On the minimum number of colors for knots

Louis H. Kauffman
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago
851 S. Morgan St., Chicago IL 60607-7045
USA
kauffman@uic.edu
and
Pedro Lopes
Department of Mathematics
Instituto Superior Técnico
Technical University of Lisbon
Av. Rovisco Pais
1049-001 Lisbon
Portugal
pelopes@math.ist.utl.pt

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Abstract

In this article we take up the calculation of the minimum number of colors needed to produce a non-trivial coloring of a knot. This is a knot invariant and we use the torus knots of type $(2, n)$ as our case study. We calculate the minima in some cases. In other cases we estimate upper bounds for these minima leaning on the features of modular arithmetic. We introduce a sequence of transformations on colored diagrams called Teneva transformations. Each of these transformations reduces the number of colors in the diagrams by one (up to a point). This allows us to further decrease the upper bounds on these minima. We conjecture on the value of these minima. We apply these transformations to rational knots.

Keywords: Knots, colorings, colors, Teneva transformations

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1 Introduction

The colorings we are concerned with are the so-called Fox colorings, [4, 7]. Given a knot diagram and an integer r we consider the integers $0, 1, 2, \ldots, r-1$ mod r, whose set will be denoted $\mathbb{Z}_r$. We assign one integer (call it a color) to each arc of the diagram so that at each crossing the sum of the integers at the under-arcs minus twice the integer at the over-arc equals zero mod r. In this way, we set up a system of equations over $\mathbb{Z}_r$. Each of the solutions of this system of equations is called an r-coloring of the knot under consideration. There are always at least r solutions, regardless of the knot diagram or the integer r we are choosing; each of these r solutions is obtained by assigning the same color to each arc of the diagram. These are called the trivial solutions. A one-to-one correspondence of the solutions of the systems of equations before and after the performance of each of the Reidemeister moves is presented in [9]. In this correspondence, the trivial solutions go over to trivial solutions and non-trivial solutions go over to non-trivial solutions. In particular, the number of solutions for each r (also referred to as the

number of r-colorings), is an invariant of the knot under study. Given an integer $r > 1$ and a link K, we let $\#col_r K$ stand for the number of r-colorings of K.

The efficiency of this invariant in distinguishing prime knots up to ten crossings is illustrated in [3]. The invariant we there associate to each knot is in fact the *color spectrum* of the knot i.e., the sequence of numbers of r-colorings. We remark that the notion of *quandles* generalizes the notion of r-colorings, see [6, 10].

The importance of knowing the color spectrum of the knot K under consideration, is that it tells us immediately whether K is interesting or not for our current research and whether there is significant topological information in the spectrum. In fact, for any r, the number of trivial colorings is r. Hence if the number of colorings for a given r is greater than r, there exist then non-trivial colorings for the knot under consideration. As an example, the trefoil exhibits nine 3-colorings. It has then non-trivial 3-colorings.

Definition 1.1 (Minimum number of colors) *Given an integer $r > 1$, assume there are non-trivial r-colorings on a given knot K. Assume further that D is a diagram of K. We let $n_{r,K}(D)$ denote the minimum number of distinct colors assigned to the arcs of D it takes to produce a non-trivial r-coloring on D. We denote* $\mathrm{mincol}_r K$ *the minimum of these minima over all diagrams D of K:*

$$\mathrm{mincol}_r K := \min\{n_{r,K}(D) \mid D \text{ is a diagram of } K\}$$

For each K, we call $\mathrm{mincol}_r K$ *the* **minimum number of colors of K, mod** r. *In the sequel, we will drop the "mod r" whenever it is clear which r we are referring to. Note that* $\mathrm{mincol}_r K$ *is tautologically a topological invariant of K.*

Apparently, in order to calculate $\mathrm{mincol}_r K$, we have to consider a diagram of K, and find the minimum number of colors it takes to construct a non-trivial r-coloring. This operation should then be repeated for all diagrams of K, presenting the minimum for each of these diagrams. Finally, the minimum of these minima is the $\mathrm{mincol}_r(K)$. In this article we present techniques that allow us to calculate $\mathrm{mincol}_r(K)$ in infinitely many cases and in other cases to estimate its upper bounds. We regard the torus knots of type $(2, n)$ as our case study. The features of modular arithmetic allow us to calculate $\mathrm{mincol}_r T(2, n)$ exactly for some combinations of n and r. The introduction of certain transformations on diagrams (Teneva transformations) allows us to better estimate the upper bounds on $\mathrm{mincol}_r T(2, n)$ for other combinations of n and r.

This invariant was first introduced in [5]. Considerations of the authors about it led them to set forth the Kauffman-Harary Conjecture which has already been proven to be true for rational knots, [8], and for Montesinos links, [1].

Conjecture 1.1 (Kauffman-Harary, [5]) *Let p be a prime integer and assume K is an alternating knot of determinant p. Then, any non-trivial p-coloring on any minimal diagram of K assigns different colors to different arcs of the diagram.*

We note that the Kauffman-Harary Conjecture deals only with a specific r per knot. Moreover, the knots under consideration in this conjecture are all alternating knots of prime determinant (and the specific r is precisely this determinant, for each of these knots). In this article we adopt a broader point of view by not specifying r. In fact, we would like to develop computational tools that would allow us to calculate the $\mathrm{mincol}_r(K)$ for any r, and for any K. Here we content ourselves on studying the class of torus knots of type $(2, n)$.

Given two positive integers l and m we let (l, m) stand for the greatest common divisor of l and m and $\langle l, m\rangle$ stand for 1 if $(l, m) = 1$, and for the least common prime divisor of l and m otherwise.

In this article we prove the following Theorem:

Main Theorem Suppose r and n are positive integers such that $(n, r) > 1$.

- If $\langle n, r\rangle \in \{2, 3\}$ then
$$\text{mincol}_r T(2, n) = \langle n, r\rangle$$
- If $\langle n, r\rangle = 5$ then
$$\text{mincol}_r T(2, n) = 4$$
- If $\langle n, r\rangle = 2k + 1$ (for some integer $k > 2$) then
$$3 < \text{mincol}_r T(2, n) \leq k + 2$$

Moreover, we conjecture that

Conjecture Suppose r and n are positive integers such that $(n, r) > 1$. Let $\langle n, r\rangle = 2k + 1$, for some integer $k > 2$. Then:
$$\text{mincol}_r T(2, n) = k + 2$$

The organization of this article is as follows. In Section 2 we calculate the number of r colorings for each $T(2, n)$ and estimate upper bounds on the minimum number of colors for each $T(2, n)$. In Section 3 we prove the Main Theorem. In Section 4 we present applications to rational knots. In Section 5 we consider directions for further work.

