On tensors of border rank l in $\mathbb{C}^{m\times n\times l}$

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Abstract

We study tensors in $\mathbb{C}^{m\times n\times l}$ whose border rank is l. We give a set-theoretic characterization of tensors in $\mathbb{C}^{3\times 3\times 4}$ and in $\mathbb{C}^{4\times 4\times 4}$ of border rank 4 at most.

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1 Introduction

Denote by $\mathbb{C}^{m\times n}, \mathrm{S}(m,\mathbb{C}), \mathbb{C}^{m\times n\times l}$ the linear spaces of $m\times n$ matrices, $m\times m$ symmetric matrices and 3-tensors $\mathcal{T}=[t_{i,j,k}]_{i=j=k=1}^{m,n,l}$ of dimension $m\times n\times l$ over the field of complex numbers $\mathbb{C}$ respectively. We identify $\mathbb{C}^{m\times n\times l}$ with $\mathbb{C}^m\otimes\mathbb{C}^n\otimes\mathbb{C}^l$. A rank one tensor is $\mathbf{u}\otimes\mathbf{v}\otimes\mathbf{w}=[u_iv_jw_k]\in\mathbb{C}^{m\times n\times l}$, where $\mathbf{u}=(u_1,\ldots,u_m)^\top\in\mathbb{C}^m, \mathbf{v}=(v_1,\ldots,v_n)^\top\in\mathbb{C}^n, \mathbf{w}=(w_1,\ldots,w_l)^\top\in\mathbb{C}^l$, and $\mathbf{u},\mathbf{v},\mathbf{w}$ are nonzero vectors. The rank of a nonzero tensor $\mathcal{T}$ is the minimal number $r:=\operatorname{rank}\mathcal{T}$, such that $\mathcal{T}=\sum_{i=1}^r\mathbf{u}_i\otimes\mathbf{v}_i\otimes\mathbf{w}_i$. The *border* rank of $\mathcal{T}\neq 0$, denoted by $\operatorname{brank}\mathcal{T}$, is a positive integer q, such that the following conditions hold. First, $\mathcal{T}$ is a limit of a sequence of tensors $\mathcal{T}_\nu\in\mathbb{C}^{m\times n\times l}, \operatorname{rank}\mathcal{T}_\nu=q, \nu\in\mathbb{N}$. Second, $\mathcal{T}$ is not a limit of any sequence of tensors, such that each tensor in the sequence has rank $q-1$ at most. For $m,n,l\ge 2$ there exists tensors with $\operatorname{rank}\mathcal{T}>\operatorname{brank}\mathcal{T}$. (This inequality does not hold for matrices, i.e. $\mathbb{C}^{m\times n}$.) The maximal border rank of $\mathcal{T}\in\mathbb{C}^{m\times n\times l}$ is called the generic rank of $\mathbb{C}^{m\times n\times l}$ and is denoted by $\operatorname{grank}(m,n,l)$. The value of $\operatorname{grank}(m,n,l)$ is known for many triples of (m,n,l). The conjectural value of $\operatorname{grank}(m,n,l)$ is given in [5]. For $r\le\operatorname{grank}(m,n,l)$ denote by $V_r(m,n,l)\subset\mathbb{C}^{m\times n\times l}$ the set of all 3-tensors of border rank r at most. It is easy to see that $V_r(m,n,l)$ is an irreducible variety in $\mathbb{C}^{m\times n\times l}$, which is a zero set of a number of homogeneous polynomials. In fact, its projectivization is the r-secant variety of $\mathbb{P}^{m-1}\times\mathbb{P}^{n-1}\times\mathbb{P}^{l-1}$.

A general problem is to characterize $V_r(m,n,l)$ in one of the following ways: set-theoretic, scheme theoretic and ideal theoretic. An elegant result of Strassen characterizes $V_4(3,3,3)$ [16]. It is a hypersurface given by a homogeneous polynomial of degree 9. This paper solves the set-theoretic aspect of the **Challenge Problem** posed by Elizabeth S. Allman in March 2007 (http://www.dms.uaf.edu/∼eallman/): Determine the ideal defining the fourth secant variety of $\mathbb{P}^3\times\mathbb{P}^3\times\mathbb{P}^3$. See [1] for more details how this particular problem related to phylogenic ideals and varieties for general Markov models. The *Salmon conjecture* [15, Conjecture 3.24] stated that $V_4(4,4,4)$ is defined by polynomials of degree 5 and 9. A first nontrivial step in characterizing $V_4(4,4,4)$ is to characterize $V_4(3,3,4)$. It is shown in [10] that $V_4(3,3,4)$ satisfies certain polynomial equations of degree 6. (See also [11, Remark 5.7] and [14].) Hence the corrected version of the Salmon conjecture states that $V_4(4,4,4)$ is defined by polynomials of degree 5, 6 and 9 [17, §2].

The first main result of this paper shows that $V_4(3,3,4)$ is cut out by a set of polynomials of degree 9 and 16. Our second main result shows that $V_4(4,4,4)$ is cut out by a set of polynomials of degree 5, 9 and 16. Most of the results in this paper are derived from results in matrix theory and relatively basic results in algebraic geometry. Whenever we could, we stated our results in a general setting.

We first explain briefly the main steps in the set-theoretic characterization of $V_4(4,4,4)$. First observe that $\mathcal{T} \in \mathbb{C}^{4\times4\times4}$ is given by any four p-slices of $\mathcal{T}$, for $p = 1,2,3$. For example, the $(i,1)$-slice of $\mathcal{T}$ is $T_{i,1} = [t_{i,j,k}]_{j,k=1}^4 \in \mathbb{C}^{4\times4}$ for $i = 1,2,3,4$. Let $\mathbf{T}_1(\mathcal{T}) = \operatorname{span}(T_{1,1},\ldots,T_{4,1})$. Assume first the generic case that $\dim \mathbf{T}_1(\mathcal{T}) = 4$ and $\operatorname{rank} \mathcal{T} = 4$. [5, Theorem 2.4] yields that $\mathbf{T}_1(\mathcal{T}) = \operatorname{span}(\mathbf{u}_1\mathbf{v}_1^\top,\ldots,\mathbf{u}_4\mathbf{v}_4^\top)$. Assume the generic case that $\mathbf{u}_1,\ldots,\mathbf{u}_4 \in \mathbb{C}^4$ and $\mathbf{v}_1,\ldots,\mathbf{v}_4 \in \mathbb{C}^4$ are linearly independent. Let $Q \in \mathbf{GL}(4,\mathbb{C})$ satisfy $Q\mathbf{u}_i = \mathbf{w}_i, \mathbf{v}_i^\top\mathbf{w}_j = \delta_{ij}, i,j = 1,\ldots,4$. Then any two matrices in $Q\mathbf{T}_1(\mathcal{T})$ commute. (This result is well known to the experts, e.g. [4] and references therein.) This result is equivalent to the statement that for any $X,Y,Z \in \mathbf{T}_1(\mathcal{T})$ the following condition holds.

$$X(\operatorname{adj} Y)Z - Z(\operatorname{adj} Y)X = 0. \tag{1.1}$$

($\operatorname{adj} Y$ is the adjoint matrix of Y.) These identities give rise to a system of homogeneous equations of degrees 5 in the entries of X,Y,Z, which always hold if $\operatorname{brank} \mathcal{T} \le 4$. Vice versa, if the above equalities hold and $\mathbf{T}_1(\mathcal{T})$ contains an invertible matrix then the results in [7] yields that $\operatorname{brank} \mathcal{T} \le 4$.

We next consider the case where $\mathbf{T}_1(\mathcal{T}), \mathbf{T}_2(\mathcal{T}), \mathbf{T}_3(\mathcal{T})$ does not contain an invertible matrix, and every three matrices in $\mathbf{T}_i(\mathcal{T})$ satisfies (1.1) for $i = 1,2,3$. In §5 we show that either $\operatorname{brank} \mathcal{T} \le 4$ or by permuting factors in $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$, if necessary, and changing bases in the first two components of $\mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4$ $\mathcal{T}$ can be viewed as a tensor $\mathbb{C}^{3\times3\times4}$. This is Corollary 5.6 of [11]. In [11] this corollary is deduced from [11, Prop. 5.4]. Unfortunately, Proposition 5.4 is wrong, see §5.

Assume that $\mathcal{T} \in V_4(3,3,4)$ and $\operatorname{rank} \mathcal{T} = 4$. Then $\mathbf{T}_3(\mathcal{T}) \subset \mathbb{C}^{3\times3}$ is spanned by $\mathbf{u}_i\mathbf{v}_i^\top$ for $i = 1,\ldots,4$. Assume the generic case where any three vectors out of $\mathbf{u}_1,\ldots,\mathbf{u}_4$ and out of $\mathbf{v}_1,\ldots,\mathbf{v}_4$ are linearly independent. Then there exists $L,R \in \mathbf{GL}(3,\mathbb{C})$ such that $L\mathbf{T}_3(\mathcal{T})$ and $\mathbf{T}_3(\mathcal{T})R$ are 4 dimensional subspaces of the subspace of 3×3 symmetric matrices $\mathrm{S}(3,\mathbb{C})$, which are spanned by 4 rank one symmetric matrices. Furthermore L,R are unique up to a multiplication by a nonzero scalar, and

$$LR^\top = R^\top L = \frac{\operatorname{tr}(LR^\top)}{3} I_3. \tag{1.2}$$

The existence of nonzero L,R such that $L\mathbf{T}_3(\mathcal{T}), \mathbf{T}_3(\mathcal{T})R \subset \mathrm{S}(3,\mathbb{C})$ is equivalent to the condition that the corresponding system of homogeneous linear equations in the entries of L and R respectively, given respectively by the coefficient matrices $C_L(\mathcal{T}), C_R(\mathcal{T}) \in \mathbb{C}^{12\times9}$, have a nontrivial solution. (Note that L and R have 9 entries.) The entries of $C_L(\mathcal{T})$ and $C_R(\mathcal{T})$ are linear combinations of the entries of $\mathcal{T}$ with coefficients $0,1,-1$. A necessary and sufficient condition for a nontrivial solution R and L is that all 9×9 minors of $C_L(\mathcal{T})$ and $C_R(\mathcal{T})$ are zero. These gives rise to a number of polynomial equations of degree 9 that the entries of $\mathcal{T} \in \mathbb{C}^{3\times3\times4}$ must satisfy. We show that the Strassen condition corresponds to some of the above polynomial equations. If the ranks of $C_L(\mathcal{T})$ and $C_R(\mathcal{T})$ is 8 then L and R are determined uniquely up to a multiplication of a nonzero scalar. The entries of L and R are polynomials of degree 8 in the entries of $C_L(\mathcal{T})$ and $C_R(\mathcal{T})$. The condition (1.2) translates to a system of polynomial equations of degree 16 in the entries of $\mathcal{T}$.

Assume first that L is invertible. Then $L\mathbf{T}_3(\mathcal{T})$ is a 4-dimensional subspace of $\mathrm{S}(3,\mathbb{C})$. A 4-dimensional generic subspace of $\mathrm{S}(3,\mathbb{C})$ spanned by 4 rank one matrices. Hence the assumption that $L \in \mathbf{GL}(3,\mathbb{C})$ yields that $\mathcal{T} \in V_4(3,3,4)$. (This case does not require (1.2).) Assume that neither L or R are invertible. Then $LR = 0$. In that case we also show that the above conditions imply that $\mathcal{T} \in V_4(3,3,4)$, by considering a few cases. (We need (1.2) to rule out certain cases.)

We survey briefly the contents of the paper. In §2 we discuss some known results which are needed in the next sections. We recall a simple known condition that rank $\mathcal{T}$ is the dimension of the minimal subspace spanned by rank one matrices that contains the subspace spanned by p-slices of $\mathcal{T}$, denoted by $\mathbf{T}_p(\mathcal{T})$, for each $p = 1, 2, 3$. Next we discuss a simple dimension condition on a generic subspace in $\mathbf{U} \subset \mathbb{C}^{m\times n}$ which implies that $\mathbf{U}$ is spanned by rank one matrices. We translate this result to the border rank of $\mathcal{T}$. We recall the Strassen characterization of $V_4(3, 3, 3)$. Next we show that for a generic tensor $\mathcal{T} \in \mathbb{C}^{m\times m\times l}$ of rank m one can change a basis in the first factor of $\mathbb{C}^m$ such that the l 3-slices of $\mathcal{T}$ are commuting matrices. These conditions give rise to the equations of type (1.1). In §3 we characterize subspaces $\mathbf{U} \subset \mathbb{C}^{m\times m}$ such that any 3 matrices satisfy the condition (1.1) and most of the matrices in $\mathbf{U}$ have rank $m - 1$. In §4 we characterize $V_4(3, 3, 4)$. §5 we discuss necessary and sufficient conditions for a $\mathcal{T} \in \mathbb{C}^{4\times4\times4}$ to have border rank 4 at most. We analyze the case where most of the matrices in $\mathbf{T}_3(\mathcal{T})$ have rank 2. In this case the condition (1.1) holds trivially for $\mathbf{T}_3(\mathcal{T})$. We show that in some cases brank $\mathcal{T} = 5$ if we do not assume the conditions (1.1) for $\mathbf{T}_1(\mathcal{T})$. This gives a counter-example to [11, Proposition 5.4], and invalidates the proof of [11, Corollary 5.6]. On the other hand we show that if (1.1) holds for $\mathbf{T}_1(\mathcal{T})$ and $\mathbf{T}_2(\mathcal{T})$ and most of the matrices in $\mathbf{T}_3(\mathcal{T})$ have rank 2, then Corollary 5.6 of [11] holds. Most of §5 is devoted to the proof of [11, Corollary 5.6]. We need to consider the case where most of the matrices in $\mathbf{T}_p(\mathcal{T})$ have rank 3 for $p = 1, 2, 3$. Our analysis depends on the results in §3.

In §6 we outline how to translate the problem of classifying tensors in $\mathbb{C}^{m\times n\times l}$ of rank l if either $2 \le l \le m, n$ or $m = n = l - 1$ and $l \ge 4$. It turns out that in the generic case this problem boils down to the condition that a corresponding l subspace denote by $\mathbf{S}(\mathbf{T}_3(\mathcal{T})) \subset \mathrm{S}(l, \mathbb{C})$ is congruent to a subspace spanned by l diagonal matrices. Note that the a simultaneous matrix diagonalization by congruence arises naturally in finding the rank decomposition of tensors [4]. We point out how some of these results can be generalized to tensors of border rank l at most.

2 Preliminary results

We first recall a basic result on the rank of 3 tensor $\mathcal{T} = [t_{i,j,k}]_{i=j=k}^{m,n,l} \in \mathbb{C}^{n\times m\times l}$ which is well known to the experts. (See for example [5, Theorem 2.4].) By a $(k, 3)$-slice, we denote the matrix $T_{k,3}(\mathcal{T}) = T_{k,3} := [t_{i,j,k}]_{i=j=1}^{m,n} \in \mathbb{C}^{m\times n}$ for $k = 1, \ldots, l$. Let $\mathbf{T}_3(\mathcal{T}) :=$ $\mathrm{span}(T_{1,3}, \ldots, T_{l,3}) \subset \mathbb{C}^{m\times n}$. We call $\mathbf{T}_3(\mathcal{T})$ the $3 - rd$ subspace induced by $\mathcal{T}$.

Theorem 2.1 *Let $\mathcal{T} \in \mathbb{C}^{m\times n\times l}$. Then* rank $\mathcal{T}$ *is the minimal dimension of a subspace $\mathbf{U} \subset \mathbb{C}^{m\times n}$ that contains $\mathbf{T}_3(\mathcal{T})$ and is spanned by rank one matrices.*

We can define similarly $(p, 1)$ and $(q, 2)$ slices of $\mathcal{T}$ and the corresponding subspaces $\mathbf{T}_1(\mathcal{T}), \mathbf{T}_2(\mathcal{T})$. Hence Theorem 2.1 can be stated for $\mathbf{T}_1(\mathcal{T})$ and $\mathbf{T}_2(\mathcal{T})$ respectively. Also note that the space $\mathbb{C}^{m\times n\times l}$ can be identified with $\mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^l$, and hence if we permute the three tensor factors $\mathbb{C}^m, \mathbb{C}^n, \mathbb{C}^l$ we obtain 6 isomorphic spaces of tensors.

Let

$$P = [p_{i'i}] \in \mathbf{GL}(m, \mathbb{C}),\ Q = [q_{j'j}] \in \mathbf{GL}(n, \mathbb{C}),\ R = [r_{k'k}] \in \mathbf{GL}(l, \mathbb{C}).$$

Then we can change the standard bases in $\mathbb{C}^m, \mathbb{C}^n, \mathbb{C}^l$ to the bases obtained from the columns of P^{-1}, Q^{-1}, R^{-1} respectively. In the new bases $\mathcal{T}$ is represented by $\mathcal{T}' = [t_{i',j',k'}] =$ $\mathcal{T}(P, Q, R)$. So

$$\mathcal{T}(P, Q, R) = P \otimes Q \otimes R(\mathcal{T}) := \mathcal{T}' = [t'_{i',j',k'}],\ t_{i',j',k'} = \sum_{i=j=k=1}^{m,n,l} p_{i'i} q_{j'j} r_{k'k} t_{i,j,k}. \quad (2.1)$$

Clearly rank $\mathcal{T}(P, Q, R) =$ rank $\mathcal{T}$. The following lemma is derived straightforward.

Lemma 2.2 *Let $\mathcal{T} \in \mathbb{C}^{m\times n\times l}, P \in \mathbf{GL}(m,\mathbb{C}), Q \in \mathbf{GL}(n,\mathbb{C}), R \in \mathbf{GL}(l,\mathbb{C})$. Let $\mathcal{T}(P,Q,R)$ be defined as in (2.1). Then*

$$\mathbf{T}_1(\mathcal{T}(P,Q,R)) = Q\mathbf{T}_1(\mathcal{T})R^\top, \quad \mathbf{T}_2(\mathcal{T}(P,Q,R)) = P\mathbf{T}_2(\mathcal{T})R^\top,$$
$$\mathbf{T}_3(\mathcal{T}(P,Q,R)) = P\mathbf{T}_3(\mathcal{T})Q^\top.$$

For a finite dimensional space $\mathbf{W}$ of dimension N denote by $\mathrm{Gr}(p,\mathbf{W})$ the Grassmannian variety of all subspaces of dimension $1 \le p \le N$. Recall that $\dim \mathrm{Gr}(p,\mathbf{W}) = p(N-p)$. Let $\Gamma(p,\mathbb{C}^{m\times n}) \subseteq \mathrm{Gr}(p,\mathbb{C}^{m\times n})$ and $\Gamma(p,\mathrm{S}(m,\mathbb{C})) \subseteq \mathrm{Gr}(p,\mathrm{S}(m,\mathbb{C}))$ be the varieties of all p-dimensional subspaces in $\mathbb{C}^{m\times n}$ and $\mathrm{S}(m,\mathbb{C})$ that can be obtained as limit of p-dimensional subspaces in $\mathbb{C}^{m\times n}$ and $\mathrm{S}(m,\mathbb{C})$ respectively, which are spanned by rank one matrices. For integers $i \le j$ denote by $[i,j]$ the set of all integers $k, i \le k \le j$. The following result is known, e.g. [3, Prop, 3.1, (iv)].

