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Rajendra Bhatia[1], **Shmuel Friedland**[2], **Tanvi Jain**[3]

[1] *Indian Statistical Institute, New Delhi 110016, India*
rbh@isid.ac.in
[2] *Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, Chicago, 60607-7045, USA*
friedlan@uic.edu
[3] *Indian Statistical Institute, New Delhi 110016, India*
tanvi@isid.ac.in

Abstract

Given positive numbers $p_1 < p_2 < \cdots < p_n$, and a real number r let L_r be the $n \times n$ matrix with its i, j entry equal to $(p_i^r - p_j^r)/(p_i - p_j)$. A well-known theorem of C. Loewner says that L_r is positive definite when $0 < r < 1$. In contrast, R. Bhatia and J. Holbrook, (Indiana Univ. Math. J, 49 (2000) 1153-1173) showed that when $1 < r < 2$, the matrix L_r has only one positive eigenvalue, and made a conjecture about the signatures of eigenvalues of L_r for other r. That conjecture is proved in this paper.

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1 Introduction

Let f be a real-valued C^1 function on $(0, \infty)$. Let $p_1 < p_2 < \cdots < p_n$ be any n points in $(0, \infty)$. The $n \times n$ matrix

$$L_f(p_1, \ldots, p_n) = \left[\frac{f(p_i) - f(p_j)}{p_i - p_j} \right]_{i,j=1}^{n} \tag{1}$$

is called a *Loewner matrix* associated with f. It is understood that when $i = j$, the quotient in (1) represents the limiting value $f'(p_i)$. Of particular interest to us are the functions $f(t) = t^r$, $r \in \mathbb{R}$. In this case we write L_r for $L_f(p_1, \ldots, p_n)$, where the roles of n and $p_1, \ldots p_n$ can be inferred from the context. Thus L_r is the $n \times n$ matrix

$$L_r = \left[\frac{p_i^r - p_j^r}{p_i - p_j} \right]_{i,j=1}^{n}. \tag{2}$$

Loewner matrices are important in several contexts, of which we mention two that led to the present study. (The reader may see Section 4.1 of [12] for an excellent discussion of both these aspects of Loewner matrices.) The function f on $(0, \infty)$ induces, via the usual functional calculus, a matrix function $f(A)$ on the space of positive definite matrices. Let $Df(A)$ be the Fréchet derivative of this function.This is a linear map on the space of Hermitian matrices. The *Daleckii-Krein formula* describes the action of this map in terms of Loewner matrices. Choose an orthonormal basis in which $A = \text{diag}(p_1, \ldots, p_n)$. Then the formula says that for every Hermitian X

$$Df(A)(X) = L_f(p_1, \ldots, p_n) \circ X, \tag{3}$$

where $A \circ B$ stands for the entrywise product $[a_{ij}b_{ij}]$ of A and B.

The function f is said to be *operator monotone* on $(0, \infty)$ if $A \geq B > 0$ implies $f(A) \geq f(B)$. (As usual $A \geq 0$ means A is positive semidefinite.) A fundamental theorem due to Charles Loewner says that f is operator monotone if and only if all Loewner matrices associated with f (for every n and for every choice $p_1, \ldots, p_n$) are positive semidefinite. Another basic fact, again proved first by Loewner, says that $f(t) = t^r$ is operator monotone if and only if $0 \leq r \leq 1$. See [1] Chapter V.

Combining these various facts with some well-kown theorems on positive linear maps [2] one can see that if f is operator monotone, then the norm of $Df(A)$ obeys the relations

$$\|Df(A)\| = \|Df(A)(I)\| = \|f'(A)\|, \tag{4}$$

and is therefore readily computable. In particular, for the function $f(t) = t^r$ if we write DA^r for $Df(A)$, then (4) gives

$$\|DA^r\| = \|rA^{r-1}\|, \quad \text{for } 0 \leq r \leq 1. \tag{5}$$

This was first noted in [3], and used to derive perturbation bounds for the operator absolute value. Then in [8] Bhatia and Sinha showed that the relation (5) holds also for $-\infty < r < 0$ and for $2 \leq r < \infty$ but, mysteriously, not for $1 < r < \sqrt{2}$. The case $\sqrt{2} \leq r < 2$, left open in this paper, was resolved in [4] by Bhatia and Holbrook, who showed that here again the relation (5) is valid.

