Asymptotic Analysis 91 (2015) 205–231
DOI 10.3233/ASY-141259
IOS Press

A note on the transition from diffusion with the flow to diffusion against the flow, for first passage times in singularly perturbed drift–diffusion models

Charles Knessl [a] and Haishen Yao [b,*]

[a] *Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, Chicago, IL 60607-7045, USA*
E-mail: knessl@uic.edu

[b] *Department of Mathematics and Computer Science, QCC, The City University of New York, 222-05 56th Avenue, Bayside, NY 11364, USA*
E-mail: HYao@qcc.cuny.edu

Abstract. We consider some singularly perturbed ODEs and PDEs that correspond to the mean first passage time T until a diffusion process exits a domain Ω in $\mathbb{R}^n$. We analyze the limit of small diffusion relative to convection, and assume that in a part of Ω the convection field takes the process toward the exit boundary. In the remaining part the flow does not hit the exit boundary, instead taking the process toward a stable equilibrium point inside Ω. Thus Ω is divided into a part where the diffusion is with the flow and a complementary part where the diffusion is against the flow. We study such first passage problems asymptotically and, in particular, determine how T changes as we go between the two parts of the domain. We shall show that the mean first passage time may be exponentially large even in the part of Ω that is with the flow, and where a typical sample path of the process hits the exit boundary on much shorter time scales.

Keywords: first passage time, diffusion process, convection field

1. Introduction

Diffusion processes/models have been used in many applied areas, dating back to the work of Einstein on Brownian motion [4].

Subsequent early work on diffusions and stochastic differential equations was done by Smoluchowski, Langevin, Ornstein and Uhlenbeck, and Kramers (see [8,13,17,18] and the early survey in [1]). Such models have found applications in a large number of areas, including chemical kinetics, genetics, signal filtering, and mathematical finance (see the books of Schuss for more discussion of various applications [15,16]). At times diffusion models arise naturally, while at other instances they arise as continuous limits of discrete models, where the limit may be viewed as a type of functional central limit theorem. Such is often the case for problems in mathematical biology and queuing theory, where a discrete model

*Corresponding author. E-mail: HYao@qcc.cuny.edu.

may be approximated by a continuous diffusion model as the size of the (discrete) state space becomes large.

Give a diffusion process $X_d(t)$, whose state space is $\mathbb{R}^n$ or a subset thereof, one is often interested in computing the probability, $\Pr[X_d(t) \in S | X_d(0) = \mathbf{x}]$, that the process is in some set S at time t given that it started at the point $\mathbf{x}$ at time $t = 0$. Another important quantity is the first passage time until the process $X_d(t)$ reaches some set or region. Typically we would start with $X_d(0) = \mathbf{x}$, for $\mathbf{x} \in \Omega \subset \mathbb{R}^n$ ($\dim(\Omega) = n$) and ask when the process exits Ω, and thus reaches the boundary $\partial\Omega$, which will typically be some $(n-1)$-dimensional manifold. The physical significance of the first passage time depends on the particular application. For example, it may be the time for the crossing of a potential barrier leading to a chemical reaction [8], the time until a cycle slip in a phased-locked loop [20], the time until a queuing system reaches its maximum capacity of customers, or the time until a certain biological species becomes extinct [9]. More on first passage times and applications appears in the books [15,16] and [14], with the latter being devoted entirely to first passage processes, including various applications such as fractal networks and reaction–diffusion problems.

For a model that incorporates both diffusive and convective (or drift) effects, the mean first passage time

$$T(\mathbf{x}) = E\big[\tau | X_d(0) = \mathbf{x} \in \Omega\big],$$

where

$$\tau = \min\big\{t:\ X_d(t) \in \partial\Omega\big\},$$

satisfies an elliptic PDE of the form $L[T] = -1$. This is sometimes referred to as the Dyhkin equation [3], and is closely related to the backward Kolmogorov equation of Markov processes. Here L is an elliptic partial differential operator and we have also the boundary condition that $T = 0$ for $\mathbf{x} \in \partial\Omega$. For some simple models, such as a Brownian motion or an Ornstein–Uhlenbeck process, we can solve explicitly for the mean passage time, at least in one dimension. But for complicated drift/diffusion fields, complicated geometries of $\partial\Omega$, and problems in dimension $n > 1$ it is usually very difficult to solve exactly for T, and thus approximate, e.g., asymptotic or numerical, methods must be used. A popular and fruitful limit to consider is where convection dominates diffusion, and then L has a form such as $L = \varepsilon\Delta + \mathbf{a}(\mathbf{x}) \cdot \nabla$, so a small parameter multiplies the highest derivatives in L, and the problem becomes of "singular perturbation" type. Singular perturbation methods for computing the first passage times are developed in [10–12], and in the probability literature such an asymptotic limit is often treated using the theory of large deviations (see [5,19] and [2] for some classic references).

The behavior of the first passage time $T(\mathbf{x})$ as $\varepsilon \to 0^+$ is highly dependent upon the behavior of the drift field $\mathbf{a}(\mathbf{x})$ and how it interacts with the boundary $\partial\Omega$ which the process is to hit. We refer to the subcharacteristics of the PDE as the solutions to the ODE(s) $\dot{\mathbf{x}} = \mathbf{a}(\mathbf{x})$, which is an elementary dynamical system, and this corresponds to neglecting diffusion entirely and thus approximating $X_d(t)$ by a deterministic or "fluid" process. If the subcharacteristics hit the boundary we refer to the problem as "diffusion with the flow". Then the diffusive effects may be small. If, however, there is a globally stable equilibrium point $\mathbf{x}_0$ that is inside the domain Ω, then all subcharacteristics flow toward the equilibrium point and possibly none of them hit the boundary. Then the effects of diffusion become more significant and they are needed for the process to ever reach the exit boundary. This is called "diffusion against the flow", and a singular perturbation method for computing T is discussed in [10], where it was found that $T(\mathbf{x})$ is asymptotically a constant, say $C(\varepsilon)$, that is exponentially large for small ε (roughly $O(e^{K/\varepsilon})$ for some $K > 0$). Thus the passage time is independent of the starting value, since most likely the process will flow along a subcharacteristic toward the stable equilibrium at $\mathbf{x}_0$, and spend a long time in a neighborhood of this point before finally undertaking the "large deviation" to exit the domain Ω.

The purpose of this note is to study the asymptotics of the mean first passage time for problems where some subcharacteristics in Ω hit the exit boundary while others do not. Thus we study the transition from diffusion with the flow to that of against the flow. By studying some model problems in one and two dimensions, we shall gain a better understanding of this transition, for $\varepsilon \to 0^+$. We divide Ω into the two parts D_+ and D_-, where in D_+ a subcharacteristic flows towards the stable equilibrium without hitting $\partial\Omega$ while in D_- it hits the boundary. We shall show that while in D_+ the solution is asymptotically exponentially large and constant, in D_- it may be exponentially large but highly dependent on the initial value of the process. In other parts of D_- the solution may be $O(1)$ as $\varepsilon \to 0$, and there the deterministic approximation is valid. The analysis carefully studies the boundary region(s) near $\partial\Omega$, and also the transition curve $\mathcal{C}$ that separates D_+ from D_-. Near $\mathcal{C}$ we shall show that there are two nested internal layers that lead to different expansion of T; these correspond to the distance from $\mathcal{C}$ being either $O(\varepsilon^{1/3})$ or $O(\sqrt{\varepsilon})$, and we note that $\mathcal{C}$ is (a portion of) the unique subcharacteristic that becomes tangent to the boundary $\partial\Omega$. The main focus here is on the mathematical (singular perturbation) methodology.