Lemma 2.3 *For $1 < m,n \in \mathbb{N}$*

$$\Gamma(p,\mathbb{C}^{m\times n}) = \mathrm{Gr}(p,\mathbb{C}^{m\times n}) \text{ for } p \in [(m-1)(n-1)+1, mn], \tag{2.2}$$
$$\Gamma(p,\mathrm{S}(m,\mathbb{C})) = \mathrm{Gr}(p,\mathrm{S}(m,\mathbb{C})) \text{ for } p \in [\binom{m}{2}+1, \binom{m+1}{2}] \tag{2.3}$$

Proof. To prove (2.2) it is enough to show the case $p = (m-1)(n-1)+1$. Clearly $\Gamma(p,\mathbb{C}^{m\times n})$ is an irreducible variety of $\mathrm{Gr}(p,\mathbb{C}^{m\times n})$. It is left to show that $\dim \Gamma(p,\mathbb{C}^{m\times n}) = \dim \mathrm{Gr}(p,\mathbb{C}^{m\times n})$. Let $\mathbb{P}V(r,m,n) \subset \mathbb{P}^{mn-1}$ be the projectivized variety of all matrices in $\mathbb{C}^{m\times n}\setminus\{0\}$ of rank r at most. It is well known that $\dim \mathbb{P}V(r,m,n) = r(m+n-r)-1$, e.g. [6]. Hence any generic projective linear subspace of dimension $(m-1)(n-1)$ in $\mathbb{P}^{mn-1}$ will intersect the Segre variety $\mathbb{P}V(1,m,n) = \mathbb{P}^{m-1}\times\mathbb{P}^{n-1}$ at a fixed number of points [6, §6]

$$\deg \mathbb{P}^{m-1}\times\mathbb{P}^{n-1} = \binom{m+n-2}{m-1}. \tag{2.4}$$

Thus, a generic projective subspace spanned by $(m-1)(n-1)+1$ points on Segre variety, i.e. $(m-1)(n-1)+1$ rank one matrices. Therefore

$$\dim \Gamma((m-1)(n-1)+1,\mathbb{C}^{m\times n}) = ((m-1)(n-1)+1)(m+n-2) =$$
$$\dim \mathrm{Gr}((m-1)(n-1)+1,\mathbb{C}^{m\times n}) \Rightarrow \Gamma(p,\mathbb{C}^{m\times n}) = \mathrm{Gr}((m-1)(n-1)+1,\mathbb{C}^{m\times n}).$$

To prove (2.3) is enough to show the case $p = \binom{m}{2}+1$. Let $\mathbb{P}\mathrm{S}(r,m)$ be the projectivized variety of all $A \in \mathrm{S}(m,\mathbb{C})\setminus\{0\}$ of rank r at most. It is well known that $\mathrm{codim}\,\mathbb{P}\mathrm{S}(r,m) = \binom{m-r+1}{2}$ in $\mathbb{P}\mathrm{S}(m,\mathbb{C})$, e.g. [6]. Hence a generic projective linear subspace of dimension $\binom{m}{2}$ in $\mathbb{P}\mathrm{S}(m,\mathbb{C})$ intersects $\mathbb{P}\mathrm{S}(1,m)$ at a fixed number of points [6]:

$$\deg \mathbb{P}\mathrm{S}(1,m) = \prod_{j=0}^{m-2} \frac{\binom{m+j}{m-1-j}}{\binom{2j+1}{j}}. \tag{2.5}$$

Similar arguments for the previous case show that $\Gamma(\binom{m}{2}+1,\mathrm{S}(m,\mathbb{C}))$ and $\mathrm{Gr}(\binom{m}{2}+1,\mathrm{S}(m,\mathbb{C}))$ have the same dimensions. Hence the two varieties are equal. $\square$

Lemma 2.4 *Let $0 \ne \mathcal{T} \in \mathbb{C}^{m\times n\times l}$. Then $r \ge \mathrm{brank}\,\mathcal{T}$, if there exists $\mathbf{U} \in \Gamma(r,\mathbb{C}^{m\times n})$ such that $\mathbf{U} \supseteq \mathbf{T}_3(\mathcal{T})$. Furthermore, $r = \mathrm{brank}\,\mathcal{T}$ if there is no $\mathbf{V} \in \Gamma(r-1,\mathbb{C}^{m\times n})$ such that $\mathbf{V} \supseteq \mathbf{T}_3(\mathcal{T})$. In particular, $\mathrm{brank}\,\mathcal{T} \ge \dim \mathbf{T}_3(\mathcal{T})$.*

Proof. Suppose that rank $\mathcal{T} = r$. Then Theorem 2.1 yields the existence $\mathbf{U} \in \Gamma(r,\mathbb{C}^{m\times n})$ such that $\mathbf{U} \supseteq \mathbf{T}_3(\mathcal{T})$. Suppose now that $\mathcal{T}$ is a limit of $\mathcal{T}'_q, q \in \mathbb{N}$ of rank $r' \le r$. It is trivial to see that $\mathcal{T}$ is a limit of $\mathcal{T}_q, q \in \mathbb{N}$ of rank r. Then $\mathbf{T}_3(\mathcal{T}_q) \subseteq U_q \in \Gamma(r,\mathbb{C}^{m\times n})$ for each $q \in \mathbb{N}$. Take a convergent subsequence $\mathbf{U}_{q_s} \to \mathbf{U} \in \Gamma(r,\mathbb{C}^{m\times n})$. So $T_{k,3}(\mathcal{T}) \in \mathbf{U}, k = 1,\ldots,l$. Hence

$\mathbf{T}_3(\mathcal{T}) \subseteq \mathbf{U}$. If brank $\mathcal{T} = r$ there is no $\mathbf{V} \in \Gamma(r-1, \mathbb{C}^{m\times n})$ such that $\mathbf{V} \supseteq \mathbf{T}_3(\mathcal{T})$. Clearly, if $\mathbf{U} \supseteq \mathbf{T}_3(\mathcal{T})$ then $\dim \mathbf{U} \geq \dim \mathbf{T}_3(\mathcal{T})$. Hence brank $\mathcal{T} \geq \dim \mathbf{T}_3(\mathcal{T})$. $\square$

We now recall some basic results for matrices we need here. Consult for example with [12, 2]. For $m \in \mathbb{N}$ let $\langle m\rangle := \{1, \ldots, m\}$. For $k \in \langle m\rangle$ denote by $2_k^{\langle m\rangle}$ the set of subsets of $\langle m\rangle$ of cardinality k. Then $\alpha \in 2_k^{\langle m\rangle}$ is viewed as $\alpha = \{\alpha_1, \ldots, \alpha_k\}$, where $1 \leq \alpha_1 < \ldots < \alpha_k \leq m$. Denote $\|\alpha\| := \sum_{j=1}^k \alpha_j, \alpha^c := \langle m\rangle\backslash\alpha$. For $A = [a_{i,j}]_{i=j=1}^{m,n} \in \mathbb{C}^{m\times n}, \alpha \in 2_k^{\langle m\rangle}, \beta \in 2_l^{\langle n\rangle}$ denote $A[\alpha, \beta] := [a_{\alpha_i,\beta_j}]_{i=j=1}^{k,l} \in \mathbb{C}^{k\times l}$. Recall that for $p \in \langle \min(m, n)\rangle$ the $p-th$ compound of A, denoted as $\mathrm{C}_p(A) \in \mathbb{C}^{\binom{m}{p}\times\binom{n}{p}}$, is a matrix whose rows and columns are indexed by $\alpha \in 2_p^{\langle m\rangle}, \beta \in 2_p^{\langle n\rangle}$ and its (α, β) entry is given by $\det A[\alpha, \beta]$. If we view A as a linear transformation $\hat{A} : \mathbb{C}^n \to \mathbb{C}^m$ given by $\mathbf{x} \mapsto A\mathbf{x}$, then $\mathrm{C}_p(A)$ represents $\bigwedge^p \hat{A} : \bigwedge^p \mathbb{C}^n \to \bigwedge^p \mathbb{C}^m$ in the corresponding bases. Clearly

$$\mathrm{C}_p(A^\top) = \mathrm{C}_p(A)^\top, \quad \mathrm{C}_p(I_m) = I_{\binom{m}{p}}.$$

(Here I_m is $m \times m$ identity matrix.) The Cauchy-Binet formula yields

$$\mathrm{C}_p(AB) = \mathrm{C}_p(A)\mathrm{C}_p(B) \text{ for any } A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{n\times l}. \tag{2.6}$$

Let $m = n$. For $p \in \langle m-1\rangle$ denote by $\mathrm{C}_{-p}(A)$ a matrix whose rows and columns are indexed by $\alpha \in 2_p^{\langle m\rangle}, \beta \in 2_p^{\langle m\rangle}$ and its (α, β) entry is given by $(-1)^{\|\alpha\|+\|\beta\|} \det A[\alpha^c, \beta^c]$. So $\mathrm{C}_{-1}(A)^\top$ is the adjoint of A, denoted as adj A. The Laplace expansion yields that

$$\mathrm{C}_p(A)\mathrm{C}_{-p}(A)^\top = \mathrm{C}_{-p}(A)^\top\mathrm{C}_p(A) = (\det A)\ I_{\binom{m}{p}}, \quad p = 1, \ldots, m-1. \tag{2.7}$$

Hence

$$\mathrm{C}_p(A^{-1}) = \frac{1}{\det A}\mathrm{C}_{-p}(A)^\top \text{ for } A \in \mathbf{GL}(m, \mathbb{C}). \tag{2.8}$$

We now recall Strassen's result [16].

Theorem 2.5 *Let $\mathcal{T} = [t_{i,j,k}]_{i=j=k}^3 \in \mathbb{C}^{3\times3\times3}$. Denote $X_k := [t_{i,j,k}]_{i=j=1}^3 \in \mathbb{C}^{3\times3}, k = 1, 2, 3$. Let $f(X_1, X_2, X_3) := \det(X_1(\text{adj } X_2)X_3 - X_3(\text{adj } X_2)X_1)$ be a polynomial of degree 12 in the entries of the matrices X_1, X_2, X_3. Then $f(X_1, X_2, X_3) = s(X_1, X_2, X_3) \det X_2$. The variety of all $\mathcal{T}$ of border rank 4 at most is a hypersurface in $\mathbb{C}^{3\times3\times3}$ of degree 9 given by equation $s(X_1, X_2, X_3) = 0$.*

The following result is straightforward.

Lemma 2.6 . *Let $A \in \mathbb{C}^{m\times n}$ and assume that* rank $A \leq k < n$. *Fix $\alpha \in 2_k^{\langle m\rangle}, \beta = \{\beta_1, \ldots, \beta_{k+1}\} \in 2_{k+1}^{\langle n\rangle}$. Let $\mathbf{x}(\alpha, \beta) = (x_1, \ldots, x_n)^\top \in \mathbb{C}^n$ be defined as follows. $x_j = 0$ if $j \notin \beta$. If $j = \beta_i$ then $x_j = (-1)^{i-1} \det A[\alpha, \beta\backslash\{\beta_i\}]$. Then $A\mathbf{x}(\alpha, \beta) = \mathbf{0}$. Furthermore, $\mathbf{x}(\alpha, \beta) \neq \mathbf{0}$ for some α, β if and only if* rank $A = k$.

The following result is known [11].

Theorem 2.7 *Let $\mathcal{T} \in \mathbb{C}^{m\times m\times l}$. Assume that* brank $\mathcal{T} \leq m$. *Then for any $A, B, C \in \mathbf{T}_3(\mathcal{T})$ the following equalities hold.*

$$\mathrm{C}_p(A)\mathrm{C}_{-p}(B)^\top\mathrm{C}_p(C) = \mathrm{C}_p(C)\mathrm{C}_{-p}(B)^\top\mathrm{C}_p(A) \text{ for } p = 1, \ldots, m-1. \tag{2.9}$$

Proof. Assume that rank $\mathcal{T} = m$ and $\mathbf{T}_3(\mathcal{T})$ contains an invertible matrix B. So $\mathbf{U} = \text{span}(\mathbf{u}_1\mathbf{v}_1^\top, \ldots, \mathbf{u}_m\mathbf{v}_m^\top) \supset \mathbf{T}_3(\mathcal{T})$. Hence $B = \sum_{i=1}^m b_i\mathbf{u}_i\mathbf{v}_i^\top$. Since rank $B = m$ we have that $b_i \neq 0, i = 1, \ldots, m$, and $\mathbf{u}_1, \ldots, \mathbf{u}_m$ and $\mathbf{v}_1, \ldots, \mathbf{v}_m$ linearly independent. There exist $P, Q \in \mathbf{GL}(m, \mathbb{C})$ such that $P\mathbf{u}_i = Q\mathbf{v}_i = \mathbf{e}_i := (\delta_{i1}, \ldots, \delta_{im})^\top, i = 1, \ldots, m$. So any matrix in $P\mathbf{U}Q^\top$ is a diagonal matrix. Hence $PAQ(PBQ)^{-1}PCQ = PCQ(PBQ)^{-1}PAQ$, i.e.

$$AB^{-1}C = CB^{-1}A, \tag{2.10}$$

for any $A, B, C \in \mathbf{U}$. Take the $p-th$ compound of the above equality, use (2.6) and (2.8) to deduce (2.9) for any $A, B, C \in \mathbf{U}$. Since almost any $B \in \mathbf{U}$ is invertible (2.9) holds for any $A, B, C \in \mathbf{U}$. In particular it holds for any $A, B, C \in \mathbf{T}_3(\mathcal{T})$.

Assume now that brank $\mathcal{T} \le m$. Then $\mathcal{T}$ is a limit of $\mathcal{T}_q, q \in \mathbb{N}$ of rank m. Furthermore, it is easy to see that we can assume that each $\mathbf{T}_3(\mathcal{T}_q)$ contains an invertible matrix. Let $\mathbf{T}_3(\mathcal{T}_q) \subset \mathbf{U}_q$, where $\mathbf{U}_q$ is a span of m matrices of rank one. Hence any three matrices $A, B, C \in \mathbf{U}_q$ satisfy (2.9). Assume that $\mathbf{U}_q, q \in \mathbb{N}$ converges to $\mathbf{U} \in \Gamma(m, \mathbb{C}^{m\times n})$. Then any 3 matrices in $\mathbf{U}$ satisfy (2.9). The proof of Lemma 2.4 yields that $\mathbf{T}_3(\mathcal{T}) \subset \mathbf{U}$. Hence any 3 matrices in $\mathbf{T}_3(\mathcal{T})$ satisfy (2.9). $\square$

The following result is well known, e.g. [5].

$$\text{brank } \mathcal{T} \le \min(n, 2m) \text{ for any } \mathcal{T} \in \mathbb{C}^{2\times m\times n} \text{ where } 2 \le m \le n. \tag{2.11}$$

3 Some subspaces of singular matrices satisfying (1.1)

For a subspace $\mathbf{U} \subset \mathbb{C}^{m\times n}$ define mrank $\mathbf{U} = \{\max \operatorname{rank} A, A \in \mathbf{U}\}$. The following theorem analyzes the condition (1.1) for a subspace $\mathbf{U} \subset \mathbb{C}^{m\times m}$ satisfying mrank $\mathbf{U} = m - 1$.

Theorem 3.1 *Let $\mathbf{U} \subset \mathbb{C}^{m\times m}$ and assume that* mrank $\mathbf{U} = m - 1$. *Then any three matrices in $\mathbf{U}$ satisfy (1.1) if and only if one of the following mutually exclusive conditions hold.*

1. *There exists a nonzero $\mathbf{u} \in \mathbb{C}^m$ such that either $\mathbf{U}\mathbf{u} = \mathbf{0}$ or $\mathbf{u}^\top \mathbf{U} = \mathbf{0}^\top$.*
2. *$m \ge 3, \dim \mathbf{U} = k + 1 \ge 2$. There exists $P, Q \in \mathbf{GL}(m, \mathbb{C})$ such that $P\mathbf{U}Q$ has a following basis $F_0, \ldots, F_k$. The last row and column of $F_0, \ldots, F_{k-1}$ is zero, i.e. $F_i = G_i \oplus 0, G_i \in \mathbb{C}^{(m-1)\times(m-1)}, i = 0, \ldots, k-1$, $G_0 = I_{m-1}$, and*

$$F_k = \begin{bmatrix} G_k & \mathbf{f} \\ \mathbf{g}^\top & 0 \end{bmatrix},\ G_k \in \mathbb{C}^{(m-1)\times(m-1)},\ \mathbf{0} \ne \mathbf{f}, \mathbf{g} \in \mathbb{C}^{m-1},\ \mathbf{g}^\top \mathbf{f} = 0. \tag{3.1}$$

Furthermore there exists two subspace $\mathbf{X}, \mathbf{Y} \subset \mathbb{C}^{(m-1)}$ with the following properties

$$\mathbf{f} \in \mathbf{X},\ \mathbf{g} \in \mathbf{Y},\ \mathbf{g}^\top \mathbf{X} = \mathbf{f}^\top \mathbf{Y} = \mathbf{0}^\top,\ G_k \mathbf{X} \subseteq \mathbf{X},\ G_k^\top \mathbf{Y} \subseteq \mathbf{Y}, \tag{3.2}$$

$$G_i \mathbf{X} = \mathbf{0},\ G_i^\top \mathbf{Y} = \mathbf{0},\ i = 1, \ldots, k-1. \tag{3.3}$$

Proof. Let $A \in \mathbf{U}, \operatorname{rank} A = m - 1$. Then adj $A = \mathbf{u}(A)\mathbf{v}(A)^\top$ for some nonzero $\mathbf{u}(A), \mathbf{v}(A) \in \mathbb{C}^m$. Since $A(\text{adj } A) = (\text{adj } A)A = 0$ we deduce that $A\mathbf{u}(A) = A^\top \mathbf{v}(A) = \mathbf{0}$. Suppose first that $\mathbf{U}\mathbf{u} = \mathbf{0}$ for some nonzero $\mathbf{u} \in \mathbb{C}^m$. So for each $A \in \mathbf{U}, \operatorname{rank} A = m - 1$ we must have that $\operatorname{span}(\mathbf{u}(A)) = \operatorname{span}(\mathbf{u})$. So we may assume that $\mathbf{u}(A) = \mathbf{u}$. Hence for any $B \in \mathbf{U}$ $B\text{adj } (A) = 0$. Since adj $(Y) = 0$ if $\operatorname{rank} Y < m - 1$, we deduce that any three matrices in $\mathbf{U}$ satisfy (1.1). Similarly, (1.1) holds if there exists nonzero $\mathbf{u}$ such that $\mathbf{u}^\top \mathbf{U} = \mathbf{0}$.

Assume now that condition 1 does not hold. Then for most of $B \in \mathbf{U}$

$$B\mathbf{u}(A) \ne \mathbf{0}, \quad B^\top \mathbf{v}(A) \ne \mathbf{0}. \tag{3.4}$$

Assume now that (1.1) holds. Let $X = B, Y = A, Z = C$, and B, C satisfy (3.4). Then

$$\operatorname{span}(B\mathbf{u}(A)) = \operatorname{span}(C\mathbf{u}(A)) = \operatorname{span}(\mathbf{x}(A)), \operatorname{span}(B^\top \mathbf{v}(A)) = \operatorname{span}(C^\top \mathbf{v}(A)) = \operatorname{span}(\mathbf{y}(A)),$$

for some nonzero $\mathbf{x}(A), \mathbf{y}(A) \in \mathbb{C}^m$. Hence, there exists two nontrivial linear functionals $\phi, \psi : \mathbf{U} \to \mathbb{C}$, depending on A, such that

$$B\mathbf{u}(A) = \phi(B)\mathbf{x}(A), \quad B^\top \mathbf{v}(A) = \psi(B)\mathbf{y}(A) \text{ for all } B \in \mathbf{U}.$$

Using (1.1) for $X = B, Y = A, Z = C$ and the above assumptions we obtain the equality $\phi(B)\psi(C) = \phi(C)\psi(B)$. Choosing B, C satisfying (3.4) we get that $\frac{\phi(B)}{\psi(B)} = \frac{\phi(C)}{\psi(C)}$. Hence $\psi(B) = a\phi(B)$ for all $B \in \mathbf{U}$, and $a \neq 0$. By replacing $\mathbf{y}(A)$ by $a\mathbf{y}(A)$ we may assume that $\psi = \phi$. Hence for each $A \in \mathbf{U}, \operatorname{rank} A = m - 1$ we have the equality

$$B\mathbf{u}(A) = \phi_A(B)\mathbf{x}(A), \; B^\top \mathbf{v}(A) = \phi_A(B)\mathbf{y}(A) \text{ for all } B \in \mathbf{U}, \tag{3.5}$$

for a corresponding nontrivial linear functional $\phi_A : \mathbf{U} \to \mathbb{C}$.

