DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **33**, Number **11&12**, November & December **2013**

doi:10.3934/dcds.2013.33.4811

pp. **4811–4840**

SINGULARITY FORMATION AND BLOWUP OF COMPLEX-VALUED SOLUTIONS OF THE MODIFIED KDV EQUATION

Jerry L. Bona

Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago
Chicago, Il 60607, USA

Stéphane Vento and Fred B. Weissler

Université Paris 13, Sorbonne Paris Cité
CNRS UMR 7539 Laboratoire Analyse, Géométrie et Applications
99 avenue J.B. Clément - 93430 Villetaneuse, France

Abstract. The dynamics of the poles of the two–soliton solutions of the modified Korteweg–de Vries equation

$$u_t + 6u^2 u_x + u_{xxx} = 0$$

are investigated. A consequence of this study is the existence of classes of smooth, complex–valued solutions of this equation, defined for $-\infty < x < \infty$, exponentially decreasing to zero as $|x| \to \infty$, that blow up in finite time.

1. **Introduction.** Studied here is the modified Korteweg–de Vries equation

$$u_t + 6u^2 u_x + u_{xxx} = 0, \tag{1.1}$$

which has been derived as a rudimentary model for wave propagation in a number of different physical contexts. The present paper is a sequel to the recent work [9] wherein the dynamics of the complex singularities of the two–soliton solution of the Korteweg-de Vries equation,

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1.2}$$

were examined in detail.

The study of the pole dynamics of solutions of the Korteweg–de Vries equation and its near relatives began with some remarks of Kruskal [14] in the early 1970's. More comprehensive work was carried out later, see [18], [11] and [12]. One goal in the preceding paper [9] was to understand in more detail the propagation of solitons in a neighborhood of the interaction time. Another motivation was an idea to be explained presently concerning singularity formation in nonlinear, dispersive wave equations. More particularly, we are interested in both the generalized Korteweg–de Vries equation

$$u_t + u_{xxx} + u^p u_x = 0 \tag{1.3}$$

2010 *Mathematics Subject Classification.* 35Q53, 35Q51, 37K40, 35A21, 35B44.
Key words and phrases. KdV-type equations, multi-soliton solutions, asymptotic behaviour.
The third author gratefully acknowledges a Sabbatical leave during the fall semester 2011.

and coupled systems of Korteweg–de Vries type, *viz.*

$$\left\{\begin{array}{l} u_t + u_{xxx} + P(u,v)_x = 0, \\ v_t + v_{xxx} + Q(u,v)_x = 0, \end{array}\right. \tag{1.4}$$

where P and Q are, say, homogeneous polynomials.

Concerning singularity formation, it is an open question whether or not smooth, rapidly decaying, real–valued solutions of (1.3) develop singularities in finite time in the supercritical case $p \geq 5$. In the critical case $p = 4$, blowup in finite time has been established by Martel and Merle [17], [16] whilst for subcritical values $p = 1, 2, 3$, there is no singularity formation for data that lies at least in the Sobolev space $H^1(\mathbb{R})$. (However, solutions corresponding to infinitely smooth initial values lying only in $L^2(\mathbb{R})$ can develop singularities; see [7].) In the supercritical case, numerical simulations reported in [4] of solutions of (1.3) initiated with an amplitude–modified solitary wave indicate blowup in finite time. Such initial data has an analytic extension to a strip symmetric about the real axis in the complex plane. The results of Bona, Grujic and Kalisch [13], [5] and [6] indicate that blowup at time t has to be accompanied by the width of the strip of analyticity shrinking to zero at the same time. This points to the prospect of a pair of complex conjugate singular points colliding at a spatial point on the real axis, thereby producing non–smooth behavior of the real–valued solution. It was shown in [9] that, in certain cases, curves of singularities of a two-soliton solution of (1.2) do merge together. This happens at the moment of interaction of the two solitons when the amplitudes are related in a particular way. This result provides some indication that the blowup seen in the numerical simulations in the supercritical case might occur because of the coalescence of curves of complex singularities. Such ruminations seem to justify interest in the pole dynamics in the context of (1.3), even in the case where the initial data is real–valued.

If one considers instead complex–valued solutions, it was shown in [8] (and see also [10]) that, in the case of spatially periodic boundary conditions, equation (1.3) has solutions which blow up in finite time for all integers $p \geq 1$. Explicit examples of smooth, complex–valued solutions defined for $x, t \in \mathbb{R}$ of the Korteweg–de Vries equation (1.2) that blow up in finite time have been given in [2, 9, 15, 19]. One outcome of the present paper is an explicit example of a blowing–up solution of the modified Korteweg–de Vries equation (1.1).

For the system (1.4) where P and Q are homogeneous quadratic polynomials, conditions on the coefficients are known that imply global well–posedness for real–valued initial data (u_0, v_0) (see [3]). And, the pole dynamics investigated in [9] for the Korteweg–de Vries equation (1.2) itself, (KdV–equation henceforth) reveals that the choice

$$P(u,v) = u^2 - v^2 \quad \text{and} \quad Q(u,v) = 2uv \tag{1.5}$$

leads to a system (1.4) possessing solutions that develop singularities in finite time. (This is the system that obtains if complex–valued solutions of the KdV–equation are broken up into real and imaginary parts.)

The latter result is obtained by a careful study of the pole dynamics of the explicit two–soliton solution $U = U(z,t) = U(x+iy,t)$ of the KdV–equation in the complex z–plane. It transpires that as a function of time, most of the singularities of this exact solution, which are all poles, move vertically in the y–direction in the complex plane as well as propagating horizontally in the x–direction. As a consequence, by choosing y_0 appropriately, one can arrange that the function $u(x,t) = U(x+iy_0,t)$ is a complex–valued solution of the KdV–equation that, at $t = 0$, is infinitely smooth and decays to zero exponentially rapidly as $x \to \pm\infty$, but which blows up for a positive value $t > 0$. The blowup time is the precise moment that a vertically moving pole crosses the horizontal line $\operatorname{Im} z = y_0$ in the complex plane. If we write $u(x,t) = v(x,t) + iw(x,t)$, then the pair (v, w) is a solution of (1.4) with the choice (1.5) for P and Q that starts at the smooth and rapidly decaying initial data $(v_0, w_0) = (v(\cdot, 0), w(\cdot, 0))$, but which forms a singularity in finite time. It is worth noting that by an appropriate choice of the particular two–soliton solution, the initial data (v_0, w_0) can be taken to be arbitrarily small in any of the usual function spaces used in the analysis of such equations.

The motivation for the current paper is to see if the phenomena described above for the KdV–equation persist for the modified KdV–equation (1.1). First, is it possible that curves of singularities of a two-soliton solution merge together, and does their behavior provide additional insight into the possible ways a singularity might be produced in nonlinear, dispersive wave equations? Second, can finite time blowup of complex-valued solutions of the modified KdV–equation be established by reference to the vertical motion of the solution's singularities in the complex plane? And of course we are interested to understand in what ways the pole dynamics in the complex plane of the two-soliton solution reflects the behavior of the interacting solitons on the real line.

It turns out that the two phenomena just mentioned do occur. In certain cases, curves of singularities do converge together at a single point. On the other hand, the behavior of these curves is not so different from what occurs in the context of the KdV–equation, and so provides little new insight. It should be pointed out that for both (1.1) and (1.2) the merging of poles together at one point does not establish blowup at that point since at least one of the poles is moving in a purely horizontal fashion. Thus, the solution restricted to the horizontal line containing this point is always singular. On the other hand, as was the case in [9], finite time blowup of complex-valued solutions of (1.1) is shown here as a consequence of the vertical movement of singularities in the complex plane.

The rest of this paper is organized as follows. In the next section, explicit formulas for the two–soliton solutions of (1.1) are given. In fact there are two types of two–soliton solutions, corresponding to interactions of solitons of the same or opposite signs. The notation introduced in that section will be used throughout the rest of the paper. In Section 3, we establish the fact that the singularities of these solutions are all simple poles and, excluding some specific "exceptional cases", they all trace out analytic curves in the complex plane, as a function of time. See Proposition 3.1 and Definition 3.1 below. In Section 4, it is shown that the poles of the two-soliton solutions separate out, for large time (positive or negative) into two sets of poles, each of which behave asymptotically like the poles of a one–soliton solution with the appropriate speed. Section 5 concerns the detailed behavior of the poles in the "exceptional cases" just mentioned, where singularities merge together. The vertical motion of poles of two soliton solutions is studied in Section 6. Section 7 shows how this vertical motion enables one to construct complex-valued solutions of (1.1) on the real line which blowup at a single point in finite time. See Proposition 7.1. The last section describes some formal calculations describing the two-soliton solution at the moment of interaction of the two "constituant" solitons.

2. **The two–soliton solutions.** Preliminary analysis of the two–soliton solutions of the mKdV–equation

$$u_t + u_{xxx} + 6u^2 u_x = 0, \quad x \in \mathbb{C}, t \in \mathbb{R}, \tag{2.1}$$

are set forth here in preparation for the investigation of their pole dynamics. We begin with a standard transformation enabling one to express solutions of (2.1) in terms of solutions of another equation. Let $u = v_x$ where u is a solution of (2.1). Then v satisfies the equation

$$\frac{d}{dx}\left(v_t + v_{xxx} + 2v_x^3\right) = 0.$$

Assume v and its derivatives vanish at infinity and search for solutions of the latter equation of the form $v = 2\arctan(g)$. A calculation shows that v satisfies

$$v_t + v_{xxx} + 2v_x^3 = 0 \tag{2.2}$$

if and only if

$$(1+g^2)(g_t + g_{xxx}) + 6g_x(g_x^2 - g g_{xx}) = 0. \tag{2.3}$$

This yields a solution to (2.1) having the form

$$u(x,t) = 2\big(\arctan g(x,t)\big)_x = \frac{2g_x(x,t)}{1+g(x,t)^2}. \tag{2.4}$$

It is important to note that equation (2.1) and (2.3) are both invariant under change of sign. That is, u is a solution of (2.1) if and only if $-u$ is a solution and, likewise, g is a solution of (2.3) if and only if $-g$ is a solution. Also, replacing g by $1/g$ in (2.4) has the effect of multiplying the solution u by -1. More precisely, if u is given by (2.4), then

$$2\left(\arctan\frac{1}{g(x,t)}\right)_x = -2\frac{g_x(x,t)}{1+g(x,t)^2} = -u(x,t). \tag{2.5}$$

The well–known soliton solution of (2.1) has a hyperbolic secant profile. In detail, for any amplitude value $k > 0$, it is straightforward to check that

$$g(x,t) = \exp\big(-k(x-x_0) + k^3 t\big) = \exp\big(-k(x - x_0 - k^2 t)\big) \tag{2.6}$$

is a solution of (2.3). The corresponding solution u of (2.1) is the soliton solution with speed k^2 and is given explicitly as

$$\begin{aligned} u(x,t) = 2\big(\arctan g(x,t)\big)_x &= -2k\frac{e^{-k(x-x_0)+k^3 t}}{1+e^{-2k(x-x_0)+2k^3 t}} && (2.7)\\ &= -k \operatorname{sech}\big(-k(x-x_0)+k^3 t\big). && (2.8)\end{aligned}$$

Note that the choice of sign in the exponential function g in (2.6) produces the *negative* soliton. Replacing g by either $-g$ or $1/g$ will produce the positive soliton.

The two–soliton solutions are a little more complicated. The formulation presented here is based on that appearing in [1]. As just noted, there are both positive and negative soliton solutions of (2.1). Consequently, there are two types of two–soliton solutions, namely interacting solitons of the same or opposite signs.

Suppose that $0 < k_1 < k_2$. Define the functions f_j by

$$f_j(x,t) = \exp(-k_j x + k_j^3 t), \quad j = 1, 2. \tag{2.9}$$

Of course, this definition omits two arbitrary spatial translations; these will be added later. Define two auxiliary functions, g^+ and g^- by

$$g^+(x,t) = -\gamma \frac{f_1(x,t)+f_2(x,t)}{1-f_1(x,t)f_2(x,t)}, \tag{2.10}$$

$$g^-(x,t) = \gamma \frac{f_1(x,t)-f_2(x,t)}{1+f_1(x,t)f_2(x,t)}, \tag{2.11}$$

where

$$\gamma = \frac{k_2+k_1}{k_2-k_1} > 1. \tag{2.12}$$

Proposition 2.1. *The functions g^+ and g^- defined in (2.10)-(2.11) are solutions to (2.3).*

Proof. It suffices to provide the proof for g^-. Indeed, if one replaces f_1 by $-f_1$, then g^- is transformed into g^+, and all the calculations below remain valid in this case.