One ingredient of the proof in [4] is their Proposition 2.1 which says that when $1 < r < 2$, the $n \times n$ matrix L_r has just one positive eigenvalue. We have remarked earlier that when $0 < r < 1$, the matrix L_r is positive semidefinite and therefore, none of its eigenvalues is negative. This contrast as r moves from $(0, 1)$ to $(1, 2)$ is intriguing, and raises the natural question about the behaviour of eigenvalues of L_r for other values of r. Bhatia and Holbrook [4] made a conjecture about this behaviour and established a small part of it: they settled the cases $r = 1, 2, \ldots, n-1$ apart from $0 < r < 1$ and $1 < r < 2$ already mentioned. The main goal of this paper is to prove this conjecture in full. This is our Theorem 1.1.

Let A be an $n \times n$ Hermitian matrix. The *inertia* of A is the triple

$$\text{In}(A) = (\pi(A), \zeta(A), \nu(A)),$$

where $\pi(A)$ is the number of positive eigenvalues of A, $\zeta(A)$ is the number of zero eigenvalues of A, and $\nu(A)$ the number of negative eigenvalues of A. Theorem 1.1 describes the inertia of L_r as r varies over $\mathbb{R}$. It is easy to see that the inertia of L_{-r} is the opposite of the inertia of L_r; i.e. $\pi(L_{-r}) = \nu(L_r)$ and $\nu(L_{-r}) = \pi(L_r)$. So we confine ourselves to the case $r > 0$.

Theorem 1.1. *Let $p_1 < p_2 < \cdots < p_n$ and r be any positive real numbers and let L_r be the matrix defined in (2). Then*

(i) *L_r is singular if and only if $r = 1, 2, \ldots, n-1$.*

(ii) *At the points $r = 1, 2, \ldots, n$, the inertia of L_r is given as follows:*

$$r = 2k \Rightarrow \text{In}\ (L_r) \ = \ (k, n-r, k),$$

and

$$r = 2k-1 \Rightarrow \text{In}\ (L_r) \ = \ (k, n-r, k-1).$$

(iii) *If $0 < r < n$ and r is not an integer, then*

$$\lfloor r \rfloor = 2k \Rightarrow \text{In}\ (L_r) \ = \ (n-k, 0, k)$$

and

$$\lfloor r \rfloor = 2k-1 \Rightarrow \text{In}\ (L_r) \ = \ (k, 0, n-k).$$

(iv) *If $r > n-1$, then* $\text{In}\ (L_r) = \text{In}\ (L_n)$.

(v) *Every nonzero eigenvalue of L_r is simple.*

It is helpful to illustrate the theorem by a picture. Figure 1 is a diagram of the (scaled) eigenvalues of a 6×6 matrix L_r when p_i are fixed and r varies. Some of the eigenvalues are very close to zero. To be able to distinguish between them the vertical scale has been expanded.

We have already mentioned that for $0 < r < 1$, statement (iii) of Theorem 1.1 follows from Loewner's theorem, and for $1 < r < 2$ it was established in [4]. The case $2 < r < 3$ was accomplished by Bhatia and Sano in [7]. We briefly explain this work.

Let $\mathcal{H}_1$ be the space

$$\mathcal{H}_1 = \left\{ x = (x_1, \ldots, x_n) : \sum_{i=1}^{n} x_i = 0 \right\}. \tag{6}$$

An $n \times n$ Hermitian matrix A is said to be *conditionally positive definite* if $\langle x, Ax \rangle \geq 0$ for all $x \in \mathcal{H}_1$, and if $-A$ has this property, then we say that A is *conditionally negative definite.* Since $\dim\mathcal{H}_1 = n-1$, a nonsingular conditionally positive definite matrix which is not positive definite has inertia $(n-1, 0, 1)$.

