PARAMETER SPACES OF SCHUBERT VARIETIES IN HYPERPLANE SECTIONS OF GRASSMANNIANS

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Abstract. Linear sections of Grassmannians provide important examples of varieties. The geometry of these linear sections is closely tied to the spaces of Schubert varieties contained in them. In this paper, we describe the spaces of Schubert varieties contained in hyperplane sections of $G(2,n)$. The group $\mathbb{P}GL(n)$ acts with finitely many orbits on the dual of the Plücker space $\mathbb{P}^*(\bigwedge^2 V)$. The orbits are determined by the singular locus of $H \cap G(2,n)$. For H in each orbit, we describe the spaces of Schubert varieties contained in $H \cap G(2,n)$. We also discuss some generalizations to $G(k,n)$.

Contents

1. Introduction

Linear sections of Grassmannians provide examples that play an important role in many branches of algebraic geometry, including the classification of varieties, derived equivalences and mirror symmetry. For example, general codimension four linear sections of $G(2,5)$ are Del Pezzo surfaces of degree five (see [C1]) and general codimension seven linear sections of $G(2,7)$ are Calabi-Yau threefolds (see [BC], [R]). The geometry of a linear section X of a Grassmannian is closely tied to the spaces of Schubert varieties contained in X, which provide crucial information about the cohomology and Hodge structure of X (see [D] and Chapter 6 of [GH]). In this paper, we will describe the spaces of Schubert varieties contained in a hyperplane section of a Grassmannian.

Let $G(k,n)$ denote the Grassmannian parameterizing k-dimensional subspaces of a fixed n-dimensional vector space V. Let λ denote a partition whose parts satisfy

$$n-k \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0.$$

When writing a partition, the parts that are equal to zero are often omitted. For many purposes, it is more convenient to group together the parts of λ that are equal. We will write λ also as $\lambda = (\mu_1^{i_1}, \cdots, \mu_t^{i_t})$ and set $k_s = \sum_{j=1}^{s} i_j$, where $\mu_1 > \mu_2 > \cdots > \mu_t$ and

$$\mu_1 = \lambda_1 = \cdots = \lambda_{k_1}, \mu_2 = \lambda_{k_1+1} = \cdots = \lambda_{k_2}, \ldots, \mu_t = \lambda_{k_{t-1}+1} = \cdots = \lambda_k.$$

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Given a partition λ and a flag $F_\bullet : F_1 \subset F_2 \subset \cdots \subset F_n = V$, the Schubert variety $\Sigma_\lambda(F_\bullet)$ is defined as

$$\Sigma_\lambda(F_\bullet) = \{[W] \in G(k,n) \mid \dim(W \cap F_{n-k+i-\lambda_i}) \geq i\}. \tag{1}$$

We will often abuse notation by dropping the reference to the flag. When we would like to emphasize the flag elements $F_{n-k+i-\lambda_i}$ imposing rank conditions, we will write $\Sigma_\lambda(F_{n-k+1-\lambda_1} \subset \cdots \subset F_{n-\lambda_k})$. The cohomology class σ_λ of the Schubert variety depends only on the partition λ and not on the choice of flag. The Schubert classes σ_λ, as λ varies over all allowed partitions, form a $\mathbb{Z}$-basis for the cohomology of $G(k,n)$ [GH, §1.5].

The Plücker map embeds the Grassmannian $G(k,n)$ in $\mathbb{P}(\bigwedge^k V)$. Let H be a hyperplane in $\mathbb{P}(\bigwedge^k V)$. Let

$$X(\lambda, H) = \{\Sigma_\lambda(F_\bullet) \mid \Sigma_\lambda(F_\bullet) \subset G(k,n) \cap H\}$$

denote the space of Schubert varieties with class σ_λ contained in $G(k,n) \cap H$. In the next section, we will see that $X(\lambda, H)$ is a closed algebraic subset of a suitable partial flag variety ($X(\lambda, H)$ may be reducible). The purpose of this paper is to describe $X(\lambda, H)$ in detail when $k = 2$ and H is arbitrary. We will also discuss some generalizations to larger k.

There is a natural incidence correspondence

$$\mathcal{I}(\lambda) = \{(\Sigma_\lambda(F_\bullet), H) \mid \Sigma_\lambda(F_\bullet) \subset H\}$$

parameterizing pairs of a Schubert variety $\Sigma_\lambda(F_\bullet)$ and a hyperplane H in the Plücker space containing $\Sigma_\lambda(F_\bullet)$. Let π_2 denote the natural projection to $\mathbb{P}^*(\bigwedge^k V)$. The first problem we address is characterizing the image of π_2. Before stating our theorems, we recall the case of $G(2,4)$.

Example 1.1 (Spaces of Schubert varieties in $G(2,4)$). The Plücker map embeds $G(2,4)$ in $\mathbb{P}^5$ as a smooth quadric hypersurface Q. The dual of Q is a smooth quadric hypersurface in $\mathbb{P}^{5*}$ [H, §15]. Since smooth quadric hypersurfaces are homogeneous, it is easy to see that a hyperplane section of Q is either smooth or singular at exactly one point. A codimension one Schubert variety $\Sigma_1(V_2 \subset V_4)$ is singular at the point of $G(2,4)$ corresponding to V_2 (see §2). By homogeneity, we conclude that a hyperplane section of $G(2,4)$ is either smooth or a codimension one Schubert variety.

The image of a Schubert variety $\Sigma_{2,1}$ is a line on Q. Conversely, every line on Q is a Schubert variety with class $\sigma_{2,1}$. Therefore, the Fano variety $\mathcal{F}_1(Q)$ parameterizing lines on Q is isomorphic to the flag variety $F(1,3;4)$ [H, §6].

Let $X = G(2,4) \cap H$ be a smooth hyperplane section of $G(2,4)$. Then X is a smooth quadric threefold. The Fano variety $\mathcal{F}_1(X)$ parameterizing lines on X is the orthogonal Grassmannian $OG(2,5)$, which is isomorphic to $\mathbb{P}^3$.

On the other hand, let $Y = \Sigma_1(V_2 \subset V_4)$ be a singular hyperplane section of $G(2,4)$. Then Y is a cone over a smooth quadric surface whose vertex is the point corresponding to the two dimensional vector space V_2. The Fano variety $\mathcal{F}_1(Y)$ parameterizing lines on Y has two irreducible components Z_1 and Z_2. Both Z_1 and Z_2 are isomorphic to the blow-up of $\mathbb{P}^3$ along a line. The two components Z_1 and Z_2 intersect exactly along the exceptional divisors of the two blow-ups. The components Z_1 and Z_2 can be geometrically described as follows. Let $l = \Sigma_{2,1}(F_1 \subset F_3)$ be a line on $G(2,4)$. The line l is contained in Y if all the two-dimensional subspaces parameterized by l intersect V_2 defining $\Sigma_1(V_2 \subset V_4)$ non-trivially. There are two possibilities. Either $V_2 \subset F_3$ and F_1 is an arbitrary one-dimensional subspace of F_3; or F_3 is arbitrary and $F_1 = F_3 \cap V_2$. These two possibilities correspond to the two components Z_1 and Z_2.

The image of a Schubert variety $\Sigma_{1,1}$ or Σ_2 under the Plücker map is a plane on the quadric hypersurface Q. Conversely, every plane on Q is a Schubert variety of the form $\Sigma_{1,1}$ or Σ_2. These varieties are parameterized by $\mathbb{P}^{3*}$ and $\mathbb{P}^3$, respectively. By the Lefschetz Hyperplane Theorem [GH, §1.2], a smooth quadric threefold does not contain any planes. Therefore, the smooth hyperplane section X of $G(2,4)$ does not contain any Schubert varieties $\Sigma_{1,1}$ or Σ_2. On the other hand, Y is a cone over a quadric surface. Such a threefold has two one-dimensional families of planes both parameterized by $\mathbb{P}^1$. The two components are distinguished by the cohomology class of the planes they parameterize. Hence, the space of Schubert varieties of the type $\Sigma_{1,1}$ or Σ_2 on Y are both parameterized by $\mathbb{P}^1$. Notice that in these two cases the incidence correspondences $\mathcal{I}(1,1)$ and $\mathcal{I}(2)$ both have dimension $5 = \dim(\mathbb{P}^*(\bigwedge^2 V))$; however, the second projection is not surjective [H, Example 12.5].

In general, $\mathbb{P}GL(n)$ acts with finitely many orbits on $\mathbb{P}^*(\bigwedge^2 V)$ [D, §2] (see Proposition 2.5). The equation of a hyperplane H in the Plücker space $\mathbb{P}(\bigwedge^2 V)$ can be expressed as $\sum a_{i,j} e_i \wedge e_j = 0$. Therefore, H may be viewed as a skew-symmetric matrix Q_H. The dimension of the kernel of Q_H is the invariant that determines the orbits of $\mathbb{P}GL(n)$ on $\mathbb{P}^*(\bigwedge^2 V)$ [D, §2] (see the discussion in §2 preceding Proposition 2.5). The dense open orbit corresponds to hyperplanes H such that $G(2,n) \cap H$ is smooth. The dual variety $G(2,n)^*$ parameterizing hyperplanes tangent to $G(2,n)$ decomposes into finitely many orbits depending on the singular locus of $H \cap G(2,n)$. Following Lemma 2.4, we will see that, for $H \in G(2,n)^*$, the singular locus of $G(2,n) \cap H$ is a Schubert variety of the form $\Sigma_{2r,2r}$ for some $1 \leq r \leq \lfloor \frac{n-2}{2} \rfloor$ [D, §2]. Let S_r denote the locus in $\mathbb{P}^*(\bigwedge^2 V)$ parameterizing hyperplanes H such that the singular locus of $G(2,n) \cap H$ contains a Schubert variety of the form $\Sigma_{2r,2r}$. By convention, we set $S_{\lceil \frac{n-1}{2} \rceil}$ to be $\mathbb{P}^*(\bigwedge^2 V)$. We thus have

$$S_1 \subset S_2 \subset \cdots \subset S_{\lceil \frac{n-1}{2} \rceil}$$

and the $\mathbb{P}GL(n)$ orbits on $\mathbb{P}^*(\bigwedge^2 V)$ are the locally closed subsets S_r/S_{r-1} (see [D, §2] and Proposition 2.5).

