t \mathrm{e}^{-(t-\tau)(-\Delta)^{\alpha_1}}[G_0(\tau) + G_1(\tau) + G_2(\tau)]\,\mathrm{d}\tau, \tag{3.9}$$

$$z(t) = -\int_0^t \mathrm{e}^{-(t-\tau)(-\Delta)^{\alpha_2}}[K_0(\tau) + K_1(\tau) + K_2(\tau)]\,\mathrm{d}\tau, \tag{3.10}$$

where

$$\begin{aligned} G_0 = {} & \mathbb{P}[(\mathrm{e}^{-t(-\Delta)^{\alpha_1}}u_0 \cdot \nabla)u_1 + (u_1 \cdot \nabla)\mathrm{e}^{-t(-\Delta)^{\alpha_1}}u_0 + (u_1 \cdot \nabla)u_1] \\ & - \mathbb{P}[(\mathrm{e}^{-t(-\Delta)^{\alpha_1}}b_0 \cdot \nabla)b_1 + (b_1 \cdot \nabla)\mathrm{e}^{-t(-\Delta)^{\alpha_1}}b_0 + (b_1 \cdot \nabla)b_1], \\ G_1 = {} & \mathbb{P}[(\mathrm{e}^{-t(-\Delta)^{\alpha_1}}u_0 \cdot \nabla)y + (u_1 \cdot \nabla)y + (y \cdot \nabla)\mathrm{e}^{-t(-\Delta)^{\alpha_1}}u_0 + (y \cdot \nabla)u_1] \\ & - \mathbb{P}[(\mathrm{e}^{-t(-\Delta)^{\alpha_1}}b_0 \cdot \nabla)z + (b_1 \cdot \nabla)z + (z \cdot \nabla)\mathrm{e}^{-t(-\Delta)^{\alpha_1}}b_0 + (z \cdot \nabla)b_1], \\ G_2 = {} & \mathbb{P}[(y \cdot \nabla)y] - \mathbb{P}[(z \cdot \nabla)z], \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} K_0 = {} & \mathbb{P}[(\mathrm{e}^{-t(-\Delta)^{\alpha_2}}u_0 \cdot \nabla)b_1 + (u_1 \cdot \nabla)\mathrm{e}^{-t(-\Delta)^{\alpha_2}}b_0 + (u_1 \cdot \nabla)b_1] \\ & - \mathbb{P}[(\mathrm{e}^{-t(-\Delta)^{\alpha_2}}b_0 \cdot \nabla)u_1 + (b_1 \cdot \nabla)\mathrm{e}^{-t(-\Delta)^{\alpha_2}}u_0 + (b_1 \cdot \nabla)u_1], \\ K_1 = {} & \mathbb{P}[(\mathrm{e}^{-t(-\Delta)^{\alpha_2}}u_0 \cdot \nabla)z + (u_1 \cdot \nabla)z + (y \cdot \nabla)\mathrm{e}^{-t(-\Delta)^{\alpha_2}}b_0 + (y \cdot \nabla)b_1] \\ & - \mathbb{P}[(\mathrm{e}^{-t(-\Delta)^{\alpha_2}}b_0 \cdot \nabla)y + (b_1 \cdot \nabla)y + (z \cdot \nabla)\mathrm{e}^{-t(-\Delta)^{\alpha_2}}u_0 + (z \cdot \nabla)u_1], \\ K_2 = {} & \mathbb{P}[(y \cdot \nabla)z] - \mathbb{P}[(z \cdot \nabla)y]. \end{aligned} \tag{3.12}$$

Note that G_0, K_0 do not depend on y or z, G_1, K_1 are linear, and G_2, K_2 are quadratic in y and z.

Remark 3.1. Note that although the second equation in system (1.1) has no pressure, since u and b are both divergence free, the term $u \cdot \nabla b - b \cdot \nabla u$ is automatically divergence free. Hence the projector $\mathbb{P}$ acting on this term does not change the second equation and hence we can write b_1 and K_i's as described above.

In this section we show how the diffusions of plane waves interact in the generalized MHD system. These interactions are the basis for the constructions of initial data to produce the norm inflation.

3.2. Diffusion of a plane wave

Suppose $k \in \mathbb{R}^3$, $v \in \mathbb{S}^2$ and $k \cdot v = 0$. Let

$$u_0 = v \cos(k \cdot x).$$

Then $\nabla \cdot u_0 = 0$ and

$$\mathrm{e}^{-t(-\Delta)^\alpha} v \cos(k \cdot x) = \mathrm{e}^{-|k|^{2\alpha} t} v \cos(k \cdot x). \tag{3.13}$$

It is also important to notice that for $s > 0$

$$\| v \cos(k \cdot x) \|_{\dot{B}^{-s}_{\infty,\infty}} \sim |k|^{-s}.$$

3.3. Interaction of plane waves

Now we consider the interaction of two different single plane waves. Suppose $k_i \in \mathbb{R}^3$, $v_i \in \mathbb{S}^2$ and $k_i \cdot v_i = 0$, for $i = 1, 2$. Let

$$u_1 = \cos(k_1 \cdot x) v_1, \qquad u_2 = \cos(k_2 \cdot x) v_2.$$

To simplify our calculations we assume that $k_2 \cdot v_1 = \frac{1}{2}$. It then follows from a straightforward calculation that

$$\begin{aligned}
&\mathcal{B}_\alpha(\mathrm{e}^{-t(-\Delta)^\alpha} u_1, \mathrm{e}^{-t(-\Delta)^\alpha} u_2) \\
&\quad = \frac{1}{4} v_1 \sin((k_2 - k_1) \cdot x) \int_0^t \mathrm{e}^{-(|k_1|^{2\alpha} + |k_2|^{2\alpha})\tau} \mathrm{e}^{-|k_2 - k_1|^{2\alpha}(t-\tau)} \, \mathrm{d}\tau \\
&\quad\quad + \frac{1}{4} v_1 \sin((k_1 + k_2) \cdot x) \int_0^t \mathrm{e}^{-(|k_1|^{2\alpha} + |k_2|^{2\alpha})\tau} \mathrm{e}^{-|k_1 + k_2|^{2\alpha}(t-\tau)} \, \mathrm{d}\tau.
\end{aligned}$$

Therefore, the interaction of the two plane waves is small in $\dot{B}^{-s}_{\infty,\infty}$ if neither the sum nor the difference of their wave vectors is small in magnitude. In the contrary, the interaction is sizable in $\dot{B}^{-s}_{\infty,\infty}$ if either the sum or the difference of their wave vectors is small in magnitude.