2 The number of r-colorings of $T(2, n)$. Upper bounds on minimum numbers of colors.

Proposition 2.1 *For each integer $n \geq 2$,*
$$\#col_r T(2, n) = (n, r)r$$
for all $r \geq 2$.

Proof: Consider the diagram of $T(2, n)$ given by the closure of σ_1^n ($\sigma_1 \in B_2$) - see Figure 1 for the $n = 3$ instance and [2] for further information on the σ_i's. Assume further that $\widehat{\sigma_1^n}$ is endowed with an r-coloring

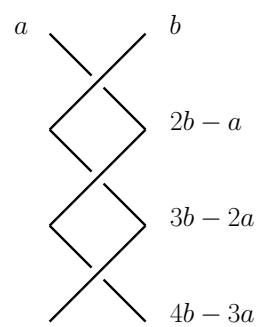

Figure 1: Coloring assignment to σ_1^3

for some integer $r \geq 2$. In order to set up the system of equations whose solutions are the r-colorings of $T(2, n)$, we will state and prove the following claim:

Claim 2.1 *Let n be an integer greater than 2 and assume σ_1^n is endowed with an r-coloring for some integer $r \geq 2$. If the top segments of the braid σ_1^n ($\sigma_1 \in B_2$) are colored (from left to right) with a, $b \in R_r$ then, for each $i = 1, ..., n$, the segment emerging from the i-th crossing will bear color $(i+1)b - ia$ (counting crossings from top to bottom).*

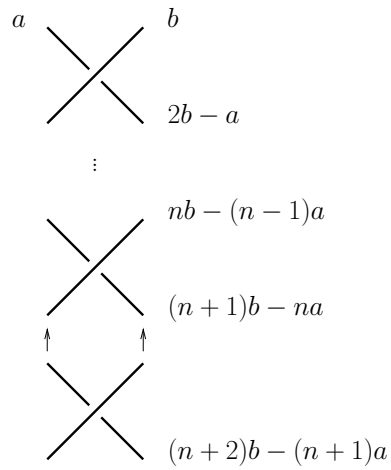

Figure 2: Juxtaposing the $(n+1)$-th crossing

Proof: By induction on n. The $n=3$ instance is clear by inspection of figure 1. Assume claim is true for a specific $n \in \mathbb{N}$. Juxtaposing another crossing to the previous n (see figure 2), it is easy to see that the induction step follows. ■

We now return to the proof of the Proposition. For n crossings, the labellings on the bottom segments become $nb-(n-1)a$ and $(n+1)b-na$ (from left to right). In order for these a and b to stand for an r-coloring, the bottom left labelling has to equal the top left labelling and the bottom right labelling has to equal the top right labelling i.e., $nb-(n-1)a =_r a$ and $(n+1)b-na =_r b$ which simplify both to

$$n(b-a) =_r 0$$

So, $b-a$ has to provide the largest factor of r which is relatively prime to n for the equation to hold; this factor is $\frac{r}{(n,r)}$. Thus $b-a$ equals $\frac{r}{(n,r)}$ or one of its multiples in $\{1,2,\ldots,r\}$ i.e., $b-a$ can be any of the following:

$$\frac{r}{(n,r)}, \quad 2\frac{r}{(n,r)}, \quad 3\frac{r}{(n,r)}, \quad \ldots \quad , \quad \big((n,r)-1\big)\frac{r}{(n,r)}, \quad (n,r)\frac{r}{(n,r)}$$

Finally, there are r ordered pairs (a,b) compatible with each of the above (n,r) possibilities. This statement is justified by the following Claim.

Claim 2.2 *For any $i \in \{0,1,2,\ldots,r-1\}$, there are exactly r pairs (a,b) which are solutions to $b-a =_r i$ $(a,b \in \{0,1,2,\ldots,r-1\})$.*

Proof: For each assignment of a value to a, b is uniquely specified by the formula $b =_r a+i$. Since r values can be assigned to a, the proof is complete. ■

Resuming the proof of the Proposition, there are (n,r) possibilities for the $b-a$ so that the equation $n(b-a) =_r 0$ holds. Each of these possibilities can be realized in r distinct ways. There are then $(n,r)r$ solutions i.e., $(n,r)r$ r-colorings of $T(2,n)$. ■

Corollary 2.1 *There are nontrivial colorings if, and only if, $(n,r) \neq 1$.*

Proof: Omitted. ■

We recall that, given two positive integers l and m, $\langle l,m\rangle$ stands for 1 if $(l,m)=1$ and stands for the least common prime factor of l and m, otherwise.

Proposition 2.2 *Let n and r be positive integers greater than 1. If $(n,r)=1$ there are only trivial r-colorings of $T(2,n)$. If $(n,r)>1$,*

$$\mathrm{mincol}_r T(2,n) \leq \langle n,r\rangle$$

Proof: That for $(n, r) = 1$ there are only trivial colorings is just a rephrasing of Corollary 2.1.

If $(n, r) > 1$ set

$$p = \langle n, r \rangle$$

The set

$$R_r^p := \left\{ 0,\ \frac{r}{p},\ 2\,\frac{r}{p},\ \ldots\ ,\ (p-1)\,\frac{r}{p} \right\}$$

endowed with the $a * b := 2b - a \pmod r$ operation ([6, 10]) is algebraically closed. As a matter of fact, it is an algebraically closed substructure of $\{0, 1, 2, \ldots r-1\}$ endowed with the same operation. It is equivalent to the set $\{0, 1, \ldots p-1\}$ endowed with the operation $a * b := 2b - a \pmod p$.