Our first theorem characterizes the image of $\pi_2(\mathcal{I}(\lambda))$ when $k = 2$.

Theorem 1.2. *Let $\lambda = (a,b)$ be a partition for $G(2,n)$. The image of the map*

$$\pi_2 : \mathcal{I}(a,b) \to \mathbb{P}^*(\bigwedge^2 V)$$

contains S_r if and only if $\lceil \frac{a+b}{2} \rceil \geq r$. In particular, the map π_2 is surjective if and only if $\lceil \frac{a+b}{2} \rceil > \frac{n-2}{2}$.

Theorem 1.2 implies that if $H \in S_r/S_{r-1}$, then $X((a,b),H)$ is not empty if and only if $\lceil \frac{a+b}{2} \rceil \geq r$. This raises the question of describing $X((a,b),H)$ in cases it is not empty. Our second theorem addresses this question.

Let Q be a skew-symmetric form on an n-dimensional vector space V. If Q is non-degenerate, then $n = 2r$ has to be even. A linear space W is called *isotropic* with respect to Q if the restriction of Q to W is identically zero. Given a vector space W, let $W^\perp$ denote the set of vectors $v \in V$ such that $v^T Q w = 0$ for every $w \in W$. If Q is non-degenerate, the variety parameterizing the k-dimensional isotropic subspaces of V is called *the isotropic Grassmannian* $SG(k,2r)$. The isotropic Grassmannian $SG(k,2r)$ is a homogeneous variety for the symplectic group $Sp(2r)$. An isotropic subspace of a non-degenerate skew-symmetric form has at most half the dimension, hence $k \leq r$.

Theorem 1.3. *Let H be a hyperplane in $\mathbb{P}(\bigwedge^2 V)$ such that $[H] \in \mathbb{P}^*(\bigwedge^2 V)$ is contained in the $\mathbb{P}GL(n)$ orbit S_r/S_{r-1}. Let F_{n-2r} be the kernel of the corresponding skew-symmetric form Q_H. Let (a,b) be a partition for $G(2,n)$ such that $\lceil \frac{a+b}{2} \rceil \geq r$. Let*

$$M = \max\left(0, n-1-a-\min(r,b)\right) \quad \textit{and} \quad N = \min\left(n-a-1, n-r-\frac{a+b+1}{2}\right).$$

(1) *Assume that $a \neq b$. Then the irreducible components Z_j of $X((a,b),H)$ are in one-to-one correspondence with integers $M \leq j \leq N$. The irreducible component Z_j parameterizes pairs $(V_{n-a-1} \subset V_{n-b})$ in $F(n-a-1, n-b; n)$ such that V_{n-a-1} is a Q_H-isotropic subspace with $\dim(V_{n-a-1} \cap F_{n-2r}) \geq j$ and V_{n-b} is a linear space $V_{n-a-1} \subset V_{n-b} \subset V_{n-a-1}^{\perp}$ with $\dim(V_{n-b} \cap F_{n-2r}) \geq 2n-2r-a-b-1-j$. The dimension of Z_j is given by*

$$\begin{aligned} \dim(Z_j) &= (a+1-b)(a+b+j-n+1) - j\frac{(4r+3a+3j-3n+4)}{2} \\ &+ \frac{(n-a-1)(3a+j-n+4)}{2}. \end{aligned}$$

(2) *Assume that $a = b$. Then $X((a,a),H)$ parameterizes Q_H-isotropic subspaces of dimension $n-a$. In particular, $X((a,a),H)$ is irreducible and*

$$\dim(X((a,a),H)) = \begin{cases} \frac{r^2+r}{2} + (n-a)(a-r) & \textit{if } n \geq a+r \\ \frac{(n-a)(3a-n+1)}{2} & \textit{if } n < a+r \end{cases}$$

Some special cases of the theorem are worth highlighting for the beauty of the geometry.

Corollary 1.4. *Let $[H] \in S_r/S_{r-1}$. Then $X((r,r),H)$ is isomorphic to the Lagrangian Grassmannian $SG(r,2r)$. In particular, $X((r,r),H)$ is irreducible of dimension $\binom{r+1}{2}$.*

Corollary 1.5. *Let $[H] \in S_r/S_{r-1}$ and $a+b+1 = 2r$, then $X((a,b),H)$ is isomorphic to the isotropic Grassmannian $SG(b,2r)$. In particular, $X((a,b),H)$ is irreducible of dimension $\frac{b(2a-b+3)}{2}$.*

Corollary 1.6. *Let $[H] \in S_r/S_{r-1}$ and $a+1 \geq 2r$. Then $X((a,0),H)$ is isomorphic to the Grassmannian $G(n-a-1, n-2r)$, hence it is irreducible of dimension $(n-a-1)(a+1-2r)$.*

Corollary 1.7. *Let a be odd. Then π_2 is a birational map from $\mathcal{I}(a,0)$ to $S_{(a+1)/2}$. In particular, when n is odd, a smooth hyperplane section of $G(2,n)$ contains a unique linear space of dimension $n-2$. Geometrically, this linear space corresponds to two-dimensional subspaces that contain the kernel of Q_H. Consequently, when n is odd, the largest dimensional linear space on a general codimension two linear section of $G(2,n)$ has dimension $n-3$.*

Corollary 1.8. *Let $[H] \in S_1$ be the hyperplane defining the Schubert variety $\Sigma_1(F_{n-2} \subset F_n)$ and let $a > b > 0$. Then $X((a,b),H)$ is the union of the following two Schubert varieties in $F(n-a-1, n-b; n)$*

(1) $\{(V_{n-a-1} \subset V_{n-b}) \mid V_{n-a-1} \subset F_{n-2}\}$,
(2) $\{(V_{n-a-1} \subset V_{n-b}) \mid \dim(V_{n-b} \cap F_{n-2}) \geq n-b-1\}$.

When $n-2 > k > 2$, $\mathbb{P}GL(n)$ no longer acts with finitely many orbits on $\mathbb{P}^*(\bigwedge^k V)$(except when $k=3$ and $n=6,7$ or 8 [D, §2.1]). It is, therefore, unrealistic to hope for as complete a classification of the spaces $X(\lambda, H)$. However, $X(\lambda, H)$ can be easily described for H in certain orbits of $\mathbb{P}GL(n)$. We will give some examples below.

Proposition 1.9. *Let λ be a partition for $G(k,n)$ such that $\lambda_1 < n-k$ and $\lambda_k = 0$. Then the image of the second projection $\pi_2(\mathcal{I}(\lambda))$ is contained in the dual variety $G(k,n)^*$. In particular, π_2 is not surjective. On the other hand, let λ be a partition such that either $\lambda_{k-1} = n-k$ and $\lambda_k > 0$; or $\lambda_1 = n-k$ and $\lambda_k = n-k-1$. Then π_2 is surjective.*

A corollary of the proof of the proposition is worth mentioning.

Corollary 1.10. *Let λ be the partition $\lambda_1 = \cdots = \lambda_{k-1} = n-k-1$ and $\lambda_k = 0$. Then $\pi_2(\mathcal{I}(\lambda)) = G(k,n)^*$.*

It is very rare to have an explicit, concrete resolution of singularities of a variety. Corollary 1.10 provides such a resolution for the dual of the Grassmannian in its Plücker embedding.

Corollary 1.11. *Let $n-2 > k > 2$. Let λ be the partition $\lambda_1 = \cdots = \lambda_{k-1} = n-k-1$ and $\lambda_k = 0$. Let $N = \binom{n}{k} - k(n-k) - 2$. Then the incidence correspondence $\mathcal{I}(\lambda)$ is a $\mathbb{P}^N$-bundle over $G(k,n)$ and is smooth. The map π_2 is a birational map from $\mathcal{I}(\lambda)$ onto $G(k,n)^*$ and gives a resolution of singularities of $G(k,n)^*$.*

Finally, we state the analogue of Corollary 1.8 for arbitrary k.

Proposition 1.12. *Let H be the hyperplane in $\mathbb{P}(\bigwedge^k V)$ defining the codimension one Schubert variety $\Sigma_1(F_{n-k} \subset F_{n-k+2} \subset \cdots \subset F_n)$. Let $\lambda = (\mu_1^{i_1}, \ldots, \mu_t^{i_t})$ be a partition. Let δ denote the Krönecker delta function. Then $X(\lambda, H)$ has $t - \delta_{0,\mu_t}$ irreducible components Z_j with $1 \leq j \leq t - \delta_{0,\mu_t}$. The component Z_j is the Schubert variety in $F(n-k+k_1-\mu_1, \ldots, n-\mu_t; n)$ parameterizing flags $(V_{n-k+k_1-\mu_1} \subset \cdots \subset V_{n-\mu_t})$ such that $\dim(V_{n-k+k_j-\mu_j} \cap F_{n-k}) \geq n-k-\mu_j+1$.*

The organization of the paper is as follows. In §2, we will recall basic facts about the geometry of Grassmannians, Schubert varieties and the dual variety to the Grassmannian in its Plücker embedding. In §3, we will prove Theorem 1.2, Proposition 1.9 and their corollaries. In §4, we will prove Theorem 1.3 and discuss its corollaries.