4. Proof of theorem 1.1

In this section we follow the ideas from [1, 4] to construct initial data that produce norm inflation for solutions to the fractional MHD system. From the discussions in section 3.3 we see that the interaction of two plane waves is not enough to show the norm inflation, which requires a large number of waves. We also make sure that the initial data is space-periodic and smooth, which ensures the local existence of a smooth periodic solution to the fractional MHD system. As we control its L^∞ norm, the solution will remain smooth until the time of the norm inflation.

4.1. Construction of initial data for the fractional MHD system

For a fixed small number $\delta > 0$ that we will specify later, the initial data will be as follows:

$$u_0 = r^{-\beta_1} \sum_{i=1}^{r} |k_i|^{\theta_1} v \cos(k_i \cdot x), \tag{4.14}$$

$$b_0 = r^{-\beta_2} \sum_{i=1}^{r} |k_i'|^{\theta_2} v' \cos(k_i' \cdot x),$$

where $\beta_1, \beta_2, \theta_1, \theta_2 > 0$ to be determined later. We expect for each i the interaction of the two plane waves $v \cos(k_i \cdot x)$ and $v' \cos(k_i' \cdot x)$ to be sizable in $\dot{B}^{-s}_{\infty,\infty}$, while the interactions of plane waves corresponding to different indexes i to be small. Hence, we choose

- *Wave vectors.* Let $\zeta = (1, 0, 0)$ and $\eta = (0, 0, 1)$. The wave vectors $k_i \in \mathbb{Z}^3$ are parallel to ζ. Let K be a large integer dependent on r. The magnitude of k_i is defined by

$$|k_i| = 2^{i-1} K, \qquad i = 1, 2, 3, \ldots, r. \tag{4.15}$$

  The wave vectors $k_i' \in \mathbb{Z}^3$ are defined by

$$k_i' = k_i + \eta. \tag{4.16}$$

- *Amplitude vectors.* Let

$$v = (0, 0, 1), \qquad v' = (0, 1, 0). \tag{4.17}$$

  Hence

$$k_i \cdot v = k_i' \cdot v' = 0$$

  which ensures that the initial data is divergence free.

The initial data construction here is not the same as the one used for the classical MHD system in [7]. We still use the fact that the interaction of the high–high frequencies waves produces low frequency wave, which is the essential part to develop the norm inflation in Besov spaces with negative indexes. However, thanks to the coupling of the velocity field and the magnetic field, we have the flexibility of having more admissible values of $\beta_1, \beta_2, \theta_1$ and θ_2, which allows us to obtain a wider range of spaces where the system (1.1) develops norm inflation (see theorem 1.1 and corollary 1.2).

We recall the following simple facts whose proof can be found in [4].

Lemma 4.1. *Let $\gamma, \theta > 0$. With the choices (4.15)–(4.17), the following holds:*

$$k_i \cdot v' = 0, \qquad k_i' \cdot v = 1, \quad \forall \quad i = 1, 2, \ldots, r, \tag{4.18}$$

$$\sum_{j<i} |k_j|^{\theta} \sim |k_{i-1}|^{\theta} \quad \text{and} \quad \sum_{j<i} |k_j'|^{\theta} \sim |k_{i-1}'|^{\theta}, \tag{4.19}$$

$$\sum_{i=1}^{r} |k_i|^{\gamma} \mathrm{e}^{-|k_i|^{2\alpha} t} \lesssim t^{-\frac{\gamma}{2\alpha}} \quad \text{and} \quad \sum_{i=1}^{r} |k_i'|^{\gamma} \mathrm{e}^{-|k_i'|^{2\alpha} t} \lesssim t^{-\frac{\gamma}{2\alpha}}. \tag{4.20}$$

Next we estimate the norms of the initial data.

Lemma 4.2. *Let (u_0, b_0) be given in (4.14) and $\theta_1, \theta_2 > 0$. Then*

$$\|u_0\|_{\dot{B}^{-\theta_1}_{\infty,\infty}} \lesssim r^{-\beta_1}, \qquad \|b_0\|_{\dot{B}^{-\theta_2}_{\infty,\infty}} \lesssim r^{-\beta_2}. \tag{4.21}$$

Proof. Due to (3.13), we have that,

$$e^{-t(-\Delta)^{\theta_1}}u_0 = r^{-\beta_1}\sum_{i=1}^{r}|k_i|^{\theta_1}v\cos(k_i\cdot x)e^{-|k_i|^{2\theta_1}t}. \tag{4.22}$$

Hence by lemma 4.1,

$$\|u_0\|_{\dot{B}^{-\theta_1}_{\infty,\infty}} \sim r^{-\beta_1}\sup_{0<t<1} t^{\frac{1}{2}}\sum_{i=1}^{r}|k_i|^{\theta_1}e^{-|k_i|^{2\theta_1}t} \lesssim r^{-\beta_1}.$$

The estimate of b_0 in $\dot{B}^{-\theta_2}_{\infty,\infty}$ can be obtained similarly. □

Lemma 4.3. *Let* (u_0, b_0) *be given in (4.14) and* $\alpha > 0$. *Then*

$$\|e^{-t(-\Delta)^{\alpha}}u_0\|_\infty \lesssim r^{-\beta_1}t^{-\frac{\theta_1}{2\alpha}}, \qquad \|e^{-t(-\Delta)^{\alpha}}b_0\|_\infty \lesssim r^{-\beta_2}t^{-\frac{\theta_2}{2\alpha}}.$$

Proof. By (4.22) and lemma 4.1, we infer that

$$\|e^{-t(-\Delta)^{\alpha}}u_0\|_\infty \lesssim r^{-\beta_1}\sum_{i=1}^{r}|k_i|^{\theta_1}e^{-|k_i|^{2\alpha}t} \lesssim r^{-\beta_1}t^{-\frac{\theta_1}{2\alpha}}.$$