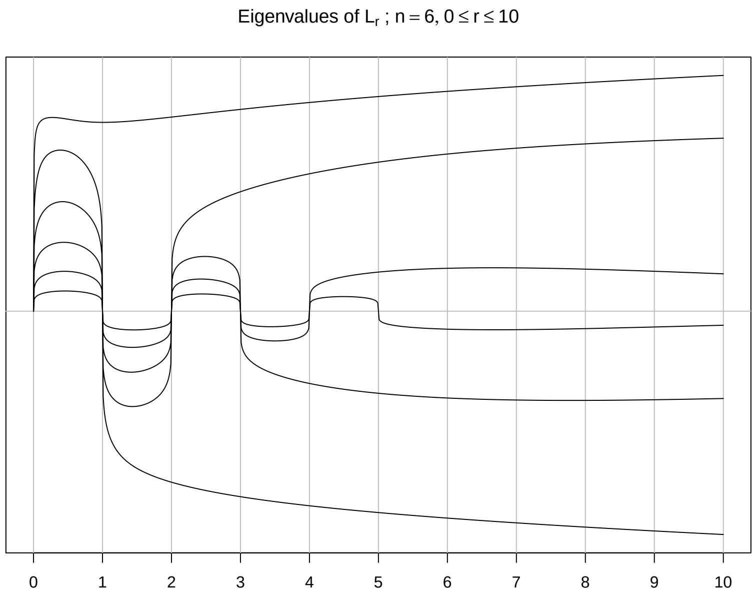

Figure 1:

In [7] it was shown that when $1 < r < 2$, the matrix L_r is nonsingular and conditionally negative definite. It follows that In $(L_r) = (1, 0, n-1)$, a fact established earlier in [4]. It was also shown in [7] that when $2 < r < 3$, the matrix L_r is nonsingular and conditionally positive definite. From this it follows that In $(L_r) = (n-1, 0, 1)$.

More generally, Bhatia and Sano [7] showed that f on $(0, \infty)$ is *operator convex* if and only if all Loewner matrices L_f are conditionally negative definite. This is a characterisation analogous to Loewner's for operator monotone functions. It is well-known that $f(t) = t^r$ is operator convex for $1 \leq r \leq 2$.

The proof of Theorem 1.1 is given in Section 2. We also indicate how the proofs for the parts already given in [4] and [7] can be considerably simplified. The inertia of the matrix $[(p_i + p_j)^r]$ has been studied by Bhatia and Jain in [5]. Some ideas in our proofs are similar to the ones used there.

2 Proofs and Remarks

Let X be an $n \times n$ nonsingular matrix. The transformation $A \mapsto X^*AX$ on Hermitian matrices is called a *congruence*. The *Sylvester Law of Inertia* says that

$$\text{In}\,(X^*AX) = \text{In}\,A \quad \text{for all} \quad X \in GL(n). \tag{7}$$

Let D be the diagonal matrix

$$D = \mathrm{diag}\,(p_1, \ldots, p_n). \tag{8}$$

Then for every r

$$L_{-r} = -D^{-r} L_r D^{-r}. \tag{9}$$

Hence by Sylvester's Law

$$\mathrm{In}\ L_r = (i_1, i_2, i_3) \Leftrightarrow \mathrm{In}\ L_{-r} = (i_3, i_2, i_1). \tag{10}$$

Thus all statements about In L_r for $r > 0$ give information about In L_{-r} as well.

Make the substitution $p_i = e^{2x_i}$, $x_i \in \mathbb{R}$. A simple calculation shows that

$$L_r = \left[\frac{e^{rx_i}}{e^{x_i}}\ \frac{\sinh r(x_i - x_j)}{\sinh(x_i - x_j)}\ \frac{e^{rx_j}}{e^{x_j}} \right].$$

In other words,

$$L_r = \Delta \widetilde{L}_r \Delta, \tag{11}$$

where $\Delta = \mathrm{diag}\,(e^{(r-1)x_1}, \ldots, e^{(r-1)x_n})$, and

$$\widetilde{L}_r = \left[\frac{\sinh r(x_i - x_j)}{\sinh(x_i - x_j)} \right]. \tag{12}$$

By Sylvester's Law In L_r = In $\widetilde{L}_r$. Several properties of L_r can be studied via $\widetilde{L}_r$, and vice versa. This has been a very effective tool in deriving operator inequalities; see, the work of Bhatia and Parthasarathy [6] and that of Hiai and Kosaki [9, 10, 11, 14].