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2. Preliminaries about the geometry of Grassmannians

In this section, we recall some basic facts about the geometry of Grassmannians and Schubert varieties. For the reader's convenience, we sketch the proofs of some classical facts about $G(2,n)^*$. We refer the reader to [GH] and [H] for facts about Grassmannians and Schubert varieties, to [D] and [PVdV] for facts about the dual variety $G(2,n)^*$, to [BL] and [C2] for facts about singularities of Schubert varieties.

In order to minimize confusion, we will denote the point in the Grassmannian $G(k,n)$ corresponding to a k-dimensional subspace W by $[W]$.

Parameter spaces of Schubert varieties. Although it is standard in the literature to define a Schubert variety by Equation (1), the Schubert variety does not determine the flag. In fact, the Schubert variety does not even determine the elements of the flag $F_{n-k+i-\lambda_i}$ that impose the rank conditions defining the Schubert variety.

For example, $\Sigma_{1,1}(F_2 \subset F_3)$ and $\Sigma_{1,1}(F_2' \subset F_3)$ define the same Schubert variety in $G(2,4)$ for any two F_2 and F_2', two-dimensional subspaces contained in F_3. Once a two-dimensional subspace W is contained in F_3, then W automatically intersects any two dimensional subspace of F_3 non-trivially.

In order to characterize the flags that define the same Schubert variety, it is more convenient to group the repeated parts in the partition λ. Recall that we express λ as $\lambda = (\mu_1^{i_1}, \dots, \mu_t^{i_t})$, where

$$\lambda_1 = \cdots = \lambda_{i_1} = \mu_1, \lambda_{i_1+1} = \cdots \lambda_{i_1+i_2} = \mu_2, \cdots, \lambda_{i_1+\cdots+i_{t-1}+1} = \cdots = \lambda_k = \mu_t$$

and

$$n-k \geq \mu_1 > \mu_2 > \cdots > \mu_t \geq 0.$$

For simplicity, set $k_s = \sum_{j=1}^{s} i_j$. In particular, $k_t = k$. The Schubert variety $\Sigma_\lambda(F_\bullet)$ can equivalently be defined as

$$\Sigma_\lambda(F_\bullet) = \{[W] \in G(k,n) \mid \dim(W \cap F_{n-k+k_j-\mu_j}) \geq k_j \text{ for } 1 \leq j \leq t\}. \tag{2}$$

Once W intersects $F_{n-k+k_s-\mu_s}$ in a k_s-dimensional subspace, it intersects $F_{n-k+k_s-\mu_s-j}$ in a subspace of dimension at least $k_s - j$. Consequently, the rank conditions in Equation (2) imply all the rank conditions in Equation (1). Conversely, it is easy to see that the Schubert variety determines the linear spaces $F_{n-k+k_s-\mu_s}$ for $1 \leq s \leq t$. Consequently, we can use the partial flag variety $F(n-k+k_1-\mu_1, \dots, n-\mu_t; n)$ as a parameter space for Schubert varieties in $G(k,n)$ with cohomology class σ_λ. The space $X(\lambda, H)$ is then naturally a closed algebraic subset of $F(n-k+k_1-\mu_1, \dots, n-\mu_t; n)$.

We have a natural incidence correspondence $\mathcal{I}(\lambda)$

$$\begin{array}{ccc} & \mathcal{I}(\lambda) = \{(\Sigma_\lambda(F_\bullet), H) \mid \Sigma_\lambda(F_\bullet) \subset H\} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ F(n-k+k_1-\mu_1, \dots, n-\mu_t; n) & & \mathbb{P}^*(\bigwedge^k V) \end{array}$$

consisting of pairs of a Schubert variety $\Sigma_\lambda(F_\bullet)$ and a hyperplane containing it. The first projection π_1 realizes $\mathcal{I}(\lambda)$ as a projective bundle over the partial flag variety $F(n-k+k_1-\mu_1, \dots, n-\mu_t; n)$. The fibers are isomorphic to $\mathbb{P}H^0(I_{\Sigma_\lambda}(1))$, where I_{Σ_λ} denotes the ideal sheaf of Σ_λ, and are all projective spaces of the same dimension. Consequently, $\mathcal{I}(\lambda)$ is irreducible and smooth [S, Theorem I.6.8]. Note, however, that the second projection π_2 is rarely flat and much harder to understand.

The Plücker embedding of the Grassmannian. The Grassmannian $G(k,n)$ is a smooth, projective variety of dimension $k(n-k)$. The Plücker map embeds $G(k,n)$ into $\mathbb{P}(\bigwedge^k V)$. The image of the Grassmannian under this embedding is the space of totally decomposable wedges. In the Plücker embedding, the linear subspaces of $G(k,n)$ have a concrete description. A line on $G(k,n)$ corresponds to a family of k-dimensional subspaces of V that contain a fixed $(k-1)$-dimensional subspace and are contained in a fixed $(k+1)$-dimensional subspace. More generally, a linear space of dimension s on $G(k,n)$ corresponds to either a family of k-dimensional subspaces that contain a fixed $(k-1)$-dimensional space and are contained in a fixed $(k+s)$-dimensional subspace; or a family of k-dimensional subspaces that are contained in a fixed $(k+1)$-dimensional subspace and contain a fixed $(k-s)$-dimensional subspace [H, §6]. Of course, the first possibility can only exist when $k+s \leq n$ and the second possibility can only exist when $s \leq k$.

The tangent space $T_{[W]}G(k,n)$ is naturally isomorphic to $\mathrm{Hom}(W, V/W)$ [H, §16]. We can also explicitly describe the projectivized tangent space to $G(k,n)$ in the Plücker embedding. Choose a basis $e_1, \ldots, e_n$ for V so that W is given as the span of the vectors $e_1, \ldots, e_k$. Then under the Plücker embedding, the image of $[W]$ is $e_1 \wedge e_2 \wedge \cdots \wedge e_k$. Let $i_1 < \cdots < i_k$ be a set of indices such that the cardinality of the set $\{i_1, \ldots, i_k\} - \{1, 2, \ldots, k\}$ is at most one. Since we can replace any of the elements $1 \leq i \leq k$ by one of the elements $k < j \leq n$, there are $k(n-k)+1$ such sets. The projectivized tangent space to $G(k,n)$ at W is spanned by the $k(n-k)+1$ points $e_{i_1} \wedge \cdots \wedge e_{i_k}$ in $\mathbb{P}(\bigwedge^k V)$ defined by setting all the Plücker coordinates but $p_{i_1,\ldots i_k}$ equal to zero. To prove this description of the tangent space, observe that the line spanned by $e_{i_1} \wedge \cdots \wedge e_{i_k}$ and $e_1 \wedge \cdots \wedge e_k$ is contained in the Grassmannian $G(k,n)$. Since the tangent space at $[W]$ contains every line in $G(k,n)$ passing through $[W]$, we conclude that the projectivized tangent space at $[W]$ contains the span of the points $e_{i_1} \wedge \cdots \wedge e_{i_k}$. Since both these projective spaces have dimension $k(n-k)$, we conclude that they are equal.

Let Σ_λ be a Schubert variety in $G(k,n)$. Then Σ_λ is cut out on $G(k,n)$ by hyperplanes. These hyperplanes can be explicitly written as follows. Suppose we choose our flag so that F_i is the span of the vectors $e_1, \ldots, e_i$. Then the Schubert variety $\Sigma_\lambda(F_\bullet)$ is cut out by the Plücker coordinates $p_{i_1,\ldots,i_k} = 0$, where at least for one j, $i_j > n-k+j-\lambda_j$ [HP]. Specializing to the case $k=2$, we obtain the following lemma.

Lemma 2.1. *The dimension of the vector space of linear spaces containing a Schubert variety $\Sigma_{a,b}$ in $G(2,n)$ is given by*

$$h^0(I_{\Sigma_{a,b}}(1)) = \binom{n}{2} - \binom{n-b}{2} + \binom{a-b+1}{2}.$$

Applying the Theorem on the Dimension of Fibers [S, Theorem I.6.7] to the first projection $\pi_1 : \mathcal{I}(a,a) \to F(n-a-1, n-b; n)$, we obtain the following corollary.

Corollary 2.2. *If $a = b$, then the first projection*

$$\pi_1 : \mathcal{I}(a,a) \to F(n-a; n) = G(n-a, n)$$

exhibits $\mathcal{I}(a,a)$ as a projective space bundle over $G(n-a,n)$ with fibers of dimension

$$\binom{n}{2} - \binom{n-a}{2} - 1.$$

In particular, $\mathcal{I}(a,a)$ is smooth, irreducible and

$$\dim(\mathcal{I}(a,a)) = \frac{a(4n-3a-1)}{2} - 1.$$

If $a > b$, then the first projection

$$\pi_1 : \mathcal{I}(a,b) \to F(n-a-1, n-b; n)$$

exhibits $\mathcal{I}(a,b)$ as a projective space bundle over $F(n-a-1, n-b; n)$ with fibers of dimension

$$\binom{n}{2} - \binom{n-b}{2} + \binom{a-b+1}{2} - 1.$$

In particular, $\mathcal{I}(a,b)$ is smooth, irreducible and

$$\dim(\mathcal{I}(a,b)) = n(a+b+1) - \frac{a^2+3a}{2} - b^2 - 2.$$

Singularities of Schubert varieties. Given a partition λ, a singular partition λ^s associated to λ is obtained by adding a hook to the partition λ. More explicitly, if $\lambda = (\mu_1^{i_1}, \dots, \mu_t^{i_t})$, then λ^s is any of the partitions

$$(\mu_1^{i_1}, \dots, \mu_{u-2}^{i_{u-2}}, (\mu_{u-1}+1)^{i_{u-1}+1}, \mu_u^{i_u-1}, \mu_{u+1}^{i_{u+1}}, \dots, \mu_t^{i_t})$$

provided that they are admissible for $G(k,n)$, where it is understood that if $\mu_{u-1}+1 = \mu_{u-2}$ those parts have to be grouped together. For example, if $(5,3,2,2,1)$ is a partition for $G(5,11)$, then the singular partitions are $(6,6,2,2,1), (5,4,4,2,1)$ and $(5,3,3,3,3)$. The singular locus of the Schubert variety $\Sigma_\lambda(F_\bullet)$ is the union of $\Sigma_{\lambda^s}(F_\bullet)$ as λ^s varies over all allowable singular partitions associated to λ. In particular, $\Sigma_{a,b}$ in $G(2,n)$ is smooth if and only if $a = n-2$ or $a = b$. Otherwise, the singular locus of $\Sigma_{a,b}(F_{n-1-a} \subset F_{n-b})$ is $\Sigma_{a+1,a+1}(F_{n-2-a} \subset F_{n-1-a})$ [BL].