The estimate for b_0 can be obtained similarly. □

4.2. Analysis of u_1 and b_1

As demonstrated in section 3.1 we consider the decomposition

$$\begin{aligned} u &= e^{-t(-\Delta)^{\alpha_1}}u_0 - u_1 + y,\\ b &= e^{-t(-\Delta)^{\alpha_2}}b_0 - b_1 + z. \end{aligned}$$

Recall the definition (3.7) and (3.8)

$$\begin{aligned} u_1(x,t) = {} & \mathcal{B}_{\alpha_1}(e^{-t(-\Delta)^{\alpha_1}}u_0(x), e^{-t(-\Delta)^{\alpha_1}}u_0(x))\\ & - \mathcal{B}_{\alpha_1}(e^{-t(-\Delta)^{\alpha_1}}b_0(x), e^{-t(-\Delta)^{\alpha_1}}b_0(x)), \end{aligned}$$

$$\begin{aligned} b_1(x,t) = {} & \mathcal{B}_{\alpha_2}(e^{-t(-\Delta)^{\alpha_2}}u_0(x), e^{-t(-\Delta)^{\alpha_2}}b_0(x))\\ & - \mathcal{B}_{\alpha_2}(e^{-t(-\Delta)^{\alpha_2}}b_0(x), e^{-t(-\Delta)^{\alpha_2}}u_0(x)). \end{aligned}$$

By the fact that $k_i\cdot v = k_i'\cdot v' = 0$ for all $i = 1, 2, \ldots, r$, it is immediately seen

$$\left(e^{-t(-\Delta)^{\alpha_1}}u_0\cdot\nabla\right)e^{-t(-\Delta)^{\alpha_1}}u_0 = \left(e^{-t(-\Delta)^{\alpha_1}}b_0\cdot\nabla\right)e^{-t(-\Delta)^{\alpha_1}}b_0 = 0,$$

hence $u_1 \equiv 0$. Again since $k_i\cdot v' = 0$ for all $i = 1, 2, \ldots, r$ by (4.18), it follows

$$\left(e^{-t(-\Delta)^{\alpha_2}}b_0\cdot\nabla\right)e^{-t(-\Delta)^{\alpha_2}}u_0 = 0,$$

hence

$$b_1(x,t) = \mathcal{B}_{\alpha_2}(e^{-t(-\Delta)^{\alpha_2}}u_0(x), e^{-t(-\Delta)^{\alpha_2}}b_0(x)). \tag{4.23}$$

By (4.17), (4.18), (4.22) and a straightforward calculation, it follows that

$$
\begin{aligned}
&\left(\mathrm{e}^{-t(-\Delta)^{\alpha_2}}u_0\cdot\nabla\right)\mathrm{e}^{-t(-\Delta)^{\alpha_2}}b_0\\
&\quad= -r^{-\beta_1-\beta_2}\sum_{i=1}^{r}\sum_{j=1}^{r}|k_i|^{\theta_1}|k_j'|^{\theta_2}\mathrm{e}^{-(|k_i|^{2\alpha_2}+|k_j'|^{2\alpha_2})t}v'\cos(k_i\cdot x)\sin(k_j'\cdot x)\\
&\quad= -\frac{r^{-\beta_1-\beta_2}}{2}\sum_{i=1}^{r}|k_i|^{\theta_1}|k_i'|^{\theta_2}\mathrm{e}^{-(|k_i|^{2\alpha_2}+|k_i'|^{2\alpha_2})t}\sin(\eta\cdot x)v'\\
&\qquad -\frac{r^{-\beta_1-\beta_2}}{2}\sum_{i\neq j}^{r}|k_i|^{\theta_1}|k_j'|^{\theta_2}\mathrm{e}^{-(|k_i|^{2\alpha_2}+|k_j'|^{2\alpha_2})t}\sin((k_j'-k_i)\cdot x)v'\\
&\qquad -\frac{r^{-\beta_1-\beta_2}}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}|k_i|^{\theta_1}|k_j'|^{\theta_2}\mathrm{e}^{-(|k_i|^{2\alpha_2}+|k_j'|^{2\alpha_2})t}\sin((k_j'+k_i)\cdot x)v'\\
&\quad := E_0+E_1+E_2,
\end{aligned}
\tag{4.24}
$$

where we used the formula $\cos x\sin y=[\sin(x+y)-\sin(x-y)]/2$.

Recall that $\eta\cdot v'=0$, $(k_j'+k_i)\cdot v'=0$ and $(k_j'-k_i)\cdot v'=0$ for all i, j due to (4.18). Hence E_0, E_1 and E_2 are divergence free vectors. Thus we can write

$$
\begin{aligned}
b_1 &= \int_0^t \mathrm{e}^{-(t-\tau)(-\Delta)^{\alpha_2}}E_0(\tau)\,\mathrm{d}\tau+\int_0^t \mathrm{e}^{-(t-\tau)(-\Delta)^{\alpha_2}}E_1(\tau)\,\mathrm{d}\tau\\
&\quad+\int_0^t \mathrm{e}^{-(t-\tau)(-\Delta)^{\alpha_2}}E_2(\tau)\,\mathrm{d}\tau := b_{10}+b_{11}+b_{12}.
\end{aligned}
\tag{4.25}
$$

We have the following estimates.