Also note that, in this $p = \langle n, r \rangle > 1$ case, the σ_1^n braid ($\sigma_1 \in B_2$), whose closure gives a knot diagram for $T(2, n)$, can be regarded as a product of $\frac{n}{p}$ σ_1^p's:

$$\underbrace{\sigma_1^p \cdot \cdots \cdot \sigma_1^p}_{\frac{n}{p}\text{ factors}}$$

This reflects on the coloring equation:

$$0 =_r n(b-a) =_r \frac{n}{p} \cdot p(b-a)$$

Hence, any two a, b from R_r^p yield an r-coloring of $\widehat{\sigma_1^n}$. We have, thus, p^2 r-colorings with colors from R_r^p. Note, also, that any of these a, b from R_r^p yield an r-coloring of each $\widehat{\sigma_1^p}$, since they satisfy

$$0 =_r p(b-a)$$

We can, thus, regard some non-trivial r-colorings of $\widehat{\sigma_1^n}$ as stackings of $\frac{n}{p}$ p-colorings of $\widehat{\sigma_1^p}$ (see Figure 3). Now, each of these non-trivial p-colorings of $\widehat{\sigma_1^p}$ uses exactly p distinct colors from R_r^p thanks to the

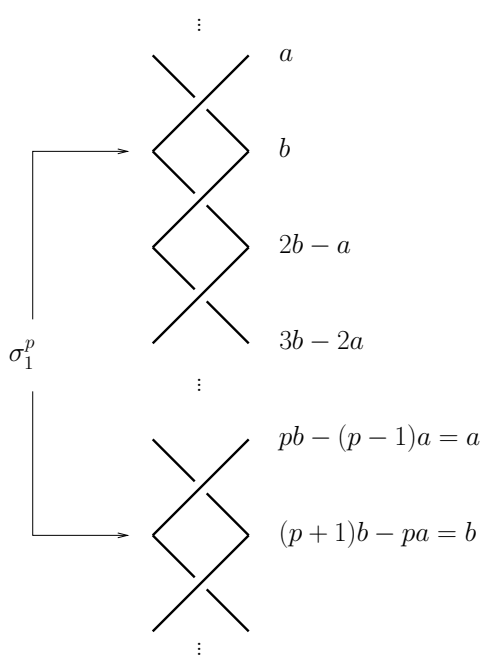

Figure 3: Coloring of σ_1^p inside coloring of σ_1^n

following claim:

Claim 2.3 *If p is prime then each non-trivial p-coloring of $\widehat{\sigma_1^p}$ uses exactly p distinct colors.*

Proof: Pick a, $b \in \{0, 1, \ldots, p-1\}$, distinct. Suppose there exist $i, j \in \{0, \ldots, p-1\}$ such that $ib - (i-1)a =_p jb - (j-1)a$. Then $(i-j)(b-a) =_p 0$ which is equivalent to saying that $i = j$, for $\mathbb{Z}_p$ is a field.■

Hence, given positive integers n and r such that $(n, r) > 1$ we constructed a non-trivial r-coloring of $T(2, n)$ which uses exactly $\langle n, r\rangle$ colorings. $\langle n, r\rangle$ is then an upper bound for $\text{mincol}_r T(2, n)$. This concludes the proof. ■

3 Proof of the Main Theorem.

In this Section we prove the Main Theorem. In Subsection 3.1 we prove the $\langle n, r\rangle \in \{2, 3\}$ instance. In Subsection 3.2 we prove $k+2$ is an upper bound on the minimum number of colors of the $\langle n, r\rangle = 2k+1$ instance after introducing the Teneva transformations. In Subsection 3.3 we prove the $\langle n, r\rangle = 5$ instance and that 3 is a lower bound on the minimum number of colors of the $\langle n, r\rangle = 2k+1$ instance.

3.1 The $\langle n, r\rangle \in \{2, 3\}$ instance.

Proposition 3.1 *For even positive integers n and r*

$$\text{mincol}_r\, T\big(2, n\big) = 2$$

Proof: For even positive integers n and r, $\langle n, r\rangle = 2$. Then, by Proposition 2.2,

$$\text{mincol}_r\, T\big(2, n\big) \leq 2$$

Since a nontrivial coloring has to use at least two distinct colors, the result follows. ■

We remind again the reader that we call trivial coloring any coloring which assigns the same color to each arc of the diagram under study. We remark that this implies that a trivial knot of more than one component can be assigned non-trivial colorings. In order to see this, consider a diagram of this knot where one component lies in a neighborhood which is disjoint from a neighborhood which contains the rest of the diagram. Color the singled out component with color a and the rest of diagram with color $b(\neq a)$ - this is a non-trivial coloring of this trivial knot. Note further that this type of phenomenon does not occur for knots. In fact, a trivial knot has a diagram with no crossings. Any coloring of this diagram can only have one color - hence any other of this knot's diagrams is colored with only one color.

Proposition 3.2 *For positive integers n and r such that $\langle n, r\rangle = 3$,*

$$\text{mincol}_r\, T\big(2, n\big) = 3$$

Proof: Since $\langle n, r\rangle = 3$, then there is an r-coloring of $T(2, n)$ with as few as three colors by the proof of Proposition 2.2. We next prove that two distinct colors are not enough to produce a non-trivial coloring in the $\langle n, r\rangle = 3$ case. We consider two possibilities: odd r and even r.

Suppose r is odd and $a, b \in \{0, 1, \ldots, r-1\}$. If $b = 2b - a$ then $a = b$; if $a = 2b - a$ then $2b = 2a$ which is equivalent to saying $a = b$ since r is odd (hence 2 is invertible). Thus if we choose distinct a, b from $\{0, 1, \ldots, r-1\}$ then $\#\{a, b, 2b-a\} = 3$ and so any non-trivial r-coloring of $T(2, n)$, for odd r, has at least three colors. This concludes the proof for odd r.

Suppose r is even; then n is odd for otherwise $\langle n, r\rangle = 2$. Assume there is an r-coloring of a diagram D of $T(2, n)$ which uses exactly two distinct colors, say a, b. At some crossing of the diagram the two colors meet. The possibilities for the color on the emerging arc are as follows. Either $b = 2b - a$ i.e., $a = b$ which is contrary to the assumption; or $a = 2b - a$ i.e., $a = b + \frac{r}{2}$. Hence, $2a - b = 2(b + \frac{r}{2}) - b = b$ mod r. Thus, for even r, b and $a = b + \frac{r}{2}$ generate an algebraically closed structure with respect to the $x * y := 2y - x$ operation ([6, 10]). This structure is formed precisely by b and $a = b + \frac{r}{2}$. Consider again the diagram D endowed with the indicated coloring which uses only the two colors a and b. By performing Reidemeister moves and consistently changing the colors after each move (cf. [9]) we obtain a new diagram of $T(2, n)$ endowed with a coloring which uses only the two colors a and b - because $\{a, b\}$ constitute an algebraically closed set with respect to the $*$-operation, as was seen above.