The remainder of the paper is organized as follows. In Section 2 we consider one-dimensional models, that may be solved exactly, and make some observations about the structure of the first passage time as $\varepsilon \to 0$. In Section 3 we analyze in detail a two-dimensional problem, that cannot be analyzed exactly, and that will show the basic asymptotic approach. A brief discussion, and possible generalizations, appears in Section 4.

2. One-dimensional passage times

We consider a Brownian particle $X(t)$ moving in a potential field $U(x)$, with a small diffusion coefficient ε. Then we define $\tau = \min\{t\colon X(t) = x_0\}$ to be the first passage time to the point x_0. The mean value

$$T(x) = E\big[\tau \big| X(0) = x \leqslant x_0\big] \tag{2.1}$$

satisfies the backward Kolmogorov equation

$$\varepsilon T''(x) - U'(x)T'(x) = -1, \quad x < x_0 \tag{2.2}$$

with the "absorbing" boundary condition

$$T(x_0) = 0. \tag{2.3}$$

We assume that the potential $U(x)$ satisfies $U'(x) < 0$ for $x < x_2$ or $x > x_1$, $U'(x) > 0$ for $x_2 < x < x_1$, for some $x_2 < x_1 < x_0$, $U(-\infty) = \infty$, $U'(x_2) = 0 = U'(x_1)$ and $U''(x_2) > 0$, $U''(x_1) < 0$. Hence, x_2 is a local minimum of the potential while x_1 is a local maximum. Equivalently, x_2 (resp., x_1) is a stable (resp., unstable) equilibrium point of the flow $\dot{x} = a(x) \equiv -U'(x)$, which is a deterministic, or "fluid", approximation to the stochastic process $X(t)$. We also assume that $U(x)$ is smooth for all $x \leqslant x_0$, and in Fig. 1 we sketch a typical potential.

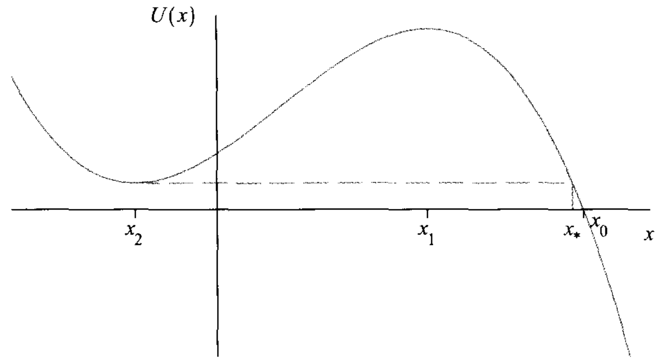

Fig. 1. A sketch of a typical potential function $U(x)$. (Colors are visible in the online version of the article; http://dx.doi.org/10.3233/ASY-141259.)

The mean first passage time $T(x)$ depends also on ε and the exit point x_0, and we evaluate it in the limit of small diffusion ($\varepsilon \to 0^+$) and various ranges of x and x_0. The ODE in (2.2) is easily solved to give:

Proposition 1. *The mean first passage time has the integral representation*

$$T(x) = \frac{1}{\varepsilon}\int_x^{x_0}\int_{-\infty}^{\eta} \exp\left[\frac{U(\eta)-U(\xi)}{\varepsilon}\right] \mathrm{d}\xi\, \mathrm{d}\eta, \quad x \leqslant x_0. \tag{2.4}$$

For $\varepsilon \to 0$ and initial conditions $x < x_1$ the drift takes the process toward the stable equilibrium at $x = x_2$, where it spends a large amount of time before finally exiting at x_0. This type of behavior is essentially independent of the starting point x, unless x becomes very close to the unstable equilibrium at x_1, and is referred to as "small diffusion against the flow". In contrast, if $x > x_1$ then $U'(x) < 0$ for $x \in (x_1, x_0]$ and the drift $a(x) = -U'(x)$ takes the process toward the exit point x_0. This is called "small diffusion with the flow", and then we might expect that for $\varepsilon \to 0$, (2.2) and (2.3) may be approximated by the first order equation

$$-U'(x)T'(x) = -1, \quad T(x_0) = 0 \tag{2.5}$$

whose solution is

$$T(x) = \int_x^{x_0} \frac{1}{-U'(\eta)}\,\mathrm{d}\eta, \quad x_1 < x \leqslant x_0. \tag{2.6}$$

This means that we approximate the stochastic process by its fluid limit and thus neglect diffusion completely. But, we show below that (2.6) may hold only in a certain subset of the interval $[x_1, x_0]$, and in fact it may *never* hold. Below we summarize several asymptotic results for (2.4) as $\varepsilon \to 0$ and thus identify precisely the conditions under which (2.6) holds.

Proposition 2. *For* $\varepsilon \to 0$, $T(x)$ *in* (2.4) *has the following asymptotic behaviors*:

(i) $x < x_1$

$$T(x) \sim \frac{2\pi}{\sqrt{U''(x_2)[-U''(x_1)]}} \exp\left[\frac{U(x_1)-U(x_2)}{\varepsilon}\right], \tag{2.7}$$

(ii) $x - x_1 = \sqrt{\varepsilon}\beta = \mathrm{O}(\sqrt{\varepsilon})$

$$T(x) \sim \frac{\sqrt{2\pi}}{\sqrt{U''(x_2)}} \exp\left[\frac{U(x_1) - U(x_2)}{\varepsilon}\right] \int_\beta^\infty \mathrm{e}^{-|U''(x_1)|v^2/2}\,\mathrm{d}v, \tag{2.8}$$

(iii) $x_1 < x < x_0$ *and* $U(x_0) > U(x_2)$

$$T(x) \sim \frac{\sqrt{2\pi\varepsilon}}{-U'(x)\sqrt{U''(x_2)}} \exp\left[\frac{U(x) - U(x_2)}{\varepsilon}\right], \tag{2.9}$$

(iv) $x - x_0 = -\varepsilon z = \mathrm{O}(\varepsilon)$ *and* $U(x_0) > U(x_2)$

$$T(x) \sim \frac{\sqrt{2\pi\varepsilon}}{-U'(x_0)\sqrt{U''(x_2)}} \exp\left[\frac{U(x_0) - U(x_2)}{\varepsilon}\right] \left(\mathrm{e}^{|U'(x_0)|z} - 1\right), \tag{2.10}$$

(v) $x_* < x < x_0$ *and* $U(x_0) < U(x_2)$, *with* x_* *defined by the condition* $U(x_2) = U(x_*)$

$$T(x) \sim \int_x^{x_0} \frac{1}{-U'(\eta)}\,\mathrm{d}\eta; \tag{2.11}$$

for $x_1 < x < x_*$ (2.9) *holds, and for* $x \approx x_*$ $T(x)$ *is asymptotic to the sum of* (2.9) *and* (2.11), *expanded about* $x = x_*$. *More precisely, if* $x - x_* = \varepsilon\alpha = \mathrm{O}(\varepsilon)$, *then*

$$T(x) = \int_{x_*}^{x_0} \frac{1}{-U'(\eta)}\,\mathrm{d}\eta + \frac{\sqrt{2\pi\varepsilon}}{-U'(x_*)\sqrt{U''(x_2)}} \mathrm{e}^{U'(x_*)\alpha} + \mathrm{O}(\varepsilon). \tag{2.12}$$

These results show that if $U(x_0) > U(x_2)$, i.e., the value of the potential at the exit boundary exceeds that at the stable equilibrium, the "fluid" behavior in (2.6) is never observed, and the first passage time is always exponentially large in ε. In such cases $T(x)$ is exponentially large and independent of x for $x < x_1$ and undergoes the transition, as x increases through x_1, to the behavior in (2.9) which does depend significantly on the starting point x. The expression in (2.10) may be viewed as a "boundary layer" correction to (2.9), for starting points near the exit boundary. Note that in order for $T(x)$ to become roughly O(1), $z = (x_0 - x)/\varepsilon$ would need to be *exponentially small*, of the order $\mathrm{O}(\mathrm{e}^{[U(x_2) - U(x_0)]/\varepsilon})$.