Temporarily, set $g = g^-$. Notice that $f_{jx} = -k_j f_j$ and $f_{jt} = k_j^3 f_j$ for $j = 1,2$. Thus, the quantities g_t, g_x, g_{xx} and g_{xxx} may be expressed in terms of f_j and k_j. First, differentiate with respect to time and come to the expression

$$g_t = \gamma \frac{k_1^3 f_1 - k_2^3 f_2 + k_1^3 f_1 f_2^2 - k_2^3 f_1^2 f_2}{(1+f_1 f_2)^2}.$$

Similarly, the derivative with respect to x is

$$g_x = \gamma \frac{k_2 f_2 - k_1 f_1 - k_1 f_1 f_2^2 + k_2 f_1^2 f_2}{(1+f_1 f_2)^2}.$$

Differentiating the latter expression leads to

$$g_{xx} = \frac{\gamma}{(1+f_1f_2)^3}\Big[k_1^2 f_1 - k_2^2 f_2 + (k_1^2+4k_1k_2+k_2^2)(f_1f_2^2 - f_1^2 f_2) - k_1^2 f_1^2 f_2^3 + k_2^2 f_1^3 f_2^2\Big].$$

Differentiating once more and simplifying gives

$$\begin{aligned} g_{xxx} = \frac{\gamma}{(1+f_1f_2)^4}\Big[&k_2^3 f_2 - k_1^3 f_1 - (k_1^3 + 4k_2^3 + 6k_1^2 k_2 + 12 k_1 k_2^2)(f_1 f_2^2 + f_1^3 f_2^2) \\ &+ (4k_1^3 + k_2^3 + 6k_1k_2^2 + 12k_1^2k_2)(f_1^2 f_2 + f_1^2 f_2^3) - k_1^3 f_1^3 f_2^4 + k_2^3 f_1^4 f_2^3)\Big]. \end{aligned}$$

It follows that

$$g_t + g_{xxx} = \frac{6\gamma(k_1+k_2)^2}{(1+f_1f_2)^4}\left[k_1 f_1^2 f_2 - k_2 f_1 f_2^2 + k_1 f_1^2 f_2^3 - k_2 f_1^3 f_2^2\right]$$

on the one hand, and

$$\begin{aligned} g_x^2 - g g_{xx} = \frac{\gamma^2}{(1+f_1f_2)^4}\big[&(k_1-k_2)^2(f_1f_2 + f_1^3 f_2^3) - \\ &8k_1k_2 f_1^2 f_2^2 + (k_1+k_2)^2(f_1^3 f_2 + f_1 f_2^3)\big] \end{aligned}$$

on the other. It is now straightforward to check that g satisfies equation (2.3). □

The above proposition implies that

$$u^\pm(x,t) = 2\big(\arctan g^\pm(x,t)\big)_x = \frac{2g_x^\pm(x,t)}{1+(g^\pm)^2(x,t)}$$

are solutions to (2.1). A further calculations reveals that

$$u^+ = 2\gamma \frac{G^+}{F^+} \tag{2.13}$$

and

$$u^- = 2\gamma \frac{G^-}{F^-} \tag{2.14}$$

where the new combinations

$$G^+ = k_1 f_1(1+f_2^2) + k_2 f_2(1+f_1^2), \tag{2.15}$$

$$G^- = -k_1 f_1(1+f_2^2) + k_2 f_2(1+f_1^2), \tag{2.16}$$

$$F^+ = (1-f_1f_2)^2 + \gamma^2(f_1+f_2)^2, \tag{2.17}$$

$$F^- = (1+f_1f_2)^2 + \gamma^2(f_1-f_2)^2, \tag{2.18}$$

have been introduced. Note that the functions with a superscript "+" are obtained from the functions with a superscript "−" simply by replacing f_1 by $-f_1$. If every occurrence of k_1 is replaced by $-k_1$ in formula (2.14) for u^-, then f_1 is replaced by $1/f_1$ and γ is replaced by $1/\gamma$. Simplifying the resulting expression gives exactly the formula (2.13) for u^+. In other words, the formulas for u^+ and u^- can be obtained from each other by replacing every occurrence of k_1 by $-k_1$.

The function u^+ is a two–soliton solution of (2.1) having two positive interacting solitons whist u^- is a two–soliton solution with two interacting solitons of opposite sign, the faster one being the positive one. One can see this graphically using MAPLE or Mathematica, for example. Analytically, there is an explicit relationship between the formulas for $u^\pm$ given by (2.13) and (2.14) and the single soliton solutions of speeds k_1^2 and k_2^2. As in (2.7), let u_j be the centered, positive, soliton solution

$$u_j(x,t) = k_j \operatorname{sech}(-k_j x + k_j^3 t) = \frac{2k_j f_j(x,t)}{1+f_j(x,t)^2} \tag{2.19}$$

of speed $k_j^2, j = 1, 2$. If the formulas for (2.13) and (2.14) are both divided by $(1+f_1^2)(1+f_2^2)$, there obtains

$$u^+ = \gamma \frac{u_1 + u_2}{D^+} \tag{2.20}$$

and

$$u^- = \gamma \frac{-u_1 + u_2}{D^-} \tag{2.21}$$

where

$$D^+ = \frac{(1-f_1f_2)^2 + \gamma^2(f_1+f_2)^2}{(1+f_1^2)(1+f_2^2)},$$

$$D^- = \frac{(1+f_1f_2)^2 + \gamma^2(f_1-f_2)^2}{(1+f_1^2)(1+f_2^2)}.$$

Formulas (2.20) and (2.21) show in particular that $u^+(x,t) > 0$ for all $x \in \mathbb{R}$ and that $u^-(x,t) > 0$ precisely for those $x \in \mathbb{R}$ and $t \in \mathbb{R}$ for which $u_2(x,t) > u_1(x,t)$.

In Section 4, the asymptotic behavior of the singularities of $u^\pm$ are examined for large positive and negative time. We will see that they separate into two groups, corresponding to the two single solitons. More remarks on the shape of the two–soliton solutions during their interaction are to be found in Section 8.

Before ending this section, we return to the issue of spatial shifts. For $0 < k_1 < k_2$ and $x_1, x_2 \in \mathbb{R}$, let

$$\tilde{f}_1(x,t) = \exp(-k_1(x-x_1) + k_1^3 t),$$
$$\tilde{f}_2(x,t) = \exp(-k_2(x-x_2) + k_2^3 t),$$

and

$$\widetilde{g^+}(x,t) = -\frac{\tilde{f}_1(x,t) + \tilde{f}_2(x,t)}{1 - \gamma^{-2}\tilde{f}_1(x,t)\tilde{f}_2(x,t)},$$
$$\widetilde{g^-}(x,t) = \frac{\tilde{f}_1(x,t) - \tilde{f}_2(x,t)}{1 + \gamma^{-2}\tilde{f}_1(x,t)\tilde{f}_2(x,t)}.$$

Let $\widetilde{u^\pm}$ denote the corresponding two–soliton solutions of (2.1), obtained from $\widetilde{g^\pm}$ by the relationship (2.4). Define the *interaction time* t_0 and the *interaction center* x_0 of $\widetilde{u^\pm}(x,t)$ to be

$$t_0 = -\frac{x_2 - x_1}{k_2^2 - k_1^2} - \frac{1}{(k_2+k_1)k_1k_2}\log\gamma \quad \text{and} \tag{2.22}$$

$$x_0 = \frac{k_2^2 x_1 - k_1^2 x_2}{k_2^2 - k_1^2} - \frac{k_1^2 + k_1k_2 + k_2^2}{(k_2+k_1)k_1k_2}\log\gamma, \tag{2.23}$$

respectively.

Proposition 2.2. *Let t_0 and x_0 be the interaction time and interaction center defined by (2.22) and (2.23). Then, for any $t \in \mathbb{R}$ and $x \in \mathbb{C}$, we have*

$$\widetilde{g^\pm}(x,t) = g^\pm(x - x_0, t - t_0), \tag{2.24}$$

where $g^\pm$ are defined in (2.10)-(2.11).
Moreover, the functions $\widetilde{u^\pm}(\cdot, t_0) = 2(\arctan \widetilde{g^\pm}(\cdot, t_0))_x$ are symmetric about the point x_0 for both real and complex values of the spatial variable.

Proof. It suffices to find $(x_0, t_0) \in \mathbb{R}^2$ such that $\tilde{f}_j(x,t) = \gamma f_j(x - x_0, t - t_0)$ $(j = 1, 2)$ for all $t \in \mathbb{R}$, $x \in \mathbb{C}$. Equivalently, (2.24) will be satisfied if

$$\begin{cases} \gamma e^{k_1 x_0 - k_1^3 t_0} = e^{k_1 x_1}, \\ \gamma e^{k_2 x_0 - k_2^3 t_0} = e^{k_2 x_2}. \end{cases}$$

Since (x_0, t_0) given by (2.22)-(2.23) is the solution for this system, the first assertion is proved. To see the symmetry of the functions $\widetilde{u^\pm}(\cdot, t_0)$ about x_0, it is only necessary to deduce from (2.24) that $u^\pm(\cdot, 0) = 2(\arctan(g^\pm(\cdot, 0))_x$ is an even function. This is easily verified since $g^\pm(-x, 0) = -g^\pm(x, 0)$ for all $x \in \mathbb{C}$. □

Since (mKdV) is invariant under time– and space–translation, it is concluded from Proposition 2.2 that the functions $\widetilde{u^\pm}(x,t) = 2(\arctan(\widetilde{g^\pm}(x,t))_x$ are also solutions to (2.1). The point is that, for a general two–soliton solution, the time and place of the interaction are given by t_0 and x_0 in (2.22) and (2.23), respectively. In particular, the solutions $u^\pm$ are already normalized so that the interaction time is $t = 0$, and the center of the interaction is at $x = 0$. It is interesting to note that the values of t_0 and x_0 in the above proposition are precisely the same as for the two–soliton solution of the Korteweg-de Vries equation, as given in Theorem 1 of [9].

We alert the reader to the fact that the notation established above shall be used throughout the rest of this paper. In particular, $u^{\pm}$, $G^{\pm}$ and $F^{\pm}$ are given by (2.13)–(2.18) and γ is given by (2.12).

3. **Singularities of the two–soliton solutions in the complex plane.** As mentioned in the introduction, the two–soliton solutions $u(x,t)$ of (2.1) are viewed as meromorphic functions in the complex variable x. We aim to investigate how the dynamics of the singularities in $\mathbb{C}$ reflect the behavior of $u(x,t)$ when x is restricted to the real axis. The main result of this section, Proposition 3.1, is that with the exception of certain well-defined situations, these singularities are all given by analytic curves in $\mathbb{C}$.

Consider first the (one)–soliton solution, given by (2.7). It is immediate that the singularities of these solutions are precisely

$$x = x_0 + k^2 t + \frac{m\pi i}{2k} \tag{3.1}$$

where m runs through the odd integers. Moreover, these singularities of u are all simple poles. Thus, the speed of the soliton is exactly the speed of its poles in the complex plane, while the position of the maximum point of the soliton at time t is the real part of the position of these poles. The imaginary part of the singularity remains constant in time of course.

The poles of the two–solitons solutions (2.13) and (2.14), correspond to the zeros of the functions $F^{\pm}$ defined in by (2.17) and (2.18). It will turn out that the singularities of $u^{\pm}$ are all simple poles, just as for the one–solitons. In all but a specific class of exceptional cases, these poles correspond to simple zeros of $F^{\pm}$.

Recall that k_1 and k_2 are called *commensurable* if there exist positive integers p_1 and p_2 such that

$$\frac{k_2}{k_1} = \frac{p_2}{p_1}. \tag{3.2}$$

Without loss of generality, it will always be presumed that $\gcd(p_1,p_2) = 1$. Thus, in what follows, we will always assume that the integers p_1 and p_2 appearing in (3.2) are without common prime factors. In this case, $F^{\pm}(x,t)$ and $G^{\pm}(x,t)$ (and thus $u^{\pm}(x,t)$) are periodic in x with minimal imaginary period $2\pi\lambda i$ where

$$\lambda = \frac{p_1}{k_1} = \frac{p_2}{k_2}. \tag{3.3}$$

Since we have $0 < k_1 < k_2$, it follows that $p_2 > p_1$.

Lemma 3.1. *For any $t \in \mathbb{R}$, the zeros of $F^{\pm}(\cdot,t)$ are simple, except for the following special case. If k_1 and k_2 are commensurable, and if $p_1, p_2 \in \mathbb{N}$ and $\lambda > 0$ are given by (3.2) and (3.3), with $p_2 \pm p_1 \in 4\mathbb{N}$, then there is a fourth-order zero of $F^{\pm}(\cdot,0)$ at $x = (\frac{1}{2}+q)\lambda\pi i$ for all $q \in \mathbb{Z}$.*

Proof. From (2.18), we infer

$$F^-_x = 2\big[(1+f_1f_2)(f_{1x}f_2 + f_1f_{2x}) + \gamma^2(f_1-f_2)(f_{1x}-f_{2x})\big].$$

Noticing that $f_{jx} = -k_jf_j$, $(j=1,2)$ and setting $X = f_1$ and $Y = f_2$, it follows that if x is a zero of $F^-(\cdot,t)$ of order greater than or equal to 2, then

$$\begin{cases} (1+XY)^2 + \gamma^2(X-Y)^2 = 0 \quad \text{and} \\ -(k_1+k_2)(1+XY)XY + \gamma^2(X-Y)(k_2Y-k_1X) = 0. \end{cases} \tag{3.4}$$

Let (X, Y) be a solution of this system. Then, from the first equation it is deduced that

$$1 + XY = i\varepsilon\gamma(X - Y), \tag{3.5}$$

where ϵ is either 1 or -1. Inserting this into the second equation, it is found that

$$i\varepsilon(k_2 - k_1)XY = k_2 Y - k_1 X,$$

provided $X \neq Y$. Extract the product XY and inject it in (3.5) to obtain the linear relation

$$Y = \frac{k_2}{k_1} X + i\varepsilon \frac{k_2 - k_1}{k_1}.$$

Then formula (3.5) implies that X must satisfy

$$1 + X\left(\frac{k_2}{k_1} X + i\varepsilon \frac{k_2 - k_1}{k_1}\right) = i\varepsilon\gamma\left(\left(1 - \frac{k_2}{k_1}\right) X - i\varepsilon \frac{k_2 - k_1}{k_1}\right),$$

which simplifies to

$$X^2 + 2i\varepsilon X - 1 = 0.$$

It follows that $X = Y = \pm i$. Thus, it must be that $e^{-k_1 x + k_1^3 t} = e^{-k_2 x + k_2^3 t} = \pm i$, from which we deduce that $-k_1 x + k_1^3 t$ and $-k_2 x + k_2^3 t$ are both purely imaginary. Therefore, $t = 0$ and x is purely imaginary. Additionally, we have

$$\begin{cases} k_1 x = (\frac{1}{2} + q_1)\pi i, \\ k_2 x = (\frac{1}{2} + q_2)\pi i, \end{cases} \quad \text{for some } q_1, q_2 \in \mathbb{Z}.$$

Thus k_1 and k_2 are commensurable and in fact, $k_2/k_1 = r_2/r_1$ with $r_j = 1 + 2q_j$. Noticing that $q_2 - q_1 \in 2\mathbb{Z}$ (because $k_1 x - k_2 x \in 2i\pi\mathbb{Z}$), it transpires that $r_2 - r_1 = 4k$ for some $k \in \mathbb{Z}$. It follows that $d = \gcd(r_1, r_2)$ divides $4k$ and therefore d divides k since d is necessarily an odd integer. Define p_1 and p_2 by $k_2/k_1 = p_2/p_1$ with $\gcd(p_1, p_2) = 1$. With this definition, $p_2 - p_1 = (r_2 - r_1)/d = 4k/d \in 4\mathbb{N}$. There are precisely two such complex numbers in the fundamental strip $S = \{x \in \mathbb{C}, -\lambda\pi < \operatorname{Im} x < \lambda\pi\}$, and they are $x = \pm\frac{\lambda}{2}\pi i$.