In particular, we could transform D into $\widehat{\sigma_1^n}$ upon performance of Reidemeister moves and the associated coloring (obtained by consistently changing the colorings after each Reidemeister move) would use

exactly two colors. We now prove that $\widehat{\sigma_1^n}$ cannot be colored with only two colors - because n is odd. As a matter of fact, starting with distinct a and b from $\{0, 1, \ldots, r-1\}$ at the top segments of σ_1^n (from left to right) we obtain, using induction, b, a (from left to right) after an odd number of crossings. This concludes the proof. ■

We remark that the proof of Proposition 3.2 yields results stronger than the statement of the Proposition. Specifically,

Corollary 3.1 *Let r be an integer greater than 1 and assume K is not splittable.*

- *If r is odd then* $\text{mincol}_r K > 2$
- *If r is even then,*
  - *a specific diagram of K admits an r-coloring with exactly two colors if, and only if, any other diagram of K admits an r-coloring with exactly two colors*

Proof: Omitted. ■

In particular,

Corollary 3.2 *If $\langle n, r\rangle$ is an odd prime then:*

$$\text{mincol}_r T(2, n) > 2$$

Proof: If r is odd, the statement of this Corollary is a particular case of the first statement of Corollary 3.1. If r is even, then n is odd. In particular, $T(2, n)$ is a knot and the second statement of Corollary 3.1 applies. Repeating an argument used in the proof of Proposition 3.2 we see that the closure of σ_1^n cannot have an r-coloring with just two colors, since n is odd. Whence no diagram of $T(2, n)$ can have an r-coloring with just two colors. This concludes the proof. ■

3.2 Teneva transformations.

The results we obtained on $\text{mincol}_r T(2, n)$ so far, relied on the features of modular arithmetic. We will now come up with better estimates for $\text{mincol}_r T(2, n)$ by making use of Reidemeister moves. In particular, we will obtain diagrams endowed with non-trivial colorings that use less colors than the ones considered so far, although these diagrams have more arcs than the $\widehat{\sigma_1^n}$'s. In order to obtain these diagrams we will use what we call Teneva transformations. This is a particular sequence of Reidemeister moves, starting from the $\widehat{\sigma_1^n}$ diagram of $T(2, n)$ endowed with a non-trivial coloring and consistently coloring the diagrams after each move, in the sense introduced in [9]. This formalizes and generalizes a particular case due to Irina Teneva presented in [5].

Specifically, we now establish that, for any odd prime $p = 2k + 1$ (for some positive integer k),

$$\text{mincol}_p T(2, p) \leq k + 2$$

In order to do this we will prove that, for any positive integer k, $\text{mincol}_{2k+1} T(2, 2k + 1) \leq k + 2$. Of course, for non-prime $2k + 1$, Proposition 2.2 presents a strictly smaller upper bound but in this way we are able to use induction on k thus establishing the $k + 2$ upper bound also for prime $p = 2k + 1$. This is a better estimate than the Proposition's, in the prime p situation. We remark, in passing, that, for any positive integer k, $k + 2$ is also an upper bound for $\text{mincol}_{2k} T(2, 2k)$; again in this case, Proposition 2.2 presents a strictly smaller upper bound.

In order to prove that $k + 2$ is an upper bound, as referred to above, we consider the diagram of the torus knot $T(2, 2k + 1)$ (for $k > 1$) as given by the braid closure of σ_1^{2k+1} ($\sigma_1 \in B_2$) and endowed with a non-trivial $(2k + 1)$-coloring. This coloring is represented by assigning $a, b \in \{0, 1, \ldots, 2k\}$ ($a \neq b$) to the top segments of the braid σ_1^{2k+1} in the usual manner. Since $2k + 1$ is the number of arcs of $\widehat{\sigma_1^{2k+1}}$ there are, at most, $2k+1$ colors in this coloring. If $a = 0$ and $b = 1$ then there are exactly $2k+1$ colors. If $2k+1$ is prime then there are exactly $2k+1$ colors whenever $a \neq b$, thanks to Claim 2.3. We, thus, assume that,

for each k, the coloring in point uses $2k+1$ colors. We then perform $1+k$ Reidemeister moves on the diagrams, consistently changing the coloring assignments after each move (cf. [9]), eventually obtaining a coloring assignment with $k+2$ colors.

We exemplify for $k=2$ starting from the diagram of $T(2,5)$ given by the closure of $\sigma_1^{2\cdot 2+1}$ and endowed with an r-coloring as shown (see Figure 4). The dotted lines indicate where the arc in point is taken to by the next move. There are $1+2=3$ moves. The first one is a type I Reidemeister move on the arc labeled a at the bottom right of the first diagram in figure 4; the two other moves are type III Reidemeister moves. The first of these two type III Reidemeister moves moves the arc labeled a (which stems from the bottom left of the second diagram) over a crossing above it in the manner indicated in figure 4 (dotted lines in the second diagram). Color $2b-a$ is introduced with this move but since it was already part of the coloring, the number of colors remains the same. The second type III Reidemeister move moves the arc we just referred to over the crossing right above it. With this move color $3b-2a$ is introduced and color $4b-3a$ is removed. Since color $3b-2a$ was already part of the coloring and there was only one arc assigned color $4b-3a$, this move reduces the number of colors by one. We end up thus with $5-1=k+2$ colors for $k=2$ as announced.