If $U(x_0) < U(x_2)$ then (2.6) applies but only in the range $x \in [x_*, x_0]$, and for $x \in [x_1, x_*)$ the first passage time remains exponentially large. The scale $x - x_* = \mathrm{O}(\varepsilon)$ represents the transition from $T(x)$ being exponentially large to being O(1), with the deterministic approximation (2.6) taking hold as $\alpha \to +\infty$.

The results in Proposition 2 may be explained intuitively as follows. For initial conditions x such that $x_1 < x < \min\{x_0, x_*\}$ a typical sample path of $X(t)$ is taken toward x_0 by the drift, but there is a very small (in fact, exponentially small) fraction of sample paths that reach the range $x < x_1$, which is the domain of attraction of the stable equilibrium point at x_2. While this fraction of paths is extremely small they tend to lead to exponentially large exit times (similarly to the case $x < x_1$). So, on the one hand a fraction close to one leads to the O(1) exit times in (2.7), but on the other hand a very small fraction leads to exponentially large times. Our analysis shows that the former sample paths dominate the mean exit time for $x \in (x_*, x_0)$ (if $U(x) = U(x_2)$ has a solution for $x \in (x_1, x_0)$) while the latter paths dominate for $x \in (x_1, x_*)$, and possibly for all $x < x_0$. We also note that neither of the expansions in (2.9) and (2.11) exhibit any singular behavior as $x \to x_*$, and (2.12) shows that for $x \approx x_*$ the two classes of sample paths contribute roughly equally to the expansion of $T(x)$.

Below we sketch only briefly the derivation of Proposition 2, since it follows from a standard application of the Laplace method to the integral in (2.4). We write $T(x) = \varepsilon^{-1} \int_x^{x_0} I(\eta;\varepsilon)\,\mathrm{d}\eta$ with

$$I(\eta;\varepsilon) = \int_{-\infty}^{\eta} \exp\left[\frac{U(\eta) - U(\xi)}{\varepsilon}\right] \mathrm{d}\xi. \tag{2.13}$$

If $\eta < x_2$ then the minimum of $U(\xi)$ in (2.13) occurs at the upper limit $\xi = \eta$ and by the Laplace method

$$I(\eta;\varepsilon) \sim \frac{\varepsilon}{-U'(\eta)}, \quad \eta < x_2. \tag{2.14}$$

If $\eta \approx x_2$ we set $\eta - x_2 = \sqrt{\varepsilon}\omega = \mathrm{O}(\sqrt{\varepsilon})$ and expand $U(\xi)$ about $\xi = \eta$ to obtain

$$I(\eta;\varepsilon) \sim \frac{\sqrt{\varepsilon}}{\sqrt{U''(x_2)}} \int_0^{\infty} \mathrm{e}^{-v^2/2} \exp\bigl(\sqrt{U''(x_2)}\omega v\bigr), \quad \omega = \frac{\eta - x_2}{\sqrt{\varepsilon}}. \tag{2.15}$$

If $x_2 < \eta < x_1$ the minimum of $U(\xi)$ is attained at the interior points $\xi = x_2$ and we have

$$I(\eta;\varepsilon) \sim \frac{\sqrt{2\pi\varepsilon}}{\sqrt{U''(x_2)}} \exp\left[\frac{U(\eta) - U(x_2)}{\varepsilon}\right], \quad x_2 < \eta < x_1. \tag{2.16}$$

If $\eta > x_1$, $U(\xi)$ has local minima both at $\xi = x_2$ and at the upper limit $\xi = \eta$. The relative sizes of these two contributions depend on whether $U(\eta) \gtrless U(x_2)$. Thus if $U(\eta) = U(x_2)$ has the solution $\eta = x_*$ ($> x_1$) then for $\eta < x_*$ (2.16) holds, while (2.14) applies for $\eta > x_*$. For $\eta \approx x_*$ we simply add (2.14) to (2.16).

Now we integrate $\varepsilon^{-1} I$ from $\eta = x$ to $\eta = x_0$, noting that different approximations (out of (2.13)–(2.16)) may apply in different subintervals. Suppose first that $x \in (x_1, x_0)$ and $U(x_0) > U(x_2)$. Then (2.16) applies over $\eta \in [x, x_0]$ and the integral over η is a Laplace integral with the major contribution coming from the lower limit $\eta = x$; this leads to the expression in (2.9). If $U(x_0) > U(x_2)$ and $x \approx x_0$ (more precisely $x - x_0 = -\varepsilon z = \mathrm{O}(\varepsilon)$) then the interval of integration is small, but we may still approximate the integrand using (2.16). This leads to

$$T(x) \sim \int_{x_0 - \varepsilon z}^{x_0} \sqrt{\frac{2\pi}{\varepsilon U''(x_2)}} \mathrm{e}^{[U(x_0) - U(x_2)]/\varepsilon} \exp\left[\frac{U'(x_0)}{\varepsilon}(\eta - x_0)\right] \mathrm{d}\eta, \tag{2.17}$$

where we expanded (2.16) about $\eta = x_0$. Evaluating the integral in (2.17) leads to the expression in (2.10). If $U(x_0) < U(x_2)$ and $x \in (x_*, x_0)$, (2.14) applies over the entire range of integration and we thus obtain (2.11). But, if $x \in (x_1, x_*)$, (2.16) applies and we again obtain (2.9). If $x < x_2$, regardless of the sign of $U(x_0) - U(x_2)$, we use (2.14)–(2.16) to conclude that the main contribution to the η-integral comes from $\eta = x_1$, where $U(\eta)$ has a local maximum; we thus obtain the constant in (2.7). If $x \approx x_1$, specifically $x - x_1 = \sqrt{\varepsilon}\beta = \mathrm{O}(\sqrt{\varepsilon})$ then we expand (2.16) about $\eta = x_1$ to obtain

$$T(x) \sim \frac{\sqrt{2\pi}}{\sqrt{\varepsilon U''(x_2)}} \exp\left[\frac{U(x_1) - U(x_2)}{\varepsilon}\right] \int_{x_1+\sqrt{\varepsilon}\beta}^{\infty} \mathrm{e}^{U''(x_1)(\eta - x_1)^2/(2\varepsilon)}\,\mathrm{d}\eta \tag{2.18}$$

and this leads to (2.8). This completes the derivations.

3. A two-dimensional example

Here we analyze a singularly perturbed two-dimensional diffusion model, which will illustrate some asymptotic phenomena similar to those in Section 2, but also lead to new complications. Since this model does not seem exactly solvable, we shall apply a singular perturbation approach.