Choose $P, Q \in \mathbf{GL}(m, \mathbb{C})$ such that $F_0 = I_{m-1} \oplus 0 \in \mathbf{U}' = P\mathbf{U}Q$. Note that $\mathbf{u}(F_0) = \mathbf{v}(F_0) = \mathbf{e}_m = (\delta_{1m}, \ldots, \delta_{mm})^\top$. Then there exists a nonzero linear functional $\phi_0 : \mathbf{U}' \to \mathbb{C}$ such that $B\mathbf{e}_m = \phi_0(B)\mathbf{x}_0, B^\top \mathbf{e}_m = \phi_0(B)\mathbf{y}_0$ for some nonzero $\mathbf{x}_0, \mathbf{y}_0 \in \mathbb{C}^m$ and $B \in \mathbf{U}'$. Observe that for any $B \in \mathbb{C}^{m\times m}$ we have $\det(F_0 + tB) = tb + O(t^2)$, where b is the (m, m) entry of B. Hence $b = 0$ for $B \in \mathbf{U}'$. Since (m, m) entry of B is zero it follows that $\mathbf{x}_0, \mathbf{y}_0$ have the last coordinate zero, i.e. $\mathbf{x}_0^\top = (\mathbf{f}^\top, 0), \mathbf{y}_0^\top = (\mathbf{g}^\top, 0)$. Consider next the strict subspace of $\mathbf{U}_0 \subset \mathbf{U}'$ satisfying $\phi_0(B) = 0$. For $B \in \mathbf{U}_0$ we have that $B\mathbf{e}_m = B^\top \mathbf{e}_m = \mathbf{0}$, i.e. the last row and column matrices in $\mathbf{U}_0$ are zero. Clearly $F_0 \in \mathbf{U}_0$. Let $F_0, \ldots, F_{k-1}$ be a basis in $\mathbf{U}_0$. Let $\phi_0(F_k) = 1$. The assumption that $\phi_0(F_k) = 1$ yields that F_k is of the form given in (3.1). (We will show the condition $\mathbf{g}^\top \mathbf{f} = 0$ later.) So $F_0, \ldots, F_k$ is a basis of $\mathbf{U}'$. Let

$$F(\mathbf{z}) = F_0 + \sum_{i=1}^{k} z_i F_i, \quad G(\mathbf{z}) = I_{m-1} + \sum_{i=1}^{k} z_i G_i, \quad \mathbf{z} = (z_1, \ldots, z_k)^\top \in \mathbb{C}^k.$$

Assume that $G(\mathbf{z})$ is invertible. A straightforward calculation shows

$$\begin{aligned}&\mathbf{u}(F(\mathbf{z}))^\top = (z_k \mathbf{f}(\mathbf{z})^\top, -1), \; \mathbf{v}(F(\mathbf{z}))^\top = (\det G(\mathbf{z}))(z_k \mathbf{g}(\mathbf{z})^\top, -1),\\ &\text{where } \mathbf{f}(\mathbf{z}) = G(\mathbf{z})^{-1}\mathbf{f}, \; \mathbf{g}(\mathbf{z})^\top = \mathbf{g}^\top G(\mathbf{z})^{-1}, \text{ and } \mathbf{g}^\top G(\mathbf{z})^{-1}\mathbf{f} = 0.\end{aligned} \tag{3.6}$$

Indeed, the equalities $F(\mathbf{z})\mathbf{u}(\mathbf{z}) = F(\mathbf{z})^\top \mathbf{v}(\mathbf{z}) = \mathbf{0}$ are verified straightforward. The equality $\mathbf{g}^\top G(\mathbf{z})^{-1}\mathbf{f} = 0$ must hold if $z_k \neq 0$. The continuity argument yields this condition for $z_k = 0$ if $G(\mathbf{z})$ is invertible. Note that the condition $\mathbf{g}^\top G(\mathbf{z})^{-1}\mathbf{f} = 0$ for $\mathbf{z} = \mathbf{0}$ yields that $\mathbf{g}^\top \mathbf{f} = 0$. To see that $\operatorname{adj} F(\mathbf{z}) = \mathbf{u}(\mathbf{z})\mathbf{v}(\mathbf{z})^\top$ just observe that the (m, m) entry of $\operatorname{adj} F(\mathbf{z}) = \det G(\mathbf{z})$. Observe next that (3.5) yields

$$G(\mathbf{w})\mathbf{u}(\mathbf{z}) = \phi_{\mathbf{z}}(\mathbf{w})\mathbf{x}(\mathbf{z}), \tag{3.7}$$

for some nonzero affine functional $\phi_{\mathbf{z}}(\mathbf{w})$. ($\phi_{\mathbf{z}}(\mathbf{w})$ is affine since in the definition of $F(\mathbf{z})$ the coefficient of A_0 is 1.) If $z_k \neq 0$ we claim that we can choose

$$\mathbf{x}(\mathbf{z})^\top = (\mathbf{f}(\mathbf{z})^\top, 0) \tag{3.8}$$

Indeed, chose $\mathbf{w} = \mathbf{0}$ so $G(\mathbf{0}) = I_{m-1}$. Clearly $F(\mathbf{0})\mathbf{u}(\mathbf{z}) = (z_k \mathbf{f}(\mathbf{z})^\top, 0)^\top = z_k(\mathbf{f}(\mathbf{z})^\top, 0)^\top$. Let $\hat{\mathbf{w}} = (w_1, \ldots, w_{k-1}, 0)^\top$. Then we get the equality

$$F(\hat{\mathbf{w}})\mathbf{u}(\mathbf{z}) = \begin{bmatrix} z_k G(\hat{\mathbf{w}})\mathbf{f}(\mathbf{z}) \\ 0 \end{bmatrix} = \phi_z(\hat{\mathbf{w}}) \begin{bmatrix} \mathbf{f}(\mathbf{z}) \\ 0 \end{bmatrix}. \tag{3.9}$$

If $z_k \neq 0$ then $\mathbf{f}(\mathbf{z})$ is an eigenvector of $G(\hat{\mathbf{w}})$ for each $\hat{\mathbf{w}}$. Assume that $G(\mathbf{z})$ is invertible for any $\mathbf{z}$ satisfying satisfying $\|\mathbf{z}\|_{\max} \leq r$ for some $r > 0$. Use the continuity argument to deduce that $\mathbf{f}(\mathbf{z})$ is an eigenvector of $G(\hat{\mathbf{w}})$ for any $\mathbf{z}$ satisfying satisfying $\|\mathbf{z}\|_{\max} \leq r$. Letting $\mathbf{z} = \mathbf{0}$ we get that $\mathbf{f} = \mathbf{f}(\mathbf{0})$ is an eigenvector for each $G(\hat{\mathbf{w}})$. Hence $G_i\mathbf{f} = \lambda_i\mathbf{f}$ for

$i = 0, \ldots, k-1$, where $\lambda_0 = 1$. By replacing G_i with $G_i - \lambda_i I_{m-1}$ we may assume without loss of generality that $G_i \mathbf{f} = 0$ for $i = 1, \ldots, k-1$. Let $\|\mathbf{z}\|_{\max} \le r$. Since $f(\mathbf{z})$ is an eigenvector of G_i and $\mathbf{f}(0)$ corresponds to the zero eigenvalue of G_i it follows $G_i \mathbf{f}(\mathbf{z}) = \mathbf{0}$, were $i = 1, \ldots, k-1$. Let $\mathbf{z} = (0, \ldots, 0, z_k)^\top$ and $|z_k| < r$. So

$$\mathbf{f}(\mathbf{z}) = (I_{m-1} + z_k G_k)^{-1} \mathbf{f} = \sum_{j=0}^{\infty} (-z_k)^j G_k^j \mathbf{f}.$$

Let $\mathbf{X}, \mathbf{Y}$ be the cyclic subspaces spanned by $G_k^j \mathbf{f}, j = 0, \ldots,$ and $(G_k^\top)^j \mathbf{g}, j = 0, \ldots,$ respectively. Clearly, $G_k \mathbf{X} \subseteq \mathbf{X}, G_k^\top \mathbf{Y} \subseteq \mathbf{Y}$. The condition that $G_i \mathbf{f}(\mathbf{z}) = \mathbf{0}$ yields that $G_i \mathbf{X} = \mathbf{0}$ for $i = 1, \ldots, k-1$. So $G(\hat{\mathbf{w}})\mathbf{f}(\mathbf{z}) = \mathbf{f}(\mathbf{z})$. The condition $\mathbf{g}^\top \mathbf{f}(\mathbf{z}) = 0$ yields that $\mathbf{g}^\top \mathbf{X} = \mathbf{f}^\top \mathbf{Y} = \mathbf{0}^\top$. Observe next that (3.9) yields that $\phi_{\mathbf{z}}(\hat{\mathbf{w}}) = z_k$. In view of (3.5) it follows that

$$F(\hat{\mathbf{w}})^\top \mathbf{v}(\mathbf{z}) = z_k \mathbf{y}(\mathbf{z}), \quad \mathbf{y}(\mathbf{z})^\top = a(\mathbf{z})(\mathbf{g}(\mathbf{z})^\top, 0) \text{ for some } 0 \ne a(\mathbf{z}) \in \mathbb{C}.$$

Hence $G_i^\top \mathbf{g}(\mathbf{z}) = 0$ and $G_i^\top \mathbf{Y} = \mathbf{0}$ for $i = 1, \ldots, k-1$. This establishes the conditions *2* of the theorem.

Vice versa, suppose that the conditions *2* of the theorem hold. Let $\mathbf{U}' = P\mathbf{U}Q$. Define $F(\mathbf{z}), G(\mathbf{z}), \mathbf{u}(\mathbf{z}), \mathbf{v}(\mathbf{z}), \mathbf{f}(\mathbf{z}), \mathbf{g}(\mathbf{z})$ as above. It is enough to show the condition (3.5), where $A = F(\mathbf{z}), B = F(\mathbf{w}), C = F(\mathbf{w}'))$ and $\det G(\mathbf{z}) \ne 0$. Observe next that (3.2-3.3) yield that

$$f(\mathbf{z}) = (I_{m-1} + z_k G_k)^{-1} \mathbf{f}, \ \mathbf{g}(\mathbf{z})^\top = \mathbf{g}^\top (I_{m-1} + z_k G_k)^{-1}, \mathbf{g}^\top \mathbf{f}(\mathbf{z}) = \mathbf{f}^\top \mathbf{g}(\mathbf{z}) = 0.$$

Then

$$\begin{aligned} F(\mathbf{w})\mathbf{u}(\mathbf{z}) = ((z_k(I_{m-1} + w_k G_k)(I_{m-1} + z_k G_k)^{-1} \mathbf{f} - w_k \mathbf{f})^\top, 0)^\top = \\ (z_k - w_k)(\mathbf{f}(\mathbf{z})^\top, 0)^\top. \end{aligned}$$

Similarly,

$$\mathbf{v}(\mathbf{z})^\top F(\mathbf{w}') = (z_k - w_k') \det G(\mathbf{z}) (\mathbf{g}(\mathbf{z})^\top, 0).$$

Hence the condition (1.1) holds for $Y = F(\mathbf{z}), X = F(\mathbf{w}), Z = F(\mathbf{w}')$ when $\det G(\mathbf{z}) \ne 0$. Since most of $F(\mathbf{z}) \in \mathbf{U}'$ satisfy the condition that $\det G(\mathbf{z}) \ne 0$ we deduce that (1.1) for each $X, Y, Z \in \mathbf{U}'$. □

Theorem 3.2 *A tensor $\mathcal{T} \in \mathbb{C}^{3\times3\times3}$ has border rank 3 at most if and only if $T_p(\mathcal{T})$ satisfies the condition (1.1) for some $p \in \{1, 2, 3\}$.*

Proof. Theorem 2.7 implies that each $\mathbf{T}_p(\mathcal{T})$ satisfies the condition (1.1). It is enough to consider the case where $\mathbf{T}_3(\mathcal{T}) \ne \{0\}$ satisfies the condition (1.1). Suppose first that $\dim \mathbf{T}_p(\mathcal{T}) \le 2$ for some $p \in \{1, 2, 3\}$. Then by changing basis in the p-th component of $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ and interchanging the first and the p-th component, we can assume that $\mathcal{T}$ as $2 \times 3 \times 3$ tensor. (2.11) yields that $\operatorname{brank} \mathcal{T} \le 3$.

Assume now that $\dim \mathbf{T}_p(\mathcal{T}) = 3$ for $p = 1, 2, 3$. Suppose first that $\operatorname{mrank} \mathbf{T}_3(\mathcal{T}) = 1$. So $\mathbf{T}_3(\mathcal{T})$ has a basis consisting of rank one matrices. Theorem 2.1 implies that $\operatorname{rank} \mathcal{T} = 3$, hence $3 \ge \operatorname{brank} \mathcal{T}$.

Assume now that $\operatorname{mrank} \mathbf{T}_3(\mathcal{T}) = 3$, i.e. there exists an invertible $Y \in \mathbf{T}_3(\mathcal{T})$. By considering $P = Y^{-1}$ and changing a basis in the first factor of $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ we may that $Y = I \in \mathbf{T}_3(\mathcal{T})$. Let $\mathbf{T}_3 = \operatorname{span}(I, A_1, A_2)$. So $A_1 A_2 = A_2 A_1$. Recall that the variety of all commuting pairs $(X_1, X_2) \in (\mathbb{C}^{3\times3})^2$ is irreducible [13]. Hence a pair (A_1, A_2) is a limit of generic commuting pairs (X_1, X_2). For a generic pair, X_1 has 3 distinct eigenvalues. So X_2 is a polynomial in X_1. Thus, there exists $Q \in \mathbf{GL}(3, \mathbb{C})$ such that $Q^{-1}\operatorname{span}(X_1, X_2)Q$ is a two dimensional subspace of 3×3 diagonal matrices $\mathbf{D} \subset \mathbb{C}^{3\times3}$. Clearly $I \in \mathbf{D}$ and $\mathbf{D}$ is spanned by 3 rank one diagonal matrices. Hence $\operatorname{span}(I, X_1, X_2) \subseteq Q^{-1}\mathbf{D}Q \in \Gamma(3, \mathbb{C}^{3\times3})$. Thus $\mathbf{T}_3(\mathcal{T}) \subseteq \mathbf{U} \in \Gamma(3, \mathbb{C}^{3\times3})$, and $\operatorname{brank} \mathcal{T} \le 3$.

Assume now that mrank $\mathbf{T}_3(\mathcal{T}) = 2$. We claim that there is no nonzero $\mathbf{u} \in \mathbb{C}^3$ such that either $\mathbf{T}_3(\mathcal{T})\mathbf{u} = \mathbf{0}$ or $\mathbf{u}^\top \mathbf{T}_3(\mathcal{T}) = \mathbf{0}^\top$. Assume to the contrary that $\mathbf{u}^\top \mathbf{T}_3(\mathcal{T}) = \mathbf{0}^\top$ for some nonzero $\mathbf{u}$. By change of basis in the first component of $\mathbb{C}^3$ we may assume that $\mathbf{u} = \mathbf{e}_3$. Hence the third row of each matrix in $\mathbf{T}_3(\mathcal{T})$ is zero. Hence $T_{3,1} = 0$, i.e. $\dim \mathbf{T}_1(\mathcal{T}) \le 2$ contradicting our assumptions. Similarly there is no nonzero $\mathbf{u}$ such that $\mathbf{T}_3(\mathcal{T})\mathbf{u} = \mathbf{0}$. Hence $\mathbf{U} = \mathbf{T}_3(\mathcal{T})$ satisfies the condition of part *2* of Theorem 3.1. Since $m = 3$ it follows that the subspaces $X, Y \subset \mathbb{C}^3$ are one dimensional. By changing bases in $\mathbb{C}^2$ we can assume that $\mathbf{f} = \mathbf{e}_1, \mathbf{g} = \mathbf{e}_2$. Hence the three 3-slices of $\mathcal{T}$ are

$$F_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; F_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; F_3 = \begin{bmatrix} * & * & 1 \\ 0 & * & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Do the following elementary row and column operations on F_2 to bring it to the form $F_2' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. First subtract a mulitple of the third row from the second and first row. Then subtract a multiple of the third column from the first column. Apply the same row and column operations on F_0 and F_1 to obtains the three 3-slices F_0, F_1, F_2' of the tensor $\mathcal{T}'$. Consider the three 2-slices $\mathcal{T}'$:

$$T_{1,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; T_{2,2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; T_{3,2} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Interchange the first two columns in each of the above matrices to obtain the matrices A_1, I, A_2. Note that $A_1 A_2 = A_2 A_1 = 0$. The previous arguments show that brank $\mathcal{T} =$ brank $\mathcal{T}' \le 3$. □

We conclude this section with the following proposition.

Proposition 3.3 *Let $\mathcal{T} \in \mathbb{C}^{m \times m \times m}$. Assume that $\dim T_3(\mathcal{T}) \le m - 1$. Then any three matrices $A, B, C \in \mathbf{T}_k(\mathcal{T})$ satisfy the conditions (2.9) for $p = 1, m-1$ and $k = 1, 2$. For $m = 4$ the condition (2.9) holds also for $p = 2$ and $k = 1, 2$.*

Proof. By changing a basis in the last component of $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ we may assume that $T_{m,3} = 0$. Hence the last row of each $T_{i,1}$ and the last column of each $T_{i,2}$ is zero. Theorem 3.1 yields that any three matrices in $\mathbf{T}_1(\mathcal{T}), \mathbf{T}_2(\mathcal{T})$ satisfy the conditions (1.1). Note that for any two matrices $A, B \in \mathbf{T}_k(\mathcal{T})$ either $A(\mathrm{adj}\ B) = 0$ or $(\mathrm{adj}\ B)A = 0$. Hence for any three matrices in $\mathbf{T}_k(\mathcal{T})$ (2.9) holds for $p = m - 1$ and $k = 1, 2$.

Assume now that $m = 4$. We now show that for any three matrices in $\mathcal{T}_k(\mathcal{T})$ and $k = 1, 2$ we have the equality $\mathrm{C}_2(A)\mathrm{C}_{-2}(B)^\top \mathrm{C}_2(C) = 0$. It is enough to show this identity for $k = 1$. Since the last row of $A \in \mathbf{T}_1(\mathcal{T})$ is zero, it follows that $\mathrm{C}_2(A)$ has three zero rows labeled $(1,4), (2,4), (3,4)$. Hence the zero rows of $\mathrm{C}_{-2}(B)$ are the rows $(1,2), (1,3), (2,3)$. So $\mathrm{C}_{-2}(B)^\top$ has three zero columns $(1,2), (1,3), (2,3)$. A straightforward calculation shows that $\mathrm{C}_2(A)\mathrm{C}_{-2}(B)^\top \mathrm{C}_2(C) = 0$. Hence $\mathrm{C}_2(A)\mathrm{C}_{-2}(B)^\top \mathrm{C}_2(C) = \mathrm{C}_2(C)\mathrm{C}_{-2}(B)^\top \mathrm{C}_2(A) = 0$. □

For $m = 4$ it seems to us that the condition (1.1) always implies the conditions (2.9) for $p = 2, 3$.

4 Tensors in $\mathbb{C}^{(m-1)\times(m-1)\times m}$ of border rank m

Let $\mathcal{T} \in \mathbb{C}^{(m-1)\times(m-1)\times m}$ be of rank m. So

$$\mathcal{T} = \sum_{i=1}^{m} \mathbf{u}_i \otimes \mathbf{v}_i \otimes \mathbf{w}_i, \quad \mathbf{u}_i, \mathbf{v}_i \in \mathbb{C}^{m-1}, \mathbf{w}_i \in \mathbb{C}^m, i = 1, \dots, m.$$

We call $\mathcal{T}$ of rank m *generic* if any $m-1$ vectors out $\mathbf{u}_1,\ldots,\mathbf{u}_m$ and $\mathbf{v}_1,\ldots,\mathbf{v}_m$ are linearly independent.