Lemma 4.4. *Let b_{10} be defined in (4.25) and $s>0$. Assume $\theta_1+\theta_2=2\alpha_2$. Then*

$$
\begin{aligned}
&\|b_{10}(\cdot,t)\|_{\dot B^{-s}_{\infty,\infty}}\gtrsim r^{1-\beta_1-\beta_2},\quad \text{for}\quad |k_1|^{-2\alpha_2}\leqslant t\leqslant T,\\
&\|b_{10}(\cdot,t)\|_\infty\lesssim r^{1-\beta_1-\beta_2},\quad \text{for all}\quad t>0.
\end{aligned}
$$

Proof. From (4.24) and (4.25) it follows by a straightforward calculation

$$
\begin{aligned}
b_{10} &= -\frac{r^{-\beta_1-\beta_2}}{2}\int_0^t\sum_{i=1}^{r}|k_i|^{\theta_1}|k_i'|^{\theta_2}\mathrm{e}^{-(|k_i'|^{2\alpha_2}+|k_i|^{2\alpha_2})\tau}\mathrm{e}^{-|\eta|^{2\alpha_2}(t-\tau)}\sin(\eta\cdot x)v'\,\mathrm{d}\tau\\
&= -\frac{r^{-\beta_1-\beta_2}}{2}\sin(\eta\cdot x)v'\sum_{i=1}^{r}|k_i|^{\theta_1}|k_i'|^{\theta_2}\mathrm{e}^{-t}\frac{1-\mathrm{e}^{-(|k_i'|^{2\alpha_2}+|k_i|^{2\alpha_2}-1)t}}{|k_i'|^{2\alpha_2}+|k_i|^{2\alpha_2}-1}\\
&\sim -\frac{r^{-\beta_1-\beta_2}}{2}\sin(\eta\cdot x)v'\sum_{i=1}^{r}\mathrm{e}^{-t}(1-\mathrm{e}^{-|k_i|^{2\alpha_2}t}),
\end{aligned}
$$

where the assumption $\theta_1+\theta_2=2\alpha_2$ implies

$$
|k_i|^{\theta_1}|k_i'|^{\theta_2}\sim|k_i'|^{2\alpha_2}+|k_i|^{2\alpha_2}-1.
$$

Hence for $|k_1|^{-2\alpha_2}\leqslant t\leqslant T$ and $s>0$,

$$
\|b_{10}(\cdot,t)\|_{\dot B^{-s}_{\infty,\infty}}\gtrsim r^{-\beta_1-\beta_2}\cdot r\sup_{0<t<1}t^{\frac{s}{2\alpha}}\mathrm{e}^{-|\eta|^{2\alpha}t}\gtrsim r^{1-\beta_1-\beta_2}.
$$

On the other hand,

$$
\|b_{10}(\cdot,t)\|_\infty\lesssim\frac{r^{-\beta_1-\beta_2}}{2}\cdot r\lesssim r^{1-\beta_1-\beta_2},
$$

for all $t>0$. □

Lemma 4.5. *Let b_{11} and b_{12} be defined in (4.25). Then*

$$\|b_{11}(\cdot,t)\|_\infty + \|b_{12}(\cdot,t)\|_\infty \lesssim r^{-\beta_1-\beta_2} t^{1-(\theta_1+\theta_2)/2\alpha_2},$$

for all $t > 0$.

Proof. Thanks to (4.24) and (4.25), it follows that

$$\begin{aligned}
b_{11} &= \frac{r^{-\beta_1-\beta_2}}{2}\int_0^t \sum_{i\neq j}^{r} |k_i|^{\theta_1}|k_j'|^{\theta_2} \mathrm{e}^{-(|k_i|^{2\alpha_2}+|k_j'|^{2\alpha_2})\tau}\mathrm{e}^{-|k_j'-k_i|^{2\alpha_2}(t-\tau)} \cdot \sin((k_j'-k_i)\cdot x)v'\,\mathrm{d}\tau \\
&\sim \frac{r^{-\beta_1-\beta_2}}{2}\sum_{i=1}^{r}\sum_{j<i} |k_i|^{\theta_1}|k_j'|^{\theta_2}\mathrm{e}^{-|k_i-k_j'|^{2\alpha_2}t}\frac{1-\mathrm{e}^{-(|k_i|^{2\alpha_2}+|k_j'|^{2\alpha_2}-|k_i-k_j'|^{2\alpha_2})t}}{|k_i|^{2\alpha_2}+|k_j'|^{2\alpha_2}-|k_i-k_j'|^{2\alpha_2}} \\
&\quad\cdot \sin((k_j'-k_i)\cdot x)v' \\
&\sim \frac{r^{-\beta_1-\beta_2}}{2}\sum_{i=1}^{r}\sum_{j<i} |k_i|^{\theta_1}|k_j'|^{\theta_2} t e^{-|k_i|^{2\alpha_2}t}\sin((k_j'-k_i)\cdot x)v',
\end{aligned}$$

where we used the fact that $\frac{1-\mathrm{e}^{-x}}{x}$ is bounded for $x > 0$. Hence, by (4.19) and (4.20) we infer that

$$\begin{aligned}
\|b_{11}(\cdot,t)\|_\infty &\lesssim r^{-\beta_1-\beta_2}\sum_{i=1}^{r}\sum_{j<i} |k_i|^{\theta_1}|k_j'|^{\theta_2} t\mathrm{e}^{-|k_i|^{2\alpha_2}t} \\
&\lesssim r^{-\beta_1-\beta_2}\sum_{i=1}^{r} |k_i|^{\theta_1}|k_i'|^{\theta_2} t\mathrm{e}^{-|k_i|^{2\alpha_2}t} \\
&\lesssim r^{-\beta_1-\beta_2}\sum_{i=1}^{r} |k_i|^{\theta_1+\theta_2} t\mathrm{e}^{-|k_i|^{2\alpha_2}t} \\
&\lesssim r^{-\beta_1-\beta_2} t^{1-(\theta_1+\theta_2)/2\alpha_2}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
b_{12} &= \frac{r^{-\beta_1-\beta_2}}{2}\int_0^t \sum_{i=1}^{r}\sum_{j=1}^{r} |k_i|^{\theta_1}|k_j'|^{\theta_2} \mathrm{e}^{-(|k_i|^{2\alpha_2}+|k_j'|^{2\alpha_2})\tau}\mathrm{e}^{-|k_i+k_j'|^{2\alpha_2}(t-\tau)} \cdot \sin((k_i+k_j')\cdot x)v'\,\mathrm{d}\tau \\
&= \frac{r^{-\beta_1-\beta_2}}{2}\sum_{i=1}^{r}\sum_{j=1}^{r} |k_i|^{\theta_1}|k_j'|^{\theta_2}\mathrm{e}^{-(|k_i|^{2\alpha_2}+|k_j'|^{2\alpha_2})t}\frac{1-\mathrm{e}^{-(|k_i+k_j'|^{2\alpha_2}-|k_i|^{2\alpha_2}-|k_j'|^{2\alpha_2})t}}{|k_i+k_j'|^{2\alpha_2}-|k_i|^{2\alpha_2}-|k_j'|^{2\alpha_2}} \\
&\quad\cdot \sin((k_i+k_j')\cdot x)v' \\
&\sim r^{-\beta_1-\beta_2}\sum_{i=1}^{r}\sum_{j\leqslant i} |k_i|^{\theta_1}|k_j'|^{\theta_2} t\mathrm{e}^{-(|k_i|^{2\alpha_2}+|k_j'|^{2\alpha_2})t}\sin((k_i+k_j')\cdot x)v'.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|b_{12}(\cdot,t)\|_\infty &\lesssim r^{-\beta_1-\beta_2}\sum_{i=1}^{r}\sum_{j\leqslant i} |k_i|^{\theta_1}|k_j'|^{\theta_2} t\mathrm{e}^{-|k_i|^{2\alpha}t} \\
&\lesssim r^{-\beta_1-\beta_2}\sum_{i=1}^{r} |k_i|^{\theta_1+\theta_2} t\mathrm{e}^{-|k_i|^{2\alpha}t} \\
&\lesssim r^{-\beta_1-\beta_2} t^{1-(\theta_1+\theta_2)/2\alpha_2}.
\end{aligned}$$