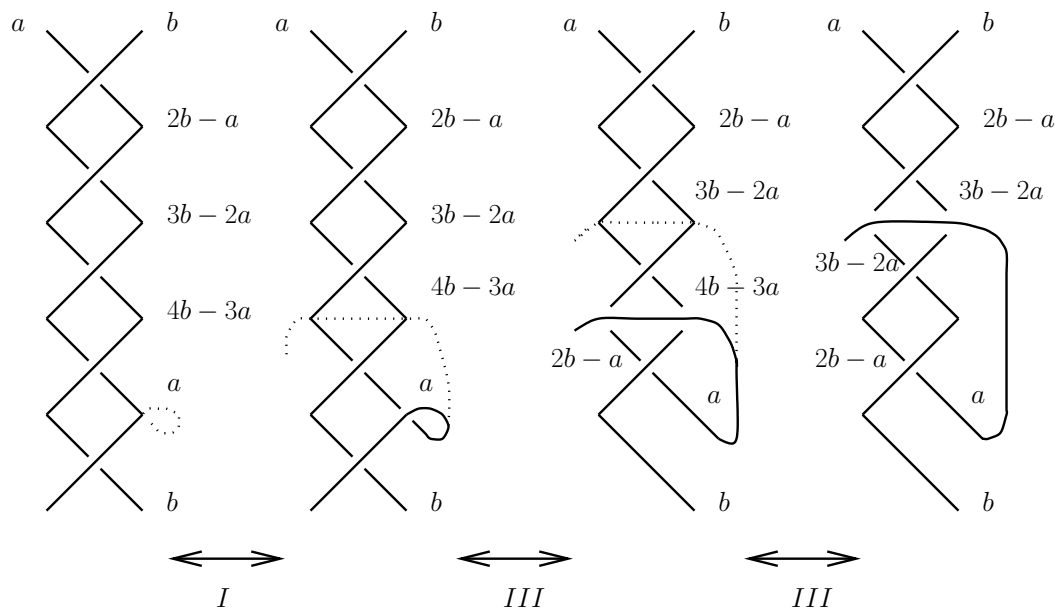

Figure 4: Colored T(2, 5): from 5 to 4 colors

For a general integer $k>1$, there are $1+k$ moves on σ_1^{2k+1} with the first three as indicated for $T(2,5)$, in figure 4. As of the third, each move will be analogous to the previous one, the arc in point being pulled up over one crossing at a time. As with $T(2,5)$, the first two moves will not affect the number of colors used. The third move will remove color $(2k)b-(2k-1)a$, the fourth move will remove color $(2k-1)b-(2k-2)a$, ... , the k-th move will remove color $(k+3)b-(k+2)a$, and the $(k+1)$-th move will remove color $(k+2)b-(k+1)a$. Analogous remarks apply for the even $n=2k$ case. We then state:

Proposition 3.3 (Teneva reduction) *For any integer $k>1$,*

$$\mathrm{mincol}_{2k+1}T(2,2k+1)\leq k+2$$

Before embarking on the proof of this proposition we will first state and prove a technical lemma which will help us deal with it.

Lemma 3.1 (Teneva transformation) *For each integer $n\geq 3$, consider σ_1^n ($\sigma_1\in B_2$) endowed with an r-coloring as in Claim 2.1. Then, $n-1$ type III Reidemeister moves are performed, taking the bottom left arc of the braid over each crossing above it until the top of the braid is reached. These type III Reidemeister moves are preceded by a type I Reidemeister move which prepares the setting for the*

subsequent moves (see left-most braid in figure 4, for the $n = 5$ instance). The diagrams are consistently colored after each move.

The first type III Reidemeister move introduces color $(n+2)b-(n+1)a$ on the left-hand side of the diagram. For $2 \leq i \leq n-1$, the i-th type III Reidemeister move removes color $(n+1-i)b-(n-i)a$ from the right-hand side of the diagram and introduces color $(n+1+i)b-(n+i)a$ on the left-hand side of the diagram.

Proof: We use induction on n. For $n = 3$, see figure 5.

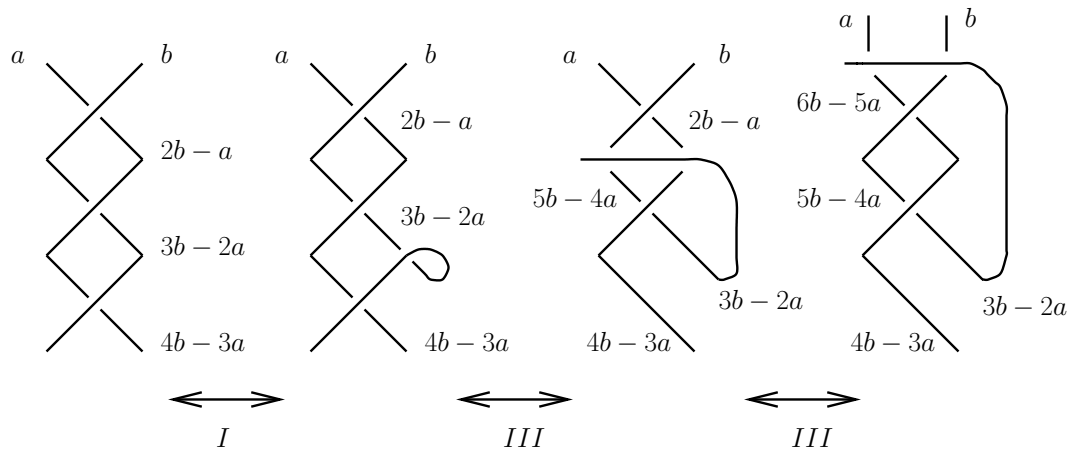

Figure 5: The $n = 3$ instance

Now assume the claim is true for a specific integer $n \geq 3$ and consider the $n+1$ situation (see figure 6). Neglecting the top σ_1, there is a string of n σ_1's with the top segments colored with $a' = b$ and

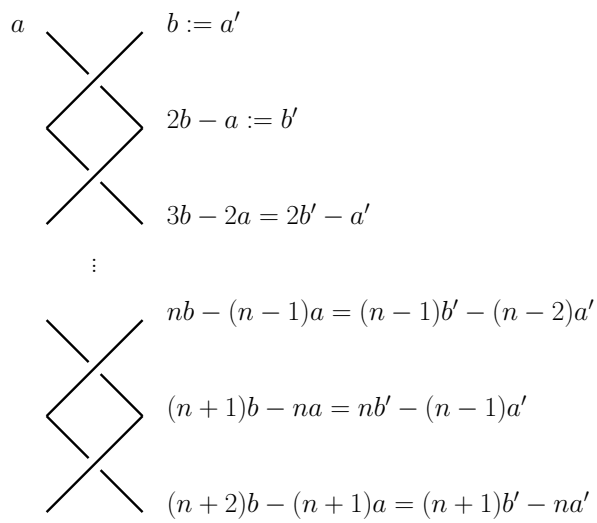

Figure 6: The assignment of colors to the σ_1^{n+1} braid

$b' = 2b - a$. Hence we can apply the induction hypothesis to say that the first type III Redemeister move introduces color $(n+2)b' - (n+1)a'$ $\big(= (n+3)b - (n+2)a \big)$ and, for $2 \leq i \leq n-1$, the i-th type III Reidemeister move removes color $(n+1-i)b' - (n-i)a'$ $\big(= (n+2-i)b - (n+1-i)a \big)$ from the right-hand side of the diagram and introduces color $(n+1+i)b' - (n+i)a'$ $\big(= (n+2+i)b - (n+1+i)a \big)$. The n-th type III Reidemeister move removes color $2b - a$ and introduces color $(2n+2)b - (2n+1)a$. Hence, the claim follows. ■