Lemma 2.3. *Let H be a hyperplane in $\mathbb{P}(\bigwedge^k V)$. Let V_1 be a linear space with $\dim(V_1) \geq k$ such that $H \cap G(k,n)$ is singular at every $[W] \in G(k,n)$ such that $W \subset V_1$. Then for any linear space U such that $\dim(U \cap V_1) \geq k-1$, $[U] \in G(k,n) \cap H$.*

Proof. If $U \subset V_1$, the lemma is immediate by assumption. Observe that if a line l on $G(k,n)$ intersects the singular locus of $H \cap G(k,n)$, then by Bezout's Theorem [Ha, I.7.7], l is contained in $H \cap G(k,n)$. We may assume that $U \not\subset V_1$. Let $F_{k-1} = U \cap V_1$ and let W be a k-dimensional subspace of V_1 containing F_{k-1}. Then the k-dimensional subspaces contained in $Span(U,W)$ and containing F_{k-1} are parameterized by a line l in $G(k,n)$. The line l contains $[W]$ which is a singular point of $H \cap G(k,n)$ by assumption. Hence $l \subset H \cap G(k,n)$. Since $[U]$ is also a point on l, we conclude that $[U] \in H \cap G(k,n)$. This concludes the proof of the lemma. ☐

Lemma 2.4. *Let H be a hyperplane in $\mathbb{P}(\bigwedge^2 V)$. Let V_1, V_2 be two linear subspaces of V such that $\dim(V_i) \geq 2$. Assume that $H \cap G(2,n)$ is singular along every two-dimensional subspace contained in V_i, $1 \leq i \leq 2$. Then $H \cap G(2,n)$ contains every two-dimensional subspace that intersects $Span(V_1,V_2)$ non-trivially and is singular along every two-dimensional subspace that is contained in $Span(V_1,V_2)$.*

Proof. Let W be a two-dimensional subspace that intersects $Span(V_1,V_2)$ in a one-dimensional subspace F_1. Then there exists a two-dimensional subspace W' such that $F_1 \subset W'$, $W' \cap V_1 \neq 0$ and $W' \cap V_2 \neq 0$. To construct W', take the span of two one-dimensional subspaces $G_1 \subset V_1 \cap Span(F_1,V_2)$ and $G_1' \subset V_2 \cap Span(F_1,G_1)$. Let $F_3 = Span(W,W')$. The two-dimensional subspaces contained in F_3 are parameterized by a plane P in $G(2,n)$. There are two special lines l_1 and l_1' on the plane P, parameterizing two-dimensional subspaces containing G_1, respectively, G_1' and contained in F_3. Since each of these two-dimensional spaces intersects V_1 or V_2 non-trivially, by Lemma 2.3, l and l' are contained in $H \cap G(2,n)$. By Bezout's Theorem, we conclude that $P \subset H \cap G(2,n)$. Therefore, $[W] \in H \cap G(2,n)$. Since $H \cap G(2,n)$ is projective and contains the dense open subset of the Schubert variety of $[W]$ such that $\dim(W \cap Span(V_1,V_2)) = 1$, we conclude that $H \cap G(2,n)$ contains every $[W]$ such that $W \cap Span(V_1,V_2) \neq 0$. This proves the first part of the lemma.

Next, we prove that a hyperplane section of $G(2,n)$ that contains a Schubert variety of the form $\Sigma_{a,0}(F_{n-1+a} \subset F_n)$ is singular along a Schubert variety of the form $\Sigma_{a+1,a+1}(F_{n-2+a} \subset$

F_{n-1+a}). This will conclude the proof of the second part of the lemma. Let $v \wedge w$ represent the Plücker point of a two-dimensional subspace contained in F_{n-1+a}. Choose coordinates for V so that F_{n-1+a} is spanned by $e_1, \ldots, e_{n-1+a}$ with $e_1 = v$ and $e_2 = w$. Then a hyperplane containing $\Sigma_{a,0}$ is a linear combination of the Plücker coordinates $p_{i,j}$ with $n-1+a < i < j \leq n$. The tangent space to $G(2,n)$ in its Plücker embedding at the point $e_1 \wedge e_2$ is given by the span of the points $e_1 \wedge e_i$ and $e_2 \wedge e_j$ with $2 \leq i \leq n$ and $3 \leq j \leq n$. All the Plücker coordinates containing $\Sigma_{a,0}$ vanish at all these points spanning the tangent space to the Grassmannian. Hence, all these hyperplanes contain the tangent space at all the points of $\Sigma_{a+1,a+1}$. We conclude that the linear section $H \cap G(2,n)$ is singular along $v \wedge w$. By homogeneity, it follows that $H \cap G(2,n)$ is singular along $\Sigma_{a+1,a+1}$. This concludes the proof of the lemma. □

One can also prove the previous lemma using the correspondence between hyperplanes and skew-symmetric forms. By assumption, V_1 and V_2 are in the kernel of the skew-symmetric form Q_H. Therefore, the span of V_1 and V_2 is also in the kernel. The lemma then follows by observing that $H \cap G(2,n)$ is singular along $[W]$, where W is in the kernel of Q_H.

It follows from Lemma 2.4 that the singular locus of a hyperplane section $H \cap G(2,n)$ is either empty or a Schubert variety of the form $\Sigma_{a,a}$ parameterizing two-dimensional subspaces contained in a vector space F_{n-a}. Simply let F_{n-a} be the span of all the two dimensional subspaces W such that $[W]$ is a singular point of $G(2,n) \cap H$. Furthermore, a has to be even. To see this we use the correspondence between the hyperplane H and the skew-symmetric form Q_H. The codimension of the kernel of a skew-symmetric form is even since the restriction of the skew-symmetric form to a complementary linear space is non-degenerate. Hence, a has to be even. Conversely, every $\Sigma_{2r,2r}$ occurs as the singular locus of some hyperplane section of $G(2,n)$. This can be seen by explicitly writing the skew-symmetric form $e_1 \wedge e_2 + e_3 \wedge e_4 + \cdots + e_{2r-1} \wedge e_{2r}$, whose kernels has codimension $2r$. Finally, Darboux's Theorem [MS, §2] guarantees that the hyperplanes corresponding to the skew-symmetric forms with the same dimensional kernel form one orbit under $\mathbb{P}GL(n)$. This recalls the proof of the following beautiful statement from Ron Donagi's paper [D] alluded to in the Introduction.

Proposition 2.5. *(*[D, §2]*) The group $\mathbb{P}GL(n)$ acts with finitely many orbits on $\mathbb{P}^*(\bigwedge^2 V)$. The orbits are indexed by an integer $1 \leq r \leq \lceil \frac{n-1}{2} \rceil$. The orbit corresponding to $r < \lceil \frac{n-1}{2} \rceil$ consists of hyperplanes H such that the singular locus of $H \cap G(2,n)$ is a Schubert variety of the form $\Sigma_{2r,2r}$. The open orbit corresponding to $r = \lceil \frac{n-1}{2} \rceil$ is the complement of the dual variety $G(2,n)^*$ parameterizing hyperplanes H such that $H \cap G(2,n)$ is smooth.*

Let $r \leq \frac{n-2}{2}$. Since a hyperplane $[H] \in S_r/S_{r-1}$ is singular along $\Sigma_{2r,2r}$ parameterizing linear spaces contained in F_{n-2r}, by Lemma 2.3, $H \cap G(2,n)$ contains the Schubert variety $\Sigma_{2r-1,0}$ parameterizing linear spaces intersecting F_{n-2r}. Conversely, we saw in the proof of Lemma 2.4 that a hyperplane containing $\Sigma_{2r-1,0}(F_{n-2r} \subset F_n)$ is singular along the Schubert variety $\Sigma_{2r,2r}$ parameterizing linear spaces that are contained in F_{n-2r}. We conclude that H contains a unique $\Sigma_{2r-1,0}$. In particular, the map $\pi_2 : \mathcal{I}(2r-1,0) \to S_r$ is birational and a resolution of singularities of S_r. Furthermore, the Theorem on the Dimension of Fibers and Corollary 2.2 then imply the following corollary.

Corollary 2.6. *(*[§2][D]*) The codimension of S_r in $\mathbb{P}^*(\bigwedge^2 V)$ is $\binom{n-2r}{2}$.*

In particular, we have the following well-known corollary.