When $n = 2$ we have

$$\widetilde{L}_r = \begin{bmatrix} r & \frac{\sinh r(x_1 - x_2)}{\sinh(x_1 - x_2)} \\ \frac{\sinh r(x_1 - x_2)}{\sinh(x_1 - x_2)} & r \end{bmatrix}.$$

So $\det \widetilde{L}_r = r^2 - \frac{\sinh^2 r(x_1 - x_2)}{\sinh^2(x_1 - x_2)}$. Thus $\det \widetilde{L}_r$ is positive for $0 < r < 1$, zero for $r = 1$, and negative for $r > 1$. One eigenvalue of $\widetilde{L}_r$ is always positive, and this shows that the second eigenvalue is positive, zero, or negative depending on whether $0 < r < 1$, $r = 1$, or $r > 1$, respectively. This establishes Theorem 1.1 in the simplest case $n = 2$.

An interesting corollary can be deduced at this stage. According to the two theorems of Loewner mentioned in Section 1, f is operator monotone if and only if all Loewner matrices L_f are positive semidefinite, and $f(t) = t^r$ is operator monotone if and only if $0 \leq r \leq 1$. Consequently, if $r > 1$, then there exists an n, and positive numbers $p_1, \ldots, p_n$ such that the associated Loewner matrix (2) is not positive definite. We can assert more:

Proposition 2.1. *Let $r > 1$. Then for every $n \geq 2$, and for every choice of $p_1, \ldots, p_n$, the matrix L_r defined in (2) has at least one negative eigenvalue.*

Proof Consider the 2×2 top left submatrix of L_r. This is a Loewner matrix. By Theorem 1.1 it has one negative eigenvalue. So, by Cauchy's interlacing principle, the $n \times n$ matrix L_r has at least one negative eigenvalue. $\square$

The Sylvester Law has a generalisation that is useful for us. Let $n \geq r$, and let A be an $r \times r$ Hermitian matrix and X an $r \times n$ matrix of rank r. Then

$$\text{In } X^*AX = \text{In } A + (0, n-r, 0). \tag{13}$$

A proof of this may be found in [5]. This permits a simple transparent proof of Part (ii) of Theorem 1.1. (This part has already been proved in [4].) When r is a positive integer we have

$$L_r = \left[p_i^{r-1} + p_i^{r-2}p_j + \cdots + p_j^{r-1}\right] = W^*VW,$$

where W is the $r \times n$ Vandermonde matrix

$$W = \begin{bmatrix} 1 & 1 & \dots & 1 \\ p_1 & p_2 & \dots & p_n \\ \cdot & \cdot & \dots & \cdot \\ p_1^{r-1} & p_2^{r-1} & \dots & p_n^{r-1} \end{bmatrix},$$

and V is the $r \times r$ antidiagonal matrix with all entries 1 on its sinister diagonal and all its other entries equal to 0. If $r = 2k$, the matrix V has k of its eigenvalues equal to 1, and the other k equal to -1. If $r = 2k-1$, then k of its eigenvalues are equal to 1, and $k-1$ are equal to -1. So, statement (ii) of Theorem 1.1 follows from the generalised Sylvester's Law (13). Next we prove statement (i).

Let $c_1, c_2, \dots, c_n$ be real numbers, not all of which are zero. Let f be the function on $(0, \infty)$ defined as

$$f(x) = \sum_{j=1}^{n} c_j \frac{x^r - p_j^r}{x - p_j}. \tag{14}$$

Theorem 2.2. *Let r be a positive real number not equal to $1, 2, \dots, n-1$. Then the function f defined in (14) has at most $n-1$ zeros in $(0, \infty)$.*

Proof Let $r_1 < r_2 < \cdots < r_m$, and let $a_1, \dots, a_m$ be real numbers not all of which are zero. Then the function

$$g(x) = \sum_{j=1}^{m} a_j x^{r_j}, \tag{15}$$

has at most $m-1$ zeros in $(0, \infty)$. This is a well-known fact, and can be found in e.g., [16], p.46.