Proof (of Proposition 3.3): For some integer $k > 1$, consider a non-trivial $(2k+1)$-coloring of $\widehat{\sigma_1^{2k+1}}$, using $2k+1$ distinct colors. Lemma 3.1, specialized to $n = 2k+1$ and reading colors mod $2k+1$, implies

that, for $2 \leq i \leq 2k$, the i-th type III Reidemeister move removes color $(2k+2-i)b-(2k+1-i)a$ and introduces color $(i+1)b-ia$. Write the set of colors in the original coloring of σ_1^{2k+1} in the following way:

$$\{a,b\} \quad \cup \quad \{2b-a\} \quad \cup \quad \{3b-2a, 4b-3a, \ldots, kb-(k-1)a, (k+1)b-ka\} \quad \cup$$
$$\cup \quad \{(k+2)b-(k+1)a, (k+3)b-(k+2)a, \ldots, (2k-1)b-(2k-2)a, 2kb-(2k-1)a\}$$

The set $\{3b-2a, 4b-3a, \ldots, kb-(k-1)a, (k+1)b-ka\}$ contains the colors that are introduced from the performance of type III Reidemeister moves $i=2$ through k; these colors were already in the diagram so after these $k-1$ moves there are two of each. The set $\{(k+2)b-(k+1)a, (k+3)b-(k+2)a, \ldots, (2k-1)b-(2k-2)a, 2kb-(2k-1)a\}$ contains the colors that are removed from the performance of type III Reidemeister moves $i=2$ through k. Only one of each of the colors in this latter set was in the coloring of the diagram before the performance of these $k-1$ moves. Then, after these $k-1$ moves, there are none of these colors in the coloring of the resulting diagram. In this way, after the type III Reidemeister move corresponding to $i=k$ is performed there are $k-1 = \#\{(k+2)b-(k+1)a, (k+3)b-(k+2)a, \ldots, (2k-1)b-(2k-2)a, 2kb-(2k-1)a\}$ less colors than in the original diagram i.e., there are only $2k+1-(k-1)=k+2$ colors left. The result follows. (Note, in passing, that, as of this $i=k$ type III Reidemeister move, the colors removed are the ones that there are in two's and the colors introduced are the ones that were previously removed, so as of this move the number of colors in the colorings of the diagrams increases). ■

We remark that, for prime $p=2k+1$ and r divisible by p then

$$\text{mincol}_r T(2,p) \leq k+2$$

since $\left\{0,\ \frac{r}{p},\ 2\,\frac{r}{p},\ \ldots\ ,\ (p-1)\,\frac{r}{p}\right\}$ endowed with the $a*b = 2b-a \pmod r$ operation is algebraically closed. It is, in fact, an algebraic closed substructure of $\{0,1,\ldots,r-1\}$ endowed with the same operation. Moreover, if $(n,r)>1$ and $\langle n,r\rangle = p$ $(=2k+1$, for some positive integer $k)$, then we regard $T(2,n)$ as the closure of a stacking of σ_1^p's, each one of which is non-trivially colored using colors from $\left\{0,\ \frac{r}{p},\ 2\,\frac{r}{p},\ \ldots\ ,\ (p-1)\,\frac{r}{p}\right\}$. Teneva reduction (Proposition 3.3) can now be applied to each σ_1^p to reduce the number of colors from $2k+1$ to $k+2$. In this way:

Proposition 3.4 *Suppose r and n are positive integers such that $\langle n,r\rangle = 2k+1$, for some integer $k>1$. Then:*

$$\text{mincol}_r T(2,n) \leq k+2$$

Proof: Omitted. ■

Definition 3.1 *We call Teneva transformation a finite sequence of moves on knot diagrams endowed with colorings as described in Lemma 3.1. This transformation introduces some colors and removes other colors in the colorings. We call this transformation Teneva reduction (Proposition 3.3) when the net effect of the Teneva transformation is to decrease the number of colors used in the coloring.*

Note that a Teneva transformation can be applied to a portion of a knot diagram endowed with a coloring such that this portion of the diagram looks like a σ_1^n $(\sigma_1 \in B_2)$. If the net effect of the Teneva transformation on the braid-like portion of the diagram is to decrease the number of colors used in the whole diagram, we call it also Teneva reduction (see Section 4). The number of type III Reidemeister moves should be adapted to each colored knot diagram whose number of colors we want to reduce in order to maximize this reduction. For the case of the $T(2,n)$ endowed with a non-trivial r-coloring such that $\langle n,r\rangle = 2k+1$, $k-1$ of those type Reidemeister moves maximize the reduction.

3.3 The $\langle n,r\rangle = 5$ instance.

Our proof of this instance of Proposition 3.4 relies on understanding how the multiplication table with respect to the $a*b := 2b-a \mod r$ operation ([6, 10]) of a subset of three distinct elements from

$\{0, 1, \ldots, r-1\}$ can be realized for a general integer $r > 2$. For each of the realizations we will then inquire into whether this subset with the indicated table can give rise to a coloring of a $T(2, n)$ or not.

Suppose we are given distinct $a, b, c \in \{0, 1, 2, \ldots, r-1\}$. We construct their multiplication table by considering the distinct solutions of the equation $2x - y = z$ in $\{a, b, c\}$. We start by remarking that the equation $2x - y = x$ has only solutions satisfying $x = y$ and will thus be systematically discarded. We remark also that equalities and belonging to sets will be understood modulo r.

1. Suppose first that the equation $2x - y = z$ has a solution where no two variables take on the same value say, $2b - a = c$, possibly after relabelling.

   (a) Assume further that $2a - c = b$ also holds. Then, adding these two expressions we obtain $2c - b = a$. The multiplication table is then shown in Table 3.1. Moreover, substituting the

	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

Table 3.1: Multiplication table for the $2b - a = c$, and $2c - a = b$ case

   last expression in either of the first two we obtain

$$3(b - a) = 0$$

   which implies that $3|r$ in order to be possible for b and a to be distinct.