Corollary 2.7. *(*[§2][D] or [PVdV]*) When n is even, then the dual $G(2,n)^*$ is a hypersurface. When n is odd $G(2,n)^*$ has codimension three.*

Finally, recall that if $n-2 > k > 2$, then the dual of $G(k,n)$ in its Plücker embedding is a hypersurface and at a general point $[H] \in G(k,n)^*$ the singular locus of $H \cap G(k,n)$ consists of one singular point. For the convenience of the reader, we briefly sketch an elementary proof. Since $G(k,n)$ is isomorphic to $G(n-k,n)$, we may further assume that $2k \leq n$. First, observe that the projective tangent spaces $\mathbb{P}T_{[W_1]} \cap \mathbb{P}T_{[W_2]} = \emptyset$ if $\dim(W_1 \cap W_2) < k-2$, $\mathbb{P}T_{[W_1]} \cap \mathbb{P}T_{[W_2]} = \mathbb{P}^3$ if $\dim(W_1 \cap W_2) = k-2$ and $\mathbb{P}T_{[W_1]} \cap \mathbb{P}T_{[W_2]} = \mathbb{P}^{n-1}$ if $\dim(W_1 \cap W_2) = k-1$. Let $U = G(k,n) \times G(k,n) - \Delta$ be the complement of the diagonal Δ in $G(k,n) \times G(k,n)$. Consider the incidence correspondence

$$J = \{([W_1],[W_2],H) \mid \mathbb{P}T_{[W_1]}, \mathbb{P}T_{[W_2]} \subset H\}$$

consisting of a point $([W_1],[W_2])$ in U and a hyperplane H containing the projective tangent spaces to $G(k,n)$ at both points. Let π_1 and π_2 denote the projection to U and $\mathbb{P}^*(\bigwedge^k V)$, respectively.

Let U_1 be the locus in U parameterizing $\{([W_1],[W_2]) \mid \dim(W_1 \cap W_2) < k-2\}$. Then the fibers of π_1 over U_1 are projective spaces of dimension $\binom{n}{k} - 2k(n-k) - 3$. Since U_1 has dimension $2k(n-k)$, the Theorem on the Dimension of Fibers implies that $\pi_2(\pi_1^{-1}(U_1))$ has codimension at least two in $\mathbb{P}^*(\bigwedge^k V)$.

Let U_2 be the locus in U parameterizing $\{([W_1],[W_2]) \mid \dim(W_1 \cap W_2) = k-2\}$. Then the fibers of π_1 over U_2 are projective spaces of dimension $\binom{n}{k} - 2k(n-k) + 1$. Since U_2 has dimension $k(n-k) + 2n - 4$, the Theorem on the Dimension of Fibers implies that $\pi_1^{-1}(U_2)$ has dimension $\binom{n}{k} - k(n-k) + 2n - 3$. Hence, $\pi_2(\pi_1^{-1}(U_2))$ has codimension at least two in $\mathbb{P}^*(\bigwedge^k V)$ if $k \geq 4$ or if $k = 3$ and $n \geq 9$. If $k = 3$ and $n = 6, 7$ or 8, we observe that the dimension of the fibers of π_2 on $\pi_1^{-1}(U_2)$ is at least 6,4 and 2, respectively. Let $W_1 = Span(e_1, e_2, e_3)$ and let $W_2 = Span(e_1, e_4, e_5)$. A hyperplane H containing $\mathbb{P}T_{[W_1]}$ and $\mathbb{P}T_{[W_2]}$ can be expressed as $\sum_{i=6}^n (a_i p_{24i} + b_i p_{34i} + c_i p_{25i} + d_i p_{35i}) = 0$ in Plücker coordinates. Consider two-dimensional subspaces Y in $Span(e_2, e_3, e_4, e_5)$ that satisfy $a_i e_2 \wedge e_4 + \cdots + d_i e_3 \wedge e_5 = 0$ for $6 \leq i \leq n$. Then H contains the tangent space to the three-dimensional subspace $Span(e_1, Y)$. The claim about the fiber dimension of π_2 follows. Hence, $\pi_2(\pi_1^{-1}(U_2))$ has codimension at least two in $\mathbb{P}^*(\bigwedge^k V)$ in these cases as well.

Let U_3 be the locus in U parameterizing $\{([W_1],[W_2]) \mid \dim(W_1 \cap W_2) = k-1\}$. Then the fibers of π_1 over U_2 are projective spaces of dimension $\binom{n}{k} - 2k(n-k) + n - 3$. The locus U_2 consists of pairs of points $([W_1],[W_2])$ such that the line spanned by them is contained in $G(k,n)$. Hence, $\dim(U_2) = 2k + (k+1)(n-k-1)$. Note that if a hyperplane H is tangent to $G(k,n)$ at both W_1 and W_2, then it is tangent at all points along the line spanned by $[W_1]$ and $[W_2]$. Consequently, the fibers of π_2 over $\pi_1^{-1}(U_2)$ have dimension at least two. By the Theorem on the Dimension of Fibers, the codimension of $\pi_2(\pi_1^{-1}(U_2))$ will be less than two if $2k + (k+1)(n-k-1) - 2k(n-k) + n - 2 > 0$. Rewriting this inequality, $0 > (k-2)n - k^2 + 3$. Using $n \geq 2k$, we immediately see that this inequality cannot be satisfied if $k \geq 4$. When $k = 3$, the inequality becomes $6 > n$. Hence, we conclude that the inequality is not satisfied for $k \geq 3$ and $n \geq 2k$. It follows that if $n-2 > k > 2$, $G(k,n)^*$ is a hypersurface and a general tangent hyperplane is tangent at a unique point. We have proved the following well-known fact for which we could not find a convenient reference.

Proposition 2.8. *If $2 < k < n-2$, then $G(k,n)^*$ in $\mathbb{P}^*(\bigwedge^k V)$ is a hypersurface. Furthermore, a general hyperplane parameterized by $G(k,n)^*$ is tangent to $G(k,n)$ at one point.*

3. The proof of Theorem 1.2

In this section, we prove Theorem 1.2 and discuss its generalizations to $G(k,n)$.

Proof of Theorem 1.2. Let $\Sigma_{a,b}(F_{n-1-a} \subset F_{n-b})$ be a Schubert variety with class $\sigma_{a,b}$ in $G(2,n)$. Suppose that H is a hyperplane in $\mathbb{P}(\bigwedge^2 V)$ containing $\Sigma_{a,b}(F_{n-1-a} \subset F_{n-b})$. Notice that $\Sigma_{a,b}(F_{n-1-a} \subset F_{n-b}) \subset G(2, F_{n-b})$. There are two possibilities. Either $G(2, F_{n-b}) \subset H$; or $H \cap G(2, F_{n-b})$ is a hyperplane section of $G(2, F_{n-b})$ that contains $\Sigma_{a,b}(F_{n-1-a} \subset F_{n-b})$. We will now analyze each of these possibilities.

First, assume that $H \cap G(2, F_{n-b})$ is a hyperplane section of $G(2, F_{n-b})$. A linear embedding $V' \hookrightarrow V$, induces an embedding $G(2, V') \hookrightarrow G(2, V)$. The following lemma analyzes the relation between the singular loci of $H \cap G(2, V)$ and $H \cap G(2, V')$.

Lemma 3.1. *Let $G(2,n) \hookrightarrow G(2,n+1)$ be the embedding induced by the embedding of $V_n \hookrightarrow V_{n+1}$. Let $H \cap G(2,n)$ be a linear section of $G(2,n)$ in $\mathbb{P}(\bigwedge^2 V_n)$ with singular locus $\Sigma_{2k,2k}$. Let H' be a general hyperplane in $\mathbb{P}(\bigwedge^2 V_{n+1})$ such that $H' \cap G(2,n+1)$ restricts to $H \cap G(2,n)$. Then the singular locus of $H' \cap G(2,n+1)$ is $\Sigma_{2(k+1),2(k+1)}$.*

Proof. Pick a basis $e_1, \dots, e_{n+1}$ of V_{n+1} such that V_n is spanned by the first n vectors and the singular locus of $H \cap G(2,n)$ parameterizes two-dimensional subspaces contained in the span F_{n-2k} of the first $n-2k$ vectors. Then H is defined by a linear equation $L(p_{i,j}) = 0$, where L is a linear combination of the Plücker coordinates $p_{i,j}$ for $i < j$ and $n - 2k < j \leq n$. A hyperplane in $\mathbb{P}(\bigwedge^2 V_{n+1})$ that contains H may be expressed as $L(p_{i,j}) + \sum_{i=1}^n a_i p_{i,n+1} = 0$.

By Bertini's Theorem [Ha, III.10.9], the singular locus of $H' \cap G(2, n+1)$ for a general hyperplane containing H is contained in $H \cap G(2,n)$. Let W be the $(n-2k-1)$-dimensional linear space cut out on F_{n-2k} by the linear equation $\sum_{i=1}^n a_i x_i = 0$. Then $H' \cap G(2, n+1)$ contains the tangent space to $G(2, n+1)$ at any two-dimensional space contained in W. At a point, $u \wedge v$ with $u, v \in W$, the tangent space is spanned by replacing at most one of u or v by elements of a basis. All the Plücker coordinates defining H' clearly vanish at all these points. Hence, $H' \cap G(2, n+1)$ is singular along two-dimensional subspaces contained in W. We conclude that the singular locus of $H' \cap G(2, n+1)$ contains a $\Sigma_{2(k+1),2(k+1)}$ of two-dimensional subspaces contained in W. Conversely, for a two-dimensional space not contained in that hyperplane, there exists a vector v such that $\sum a_i v_i \neq 0$. Hence, the point $v \wedge e_{n+1}$ is not contained in H', but it is contained in the tangent space to a line $w \wedge v$. Hence, the singular locus does not contain all of $\Sigma_{k,k}$. The lemma follows. $\square$