3.1. Problem statement

Let Ω be a domain in $\mathbb{R}^2$ and let τ be the random time until a diffusion process $(X(t), Y(t))$ exits Ω by hitting the boundary $\partial\Omega$, i.e., $\tau = \min\{t\colon (X(t), Y(t)) \in \partial\Omega\}$. Taking the drift vector as $(a(x,y), b(x,y))$, and again assuming a small diffusion coefficient ε, then mean first passage time

$$T(x,y) = E\big[\tau \big| \big(X(0), Y(0)\big) = (x,y) \in \Omega\big] \tag{3.1}$$

satisfies

$$\varepsilon[T_{xx} + T_{yy}] + a(x,y)T_x + b(x,y)T_y = -1, \quad (x,y) \in \Omega, \tag{3.2}$$

$$T(x,y) = 0, \quad (x,y) \in \partial\Omega. \tag{3.3}$$

We shall assume that there is a unique stable equilibrium point at $(x,y) = (0,0)$ that is inside Ω and consider the drift field

$$a(x,y) = -ax, \qquad b(x,y) = -y, \tag{3.4}$$

where a is a constant. The boundary will be the straight line $x + y = 1$ so that

$$\Omega = \big\{(x,y)\colon\ x + y \leqslant 1\big\}. \tag{3.5}$$

For simplicity we shall also take $a = 2$, but the analysis is virtually unchanged for any $a > 1$. In Fig. 2 we sketch the domain Ω.

If $\varepsilon = 0$ the problem in (3.2) becomes a deterministic one, and T becomes the time it takes for a "subcharacteristic" curve to reach the line that is $\partial\Omega$. The subcharacteristics are defined as the solutions to $(\dot{x}, \dot{y}) = (-ax, -y)$ so we get the curves $y = C|x|^{1/a}$ and if $a = 2$ these are the parabolas $y^2 = kx$, $k \in \mathbb{R}$. The origin is thus a stable improper node of this simple dynamical system. Through each point $(x,y) \neq (0,0)$ passes a unique subcharacteristic that eventually approaches the origin. But a subcharac-

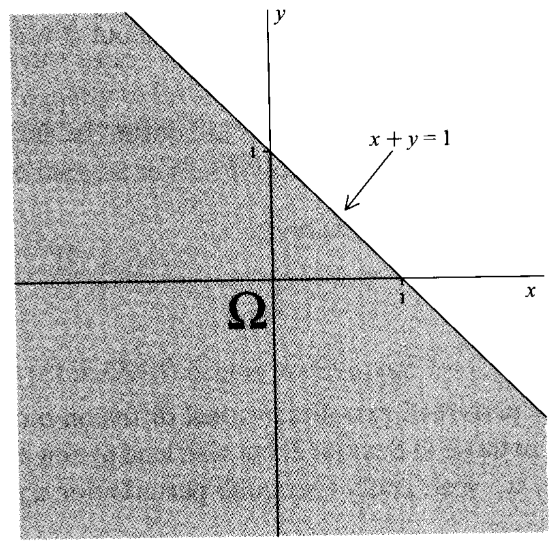

Fig. 2. A sketch of the domain Ω. (Colors are visible in the online version of the article; http://dx.doi.org/10.3233/ASY-141259.)

teristic may hit the exit boundary $x + y = 1$ before reaching the origin. Thus we may naturally divide Ω into two main parts, writing

$$\Omega = D_+ \cup D_- \cup \mathcal{C}, \tag{3.6}$$

where

$$D_+ = \{(x, y)\colon\ y < 2\sqrt{-x}, x < -1 \text{ or } y < 1 - x, x \geqslant -1\} \tag{3.7}$$

and

$$D_- = \{(x, y)\colon\ y > 2\sqrt{-x}, x < -1\}. \tag{3.8}$$

The curve $\mathcal{C} = \{(x, y)\colon\ y = 2\sqrt{-x}, x < -1\}$ is the unique subcharacteristic that becomes tangent to the boundary (at the point $(-1, 2)$) before continuing toward the origin.

Thus for $(x, y) \in D_+$ the drift field takes the process toward the stable equilibrium before hitting the boundary, while for $(x, y) \in D_-$ we hit the line $x + y = 1$. We would hence expect $T(x, y)$ to be, for small ε, very large and nearly constant in D_+, but be $O(1)$ for $(x, y) \in D_-$. Neglecting diffusion entirely in (3.2) and solving the elementary PDE leads to

$$T(x, y) = \log\left[\frac{y - \sqrt{y^2 + 4x}}{2}\right], \quad (x, y) \in D_- \tag{3.9}$$

and we note that the above satisfies $T(x, 1 - x) = 0$ for $x \leqslant -1$. As y decreases through $\mathcal{C}$ the expression in (3.9) becomes complex and thus clearly invalid. However, we shall show that (3.9) is correct only in a subset of D_-, just as (2.11) was only valid for $x \in (x_*, x_0) \subset (x_1, x_0)$. We shall also obtain the correct form of $T(x, y)$ in the remaining part of D_-, and study carefully the asymptotic transition(s) along the critical parabola $y^2 + 4x = 0$ ($y > 2$, $x < -1$), and near the "corner point" $(2, -1)$ where the curve is tangent to the exit boundary; it is at this corner where the structure of $T(x, y)$ is the most complicated.

3.2. Summary of results

An asymptotic analysis of (3.5) leads to the following results.

Proposition 3. *The asymptotic expansions of* $T(x,y)$ *as* $\varepsilon \to 0$ *are*

(i) $(x,y) \in D_+$

$$T(x,y) \sim C(\varepsilon) \equiv \frac{3\sqrt{3\pi\varepsilon}}{4}\exp\left(\frac{1}{3\varepsilon}\right), \tag{3.10}$$

(ii) $1-x-y = \varepsilon\zeta = \mathrm{O}(\varepsilon)$, $x > -1$ (*where exit boundary meets* D_+)

$$T(x,y) \sim C(\varepsilon)\left[1-\exp\left(-\frac{x+1}{2}\zeta\right)\right], \tag{3.11}$$

(iii) $1-x-y = \varepsilon^{2/3}u = \mathrm{O}(\varepsilon^{2/3})$, $x+1 = \frac{1}{2}\tau\varepsilon^{1/3} = \mathrm{O}(\varepsilon^{1/3})$ (*where* D_+, D_- *and* $\partial\Omega$ *all meet*; $u>0$, $-\infty<\tau<\infty$)

$$T(x,y) \sim C(\varepsilon)\exp\left(\frac{\tau^3}{384}-\frac{u\tau}{8}\right)\widetilde{S}(u_1,\tau_1), \tag{3.12}$$

where $\tau_1 = 2^{-7/3}\tau$, $u_1 = 2^{-2/3}u$ *and*

$$\widetilde{S}(u_1,\tau_1) = \sum_{k=0}^{\infty} \mathrm{e}^{-\tau_1 r_k}\frac{\mathrm{Ai}(u_1+r_k)}{[\mathrm{Ai}'(r_k)]^2}, \quad \tau_1 < 0. \tag{3.13}$$