□

4.3. Analysis of y and z

In this section we analyse the parts y and z of the solution. The idea is to control y, z using the estimate (2.4) of the bilinear operator $\mathcal{B}_\alpha$ in the space L^∞.

Recall from section 3.1 that

$$y(t) = -\int_0^t \mathrm{e}^{-(t-\tau)(-\Delta)^{\alpha_1}}[G_0(\tau) + G_1(\tau) + G_2(\tau)]\,\mathrm{d}\tau, \quad t \in [0, T]. \tag{4.26}$$

$$z(t) = -\int_0^t \mathrm{e}^{-(t-\tau)(-\Delta)^{\alpha_2}}[K_0(\tau) + K_1(\tau) + K_2(\tau)]\,\mathrm{d}\tau, \qquad t \in [0, T]. \tag{4.27}$$

Lemma 4.6. *Let $\alpha_1, \alpha_2 \in [1, \infty)$ and $\beta_1, \beta_2 \in (\frac{1}{3}, \frac{1}{2})$. Assume additionally that*

$$\begin{cases} \theta_1 + \theta_2 = 2\alpha_2, \\ 1 \leqslant \theta_1 \leqslant 2\alpha_1 - 1, \\ 1 \leqslant \theta_2 \leqslant 2\alpha_1 - 1. \end{cases} \tag{4.28}$$

Then

$$\begin{aligned} \|y(t)\|_\infty + \|z(t)\|_\infty \lesssim\, & r^{1-\beta_1-2\beta_2} t^{1-\frac{1}{2\alpha_1}-\frac{\theta_2}{2\alpha_1}} + r^{2(1-\beta_1-\beta_2)} t^{1-\frac{1}{2\alpha_1}} \\ & + r^{1-2\beta_1-\beta_2} t^{1-\frac{1}{2\alpha_2}-\frac{\theta_1}{2\alpha_2}}, \end{aligned} \tag{4.29}$$

for all $0 \leqslant t \leqslant T$, provided T is small and r is large enough.

Proof. Recall that $u_1 = 0$. It follows from (3.11) and (4.26) that

$$\begin{aligned} \|y(t)\|_\infty \lesssim\, & \|\mathcal{B}_{\alpha_1}(\mathrm{e}^{-t(-\Delta)^{\alpha_1}} b_0, b_1(t))\|_\infty + \|\mathcal{B}_{\alpha_1}(b_1(t), b_1(t))\|_\infty \\ & + \|\mathcal{B}_{\alpha_1}(\mathrm{e}^{-t(-\Delta)^{\alpha_1}} u_0, y(t))\|_\infty + \|\mathcal{B}_{\alpha_1}(\mathrm{e}^{-t(-\Delta)^{\alpha_1}} b_0, z(t))\|_\infty \\ & + \|\mathcal{B}_{\alpha_1}(b_1(t), z(t))\|_\infty + \|\mathcal{B}_{\alpha_1}(y(t), y(t))\|_\infty + \|\mathcal{B}_{\alpha_1}(z(t), z(t))\|_\infty. \end{aligned}$$

Applying the bilinear estimate (2.4), lemmas 4.3, 4.4, and 4.5 we infer

$$\begin{aligned} \|\mathcal{B}_{\alpha_1}(\mathrm{e}^{-t(-\Delta)^{\alpha_1}} b_0, b_1(t))\|_\infty & \lesssim \int_0^t \frac{1}{(t-\tau)^{1/2\alpha_1}} \|\mathrm{e}^{-\tau(-\Delta)^{\alpha_1}} b_0\|_\infty \|b_1(\tau)\|_\infty \,\mathrm{d}\tau \\ & \lesssim r^{1-\beta_1-2\beta_2} \int_0^t (t-\tau)^{-\frac{1}{2\alpha_1}} \tau^{-\frac{\theta_2}{2\alpha_1}} \,\mathrm{d}\tau \\ & \lesssim r^{1-\beta_1-2\beta_2} t^{1-\frac{1}{2\alpha_1}-\frac{\theta_2}{2\alpha_1}}, \end{aligned}$$

where we used the boundedness of Beta function for $\alpha > 1/2$ and $0 < \frac{\theta_2}{2\alpha_1} < 1$:

$$\int_0^t (t-\tau)^{-\frac{1}{2\alpha_1}} \tau^{-\frac{\theta_2}{2\alpha_1}} \,\mathrm{d}\tau = t^{1-\frac{1}{2\alpha_1}-\frac{\theta_2}{2\alpha_1}} B(1 - \frac{\theta_2}{2\alpha_1}, 1 - \frac{1}{2\alpha_1}) \leqslant C t^{1-\frac{1}{2\alpha_1}-\frac{\theta_2}{2\alpha_1}}.$$