Lemma 4.1 *Let $\mathcal{T}=[t_{i,j,k}]\in\mathbb{C}^{(m-1)\times(m-1)\times m}$ be a generic rank m tensor. Then there exists unique $L,R\neq 0$ (up to a nonzero scalars), such that*

$$L\mathbf{T}_3(\mathcal{T})\subset \mathrm{S}(m-1,\mathbb{C}),\ \mathbf{T}_3(\mathcal{T})R\subset \mathrm{S}(m-1,\mathbb{C}), \tag{4.1}$$

$$LR^\top=R^\top L=(\frac{1}{m-1}\mathrm{trace}\,(LR^\top))I_{m-1}. \tag{4.2}$$

Furthermore $L,R\in\mathbf{GL}(m-1,\mathbb{C})$.

Proof. Let $U,V\in\mathbf{GL}(m-1,\mathbb{C})$ such that $U\mathbf{u}_i=V\mathbf{v}_i=\mathbf{e}_i, i=1,\ldots,m-1$. Let $U\mathbf{u}_m=\mathbf{x}=(x_1,\ldots,x_{m-1})^\top, V\mathbf{v}_m=\mathbf{y}=(y_1,\ldots,y_{m-1})^\top$. The assumption that any $m-1$ vectors from $\mathbf{u}_1,\ldots,\mathbf{u}_m$ and $\mathbf{v}_1,\ldots,\mathbf{v}_m$ are linearly independent imply that all the coordinates of $\mathbf{x}$ and $\mathbf{y}$ are nonzero. Hence there exists a diagonal $D\in\mathbf{GL}(m-1,\mathbb{C})$ such that $D\mathbf{x}=D^{-1}\mathbf{y}$. So $(D\mathbf{e}_i)(D^{-1}\mathbf{e}_i)^\top, i=1,\ldots,m-1$ are $m-1$ commuting diagonal matrices. Furthermore the matrix $(D\mathbf{x})(D^{-1}\mathbf{y})^\top$ is symmetric. Hence $DU\mathbf{T}_3(\mathcal{T})V^\top D^{-1}\subset \mathrm{S}(m-1,\mathbb{C})$. Thus $L=V^{-1}D^2U$ and $R=(L^{-1})^\top$ will satisfy the conditions of the lemma. It is left to show that L and R are unique up to a multiple of a nonzero constant. For that we may assume already that $\mathbf{T}_3(\mathcal{T})$ is spanned by $\mathbf{e}_i\mathbf{e}_i^\top, i=1,\ldots,m-1$ and $\mathbf{z}\mathbf{z}^\top$ for some $\mathbf{z}$ with nonzero coordinates. The assumptions that $L\mathbf{e}_i\mathbf{e}_i^\top$ is symmetric for $i=1,\ldots,m-1$ yields that L is a diagonal matrix. The assumption that $L\mathbf{z}\mathbf{z}^\top$ is symmetric implies that $L=dI_{m-1}$. So if $L\neq 0$ then it is a nonzero multiple of I_{m-1}. Similar results hold for R. In particular, $R^\top$ is an inverse of L times a nonzero constant. $\square$

Lemma 4.2 *Let $\mathcal{T}=[t_{i,j,k}]\in\mathbb{C}^{(m-1)\times(m-1)\times m}$ be a border rank m at most. Then there exist $L,R\in\mathbb{C}^{(m-1)\times(m-1)}\backslash\{0\}$ such that* (4.1) - (4.2) *hold.*

Proof. There exist a sequence of $\mathcal{T}_k\in\mathbb{C}^{(m-1)\times(m-1)\times m}$ of rank m at most that converge to $\mathcal{T}$. By perturbing each $\mathcal{T}_k$ we can assume that each $\mathcal{T}_k$ ia generic tensor of rank m. So there exists $L_k,R_k\in\mathbf{GL}(m-1,\mathbb{C})$ satisfying (4.2). Normalize L_k,R_k to have $\mathrm{trace}\,(L_kL_k^*)=\mathrm{trace}\,(R_kR_k^*)=1$. Since the set $\{A\in\mathbb{C}^{(m-1)\times(m-1)},\mathrm{trace}\,(AA^*)=1\}$ is compact, there exists a subsequence $k_p,p\in\mathbb{N}$, such that $L_{k_p}\to L, R_{k_p}\to R$ and $\mathbf{T}_3(\mathcal{T}_{k_p})$ converges to $\mathbf{U}\in\Gamma(m,\mathbb{C}^{m\times m})$. Clearly $L\mathbf{U},\mathbf{U}R\subset \mathrm{S}(m,\mathbb{C})$, and L,R satisfy the equality in (4.1)-(4.2). As $\mathbf{U}\supseteq\mathbf{T}_3(\mathcal{T})$ we deduce the lemma. $\square$

Lemma 4.3 *Let $\mathcal{T}=[t_{i,j,k}]\in\mathbb{C}^{(m-1)\times(m-1)\times r}$, where $3\le r$. Denote by $T_k:=[t_{i,j,k}]_{i=j=1}^{m-1}$ the $(k,3)$-slice of $\mathcal{T}$ for $k=1,\ldots,r$. Then the two systems*

$$T_kR-R^\top T_k^\top=0,\ k=1,\ldots,r, R\in\mathbb{C}^{(m-1)\times(m-1)}, \tag{4.3}$$

$$LT_k-T_k^\top L^\top=0,\ k=1,\ldots,r, L\in\mathbb{C}^{(m-1)\times(m-1)} \tag{4.4}$$

have nontrivial solutions R,L if and only if the following conditions hold. Let $C_R(T_1,\ldots,T_r)$, $C_L(T_1,\ldots,T_r)\in\mathbb{C}^{r(m-1)^2\times(m-1)^2}$ be the coefficient matrices of the systems (4.3) and (4.4) in $(m-1)^2$ variables, (the entries of R and L respectively), and $r\binom{m-1}{2}$ equations. Then $\mathrm{rank}\,C_R(T_1,\ldots,T_r)<(m-1)^2$ *and* $\mathrm{rank}\,C_L(T_1,\ldots,T_r)<(m-1)^2$. *Equivalently any $(m-1)^2\times(m-1)^2$ minors of $C_R(T_1,\ldots,T_r)$ and $C_L(T_1,\ldots,T_r)$ vanishes. This assumption is equivalent to the assumption that the entries of $T_1,\ldots,T_r$ satisfy corresponding system of $2\binom{\frac{r(m-1)(m-2)}{2}}{(m-1)^2}$ homogeneous polynomial equations of degree $(m-1)^2$.*

Assume furthermore that $\mathrm{rank}\,C_R(T_1,\ldots,T_r)=\mathrm{rank}\,C_L(T_1,\ldots,T_r)=(m-1)^2-1$. *Then nonzero solutions R,L of (4.3) and (4.4) are unique up to multiples by nonzero constants. The equalities (4.2) are equivalent to $2\binom{\frac{r(m-1)(m-2)}{2}}{(m-1)^2-1}^2(m-1)^2$ homogeneous polynomial equations of degree $2((m-1)^2-1)$.*

Proof. As $X - X^\top$ is a skew symmetric matrix, the condition that $X \in \mathrm{S}(m-1,\mathbb{C})$ is equivalent to the fact that the entries of X satisfy $\binom{m-1}{2}$ linearly independent conditions. So $C_L(T_1,\ldots,T_r), C_R(T_1,\ldots,T_r) \in \mathbb{C}^{r\binom{m-1}{2}\times(m-1)^2}$. Note that any element of $C_R(T_1,\ldots,T_r)$ and $C_L(T_1,\ldots,T_r)$ is a linear function in the entries of some matrix T_k. Hence any $(m-1)^2\times(m-1)^2$ minor of $C_R(T_1,\ldots,T_l)$ and $C_L(T_1,\ldots,T_l)$ is a polynomial of degree $(m-1)^2$ in entries of $\mathcal{T}$. There are $\binom{\frac{r(m-1)(m-2)}{2}}{(m-1)^2}$ distinct minors of order $(m-1)^2$ of $C_R(T_1,\ldots,T_r)$ and $C_L(T_1,\ldots,T_r)$ respectively, which corresponds to a choice of $(m-1)^2$ rows from $r\binom{m-1}{2}$ rows. Hence the total number of polynomial conditions for the existence of nonzero solution of (4.3) and (4.4) is equivalent to the vanishing of all $2\binom{\frac{r(m-1)(m-2)}{2}}{(m-1)^2}$ minors of $C_R(T_1,\ldots,T_r)$ and $C_L(T_1,\ldots,T_r)$ of order $(m-1)^2$.

Suppose now that rank $C_R(T_1,\ldots,T_r) =$ rank $C_L(T_1,\ldots,T_r) = (m-1)^2-1$. Choose a solution for L and R as in Lemma 2.6. If either L or R are zero matrices then (4.2) holds trivially. If $R, L \neq 0$ then the conditions (4.2) are equivalent to $2(m-1)^2$ polynomial identities of degree $2((m-1)^2-1)$ in the entries of $T_1,\ldots,T_r$. The number of choices of L and R as described in Lemma 2.6 is $\binom{\frac{r(m-1)(m-2)}{2}}{(m-1)^2-1}^2$ respectively. $\square$

We now discuss in detail the cases $m=4$ and $r=3,4$. The case $r=3$ is the Strassen condition.

Theorem 4.4 *Let $\mathcal{T} = [t_{i,j,k}]_{i=j=k}^3 \in \mathbb{C}^{3\times3\times3}$. Denote by $T_1,T_2,T_3 \in \mathbb{C}^{3\times3}$ the three 3-slices of $\mathcal{T}$. Let $C_R(T_1,T_2,T_3), C_L(T_1,T_2,T_3) \in \mathbb{C}^{9\times9}$ be the matrix coefficients of the systems (4.3) and (4.4) in the 9 entries of R and L respectively. Then the border rank of $\mathcal{T}$ is 4 at most if and only if one of the following condition hold.*

1. $\det C_R(T_1,T_2,T_3) = 0$.
2. $\det C_L(T_1,T_2,T_3) = 0$.

Equivalently, for any $\mathcal{T} \in \mathbb{C}^{3\times3\times3}$

$$\det C_R(T_1,T_2,T_3) = as(T_1,T_2,T_3), \quad \det C_L(T_1,T_2,T_3) = bs(T_1,T_2,T_3) \tag{4.5}$$

for some nonzero $a,b\in\mathbb{C}$, where $s(T_1,T_2,T_3)$ is the Strassen polynomial described in Theorem 2.5.

Proof. Suppose first that $T_1,T_2,T_3 \in \mathrm{S}(3,\mathbb{C})$. Assume that T_1,T_2,T_3 are three generic matrices. Add a generic matrix $T_4 \in \mathrm{S}(3,\mathbb{C})$. The proof of Lemma 2.3 yields that $\mathrm{span}(T_1,T_2,T_3,T_4)$ is spanned by 4 rank one symmetric matrices. Theorem 2.1 yields that rank $\mathcal{T} \le 4$. Assume that $T_1(\mathrm{adj}\, T_2)T_3 \neq T_3(\mathrm{adj}\, T_2)T_1$. Theorem 2.7 implies that brank $\mathcal{T} \ge 4$. Hence rank $\mathcal{T} = 4$. Since any $T_1,T_2,T_3 \in \mathrm{S}(3,\mathbb{C})$ can be approximated by three symmetric matrices in general position we deduce that brank $\mathcal{T} \le 4$ if the three 3-slices of $\mathcal{T}$ are symmetric matrices. Thus if (4.3) has a solution $R \in \mathbf{GL}(3,C)$ then brank $\mathcal{T} \le 4$.

We now show that there exists $T_1,T_2,T_3 \in \mathbb{C}^{3\times3}$ such that (4.3) has only the trivial solution $R=0$. Let $T_1 = I$, T_2 a diagonal matrix with 3 distinct eigenvalues and $T_3 = [s_{ij}]_{i=j=1}^3$, were all $s_{ij} \neq 0$. The first condition of (4.3) yields that $R \in \mathrm{S}(3,\mathbb{C})$. The second condition of (4.3) imply that R commutes with T_2. Hence R is a diagonal matrix $\mathrm{diag}(r_1,r_2,r_3)$. The third condition of (4.3) is the condition $r_i s_{ij} = s_{ji} r_j, i,j = 1,\ldots,3$. So if $s_{12} = s_{21}, s_{13} = s_{32}$ and $s_{23} \neq s_{32}$ it follows that $r_1 = r_2 = r_3 = 0$.

On the other hand if T_3 is also a symmetric matrix with nonzero entries, then (4.3) implies that $R = rI_3$. Hence the condition $\det C_R(T_1,T_2,T_3) = 0$ yield in the generic case, i.e. $\det R \neq 0$, that brank $\mathcal{T} \le 4$. By Strassen's theorem the set of $\mathcal{T} \in \mathbb{C}^{3\times3\times3}$ of border rank 4 is a hypersurface given by the equation $s(T_1,T_2,T_3) = 0$. Hence $\det C_R(T_1,T_2,T_3) = aS(T_1,T_2,T_3)$ for some $a \neq 0$. Similar results apply to $C_L(T_1,T_2,T_3)$. $\square$

Assume $\mathcal{T} \in \mathbb{C}^{3\times 3\times 3}$ has border rank 4 at most. Then $\det C_R(T_1,T_2,T_3)=0$, and $\det C_L(T_1,T_2,T_3)=0$. Then the choice of R and L given by Lemma 2.6 is a column of adj C_R and adj C_L respectively. So the entries of R and L are homogeneous polynomials of degree 8 in the entries of $\mathcal{T}$. Assume the generic case $\det R \neq 0$. Then the arguments in the proof of Theorem 4.4 show that (4.2) hold. Note that since each entry of R and L are polynomials of degree 8 in the entries of $\mathcal{T}$. So (4.2) are 18 polynomial equations of degree 16. Since the only condition for $\mathcal{T} \in \mathbb{C}^{3\times 3\times 3}$ is the vanishing of the Strassen polynomial, we deduce that each polynomial equation of (4.2) is given by the Strassen polynomial times a homogeneous polynomial of degree 7. In conclusion, in this case, (4.2) do not give any additional restriction on $\mathcal{T}$.

We now discuss the case $m=r=4$. So $\mathcal{T}=[t_{i,j,k}] \in \mathbb{C}^{3\times 3\times 4}$. We have four 3-slices $T_k=[t_{i,j,k}]_{i=j=1}^3 \in \mathbb{C}^{3\times 3}, k=1,\ldots,4$. Let $R=[x_{ij}]_{i=j=1}^3, L=[y_{ij}]_{i=j=1}^3$ be 3×3 matrices with unknown entries. Then (4.3) and (4.4) are 12 equations homogeneous equations in 9 variables $x_{11},\ldots,x_{33}$ and $y_{11},\ldots,y_{33}$, which are given by the coefficient matrices $C_R(T_1,T_2,T_3,T_4), C_L(T_1,T_2,T_3,T_4) \in \mathbb{C}^{12\times 9}$ respectively. The condition that there exists nonzero R and L satisfying (4.3) and (4.4) respectively, are equivalent to the conditions rank $C_R(T_1,T_2,T_3,T_4) \le 8$, rank $C_L(T_1,T_2,T_3,T_4) \le 8$. So each 9×9 minor of $C_R(T_1,T_2,T_3,T_4)$, $C_L(T_1,T_2,T_3,T_4)$ is zero. The number of these conditions is $2\binom{12}{9}=440$ polynomial equations of degree 9. Fix submatrices $A,B\in\mathbb{C}^{9\times 9}$ of $C_R(T_1,T_2,T_3,T_4), C_L(T_1,T_2,T_3,T_4)$ respectively. Then each column of adj A, adj B respectively, represents a solution R,L of (4.3) and (4.4) respectively. If rank $A<8$ then $R=0$. If rank $A=8$, then one of the 9 columns of adj A is nonzero. Similar conditions hold for B. So the number of the above choices of R and L is $9\times 220=1980$ for each of them. Hence the total number of the above choices of pairs R,L is 1980^2. For each choice of R,L we assume that 18 conditions given by (4.2) hold. (To be precise, since $\operatorname{tr} LR^\top = \operatorname{tr} R^\top L$ we need at most 17 equations of (4.2).) It is not known the the author if the conditions that rank $C_R(T_1,T_2,T_3,T_4)\le 8$, rank $C_L(T_1,T_2,T_3,T_4)\le 8$ imply (4.2), as in the case of $\mathcal{T}\in\mathbb{C}^{3\times 3\times 3}$.

Theorem 4.5 *$\mathcal{T}=[t_{i,j,k}]_{i=j=k=1}^{3,3,4}\in\mathbb{C}^{3\times 3\times 4}$ has a border rank 4 at most if and only the following conditions hold.*

1. *Let $T_k := [t_{i,j,k}]_{i=j=1}^3 \in \mathbb{C}^{3\times 3}, k=1,\ldots,4$ be the four 3-slices of $\mathcal{T}$. Then the ranks of $C_L(T_1,\ldots,T_4), C_R(T_1,\ldots,T_4)$ are less than 9. (Those are $9-th$ degree equations.)*

2. *Let R,L be solutions of (4.3) and (4.4) respectively as given in Lemma 2.6, (as described above). Then (4.2) holds. (Those are $16-th$ degree equations.)*

Proof. Lemma 4.2 implies that if brank $\mathcal{T}\le 4$ then the conditions 1-2 hold. We now assume that the conditions 1-2 hold. Let $\mathbf{U} := \mathbf{T}_3(\mathcal{T})$. Suppose first that $\dim \mathbf{U} \le 3$. Pick $A_1,A_2,A_3\in\mathbf{U}$ such that $\operatorname{span}(A_1,A_2,A_3)=\mathbf{U}$. Since each A_i is a linear combination of $T_1,\ldots,T_4$, our assumption implies that there exists nonzero R such that $A_iR-R^\top A_i^\top=0$ for $i=1,2,3$. Hence $\det C_R(A_1,A_2,A_3)=0$ which is equivalent to the Strassen condition $s(A_1,A_2,A_3)=0$. Strassen's theorem implies that rank $\mathcal{T}_3\le 4$.

Assume now that $\dim\mathbf{U}=4$. Lemma 2.4 implies that brank $\mathcal{T}\ge 4$. Let $R\in\mathbb{C}^{3\times 3}\backslash\{0\}$ be a solution of (4.3) for $m=4$. If $R\in\mathbb{C}^{3\times 3}$ has rank 3 then $T_1' := T_1R,\ldots,T_4' := T_4R$ are 4 linearly independent symmetric matrices. Use Lemma 2.3 to deduce that $\mathcal{T}'\in\mathbb{C}^{3\times 3\times 4}$ has border rank 4. Similar results hold if rank $L=3$. It is left to consider the case where max(rank R, rank L) ≤ 2. We now consider a number of cases.

A: rank $C_R(T_1,\ldots,T_4)=$ rank $C_L(T_1,\ldots,T_4)=8$

I: rank $L=$ rank $R=1$. So after change of basis we can assume that $L=\mathbf{e}_3\mathbf{e}_3^\top$. Then the condition that $LT-T^\top L^\top=0$ is equivalent to $T^\top\mathbf{e}_3=t\mathbf{e}_3$ for any $T\in\mathbf{U}$. We now consider the following mutually exclusive subcases.

1: $T_i^\top\mathbf{e}_3=0$ for $i=1,\ldots,4$. Hence $\mathcal{T}$ can be viewed as a tensor in $\mathbb{C}^{2\times 3\times 4}$. (2.11) implies that brank $\mathcal{T}\le 4$.

2: $\mathbf{U}$ contains $F_4 := \mathbf{e}_3\mathbf{e}_3^\top$. So we can choose a basis F_1, F_2, F_3, F_4 such that $F_i^\top \mathbf{e}_3 = \mathbf{0}, i = 1,2,3$. Hence the tensor $\mathcal{T}' \in \mathbb{C}^{3\times 3\times 3}$, whose three 3-slices are F_1, F_3, F_3, can be viewed as a tensor in $\mathbb{C}^{2\times 3\times 3}$. (2.11) implies that brank $\mathcal{T}' \le 3$ and and the border rank of $\mathcal{T}$ is 4 at most.