Here $\mathrm{Ai}(\cdot)$ *is the Airy function, and the* r_k *are the roots of* $\mathrm{Ai}(\cdot)$, *ordered as* $r_0 > r_1 > r_2 > \cdots$, *so that* $r_0 = -2.3381\ldots$. *A representation of* $\widetilde{S}$ *that applies for all* τ_1 *is given by*

$$\widetilde{S}(u_1,\tau_1) = \mathrm{e}^{-\tau_1^3/3}\mathrm{e}^{\tau_1 u_1} - \overline{S}(u_1,\tau_1), \tag{3.14}$$

$$\begin{aligned}\overline{S}(u_1,\tau_1) = {} & \int_0^\infty \mathrm{Ai}(u_1+\xi)\mathrm{e}^{-\tau_1\xi}\,\mathrm{d}\xi \\ & + \int_0^\infty \frac{\mathrm{Ai}(u_1+\omega\xi)}{\mathrm{Ai}(\omega\xi)}\mathrm{Ai}(\xi)\mathrm{e}^{-\tau_1\omega\xi}\,\mathrm{d}\xi \\ & + \int_0^\infty \frac{\mathrm{Ai}(u_1+\omega^2\xi)}{\mathrm{Ai}(\omega^2\xi)}\mathrm{Ai}(\xi)\mathrm{e}^{-\tau_1\omega^2\xi}\,\mathrm{d}\xi,\end{aligned} \tag{3.15}$$

where $\omega = \mathrm{e}^{2\pi\mathrm{i}/3}$ *and* ω^2 *are cube roots of unity.*

(iv) $(x,y) \in D_-$

$$\begin{aligned}T(x,y) \sim {} & \log\left[\frac{y-\sqrt{y^2+4x}}{2}\right] \\ & + C(\varepsilon)\varepsilon^{1/6}\exp\left[\frac{1}{\varepsilon}\psi(x,y)\right]\exp\left[\frac{1}{\varepsilon^{1/3}}\psi_1(s)\right]\frac{\mathrm{e}^{(3/2)t}K_0(s)}{|\Delta_0(s,t)|^{1/2}},\end{aligned} \tag{3.16}$$

where

$$\psi(x,y) = \psi_0(s) - \frac{A^2}{4}\left(1 - \mathrm{e}^{-4t}\right) - \frac{B^2}{2}\left(1 - \mathrm{e}^{-2t}\right), \tag{3.17}$$

$$A = s + \frac{1}{\sqrt{8}}\sqrt{1 - 2s + 5s^2}, \qquad B = \frac{1-s}{2} - \frac{1}{\sqrt{8}}\sqrt{1 - 2s + 5s^2}, \tag{3.18}$$

$$\begin{aligned}\psi_0(s) = {} & \frac{3}{4}s^2 - \frac{1}{2}s - \frac{1}{20} + \frac{5s-1}{10\sqrt{2}}\sqrt{1 - 2s + 5s^2} \\ & + \frac{\sqrt{2}}{5\sqrt{5}}\log\left[\frac{5s - 1 + \sqrt{5}\sqrt{1 - 2s + 5s^2}}{2\sqrt{10} - 6}\right],\end{aligned} \tag{3.19}$$

and (s,t) *are related to* (x,y) *via the mapping*

$$x = \frac{1}{2}A\mathrm{e}^{-2t} + \left(s - \frac{A}{2}\right)\mathrm{e}^{2t}, \qquad y = B\mathrm{e}^{-t} + (1 - s - B)\mathrm{e}^{t}. \tag{3.20}$$

The range $t > 0$ *and* $s < -1$ *corresponds to* $(x,y) \in D_-$. *Furthermore,*

$$\psi_1(s) = \frac{r_0}{\sqrt{2}}\int_s^{-1} \frac{(-3v-1)^{2/3}}{\sqrt{1 - 2v + 5v^2}}\,\mathrm{d}v, \quad s \leqslant -1, \tag{3.21}$$

$$\begin{aligned}K_0(s) &= \frac{1}{\sqrt{2\pi}}\left[\mathrm{Ai}'(r_0)\right]^{-2}(-3s-1)^{1/6}F(s), \\ F(s) &= \left(\frac{-3s-1}{2}\right)^{1/3}\sqrt{\frac{8}{1 - 2s + 5s^2}}\left(\frac{2\sqrt{10} - 6}{5s - 1 + \sqrt{5}\sqrt{1 - 2s + 5s^2}}\right)^{\sqrt{10}/2}\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}\Delta_0(s,t) &= \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} \\ &= \frac{-3s-1}{16\sqrt{1 - 2s + 5s^2}}\Big[\left(\sqrt{2} - 5\sqrt{2}s + 3\sqrt{1 - 2s + 5s^2}\right)\mathrm{e}^{3t} \\ &\quad - \left(\sqrt{2} - 5\sqrt{2}s + \sqrt{1 - 2s + 5s^2}\right)\mathrm{e}^{t} \\ &\quad + \left(-\sqrt{2} + 5\sqrt{2}s + \sqrt{1 - 2s + 5s^2}\right)\mathrm{e}^{-t} \\ &\quad - \left(-\sqrt{2} + 5\sqrt{2}s + 3\sqrt{1 - 2s + 5s^2}\right)\mathrm{e}^{-3t}\Big]\end{aligned} \tag{3.23}$$

is the Jacobian associated with (3.20). *The equation*

$$\psi(x,y) + \frac{1}{3} = 0, \quad x \leqslant -1 \tag{3.24}$$

defines a curve Γ *in the* (x,y) *plane, which crosses the exit boundary at* $(x,y) \cong (-4.615, 5.615)$, *that divides* D_- *into the two parts* $D_- = D_* \cup D_*^C$ *where* D_* *corresponds to* $y > 2\sqrt{x}$ *with*

y below the curve Γ. Then in D_* the second term in (3.16) (*proportional to* $C(\varepsilon)$) *dominates and $T(x,y)$ is exponentially large in $1/\varepsilon$. In the complement D_*^C the first term dominates and $T(x,y) \sim \log\{\frac{1}{2}[y - \sqrt{y^2+4x}]\}$ for y above the curve Γ.*

(v) $1 - x - y = \varepsilon^{2/3}u = \mathrm{O}(\varepsilon^{2/3})$, $u > 0$, $-4.615\ldots < x < -1$

$$T(x,y) \sim C(\varepsilon)\exp\left[\frac{1}{\varepsilon}\psi_0(x)\right]\exp\left[\frac{1}{\varepsilon^{1/3}}\psi_1(x)\right] \times \left[\mathrm{Ai}'(r_0)\right]^{-2}\mathrm{e}^{-u\psi_y(x,1-x)/\varepsilon^{1/3}}F(x)\,\mathrm{Ai}\left(\frac{(-3x-1)^{1/3}}{2}u + r_0\right), \tag{3.25}$$

with

$$\psi_y(x, 1-x) = B(x) = \frac{1-x}{2} - \frac{1}{\sqrt{8}}\sqrt{1-2x+5x^2} \tag{3.26}$$

and $\psi_0(x)$, $\psi_1(x)$ and $F(x)$ are obtained by replacing s by x in (3.19), (3.21) *and* (3.23) (*in fact $s(x,1-x) = x$ by* (3.20) *with $t = 0$*). *When $x < -4.615\ldots$,* (3.9) *is asymptotically valid near the exit boundary $x + y = 1$.*