Similarly, using the estimates obtained in previous two subsections, we obtain

$$\|\mathcal{B}_{\alpha_1}(b_1(t), b_1(t))\|_\infty \lesssim \int_0^t \frac{1}{(t-\tau)^{1/2\alpha_1}} \|b_1(\tau)\|_\infty^2 \,\mathrm{d}\tau \lesssim r^{2(1-\beta_1-\beta_2)} t^{1-\frac{1}{2\alpha_1}},$$

$$\begin{aligned} \|\mathcal{B}_{\alpha_1}(\mathrm{e}^{-t(-\Delta)^{\alpha_1}} u_0, y(t))\|_\infty & \lesssim \int_0^t \frac{1}{(t-\tau)^{1/2\alpha_1}} \|\mathrm{e}^{-\tau(-\Delta)^{\alpha_1}} u_0\|_\infty \|y(\tau)\|_\infty \,\mathrm{d}\tau \\ & \lesssim r^{-\beta_1} \int_0^t (t-\tau)^{-\frac{1}{2\alpha_1}} \tau^{-\frac{\theta_1}{2\alpha_1}} \,\mathrm{d}\tau \sup_{0<\tau<t} \|y(\tau)\|_\infty \\ & \lesssim r^{-\beta_1} t^{1-\frac{1}{2\alpha_1}-\frac{\theta_1}{2\alpha_1}} \sup_{0<\tau<t} \|y(\tau)\|_\infty, \end{aligned}$$

$$\begin{aligned}\|\mathcal{B}_{\alpha_1}(\mathrm{e}^{-t(-\Delta)^{\alpha_1}}b_0, z(t))\|_\infty &\lesssim \int_0^t \frac{1}{(t-\tau)^{1/2\alpha_1}}\|\mathrm{e}^{-\tau(-\Delta)^{\alpha_1}}b_0\|_\infty\|z(\tau)\|_\infty\,\mathrm{d}\tau\\ &\lesssim r^{-\beta_2}\int_0^t (t-\tau)^{-\frac{1}{2\alpha_1}}\tau^{-\frac{\theta_2}{2\alpha_1}}\,\mathrm{d}\tau \sup_{0<\tau<t}\|z(\tau)\|_\infty\\ &\lesssim r^{-\beta_2}t^{1-\frac{1}{2\alpha_1}-\frac{\theta_2}{2\alpha_1}}\sup_{0<\tau<t}\|z(\tau)\|_\infty,\end{aligned}$$

$$\begin{aligned}\|\mathcal{B}_{\alpha_1}(b_1(t), z(t))\|_\infty &\lesssim \int_0^t \frac{1}{(t-\tau)^{1/2\alpha_1}}\|b_1(\tau)\|_\infty\|z(\tau)\|_\infty\,\mathrm{d}\tau\\ &\lesssim r^{1-\beta_1-\beta_2}\int_0^t (t-\tau)^{-\frac{1}{2\alpha_1}}\,\mathrm{d}\tau \sup_{0<\tau<t}\|z(\tau)\|_\infty\\ &\lesssim r^{1-\beta_1-\beta_2}t^{1-\frac{1}{2\alpha_1}}\sup_{0<\tau<t}\|z(\tau)\|_\infty,\end{aligned}$$

$$\begin{aligned}&\|\mathcal{B}_{\alpha_1}(y(t), y(t))\|_\infty + \|\mathcal{B}_{\alpha_1}(z(t), z(t))\|_\infty\\ &\quad\lesssim \int_0^t \frac{1}{(t-\tau)^{1/2\alpha_1}}\left(\|y(\tau)\|_\infty^2 + \|z(\tau)\|_\infty^2\right)\mathrm{d}\tau\\ &\quad\lesssim t^{1-\frac{1}{2\alpha_1}}\left(\sup_{0<\tau<t}\|y(\tau)\|_\infty^2 + \sup_{0<\tau<t}\|z(\tau)\|_\infty^2\right).\end{aligned}$$

Thus we have

$$\begin{aligned}\|y(t)\|_\infty &\lesssim r^{1-\beta_1-2\beta_2}t^{1-\frac{1}{2\alpha_1}-\frac{\theta_2}{2\alpha_1}} + r^{2(1-\beta_1-\beta_2)}t^{1-\frac{1}{2\alpha_1}}\\ &\quad + r^{-\beta_1}t^{1-\frac{1}{2\alpha_1}-\frac{\theta_1}{2\alpha_1}}\sup_{0<\tau<t}\|y(\tau)\|_\infty + r^{-\beta_2}t^{1-\frac{1}{2\alpha_1}-\frac{\theta_2}{2\alpha_1}}\sup_{0<\tau<t}\|z(\tau)\|_\infty\\ &\quad + r^{1-\beta_1-\beta_2}t^{1-\frac{1}{2\alpha_1}}\sup_{0<\tau<t}\|z(\tau)\|_\infty\\ &\quad + t^{1-\frac{1}{2\alpha_1}}\left(\sup_{0<\tau<t}\|y(\tau)\|_\infty^2 + \sup_{0<\tau<t}\|z(\tau)\|_\infty^2\right).\end{aligned}\tag{4.30}$$

By (3.12), (4.27) and the fact $u_1 = 0$, we have

$$\begin{aligned}\|z(t)\|_\infty &\lesssim \|\mathcal{B}_{\alpha_1}(\mathrm{e}^{-t(-\Delta)^{\alpha_2}}u_0, b_1(t))\|_\infty + \|\mathcal{B}_{\alpha_2}(\mathrm{e}^{-t(-\Delta)^{\alpha_2}}u_0, z(t))\|_\infty\\ &\quad + \|\mathcal{B}_{\alpha_2}(\mathrm{e}^{-t(-\Delta)^{\alpha_2}}b_0, y(t))\|_\infty + \|\mathcal{B}_{\alpha_2}(b_1(t), y(t))\|_\infty + \|\mathcal{B}_{\alpha_2}(y(t), z(t))\|_\infty.\end{aligned}$$