It is straightforward to ascertain that for such values of x, $F^-_{xx}(x, 0) = F^-_{xxx}(x, 0) = 0$ and $F^-_{xxxx}(x, 0) \neq 0$. The same arguments hold for F^+. □

Lemma 3.2. *For any $t \in \mathbb{R}$, the zeros of $F^-(\cdot, t)$ and $G^-(\cdot, t)$ are distinct, except the following special case. If k_1 and k_2 are commensurable, and if $p_1, p_2 \in \mathbb{N}$ and $\lambda > 0$ are given by (3.2) and (3.3), with $p_2 - p_1 \in 4\mathbb{N}$, then there is a third-order zero of $G^-(\cdot, 0)$ at $x = (\frac{1}{2} + q)\lambda\pi i$ for all $q \in \mathbb{Z}$. The same statement is true with F^- and G^- replaced by F^+ and G^+, and the condition $p_2 - p_1 \in 4\mathbb{N}$ replaced by $p_2 + p_1 \in 4\mathbb{N}$.*

Proof. In the notation of the proof of Lemma 3.1, the result follows if the system

$$\begin{cases} (1 + XY)^2 + \gamma^2(X - Y)^2 = 0, \\ -k_1 X(1 + Y^2) + k_2 Y(1 + X^2) = 0, \end{cases} \tag{3.6}$$

admits as its only solution $X = Y = \pm i$. As before, if (X, Y) is a solution, there exists $\varepsilon \in \{-1, 1\}$ for which

$$1 + XY = i\varepsilon\gamma(X - Y).$$

Extracting from this the variable Y and computing $1 + Y^2$ leads to

$$Y = \frac{i\varepsilon\gamma X - 1}{X + i\varepsilon\gamma}$$

and

$$1+Y^2=(1-\gamma^2)\frac{1+X^2}{(X+i\varepsilon\gamma)^2}.$$

Now insert this into the second equation of the system (3.6) and, assuming by contradiction that $1+X^2$ is not zero, simplify the outcome. It follows that

$$\frac{k_1}{k_2}(1-\gamma^2)X=(i\varepsilon\gamma X-1)(X+i\varepsilon\gamma).$$

After further simplifications, this becomes

$$X^2+2i\varepsilon X-1=0,$$

and the claim follows. □

Definition 3.1. The values of k_1 and k_2 for which $F^{\pm}(\cdot,0)$ has multiple zeros are collectively referred to as the exceptional case. This occurs when k_1 and k_2 are commensurable, p_1 and p_2 are odd, and

$$p_2-p_1\in 4\mathbb{N}$$

when considering F^-, and

$$p_2+p_1\in 4\mathbb{N}$$

when considering F^+.

Lemmas 3.1 and 3.2 show that all singularities of $u^{\pm}(\cdot,t)$ are simple poles. In all but the exceptional case, and in the exceptional case when $t\neq 0$, they correspond to simple zeros of $F^{\pm}$. In the exceptional case, while there can be simple zeros of $F^{\pm}(\cdot,0)$, there are also singularities of $u^{\pm}(\cdot,0)$ which are produced by a third-order zero of $G^{\pm}(\cdot,0)$ coinciding with a fourth-order zero of $F^{\pm}(\cdot,0)$. In this last case, it follows from Rouché's Theorem that four simple poles converge, as $t\to 0$, to the simple pole at $(\frac{1}{2}+q)\lambda\pi i$ for all $q\in\mathbb{Z}$. Moreover, by the residue theorem, the sum of the residues of $u^{\pm}(\cdot,t)$ at these four poles converge, as $t\to 0$, to the residue of the simple pole at $(\frac{1}{2}+q)\lambda\pi i$ for all $q\in\mathbb{Z}$.

Proposition 3.1. *In all but the exceptional case, the poles of $u^{\pm}(\cdot,t)$, i.e. the zeros of $F^{\pm}(\cdot,t)$, are described by analytic curves $x:\mathbb{R}\to\mathbb{C}$. In the exceptional case, these curves are defined and analytic separately for $t<0$ and $t>0$. However, in the exceptional case, if a zero of $F^{\pm}(\cdot,0)$ is not a fourth-order zero as described in Lemma 3.1, then it is part of an analytic curve $x:\mathbb{R}\to\mathbb{C}$ of zeros of $F^{\pm}(\cdot,t)$.*

Proof. Consider the case of u^+ and F^+. Since all the zeros of $F^+(\cdot,t)$ are simple, the implicit–function theorem shows that, for a fixed time t_0, a zero of $F^+(\cdot,t_0)$ can be locally and uniquely continued as an analytic curve $x(t)$ such that $F^+(x(t),t)=0$. Such a curve $x(t)$ can be continued as long as it remains in a bounded region of $\mathbb{C}$. Thus, we need to show that $|x(t)|$ must stay bounded as long as t remains in a bounded interval of $\mathbb{R}$. First, it follows from Proposition 6.3 in Section 6 below that the imaginary part of $x(t)$ must remain bounded. Furthermore, if $\operatorname{Re}x(t)\to\infty$ in finite time, then the equation $F^+(x(t),t)=0$ implies $1=0$. If $\operatorname{Re}x(t)\to-\infty$ in finite time, then the equation $\frac{F^+(x(t),t)}{(1-f_1(x(t),t)f_2(x(t),t))^2}=0$ also implies that $1=0$.

A similar argument works for u^- and F^-. □

This section closes with a specific example of a solution lying in the exceptional case, namely $k_1 = 1$, $k_2 = 5$ (calculations done with MAPLE). In this case, $\lambda = 1$ and F^- and G^- may be rewritten as

$$G^-(x,t) = -e^{251t}y^{11} + 5e^{127t}y^7 + 5e^{125t}y^5 - e^t y \text{ and}$$
$$F^-(x,t) = e^{252t}y^{12} + \frac{9}{4}e^{250t}y^{10} - \frac{5}{2}e^{126t}y^6 + \frac{9}{4}e^{2t}y^2 + 1,$$

where $y = e^{-x}$. Note that at time $t = 0$, F^- and G^- are symmetric polynomials in y, and that i and $-i$ are third-order zeros of G^-, and fourth-order zeros of F^-. All the other zeros are simple. We also have the decomposition

$$u^-(x,0) = 3\frac{G^-(x,0)}{F^-(x,0)} = \frac{-2}{y-i} - \frac{2}{y+i} + \frac{y-4i}{2y^2 - iy - 2} + \frac{y+4i}{2y^2 + iy - 2}.$$

4. **Large–time asymptotic behavior of the singularities.** In this section we show that the poles of $u^\pm$, that is the zeros of $F^\pm$, separate out into two groups, as $t \to \pm\infty$, behaving asymptotically as poles of single solitons, of speeds k_1^2 and k_2^2, respectively, as described by formula (3.1). This, of course, reflects the fact that $u^\pm$ are in fact "two–soliton" solutions of (2.1).

The situation is nearly identical to that obtaining for the two–soliton solution for the Korteweg–de Vries equation (compare the following theorem with Theorem 2 in [9]). In particular, the backward shift of the slower wave, and the corresponding forward shift of the faster wave, well–known in the case of the KdV–solitons, are also present for the modified KdV–equation.

Theorem 4.1. *The asymptotic behaviors as $t \to \pm\infty$ of the curves $x(t)$ of zeroes of $F^\pm(x,t)$ whose existence was determined in Proposition 3.1 are completely described as follows.*

1. *For every odd integer $m \in \mathbb{Z}$, there exists a unique curve $x_{m,s^-}(t)$ of zeros of $F^\pm(\cdot,t)$ such that*
$$x_{m,s^-}(t) = k_1^2 t + \frac{1}{k_1}\log\gamma + \frac{m\pi i}{2k_1} + o(1),$$
as $t \to -\infty$.
2. *For every odd integer $m \in \mathbb{Z}$, there exists a unique curve $x_{m,s^+}(t)$ of zeros of $F^\pm(\cdot,t)$ such that*
$$x_{m,s^+}(t) = k_1^2 t - \frac{1}{k_1}\log\gamma + \frac{m\pi i}{2k_1} + o(1),$$
as $t \to \infty$.
3. *For every odd integer $n \in \mathbb{Z}$, there exists a unique curve $x_{n,f^-}(t)$ of zeros of $F^\pm(\cdot,t)$ such that*
$$x_{n,f^-}(t) = k_2^2 t - \frac{1}{k_2}\log\gamma + \frac{n\pi i}{2k_2} + o(1),$$
as $t \to -\infty$.
4. *For every odd integer $n \in \mathbb{Z}$, there exists a unique curve $x_{n,f^+}(t)$ of zeros of $F^\pm(\cdot,t)$ such that*
$$x_{n,f^+}(t) = k_2^2 t + \frac{1}{k_2}\log\gamma + \frac{n\pi i}{2k_2} + o(1),$$
as $t \to \infty$.

To prove Theorem 4.1, it is convenient to study the zeros of $F^{\pm}(\cdot, t)$ with respect to frames of reference which move at the speed of each constituant soliton. As in Section 4 of [9], define

$$z = x - k_1^2 t, \tag{4.1}$$

$$w = x - k_2^2 t, \tag{4.2}$$

$$r = \exp(k_2(k_2^2 - k_1^2)t), \tag{4.3}$$

$$s = \exp(k_1(k_2^2 - k_1^2)t). \tag{4.4}$$

If

$$H^{\pm}(z, r) = (1 \mp re^{-(k_1+k_2)z})^2 + \gamma^2(e^{-k_1 z} \pm re^{-k_2 z})^2 \tag{4.5}$$

and

$$I^{\pm}(w, s) = (s \mp e^{-(k_1+k_2)w})^2 + \gamma^2(e^{-k_1 w} \pm se^{-k_2 w})^2, \tag{4.6}$$

then

$$F^{\pm}(x, t) = H^{\pm}(z, r) = s^{-2} I^{\pm}(w, s).$$

Zeros $z(r)$ of $H^{\pm}(\cdot, r)$ which remain bounded in $\mathbb{C}$ as $r \to 0$ and as $r \to \infty$ correspond to zeros $x(t)$ of $F^{\pm}(\cdot, t)$ asymptotically traveling at speed k_1^2 as $t \to \pm\infty$. Likewise, zeros $w(s)$ of $I^{\pm}(\cdot, s)$ which remain bounded in $\mathbb{C}$ as $s \to 0$ and as $s \to \infty$ correspond to zeros $x(t)$ of $F^{\pm}(\cdot, t)$ asymptotically traveling at speed k_2^2 as $t \to \pm\infty$. The following is true of the curves $z(r)$ and $w(s)$.

Proposition 4.1. *For every odd integer $m \in \mathbb{Z}$, there exists a smooth curve $z_m^{\pm}(r)$ defined in some interval of $r \geq 0$, such that $H^{\pm}(z_m^{\pm}(r), r) = 0$ and*

$$z_m^{\pm}(r) = \frac{1}{k_1} \log \gamma + \frac{m\pi i}{2k_1} \pm i^m \frac{4k_2}{k_2^2 - k_1^2} \gamma^{-k_2/k_1} e^{-\frac{k_2}{2k_1} m\pi i} r + o(r) \tag{4.7}$$

as $r \to 0^+$. For every odd integer $n \in \mathbb{Z}$, there exists a smooth curve $w_n^{\pm}(s)$ defined in some interval of $s \geq 0$, such that $I^{\pm}(w_n^{\pm}(s), s) = 0$ and

$$w_n^{\pm}(s) = -\frac{1}{k_2} \log \gamma + \frac{n\pi i}{2k_2} \pm i^n \frac{4k_1}{k_2^2 - k_1^2} \gamma^{-k_1/k_2} e^{\frac{k_1}{2k_2} n\pi i} s + o(s) \tag{4.8}$$

as $s \to 0^+$.