(vi) $y = 2\sqrt{-x} + \varepsilon^{1/3}\eta$, $\eta > 0$, $x < -1$ (*near curve $\mathcal{C}$*)

$$\begin{aligned} T(x,y) \sim{}& C(\varepsilon)\varepsilon^{1/6}\exp\left[\frac{\eta^2}{2\varepsilon^{1/3}}\psi_{yy}(x,2\sqrt{-x}) + \frac{\eta^3}{6}\psi_{yyy}(x,2\sqrt{-x})\right] \\ &\times \frac{2(-x)}{\sqrt{-1-x}\sqrt{1-3x}}\frac{2^{1/6}}{\sqrt{2\pi}} \\ &\times \sum_{k=0}^{\infty}\left[\mathrm{Ai}'(r_k)\right]^{-2}\exp\left[\frac{r_k\eta(-x)^{3/2}2^{5/3}}{(-1-x)(1-3x)}\right], \end{aligned} \tag{3.27}$$

where r_k are again the Airy roots, and

$$\psi_{yy}(x,2\sqrt{-x}) = \frac{2x}{(x+1)(3x-1)} < 0, \tag{3.28}$$

$$\psi_{yyy}(x,2\sqrt{-x}) = -\frac{\sqrt{-x}(43x^4+18x^2+3)}{(3x^2+2x-1)^3} < 0. \tag{3.29}$$

(vii) $y = 2\sqrt{-x} + \varepsilon^{1/2}\Delta$, $-\infty < \Delta < \infty$, $x < -1$ (*even nearer curve $\mathcal{C}$*)

$$T(x,y) \sim \frac{C(\varepsilon)}{\sqrt{\pi}}\int_{\Delta/f(x)}^{\infty}\mathrm{e}^{-v^2}\,\mathrm{d}v, \quad f(x) = \sqrt{\frac{(-1-x)(1-3x)}{-x}}. \tag{3.30}$$

The results in Proposition 3 show that the asymptotic structure of $T(x,y)$ is in fact quite intricate, even in the simple case of the linear drift functions in (3.4) and the linear exit boundary. Later we discuss possible generalizations. For now we note that $T(x,y)$ has essentially different behaviors in the three regions D_+, D_* and D_*^C. In D_+ the mean first passage time is exponentially large and independent of the starting point (x, y). In D_* it is also exponentially large in $1/\varepsilon$, with an additional subexponential factor that is $\exp[\mathrm{O}(\varepsilon^{-1/3})]$ (cf. (3.16)), and now depends intricately upon (x, y). In D_*^C we have $T = \mathrm{O}(1)$ and there (3.9) applies asymptotically, which neglects diffusion entirely.

The remaining five ranges in the (x, y) plane treated in Proposition 3 represent boundary layer and internal layer expansions. Cases (ii) and (v) apply near the exit boundary and (ii), which applies for $1 - x - y = \mathrm{O}(\varepsilon)$, gives a boundary layer correction to (3.10), while (v) applies in the thicker range $1 - x - y = \mathrm{O}(\varepsilon^{2/3})$ and gives a boundary layer correction to the second term in the right-hand side of (3.6). For values of $x < -4.615\ldots$ no boundary layer correction is needed as (3.9) satisfies the boundary condition for all $x < -1$.

The expressions in (3.27) and (3.30) represent internal layers that connect the expansions in D_* and D_+. As $\eta \to +\infty$ the $k = 0$ term in the sum in (3.27) dominates the others, and then we can easily show that this result agrees with (3.16) as $y \downarrow 2\sqrt{-x}$ (corresponding to $s \uparrow -1$). The matching between (3.27) and (3.30) occurs in an intermediate limit where $\eta \to 0^+$ and $\Delta = \varepsilon^{-1/6}\eta \to +\infty$. Now the integral in (3.30) can be approximated by

$$\frac{f(x)}{2\Delta} \exp\left[-\frac{\Delta^2}{f^2(x)}\right] = \frac{f(x)\varepsilon^{1/6}}{\eta} \exp\left[-\frac{\eta^2}{\varepsilon^{1/3} f^2(x)}\right], \tag{3.31}$$

and for $\eta \to 0^+$ the terms with $k \gg 1$ in the sum in (3.27) dominate. For $k \to \infty$ the Airy roots r_k can be approximated by $r_k \sim -(\frac{3}{2}k\pi)^{2/3}$ and the sum over k approximated by an integral, via the Euler–MacLaurin formula. Then the Airy functions disappear completely and, in view of (3.31), the matching between (3.27) and (3.30) follows. Note also that, by (3.28), $\psi_{yy}(x, 2\sqrt{-x}) = -2[f(x)]^{-2}$.

For $\Delta \to +\infty$, the right side of (3.30) approaches the constant $C(\varepsilon)$ and this agrees with (3.10), which applies in the interior of D_+. Thus the condition that $\Delta/f(x) = [y - 2\sqrt{x}]/[f(x)\sqrt{\varepsilon}] \gg 1$ is needed for $T(x, y)$ to become essentially independent of the starting point (x, y). With some work we can also verify the matching of the internal layers in (3.27) and (3.30), as $x \uparrow -1$, to the "corner layer" expression(s) in (3.12)–(3.15). Also, for $u \to 0$, $\tau \to +\infty$ with $u\tau$ fixed, this corner layer will match to the boundary layer in (3.11), as it is expanded for $x \downarrow -1$, $\zeta \to \infty$ with $(x + 1)\zeta$ fixed. For u fixed and $\tau \to -\infty$, (3.12) will match to the boundary layer in (3.25), for $x \uparrow -1$, as then $\frac{1}{2}(-3x - 1)^{1/3}u \sim 2^{-2/3}u = u_1$. More discussion of the various asymptotic matchings occurs within the derivations in Section 3.3.

In Fig. 3 we sketch the three main regions D_+, D_* and D_*^C, also indicating the separating curves Γ and $\mathcal{C}$ ($y = 2\sqrt{-x}$).

3.3. Derivations

We establish Proposition 3, via a singular perturbation analysis of (3.2) with (3.4) and (3.3). Thus we analyze the PDE

$$\varepsilon[T_{xx} + T_{yy}] - 2xT_x - yT_y = -1, \quad x + y < 1. \tag{3.32}$$

In the range D_+, where T is asymptotically large and independent of (x, y), the method of Matkowsky and Schuss [10,11] applies, which yields the results in (3.10) and (3.11). We merely sketch the main points. If $C(\varepsilon) \to \infty$ and $\varepsilon \to 0$, $T(x, y) \sim C(\varepsilon)$ is a possible asymptotic solution in D_+, as all subcharacteristics reach the origin. But this solution does not satisfy the boundary condition in (3.3).

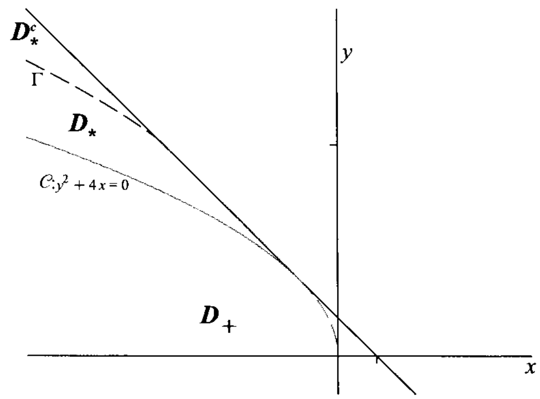

Fig. 3. A sketch of the three main regions and the curves that separate them. (Colors are visible in the online version of the article; http://dx.doi.org/10.3233/ASY-141259.)