   (b) Assume now there is only one solution of $2x - y = z$ (modulo permutation of y and z) with no two variables taking on the same value. This solution is $2b - a = c$.

      i. Assume then $2a - c = c$. Then $2(a - c) = 0$ which implies that r is even and $c = a + \frac{r}{2}$. If $2a - b = b$ then again $b = a + \frac{r}{2} = c$ which is a contradiction (a, b, c are distinct). Then $2a - b \notin \{a, b, c\}$. Further,

$$2c - b = 2\left(a + \frac{r}{2}\right) - b = 2a - b \notin \{a, b, c\}$$

      The multiplication table is then shown in Table 3.2. In Table 3.2 an "X" means the

	a	b	c
a	a		
b	X	b	X
c			c

Table 3.2: Multiplication table for the $2a - b$ and $2c - b$ not in $\{a, b, c\}$ case

      corresponding entry does not belong to $\{a, b, c\}$ and an empty entry means the specification of that entry is not relevant.

      ii. Finally assume $2a - c \notin \{a, b, c\}$. Then $2a - c \neq c$ which implies that $2c - a \notin \{a, b, c\}$. Further, if $2c - b = b$ then subtracting this expression from $2c - a = e \notin \{a, b, c\}$ implies $2b - a = e \notin \{a, b, c\}$ which contradicts the standing assumption $2b - a = c$. Then $2c - b \notin \{a, b, c\}$. Also, if $2a - b = b$ then $2b - a = a \neq c$ so $2a - b \notin \{a, b, c\}$. The multiplication table for this case looks like Table 3.2.

2. Assume now there is no solution of $2x - y = z$ in $\{a, b, c\}$ where no two variables take on the same value.

   (a) Suppose the equation $2x - y = y$ has solutions satisfying $x \neq y$. Then r is even. If it has more than one such solutions (modulo permutation of x and y), say $2(b - a) = 0 = 2(b - c)$, possibly

after relabelling, then $a = b + \frac{r}{2} = c$ which is impossible. Then there is at most one solution of $2x - y = y$ with $x \neq y$ (modulo permutation of x and y). Suppose it is $2a - c = c$. Then $2a - b \notin \{a, b, c\}$ and $2c - b \notin \{a, b, c\}$. The multiplication table for this case looks like Table 3.2.

(b) If there are no solutions of $2x - y = y$ satisfying $x \neq y$ then the multiplication table again looks like Table 3.2.

There are then two possibilities for r-colorings involving exactly three distinct colors a, b, c. Either when two distinct colors meet at a crossing the third one emerges from that crossing. This possibility corresponds to Table 3.1 and further implies that $3|r$, as it was seen above. Or there is always a color which cannot be assigned to an under-arc which ends at a crossing. This possibility corresponds to Table 3.2 where the special color is b. But with knots (or non-splittable links) any under-arc ends at a crossing. These knots and links can therefore only be colored with three distinct colors complying with the former possibility. This is the case with the $T(2, n)$ since for odd n they are knots and for even n they are links made of two linked components. We can thus state:

Proposition 3.5 *Assume K is non-splittable. If $3 \nmid r$, there is no r-coloring of a diagram of K which uses exactly three colors from $\{0, 1, \ldots, r-1\}$.*

Proof: Omitted. ■

Proposition 3.6 *If $\langle n, r\rangle = 5$ then*

$$\text{mincol}_r T(2, n) = 4$$

Proof: We recall that, via the proof of Proposition 2.2, there is a coloring that uses exactly 4 colors from R_r whenever $\langle n, r\rangle = 5$. Corollary 3.2 shows us that $\text{mincol}_r T(2, n)$ here cannot be 2. We will now show that $\text{mincol}_r T(2, n)$ cannot be 3 thus concluding the proof. Since $T(2, n)$ is either a knot or a link with two linked components then an r-coloring involving exactly three colors corresponds to the situation described by Table 3.1 according to the discussion preceding Proposition 3.5. In particular, $3|r$. Also, these three colors are algebraically closed under the given operation. This implies that if there is a diagram of $T(2, n)$ which is colored by these three colors then any other diagram of $T(2, n)$ is colored by these three colors. On the other hand, the closure of σ_1^n is colored by three colors only if $3|n$ (adapt the argument used in the end of the proof of Proposition 3.2). But if $3|r$ and $3|n$ then $\langle n, r\rangle \neq 5$, which contradicts the hypothesis. This concludes the proof.

■

Corollary 3.3 *Suppose $\langle n, r\rangle = 2k + 1$ with $k > 1$. Then*

$$3 < \text{mincol}_r T(2, n)$$

Proof: Omitted. ■

4 Illustrative examples of Teneva reduction on rational knots.

We will now consider applications of Teneva reduction to rational links. For facts and notation on rational knots we refer to [8].

We consider the rational knot $N\big[[8], [-6]\big]$ endowed with a non-trivial 7-coloring, using all 7 colors, view Figure 7.

The two portions of Figure 7 boxed by dotted lines can be regarded as σ_1^4 - the one on the left - and σ_1^{-4} - the one on the right. In Figure 8 we show an instance of Teneva reduction applied to this σ_1^{-4}: Note that had we continued performing type III Reidemeister moves in the Teneva reduction we would have increased the number of the colors in the coloring of the rational knot.

We remark that Teneva reduction can be applied in a similar way to the σ_1^4 in Figure 7. With these reductions the coloring on $N\big[[8], [-6]\big]$ depicted in Figure 7 has changed to the equivalent coloring shown in Figure 9, using only 5 colors - less 2 than the 7 colors used in Figure 7.