We are now ready to prove the theorem in the case H does not contain $G(2, F_{n-b})$. There are two cases that we need to analyze separately. First, assume that $a = n-2$. Since the Grassmannian contains linear spaces of the form $\Sigma_{n-2,0}$, any hyperplane section contains linear spaces $\Sigma_{n-2,1}$ of one smaller dimension. Hence, π_2 is surjective for $\lambda = (n-2, i)$ for $i > 0$. We now have to analyze the case $\lambda = (n-2, 0)$. In this case, the flag variety $F(1, n; n)$ is isomorphic to $\mathbb{P}^{n-1}$. Hence, $\dim(\mathcal{I}) = \binom{n}{2} - 1$. If n is even, then the general singular hyperplane section X of $G(2,n)$ is singular along a point $[\Lambda] \in G(2,n)$. Furthermore, in this case the dual variety $G(2,n)^*$ is a hypersurface, hence has dimension $\binom{n}{2} - 2$. By Lemma 2.4, if $F_1 \subset \Lambda$, then every two-dimensional subspace containing F_1 is contained in X. Since the space of one-dimensional subspaces of Λ is isomorphic to $\mathbb{P}^1$, the general fiber of π_2 over $G(2,n)^*$ has dimension greater than or equal to one. By the Theorem on the Dimension of Fibers, $\dim(\pi_2^{-1}(G(2,n)^*) \geq \binom{n}{2} - 1$. However, since $\pi_2^{-1}(G(2,n)^*) \subset \mathcal{I}(n-2, 0)$, $\dim(\pi_2^{-1}(G(2,n)^*) \leq \binom{n}{2} - 1$. We conclude that $\pi_2^{-1}(G(2,n)^*) = \mathcal{I}(n-2, 0)$ and consequently, π_2 is not surjective.

If n is odd, then the skew-symmetric form Q_H corresponding to any hyperplane H in $\mathbb{P}(\bigwedge^2 V)$ must have non-trivial kernel. Let v be a vector in the kernel of Q_H. Then any two-dimensional subspace W such that $v \in W$ is isotropic with respect to Q_H. Consequently, H contains the Schubert variety $\Sigma_{n-2,0}$ parameterizing the two-dimensional subspaces containing v. For a general hyperplane H, the kernel of Q_H is one-dimensional and H contains a unique Schubert variety of the form $\Sigma_{n-2,0}$.

Now we can discuss the case $\Sigma_{a,0}$ with $a < n-2$. Let X be a general hyperplane section containing $\Sigma_{a,0}$. Then the singular locus of X contains $\Sigma_{a+1,a+1}$. If a is odd, then by Proposition 2.5, there exists hyperplane sections whose singular locus is $\Sigma_{a+1,a+1}$. By Lemma 2.3, such hyperplane sections contain a Schubert variety of the form $\Sigma_{a,0}$. We conclude that $\pi_2(\mathcal{I}(a,0)) = S_{(a+1)/2}$. If a is even, then $a+1$ is odd. Since, by Proposition 2.5, the singular locus of a hyperplane section has the form $\Sigma_{2k,2k}$, the singular locus of X contains but cannot equal $\Sigma_{a+1,a+1}$. We conclude that the singular locus of X has to contain a larger Schubert variety of the form $\Sigma_{a,a}$. Conversely, a hyperplane section whose singular locus has the form $\Sigma_{a,a}$ contains a Schubert variety of the form $\Sigma_{a,0}$. We conclude that the image of π_2 is $S_{a/2}$.

Returning to the original argument, if $b > 0$, then $\Sigma_{a,b}$ is a Schubert variety with class $\sigma_{a-b,0}$ in $G(2, n-b)$. Hence, any hyperplane section of $G(2, n-b)$ containing $\sigma_{a-b,0}$ is singular along a Schubert variety of the form $\Sigma_{a-b+1,a-b+1}$ if $a-b$ is odd or $\Sigma_{a-b,a-b}$ if $a-b$ is even. Using Lemma 3.1 b-times, we conclude that if $a-b$ is even, then the general hyperplane containing $\Sigma_{a,b}$ is smooth if $a+b > n-3$ or singular along a Schubert variety of the form $\Sigma_{a+b+1,a+b+1}$ when $a+b \leq n-3$. Similarly, when $a-b$ is odd, then a hyperplane section of $G(2, n-b)$ containing $\Sigma_{a-b,0}$ is singular along $\Sigma_{a-b,a-b}$. Using Lemma 3.1 b-times, we conclude that a general hyperplane containing $\Sigma_{a,b}$ is smooth when $a+b > n-2$ or singular along $\Sigma_{a+b,a+b}$ when $a+b \leq n-2$.

Finally, we analyze the cases when the hyperplane contains $G(2, n-b)$ or when $a = b$. The first observation is that the only hyperplanes containing a Schubert variety of the form $\Sigma_{1,1}(F_{n-2} \subset F_{n-1})$ are Schubert varieties $\Sigma_1(G_{n-2} \subset G_n)$. The flag variety $F(n-1; n) \cong (\mathbb{P}^{n-1})^*$, hence has dimension $n-1$. The fiber dimension of π_1 over a point in $F(n-1;n)$ is $n-2$. Hence the dimension of $\mathcal{I}(1,1)$ is $2n-3$. The locus of Schubert varieties in $\mathbb{P}^*(\bigwedge^2 V)$ has dimension $2(n-2)$. If F_{n-1} contains G_{n-2}, then $\Sigma_{1,1}(F_{n-2} \subset F_{n-1}) \subset \Sigma_1(G_{n-2} \subset G_n)$. Hence, the fiber of π_2 over a hyperplane corresponding to a Schubert variety has dimension at least one. We conclude that $\dim(\pi_2^{-1}(S_1)) = 2n-3 = \dim(\mathcal{I}(1,1))$. Hence, $\pi_2(\mathcal{I}(1,1)) = S_1$ and every hyperplane containing a Schubert variety $\Sigma_{1,1}$ is a Schubert variety Σ_1. Applying Lemma 3.1 $(b-1)$-times, we conclude that a general hyperplane section containing $\Sigma_{b,b}$ is smooth if $2b > n-2$ or singular along a Schubert variety of the form $\Sigma_{2b,2b}$ if $2b \leq n-2$. This also concludes the discussion of the case $a \neq b$. Let H and H' be two hyperplanes containing $\Sigma_{a,b}$. If $G(2, F_{n-b}) \subset H$ and $G(2, F_{n-b}) \not\subset H'$, then we have just proved that the dimension of the singular locus of $G(2,n) \cap H$ is greater than or equal to the dimension of the singular locus of $H' \cap G(2,n)$. This concludes the proof of the theorem. $\square$

Since the proof of Proposition 1.9 uses similar techniques, we include it in this section.

Proof of Proposition 1.9. Let λ be a partition of the form $\lambda_1 = \lambda_{k-1} = n-k$ and $\lambda_k > 0$, then the Plücker image of Σ_λ is a linear space. Since the Grassmannian contains linear spaces with cohomology class σ_μ, where $\mu = ((n-k)^{k-1}, 0)$, every hyperplane section contains linear spaces with cohomology class σ_λ. The same argument applies for a partition λ with $\lambda_1 = n-k$ and $\lambda_k \geq n-k-1$ by considering linear spaces with cohomology class σ_ν, where $\nu = ((n-k-1)^k)$. This proves the second part of the proposition.

To prove the first part of the proposition, we will show that if λ is a partition such that $\lambda_1 < n-k$ and $\lambda_k = 0$, then any hyperplane H containing Σ_λ is singular. Fix a basis $e_1, \dots, e_n$ of V. Let $F_\bullet$ be the flag where the flag element F_i is the span of the basis vectors $e_1, \dots, e_i$. Let H be a hyperplane containing $\Sigma_\lambda(F_\bullet)$. Then the equation defining H must be a linear combination of the Plücker coordinates defining $\Sigma_\lambda(F_\bullet)$. Recall that the Plücker coordinates vanishing on $\Sigma_\lambda(F_\bullet)$ are $p_{i_1,\dots,i_k}$ with $i_1 < \cdots < i_k$ such that $i_j > n-k+j-\lambda_j$ for at least one j. Since by assumption $\lambda_k = 0$ and we cannot have $i_k > n$, there must exist $j < k$ such that $i_j > n-k+j-\lambda_j$. In particular, $i_{k-1} > n-k+j-\lambda_j+k-j-1 = n-\lambda_j > k$.

It follows that $H \cap G(k,n)$ is singular at the point $p = e_1 \wedge e_2 \wedge \cdots \wedge e_k$. The tangent space to $G(k,n)$ at p is spanned by Plücker coordinates $p_{i_1,\dots,i_k}$ where the set $\{i_1, \dots, i_k\}$ differs from $\{1, \dots, k\}$ in at most one element. On the other hand, the Plücker coordinates occurring in the equation of H have indices that differ from $\{1, \dots, k\}$ in at least two elements. Hence, H vanishes at all the points spanning the tangent space to $G(k,n)$ at p. We conclude that $H \cap G(k,n)$ is singular at p. This concludes the proof of the proposition. $\square$

Proofs of Corollaries 1.10 and 1.11. When λ is the partition $\lambda_1 = \cdots = \lambda_{k-1} = n-k-1$ and $\lambda_k = 0$, then, by Proposition 1.9, for any hyperplane H containing Σ_λ the hyperplane section $H \cap G(k,n)$ is singular at a point. Conversely, if $H \cap G(k,n)$ is singular at a point $p = e_1 \wedge \cdots \wedge e_k$, then by Lemma 2.3 the Schubert variety Σ_λ parameterizing k-dimensional subspaces that intersect $Span(e_1, \dots, e_k)$ in a subspace of dimension at least $k-1$ is contained in H. In this case, we conclude that the image of $\pi_2(\mathcal{I}(\lambda))$ is precisely the dual variety.