Setting $z = x + y$ and $w = x - y$ the PDE in (3.32) becomes

$$2\varepsilon[T_{zz} + T_{ww}] - \frac{1}{2}(3z + w)T_z - \frac{1}{2}(z + 3w)T_w = -1, \quad z < 1 \tag{3.33}$$

and (3.3) becomes $T|_{z=1} = 0$. We can easily construct a boundary layer correction to the expansion in D_+ by setting $z = 1 - \varepsilon\zeta$, so that $w = 2x - 1 + \varepsilon\zeta \sim 2x - 1$. Then if $T \sim C(\varepsilon)T_0(\zeta, w)$ we obtain to leading order from (3.33)

$$2T_{0,\zeta\zeta} + \frac{3 + w}{2}T_{0,\zeta} = 0, \quad \zeta > 0, T_0|_{\zeta=0} = 0. \tag{3.34}$$

Solving (3.34) subject to the matching condition $T_0 \to 1$ as $\zeta \to \infty$ gives $T_0(\zeta, w) = 1 - \exp[-(w + 3)\zeta/4]$, which is equivalent to (3.11), as $w + 3 \sim 2(x + 1)$. This analysis only applies for $x > -1$, i.e., where the boundary $\partial\Omega$ meets D_+. To determine $C(\varepsilon)$ we multiply [10,11] (3.3) by $\exp[-\varepsilon^{-1}(x^2 + y^2/2)]$ and integrate over all of Ω. Then by the Laplace method

$$\iint_{x+y<1} \exp\left[-\frac{1}{\varepsilon}\left(x^2 + \frac{y^2}{2}\right)\right] \mathrm{d}x\,\mathrm{d}y \sim \sqrt{2}\pi\varepsilon$$

and after some integration by parts we are led to

$$\varepsilon\int_{-\infty}^{\infty}\big[T_x(1 - u, u) + T_y(1 - u, u)\big] \exp\left\{-\frac{1}{\varepsilon}\left[\frac{u^2}{2} + (1 - u)^2\right]\right\} du \sim -\sqrt{2}\pi\varepsilon. \tag{3.35}$$

The main contribution to the integral comes from $u = 2/3$, and using the boundary layer expansion $T \sim C(\varepsilon)T_0(\zeta, w)$ we have $T_x + T_y \sim 2C(\varepsilon)T_{0,z} \sim -2\varepsilon^{-1}C(\varepsilon)T_{0,\zeta}$ so that $(T_x + T_y)(1 - u, u)|_{u=2/3} \sim -\frac{4}{3}\varepsilon^{-1}C(\varepsilon)$. By the Laplace method the left-hand side of (3.35) becomes

$$-\frac{4}{3}\sqrt{\frac{2\pi\varepsilon}{3}}\exp\left(-\frac{1}{3\varepsilon}\right)C(\varepsilon) \tag{3.36}$$

and thus $C(\varepsilon)$ is as in (3.10), and the boundary layer correction near $D_+ \cap \partial\Omega$ is as in (3.11).

We next consider D_-, where the subcharacteristics reach the exit boundary $x + y = 1$, at points that have $x > -1$, so that (3.11) does not apply. We employ the ray method of geometrical optics, where we seek an asymptotic solution of (3.32) in the form

$$T(x,y) \sim \varepsilon^{\nu} C(\varepsilon) \exp\left[\frac{1}{\varepsilon}\psi(x,y)\right] \exp\left[\frac{1}{\varepsilon^{1/3}}\psi_1(x,y)\right] K(x,y). \tag{3.37}$$

Here ν is a constant and for convenience we included the multiplicative factor $C(\varepsilon)$, to facilitate the asymptotic matching between the regions D_- and D_+. As long as $C(\varepsilon)\mathrm{e}^{\psi/\varepsilon}$ is asymptotically large, which, in view of (3.10), requires that $\psi(x,y) + 1/3 > 0$, the term -1 in the right-hand side of (3.32) will be negligible compared to the left-hand side. But, if $\psi(x,y) + 1/3 < 0$ then we can construct an expansion of the solution of (3.32) that remains O(1) as $\varepsilon \to 0$, and this yields precisely (3.9) as the leading term. By linear superposition, $T(x,y)$ in D_- will be asymptotic to the sum of (3.9) and (3.37), with the condition $\psi(x,y) + 1/3 = 0$ determining a curve Γ in the (x,y) plane, above (resp., below) which (3.9) (resp., (3.37)) is the dominant part. Of course, we have yet to determine $\psi(x,y)$.

Using (3.37) in (3.32) we obtain the eiconal and transport equations

$$2x\psi_x + y\psi_y - \psi_x^2 - \psi_y^2 = 0, \tag{3.38}$$

$$[2\psi_x - 2x]K_x + [2\psi_y - y]K_y + [\psi_{xx} + \psi_{yy}]K = 0. \tag{3.39}$$

In addition, the sub-exponential $(\mathrm{O}[\exp(\varepsilon^{-1/3})])$ factor in (3.37) must satisfy

$$[2\psi_x - 2x]\psi_{1,x} + [2\psi_y - y]\psi_{1,y} = 0. \tag{3.40}$$

The factor involving ψ_1 is needed in order to accomplish the asymptotic matching between (3.37) and the boundary layer near $x + y = 1$, $x < -1$, as well as the internal layer(s) that we shall construct near the transition curve $y = 2\sqrt{-x}$, $x < -1$. But since $\varepsilon^{1/3}$ does not appear in (3.32), ψ_1 satisfies a fairly simple homogeneous linear PDE, once ψ is computed.

Equation (3.38) admits many different solutions, depending on the "initial manifold" on which the values of ψ are given. The characteristic curves, or rays, for the PDE are obtained by solving the system

$$\frac{\mathrm{d}\psi_x}{\mathrm{d}t} \equiv \dot{\psi}_x = -2\psi_x, \qquad \dot{\psi}_y = -\psi_y, \tag{3.41}$$

$$\dot{x} = 2x - 2\psi_x, \qquad \dot{y} = y - 2\psi_y \tag{3.42}$$

and the solution ψ follows from

$$\dot{\psi} = \psi_x\dot{x} + \psi_y\dot{y} = -\psi_x^2 - \psi_y^2. \tag{3.43}$$

Here "$\cdot$" denotes the directional derivative along a ray. We choose the initial manifold to be the exit boundary, where $(x,y) = (s, 1-s)$, $s < -1$, so a ray hits the exit boundary where $x = s$. Furthermore, we need the solution where each ray is *tangent* to the exit boundary. Then the exit boundary will become a *caustic* of these rays. That this is the appropriate ray family for this problem can be argued by examining the structure of the corner layer, where $(x,y) \approx (-1,2)$ and (3.12) applies, as this solution involves the Airy functions that are characteristic of caustic curves. By expanding (3.12) as we go away from

$(-1, 2)$ into D_-, we can also argue that (3.37) is the appropriate form of the expansion, and in particular that the sub-exponential factor must be included. We shall first analyze D_-, then the boundary $\partial\Omega$ for $x < -1$, and finally the corner region near $(-1, 2)$. However, the singular perturbation analysis could also be done by considering the regions in a different order.