Proof. The relation $H^{\pm}(z, 0) = 0$ is satisfied if and only if there exists an odd integer $m \in \mathbb{Z}$ such that

$$z = \frac{1}{k_1} \log \gamma + \frac{m\pi i}{2k_1},$$

and similarly, $I^{\pm}(w, 0) = 0$ is equivalent to

$$w = -\frac{1}{k_2} \log \gamma + \frac{n\pi i}{2k_2}$$

for some odd integer $n \in \mathbb{Z}$. Applying the implicit function theorem, there exist smooth curves $z_m^{\pm}(r)$ and $w_n^{\pm}(s)$ defined in a neighborhood of $z_0 = 1/k_1 \log \gamma + m\pi i/2k_1$ and $w_0 = -1/k_2 \log \gamma + n\pi i/2k_2$, respectively, such that $H^{\pm}(z_m^{\pm}(r), r) = 0$, $z_m^{\pm}(0) = z_0$ and $I(w_n^{\pm}(s), s) = 0$, $w_n^{\pm}(0) = w_0$. It remains to calculate $(z_m^{\pm})'(r)$ and $(w_n^{\pm})'(r)$. Differentiating the equation $H^{\pm}(z_m^{\pm}(r), r) = 0$ with respect to r yields

$$\begin{aligned}\big[-(k_1+k_2)(1 \mp re^{-(k_1+k_2)z})re^{-(k_1+k_2)z} + \gamma^2(e^{-k_1 z} \pm re^{-k_2 z})(k_2 re^{-k_2 z} \pm k_1 e^{-k_1 z})\big] z' \\ = -(1 \mp re^{-(k_1+k_2)z})e^{-(k_1+k_2)z} + \gamma^2(e^{-k_1 z} \pm re^{-k_2 z})e^{-k_2 z}\end{aligned}$$

where $z = z_m^{\pm}(r)$. Taking $r = 0$ gives

$$\pm k_1\gamma^2 e^{-2k_1 z_m(0)}(z_m^{\pm})'(0) = (\gamma^2 - 1)e^{-(k_1+k_2)z_m^{\pm}(0)},$$

or equivalently,

$$(z_m^{\pm})'(0) = \pm\frac{\gamma^2 - 1}{k_1\gamma}e^{(k_1-k_2)z_m^{\pm}(0)}.$$

Using that

$$e^{k_1 z_m^{\pm}(0)} = e^{\log\gamma + m\pi i/2} = \gamma i^m$$

and

$$e^{-k_2 z_m^{\pm}(0)} = \gamma^{-k_2/k_1}e^{-\frac{k_2}{2k_1}m\pi i}$$

leads to

$$(z_m^{\pm})'(0) = \pm i^m\frac{4k_2}{k_2^2 - k_1^2}\gamma^{-k_2/k_1}e^{-\frac{k_2}{2k_1}m\pi i}. \tag{4.9}$$

This establishes the claimed asymptotic behavior of $z_m^{\pm}(r)$ as $r \to 0$.

The derivative $(w_n^{\pm})'(0)$ is similarly calculated. Differentiating the equation $I^{\pm}(w_n^{\pm}(s), s) = 0$ with respect to s leads to

$$\begin{aligned}[\pm(k_1+k_2)(s\mp e^{-(k_1+k_2)w})e^{-(k_1+k_2)w}+\gamma^2(e^{-k_1w}\pm se^{-k_2w})(\mp k_2se^{-k_2w}-k_1e^{-k_1s})]w'\\ = -(s\mp e^{-(k_1+k_2)w})\mp\gamma^2(e^{-k_1w}\pm se^{-k_2w})e^{-k_2w}.\end{aligned}$$

At $s = 0$, there obtains

$$[-(k_1+k_2)e^{-2(k_1+k_2)w_n^{\pm}(0)} - k_1\gamma^2 e^{-2k_1w_n^{\pm}(0)}](w_n^{\pm})'(0) = \mp(\gamma^2-1)e^{-(k_1+k_2)w_n^{\pm}(0)},$$

or equivalently,

$$[(k_1+k_2) + k_1\gamma^2 e^{2k_2w_n^{\pm}(0)}](w_n^{\pm})'(0) = \pm(\gamma^2-1)e^{(k_1+k_2)w_n^{\pm}(0)}.$$

Since $e^{2k_2w_n^{\pm}(0)} = e^{-2\log\gamma + n\pi i} = -1/\gamma^2$ (recall n is odd), the above is further equivalent to

$$k_2(w_n^{\pm})'(0) = \pm(\gamma^2-1)e^{(k_1+k_2)w_n^{\pm}(0)}.$$

Using $e^{k_2w_n^{\pm}(0)} = e^{-\log\gamma + n\pi i/2} = i^n/\gamma$ we conclude that

$$(w_n^{\pm})'(0) = \pm i^n\frac{4k_1}{k_2^2 - k_1^2}\gamma^{-k_1/k_2}e^{\frac{k_1}{2k_2}n\pi i}.$$

This establishes the claimed asymptotic behavior of $w_n^{\pm}(s)$ as $s \to 0$. □

A proof of Theorem 4.1 is now readily available. The previous result provides all the required curves of zeros of $F^{\pm}(\cdot, t)$ for large negative t, i.e. $x_{m,s^-}(t)$ and $x_{n,s^-}(t)$. On the other hand, since

$$F^{\pm}(-x, -t) = e^{2(k_1+k_2)x - 2(k_1^3+k_2^3)t}F^{\pm}(x, t),$$

it follows that $F^{\pm}(x, t) = 0$ if and only if $F^{\pm}(-\overline{x}, -t) = 0$. Thus we can set $x_{m,s^+}(t) = -\overline{x}_{m,s^-}(-t)$ and $x_{n,s^+}(t) = -\overline{x}_{n,s^-}(-t)$ for $t > 0$ sufficiently large.

The only remaining issue is to prove that there are no additional zeros of $F^{\pm}(\cdot, t)$. If k_1 and k_2 are commensurable, this is straightforward. For each $t \in \mathbb{R}$, $F^{\pm}(\cdot, t)$ is a polynomial in $e^{-x\lambda}$ of degree $2(p_1 + p_2)$, and thus must have precisely $2(p_1 + p_2)$ zeros in the complex plane. The zeros of $F^{\pm}(\cdot, t)$ described in Theorem 4.1 account for all of them, for large $|t|$, and so no other zeros can exist. If k_1 and k_2 are not commensurable, the desired result can be obtained by approximating k_1 and k_2 by sequences $\{k_1^{\nu}\}_{\nu=1}^{\infty}$ and $\{k_2^{\nu}\}_{\nu=1}^{\infty}$, which are commensurable.

This section concludes with a proposition, whose proof depends on the proof of Proposition 4.1, and which will be important for establishing blowup in Section 7. Recall that if k_1 and k_2 are commensurable, then the relatively prime integers p_1 and p_2 are given by (3.2) and (3.3).

Proposition 4.2. *Let m and n be odd integers The curves $x_{m,s^-}(t)$ and $x_{n,f^-}(t)$, defined for large negative t, and the curves $x_{m,s^+}(t)$ and $x_{n,f^+}(t)$, defined for large $t > 0$, constructed by Theorem 4.1, have the following properties :*

- $\operatorname{Im}(x_{m,s^-})'(t) \neq 0$ *for all sufficiently large negative t, except possibly in the case where k_1 and k_2 are commensurable, p_2 is odd, and m is an odd multiple of p_1.*
- $\operatorname{Im}(x_{n,s^-})'(t) \neq 0$ *for all sufficiently large negative t, except possibly in the case where k_1 and k_2 are commensurable, p_1 is odd, and m is an odd multiple of p_2.*
- $\operatorname{Im}(x_{m,f^+})'(t) \neq 0$ *for all sufficiently large $t > 0$, except possibly in the case where k_1 and k_2 are commensurable, p_2 is odd, and m is an odd multiple of p_1.*
- $\operatorname{Im}(x_{n,s^+})'(t) \neq 0$ *for all sufficiently large negative t, except possibly in the case where k_1 and k_2 are commensurable, p_1 is odd, and m is an odd multiple of p_2.*

Proof. Since $x_{m,s^+}(t) = -\overline{x}_{m,s^-}(-t)$ and $x_{n,s^+}(t) = -\overline{x}_{n,s^-}(-t)$, the first and third cases are equivalent, as are the second and fourth. We give the proof only in the first case. The second case follows similarly.

For large negative t, we know that

$$x_{m,s^-}(t) = z_m^{\pm}(e^{k_2(k_2^2-k_1^2)t}),$$

where the curves $z_m^{\pm}$ are as constructed in Proposition 4.1. Therefore,

$$\operatorname{Im}(x_{m,s^-})'(t) = k_2(k_2^2 - k_1^2)\operatorname{Im}(z_m^{\pm})'(e^{k_2(k_2^2-k_1^2)t}),$$

for large negative t. Furthermore, formula (4.9) in the proof of Proposition 4.1 yields that

$$(z_m^{\pm})'(0) = \pm i^m \frac{4k_2}{k_2^2 - k_1^2}\gamma^{-k_2/k_1}e^{-\frac{k_2}{2k_1}m\pi i}.$$

Thus, $\operatorname{Im}(z_m^{\pm})'(0) \neq 0$ except precisely when

$$i^m e^{-\frac{k_2}{2k_1}m\pi i} = e^{-\frac{(k_2-k_1)}{2k_1}m\pi i}$$

is a real number, which must be ± 1. For this, it is necessary and sufficient that $\frac{m(k_2-k_1)}{2k_1}$ be an integer. If k_1 and k_2 are not commensurable, this can never happen. In the commensurable case, using (3.2) and (3.3), this condition is equivalent to saying that

$$\frac{m(p_2-p_1)}{2p_1} = \frac{mp_2}{2p_1} - \frac{m}{2}$$

be an integer, *i.e.* that $\frac{mp_2}{p_1}$ be an odd integer (since by assumption m is an odd integer). Since p_1 and p_2 are relatively prime, this can only happen if m is an odd multiple of p_1 and p_2 is odd.

In other words, in all cases, except where

- k_1 and k_2 are commensurable,
- p_2 is odd,
- m is an odd multiple of p_1,

it follows that $\mathrm{Im}(z_m^{\pm})'(0) \neq 0$. Therefore, $\mathrm{Im}(z_m^{\pm})'(r) \neq 0$ for sufficiently small $r > 0$, and thus, $\mathrm{Im}(x_{m,s^-})'(t) \neq 0$ for sufficiently large negative t. $\square$

5. **The nature of the singularity in the exceptional case.** In this section a more detailed analysis is undertaken of the singularity of $u^{\pm}$ in the exceptional case (see Definition 3.1). As described just after this definition, the singularity at $(\frac{1}{2} + q)\lambda\pi i$, for any $q \in \mathbb{Z}$, corresponds to a fourth-order zero of $F^{\pm}(\cdot, 0)$ and is approached by four simple zeros of $F^{\pm}$ as $t \to 0$. The goal is to understand the behavior of a smooth curve of zeros of $F(\cdot, t)$ in a neighborhood of such a fourth-order zero.

We claim it suffices to analyse the singularity at $\frac{\lambda}{2}\pi i$, i.e. the case $q = 0$. Indeed, since $F^{\pm}(\cdot, t)$ is $2\pi\lambda i$ periodic, it is enough to consider $q = -1, 0$. Furthermore, $F^{\pm}(\overline{x}, t) = \overline{F^{\pm}(x, t)}$, and so $F^{\pm}(x(t), t) = 0$ if and only if $F^{\pm}(\overline{x}(t), t) = 0$. Hence, only the case $q = 0$ need be examined.

Remark that

$$F^{\pm}(x, t) = e^{-2(k_1+k_2)x+2(k_1^3+k_2^3)t} F^{\pm}(-x, -t).$$

From this, it is deduced that $F^{\pm}(x(t), t) = 0$ if and only if $F^{\pm}(-\overline{x}(t), -t) = 0$. In other words, if $x(t)$ is a curve of zeros approaching $\frac{\lambda}{2}\pi i$ as $t \nearrow 0$, then $-\overline{x}(t)$ is a curve of zeros approaching $\frac{\lambda}{2}\pi i$ as $t \searrow 0$.

Theorem 5.1. *Suppose we are in the exceptional case wherein k_1 and k_2 are commensurable and the odd integers $p_1, p_2 \in \mathbb{N}$ and $\lambda > 0$ are as in (3.2) and (3.3). In the case of F^+, it is assumed that $p_1 + p_2 \in 4\mathbb{N}$ and in the case of F^-, it is presumed that $p_2 - p_1 \in 4\mathbb{N}$. Let $x(t)$ be a smooth curve, defined for t close to 0, such that $F^{\pm}(x(t), t) = 0$ and $x(t) \neq \frac{\lambda}{2}\pi i$ but $x(t) \to \frac{\lambda}{2}\pi i$ as $t \to 0$. Then, either*

$$\lim_{t\to 0} \frac{(x(t) - \frac{\lambda}{2}\pi i)^3}{t} = -12, \tag{5.1}$$

or

$$\lim_{t\to 0} \frac{x(t) - \frac{\lambda}{2}\pi i}{t} = k_1^2 + k_2^2. \tag{5.2}$$

Remark 5.1. The similarity of the result in Theorem 5.1 with the "exceptional case" for the two–soliton solution of the KdV–equation is quite striking. The behavior of the curves described in (5.1) is exactly the same as given by Proposition 4.9 in [9]. However, the exceptional case for the KdV–equation corresponds to p_1 being an odd integer, and p_2 being an even integer. For the mKdV–equation, the "exceptional pole" is located at $\frac{\lambda}{2}\pi i$, rather than at $\lambda\pi i$ as it is for the KdV–equation.

Also, how does one explain the behavior described by (5.2)? In the case of the two–soliton solution of the KdV–equation, if p_1 is even and p_2 is odd, there is a horizontally moving pole approaching the singularity at $\pi\lambda i$ exactly as described by (5.2). This calculation was not carried out in [9], but can be obtained by a simple modification of the proof of Proposition 4.9 in [9]. Thus, it appears that the exceptional case for the two–soliton solution of the mKdV–equation (2.1) includes the behavior of curves of singularities from two different cases of the two–soliton solution of the KdV–equation, namely the "exceptional case", where p_1 is odd and p_2 is even, as well as the case where p_1 is even and p_2 is odd. It is precisely in these two cases that there is a pole located at the same place $\pi\lambda i$ at time 0.