3: Let T_k' be obtained from T_k by deleting the last row for $k = 1,2,3,4$. We claim that $T_1', \ldots, T_4'$ are linearly independent. Otherwise, there is a nontrivial combination $F \in \mathbb{C}^{3\times 3}$ of $T_1, \ldots, T_4$ such that the first two rows of F are zero rows. Since $T_1, \ldots, T_4$ are linearly independent $F \ne 0$. As $F^\top \mathbf{e}_3 = t\mathbf{e}_3$ it follows that $F = t\mathbf{e}_3\mathbf{e}_3^\top, t \ne 0$. This contradicts our assumption that the case *2* does not hold. We now use the assumption that $TR - R^\top T^\top = 0$ and $R = \mathbf{x}\mathbf{y}^\top, \mathbf{x} = (x_1, x_2, x_3)^\top, \mathbf{y} = (y_1, y_2, y_3)^\top$ for each $T \in \mathbf{U}$. So $T_k\mathbf{x} = s_k\mathbf{y}$ for $k = 1, \ldots, 4$. Suppose first that that all $s_k = 0$. Then we are done as in the case *1*. So we assume that $s_i \ne 0$ for some i. Since R and L have rank one, it follows that the condition (4.2) implies that $R^\top L = LR^\top = 0$. Hence $x_3 = \mathbf{x}^\top \mathbf{e}_3 = 0, y_3 = \mathbf{e}_3^\top \mathbf{y} = 0$. Let $\hat{T}_k \in \mathbb{C}^{2\times 2}$ obtained from T_k by erasing the last row and column for $k = 1, \ldots, 4$. Let $\hat{\mathbf{x}} \in \mathbb{C}^2$ be obtained from $\mathbf{x} \in \mathbb{C}^3$ by deleting the the last coordinate. Then $\hat{T}_k\hat{\mathbf{x}} = s_k\hat{\mathbf{x}}$. So by changing the coordinates in $\mathbb{C}^2$ we may assume that $\hat{\mathbf{x}} = (0,1)^\top$. Combine the above conditions with the conditions that $T_i^\top \mathbf{e}_3 = t_i\mathbf{e}_3, i = 1, \ldots, 4$ to deduce that there exists $P, Q \in \mathbf{GL}(3, \mathbb{C})$ with the following properties. Let $\tilde{T}_k = PT_kQ = [\tilde{t}_{i,j,k}]_{i=j=1}^3 \in \mathbb{C}^{3\times 3}, k = 1, \ldots, 4$. Then

$$\tilde{t}_{i,j,k} = 0 \text{ for } (i,j) = (1,2),\ (i,j) = (3,1),\ (i,j) = (3,2) \text{ and } k = 1, \ldots, 4. \tag{4.6}$$

Take a generic subspace $\mathbf{V} \subset \mathbb{C}^{3\times 3}$ of dimension 4 whose entries are zero at the places (i,j) given by (4.6). We claim that $\mathbf{V} \in \Gamma(4, \mathbb{C}^{3\times 3})$. First take a matrix $D = [d_{ij}]_{i=j=1}^3 \in \mathbf{V}$ such that $d_{ij} = 0$ for $(1,1), (2,1), (2,2)$. Generically there would one matrix, up to multiplication by a scalar, such that $d_{33} \ne 0$. D has rank one. Now consider the 3-dimensional subspace of $\mathbf{V}$ where the $(3,3)$ entry of each matrix is zero. Then $\mathbf{V}$ can be viewed as a 3-dimensional subspace in $\tilde{\mathbf{V}} \subset \mathbb{C}^{2\times 3}$. By Lemma 2.3 $\tilde{\mathbf{V}} \in \Gamma(3, \mathbb{C}^{2\times 3})$. Hence $\mathbf{V} \in \Gamma(4, \mathbb{C}^{3\times 3})$ and brank $\mathcal{T} \le 4$.

II: $\max(\operatorname{rank} L, \operatorname{rank} R) = 2$. By considering $\mathbf{U}^\top$ if necessary we may assume that rank $L = 2$. So there exist $P, Q \in \mathbf{GL}(3, \mathbb{C})$ such that $PLQ = \operatorname{diag}(1,1,0)$. Without loss of generality we may assume that $P = Q = I$. Then each LT_k is symmetric. In particular $T_k\mathbf{e}_3 = t_k\mathbf{e}_3$ and the 2×2 submatrix $[t_{i,j,k}]_{i,j=1}^2$ is symmetric. We now claim that any four dimensional subspace $\mathbf{V} \subset \mathbb{C}^{3\times 3}$, such that each $T = [t_{ij}]_{i,j=1}^3 \in \mathbf{V}$ satisfies $t_{12} = t_{21}, t_{13} = t_{23} = 0$, is in $\Gamma(4, \mathbb{C}^{3\times 3})$. As $\dim \mathbf{V} = 4$ there exists $0 \ne S = [s_{ij}]_{i=j=1}^3 \in \mathbf{V}$ such that $0 = s_{11} = s_{22} = s_{12} (= s_{21})$. Hence rank $S = 1$. For a generic $\mathbf{V}$ satisfying the above conditions $s_{33} \ne 0$. Consider now the 3 -dimensional subspace $\mathbf{W}$ of $\mathbf{V}$ with $t_{33} = 0$. Since $\mathbf{W}$ can be viewed as a 3-dimensional subspace of $C^{3\times 2}$, Lemma 2.3 yields that $W \in \Gamma(3, \mathbb{C}^{3\times 3})$. Hence $\mathbf{V} \in \Gamma(4, \mathbb{C}^{3\times 3})$ and brank $\mathcal{T} \le 4$.

B: $\min(\operatorname{rank} C_R(T_1, \ldots, T_4), \operatorname{rank} C_L(T_1, \ldots, T_4)) < 8$. By considering $\mathbf{U}^\top$ if necessary we can assume that rank $C_L(T_1, \ldots, T_4) < 8$. So there exist at least two linearly independent matrices $L_1, L_2 \in \mathbb{C}^{3\times 3}$ such that (4.4) holds. If $\max(\operatorname{rank} L_1, \operatorname{rank} L_2) = 3$ we deduce that brank $\mathcal{T} \le 4$ as in the beginning of our proof. If $\max(\operatorname{rank} L_1, \operatorname{rank} L_2) = 2$ we deduce that brank $\mathcal{T} \le 4$ as in the case A.II. So it is left to consider the case where L_1 and L_2 are rank one matrices such any their linear combination is also rank a one matrix. It is easy to show that we can choose $P, Q \in \mathbf{GL}(3, \mathbb{C})$ such that $PL_1Q = \mathbf{e}_3\mathbf{e}_3^\top$ and PL_2Q is either $\mathbf{e}_2\mathbf{e}_3^\top$ or $\mathbf{e}_3\mathbf{e}_2^\top$. So we have two cases.

I: $L_1 = \mathbf{e}_3\mathbf{e}_3^\top, L_2 = \mathbf{e}_2\mathbf{e}_3^\top$. The condition (4.4) for L_1 yields that $T_k^\top \mathbf{e}_3 = t_k\mathbf{e}_3$ for $k = 1,2,3,4,$. I.e. any $T \in \mathbf{U}$ has zero entries at the places $(3,1), (3,2)$. The condition (4.4) for L_1 yields $T_k^\top \mathbf{e}_3 = t_k'\mathbf{e}_2$. Hence $t_k = t_k' = 0$. Thus the third row of each T_k is zero. So $\mathcal{T} \in \mathbb{C}^{2\times 3\times 4}$ and (2.11) yields that brank $\mathcal{T} \le 4$.

II: $L_1 = \mathbf{e}_3\mathbf{e}_3^\top, L_2 = \mathbf{e}_3\mathbf{e}_2^\top$. The condition (4.4) for L_1 yields that any $T \in \mathbf{U}$ has zero entries at the places $(3,1), (3,2)$. The condition (4.4) for L_2 yields that $T_k^\top \mathbf{e}_2 = t_k'\mathbf{e}_3$ for $k = 1,2,3,4$. So the entries $(2,1), (2,2)$ are zero for each $T \in \mathbf{U}$. Take a nonzero $T_4' \in \mathbf{U}$

whose first row is zero. It is a rank one matrix. Then either $(3,3)$ entry or $(2,2)$ entry of T_4' is not equal to zero. Assume for simplicity of the argument that $(3,3)$ entry of T_4' is nonzero. Hence $\mathbf{U}$ contains a three dimensional subspace $\mathbf{U}'$ whose last row is zero. Since a generic 3 dimensional subspace of 2×3 matrices is spanned by rank one matrices it follows that $\mathcal{T}$ has border rank 4 at most. $\square$

Note that in the proof of Theorem 4.5 we used the condition 2, which are degree 16 polynomial equations, only in the proof of the case A.I.3. Thus one can eliminate the use of degree 16 polynomial equations, if one can show directly that a generic 4-dimensional subspace of matrices satisfying (4.3) and (4.4) for R and L of rank one, such that $RL^\top \neq 0$, is in $\Gamma(4, \mathbb{C}^{3\times 3})$. As an example, consider the case where $R = L = \mathbf{e}_3\mathbf{e}_3^\top$, which is ruled out by (4.2). Then the conditions (4.3) and (4.4) are equivalent to the assumption that $\mathbf{T}_3(\mathcal{T})$ is a four dimensional subspace of block diagonal matrices of the form

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}. \tag{4.7}$$

Hence Theorem 4.5 yields that $\mathbf{T}_3(\mathcal{T}) \not\in \Gamma(4, \mathbb{C}^{3\times 3})$. It was shown in [14] that the corresponding $\mathcal{T}$ does not satisfy the degree 6 polynomial equations found in [10].

5 Tensors in $\mathbb{C}^{4\times 4\times 4}$ of border rank 4 at most

Theorem 5.1 *$\mathcal{T} = [t_{i,j,k}]_{i=j=k=1}^4 \in \mathbb{C}^{4\times 4\times 4}$ has a border rank 4 at most if and only the following conditions hold.*

1. *Any three matrices in $\mathbf{T}_1(\mathcal{T}), \mathbf{T}_2(\mathcal{T}), \mathbf{T}_3(\mathcal{T})$ satisfy the conditions (1.1). (These are $5-th$ order degree equations on entries of X, Y, Z.)*
2. *For each $P_1, P_2, P_3 \in \mathbb{C}^{4\times 4}$ let $\mathcal{T}(P_1, P_2, P_3) = [t_{i,j,k}(P_1, P_2, P_3)]_{i=j=k=3}^4 \in \mathbb{C}^{4\times 4\times 4}$ be the tensor given by (2.1). Let $S_{i_p,p}(P_1, P_2, P_3) \in \mathbb{C}^{4\times 4}, i_p = 1, \ldots, 4$ be the four p-slices of $\mathcal{T}(P_1, P_2, P_3)$ for $p = 1, 2, 3$. (The entries of $S_{i_p,p}(P_1, P_2, P_3) \in \mathbb{C}^{4\times 4}$ are given by $t_{i_1,i_2,i_3}(P_1, P_2, P_3)$, where i_p is fixed for a given $p \in \{1, 2, 3\}$ and $i_p \in \{1, 2, 3, 4\}$.) Denote by $T_{i_p,p}(P_1, P_2, P_3) \in \mathbb{C}^{3\times 3}$ the submatrix obtained from $S_{i_p,p}(P_1, P_2, P_3)$ by deleting the last row and column, for $i_p = 1, 2, 3, 4$. Then*

$$\operatorname{rank} C_L(T_{1,p}(P_1, P_2, P_3), \ldots, T_{4,p}(P_1, P_2, P_3)) \leq 8, \tag{5.1}$$
$$\operatorname{rank} C_R(T_{1,p}(P_1, P_2, P_3), \ldots, T_{4,p}(P_1, P_2, P_3)) \leq 8, \tag{5.2}$$

   for $p = 1, 2, 3$. (Those are degree $9-th$ degree equations.) Moreover the following conditions are satisfied for each $p \in \{1, 2, 3\}$. Let $R_p(P_1, P_2, P_3), L_p(P_1, P_2, P_3)$ be solutions of (4.3) and (4.4) respectively as given in Lemma 2.6. Then (4.2) holds. (Those are degree $16-th$ degree equations.)

To prove Theorem 5.1 we need to prove Corollary 5.6 of [11].

Theorem 5.2 *Let $\mathcal{T} \in \mathbb{C}^{4\times 4\times 4}$ and assume that any three matrices X, Y, Z in $\mathbf{T}_p(\mathcal{T})$ satisfy (1.1) for $p = 1, 2, 3$. Then either* $\operatorname{brank} \mathcal{T} \leq 4$ *or or* $\dim \mathbf{T}_p(\mathcal{T}) \leq 3, \dim \mathbf{T}_q(\mathcal{T}) \leq 3$ *for two integers $1 \leq p < q \leq 3$. Equivalently, by permuting factors in $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$, if necessary, and changing bases in the first two components of $\mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4$ the tensor $\mathcal{T}$ can be viewed as a tensor $\mathbb{C}^{3\times 3\times 4}$.*

The proof of this theorem is completed by considering a number of lemmas.

Lemma 5.3 *Let $\mathcal{T} = [t_{i,j,k}] \in \mathbb{C}^{4\times 4\times 4}$ and $p \in \{1, 2, 3\}$. Assume that $\mathbf{T}_p(\mathcal{T})$ contains an invertible matrix. Then the condition (1.1) for any three matrices in $\mathbf{T}_p(\mathcal{T})$ implies that* $\operatorname{brank} \mathcal{T} \leq 4$.

Proof. It is enough to consider the case $p = 3$. Assume that $Y \in \mathbf{T}_3(\mathcal{T})$ is invertible. By considering $P = Y^{-1}$ and changing a basis in the first factor of $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ we may that $Y = I \in \mathbf{T}_3(\mathcal{T})$. Let $\mathbf{T}_3 = \mathrm{span}(I, A_1, A_2, A_3)$. So $A_iA_j = A_jA_i$ for $i, j = 1, 2, 3$. Recall that the variety of all $(X_1, X_2, X_3) \in (\mathbb{C}^{4\times 4})$ such that $X_iX_j = X_jX_i, i, j = 1, 2, 3$ is irreducible [7]. Hence a triple (A_1, A_2, A_3) is a limit of generic commuting triples (X_1, X_2, X_3). For a generic triple, X_1 has 4 distinct eigenvalues. So X_2, X_3 are polynomial in X_1. Thus, there exists $Q \in \mathbf{GL}(4, \mathbb{C})$ such that $Q^{-1}\mathrm{span}(X_1, X_2, X_3)Q$ is a three dimensional subspace of 4×4 diagonal matrices $\mathbf{D} \subset \mathbb{C}^{4\times 4}$. Clearly $I \in \mathbf{D}$ and $\mathbf{D}$ is spanned by 4 rank one diagonal matrices. Hence $\mathrm{span}(X_1, X_2, X_3, I) \subseteq Q^{-1}\mathbf{D}Q \in \Gamma(4, \mathbb{C}^{4\times 4})$. Thus $\mathbf{T}_3(\mathcal{T}) \subseteq \mathbf{U} \in \Gamma(4, \mathbb{C}^{4\times 4})$, and brank $\mathcal{T} \le 4$. $\square$

In view of Lemma 5.3 we need to show Theorem 5.2 only in the case mrank $\mathbf{T}_p(\mathcal{T}) \le 3$ for $p = 1, 2, 3$. Clearly, it is enough to assume that $\mathcal{T} \ne 0$. If mrank $\mathbf{T}_p(\mathcal{T}) = 1$ for some $p \in \{1, 2, 3\}$, then $\mathcal{T}_p(\mathcal{T})$ spanned by rank one matrices. Theorem 2.1 implies that rank $\mathcal{T} \le 4$. Thus we need to consider the case

$$2 \le \mathrm{mrank}\ \mathbf{T}_p(\mathcal{T}) \le 3 \text{ for } p = 1, 2, 3. \tag{5.3}$$

We now consider the case mrank $\mathbf{T}_3(\mathcal{T}) = 2$.

Lemma 5.4 *Let $\mathcal{T} \in \mathbb{C}^{4\times 4\times 4}$. Suppose that the (i, j) entry of each 3-slice $T_{k,3}$ is zero if* $\min(i, j) \ge 2$. *Then* mrank $\mathbf{T}_3(\mathcal{T}) \le 2$ *and any three matrices $A, B, C \in \mathbf{T}_3(\mathcal{T})$ satisfy (2.9) for* $p = 1, 2, 3$. *(In particular (1.1) holds for $\mathbf{T}_3(\mathcal{T})$.) For generic choices of the four 3-slices $T_{1,3}, \ldots, T_{4,3}$ of the above form* brank $\mathcal{T} = 5$. *Furthermore, if (1.1) holds for $\mathbf{T}_1(\mathcal{T})$ and $\mathbf{T}_2(\mathcal{T})$, then* brank $\mathcal{T} \le 4$.

Proof. Let $A = [a_{ij}]_{i=j=1}^4 \in \mathbb{C}^{4\times 4}$. Assume that $a_{ij} = 0$ if $\max(i, j) \ge 2$. So the nonzero entries of A are on the first row and column. Clearly rank $A \le 2$. Hence adj $A = 0$. This implies that any three matrices $A, B, C \in \mathbf{T}_3(\mathcal{T})$ satisfy (2.9) for $p = 1, 3$. Next observe that $\mathrm{C}_2(A)$ has zero $(2, 3), (2, 4), (3, 4)$ rows and columns. So $\mathrm{C}_{-2}(B)$ has zero $(1, 2), (1, 3), (1, 4)$ rows and columns. Hence $\mathrm{C}_2(A)\mathrm{C}_{-2}(B)^\top \mathrm{C}_2(C) = 0$, and (2.9) holds for $p = 2$.

Assume now

$$T_{i,3} = \begin{bmatrix} a_i & b_i & c_i & d_i \\ e_i & 0 & 0 & 0 \\ f_i & 0 & 0 & 0 \\ g_i & 0 & 0 & 0 \end{bmatrix}, \quad i = 1, 2, 3, 4.$$

Consider now $T_{1,1}, \ldots, T_{4,1}$. Note that $T_{1,1}$ is a full matrix, while $T_{i,1}$ has a full first column, while the other 3 columns are equal to zero. Suppose first that $\det T_{i,1} \ne 0$. (This is true of $T_{1,3}, \ldots, T_{4,3}$ are generic.) Consider the $A_i = T_{1,1}^{-1}T_{i,1}$ for $i = 1, 2, 3, 4$. So $A_1 = I_4$ and A_i has the first nonzero column $\mathbf{a}_i$, i.e. $A_i = \mathbf{a}_i\mathbf{e}_1^\top$ for $i = 2, 3, 4$. The commutation condition (1.1) with $X = A_i, Y = I_4, Z = A_j$ for $2 \le i < j \le 4$ is equivalent to $(\mathbf{a}_j\mathbf{e}_1^\top)\mathbf{a}_i\mathbf{e}_1^\top = (\mathbf{a}_i\mathbf{e}_1^\top)\mathbf{a}_j\mathbf{e}_1^\top$. Assuming that $(\mathbf{e}_1^\top\mathbf{a}_j)(\mathbf{e}_1^\top\mathbf{a}_i) \ne 0$ we deduce that the commutation condition holds of and only if $\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ are colinear. The assumption that $T_{1,3}, \ldots, T_{4,3}$ are generic matrices with nonzero entries in the first row and column yield that $\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ are 3 generic vectors. Hence $\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ are not colinear, and the commutation condition (1.1) does not hold for $\mathbf{T}_1(\mathcal{T})$. Therefore brank $\mathcal{T} \ge 5$.

To show that brank $\mathcal{T} = 5$ we add to the space spanned by $I_4, \mathbf{a}_1\mathbf{e}_1^\top, \mathbf{a}_2\mathbf{e}_1^\top, \mathbf{a}_3\mathbf{e}_1^\top$ the rank one matrix $\mathbf{e}_1\mathbf{e}_1^\top$. Let $\mathbf{a}_i' = \mathbf{a}_i - (\mathbf{e}_1^\top\mathbf{a}_i)\mathbf{e}_1, i = 1, 2, 3$. Then the three matrices $A_i' = \mathbf{a}_i'\mathbf{e}_1^\top = A_i - (\mathbf{e}_1^\top\mathbf{a}_i)\mathbf{e}_1\mathbf{e}_1^\top, i = 1, 2, 3$ commute. Let I_4, A_1', A_2', A_3' be the four 3-slices of $\mathcal{T}' \in \mathbb{C}^{4\times 4\times 4}$. The proof of Lemma 5.3 yields that brank $\mathcal{T}' \le 4$. Hence brank $\mathcal{T} \le 5$ and we conclude that brank $\mathcal{T} = 5$.