Note that $h^0(I_{\Sigma_\lambda}(1)) = \binom{n}{k} - k(n-k) - 1 = N$. Hence, the incidence correspondence $\mathcal{I}(\lambda)$ is a projective space bundle over $G(k,n)$ with fibers of dimension $N-1$. In particular, $\dim(\mathcal{I}(\lambda)) = \binom{n}{k} - 2$. When $n-2 > k > 2$, the dual variety $G(k,n)^*$ is a hypersurface and the general tangent hyperplane to $G(k,n)$ is tangent at a unique point. Therefor, π_2 is a birational map. Hence, $\pi_2 : \mathcal{I}(\lambda) \to G(k,n)^*$ gives a resolution of singularities of $G(k,n)^*$. This concludes the proofs of Corollary 1.10 and Corollary 1.11. $\square$

4. The proof of Theorem 1.3

In this section, we prove Theorem 1.3 and discuss some generalizations to $G(k,n)$.

Proof of Theorem 1.3. Let H be a hyperplane in $\mathbb{P}(\bigwedge^2 V)$ such that $[H] \in S_r/S_{r-1}$. Then $H \cap G(2,n)$ is singular along a Schubert variety $\Sigma_{2r,2r}$ parameterizing two-dimensional subspaces of V contained in a linear subspace F_{n-2r}. First, suppose that $a \neq b$. Let $(V_{n-a-1} \subset V_{n-b})$ be the partial flag defining a Schubert variety $\Sigma_{a,b} \subset H \cap G(2,n)$. Suppose that $\dim(V_{n-a-1} \cap F_{n-2r}) = j$. Then clearly

$$0 \leq j \leq \min(n-a-1, n-2r).$$

Consider the restriction of H to $G(2, V_{n-b})$. Either H identically vanishes on $G(2, V_{n-b})$; or H defines a hyperplane section of $G(2, V_{n-b})$.

If H identically vanishes on $G(2, V_{n-b})$, then both V_{n-a-1} and V_{n-b} are Q_H-isotropic. Hence, trivially $V_{n-a-1} \subset V_{n-b} \subset V_{n-a-1}^\perp$. Take a linear space S_{2r} of dimension $2r$ complementary to F_{n-2r}. Then the restriction of Q_H to S_{2r} is non-degenerate. Since $Span(V_{n-a-1}, F_{n-2r}) \cap S_{2r}$ is isotropic with respect to the restriction of Q_H to S_{2r}, its dimension $n-a-1-j$ must be less than or equal to r. Equivalently, $n-a-1-r \leq j$. Similarly, since V_{n-b} is isotropic, $n-b \leq n-r$. In particular, $b \geq r$. Hence, the inequality $n-a-1-\min(r,b) \leq j$ holds.

Next, suppose that H defines a hyperplane section of $G(2, V_{n-b})$. By our assumption that $\Sigma_{a,b}(V_{n-a-1} \subset V_{n-b}) \subset H \cap G(2,n)$, we must have that $[W] \in H \cap G(2,n)$ for every two-dimensional subspace W that intersects V_{n-a-1} non-trivially and is contained in V_{n-b}. In particular, $[W]$ is contained in $H \cap G(2,n)$ for every two-dimensional subspace W contained in V_{n-a-1}. We conclude that the skew-symmetric form Q_H vanishes identically on V_{n-a-1}. Hence, V_{n-a-1} is Q_H-isotropic. Hence, $Span(V_{n-a-1}, F_{n-2r})$ is also Q_H-isotropic. The dimension of this vector space, which by assumption is $n-a-1+n-2r-j$, has to be less than or equal to $n-r$. We conclude that $n-a-1-r \leq j$.

Finally, since the restriction of Q_H to V_{n-b} must contain V_{n-a-1} in its kernel, we must have that $V_{n-b} \subset V_{n-a-1}^{\perp}$. By assumption, the dimension of $V_{n-a-1}^{\perp}$ is $n-1-a-j$. Hence, $n-a-1-j \leq b$. Combining all these inequalities, yields the inequality

$$\max(0, n-a-1-\min(b,r)) \leq j \leq \min(n-a-1, n-2r).$$

Note that by assumption $2r \leq a+b+1$, so for j satisfying the assumptions of the theorem, these inequalities hold.

Conversely, suppose j satisfies the inequalities

$$\max(0, n-a-1-\min(b,r)) \leq j \leq \min(n-a-1, n-2r).$$

Then every Schubert variety $\Sigma_{a,b}(V_{n-a-1} \subset V_{n-b})$ is contained in $H \cap G(2,n)$ provided V_{n-a-1} is Q_H isotropic and $V_{n-b} \subset V_{n-a-1}^{\perp}$. This is clear since the kernel of Q_H restricted to $V_{n-a-1}^{\perp}$ contains V_{n-a-1}. Hence, every two-dimensional space intersecting V_{n-a-1} non-trivially is Q_H isotropic.

Furthermore, there exists flags $(V_{n-a-1} \subset V_{n-b})$ such that $\dim(V_{n-a-1} \cap F_{n-2r}) = j$. To construct such a flag, let S_{2r} be a linear space complementary to F_{n-2r}. Pick a Q_H isotropic subspace W of dimension $n-a-1-j$ in S_{2r}. This is possible since $n-a-1-j \leq r$. Pick a j-dimensional subspace W' of F_{n-2r}. Let $V_{n-a-1} = Span(W, W')$. Then V_{n-a-1} is isotropic and has dimension $n-a-1$. Next, consider $V_{n-a-1}^{\perp}$, which has dimension $a+1+j$. Since by assumption $n-a-1-b \leq j$, $n-b \leq a+1+j$. Therefore, there exists $n-b$ dimensional subspaces of $V_{n-a-1}^{\perp}$ containing V_{n-a-1}.

Let Z_j denote the locus of two-step flags $(V_{n-a-1} \subset V_{n-b})$ in $F(n-a-1, n-b; n)$ such that V_{n-a-1} is Q_H isotropic, $\dim(V_{n-a-1} \cap F_{n-2r}) \geq j$ and $V_{n-b} \subset V_{n-a-1}^{\perp}$ and $\dim(V_{n-b} \cap F_{n-2r}) \geq 2n-2r-a-b-1-j$. It is clear from the construction in the previous paragraph that Z_j is irreducible. Recall the following definitions from the statement of the theorem

$$M = \max\left(0, n-1-a-\min(r,b)\right) \quad \text{and} \quad N = \min\left(n-a-1, n-r-\frac{a+b+1}{2}\right).$$

We have shown that

$$X((a,b),H) = \bigcup_{j=M}^{\min(n-a-1,n-2r)} Z_j$$

and in this range each Z_j is non-empty. Finally, there remains to check that Z_j is an irreducible component of $X((a,b),H)$ if $M \leq j \leq N$ and $X((a,b),H) = \bigcup_{j=M}^{N} Z_j$.

The dimension $\dim(V_{n-a-1} \cap F_{n-2r})$ is an upper-semi-continuous function. Consequently, if $j_1 > j_2$, then linear spaces intersecting F_{n-2r} in a j_1-dimensional subspace cannot specialize to linear spaces intersecting F_{n-2r} in a j_2-dimensional subspace. Therefore, Z_{j_2} cannot be contained in Z_{j_1}. On the other hand, $\dim(V_{n-b} \cap F_{n-2r})$ is also an upper-semi-continuous function. By construction, for a general point (V_{n-a-1}, V_{n-b}) in Z_j, $\dim(V_{n-b} \cap F_{n-2r}) = \max(j, 2n-2r-a-b-1-j)$ since V_{n-b} is an arbitrary linear space containing V_{n-a-1} and contained in the $(a+j+1)$-dimensional space $V_{n-a-1}^{\perp}$. Suppose $n-r-\frac{a+b+1}{2} \geq j_1 > j_2$, then the dimension of $V_{n-b} \cap F_{n-2r}$ for a general point in Z_{j_1}, respectively, Z_{j_2} is given by $2n-2r-a-b-1-j_1 < 2n-2r-a-b-1-j_2$. Hence, Z_{j_1} cannot be contained in Z_{j_2}. We conclude that for $M \leq j \leq N$, Z_j form irreducible components of $X((a,b),H)$.

There remains to show that when $2j > 2n-2r-a-b-1$, then Z_j is contained in Z_{j-1}. Let $(V_{n-a-1} \subset V_{n-b})$ be a point of Z_j such that $\dim(V_{n-a-1} \cap F_{n-2r}) = \dim(V_{n-b} \cap F_{n-2r}) = j$. Let E be a codimension one linear space in V containing the vector space $Span(V_{n-b}, F_{n-2r})$. By assumption,

$$\dim(Span(V_{n-b}, F_{n-2r})) = 2n-2r-b-j < a+1+j \leq n.$$

Hence, we can always find a codimension one linear space E containing $Span(V_{n-b}, F_{n-2r})$. Since a non-degenerate skew-symmetric form can only exist in an even dimensional vector space, the dimension of the kernel of Q_H restricted to E has to have dimension greater than or equal to $n-2r+1$. Denote this kernel by K_E. Let V_{a+1-b} be a general subspace in V_{n-b} complementary to V_{n-a-1}. Pick a pencil of linear spaces $V_{n-a-1}(t)$ such that $V_{n-a-1}(0) = V_{n-a-1}$, $V_{n-a-1}(t) \subset K_E$ and $V_{n-a-1}(t) \not\subset F_{n-2r}$ for $t \neq 0$. Consider the pencil of flags $(V_{n-a-1}(t) \subset Span(V_{n-a-1}(t), V_{a+1-b}))$. First, notice that when $t=0$, this is simply $(V_{n-a-1} \subset V_{n-b})$. Hence, except for finitely many t, these flags are contained in $F(n-a-1, n-b; n)$. By construction, $\dim(V_{n-a-1}(t) \cap F_{n-2r}) = j-1$. Since $V_{n-a-1}(t) \subset K_E$, $Span(V_{n-a-1}(t), V_{a+1-b}) \subset V_{n-a-1}(t)^{\perp}$. Hence, the general member of this family is contained in Z_{j-1}. We conclude that $Z_j \subset Z_{j-1}$.