Proof. The proof is provided for F^-. Similar arguments apply for F^+. Let $z(r) = x(t) - k_1^2 t$ where r is defined in (4.3). It follows that $z(r)$ is a smooth curve such that $H^-(z(r),r)=0$, where H^- is as in (4.5). Notice that $z(r)\to\frac{\lambda}{2}\pi i$ as $r\to 1$. Now,

$$H^-(z,r)=0 \Leftrightarrow 1+re^{-(k_1+k_2)z}=\pm i\gamma(e^{-k_1z}-re^{-k_2z}). \tag{5.3}$$

If the minus sign is chosen on the right–hand side of (5.3), we come to

$$r=-\frac{i\gamma e^{-k_1z}+1}{e^{-(k_1+k_2)z}-i\gamma e^{-k_2z}}.$$

Differentiating this relation with respect to r, it follows that

$$\begin{aligned}1=\frac{1}{(e^{-(k_1+k_2)z}-i\gamma e^{-k_2z})^2}\Big[&i\gamma k_1e^{-k_1z}\big(e^{-(k_1+k_2)z}-i\gamma e^{-k_2z}\big)\\ &+\big(i\gamma e^{-k_1z}+1\big)(-(k_1+k_2)e^{-(k_1+k_2)z}+i\gamma k_2e^{-k_2z}\big)\Big]z'(r).\end{aligned}$$

This may be rewritten as

$$1=\frac{i\gamma k_2e^{-k_2z}\big(1+ie^{-k_1z}\big)^2}{\big(e^{-(k_1+k_2)z}-i\gamma e^{-k_2z}\big)^2}z',$$

or what is the same,

$$z'=-\frac{ie^{k_2z}\big(e^{-(k_1+k_2)z}-i\gamma e^{-k_2z}\big)^2}{k_2\gamma\big(1+ie^{-k_1z}\big)^2}. \tag{5.4}$$

Since p_1 and p_2 are both odd with $p_2-p_1\in 4\mathbb{N}$ and $z(r)\to\frac{\lambda}{2}\pi i$ as $r\to 1$, it must be that $e^{-k_1z}\to e^{-p_1\pi i/2}$ and $e^{-k_2z}\to e^{-p_2\pi i/2}$ as $r\to 1$. If p_1 and p_2 are both in $4\mathbb{N}+1$, then e^{-k_1z} and e^{-k_2z} both converge to $-i$ as $r\to 1$, while if p_1 and p_2 are both in $4\mathbb{N}+3$, then e^{-k_1z} and e^{-k_2z} both converge to i as $r\to 1$. We suppose first that p_1 and p_2 are both in $4\mathbb{N}+1$. In this case, it follows from (5.4) that

$$z'(r)\to\frac{(\gamma+1)^2}{4k_2\gamma}=\frac{k_2}{k_2^2-k_1^2} \tag{5.5}$$

as $r\to 1$, and thus

$$\lim_{r\to 1}\frac{z(r)-\frac{\lambda}{2}\pi i}{r-1}=\frac{k_2}{k_2^2-k_1^2}.$$

Turning back to $x(t)$, and using the fact that $(r-1)/t\to k_2(k_2^2-k_1^2)$ as $t\to 0$, it follows that

$$\lim_{t\to 0}\frac{x(t)-k_1^2t-\frac{\lambda}{2}\pi i}{t}=k_2^2$$

from which it is concluded that

$$\lim_{t\to 0}\frac{x(t)-\frac{\lambda}{2}\pi i}{t}=k_1^2+k_2^2.$$

If instead the plus sign is chosen on the right–hand side of (5.3), then it is immediately inferred that

$$r=\frac{i\gamma e^{-k_1z}-1}{e^{-(k_1+k_2)z}+i\gamma e^{-k_2z}}.$$

Differentiating this equation with respect to r, leads to the formula

$$\begin{aligned}1=\frac{1}{(e^{-(k_1+k_2)z}+i\gamma e^{-k_2z})^2}\Big[&-i\gamma k_1e^{-k_1z}\big(e^{-(k_1+k_2)z}+i\gamma e^{-k_2z}\big)\\ &-\big(i\gamma e^{-k_1z}-1\big)\big(-(k_1+k_2)e^{-(k_1+k_2)z}-i\gamma k_2e^{-k_2z}\big)\Big]z',\end{aligned}$$

which may be simplified to

$$1 = -\frac{i\gamma k_2 e^{-k_2 z}(1 - ie^{-k_1 z})^2}{(e^{-(k_1+k_2)z} + i\gamma e^{-k_2 z})^2} z'.$$

From this, it is inferred that

$$(1 - ie^{-k_1 z})^2 z' = \frac{ie^{k_2 z}}{k_2 \gamma}(e^{-(k_1+k_2)z} + i\gamma e^{-k_2 z})^2. \tag{5.6}$$

Since p_1 and p_2 are both in $4\mathbb{N}+1$, it is concluded that

$$(1 - ie^{-k_1 z})^2 z' \to -\frac{(\gamma - 1)^2}{k_2 \gamma} = \frac{-4k_1^2}{k_2(k_2^2 - k_1^2)} \tag{5.7}$$

as $r \to 1$. A consequence of this is that

$$\frac{d}{dr}(1 - ie^{-k_1 z})^3 = 3(1 - ie^{-k_1 z})^2 (ik_1 e^{-k_1 z})z' \to \frac{-12k_1^3}{k_2(k_2^2 - k_1^2)}.$$

L'Hopital's rule comes to the rescue and it is found that

$$\lim_{r \to 1} \frac{(1 - ie^{-k_1 z})^3}{r - 1} = \frac{-12k_1^3}{k_2(k_2^2 - k_1^2)}.$$

Since

$$\lim_{z \to \frac{\lambda}{2}\pi i} \frac{1 - ie^{-k_1 z}}{z - \frac{\lambda}{2}\pi i} = -i \lim_{z \to \frac{\lambda}{2}\pi i} \frac{e^{-k_1 z} - e^{-k_1 \frac{\lambda}{2}\pi i}}{z - \frac{\lambda}{2}\pi i} = -ik_1 e^{-k_1 \frac{\lambda}{2}\pi i} = k_1,$$

it follows that

$$\lim_{r \to 1} \frac{(z(r) - \frac{\lambda}{2}\pi i)^3}{r - 1} = \frac{-12}{k_2(k_2^2 - k_1^2)}.$$

Reverting to the original variable $x(t)$ and using again that $(r-1)/t \to k_2(k_2^2 - k_1^2)$ as $t \to 0$, it follows that

$$\lim_{t \to 0} \frac{(x(t) - k_1^2 t - \frac{\lambda}{2}\pi i)^3}{t} = -12$$

and thus

$$\lim_{t \to 0} \frac{(x(t) - \frac{\lambda}{2}\pi i)^3}{t} = -12.$$

It remains to treat the situation where $p_1, p_2 \in 4\mathbb{N}+3$, in which case $e^{-k_j z(r)} \to i$ as $r \to 1$. In fact, this case is "dual" to the one just treated and the same calculations lead to the result. To see this, assume first that the minus sign obtains on the right–hand side of (5.3). Then (5.4) implies that

$$(1 + ie^{-k_1 z})^2 z' = -\frac{ie^{k_2 z}}{k_2 \gamma}(e^{-(k_1+k_2)z} - i\gamma e^{-k_2 z})^2 \to -\frac{(\gamma - 1)^2}{k_2 \gamma} = \frac{-4k_1^2}{k_2(k_2^2 - k_1^2)}$$

as $r \to 1$. On the other hand, it is straightforward that

$$\lim_{z \to \frac{\lambda}{2}\pi i} \frac{1 + ie^{-k_1 z}}{z - \frac{\lambda}{2}\pi i} = k_1.$$

Following the same line of development as pursued for the positive sign in the previous situation where both p_1 and p_2 lie in $4\mathbb{N}+1$ leads to

$$\lim_{t \to 0} \frac{(x(t) - \frac{\lambda}{2}\pi i)^3}{t} = -12.$$

Now assume we have the positive sign on the right–hand side of (5.3). From (5.6) we deduce

$$z' = \frac{ie^{k_2 z}}{k_2\gamma}\frac{(e^{-(k_1+k_2)z} + i\gamma e^{-k_2 z})^2}{(1-ie^{-k_1 z})^2} \to \frac{(\gamma+1)^2}{4k_2\gamma} = \frac{k_2}{k_2^2 - k_1^2}$$

as $r \to 1$. Then, it is clear that

$$\lim_{t\to 0}\frac{x(t) - \frac{\lambda}{2}\pi i}{t} = k_1^2 + k_2^2.$$

□

6. **Vertical movement of poles.** In this section, the vertical motion of the singularities of the two–soliton solutions $u^{\pm}$ of (2.1) is studied. As seen in Section 2, this comes down to studying the zeroes of $F^{\pm}$ given by (2.17)–(2.18). Recall the solution u^+, given by (2.13), represents two interacting solitons of the same sign, while u^-, given by (2.14), represents two interacting solitons of opposite sign. Also, as we've already shown (Proposition 3.1), these singularities are described by analytic curves $x(t)$, except for certain singularities at $t = 0$ in the exceptional case. By vertical movement, we mean that $\operatorname{Im} x'(t) \neq 0$. This contrasts with the movement of poles of the single soliton solution, which is entirely horizontal : $\operatorname{Im} x'(t) \equiv 0$, as deduced from (3.1).

It follows from Proposition 4.2 that all the two–soliton solutions $u^{\pm}$ of (2.1) have some singularities with nonzero vertical movement for large (positive and negative) time. Our goal in this section is to study this phenomenon more carefully. In fact we prove below, in the noncommensurable case and in the commensurable case with p_1 and p_2 of opposite parity, that if $x(t)$ is a smooth curve of singularities of $u^{\pm}$, and if either $t \neq 0$ or if $\operatorname{Re} x(t) \neq 0$, then $\operatorname{Im} x'(t) \neq 0$. See Corollary 6.1. In the commensurable case where p_1 and p_2 are both odd, but not the exceptional case, the same is true, except for certain poles which move only horizontally. See Corollary 6.2. The situation in the exceptional case (Definition 3.1) is more delicate, and thus (in order to minimize tedious calculations) is not treated here.

To carry out the analysis needed to establish these results, we begin with the observation that

$$F^+(x,t) = F_1^+(x,t)F_2^+(x,t) \tag{6.1}$$

$$F^-(x,t) = F_1^-(x,t)F_2^-(x,t) \tag{6.2}$$

where

$$F_1^+ = 1 + i\gamma f_1 + i\gamma f_2 - f_1 f_2 \tag{6.3}$$

$$F_2^+ = 1 - i\gamma f_1 - i\gamma f_2 - f_1 f_2 \tag{6.4}$$

$$F_1^- = 1 + i\gamma f_1 - i\gamma f_2 + f_1 f_2 \tag{6.5}$$

$$F_2^- = 1 - i\gamma f_1 + i\gamma f_2 + f_1 f_2 \tag{6.6}$$

and f_1 and f_2 are as in (2.9). Note the similarity in form between the above functions and the function F defined by formula (2.13) in [9].

Since $F_1^+(x,t) = 0$ if and only if $F_2^+(\overline{x},t) = 0$, to study the zeros of F^+, it suffices to study the zeros of F_1^+. Similarly, since $F_1^-(x,t) = 0$ if and only if $F_2^-(\overline{x},t) = 0$, to study the zeros of F^-, it suffices to study the zeros of F_1^-.

As before, we distinguish between the cases where k_1 and k_2 are commensurable and the cases where they are not. In the former case, we use the same notation already established : p_1 and p_2 are the relatively prime positive integers such that (3.2) holds, and let λ be defined as in (3.3). Hence, if k_1 and k_2 are commensurable, the functions f_1 and f_2 are $2\pi\lambda i$ periodic in x. Thus, in the commensurable case, all four of the functions $F_1^{\pm}, F_2^{\pm}$, must also be $2\pi\lambda i$ periodic in x.

Proposition 6.1. *The zeroes of F^+ and F^- in the complex plane lie off the real axis. Moreover, in the commensurable case, if either $F^+(x,t)=0$ or $F^-(x,t)=0$, then $\operatorname{Im} x \neq 2m\pi\lambda$ for all $m\in\mathbb{Z}$.*

Proof. Suppose $x\in\mathbb{R}$ and $F^+(x,t)=0$. It follows that $F_1^+(x,t)=F_2^+(x,t)=0$. Taking real and imaginary parts produces the coupled system

$$\begin{aligned} 1 &= f_1 f_2, \\ f_1 &= -f_2, \end{aligned}$$

from which it follows that $f_1(x,t)^2=-1$, which is impossible. If $x\in\mathbb{R}$ and $F^-(x,t)=0$, it similarly follows that

$$\begin{aligned} 1 &= -f_1 f_2, \\ f_1 &= f_2, \end{aligned}$$

from which one again observes that $f_1(x,t)^2=-1$, which is again impossible.

The last statement follows from the $2\pi\lambda i$ periodicity which holds in the commensurable case. ☐

If k_1 and k_2 are commensurable, further information about the location of the zeros of $F^{\pm}$ can be obtained.