We now consider non-generic $\mathcal{T} \in \mathbb{C}^{4\times 4\times 4}$ satisfying the conditions of our lemma. Suppose first that $\dim \mathbf{T}_3(\mathcal{T}) \le 2$. By changing a basis in the last component of $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ we may assume that that $T_{3,3} = T_{4,3} = 0$. Then $\mathcal{T}$ can be viewed as a tensor in $\mathbb{C}^{4\times 4\times 2}$. (2.11) yields that brank $\mathcal{T} \le 4$. Assume that $\dim \mathbf{T}_3(\mathcal{T}) = 3$. By changing a basis in the last component of $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ we may assume that $T_{4,3} = 0$. Assume now that the entries on the first row and column of $T_{1,3}, T_{2,3}, T_{3,3}$ are in general position. Then there exists $P = [1] \oplus P_1, Q = [1] \oplus Q_1$, where $P_1, Q_1 \in \mathbf{GL}(3, \mathbb{C})$ we may assume that $T'_{i,3} = PT_{i,3}Q = x_i \mathbf{e}_1 \mathbf{e}_1^\top + \mathbf{e}_1 \mathbf{e}_{i+1}^\top + \mathbf{e}_{i+1} \mathbf{e}_1^\top$ for $i = 1, 2, 3$. Let $T'_{4,3} = \mathbf{e}_1 \mathbf{e}_1^\top$, and denote by $\mathcal{T}' \in \mathbb{C}^{4\times4\times4}$ the tensor whose four 3-slices are $T'_{i,3}, i = 1, 2, 3, 4$. We claim that $\operatorname{brank} \mathcal{T}' \le 4$. Consider the following basis in $\mathbf{T}_3(\mathcal{T}')$: $\mathbf{e}_1\mathbf{e}_1^\top$ and $\mathbf{e}_1\mathbf{e}_i^\top + \mathbf{e}_i\mathbf{e}_1^\top$ for $i = 2, 3, 4$. The above arguments show that $\mathbf{T}_1(\mathcal{T}')$ is a 4 dimensional subspace of commuting matrices, which contain I_4. Hence by Lemma 5.3 $\operatorname{brank} \mathcal{T}' \le 4$. Hence $\operatorname{brank} \mathcal{T} \le 4$. Since any three matrices $T_{1,3}, T_{2,3}, T_{3,3}$ with zero entries in the position (i, j) for $\min(i, j) \ge 2$ can be approximated by generic matrices of this kind, we deduce that $\operatorname{brank} \mathcal{T} \le 4$.

We now assume that $\dim \mathbf{T}_3(\mathcal{T}) = 4$. Assume now that $\mathbf{T}_1(\mathcal{T})$ and $\mathbf{T}_2(\mathcal{T})$ satisfy the condition (1.1). If either $\mathbf{T}_1(\mathcal{T})$ or $\mathbf{T}_2(\mathcal{T})$ contain an invertible matrix then by Lemma 5.3 $\operatorname{brank} \mathcal{T} \le 4$. So assume that $\operatorname{mrank} \mathbf{T}_1(\mathcal{T}), \operatorname{mrank} \mathbf{T}_2(\mathcal{T}) \le 3$. Hence, the four first rows and columns of $T_{1,3}, \ldots, T_{4,1}$ are linearly dependent. By changing a basis in the last component of $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ we may assume that the first row of $T_{4,3}$ is zero. As $\dim \mathbf{T}_3(\mathcal{T}) = 4$ it follows that that $T_{4,3}$ has zero first row and nonzero first column. Apply elementary row operations to the last three rows of $T_{4,1}$ to assume that $T_{4,1} = \mathbf{e}_4\mathbf{e}_1^\top$. Apply the same elementary row operations to $T_{1,3}, T_{2,3}, T_{3,3}$. Hence, we can assume without loss of generality that $T_{4,1} = \mathbf{e}_4\mathbf{e}_1^\top$. By considering $T_{i,3} - t_i T_{4,3}$ for $i = 1, 2, 3$ we may assume the $(4, 1)$ entry of $T_{i,3}$ is zero for $i = 1, 2, 3$. Consider again the column space spanned by the first three columns of $T_{1,3}, T_{2,3}, T_{3,3}$. It must be two dimensional, otherwise the column space $\mathcal{T}$ is four dimensional and $\mathbf{T}_2(\mathcal{T})$ will contain an invertible matrix. So by changing a basis in $\operatorname{span}(T_{1,3}, T_{2,3}, T_{3,3})$ we can assume that the first column of $T_{3,3}$ is zero. As $\dim \mathbf{T}_3(\mathcal{T}) = 4$ the first row of $T_{3,3}$ is nonzero. Apply elementary column operations to the last three columns of $T_{3,3}$ we may assume that $T_{3,3} = \mathbf{e}_1\mathbf{e}_4^\top$. Apply the same elementary column operations to $T_{1,3}, T_{2,3}, T_{4,3}$. We still have that $T_{4,3} = \mathbf{e}_4\mathbf{e}_1^\top$. Apply the above arguments to deduce that without loss of generality we may assume that the last row and column of $T_{1,3}, T_{2,3}$ are zero. Consider first $T_{i,2}, i = 1, 2, 3, 4$. Observe that

$$T_{1,2} = \begin{bmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_{2,2} = \begin{bmatrix} * & * & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$T_{3,2} = \begin{bmatrix} * & * & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, T_{4,2} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The assumption that $T_{1,2} + T_{4,2}$ is singular yields that the second and the third row of $T_{1,2}$ are linearly dependent. Do elementary row operations on the second and the third row of $T_{1,2}$ to obtain that a zero third row. Do the same elementary row operations on $T_{i,2}, i = 2, 3, 4$ to deduce that we may assume that each $T_{i,2}$ has zero third row. Translating back to $\mathbf{T}_1(\mathcal{T})$ we deduce that we may assume in addition to all our above assumptions on $T_{1,3}, \ldots, T_{4,3}$ the third row of $T_{1,3}, T_{2,3}$ are zero. So all matrices in $\mathbf{T}_3(\mathcal{T})$ have zero third row. Consider now $T_{i,1}$ for $i = 1, 2, 3, 4$. Observe that $T_{3,1} = 0$. Apply the same arguments as for $T_{i,2}$ for $i = 1, 2, 3, 4$ to deduce that we can assume that the third column of $T_{1,1}$ is zero. This implies that in this case we can assume that after suitable change of basis in the first two components of $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$, in addition to the assumption that all (i, j) entries of matrices in $\mathbf{T}_3(\mathcal{T})$ are zero if $\min(i, j) \ge 3$, the third row and column of each matrix in $\mathbf{T}_3(\mathcal{T})$ is zero.

It is left to show that $\operatorname{brank} \mathcal{T} \le 4$. Observe that the last two rows and columns of $T_{1,3}, T_{2,3}$ are zero. Let $\mathcal{T}' \in \mathbb{C}^{4\times4\times2}$ be the tensor whose two 3-slices are $T_{1,3}, T_{2,3}$. So $\mathcal{T}'$ can be viewed as a tensor in $\mathbb{C}^{2\times2\times2}$. (2.11) yields that $\operatorname{brank} \mathcal{T}' \le 2$. As $T_{3,4} = \mathbf{e}_1\mathbf{e}_4^\top, T_{4,4} = \mathbf{e}_4\mathbf{e}_1^\top$ we deduce that $\operatorname{brank} \mathcal{T} \le 4$. $\square$

We remark that Lemma 5.4 refutes Proposition 5.4 of [11], which claims that for any tensor $\mathcal{T} \in \mathbb{C}^{4\times4\times4}$ for which $\mathbf{T}_3(\mathcal{T})$ satisfies (1.1) either brank $\mathcal{T} \le 4$ or there exists nonzero $\mathbf{u} \in \mathbb{C}^4$ such that either $T_3(\mathcal{T})\mathbf{u} = \mathbf{0}$ or $\mathbf{u}^\top \mathbf{T}_3(\mathcal{T}) = \mathbf{0}^\top$. Indeed, if we assume as in the first part of Lemma 5.4 that $\mathcal{T} = [t_{i,j,k}] \in \mathbb{C}^{4\times4\times4}$ is a generic tensor such that $t_{i,j,k} = 0$ if $\min(i,j) \ge 2$, then there is no nonzero $\mathbf{u} \in \mathbb{C}^4$ such that either $T_3(\mathcal{T})\mathbf{u} = \mathbf{0}$ or $\mathbf{u}^\top \mathbf{T}_3(\mathcal{T}) = \mathbf{0}^\top$, and by this lemma brank $\mathcal{T} = 5$.

Lemma 5.5 *Let $\mathcal{T}' \in \mathbb{C}^{4\times4\times4}$ and assume that* mrank $\mathbf{T}_3(\mathcal{T}') = 2$. *Then either* brank $\mathcal{T}' \le 4$ *or it is possible to change bases in the first two components of $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ to obtain $\mathcal{T} \in \mathbb{C}^{4\times4\times4}$, which satisfies one the following two conditions. Either $\mathcal{T} \in \mathbb{C}^{4\times4\times4}$ satisfies the conditions of Lemma 5.4, or the last row and column of each matrix in $\mathbf{T}_3(\mathcal{T})$ is zero. (In the last case, in addition every submatrix of $A \in \mathbf{T}_3(\mathcal{T})$ based on the first three rows and columns is singular.)*

Proof. By changing a basis in the first and second component of $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ we can assume that $T'_{1,3} = I_2 \oplus 0$. Since mrank $\mathbf{T}_3(\mathcal{T}') = 2$ it follows that the four entries $(3,3), (3,4), (4,3), (4,4)$ of each matrix in $\mathbf{T}_3(\mathcal{T}')$ are zero. So any matrix in $A \in \mathbf{T}_3(\mathcal{T}')$ has the block form $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix}$, where each $A_{ij} \in \mathbb{C}^{2\times2}$. Consider $A + tT'_{1,3}$. Assume that $\det(A_{11} + tI_2) \ne 0$. Then rank $(A + tT'_{1,3}) = 2$ if and only if $A_{21}(A_{11} + tI_2)^{-1}A_{12} = 0$. Assume that rank $A_{12} = 2$ for some $A \in \mathbf{T}_3(\mathcal{T}')$. Then for most of $A \in \mathbf{T}_3(\mathcal{T}')$ rank $A_{12} = 2$. Hence for most A's $A_{12} = 0$. So the last two rows of each $A \in \mathbf{T}_3(\mathcal{T}')$ are zero. Hence $\mathcal{T}'$ can be viewed as a tensor in $\mathbb{C}^{2\times4\times4}$ and brank $\mathcal{T}' \le 4$. Similarly if for rank $A_{21} = 2$ for some $A \in \mathbf{T}_3(\mathcal{T}')$ we deduce that brank $\mathcal{T}' \le 4$. It is left to assume that rank A_{12}, rank $A_{21} \le 1$ for each $A \in \mathbf{T}_3(\mathcal{T}')$, and rank $A_{12} =$ rank $A_{21} = 1$ for some matrix A. It is easy to see that A_{12} is either $\mathbf{x}(A_{12})\mathbf{u}^\top$ or $\mathbf{u}\mathbf{x}(A_{12})^\top$, and A_{21} is either $\mathbf{y}(A_{21})\mathbf{v}^\top$ or $\mathbf{v}\mathbf{y}(A_{21})^\top$. Here $\mathbf{u}, \mathbf{v} \in \mathbb{C}^2 \setminus \{\mathbf{0}\}$ and $\mathbf{x}(A_{12}), \mathbf{y}(A_{21}) \in \mathbb{C}^2$ are not identically zero vectors which depend linearly on the entries of A_{12} and A_{21} respectively. By using elementary row or column operations on the first two rows and columns, and then on the last two rows and columns we may assume that $\mathbf{u} = \mathbf{v} = \mathbf{e}_1$. We now obtain the four 3-slices of $\mathcal{T}$. Note that $T_{1,3} = B \oplus 0$, where $B \in \mathbb{C}^{2\times2}$ is invertible.

Suppose first that $A_{12} = \mathbf{e}_1\mathbf{x}(A_{12})^\top, A_{21} = \mathbf{y}(A_{21})\mathbf{e}_1^\top$. Since $\mathbf{x}(A_{12})$ and $\mathbf{y}(A_{21})$ are not zero identically, and rank $A \le 2$ we deduce that the $(2,2)$ entry of each A is zero. (For example if $(3,1)$ and $(1,3)$ entries of A are nonzero then consider the minor with the first three rows and columns.) Hence $\mathbf{T}_3(\mathcal{T})$ is of the form given by Lemma 5.4.

Suppose next that $A_{12} = \mathbf{x}(A_{12})\mathbf{e}_1^\top, A_{21} = \mathbf{e}_1\mathbf{y}(A_{21})^\top$. Then the last row and column of each matrix in $\mathbf{T}_3(\mathcal{T})$ is zero. Moreover, each 3×3 submatrix of $\mathbf{T}_3(\mathcal{T})$ is singular.

The next case is $A_{12} = \mathbf{x}(A_{12})\mathbf{e}_1^\top, A_{21} = \mathbf{y}(A_{21})\mathbf{e}_1^\top$. Thus the last column of each $A \in \mathbf{T}_3(\mathcal{T})$ is zero. Since $\mathbf{x}(A_{21})$ is not identically zero, and mrank $\mathbf{T}_2(\mathcal{T}) = 2$ we deduce the minor based on rows $1,2$ and columns $2,3$ must be identically zero. So we have two possibilities. First possibility: by elementary row operations on the first two rows of matrices on $\mathbf{T}_3(\mathcal{T})$ we can bring $\mathbf{T}_3(\mathcal{T})$ to matrices of the form given by Lemma 5.4 with an addition condition, that the last column of all these matrices is zero. Second possibility: by elementary column operations we can achieve that also the third column of all matrices in $\mathbf{T}_3(\mathcal{T})$ are zero. So $\mathcal{T}$ can be viewed as a tensor in $\mathbb{C}^{4\times2\times4}$. Hence brank $\mathcal{T} =$ brank $\mathcal{T}' \le 4$. Similar results hold for the last case $A_{12} = \mathbf{e}_1\mathbf{x}(A_{12})^\top, A_{21} = \mathbf{e}_1\mathbf{y}(A_{21})^\top$. $\square$

We state the precise version of Theorem 3.1 for $m = 4$.

Lemma 5.6 *Let $\mathbf{U} \subset \mathbb{C}^{4\times4}$ and assume that* mrank $\mathbf{U} = 3$. *Then any three matrices in* $\mathbf{U}$ *satisfy (1.1) if and only if one of the following mutually exclusive conditions hold.*

1. *There exists a nonzero $\mathbf{u} \in \mathbb{C}^4$ such that either $\mathbf{U}\mathbf{u} = \mathbf{0}$ or $\mathbf{u}^\top\mathbf{U} = \mathbf{0}^\top$.*

2. $\dim \mathbf{U} = k+1 \ge 2$. *There exists $P, Q \in \mathbf{GL}(4,\mathbb{C})$ such that $P\mathbf{U}Q$ has a following basis $F_0, \ldots, F_k$. The last row and column of $F_0, \ldots, F_{k-1}$ is zero, i.e. $F_i = G_i \oplus 0, G_i \in$*

$\mathbb{C}^{3\times 3}, i = 0,\dots,k-1,$ $G_0 = I_3,$ *and*

$$F_k = \begin{bmatrix} G_k & \mathbf{e}_1 \\ \mathbf{e}_2^\top & 0 \end{bmatrix}, \; G_k \in \mathbb{C}^{3\times 3}. \tag{5.4}$$

Furthermore $G_k, G_1, \dots, G_{k-1}$ *have one the following possible three forms.*

(a)

$$G_k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; G_i = \begin{bmatrix} 0 & a_i & b_i \\ 0 & 0 & 0 \\ 0 & c_i & d_i \end{bmatrix}, i = 1,\dots,k-1. \tag{5.5}$$

(b)

$$G_k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ h & 0 & 0 \end{bmatrix}, \; G_i = \begin{bmatrix} 0 & a_i & 0 \\ 0 & 0 & 0 \\ 0 & c_i & 0 \end{bmatrix}, i = 1,\dots,k-1. \tag{5.6}$$

(c)

$$G_k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & f \\ 0 & 0 & 0 \end{bmatrix}, \; G_i = \begin{bmatrix} 0 & a_i & b_i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, i = 1,\dots,k-1. \tag{5.7}$$

Proof. By changing a basis in $\mathbb{C}^3$ we can assume that $\mathbf{f}, \mathbf{g} \in \mathbb{C}^3$ appearing in Theorem 3.1 for $m = 4$ are of the form $\mathbf{f} = \mathbf{e}_1, \mathbf{g} = \mathbf{e}_2$. Next we observe that G_k can be always assumed to be of the form

$$G_k = \begin{bmatrix} 0 & 0 & 0 \\ g & 0 & f \\ h & 0 & 0 \end{bmatrix}. \tag{5.8}$$

Indeed, first replace F_k by $F_k' = F_k - tF_0$ such that the $(3,3)$ entry of F_k is zero. Next, use the following elementary row and column operations without changing the form of $F_0, \dots, F_{k-1}$. Subtract a multiple of a row four from row i for $i = 1,2,3$. Similarly, subtract a multiple of a column four from column i for $i = 1,2,3$. The exact forms of $G_k, G_1, \dots, G_{k-1}$ are obtained by choosing subspaces X, Y appearing in Theorem 3.1 to be of the following forms: $X = \mathrm{span}(\mathbf{e}_1), Y = \mathrm{span}(\mathbf{e}_2)$; $X = \mathrm{span}(\mathbf{e}_1, \mathbf{e}_3), Y = \mathrm{span}(\mathbf{e}_2)$; $X = \mathrm{span}(\mathbf{e}_1), Y = \mathrm{span}(\mathbf{e}_2, \mathbf{e}_3)$.

Consider now the choice $X = \mathrm{span}(\mathbf{e}_1, \mathbf{e}_3), Y = \mathrm{span}(\mathbf{e}_2, \mathbf{e}_3)$. Then G_i is of the form given by (5.5) where $b_i = c_i = d_i = 0$ for $i = 1,\dots,k-1$. Furthermore $G_k = 0$. This can be considered as a special case of 2a. $\square$

Proof of Theorem 5.2. In view of Theorem 2.1 and Lemmas 5.3, 5.5, 5.4 we need only to consider the case mrank $\mathbf{T}_p(\mathcal{T}) = 3$ for $p = 1,2,3$. Assume first that $\dim \mathbf{T}_p(\mathcal{T}) \le 2$ for some $p \in \{1,2,3\}$. Then by changing a basis in the p-th factor of $C^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ and permuting the factors, we obtain that $\mathcal{T}$ can be viewed as a tensor in $\mathbb{C}^{2\times 4\times 4}$. (2.11) yields that brank $\mathcal{T} \le 4$. Hence we assume that $\dim \mathbf{T}_p(\mathcal{T}) \ge 3$ for $p = 1,2,3$. Assume next that $\dim \mathbf{T}_p(\mathcal{T}) = \dim \mathbf{T}_q(\mathcal{T}) = 3$ for some $1 \le p < q \le 3$. Then Theorem 5.2 holds.