The computation of the dimension of Z_j is standard. We have to choose a Q_H isotropic subspace V_{n-a-1} that intersects the kernel of Q_H in a subspace of dimension j. The reader can easily check that the dimension of the space of such isotropic subspaces is

$$\frac{(n-a-1)(3a+j-n+4)}{2} - j\frac{(4r+3a+3j-3n+4)}{2}.$$

Then we need to choose an $(n-b)$-dimensional subspace in the $(a+j+1)$-dimensional subspace $V_{n-a-1}^{\perp}$ containing V_{n-a-1}. The dimension of the space of such linear spaces V_{n-b} is

$$(a+1-b)(a+b+j-n+1).$$

This immediately yields the dimension formula for Z_j.

Next, suppose that $a=b$. In this case, the Schubert variety is determined by one flag element V_{n-a}. Since $\Sigma_{a,a} \subset H \cap G(2,n)$, V_{n-a} is Q_H isotropic. Conversely, if V_{n-a} is Q_H-isotropic, then $[W] \in H \cap G(2,n)$ for every two dimensional subspace $W \subset V_{n-a}$. We conclude that $X((a,a),H)$ is the space of Q_H-isotropic linear spaces of dimension $n-a$. It is standard that this space is irreducible and has the claimed dimension. □

Example 4.1. Let H be a hyperplane in $\mathbb{P}(\bigwedge^2 V_8)$ such that $[H] \in S_2$. Let $H \cap G(2,8)$ be the corresponding hyperplane section of $G(2,8)$. Then the space $X((5,4),H)$ parameterizing Schubert varieties of the form $\Sigma_{5,4}(V_2 \subset V_4)$ has two irreducible components Z_0 and Z_1. The singular locus of $H \cap G(2,8)$ consists of two-dimensional subspaces contained in a four-dimensional vector space F_4. The component Z_0 parameterizes pairs $(V_2 \subset V_4)$ such that $[V_2] \in H \cap G(2,8)$ and $V_4 \subset V_2^{\perp}$ and $\dim(V_4 \cap F_4) \geq 2$. The component Z_1 parameterizes V_2 such that $\dim(V_2 \cap F_4) \geq 1$ and $V_4 \subset V_2^{\perp}$. Note that Z_1 contains the pairs where $V_2 \subset F_4$ and $V_4 \subset V_2^{\perp}$.

The corollaries are obtained by specializing the numbers a and b.

Proof of Corollary 1.4. When $a=b=r$, we are in Case (2) of Theorem 1.3. $X((a,a),H)$ parameterizes $(n-a)$-dimensional isotropic subspaces of Q_H. These are maximal dimensional isotropic subspaces, hence they all contain the kernel F_{n-2r} of Q_H. Passing to the quotient V/F_{n-2r}, we see that $X((a,a),H)$ parameterizes r-dimensional isotropic subspaces of a $2r$-dimensional vector space under a non-degenerate skew-symmetric form. We conclude that $X((a,a),H)$ is isomorphic to $SG(r,2r)$. This variety is irreducible of dimension $\binom{r+1}{2}$. $\square$

Proof of Corollary 1.5. When $a+b+1=2r$, we are in Case (1) of Theorem 1.3. The integers a and b must satisfy the inequalities $b<r\leq a$. Hence $n-a-b-1=n-2r\leq j\leq n-r-\frac{a+b+1}{2}=n-2r$. We conclude that $j=n-2r$ and that $X((a,2r-a-1),H)$ is irreducible. The linear space V_{n-a-1} must contain the kernel of Q_H, which by assumption has dimension $n-2r=j$. Furthermore, $\dim(V_{n-a-1}^{\perp})=n-2r+a+1=n-b$. Hence, $V_{n-b}=V_{n-a-1}^{\perp}$. Therefore, $X((a,2r-a-1),H)$ can be identified with $SG(b,2r)$. $\square$

Proof of Corollary 1.6. When $b=0$, we are in Case (1) of Theorem 1.3. In this case, $n-a-1\leq j\leq n-a-1$. Hence, there is only one component and V_{n-a-1} is contained in F_{n-2r}. Therefore, in this case, $X((a,0),H)$ parameterizes linear spaces V_{n-a-1} contained in F_{n-2r}. This is the Grassmannian $G(n-a-1,n-2r)$, which has dimension $(n-a-1)(a+1-2r)$. $\square$

Proof of Corollary 1.7. If a is odd and $\lambda=(a,0)$, then we are in Case (1) of Theorem 1.3 and $j=n-a-1$. Hence, $V_{n-a-1}=F_{n-2r}$ and $V_{n-b}=V$. Hence, π_2 is a birational map from $\mathcal{I}(\lambda)$ to $S_{\frac{a+1}{2}}$. In particular, when n is odd a smooth hyperplane section contains a unique linear space of the form $\Sigma_{n-2,0}$. The rest of the corollary is obvious. $\square$

Finally, we prove Proposition 1.12, which clearly specializes to Corollary 1.8 when $k=2$.

Proof of Proposition 1.12. Let $H=\Sigma_1(F_{n-k}\subset F_{n-k+2}\subset\cdots\subset F_n)$. A Schubert variety Σ_λ is contained in H if and only if every k-dimensional subspace parameterized by Σ_λ intersects F_{n-k} non-trivially. Let $V_{n-k+k_1-\mu_1}\subset V_{n-k+k_2-\mu_2}\subset\cdots\subset V_{n-\mu_t}$ be the linear spaces defining Σ_λ. Let W be any k-dimensional subspace such that $[W]\in\Sigma_\lambda$. If for some j, $\dim(V_{n-k+k_j-\mu_j}\cap F_{n-k})\geq n-k-\mu_j+1$, then we can estimate $\dim(W\cap F_{n-k}\cap V_{n-k+k_j-\mu_j})$ as follows. $\dim(W\cap V_{n-k+k_j-\mu_j})\geq k_j$ since $[W]\in\Sigma_\lambda$. Hence, $\dim(W\cap F_{n-k}\cap V_{n-k+k_j-\mu_j})\geq k_j+n-k-\mu_j+1-(n-k+k_j-\mu_j)=1$. We conclude that $[W]\in H\cap G(k,n)$, hence $\Sigma_\lambda\subset H\cap G(k,n)$. Note that if $\mu_t=0$, then the condition $\dim(V_{n-\mu_t}\cap F_{n-k})\geq n-k+1$ is impossible to satisfy. Therefore, that case has to be treated separately.

Conversely, suppose that $\dim(V_{n-k+k_j-\mu_j}\cap F_{n-k})=n-k-\mu_j$ for every $1\leq j\leq t$. Then there exists a k_1-dimensional subspace in $V_{n-k+k_1-\mu_1}$ that does not intersect F_{n-k}. This can be extended to a k_2-dimensional subspace in $V_{n-k+k_2-\mu_2}$ that does not intersect F_{n-k}. Continuing this way, we construct a k-dimensional subspace W such that $[W]\in\Sigma_\lambda$, but $[W]\notin H\cap G(k,n)$.

Let S_j be the Schubert variety in the flag variety $F(n-k+k_1-\mu_1,\ldots,n-\mu_t;n)$ defined by

$$S_j=\{(V_{n-k+k_1-\mu_1}\subset\cdots\subset V_{n-\mu_t}\mid\dim(V_{n-k+k_j-\mu_j}\cap F_{n-k})\geq n-k-\mu_j+1\}.$$

We have shown that $X(\lambda,H)=\cup_{i=1}^{t-\delta_{0,t}}S_j$. Since the Schubert varieties $S_j\not\subset S_i$ for $i\neq j$, we conclude that the $t-\delta_{0,t}$ Schubert varieties S_j form the irreducible components of $X(\lambda,H)$. This concludes the proof of the proposition. $\square$

References

[BL] S. Billey and V. Lakshmibai, *Singularities of Schubert varieties*, Springer, 2000.

[BC] L. Borisov and A. Căldăraru, The Pfaffian-Grassmannian derived equivalence, *J. Algebraic Geom.*, **18** no. 2 (2009), 201–222.

[C1] I. Coskun, The enumerative geometry of Del Pezzo surfaces via degenerations, *Amer. Math. J.,* **128** no. 3 (2006), 751–786.

[C2] I. Coskun, Rigid and non-smoothable Schubert classes, *J. Differential Geom.*, **87** no.3 (2011), 493–514.

[D] R.Y. Donagi, On the geometry of Grassmannians, *Duke Math. J.*, **44** no.4 (1977), 795–837.

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley-Science, 1978.

[H] J. Harris, *Algebraic Geometry: A First Course*, Springer, 1992.

[Ha] R. Hartshorne, *Algebraic Geometry*, Springer, 1977.

[HP] W.V.D. Hodge and D. Pedoe, *Methods of Algebraic Geometry*, Cambridge University Press, 1994.

[MS] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Clarendon Press, Oxford, 1998.

[PVdV] J. Pintkowski and A. Van de Ven, The automorphism group of linear sections of the Grassmannian $\mathbb{G}(1,n)$, *Documenta Math.*, **4** (1999), 623–664.

[R] E.A. Rødland, The Pfaffian Calabi-Yau, its mirror, and their link to the Grassmannian $G(2,7)$, *Compositio Math.*, **122** no. 2 (2000), 135–149.

[S] I.R. Shafarevich, *Basic Algebraic Geometry I*, Springer-Verlag, 1994.

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