Proposition 6.2. *Suppose that k_1 and k_2 are commensurable. If either $F^+(x,t)=0$ or $F^-(x,t)=0$, then $\operatorname{Im} x\neq(2m+1)\pi\lambda$ for any $m\in\mathbb{Z}$.*

Proof. Because of $2\pi\lambda i$–periodicity, it suffices to consider $m=0$. Suppose $\operatorname{Im} x=\pi\lambda$ and $F_1^+(x,t)=0$. Since $x-\overline{x}=2\pi\lambda i$, it follows from $2\pi\lambda i$-periodicity that $F_1^+(\overline{x},t)=0$. But this says precisely that $F_2^+(x,t)=0$. Adding and subtracting the two equations, $F_1^+(x,t)=0$ and $F_2^+(x,t)=0$ gives

$$\begin{aligned} 1 &= f_1 f_2, \\ f_1 &= -f_2, \end{aligned}$$

from which it is concluded that $f_1(x,t)^2=-1$, i.e. $f_1(x,t)=\pm i$ and $f_2(x,t)=\mp i$. The latter system together with the definition (2.9) of the f_j's implies

$$\begin{aligned} \operatorname{Re} x &= k_1^2 t, \\ \operatorname{Re} x &= k_2^2 t, \end{aligned}$$

and so $t=\operatorname{Re} x=0$. Thus $x=\pi\lambda i$. However,

$$f_1(\pi\lambda i,0)=\exp(-k_1\pi\lambda i)=\exp(-p_1\pi i)=\pm 1,$$

since p_1 is an integer. This contradiction shows that $F_1^+(x,t)$ cannot be zero if $\operatorname{Im} x=\pi\lambda$. A similar argument applies to the other functions F_2^+, F_1^- and F_2^-. ☐

To give a complete analysis of the vertical movement of poles of the two–soliton solutions of (2.1), one would have to reproduce calculations which are similar to those found in Section 3 of [9]. Unfortunately, the calculations in [9] do not seem to directly apply to the present situation in all cases. In an effort at economy, we do not treat in detail the vertical movement of the poles in the exceptional case

(Definition 3.1), beyond what is already established in Proposition 4.2. In the other cases, we prove here that, with certain very specific exceptions, the poles in the two–soliton solution always have nonzero vertical movement. While it is not necessary to reproduce the entirety of Section 3 of [9], certain, somewhat tedious calculations, closely modeled on those in [9], cannot be avoided.

To keep the calculations from being any more cumbersome than necessary, a detailed analysis is carried out only for the solution u^+. Analogous results are easily proved for u^- using very similar calculations, as will be explained in more detail in the proof of Corollary 6.1 below.

The following notation

$$\alpha = -\operatorname{Im} x, \tag{6.7}$$

$$A_1 = e^{-k_1 \operatorname{Re} x + k_1^3 t}, \tag{6.8}$$

$$A_2 = e^{-k_2 \operatorname{Re} x + k_2^3 t}, \tag{6.9}$$

is taken from [9]. The function F_1^+ may be rewritten in this notation, *viz.*

$$F_1^+(x,t) = 1 + i\gamma A_1 e^{ik_1\alpha} + i\gamma A_2 e^{ik_2\alpha} - A_1 A_2 e^{i(k_1+k_2)\alpha}. \tag{6.10}$$

To investigate possible vertical movement of poles of u^+, we examine more closely the zeros of $F_1^+(x,t)$.

Proposition 6.3. *Suppose $F_1^+(x,t) = 0$. Then, $\cos k_1\alpha = 0$ if and only if $\cos k_2\alpha = 0$. Moreover, the relation*

$$\Big(A_2 - \frac{1}{A_2}\Big)\cos k_1\alpha + \Big(A_1 - \frac{1}{A_1}\Big)\cos k_2\alpha = 0 \tag{6.11}$$

always holds. In case $\cos k_1\alpha \neq 0$ and $\cos k_2\alpha \neq 0$, then $A_1 = 1$ if and only if $A_2 = 1$, and this can only happen if $t = 0$ and $\operatorname{Re} x = 0$.

Proof. First, multiply the equation $F_1^+(x,t) = 0$ by $1 - i\gamma A_1 e^{-ik_1\alpha}$, express this using the representation (6.10) and take the imaginary part of the result. This leads to the formula

$$\Big(A_1 + \frac{1}{A_1}\Big)\cos k_2\alpha = \frac{1}{\gamma}\sin(k_2+k_1)\alpha - \gamma\sin(k_2-k_1)\alpha. \tag{6.12}$$

In the same way, multiplying by $1 - i\gamma A_2 e^{-ik_2\alpha}$ and subsequently extracting the imaginary part yields

$$\Big(A_2 + \frac{1}{A_2}\Big)\cos k_1\alpha = \frac{1}{\gamma}\sin(k_2+k_1)\alpha + \gamma\sin(k_2-k_1)\alpha. \tag{6.13}$$

If $\cos k_2\alpha = 0$, so that in particular $\sin k_2\alpha = \pm 1$, it follows from (6.12), using the formulas for the sine of the sum and difference, that $\cos k_1\alpha = 0$. In the same way, using (6.13), if $\cos k_1\alpha = 0$, then $\cos k_2\alpha = 0$. This proves the first assertion in the proposition.

For future reference, notice that it follows by subtracting (6.12) from (6.13) that

$$\gamma\sin(k_2-k_1)\alpha = \frac{1}{2}\Big(A_2 + \frac{1}{A_2}\Big)\cos k_1\alpha - \frac{1}{2}\Big(A_1 + \frac{1}{A_1}\Big)\cos k_2\alpha. \tag{6.14}$$

Taking the real and imaginary parts of (6.10) and setting them equal to zero yields

$$1 - \gamma A_1 \sin k_1\alpha - \gamma A_2 \sin k_2\alpha - A_1A_2\cos(k_1+k_2)\alpha = 0, \tag{6.15}$$

$$\gamma A_1 \cos k_1\alpha + \gamma A_2\cos k_2\alpha - A_1A_2\sin(k_1+k_2)\alpha = 0. \tag{6.16}$$

Multiply the first equation above by $\sin(k_1+k_2)\alpha$, the second equation by $\cos(k_1+k_2)\alpha$ and form the difference of the outcomes. The formula

$$\sin(k_1+k_2)\alpha = \gamma A_2 \cos k_1\alpha + \gamma A_1 \cos k_2\alpha \tag{6.17}$$

emerges from these machinations. But, (6.16) implies

$$\sin(k_1+k_2)\alpha = \frac{\gamma}{A_2}\cos k_1\alpha + \frac{\gamma}{A_1}\cos k_2\alpha. \tag{6.18}$$

The last two equations taken together imply (6.11)

This proves the second assertion of the proposition. Finally, it is clear that $A_1 = A_2 = 1$ if and only if $t = 0$ and $\operatorname{Re} x = 0$. □

Proposition 6.4. *Suppose that* $\cos k_1\alpha = \cos k_2\alpha = 0$. *It follows that* k_1 *and* k_2 *are commensurable, that* p_1 *and* p_2 *are both odd and that* α *is an odd multiple of* $\pi\lambda/2$.

Proof. If $\cos k_1\alpha = \cos k_2\alpha = 0$, then there exist integers m and n such that

$$k_1\alpha = (2m+1)\frac{\pi}{2},$$
$$k_2\alpha = (2n+1)\frac{\pi}{2}.$$

Thus it is clear that k_1 and k_2 are commensurable and

$$\frac{p_1}{p_2} = \frac{k_1}{k_2} = \frac{2m+1}{2n+1},$$

whence

$$p_1(2n+1) = p_2(2m+1). \tag{6.19}$$

It follows that $p_2 - p_1$ is even, and since they are also relatively prime, they both must be odd. Next, equation (6.19) implies that $p_1 = (2m+1)/c$, where

$$c = \gcd(2m+1, 2n+1).$$

Consequently,

$$\alpha = \frac{2m+1}{k_1}\frac{\pi}{2} = \frac{2m+1}{p_1}\frac{\pi\lambda}{2} = c\frac{\pi\lambda}{2},$$

which concludes the proof since c is necessarily odd. □

Proposition 6.5. *Let* $x(t)$ *be a smooth curve such that* $F_1^+(x(t),t) = 0$ *and* $\partial_x F_1^+(x(t),t) \neq 0$. *It follows that* $\operatorname{Im} x'(t)$ *has the same sign as*

$$\Big(A_1 - \frac{1}{A_1}\Big)\cos k_2\alpha = -\Big(A_2 - \frac{1}{A_2}\Big)\cos k_1\alpha.$$

Proof. Let $x(t)$ be a smooth curve such that $F_1^+(x(t),t) = 0$ and $\partial_x F_1^+(x(t),t) \neq 0$. The chain rule provides the relation

$$\begin{aligned} x'(t) &= -\frac{\partial_t F_1^+(x(t),t)}{\partial_x F_1^+(x(t),t)} \\ &= \frac{k_1^3 i\gamma A_1 e^{ik_1\alpha} + k_2^3 i\gamma A_2 e^{ik_2\alpha} - (k_1^3+k_2^3)A_1A_2e^{i(k_1+k_2)\alpha}}{k_1 i\gamma A_1 e^{ik_1\alpha} + k_2 i\gamma A_2 e^{ik_2\alpha} - (k_1+k_2)A_1A_2e^{i(k_1+k_2)\alpha}} \\ &= \frac{k_1^3\gamma A_1 e^{-ik_2\alpha} + k_2^3\gamma A_2 e^{-ik_1\alpha} + i(k_1^3+k_2^3)A_1A_2}{k_1\gamma A_1 e^{-ik_2\alpha} + k_2\gamma A_2 e^{-ik_1\alpha} + i(k_1+k_2)A_1A_2} \end{aligned}$$

$$= \frac{1}{\left|k_1\gamma A_1 e^{-ik_2\alpha} + k_2\gamma A_2 e^{-ik_1\alpha} + i(k_1+k_2)A_1A_2\right|^2}\Big[\big(k_1^3\gamma A_1 e^{-ik_2\alpha} + k_2^3\gamma A_2 e^{-ik_1\alpha} + i(k_1^3+k_2^3)A_1A_2\big)\big(k_1\gamma A_1 e^{ik_2\alpha} + k_2\gamma A_2 e^{ik_1\alpha} - i(k_1+k_2)A_1A_2\big)\Big].$$

The imaginary part of the numerator in this last expression is equal to

$k_1k_2(k_2^2-k_1^2)\gamma^2 A_1A_2\sin(k_2-k_1)\alpha + k_1k_2(k_1+k_2)^2A_1A_2(A_1\cos k_2\alpha - A_2\cos k_1\alpha)$,

which is a positive multiple of, and so has the same sign as,

$$\gamma\sin(k_2-k_1)\alpha + A_1\cos k_2\alpha - A_2\cos k_1\alpha. \tag{6.20}$$

Finally, formulas (6.14) and (6.11) reveal that the expression in (6.20) is equal to

$$A_1\cos k_2\alpha - \frac{1}{A_1}\cos k_2\alpha.$$

□

Corollary 6.1. *Suppose either that k_1 and k_2 are not commensurable, or that they are commensurable with p_1 and p_2 having opposite parity. Let $x(t)$ be a smooth curve such that $F_1^+(x(t),t)=0$. If either $t\neq 0$ or if $\operatorname{Re} x\neq 0$, then $\operatorname{Im} x'(t)\neq 0$. The same is true for F_2^+, F_1^-, and F_2^-.*

Proof. We know from Lemma 3.1 that the roots of $F^+(\cdot,t)$ are all simple unless k_1 and k_2 are commensurable and p_1 and p_2 are both odd. It follows that the same is true for $F_1^+(\cdot,t)$. Thus, it must be the case that $\partial_x F_1^+(x(t),t)\neq 0$. The last three propositions can now be brought to bear to establish the claim, at least for the function F_1^+.

To establish the result for the three other functions, one notes that the calculations in the proofs of Propositions 6.3 and 6.5 treat the coefficients A_1 and A_2 as objects in and of themselves, with no relation to the variables x or t or the parameters k_1 and k_2. Moreover, the expressions for these three other functions, F_2^+, F_1^-, and F_2^-, analogous to the expression (6.10), for F_1^+ are obtained from (6.10) by formally replacing A_1 by $-A_1$ and/or A_2 by $-A_2$. Thus, these two propositions are valid for the three other functions, modulo a change of sign in either or both of A_1 and A_2. The conclusion is still that $\operatorname{Im} x'(t)\neq 0$. □

Remark 6.1. To recapitulate, it has been shown, except in the commensurable case with p_1 and p_2 both odd, that if either $t\neq 0$ or if $\operatorname{Re} x\neq 0$, then $\operatorname{Im} x'(t)\neq 0$, so the poles of the solution u^+ (and of the solution u^-) are *always* moving vertically. The same is true in the case that p_1 and p_2 are both odd, so long as the imaginary part of the pole is not an odd multiple of $\pi\lambda/2$.

Interestingly, in the commensurable case when p_1 and p_2 have opposite parity, it turns out that the poles of u^- and those of u^+ bear a very simple relationship to one another.

Proposition 6.6. *Suppose that p_1 and p_2 have opposite parity (one odd, one even). It follows that there exists $\theta\in\mathbb{R}$ such that*

$$F^+(x-i\theta,t) = F^-(x,t) \tag{6.21}$$

for all $x\in\mathbb{C}$ and $t\in\mathbb{R}$. In other words, the poles of u^- are precisely given by a vertical translation of the poles of u^+.