Assume next that $\dim \mathbf{T}_p(\mathcal{T}) = 3$ for some $p \in \{1,2,3\}$ and $\dim \mathbf{T}_q(\mathcal{T}) = 4$ for $q \in \{1,2,3\} \setminus \{p\}$. By permuting the factors of $C^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ we may assume that $\dim \mathbf{T}_1(\mathcal{T}) = \dim \mathbf{T}_2(\mathcal{T}) = 4, \dim \mathbf{T}_3(\mathcal{T}) = 3$. Observe first that there is no nonzero $\mathbf{u} \in \mathbb{C}^4$ such that either $\mathbf{T}_3(\mathcal{T})\mathbf{u} = \mathbf{0}$ or $\mathbf{u}^\top \mathbf{T}_3(\mathcal{T}) = \mathbf{0}^\top$. Indeed, assume first that $\mathbf{T}_3(\mathcal{T})\mathbf{u} = \mathbf{0}$ for some nonzero $\mathbf{u}$. By change of coordinates in the second component of $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ we can assume that $\mathbf{u} = \mathbf{e}_4$. So the fourth column of each matrix in $\mathcal{T}_3(\mathcal{T})$ is zero. Hence $T_{4,2} = 0$ which contradicts that $\dim \mathbf{T}_2(\mathcal{T}) = 4$. Similar arguments apply if $\mathbf{u}^\top \mathbf{T}_3(\mathcal{T}) = \mathbf{0}^\top$.

We now apply part *2* of Lemma 5.6. Here $k = 2$. Assume first that F_1, F_2 have the form given in *2.a*. Consider the matrix $G_1 = \begin{bmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & c & d \end{bmatrix}$. Assume the generic case $d \ne 0$ and

$ad-bc\neq 0$. Then the three eigenvalues of G_1 are $0,0,d$. The Jordan canonical form of G_1 is $J=\begin{bmatrix} 0 & e & 0\\ 0 & 0 & 0\\ 0 & 0 & d\end{bmatrix}$, where $e\neq 0$. Furthermore, $PG_1P^{-1}=J$ where and P,P^{-1} have the form

$$P=\begin{bmatrix} 1 & \alpha & \beta\\ 0 & 1 & 0\\ 0 & \gamma & 1\end{bmatrix},\quad P^{-1}=\begin{bmatrix} 1 & -\alpha+\beta\gamma & -\beta\\ 0 & 1 & 0\\ 0 & -\gamma & 1\end{bmatrix}. \tag{5.9}$$

(Note that $\mathbf{e}_1,\mathbf{e}_2^\top$ are right and left eigenvectors of G_1,J corresponding to 0 eigenvalue.) Let $Q=P\oplus[1]\in\mathbf{GL}(4,C)$. Then $QF_0Q^{-1}=F_0, QF_1Q^{-1}=J\oplus[0], QF_2D^{-1}=F_2$. Equivalently, we may assume that G_1 is equal to J. Let $F_3=\mathbf{e}_3\mathbf{e}_3^\top\in\mathbb{C}^{4\times 4}$ and consider the tensor $\mathcal{T}'\in\mathbb{C}^{4\times 4\times 4}$ whose four 3-slices are

$$T'_{1,3}=F_0-\mathbf{e}_3\mathbf{e}_3^\top,\ T'_{2,3}=\frac{1}{e}(F_1-d\mathbf{e}_3\mathbf{e}_3^\top),\ T'_{3,3}=F_2,\ T_{4,3}=\mathbf{e}_3\mathbf{e}_3^\top.$$

Let $\mathcal{T}''\in\mathbb{C}^{4\times 4\times 3}$ is obtained from $\mathcal{T}'$ by deleting the last 3-slice of $\mathcal{T}'$. We claim that brank $\mathcal{T}''\leq 3$. Observe that the three 3-slices of $\mathcal{T}''$ have zero third row and column. So we can view $\mathcal{T}''$ as $\mathcal{S}\in\mathbb{C}^{3\times 3\times 3}$ whose three slices are given as in the last part of the proof of Theorem 3.2. Hence brank $\mathcal{T}''\leq 3$ and brank $\mathcal{T}\leq 4$.

Assume now that F_1,F_2 have the form given in *2.c*. So

$$G_1=\begin{bmatrix} 0 & a & b\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, G_2=\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & f\\ 0 & 0 & 0\end{bmatrix}.$$

Assume the generic case that $b,f\neq 0$. Consider $F_i'=(P\oplus[1])F_i(P\oplus[1])^{-1}$ for $i=0,1,2$. Assume that P is of the form (5.9), where $\alpha=\beta=0,\gamma=\frac{a}{b}$. Then

$$F_0'=F_0, F_1'=G_1'\oplus[0], F_2'=\begin{bmatrix} G_2' & \mathbf{e}_1\\ \mathbf{e}_2^\top & 0\end{bmatrix}, G_1'=\begin{bmatrix} 0 & 0 & b\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, G_2'=\begin{bmatrix} 0 & 0 & 0\\ 0 & -\frac{fa}{b} & f\\ 0 & -\frac{fa^2}{b^2} & \frac{fa}{b}\end{bmatrix}.$$

Let $F_2''=F_2'-\frac{fa}{b}F_0$. Then do the following elementary column and row operations on F_0,F_1',F_2'' to obtain $\hat F_i, i=0,1,2$. Add to column one $\frac{fa}{b}$ times column four, add to row three $\frac{fa^2}{b^2}$ times row four and add to row two $\frac{2fa}{b}$ times row four. Observe that $\hat F_0,\hat F_1,\hat F_2$ are of the form F_0,F_1,F_2 we started with, and with the addional fact that $a=0$ in G_1. Since $b\neq 0$, by replacing F_1 with $\frac{1}{b}F_1$ we may assume that $b=1$. It is left to show that our tensor $\mathcal{T}\in\mathbb{C}^{4\times 4\times 3}$ has border rank 4 at most. Let $\mathcal{T}(t)\in\mathbb{C}^{4\times 4\times 3}$ be the tensor with the following three 3-slices $F_0(t)=I_3\oplus[t],F_1,F_2$. It suffices to show that for any $t\neq 0$, brank $\mathcal{T}(t)\leq 4$. Mulitply the last row of $F_0(t),F_1,F_2$ by $\frac{1}{t}$ to obtain $I_4,F_1,F_2(t)$. Let $\mathcal{T}(t)'\in\mathbb{C}^{4\times 4\times 3}$ be the tensor with the above three 3-slices. Observe that $F_1F_2(t)=F_2(t)F_1=0$. Apply the arguments in the proof of Lemma 5.3 to deduce that brank $\mathcal{T}(t)'\leq 4$.

Assume now that F_1,F_2 have the form given in *2.b*. Consider $F_0^\top=F_0,F_1^\top,F_2^\top$. Let $P=\begin{bmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$. Then $F_0=(P\oplus[1])F_0^\top(P\oplus[1])^{-1}$ and $(P\oplus[1])F_i^\top(P\oplus[1])^{-1}, i=1,2$ are of the form *2.c*. Hence brank $\mathcal{T}\leq 4$.

It is left to discuss the case where $\dim\mathbf{T}_p(\mathcal{T})=4,\text{mrank }\mathbf{T}_p(\mathcal{T})=3,p=1,2,3$. We show that this case does not exists. As above we observe that there is no nonzero $\mathbf{u}$ such that either $\mathbf{T}_p(\mathcal{T})\mathbf{u}=0$ or $\mathbf{u}^\top\mathbf{T}_p(\mathcal{T})=\mathbf{0}^\top$. We now apply Lemma 5.6 to $\mathbf{T}_3(\mathcal{T})$. We change bases in the first two components of $\mathbb{C}^4\otimes\mathbb{C}^4\otimes\mathbb{C}^4$ to obtain one in the three possibilities discussed in Lemma 5.6. We start with the case *2.a*, where $k=3$. Our assumption that $\dim\mathbf{T}_3(\mathcal{T})=4$ implies that the matrices G_1 and G_2 in (5.5) are linearly independent.

Observe next that

$$T_{1,2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, T_{2,2} = \begin{bmatrix} 0 & a_1 & a_2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & c_1 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$T_{3,2} = \begin{bmatrix} 0 & b_1 & b_2 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & d_1 & d_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, T_{4,2} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The assumption that mrank $\mathbf{T}_2(\mathcal{T}) = 3$ yields that $\det(T_{2,2} + x_1 T_{1,2} + x_3 T_{3,2} + x_4 T_{4,2}) = 0$. Hence $(a_1 + x_3 b_1)(c_2 + x_3 d_2) - (a_2 + x_3 b_2)(c_1 + x_3 d_1)$ is identically zero. Let

$$A_1 = \begin{bmatrix} a_1 & a_2 \\ c_1 & c_2 \end{bmatrix}, \ A_2 = \begin{bmatrix} b_1 & b_2 \\ d_1 & d_2 \end{bmatrix}.$$

Then there exists a nonzero $\mathbf{v} \in \mathbb{C}^2$ such that either $A_1\mathbf{v} = A_2\mathbf{v} = \mathbf{0}$ or $\mathbf{v}^\top A_1 = \mathbf{v}^\top A_2 = \mathbf{0}^\top$. The first possibility yields that there exists nonzero $\mathbf{u} \in \mathbb{C}^4$ such that $\mathbf{T}_2(\mathcal{T})\mathbf{u} = 0$ which contradicts our assumptions. Hence, there exists a nonzero $\mathbf{v} \in \mathbb{C}^2$ such that $\mathbf{v}^\top A_1 = \mathbf{v}^\top A_2 = \mathbf{0}^\top$.

The assumption that mrank $\mathbf{T}_1(\mathcal{T}) = 3$ yields that $(a_1 + x_3 c_1)(b_2 + x_3 d_2) - (a_2 + x_3 c_2)(b_1 + x_3 d_1)$ is identically zero. Let

$$B_1 = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}, \ A_2 = \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix}.$$

As above we deduce that there exists nonzero $\mathbf{w} \in \mathbb{C}^2$ such that $B_1\mathbf{w} = B_2\mathbf{w} = \mathbf{0}$. In particular we deduce that the rows $(a_1, b_1), (a_2, b_2)$ and $(c_1, d_1), (c_2, d_2)$ are linearly dependent. Hence, by choosing a new basis in $\mathrm{span}(G_1, G_2)$ we can assume that $c_1 = d_1 = 0$ and $a_2 = b_2 = 0$. So A_1, A_2 are both diagonal, singular, and have a common left zero eigenvector. So either $a_1 = b_1 = 0$ or $c_2 = d_2 = 0$. That is either $G_1 = 0$ or $G_2 = 0$ which contradicts our assumption that $\dim \mathbf{T}_3(\mathcal{T}) = 4$.

Assume the condition *2.b*. Consider the matrices $T_{1,2}, T_{2,2}, T_{3,2}, T_{4,2}$. Note that $T_{2,2} = \begin{bmatrix} 0 & a_1 & a_2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & c_1 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. The assumption that mrank $\mathbf{T}_2(\mathcal{T}) = 3$ yields that $a_1 c_2 - a_2 c_1 = 0$. So the vectors $(a_1, c_1)^\top, (a_2, c_2)^\top$ are linearly dependent. Hence F_1, F_2 are linearly dependent. This contradicts the assumption that $\dim \mathbf{T}_3(\mathcal{T}) = 4$.

Assume the condition *2.c*. Consider the matrices $T_{1,1}, T_{2,1}, T_{3,1}, T_{4,1}$. The assumption that $\det T_{1,1} = 0$ yields that the vectors $(a_1, b_1), (a_2, b_2)$ are linearly dependent. Hence F_1, F_2 are linearly dependent. This contradicts the assumption that $\dim \mathbf{T}_3(\mathcal{T}) = 4$. $\square$

Proof of Theorem 5.1. We first show that if brank $\mathcal{T} \le 4$ then the conditions 1-2 hold. Theorem 2.7 yields the condition 1. Let $P_1, P_2, P_3 \in \mathbf{GL}(4, \mathbb{C})$. Then brank $\mathcal{T}(P_1, P_2, P_3) \le 4$. Let $\mathcal{T}'(P_1, P_2, P_3) = [t_{i,j,k}(P_1, P_2, P_3)]_{i=j=k}^{3,3,4} \in \mathbb{C}^{3\times 4}$. Clearly brank $\mathcal{T}'(P_1, P_2, P_3) \le 4$. Theorem 4.5 yields the conditions (5.1)-(5.2) for $p = 3$ and (4.2) for $R_3(P_1, P_2, P_3), L_3(P_1, P_2, P_3)$. Similar arguments imply the conditions 2 for $p = 1, 2$. Since $\mathbf{GL}(4, \mathbb{C})$ is dense in $C^{4\times 4}$ we deduce the condition 2 for any $P_1, P_2, P_3 \in \mathbb{C}^{4\times 4}$.

Assume that $\mathcal{T}$ satisfies the conditions 1-2. Suppose to the contrary that brank $\mathcal{T} > 4$. Theorem 5.2 yields that there exists nonzero $\mathbf{u} \in \mathbb{C}^4$ such that either $\mathbf{T}_3(\mathcal{T})\mathbf{u} = \mathbf{0}$ or $\mathbf{u}^\top \mathbf{T}_3(\mathcal{T}) = 0$. By changing the first two factors if needed we can assume that $\mathbf{u}^\top \mathbf{T}_3(\mathcal{T}) = \mathbf{0}$. By changing a basis in the first factor we may assume that $\mathbf{u} = \mathbf{e}_4 = (0, 0, 0, 1)^\top$. Consider now $\mathbf{T}_1(\mathcal{T})$. Note that the fourth 1-slice $T_{4,1}$ is zero matrix. Since brank $\mathcal{T} > 4$

Theorem 5.2 yields that there exists nonzero $\mathbf{v} \in \mathbb{C}^4$ such that either $\mathbf{T}_1(\mathcal{T})\mathbf{v} = \mathbf{0}$ or $\mathbf{v}^\top \mathbf{T}_1(\mathcal{T}) = \mathbf{0}$. By permuting the two last factors $\mathbb{C}^4$, if needed, we may assume that $\mathbf{v}\mathbf{T}_1(\mathcal{T}) = \mathbf{0}$. By changing a basis in the second factor we can assume that $\mathbf{v} = \mathbf{e}_4$. This finally means that after permuting the factors of $\mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4$, and changing bases in the first two factors, we obtain a new tensor $\mathcal{T}'$ whose four 3-slices are matrices with the zero last row and column. Permuting back to the original factors and changing bases correspondingly using $P_1, P_2, P_3 \in \mathbf{GL}(4, \mathbb{C})$, (one of these matrices is an identity matrix), we deduce that for some $p \in \{1, 2, 3\}$, the four p-slices $S_{1,p}(P_1, P_2, P_3), \ldots, S_{4,p}(P_1, P_2, P_3)$ have zero last row and column. Combine conditions 2 with Theorem 4.5 to deduce that the tensor $\mathcal{T}' \in \mathbb{C}^{3\times3\times4}$, whose four 3-slices are $T_{1,p}(P_1, P_2, P_3), \ldots, T_{4,p}(P_1, P_2, P_3)$, has border rank 4 at most. Hence $4 \geq \mathrm{brank}\ \mathcal{T}(P_1, P_2, P_3) = \mathrm{brank}\ \mathcal{T}$ which contradicts our assumption. $\square$

We conclude this section by showing that the conditions in of Theorem 5.1 can be stated as a finite number of polynomial equations in degrees $5, 9$ and 16 in the entries of $\mathcal{T}$. First we consider the conditions 1. In the notation of 1 $\mathbf{T}_p(\mathcal{T})$ is spanned by $S_{1,p}(I, I, I), \ldots, S_{4,p}(I, I, I)$. Fix $p \in \{1, 2, 3\}$. Let

$$X = \sum_{i=1}^{4} x_i S_{i,p}(I, I, I), \quad Y = \sum_{j=1}^{4} y_j S_{j,p}(I, I, I), \quad Z = \sum_{k=1}^{4} z_k S_{k,p}(I, I, I).$$

Then $\mathrm{adj}\, Y$ is 4×4 matrix whose entries are homogeneous polynomials of degree 3 in $\mathbf{y} = (y_1, \ldots, y_4)^\top$. Note that the coefficients of the monomials of these polynomials are polynomials of degree 3 in the entries of $\mathcal{T}$. Substitute the expressions of $X, \mathrm{adj}\, Y, Z$ into the conditions (1.1) to deduce that a finite number of polynomials in $\mathbf{x}, \mathbf{y}, \mathbf{z}$ of degree 5 must vanish identically. That is, the corresponding coefficient of each monomial must be zero. This procedure gives rise to a finite number of polynomial equations of degree 5 that the entries of $\mathcal{T}$ satisfy. Clearly, if these conditions hold then the condition 1 hold.

We now discuss the conditions 2. Write $P_q = [x_{ij,q}]_{i=j=1}^4$ for $q = 1, 2, 3$. We view each $x_{ij,q}$ as a variable. So the entries of P_1, P_2, P_3 give rise to 48 variables. The entries $S_{1,p}(P_1, P_2, P_3), \ldots, S_{4,p}(P_1, P_2, P_3), p = 1, 2, 3$, are multilinear polynomials of degree 3 in 48 variables. Hence the entries of $T_{1,p}(P_1, P_2, P_3), \ldots, T_{4,p}(P_1, P_2, P_3)$, are multilinear polynomials of degree 3, whose coefficients are homogeneous polynomials of degree 1 in the entries of $\mathcal{T}$. The conditions (5.1-5.2) yield that each 9×9 minor of the two matrices in (5.1-5.2) must be identically zero. Such a minor is a homogeneous polynomial of degree $27 = 3 \times 9$ in the entries of P_1, P_2, P_3. The coefficient of each monomial is a polynomial of degree 9 in the entries of $\mathcal{T}$. Hence the coefficient of each monomial appearing in the expansion of each minor must equal to zero. (Recall that for each $p \in \{1, 2, 3\}$ we have 440 such 9×9 minors.) These conditions give rise to a finite number polynomial conditions of degree 9 on the entries of $\mathcal{T}$. Repeat the same procedure for the conditions on $R_p(P_1, P_2, P_3), L_p(P_1, P_2, P_3)$ given by (4.3-4.4) to deduce a finite number of polynomial conditions of degree 16 on the entries of $\mathcal{T}$. Clearly, these polynomial equations of degree 9 and 16 imply that the condition 2 hold for each $P_1, P_2, P_3 \in \mathbb{C}^{4\times4\times4}$.

6 Tensors in $\mathbb{C}^{m\times n\times l}$ of rank l

Let $m, n, l \geq 2$ and and assume that $\mathcal{T} = [t_{i,j,k}] \in \mathbb{C}^{m\times n\times l}$. In this section we study mainly the conditions when $\mathrm{rank}\ \mathcal{T} = \dim \mathbf{T}_3(\mathcal{T})$. We point out briefly how to state some of these conditions for tensors of border rank l at most. We consider the generic case $\dim \mathbf{T}_3(\mathcal{T}) = l$. Equivalently we study the conditions on a subspace $\mathbf{T} \subset \mathbb{C}^{m\times n}$, of dimension l, to be spanned by l linearly independent rank one matrices. First choose a basis $T_1, \ldots, T_l \in \mathbb{C}^{m\times n}$ of $\mathbf{T}$. Let $\mathbf{z} = (z_1, \ldots, z_l)^\top \in \mathbb{C}^l$ and denote $T(\mathbf{z}) = \sum_{k=1}^{l} z_k T_k$. Recall that $T(\mathbf{z})$ has rank at most rank 1 if all 2×2 minors of $T(\mathbf{z})$ are zero. Let

$$\alpha=(i_1,i_2), 1\le i_1<i_2\le m,\quad \beta=(j_1,j_2), 1\le j_1<j_2\le n.$$

As in §2 denote by $2_2^{\langle m\rangle}, 2_2^{\langle n\rangle}$ the set of all allowable α,β respectively. Then $T(\mathbf{z})[\alpha,\beta]$ is the 2×2 minor of $T(\mathbf{z})$ based on the rows i_1,i_2 and the columns j_1,j_2. Clearly $T(\mathbf{z})[\alpha,\beta]$ is a quadratic form in $\mathbf{z}$. For $A=[a_{ij}], B=[b_{ij}]\in\mathbb{C}^{m\times n}$ denote

$$b(A,B)[\alpha,\beta]=\det\begin{bmatrix} a_{i_1j_1} & b_{i_1j_2}\\ a_{i_2j_1} & b_{i_2j_2}\end{bmatrix}.$$

Then the condition that $T(\mathbf{z})$ has rank one at most is given by the system of quadratic equations

$$T(\mathbf{z})[\alpha,\beta]=\sum_{p,q=1}^{l} b(A_p,A_q)[\alpha,\beta]z_pz_q=0,\ \text{ for all } \alpha\in 2_2^{\langle m\rangle},\beta\in 2_2^{\langle n\rangle}. \tag{6.1}$$

Note the number of these equations is $\binom{m}{2}\binom{n}{2}$. With each of the above quadratic form we associate a symmetric matrix $S(\alpha,\beta)(T_1,\dots,T_l)\in \mathrm{S}(l,\mathbb{C})$, Denote by $\mathbf{S}(T_1,\dots,T_l)\subset\mathrm{S}(l\mathbb{C})$ the subspace spanned by $S(\alpha,\beta)(T_1,\dots,T_l), \alpha\in 2_2^{\langle m\rangle}, 2_2^{\langle n\rangle}$. Suppose that we change a basis of $\mathbf{T}$ from $T_1,\dots,T_l$ to $T_1',\dots,T_l'$. This is equivalent to the change of variables $\mathbf{z}=R\mathbf{y}$. Hence

$$S(\alpha,\beta)(T_1',\dots,T_l')=R^\top S(\alpha,\beta)(T_1,\dots,T_l)R, \text{ for some } R\in\mathbf{GL}(l,\mathbb{C}), \tag{6.2}$$

and $\alpha\in 2_2^{\langle m\rangle},\beta\in 2_2^{\langle n\rangle}$. Thus $\mathbf{S}(T_1',\dots,T_l')=R^\top\mathbf{S}(T_1,\dots,T_l)R$. For simplicity of notation we let $\mathbf{S}(\mathbf{T}):=\mathbf{S}(T_1,\dots,T_l)$. So the subspace $\mathbf{S}(\mathbf{T})$ is defined up to congruence.