Now let f be the function defined in (14) and let

$$g(x) = f(x) \prod_{j=1}^{n} (x - p_j). \tag{16}$$

Then g can be expressed in the form (15) with $m = 2n$ and

$$\{r_1, \ldots, r_{2n}\} = \{0, 1, \ldots, n-1, r, r+1, \ldots, r+n-1\}.$$

Further, we have $g(x) = x^r h_1(x) - h_2(x)$, where

$$h_1(x) = \sum_{i=1}^{n} c_i \prod_{j \neq i} (x - p_j), \;\; h_2(x) = \sum_{i=1}^{n} c_i p_i^r \prod_{j \neq i} (x - p_j).$$

Both h_1 and h_2 are Lagrange interpolation polynomials of degree at most $n-1$. Since not all c_i are zero, neither of these polynomials is identically zero. So, if $r \neq 1, 2, \ldots, n-1$, then g is not the zero function.

Hence the function g defined by (16) has at most $2n-1$ zeros in $(0, \infty)$. Of these, n zeros occur at $x = p_j$, $1 \leq j \leq n$. So f has at most $n-1$ zeros in $(0, \infty)$. $\square$

Corollary 2.3. *Let r be a positive real number different from $1, 2, \ldots, n-1$. Then the matrix L_r defined in (2) is nonsingular.*

Proof The matrix L_r is singular if and only if there exists a nonzero vector $c = (c_1, \ldots, c_n)$ such that $L_r(c) = 0$. In other words there exist real numbers $c_1, \ldots, c_n$, not all zero, such that

$$\sum_{j=1}^{n} c_j \frac{p_i^r - p_j^r}{p_i - p_j} = 0$$

for $i = 1, 2, \ldots, n$. But then the function $f(x)$ in (14) would have n zeros, viz., $x = p_1, \ldots, p_n$. That is not possible. $\square$

We have proved Part (i) of Theorem 1.1. Part (iv) follows from this. If the inertia of L_r were to change at some point $r_0 > n-1$, then one of the eigenvalues has to change sign at r_0. This is ruled out as L_r is nonsingular for all $r > n-1$.

Our argument shows that if $p_1 < p_2 < \cdots < p_n$ and $q_1 < q_2 < \cdots < q_n$ are two n-tuples of positive real numbers, then the matrix $\left[\frac{p_i^r - q_j^r}{p_i - q_j}\right]$ is nonsingular for every positive r different from $1, 2, \ldots, n-1$.

An $n \times n$ real matrix A is said to be *strictly sign-regular* (SSR for short) if for every $1 \leq k \leq n$, all $k \times k$ sub-determinants of A are nonzero and have the same sign. If this is true for every $1 \leq k \leq r$ for some $r < n$, then we say that A is in the class $\mathrm{SSR}_{\mathrm{r}}$. Sign-regular matrices and kernels are studied extensively in [15].

We have noted above that if r is any positive real number and k is any positive integer not greater than r, then every $k \times k$ matrix of the form $\left[\frac{p_i^r - q_j^r}{p_i - q_j}\right]$ is nonsingular. Let L_r be an $n \times n$ Loewner matrix. Let $r \neq 1, 2, \ldots, n-1$. Using a homotopy argument one can see that all $k \times k$ sub-determinants of L_r are nonzero and have the same sign. Thus L_r is an SSR matrix. If $r = 1, 2, \ldots, n-1$, then the same argument shows that for $k \leq r$ all $k \times k$ sub-determinants of L_r are nonzero and have the same sign. In other words, L_r is an $\mathrm{SSR_r}$ matrix.