Proof. Suppose $\theta \in \mathbb{R}$ is such that

$$f_1(x - i\theta, t) = f_1(x, t), \tag{6.22}$$

$$f_2(x - i\theta, t) = -f_2(x, t), \tag{6.23}$$

for all $x \in \mathbb{C}$ and $t \in \mathbb{R}$. It would then follow that (6.21) holds for all $x \in \mathbb{C}$ and $t \in \mathbb{R}$. The same would be true if we had instead

$$f_1(x - i\theta, t) = -f_1(x, t), \tag{6.24}$$

$$f_2(x - i\theta, t) = f_2(x, t). \tag{6.25}$$

For (6.22) and (6.23) to be valid, it is necessary and sufficient that

$$\exp(ik_1\theta) = \exp(ip_1\theta/\lambda) = 1, \tag{6.26}$$

$$\exp(ik_2\theta) = \exp(ip_2\theta/\lambda) = -1. \tag{6.27}$$

For these latter conditions to hold, it is necessary for there to be two integers m and n such that

$$p_1\theta/\lambda = 2m\pi, \tag{6.28}$$

$$p_2\theta/\lambda = (2n+1)\pi, \tag{6.29}$$

or, what is the same,

$$\frac{\theta}{\lambda\pi} = \frac{2m}{p_1} = \frac{2n+1}{p_2}. \tag{6.30}$$

If p_1 is even and p_2 is odd, it is clear that one may choose appropriate values of θ, m and n so that the last equation holds

In the opposite case, if p_1 is odd and p_2 is even, a similar argument shows that there exists θ, m and n such that (6.24) and (6.25) hold. □

It remains to consider the situation wherein k_1 and k_2 are commensurable, with p_1 and p_2 odd, but excluding the exceptional case. For the two-soliton solution u^+, this means $p_2 - p_1 \in 4\mathbb{N}$, and for the two-soliton solution u^-, this means $p_2 + p_1 \in 4\mathbb{N}$. In these two cases, it turns out that the movement of the poles can in fact be reduced to the movement of the poles of the two–soliton solution of the KdV–equation (1.2). The following propositions give the relationship between the functions $F^{\pm}$ and the analogous function F given by formula (2.13) in [9]. As the proofs are similar, we provide a proof only of the first of the two proposition.

Proposition 6.7. *Suppose that p_1 and p_2 are both odd. If $p_2 - p_1 \in 4\mathbb{N}$ then there exist $\theta_1, \theta_2 \in \mathbb{R}$ such that*

$$F_1^+(x - i\theta_1, t) = 1 + \gamma f_1(x,t) + \gamma f_2(x,t) + f_1(x,t)f_2(x,t) \tag{6.31}$$

and

$$F_2^+(x - i\theta_2, t) = 1 + \gamma f_1(x,t) + \gamma f_2(x,t) + f_1(x,t)f_2(x,t) \tag{6.32}$$

for all $x \in \mathbb{C}$ and $t \in \mathbb{R}$.

Proposition 6.8. *Suppose that p_1 and p_2 are both odd. If $p_2 + p_1 \in 4\mathbb{N}$ then there exist $\theta_1, \theta_2 \in \mathbb{R}$ such that*

$$F_1^-(x - i\theta_1, t) = 1 + \gamma f_1(x,t) + \gamma f_2(x,t) + f_1(x,t)f_2(x,t) \tag{6.33}$$

and

$$F_2^-(x - i\theta_2, t) = 1 + \gamma f_1(x,t) + \gamma f_2(x,t) + f_1(x,t)f_2(x,t) \tag{6.34}$$

for all $x \in \mathbb{C}$ and $t \in \mathbb{R}$.

Proof. (Proof of Proposition 6.7) Suppose $\theta_1 \in \mathbb{R}$ is such that

$$f_1(x - i\theta_1, t) = -if_1(x, t), \tag{6.35}$$
$$f_2(x - i\theta_1, t) = -if_2(x, t), \tag{6.36}$$

for all $x \in \mathbb{C}$ and $t \in \mathbb{R}$. It would follow that (6.31) holds for all $x \in \mathbb{C}$ and $t \in \mathbb{R}$. In addition, if we have

$$f_1(x - i\theta_2, t) = if_1(x, t), \tag{6.37}$$
$$f_2(x - i\theta_2, t) = if_2(x, t), \tag{6.38}$$

then (6.32) would be true.

For the system (6.35)–(6.36) to be valid, it is necessary and sufficient that

$$\exp(ik_1\theta) = \exp(ip_1\theta/\lambda) = -i, \tag{6.39}$$
$$\exp(ik_2\theta) = \exp(ip_2\theta/\lambda) = -i. \tag{6.40}$$

For this to hold, there must be two integers m and n such that

$$\frac{p_1\theta}{\lambda} = (4m - 1)\frac{\pi}{2}, \tag{6.41}$$
$$\frac{p_2\theta}{\lambda} = (4n - 1)\frac{\pi}{2}, \tag{6.42}$$

which is the same as asking for two integers m and n such that

$$\frac{2\theta}{\lambda\pi} = \frac{4m - 1}{p_1} = \frac{4n - 1}{p_2}. \tag{6.43}$$

For this to be true, it is necessary that

$$\frac{4(p_2m - p_1n)}{p_2 - p_1} = 1. \tag{6.44}$$

Since p_1 and p_2 are relatively prime, there exist integers r and s such that

$$rp_2 + sp_1 = 1;$$

hence, simply take

$$m = \frac{(p_2 - p_1)r}{4},$$
$$n = -\frac{(p_2 - p_1)s}{4}.$$

The proof for F_2^+ is similar, but with $4m+1$ replacing $4m-1$ and $4n+1$ replacing $4n - 1$. □

The expression on the right side of the four formulas (6.31), (6.32), (6.33), and (6.34), i.e.

$$1 + \gamma f_1(x, t) + \gamma f_2(x, t) + f_1(x, t)f_2(x, t), \tag{6.45}$$

is exactly the function F in formula (2.13) of [9] whose zeros correspond to the poles of the 2–soliton solution of the KdV–equation. Thus, the results of [9] may be used to describe the behavior of the poles of $u^{\pm}$ in the cases under consideration.

More precisely, we may now affirm the following.

Corollary 6.2. *In the commensurable case, if p_1 and p_2 are both odd and $p_2 - p_1 \in 4\mathbb{N}$, the solution u^+ has two poles moving horizontally on the line $\operatorname{Im} x = \pi\lambda/2$ and also on the line $\operatorname{Im} x = -\pi\lambda/2$. There are $2(p_1 + p_2 - 2)$ other poles with imaginary part between $\pm\pi\lambda$. Each such pole $x(t)$ will move vertically for all $t \neq 0$, and for all $t \in \mathbb{R}$ if $\operatorname{Re} x(0) \neq 0$. The same is true for u^- in the case p_1 and p_2 are both odd if $p_2 + p_1 \in 4\mathbb{N}$. These configurations will repeat with $2\pi\lambda i$ periodicity, so that horizontally moving poles are found with imaginary part equal to every odd multiple of $\pi\lambda/2$.*

This section closes with a proposition giving more precise information about the location of singularities of the two–soliton solutions. Its main interest is that it can be used to show that the finite time blowup which will be discussed in the following section is in fact *single point* blowup.

Proposition 6.9. *Suppose either that k_1 and k_2 are not commensurable, or that they are commensurable with p_1 and p_2 having opposite parity. Fix $t \neq 0$ and let $\alpha \in \mathbb{R}$. There is at most one $x \in \mathbb{C}$ with $\operatorname{Im} x = -\alpha$ such that $F^+(x,t) = 0$. The same is true for F^-. In other words, if $t \neq 0$, the singularities of $u^\pm$ all have different imaginary parts.*

Proof. Here is a proof for F^+. Recall that from inequality (6.1), we have $F^+(x,t) = F_1^+(x,t)F_2^+(x,t)$. We claim that $F_1^+(\cdot,t)$ and $F_2^+(\cdot,t)$ can not have zeros with the same imaginary part (for $t \neq 0$). Indeed, if one carries out the proof of Proposition 6.3, but for the case of F_2^+, the formula one obtains, analogous to formula (6.12), is

$$-\left(A_1 + \frac{1}{A_1}\right)\cos k_2\alpha = \frac{1}{\gamma}\sin(k_2+k_1)\alpha - \gamma\sin(k_2-k_1)\alpha. \tag{6.46}$$

Since the condition $t \neq 0$ implies that $\cos k_2\alpha \neq 0$ by Propositions 6.3 and 6.4, it is easily seen using (6.12) and (6.46) that roots of $F_1^+(\cdot,t)$ and $F_2^+(\cdot,t)$ must have different imaginary parts. This proves our claim.

We next show that there can not be two different zeros of $F_1^+(\cdot,t)$ with the same imaginary part. (The case of $F_2^+(\cdot,t)$ is handled similarly.) Formulas (6.12) and (6.13) show that A_1 and A_2 are explicitly determined, up to reciprocal, in terms of α. We know that neither is equal to 1 by Proposition 6.3 since $t \neq 0$. Thus, given α, there are only four possible couples (A_1, A_2). Furthermore, Proposition 6.3, in particular formula (6.11), shows that only two of the four couples are in fact possible, related by taking the reciprocals of *both* A_1 and A_2. Finally, formulas (6.8) and (6.9) imply that

$$\operatorname{Re} x = \frac{k_1^3\log A_1 - k_2^3\log A_2}{k_1k_2(k_2^2 - k_1^2)}. \tag{6.47}$$

If one replaces A_1 and A_2 by their reciprocals in the above formula, the result is to replace $\operatorname{Re} x$ by $-\operatorname{Re} x$. On the other hand, since $t \neq 0$, replacing $\operatorname{Re} x$ by $-\operatorname{Re} x$ in (6.8) and (6.9) does *not* correspond to replacing A_1 and A_2 by their reciprocals. This shows there can be only one possible value of $\operatorname{Re} x$, thus concluding the proof. □

Remark 6.2. A careful examination of the arguments in this section reveals that the above corollary is still valid in the commensurable case with p_1 and p_2 both odd, as long as α is not an odd multiple of $\pi\lambda/2$.

7. **Finite time blowup of solutions.** In this section, the results of the previous sections are shown to imply that there exist complex–valued solutions to (2.1) on $\mathbb{R}$ which blow up in finite time. Our main result, Proposition 7.1 below, exhibits regular, complex-valued solutions of (2.1) which develop a unique singularity at a given time T (i.e. "single-point blowup"), and then continue as a smooth solution after the blowup time. The rate of blowup and the blowup profile are given explicitly. These solutions are constructed from the two soliton solutions $u^{\pm}$ studied above.

To simplify notation, in what follows we let u denote either u^+, given by (2.13), or u^-, given by (2.14), a two–soliton solution of (2.1) of either type (interacting solitons of the same or opposite sign). Similarly, F denotes the corresponding function F^+ given by (2.17), in the case of u^+, or the function F^- given by (2.18), in the case of u^-. Also, in a change of notation from what has come before, let $z(t)$ denote an analytic curve of singularities of u, rather than $x(t)$.

Lemma 7.1. *Let $z(t)$ be an analytic curve of singularities of u, as described by Proposition 3.1, and fix a $T \in \mathbb{R}$ where z is defined. The function u is given locally near the singular point $(z(T), T)$ by*

$$u(x,t) = \frac{\pm i}{x - z(t)} + (x - z(t))w(x,t), \tag{7.1}$$

where w is a smooth function in a neighborhood of $(z(T), T)$.

Proof. Since $z(t)$ is a simple pole of u for each t, we necessarily have the local expansion

$$u(x,t) = \frac{r(t)}{x - z(t)} + v(x,t),$$

where r and v are regular functions. Substituting this expression into the mKdV equation (2.1), and examining the coefficients of the most singular terms, one immediately sees that $r(t)$ must be constant, and in fact equal to $\pm i$. Looking at the next most singular terms, it becomes apparent that $v(x,t)$ must have a factor of $x - z(t)$. This concludes the proof. □

Lemma 7.2. *Let $z(t)$ be an analytic curve of singularities of u, as described by Proposition 3.1. Fix $T \in \mathbb{R}$ where z is defined, and denote*

$$z(T) = x_0 - i\alpha, \tag{7.2}$$

where $x_0, \alpha \in \mathbb{R}$. Suppose that

(i) $\operatorname{Im} z'(T) \neq 0$*;*

(ii) $z(T)$ is the only singularity of u at time T with imaginary part equal to $-\alpha$.

It follows that there exists $\epsilon > 0$ such that $u(x,t)$ has no singularities on the set $\{(x,t) \in \mathbb{C} \times \mathbb{R} : |\operatorname{Im} x + \alpha| < \epsilon, |T - t| < \epsilon\}$ other than the curve $(z(t), t)$. In particular, choosing $\epsilon > 0$ perhaps smaller, if $0 < |T - t| < \epsilon$, then $u(x,t)$ has no singularities on the horizontal line $\operatorname{Im} = -\alpha$ *in* $\mathbb{C}$.

Proof. If this were not true, there would exist a sequence $t_n \to T$ and singularities $y(t_n)$ of $u(\cdot, t_n)$, i.e. $F(y(t_n), t_n) = 0$, so that $\operatorname{Im} y(t_n) \to -\alpha$, but such that the points $(y(t_n), t_n)$ are not on the curve $(z(t), t)$. The sequence $\{y(t_n)\}_{n=1}^{\infty}$ must be bounded, by the same argument as used in the proof of Proposition 3.1. Thus, a subsequence must converge as $n \to \infty$ to some point y with $\operatorname{Im} y = -\alpha$ and $F(y, T) = 0$. By hypothesis (ii), it follows that $y = z(T)$. However, by the implicit function theorem (see the proof of Proposition 3.1), all the zeros of F in a neighborhood of the point $(z(T), T)$ in $\mathbb{C} \times \mathbb{R}$ are given by the curve $(z(t), t)$. Thus, for sufficiently large n, the points $(y(t_n), t_n)$ must lie on this curve.