Proposition 6.1 *Let $T_1,\dots,T_l$ be a basis in $\mathbf{T}\subset\mathbb{C}^{m\times n}$. Then for any $P\in\mathbf{GL}(m,\mathbb{C}), Q\in\mathbf{GL}(n,\mathbb{C})$*

$$S(\alpha,\beta)(PT_1Q,\dots,PT_lQ)=\sum_{\gamma\in 2_2^{\langle m\rangle},\delta\in 2_2^{\langle n\rangle}} P[\alpha,\gamma]Q[\delta,\beta]S(\gamma,\delta)(T_1,\dots,T_l). \tag{6.3}$$

Hence $\mathbf{S}(P\mathbf{T}Q)=\mathbf{S}(\mathbf{T})$.

Proof. Recall that for any $A\in\mathbb{C}^{m\times n}$ the 2×2 minors of PAQ are given by the Cauchy-Binet formula

$$(PAQ)[\alpha,\beta]=\sum_{\gamma\in 2_2^{\langle m\rangle},\delta\in 2_2^{\langle n\rangle}} P[\alpha,\gamma]A[\gamma,\delta]Q[\delta,\beta].$$

Clearly $PT(\mathbf{z})Q=\sum_{k=1}^l z_kPT_kQ$. Apply the Cauchy-Binet formula to deduce (6.3). Hence $\mathbf{S}(P\mathbf{T}Q)\subseteq\mathbf{S}(\mathbf{T})$. Since $\mathbf{T}=P^{-1}(P\mathbf{T}Q)Q^{-1}$ we obtain that $\mathbf{S}(P\mathbf{T}Q)\supseteq\mathbf{S}(\mathbf{T})$. □

Let $\mathbf{S}(T_1,\dots,T_l)^\perp\subset\mathrm{S}(l,\mathbb{C})$ be the orthogonal complement of with respect the symmetric product on $\mathbb{C}^{l\times l}$: $\langle A,B\rangle:=\operatorname{tr}AB^\top$.

Lemma 6.2 *Let $\mathbf{T}\subset\mathbb{C}^{m\times n}$ be an l-dimensional subspace. Then $\mathbf{T}$ contains r-linearly independent rank one matrices if and only the subspace $\mathbf{S}(\mathbf{T})^\perp$ contains r linearly independent rank one symmetric matrices which are simultaneously diagonable by congruency. That is, if $T_1,\dots,T_l$ is a basis of $\mathbf{T}$, then there exist $R\in\mathbf{GL}(l,\mathbb{C})$ such that*
$R^\top\operatorname{diag}(\delta_{1k},\dots,\delta_{lk})R\in\mathbf{S}(\mathbf{T})^\perp$ for $k=1,\dots,r$.

Proof. Choose a basis in $\mathbf{T}$: $T_1,\ldots,T_l$ such that $T_1,\ldots,T_r$ are r-rank one linearly independent matrices. Hence the quadratic form $T(\mathbf{z})[\alpha,\beta]$ does not contain terms $z_1^2,\ldots,z_r^2$. Thus the diagonal entries $(1,1),\ldots,(r,r)$ are zero for each $S(\alpha,\beta)(T_1,\ldots,T_l)$. Therefore $D_k := \mathrm{diag}(\delta_{1k},\ldots,\delta_{lk}) \in \mathbf{S}(T_1,\ldots,T_l)^\perp$ for $k=1,\ldots,r$.

Assume now that $\mathbf{S}(T_1,\ldots,T_k)^\perp$ contains $R^{-1}D_k(R^{-1})^\top$ for $k=1,\ldots,r$. This is equivalent to the fact that $D_k \in \mathbf{S}(T_1',\ldots,T_l')^\perp$ for a corresponding basis $T_1',\ldots,T_l'$ of $\mathbf{T}$. Hence $T_1',\ldots,T_r'$ are r linearly independent rank one matrices in $\mathbf{T}$. $\square$

Corollary 6.3 *Let $\mathbf{T} \subset \mathbb{C}^{m\times n}$ be an l-dimensional subspace. Assume that $\mathbf{T}$ contains r-linearly independent rank one matrices. Then dimension $\mathbf{S}(\mathbf{T})$ is at most $\binom{l+1}{2}-r$.*

The data induced by $S(\alpha,\beta)(T_1,\ldots,T_l)$ for $\alpha \in 2_2^{\langle m\rangle}, \beta \in 2_2^{\langle n\rangle}$ can be arranged in the following $\binom{m}{2}\binom{n}{2} \times \binom{l+1}{2}$ matrix $C(\mathbf{T}) = C(T_1,\ldots,T_l) = [c_{(\alpha,\beta)(p,q)}]$. Here

$$c_{(\alpha,\beta)(p,q)} = b(T_p,T_q)[\alpha,\beta] + b(T_q,T_p)[\alpha,\beta],\ 1 \le p \le q \le l, \alpha \in 2_2^{\langle m\rangle}, \beta \in 2_2^{\langle n\rangle}. \tag{6.4}$$

Corollary 6.3 equivalent to the statement that rank $C(\mathbf{T}) \le \binom{l+1}{2}-r$. That is, all minors of order $\binom{l+1}{2}-r+1$ of $C(\mathbf{T})$ are zero. We now give two examples of generic subspaces $\mathbf{T} \subset \mathbb{C}^{m\times n}$ of dimension l spanned by rank one matrices which satisfy $\dim \mathrm{S}(\mathbf{T})^\perp = l$. In these cases we obtain necessary conditions for $\mathcal{T} \in \mathbb{C}^{m\times n\times l}$ to have a border rank l at most.

Lemma 6.4 *Assume that one of the following conditions hold.*

1. *$2 \le l \le m,n$ and $\mathbf{T} \subset \mathbb{C}^{m\times n}$ is an l-dimensional subspace spanned by l rank one matrices $\mathbf{u}_1\mathbf{v}_1^\top,\ldots,\mathbf{u}_l\mathbf{v}_l^\top$, where $\mathbf{u}_1,\ldots,\mathbf{u}_l \in \mathbb{C}^m$ and $\mathbf{v}_1,\ldots,\mathbf{v}_l \in \mathbb{C}^n$ are linearly independent.*

2. *$m=n=l-1 \ge 3$ and $\mathbf{T} \subset \mathbb{C}^{(l-1)\times(l-1)}$ is an l-dimensional subspace spanned by l rank one matrices $\mathbf{u}_1\mathbf{v}_1^\top,\ldots,\mathbf{u}_l\mathbf{v}_l^\top$, where any $l-1$ vectors out of $\mathbf{u}_1,\ldots,\mathbf{u}_l \in \mathbb{C}^{l-1}$ and $\mathbf{v}_1,\ldots,\mathbf{v}_l \in \mathbb{C}^{l-1}$ are linearly independent.*

Then $\dim \mathbf{S}(\mathbf{T}) = \binom{l}{2}, \dim \mathbf{S}(\mathbf{T})^\perp = l$, and $\mathbf{S}(\mathbf{T})^\perp$ spanned by l ranks one symmetric matrices which are simultaneously diagonable by congruency. Hence for any $\mathbf{T} \in \Gamma(l,\mathbb{C}^{m\times n})$ the matrix $C(\mathbf{T})$ has rank at most $\binom{l}{2}$. In particular if $\mathcal{T} = [t_{i,j,k}] \in \mathbb{C}^{m\times n\times l}$ has border rank not more than l, then the l 3-slices $T_k := [t_{i,j,k}]_{i=j=1}^{m,n} \in \mathbb{C}^{m\times n}, k=1,\ldots,l$ satisfy the identities given by the vanishing of all $\binom{l}{2}+1$ minors of $C(T_1,\ldots,T_l)$.

Proof. Let $\mathbf{e}_i := (\delta_{i1},\ldots,\delta_{im})^\top, \mathbf{f}_j := (\delta_{j1},\ldots,\delta_{jn})^\top$ for $i=1,\ldots,m$ and $j=1,\ldots,n$. Consider first that the case 1. Since $\mathbf{u}_1,\ldots,\mathbf{u}_l$ and $\mathbf{v}_1,\ldots,\mathbf{v}_l$ are linearly independent, it follows that there exist $P \in \mathbf{GL}(m,\mathbb{C}), Q \in \mathbf{GL}(n,\mathbb{C})$ such that $P\mathbf{u}_i = \mathbf{e}_i, Q\mathbf{v}_i = \mathbf{f}_i, i = 1,\ldots,l$. Let $T_i' := P(\mathbf{u}_i\mathbf{v}_i^\top)Q^\top = \mathbf{e}_i\mathbf{f}_i^\top$ for $i=1,\ldots,l$. Thus $T'(\mathbf{z}) = \mathrm{diag}(z_1,\ldots,z_l,0,\ldots,)$. The only nonzero 2×2 minors of $T'(\mathbf{z})$ are $T'(\mathbf{z})[\alpha,\alpha]$ where $\alpha=(p,q)$ and $1 \le p < q \le l$. So $\mathbf{S}(PT_1Q^\top,\ldots,PT_lQ^\top)$ consists of all symmetric matrices $A \in \mathrm{S}(l,\mathbb{C})$ with zero diagonal. Hence $\dim \mathbf{S}(\mathbf{T}) = \binom{l}{2}, \dim \mathbf{S}(\mathbf{T})^\perp = l$ and $\mathbf{S}(\mathbf{T})^\perp$ spanned by l ranks one symmetric matrices which are simultaneously diagonable by congruency.

Consider now the case 2. The arguments of the proof of Lemma 4.1 yield that we may assume that $\mathbf{u}_i = \mathbf{v}_i = \mathbf{e}_i$ for $i=1,\ldots,l-1$ and $\mathbf{u}_l = \mathbf{v}_l = \mathbf{w} = (w_1,\ldots,w_{l-1})^\top$, where $w_i \ne 0$ for $i=1,\ldots,l-1$. Let $T_i = \mathbf{u}_i\mathbf{v}_i^\top$ for $i=1,\ldots,l$. A straightforward calculation shows that $\det T(\mathbf{z})[(1,3),(1,2)] = w_2w_3z_1z_l$. Similarly, we have all the quadratic forms of the form $w_jw_kz_iz_l$ for $i=2,\ldots,l-1$, were i,j,k are three distinct elements in $\langle l-1\rangle$. By considering $\det T(\mathbf{z})[\alpha,\alpha], \alpha \in 2_2^{\langle l-1\rangle}$ we deduce that the space of quadratic polynomials spanned by $\det T(\mathbf{z})[\alpha,\beta], \alpha,\beta \in 2_2^{\langle l-1\rangle}$ contains all the monomials z_iz_j for $i \ne j$. Hence $\mathbf{S}(\mathbf{T})$ is all $A \in \mathrm{S}(l,\mathbb{C})$ with zero diagonal entries. I.e. $\dim \mathbf{S}(\mathbf{T}) = \binom{l}{2}, \dim \mathbf{S}(\mathbf{T})^\perp = l$ and $\mathbf{S}(\mathbf{T})^\perp$ spanned by l ranks one symmetric matrices which are simultaneously diagonable by congruency. Other claims of the lemma follow straightforward from the continuity argument. □

Theorem 6.5 *Assume that* $\mathcal{T} \in \mathbb{C}^{m\times n\times l}$*,* $\dim \mathbf{T}_3(\mathcal{T}) = l$ *and* $\dim \mathbf{S}(\mathbf{T}_3(\mathcal{T}))^\perp = l$*. Then*

1. $\operatorname{rank} \mathcal{T} = l$ *if and only if the following conditions hold*
   (a) $\mathbf{S}(\mathbf{T}_3(\mathcal{T}))^\perp$ *contains an invertible matrix.*
   (b) *The condition (1.1) holds for any three matrices* $A, B, C \in \mathbf{S}(\mathbf{T}_3(\mathcal{T}))^\perp$*.*
   (c) $A \operatorname{adj} B$ *have* l *distinct eigenvalues for some* $A, B \in \mathbf{S}(\mathbf{T}_3(\mathcal{T}))^\perp$*.*
2. *Assume that either* $2 \le l \le m, n$ *or* $m = n = l - 1 \ge 3$*. If* $\operatorname{brank} \mathcal{T} = l$ *and* $\dim \mathbf{S}(\mathbf{T}_3(\mathcal{T}))^\perp = l$ *then* $\mathbf{S}(\mathbf{T}_3(\mathcal{T}))^\perp \in \Gamma(l, \mathrm{S}(l,\mathbb{C}))$*. In particular, the conditions (2.9) holds for* $p = 1, \ldots, l-1$ *and any three matrices* $A, B, C \in \mathbf{S}(\mathbf{T}_3(\mathcal{T}))^\perp$*.*

Proof. Let $\mathbf{U} := \mathbf{S}(\mathbf{T}_3(\mathcal{T}))^\perp \subset \mathrm{S}(l,\mathbb{C})$. Suppose that $\operatorname{rank} \mathcal{T} = l$. Lemma 6.2 yields that $\mathbf{U}$ contains l linearly independent rank one symmetric matrices which are simultaneously diagonable by a congruency. Since $\dim \mathbf{U} = l$ we deduce that $\mathbf{U}$ has a basis of the form $R^\top \operatorname{diag}(\delta_{1k}, \ldots, \delta_{lk}) R \in \mathrm{S}(\mathbf{T})^\perp$ for $k = 1, \ldots, l$. Hence the conditions 1a-1c hold.

Suppose that the condition 1a holds. By considering $\mathcal{T}(I_m, I_n, R)$ for some $R \in \mathbf{GL}(l,\mathbb{C})$ we may assume that $I \in \mathbf{U}$. The condition 1b for $B = I$ yields that $\mathbf{U}$ is a subspace of commuting matrices. Hence, there exists a unitary matrix V such that $V\mathbf{U}V^*$ is an upper triangular matrix, e.g. [9, §24.2, Fact 3]. The assumption 1c yields that most of the matrices of the the form $A \operatorname{adj} B$ have simple eigenvalues. Choose $B \in \mathbf{GL}(l,\mathbb{C}) \cap \mathbf{U}$. So $\operatorname{adj} B = \frac{1}{\det B} B^{-1}$. Thus, most of the matrices of the form AB^{-1} have simple eigenvalues. Hence most of the matrices in $\mathbf{U}$ have simple eigenvalues. Choose $A \in \mathbf{U}$ such that A has simple eigenvalues. Thus any $C \in \mathbf{U}$ is a polynomial in A. Since $\dim \mathbf{U} = l$ it follows that $\mathbf{U}$ has a basis of l-rank one commuting matrices which are simultaneously diagonable by an orthogonal matrix. Lemma 6.2 yields that $\operatorname{rank} \mathcal{T} = l$.

Assume that $\operatorname{brank} \mathcal{T} = l$. Hence $\mathcal{T}$ is a limit of rank l tensors $\mathcal{T}_q, q \in \mathbb{N}$ satisfying the conditions of Lemma 6.4. Since $\dim \mathbf{S}(\mathbf{T}_3(\mathcal{T}))^\perp = l$ we deduce that $\mathbf{S}(\mathbf{T}_3(\mathcal{T}))^\perp$ is the limit of $\mathbf{S}(\mathbf{T}_3(\mathcal{T}_q))^\perp, q \in \mathbb{N}$. (Without this assumption we can only deduce that any convergent subsequence of subspaces in the sequence $\mathbf{S}(\mathbf{T}_3(\mathcal{T}_q))^\perp, q \in \mathbb{N}$ converges to a subspace of $\mathbf{S}(\mathbf{T}_3(\mathcal{T}))^\perp$.) Apply for each $\mathcal{T}_q$ part 1 to deduce part 2. □

Note that a simultaneous matrix diagonalization by congruence arises naturally in finding the rank decomposition of tensors [4]. As in our characterization of $V_4(3,3,4)$ we can restate the conditions (1b) of Theorem 6.5 in terms of some polynomial equations. These equations will also hold for any $\mathcal{T}$ satisfying $\operatorname{brank} \mathcal{T} \le l$.

References

[1] E.S. Allman and J.A. Rhodes, Phylogenic ideals and varieties for general Markov model, *Advances in Appl. Math.*, 40 (2008) 127-148.

[2] R.A. Brualdi, S. Friedland and A. Pothen, The sparse basis problem and multilinear algebra, *SIAM J. Matrix Anal. Appl.* 16 (1995), 1-20.

[3] J. Buczynski and J. M. Landsberg, Ranks of tensors and a generalization of secant varieties, arXiv:0909.4262.

[4] L. DeLathauwer, A link between the canonical decomposition in multilinear algebra and simultaneous matrix diagonalization , *SIAM Journal of Matrix Analysis and Applications* 28 (2006), 642-666.

[5] S. Friedland, On the generic rank of 3-tensors, arXiv:0805.3777.

[6] S. Friedland and C. Krattenthaler, 2-adic valuations of certain ratios of products of factorials and applications, *Linear Algebra and its Applications*, 426 (2007), 159-189.

[7] R.M. Guralnick and B.A. Sethuraman, Commuting pairs and triples of matrices and related varieties *Linear Algebra and its Applications*, 310 (2000), 139-148.

[8] J. Harris and L.W. Tu, On symmetric and skew-symmetric determinantal varieties, *Topology* **23** (1984), 71–84.

[9] L. Hogben, *Handbook of Linear Algebra* CRC Press, 2007.

[10] J.M. Landsberg and L. Manivel, On the ideals of secant varieties of Segre varieties, *Found. Comput. Math.* 4 (2004), 397422.

[11] J.M. Landsberg and L. Manivel, Generalizations of Strassen's equations for secant varieties of Segre varieties, *Comm. Algebra* 36 (2008), 405–422.

[12] M. Marcus and H. Minc, *A Survey of Matrix Theory and Matrix Inequalities*, Prindle, Weber & Schmidt, Boston, 1964.

[13] T. Motzkin and O. Taussky-Todd, pairs of matrices with property L. II, *Trans. AMS* 80 (1955), 387-401.

[14] L. Oeding and D.J. Bates, Toward a salmon conjecture, arXiv:1009.6181v1.

[15] L. Pachter and B. Sturmfels, *Algebraic Statistics for Computational Biology*, Cambridge University Press, 2005.

[16] V. Strassen, Rank and optimal computations of generic tensors, *Linear Algebra Appl.* 52/53: 645-685, 1983.

[17] B. Sturmfels, Open problems in algebraic statistics, in *Emerging Applications of Algebraic Geometry*, (editors M. Putinar and S. Sullivant), I.M.A. Volumes in Mathematics and its Applications, 149, Springer, New York, 2008, pp. 351-364.

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