Let A be any matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ arranged so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. The Perron theorem tells us that if A is entrywise positive, then $\lambda_1 > 0$ and λ_1 is a simple eigenvalue of A. (See[13], p. 526). Applying this to successive exterior powers, we see that all eigenvalues of an SSR matrix are simple, and the r nonzero eigenvalues of an $\mathrm{SSR_r}$ matrix of rank r are simple. This proves Part (v) of Theorem 1.1.

We now turn to proving Part (iii). Using the identity

$$\frac{p_i^r - p_j^r}{p_i - p_j} = \frac{p_i^{r-1}(p_i - p_j) + p_i(p_i^{r-2} - p_j^{r-2})p_j + (p_i - p_j)p_j^{r-1}}{p_i - p_j}$$

we see that for every $r \in \mathbb{R}$,

$$L_r = D^{r-1}E + DL_{r-2}D + ED^{r-1}, \tag{17}$$

where D is the diagonal matrix in (8) and E is the $n \times n$ matrix with all its entries equal to one.

By Loewner's Theorem L_r is positive definite for $0 < r < 1$, and because of (10) it is negative definite for $-1 < r < 0$. Now suppose $1 < r < 2$. Let x be any nonzero vector in the space $\mathcal{H}_1$ defined in (6). Note that this $(n-1)$-dimensional space is the kernel of the matrix E. Using (17) we have

$$\langle x, L_r x\rangle = \langle x, D^{r-1}Ex\rangle + \langle x, DL_{r-2}Dx\rangle + \langle x, ED^{r-1}x\rangle.$$

The first and the third term on the right hand side are zero because $Ex = 0$. So,

$$\langle x, L_r x\rangle = \langle y, L_{r-2}y\rangle,$$

where $y = Dx$. The last inner product is negative because $L_{r-2} < 0$. Thus $\langle x, L_r x\rangle < 0$ for all $x \in \mathcal{H}_1$. In other words, L_r is conditionally negative definite if $1 < r < 2$. The same argument shows that L_r is conditionally positive definite if $2 < r < 3$ (because in this case L_{r-2} is positive definite). This was proved in [7] by more elaborate arguments. In particular, we have

$$\mathrm{In}\ L_r = (1, 0, n-1), \ \text{ if } \ 1 < r < 2, \tag{18}$$

and

$$\mathrm{In}\ L_r = (n-1, 0, 1), \ \text{ if } \ 2 < r < 3. \tag{19}$$

We note here that if $n = 3$, then because of Part (iv) already proved we have $\mathrm{In}\ L_r = (2, 0, 1)$ for all $r > 2$. So the theorem is completely proved for $n = 3$.

Let $n > 3$ and suppose $3 < r < 4$. Now consider the space

$$\begin{aligned} \mathcal{H}_2 &= \left\{x : \sum x_i = 0, \ \sum p_i x_i = 0\right\} \\ &= \{x : Ex = 0, \ EDx = 0\}. \end{aligned}$$

This space is of dimension $n-2$, being the orthogonal complement of the span of the vectors $e = (1, 1, \ldots 1)$ and $p = (p_1, p_2, \ldots, p_n)$. Let $x \in \mathcal{H}_2$. Again using the relation (17) we see that

$$\langle x, L_r x\rangle = \langle y, L_{r-2} y\rangle,$$

where $y = Dx$. Since $EDx = 0$, y is in $\mathcal{H}_1$, and since $1 < r-2 < 2$, we have $\langle x, L_r x\rangle < 0$. This is true for all $x \in \mathcal{H}_2$. So, by the minmax principle L_r has at least $n-2$ negative eigenvalues. The case $n = 3$ of the theorem already proved shows that L_r has a 3×3 principal submatrix with two positive eigenvalues. So, by Cauchy's interlacing principle, L_r has at least two positive eigenvalues. Thus L_r has exactly two positive and $n-2$ negative eigenvalues. In other words,

$$\text{In } L_r = (2, 0, n-2) \text{ for } 3 < r < 4. \tag{20}$$

At this stage note that the Theorem is completely proved for $n = 4$. Now let $n > 4$, and consider the case $4 < r < 5$. Arguing as before $\langle x, L_r x\rangle > 0$ for all $x \in \mathcal{H}_2$. So L_r has at least $n-2$ positive eigenvalues. It also has a 4×4 principal submatrix with two negative eigenvalues. Hence