For $\alpha_2 > 1/2$ and $0 < \frac{\theta_1}{2\alpha_2}, \frac{\theta_2}{2\alpha_2} < 1$, a similar calculation shows that

$$\begin{aligned}\|z(t)\|_\infty &\lesssim r^{1-2\beta_1-\beta_2}t^{1-\frac{1}{2\alpha_2}-\frac{\theta_1}{2\alpha_2}} + r^{-\beta_1}t^{1-\frac{1}{2\alpha_2}-\frac{\theta_1}{2\alpha_2}}\sup_{0<\tau<t}\|z(\tau)\|_\infty\\ &\quad + r^{-\beta_2}t^{1-\frac{1}{2\alpha_2}-\frac{\theta_2}{2\alpha_2}}\sup_{0<\tau<t}\|y(\tau)\|_\infty + r^{1-\beta_1-\beta_2}t^{1-\frac{1}{2\alpha_2}}\sup_{0<\tau<t}\|y(\tau)\|_\infty\\ &\quad + t^{1-\frac{1}{2\alpha_2}}\sup_{0<\tau<t}\|y(\tau)\|_\infty\sup_{0<\tau<t}\|z(\tau)\|_\infty.\end{aligned}\tag{4.31}$$

Let $w(t) = \|y(t)\|_\infty + \|z(t)\|_\infty$ and $w = \sup_{0<\tau<t}\|y(\tau)\|_\infty + \sup_{0<\tau<t}\|z(\tau)\|_\infty$. Adding (4.30) and (4.31) yields

$$\begin{aligned}w(t) &\lesssim r^{1-\beta_1-2\beta_2}t^{1-\frac{1}{2\alpha_1}-\frac{\theta_2}{2\alpha_1}} + r^{2(1-\beta_1-\beta_2)}t^{1-\frac{1}{2\alpha_1}} + r^{1-2\beta_1-\beta_2}t^{1-\frac{1}{2\alpha_2}-\frac{\theta_1}{2\alpha_2}}\\ &\quad + \left(r^{-\beta_1}t^{1-\frac{1}{2\alpha_1}-\frac{\theta_1}{2\alpha_1}} + r^{-\beta_2}t^{1-\frac{1}{2\alpha_2}-\frac{\theta_2}{2\alpha_2}} + r^{1-\beta_1-\beta_2}t^{1-\frac{1}{2\alpha_2}}\right)\sup_{0<\tau<t}\|y(\tau)\|_\infty\\ &\quad + \left(r^{-\beta_2}t^{1-\frac{1}{2\alpha_1}-\frac{\theta_2}{2\alpha_1}} + r^{1-\beta_1-\beta_2}t^{1-\frac{1}{2\alpha_1}} + r^{-\beta_1}t^{1-\frac{1}{2\alpha_2}-\frac{\theta_1}{2\alpha_2}}\right)\sup_{0<\tau<t}\|z(\tau)\|_\infty\\ &\quad + \left(t^{1-\frac{1}{2\alpha_1}} + t^{1-\frac{1}{2\alpha_2}}\right)w^2\end{aligned}$$

$$
\begin{aligned}
&:= C_1 + C_2 \sup_{0<\tau<t} \|y(\tau)\|_\infty + C_3 \sup_{0<\tau<t} \|z(\tau)\|_\infty + C_4 w^2 \\
&\lesssim C_1 + (C_2 + C_3) w + C_4 w^2 \\
&= C_1 + (C_2 + C_3 + C_4 w)\, w.
\end{aligned} \tag{4.32}
$$

We shall choose large enough r and small enough $T > 0$, such that

$$
C_0 := C_2 + C_3 + C_1 C_4 \ll 1 \tag{4.33}
$$

for $0 \leqslant t \leqslant T$. We claim the following estimate holds:

$$
\|w(t)\|_\infty \lesssim C_1 \lesssim r^{1-\beta_1-2\beta_2} t^{1-\frac{1}{2\alpha_1}-\frac{\theta_2}{2\alpha_1}} + r^{2(1-\beta_1-\beta_2)} t^{1-\frac{1}{2\alpha_1}} + r^{1-2\beta_1-\beta_2} t^{1-\frac{1}{2\alpha_2}-\frac{\theta_1}{2\alpha_2}},
$$

for all $0 < t \leqslant T$. Indeed, since $w(0) = 0$, and $w(t)$ is continuous, $\|w(t)\|_\infty$ cannot cross $2C_1$. Otherwise, at certain time t_0, we have $w(t_0) = 2C_1$. Combining with (4.32), it implies

$$
2C_1 \lesssim C_1 + 2\,(C_2 + C_3 + 2C_1C_4)\, C_1
$$

and hence

$$
C_2 + C_3 + C_1 C_4 \gtrsim 1,
$$

which contradicts (4.33).

Now we choose r and T to guarantee (4.33). Note that the powers of t in C_0 are all non-negative by the hypothesis of the lemma. Thus,

$$
C_0 \lesssim r^{1-\beta_1-\beta_2} T^{1-\frac{1}{2\alpha_2}} + r^{1-\beta_1-\beta_2} T^{1-\frac{1}{2\alpha_1}} + r^{2(1-\beta_1-\beta_2)} T^{1-\frac{1}{2\alpha_1}} \left(T^{1-\frac{1}{2\alpha_1}} + T^{1-\frac{1}{2\alpha_2}} \right).
$$

Let $T = r^{-\gamma}$. It follows

$$
C_0 \lesssim r^{1-\beta_1-\beta_2-\gamma(1-\frac{1}{2\alpha_2})} + r^{1-\beta_1-\beta_2-\gamma(1-\frac{1}{2\alpha_1})} + r^{2(1-\beta_1-\beta_2)-\gamma(2-\frac{1}{\alpha_1})} + r^{2(1-\beta_1-\beta_2)-\gamma(2-\frac{1}{2\alpha_1}-\frac{1}{2\alpha_2})}. \tag{4.34}
$$

We choose γ such that

$$
\gamma > \frac{1-\beta_1-\beta_2}{1-1/(2\alpha_1)}, \qquad \gamma > \frac{1-\beta_1-\beta_2}{1-1/(2\alpha_2)}, \tag{4.35}
$$

which guarantees all the powers of r are negative in (4.34). Hence (4.33) is satisfied for r large enough. It proves the conclusion of the lemma. □