The last statement follows since $\operatorname{Im} z'(T) \neq 0$. $\square$

Remark 7.1. All two–soliton solutions u described in this paper exhibit singularities of the type described in Lemma 7.2, where $|T|$ is chosen sufficiently large. This follows from Theorem 4.1 and Proposition 4.2.

In addition, suppose either that k_1 and k_2 are not commensurable, or that they are commensurable with p_1 and p_2 having opposite parity. Then every curve $z(t)$ of singularities of u satisfies the conditions of Lemma 7.2 for every $T \neq 0$. This is a consequence of Corollary 6.1 and Proposition 6.9.

Similar phenomena occur in the other commensurable cases, but not for all curves of singularities. This is suggested by the Remarks 6.1 and 6.2 above. In the interests of keeping the analysis to a reasonable length, we do not carry out the detailed calculations necessary make this more precise.

Proposition 7.1. *Let $z(t)$ be an analytic curve of singularities of u satisfying the hypotheses of Lemma 7.2. In particular, $z(T) = x_0 - i\alpha$, where $x_0, \alpha \in R$. Let u_α be given by*

$$u_\alpha(x, t) = u(x - i\alpha, t). \tag{7.3}$$

Denote by $u_\alpha(t)$ the spatial function at time t, $u_\alpha(\cdot, t) = u(\cdot - i\alpha, t)$. Then u_α has the following properties.

(1) $u_\alpha : \mathbb{R} \times (T - \epsilon, T + \epsilon) \setminus \{(x_0, T)\} \to \mathbb{C}$ is a smooth solution of (2.1), where $\epsilon > 0$ is as in Lemma 7.2.

(2) $u_\alpha(t)$ decays exponentially for all $0 < |T - t| < \epsilon$ and blows up at the single point x_0 as $t \to T$, from above and below.

(3) $\lim_{t \to T}(T - t)u_\alpha(x_0 + y(T - t), t) = \frac{\pm i}{y + z'(T)}$, uniformly for $y \in \mathbb{R}$.

(4) $\|u_\alpha(t)\|_\infty \sim |(T - t) \operatorname{Im} z'(T)|^{-1}$ as $t \to T$.

Proof. Property (1) is an immediate consequence of Lemma 7.2. The formulas (2.13) and (2.14) easily imply that, for a fixed $t \in \mathbb{R}$ and a fixed value of $\operatorname{Im} x$, the function u decays exponentially as $|\operatorname{Re} x| \to \infty$. Property (4) is an immediate consequence of property (3). We therefore turn to the proof of property (3).

It follows from Lemma 7.1 and the assumptions on $z(t)$ that locally

$$u_\alpha(x, t) = u(x - i\alpha, t) = \frac{\pm i}{x - i\alpha - z(t)} + f(x, t)$$

and

$$z(t) = x_0 - i\alpha - z'(T)(T - t) + (T - t)^2 g(t),$$

where f and g are regular functions defined locally near x_0 and T. In fact, since $u(x, t)$ has no singularity other than the curve $(z(t), t)$ on a set of the form $\{(x, t) : |\operatorname{Im} x + \alpha| < \epsilon, |T - t| < \epsilon\}$, for sufficiently small ϵ, it follows from the formulas (2.13) and (2.14) that f is bounded on a set of that same form. Putting the above two formulas together yields

$$u_\alpha(x, t) = \frac{\pm i}{x - x_0 + z'(T)(T - t) - (T - t)^2 g(t)} + f(x, t)$$

If we now set $y = (x - x_0)/(T - t)$, so that $x = x_0 + y(T - t)$, the previous formula becomes

$$(T - t)u_\alpha(x_0 + y(T - t), t) = \frac{\pm i}{y + z'(T) - (T - t)g(t)} + (T - t)f(x, t),$$

from which the desired conclusion follows. □

The blowup rate given in the above proposition is not the one predicted by the scaling properties of the mKdV equation. The "scaling" blowup rate would be $\|u_\alpha(t)\|_\infty \sim C|(T - t)|^{-1/3}$.

As is the case for the two-soliton solutions of the Korteweg-de Vries equation (1.2), one expects there to be "two-point" blowup at $T = 0$ if $\operatorname{Re} z(0) \neq 0$. See Theorem 4 in [9]. Again in the interests of keeping the quantity of technical calculations to a reasonable level, we have decided to forego a proof of this.

It is interesting to note that this result of singularity formation for complex–valued solutions of the mKdV–equation can be interpreted as a blow–up result for real–valued solutions of a system of dispersive equations. Let u be a solution of (2.1) and let $r = \operatorname{Re} u$ and $s = \operatorname{Im} u$. It follows that r and s satisfy the real–valued system

$$r_t + r_{xxx} + 6(r^2 - s^2)r_x - 2rss_x = 0, \tag{7.4}$$

$$s_t + s_{xxx} + 2rsr_x + 6(r^2 - s^2)s_x = 0. \tag{7.5}$$

Thus, we have shown that this coupled, dispersive system admits real–valued solutions (exponentially decaying in space) which blow up in finite time.

8. Some formal calculations.

(All the computations in this section have been carried out using MAPLE.) As noted at the end of Section 2, the interaction time for the two–soliton solutions $u^\pm$, given in (2.13) and (2.14), is $t = 0$, the center of the interaction is $x = 0$ and at $t = 0$, the solution is even in x. The explicit formulas for $u^\pm$ allow us to observe and calculate certain aspects of these solutions at the moment of interaction. In particular, it is interesting to know whether there is a single maxima during the interaction or not, and it is likewise interesting to know the speed of the two solitons at the moment of interaction.

In the case of u^+, one observes that that the solution has one centered maximum if the ratio k_2/k_1 is large enough (bigger than around 2.6), and two symmetrically located maxima for smaller values of k_2/k_1. Partial confirmation of this can be obtained by computing $u^+_{xx}(0, 0)$. Since, by symmetry, $u^+_x(0, 0) = 0$, the sign of the second derivative will tell us if it is a local maximum or a local minimum. A MAPLE–implemented calculation shows that

$$u^+_{xx}(0, 0) = -(k_2 - k_1)(k_1^2 - 3k_1k_2 + k_2^2).$$

Therefore, if $1 < k_2/k_1 < (3+\sqrt{5})/2$, then $u^+_{xx}(0, 0) > 0$, which means that $u^+(\cdot, 0)$ has a local minimum at $x = 0$. Thus, there are (at least) two maxima at the moment of interaction. If $k_2/k_1 > (3 + \sqrt{5})/2$, then $u^+(\cdot, 0)$ has a local maximum at $x = 0$, which is consistent with there being a single maximum at the moment of interaction.

Similarly, we calculate that

$$u^-_{xx}(0, 0) = -(k_2 + k_1)(k_1^2 + 3k_1k_2 + k_2^2),$$

which means $u^-(\cdot, 0)$ has a local maximum at $x = 0$ for all values $0 < k_1 < k_2$, which is consistent with the graphical observations of the solution itself.

If we are interested in the "speed" of the two–soliton solution at the moment of interaction, one idea is to calculate the speed of the maximum or the local minimum of the solution as (x,t) approaches $(0,0)$. One might argue that the movement of the maximum is some kind of speed. It must be acknowledged that the interpretation is less clear when one is tracking a local minimum. To calculate this speed, suppose $y(t) = y_{\pm}(t)$ is a real–valued curve such that $u_x^{\pm}(y(t),t) = 0$, i.e. $y_{\pm}(t)$ is always at an extremal point of the solution $u = u^{\pm}$. Suppose also that $y_{\pm}(0) = 0$, which is to say the curve is at the interaction center at the interaction time $t = 0$. Differentiating with respect to t reveals that $u_{xx}(y(t),t)y'(t) + u_{xt}(y(t),t) = 0$, or

$$y'(t) = -\frac{u_{xt}(y(t),t)}{u_{xx}(y(t),t)}.$$

This gives, in turn,

$$y'(0) = -\frac{u_{xt}(0,0)}{u_{xx}(0,0)}, \tag{8.1}$$

which might be thought of as representing the speed of the two–soliton solution at the moment of interaction of the two solitons. The value of $y'(0)$ can be calculated explicitly from (2.13) and (2.14). The results are as follows, for both u^+ and u^-,

$$y'_+(0) = \frac{k_1^4 - 3k_1^3k_2 + 3k_1^2k_2^2 - 3k_1k_2^3 + k_2^4}{k_1^2 - 3k_1k_2 + k_2^2},$$
$$y'_-(0) = \frac{k_1^4 + 3k_1^3k_2 + 3k_1^2k_2^2 + 3k_1k_2^3 + k_2^4}{k_1^2 + 3k_1k_2 + k_2^2}.$$

For u^-, where there is always a maximum at $x = 0$ at the moment of interaction, the maximum is moving with a positive speed. What that speed represents is not entirely clear. In the case of u^+, where the midpoint is a local maximum only if $k_2/k_1 > (3+\sqrt{5})/2$, we see that for these values of $0 < k_1 < k_2$, we have indeed $y'_+(0) > 0$. On the other hand, $y'_+(0) < 0$ for at least some values of $0 < k_1 < k_2$ with $k_2/k_1 < (3+\sqrt{5})/2$. (The lower bound on k_2/k_1 for which this speed is negative is around 2.15.) This negative speed represents the speed of the local minimum, between the two maxima. It is curious that in some cases this minimum is moving backwards.

It is also interesting to do this for the two–soliton solution of the KdV–equation, a calculation which was not carried out in [9]. In this case, it is found that

$$y'(0) = \frac{k_1^4 + 2k_1^2k_2^2 - k_2^4}{3k_1^2 - k_2^2}.$$

As is known, the two–soliton solution of the KdV–equation has one maximum at the interaction time if $k_2/k_1 > \sqrt{3}$ and two maxima, symmetrically located about the interaction center, if $1 < k_2/k_1 < \sqrt{3}$. Here we see that $y'(0) < 0$ for $\sqrt{1+\sqrt{2}} < k_2/k_1 < \sqrt{3}$. In this case, of course, it is the minimum which is moving backward.

REFERENCES

[1] J. Bi, *Novel solutions of MKdV–equation with the modified Bäcklund transformation*, J. Shanghai Univ., **8** (Enlgish Edition) (2004), 286–288.

[2] B. Birnir, *An example of blow–up for the complex KdV–equation and existence beyond the blow–up*, SIAM J. Appl. Math., **47** (1987), 710–725.

[3] J. L. Bona, J. Cohen and G. Wang, *Global well posedness for a system of KdV–type equations with coupled quadratic nonlinearities*, to appear in the Nagoya Mathematical Journal.

[4] J. L. Bona, V. A. Dougalis, O. A. Karakashian and W. R. McKinney, *Conservative, high–order numerical schemes for the generalized Korteweg–de Vries equation*, Philos. Trans. Royal Soc. London, Ser. A, **351** (1995), 107–164.

[5] J. L. Bona and Z. Grujiç, *Spatial analyticity properties of nonlinear waves*, Math. Models Methods Appl. Sci., **13** (2003), 345–360.

[6] J. L. Bona, Z. Grujiç and H. Kalisch, *Algebraic lower bounds for the uniform radius of spatrial analyticity for the generalized Korteweg–de Vries equation*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire, **22** (2005), 783–797.

[7] J. L. Bona and J.-C. Saut, *Dispersive blowup of generalized Korteweg–de Vries equations*, J. Differential Equations, **103** (1993), 3–57.

[8] J. L. Bona and F. B. Weissler, *Blow–up of spatially periodic complex–valued solutions of nonlinear dispersive equations*, Indiana Univ. Math. J., **50** (2001), 759–782.

[9] J. L. Bona and F. B. Weissler, *Pole dynamics of interacting solitons and blowup of complex–valued solutions of KdV*, Nonlinearity, **22** (2009), 311–349.

[10] J. L. Bona and F. B. Weissler, *Finite time blowup of spatially periodic, complex–vlaued solutions of the Kortweg–de Vries equation*, in preparation.

[11] G. Bowtell and A. E. G. Stuart, *A particle representation of the Kortweg–de Vries soliton*, J. Math. Phys., **24** (1983), 969–981.

[12] A. C. Bryan and A. E. G. Stuart, *On the dynamics of soliton interactions for the Korteweg–de Vries equation*, Chaos, Solitons & Fractals, **2** (1992), 487–491.

[13] Z. Grujiç and H. Kalisch, *Local well–posedness of the generalized Korteweg–de Vries equation in spaces of analytic functions*, Diff. Integral Eqns., **15** (2002), 1325–1334.

[14] M. D. Kruskal, *The Korteweg–de Vries equation and related evolution equations*, *Nonlinear Wave Motion (Lectures in Applied Mathematics **15**)* (ed. A. C. Newell), (Providence, RI: American Mathematical Society), pp. 61–83.

[15] Y.-C. Li, *Simple explicit formulae for finite time blowup solutions to the complex KdV equation*, Chaos, Solitons & Fractals, **39** (2009), 369–372.

[16] Y. Martel and F. Merle, *Stability of blow–up profile and lower bounds for blow–up rate for the critical generalized KdV equation*, Ann. of Math. (2), **155** (2002), 235–280.

[17] F. Merle, *Existence of blow–up solutions in the energy space for the critical generalized KdV equation*, J. American Math. Soc., **14** (2001), 555–578.

[18] W. R. Thickstun, *A system of particles equivalent to solitons*, J. Math. Anal. Appl., **55** (1976), 335–346.

[19] J. Wu and J.-M. Yuan, *The complex KdV equation with or without dissipation*, Discrete Cont. Dynamical Sys., Ser. B, **5** (2005), 489–512.

Received November 2011; revised August 2012.

E-mail address: bona@math.uic.edu
E-mail address: vento@math.univ-paris13.fr
E-mail address: weissler@math.univ-paris13.fr