$$\text{In } L_r = (n-2, 0, 2) \ \text{ for } \ 4 < r < 5. \tag{21}$$

The argument can be continued, introducing the space

$$\begin{aligned} \mathcal{H}_3 &= \left\{x : \sum x_i = 0, \sum p_i x_i = 0, \sum p_i^2 x_i = 0\right\} \\ &= \left\{x : Ex = 0, EDx = 0, ED^2 x = 0\right\} \end{aligned}$$

at the next stage. Using this we can prove statement (iii) for $5 < r < 6$ and $6 < r < 7$. It is clear now how to complete the proof.

All parts of Theorem 1.1 have now been established. □

We end this section with a few questions.

1. Let $f(z)$ be the complex function defined as

$$f(z) = \det \left[\frac{p_i^z - p_j^z}{p_i - p_j}\right].$$

   Our analysis has shown that f has zeros at $z = 0, \pm 1, \pm 2, \ldots, \pm n-1$; these zeros have multiplicities $n, n-1, \ldots, 1$, respectively; and these are the only real zeros of f. It might be of interest to find what other zeros f has in the complex plane.

2. When $n = 3$, calculations show that

$$\det L_3 = -(p_1 - p_2)^2 (p_1 - p_3)^2 (p_2 - p_3)^2,$$

and

$$\det L_4 = -2(p_1-p_2)^2(p_1-p_3)^2(p_2-p_3)^2 \{(p_1+p_2+p_3)(p_1p_2+p_1p_3+p_2p_3)+p_1p_2p_3\}\,.$$

It might be of interest to find formulas for the determinants of the matrices L_m for integers m.

3. Two of the authors have studied the matrix $P_r = [(p_i+p_j)^r]$ in [5]. It turns out that In $P_r =$ In L_{r+1} for all $r > 0$. Why should this be so, and are there other interesting connections between these two matrix families?

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References

[1] R. Bhatia, *Matrix Analysis,* Springer, 1997.

[2] R. Bhatia, *Positive Definite Matrices,* Princeton Univ. Press, 2007.

[3] R. Bhatia, *First and second order perturbation bounds for the operator absolute value*, Linear Algebra Appl., 208 (1994) 367-376.

[4] R. Bhatia and J. A. Holbrook, *Fréchet derivatives of the power function,* Indiana Univ. Math. J., 49(2000) 1155-1173.

[5] R. Bhatia and T. Jain, *Inertia of the matrix* $[(p_i+p_j)^r]$, to appear in J. Spectral Theory.

[6] R. Bhatia and K. R. Parthasarathy, *Positive definite functions and operator inequalities*, Bull. London Math. Soc., 32 (2000) 214-228.

[7] R. Bhatia and T. Sano, *Loewner matrices and operator convexity*, Math. Ann., 344 (2009) 703-716.

[8] R. Bhatia and K. B. Sinha, *Variation of real powers of positive operators,* Indiana Univ. Math. J., 43 (1994) 913-925.

[9] F. Hiai and H. Kosaki, *Comparison of various means for operators*, J. Funct. Anal., 163 (1999) 300-323.

[10] F. Hiai and H. Kosaki, *Means for matrices and comparison of their norms*, Indiana Univ. Math. J., 48 (1999) 899-936.

[11] F. Hiai and H. Kosaki, *Means of Hilbert Space Operators*, Springer 2003.

[12] R. A. Horn, *The Hadamard product. Matrix theory and applications*, Proc. Sympos. Appl. Math., 40 (1989) 87-169.

[13] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Second edition, Cambridge Univ. Press, 2013.

[14] H. Kosaki, *Arithmetic-geometric mean and related inequalities for operators*, J. Funct. Anal., 156 (1998) 429-451.

[15] S. Karlin, *Total Positivity,* Stanford University Press, 1968.

[16] G. Pólya and G. Szegö, *Problems and Theorems in Analysis,* Volume II, 4th ed., Springer, 1971.