4.4. Finishing the proof

Now we are ready to complete the proof of theorem 1.1. Since u_0 and b_0 are smooth and space-periodic, there exists $T^* > 0$ and a smooth space-periodic solution $(u(t), b(t))$ to (1.1) on $[0, T^*)$ with $u(0) = u_0$, $b(0) = b_0$, such that either $T^* = +\infty$ or

$$
\limsup_{t \to T^*-} (\|u(t)\|_\infty + \|b(t)\|_\infty) = +\infty.
$$

Lemmas 4.4, 4.5, and 4.6 imply that $T^* > T$. Now using (3.6), we combine the imbedding estimate (2.2), lemmas 4.3, 4.4, 4.5 and 4.6 to obtain that, for $|k_1|^{-2\alpha_2} \leqslant t \leqslant T$

$$
\begin{aligned}
\|b(\cdot,t)\|_{\dot{B}^{-s}_{\infty,\infty}} &\geqslant \|b_{10}(\cdot,t)\|_{\dot{B}^{-s}_{\infty,\infty}} - \|b_{11}(\cdot,t)\|_\infty - \|b_{12}(\cdot,t)\|_\infty - \|\mathrm{e}^{-t(-\Delta)^{\alpha_2}} b_0\|_\infty - \|z(\cdot,t)\|_\infty \\
&\gtrsim r^{1-\beta_1-\beta_2} \left(1 - r^{\beta_1-1} t^{-\frac{\theta_2}{2\alpha_2}} - r^{-\beta_2} t^{1-\frac{1}{2\alpha_1}-\frac{\theta_2}{2\alpha_1}} - r^{1-\beta_1-\beta_2} t^{1-\frac{1}{2\alpha_1}} - r^{-\beta_1} t^{1-\frac{1}{2\alpha_2}-\frac{\theta_1}{2\alpha_2}} \right) \\
&\gtrsim r^{1-\beta_1-\beta_2} \left(1 - r^{\beta_1-1} |k_1|^{\theta_2} - r^{-\beta_2} T^{1-\frac{1}{2\alpha_1}-\frac{\theta_2}{2\alpha_1}} - r^{1-\beta_1-\beta_2} T^{1-\frac{1}{2\alpha_1}} \right),
\end{aligned} \tag{4.36}
$$

where in the last step we used the fact $r^{-\beta_1}t^{1-\frac{1}{2\alpha_2}-\frac{\theta_1}{2\alpha_2}} \ll \frac{1}{2}$, since the power of r is negative and the power of t is non-negative. We will choose parameters so that

$$A := r^{\beta_1-1}|k_1|^{\theta_2} + r^{-\beta_2}T^{1-\frac{1}{2\alpha_1}-\frac{\theta_2}{2\alpha_1}} + r^{1-\beta_1-\beta_2}T^{1-\frac{1}{2\alpha_1}} \leqslant 1/2. \tag{4.37}$$

Let $|k_1| = r^{\zeta}$ with positive ζ, and $T = r^{-\gamma}$ as in lemma 4.6. Then

$$A = r^{\beta_1-1+\zeta\theta_2} + r^{-\beta_2-\gamma(1-\frac{1}{2\alpha_1}-\frac{\theta_2}{2\alpha_1})} + r^{1-\beta_1-\beta_2-\gamma(1-\frac{1}{2\alpha_1})}.$$

To make (4.37) hold for large enough r, it is sufficient to choose ζ, γ such that

$$\begin{cases} 0 < \zeta < \frac{1-\beta_1}{\theta_2}, \\ \dfrac{1-\beta_1-\beta_2}{1-1/(2\alpha_1)} < \gamma < 2\alpha_2\zeta, \\ 1 - \dfrac{1}{2\alpha_1} - \dfrac{\theta_2}{2\alpha_1} \geqslant 0. \end{cases} \tag{4.38}$$

Moreover, the condition $\gamma < 2\alpha_2\zeta$ in (4.38) guarantees that $|k_1|^{-2\alpha_2} < T$, which is required in lemma 4.4. We verify that there exist $\beta_1, \beta_2, \theta_1$ and θ_2 such that the assumption (4.28) in lemma 4.6 and conditions (4.35) (4.38) are compatible. Indeed, one can take $\beta_1 = \beta_2 = \frac{1}{2} - \epsilon$ for $0 < \epsilon < \frac{1}{6}$. Since $\alpha_1, \alpha_2 \geqslant 1$, it follows

$$\frac{1-\beta_1-\beta_2}{1-1/(2\alpha_1)} \leqslant 4\epsilon, \qquad \frac{1-\beta_1-\beta_2}{1-1/(2\alpha_2)} \leqslant 4\epsilon.$$

On the other hand, due to the relation $\theta_1 + \theta_2 = 2\alpha_2$ in (4.28), we have

$$2\alpha_2\frac{1-\beta_1}{\theta_2} > 1 - \beta_1 > \frac{1}{2} + \epsilon.$$

Thus, the first two conditions in (4.38) are compatible. Straightforward computation shows all the other conditions among (4.28), (4.35) and (4.38) are also compatible.

Given any $\delta > 0$ in theorem 1.1, we now choose a suitable large r such that

$$r^{1-\beta_1-\beta_2} \gtrsim \frac{1}{\delta}.$$

Therefore, it follows from (4.36) and (4.37) that

$$\|u(\cdot, T)\|_{\dot{B}^{-s}_{\infty,\infty}} \gtrsim r^{1-\beta_1-\beta_2} \gtrsim \frac{1}{\delta}.$$

Finally, lemma 4.2 implies that the initial data u_0 satisfies

$$\|u_0\|_{\dot{B}^{-\theta_1}_{\infty,\infty}} \lesssim r^{-\beta_1} \lesssim \delta, \quad \|b_0\|_{\dot{B}^{-\theta_2}_{\infty,\infty}} \lesssim r^{-\beta_2} \lesssim \delta.$$

This competes the proof of theorem 1.1.

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