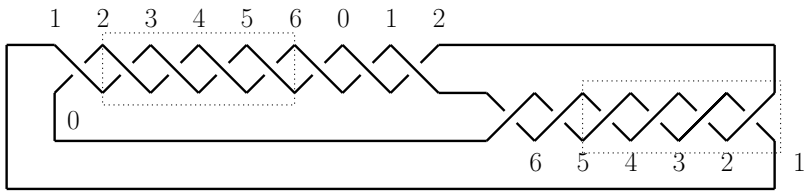

Figure 7: Non-trivial 7-coloring on $N\big[[8],[-6]\big]$ using the 7 colors

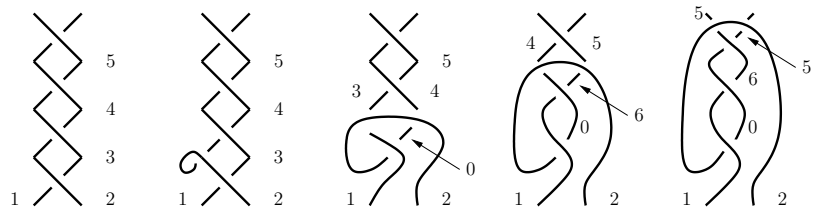

Figure 8: An instance of Teneva reduction

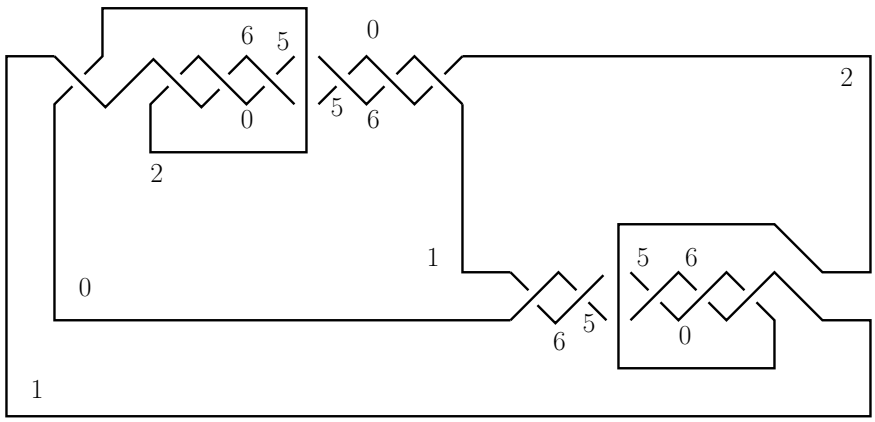

Figure 9: Reducing the colors of the non-trivial coloring from 7 to 5

We now show an example of a rational knot, R, of prime determinant, p, such that its minimum number of colors modulo p and over all diagrams is strictly less than the minimum number of colors over minimal diagrams. This rational knot is $N\big[[8],[-9]\big]$. Its determinant is the prime $73(= 9 \cdot 8 + 1)$. The Teneva reduction in this case is similar to the preceding one so we will just show the initial diagram endowed with a non-trivial 73-coloring, the portions of the diagram that will undergo Teneva reduction (inside the boxed areas), see Figure 10; and the final diagram after Teneva reduction has been performed on the indicated portions, see Figure 11.

In particular, note that the minimal alternating diagram on Figure 10 uses 17 colors to produce a non-trivial 73-coloring. This is in accordance with the Kauffman-Harary Conjecture (Conjecture 1.1) since this diagram has $9 + 8 = 17$ arcs. Notwithstanding, the diagram in Figure 11 is Reidemeister equivalent to the preceding one and uses only 12 colors to produce a non-trivial 73-coloring. In this way, our Conjecture (see Section 1) does not apply for general rational knots with "n" replaced by "Determinant of R". Moreover, this example shows that, for rational knots, the minimum number of colors predicted by the Kauffman-Harary conjecture, which considers only minimal diagrams, can be further decreased using non-minimal diagrams.

5 Final remarks

The main results of this paper are the expression of the number of r-colorings of a torus knot $T(2, n)$ for any integer $r > 2$ and the reduction of the upper bounds of the minimum number of colors necessary

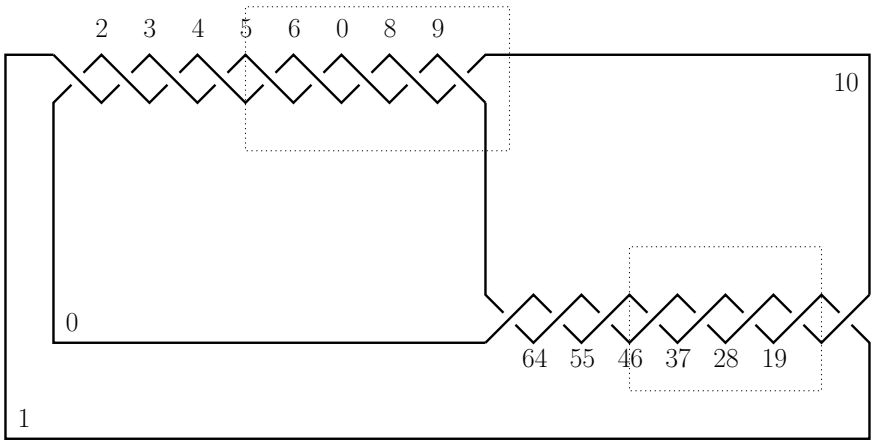

Figure 10: Non-trivial 73-coloring on $N\big[[8],[-9]\big]$ using the $17 = 8 + 9$ colors

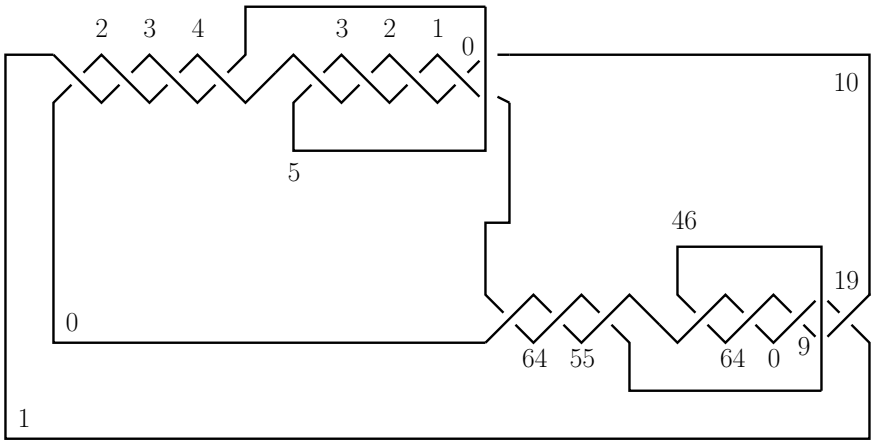

Figure 11: Non-trivial 73-coloring on $N\big[[8],[-9]\big]$ using 12 colors

to produce a non-trivial r-coloring using Teneva transformations. A first question we would like to answer concerns the truth of our conjecture on the minimum number of colors for the $T(2,n)$'s.

We regard this work as a case study which motivates us to attack more general situations. In this way, we would like to consider other classes of knots, other classes of labelling quandles, inquiring into the possibility of other sorts of transformations that allow one to reduce the number of colors of a non-trivial coloring, finding techniques to compute the minima of these colors given a diagram of a knot (at least for some classes of knots), etc. We aim to address these topics in future work.

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