The solutions to the two ODEs in (3.41) are

$$\psi_x = A\mathrm{e}^{-2t}, \qquad \psi_y = B\mathrm{e}^{-t}, \tag{3.44}$$

where A, B are constant along a ray, so that A (resp., B) is the value of ψ_x (resp., ψ_y) along $x + y = 1$, where $t = 0$. Then solving (3.42) with (3.44) subject to $(x, y)|_{t=0} = (s, 1 - s)$ leads to (3.20), which expresses (x, y) in terms of A, B, s and t. Then we solve (3.43) to get

$$\psi(x, y) = \psi_0(s) - \frac{A^2}{4}\left(1 - \mathrm{e}^{-4t}\right) - \frac{B^2}{2}\left(1 - \mathrm{e}^{-2t}\right), \tag{3.45}$$

where $\psi_0(s)$ is ψ along the exit boundary. To determine A and B we first evaluate (3.38) along $x = s$, $y = 1 - s$, and thus

$$2sA + (1 - s)B = A^2 + B^2. \tag{3.46}$$

The tangency condition translates to $\mathrm{d}y/\mathrm{d}x|_{t=0} = \dot{y}/\dot{x}|_{t=0} = -1$ so that

$$1 - s - 2B = -(2s - 2A) \quad \text{or} \quad 2(A + B) = 1 + s. \tag{3.47}$$

Solving (3.46) and (3.47) for (A, B) in terms of s yields the expressions in (3.17), and we note that $A = B = 0$ when $s = -1$. The rays are discussed further in the Appendix. By differentiating ψ along the exit boundary we obtain $\psi_x\dot{x} + \psi_y\dot{y}|_{t=0} = \psi_0'(s) = A - B$ and thus, in view of (3.17),

$$\psi_0'(s) = \frac{3s - 1}{2} + \frac{1}{\sqrt{2}}\sqrt{1 - 2s + 5s^2}. \tag{3.48}$$

Integrating (3.48) subject to $\psi_0(-1) = 0$ leads to the expression in (3.19). Note that when $s = -1$, A, B and $\psi_0(s)$ all vanish, so that $\psi(x, 2\sqrt{-x}) = 0$ for $x < -1$. When $s = -1$ the corresponding ray has $x = -\mathrm{e}^{2t}$, $y = 2\mathrm{e}^{t}$, so that $y = 2\sqrt{-x}$, which is the subcharacteristic that goes through the origin and is tangent to the boundary. Thus this curve is both a ray for (3.38) and a subcharacteristic of (3.32). In Fig. 4 we sketch several of the caustic rays, which fill the domain D_-.

Having computed ψ in (3.37) we examine the higher order terms. First, Eq. (3.40) means that $\dot{\psi}_1 = 0$ so that $\psi_1(x, y)$ is constant along a ray and we write

$$\psi_1(x, y) = \psi_1(s). \tag{3.49}$$

To integrate (3.39), which may be written as $\dot{K} = [\psi_{xx} + \psi_{yy}]K$ along a ray, we first note that

$$t_x = \frac{y_s}{\Delta_0}, \qquad t_y = -\frac{x_s}{\Delta_0}, \qquad s_x = -\frac{y_t}{\Delta_0}, \qquad s_y = \frac{x_t}{\Delta_0}, \tag{3.50}$$

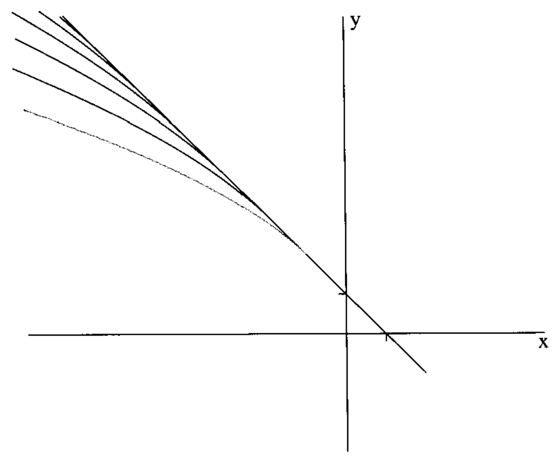

Fig. 4. A sketch of several of the caustic rays. (Colors are visible in the online version of the article; http://dx.doi.org/10.3233/ASY-141259.)

where $\Delta_0 = \Delta_0(s,t)$ is the Jacobian in (3.23). Then, by the chain rule, (3.50) and (3.42), we have

$$\begin{aligned}
\psi_{xx} + \psi_{yy} &= \psi_{xs}s_x + \psi_{xt}t_x + \psi_{ys}s_y + \psi_{yt}t_y \\
&= \frac{1}{\Delta_0}[\psi_{xt}y_s - \psi_{xs}y_t + \psi_{ys}x_t - \psi_{yt}x_s] \\
&= \frac{1}{\Delta_0}\left[\left(x - \frac{x_t}{2}\right)_t y_s - \left(x - \frac{x_t}{2}\right)_s y_t \right. \\
&\quad \left. + \left(\frac{y - y_t}{2}\right)_s x_t - \left(\frac{y - y_t}{2}\right)_t x_s\right] \\
&= \frac{1}{\Delta_0}\left[\frac{3}{2}(x_t y_s - x_s y_t) \right. \\
&\quad \left. + \frac{1}{2}(x_{ts}y_t - x_{tt}y_s + y_{tt}x_s - y_{ts}x_t)\right] \\
&= \frac{3}{2} - \frac{1}{2}\frac{\Delta_{0,t}}{\Delta_0}.
\end{aligned} \tag{3.51}$$

Thus the general solution to (3.39) is

$$K(x,y) = K_0(s)\mathrm{e}^{3t/2}\left|\Delta_0(s,t)\right|^{-1/2}. \tag{3.52}$$

It remains to determine the functions $K_0(s)$ in (3.52), $\psi_1(s)$ in (3.49), and the constant ν in (3.37). This we accomplish below, by asymptotic matching to some boundary and corner layer expansions. Some more properties of the rays and the function $\psi(x,y)$ are discussed in the Appendix.

We consider the range $x + y \approx 1$ with $x < -1$. Noting that

$$\psi(x,y) = \psi(x, 1-x) + \psi_y(x, 1-x)(y + x - 1) + \mathrm{o}(y + x - 1)$$

with $\psi(x,1-x)=\psi_0(x)$ and $\psi_y(x,1-x)=B(x)$, we introduce the new variable u with

$$1-x-y=\varepsilon^{2/3}u,\quad u\geqslant 0 \tag{3.53}$$

and for $u,x=\mathrm{O}(1)$ with $x>-1$ set

$$\begin{aligned}T(x,y)&=\varepsilon^{\nu_1}C(\varepsilon)\mathrm{e}^{\psi(x,1-x)/\varepsilon}\exp\biggl\{\frac{1}{\varepsilon^{1/3}}\bigl[\psi_1(x)-u\psi_y(x,1-x)\bigr]\biggr\}\\&\quad\times\bigl[B_0(x,u)+\varepsilon^{1/3}B_1(x,u)+\mathrm{O}\bigl(\varepsilon^{2/3}\bigr)\bigr].\end{aligned} \tag{3.54}$$

Here ν_1 is another constant that we shall determine later. The exponential factors in (3.54) must be included in order for (3.54) to have a chance (as $u\to+\infty$) of asymptotically matching to (3.37). By using (3.54) in the homogeneous version of (3.32), expanding for small ε, and noting that (3.38) holds along $x+y=1$, we obtain at the first two orders in ε the following equations:

$$2B_{0,uu}+\biggl[\frac{\sqrt{2}}{2}\sqrt{1-2x+5x^2}\psi_1'(x)+\frac{1}{4}(3x+1)u\biggr]B_0=0, \tag{3.55}$$

$$\begin{aligned}&2B_{1,uu}+\biggl[\frac{\sqrt{2}}{2}\sqrt{1-2x+5x^2}\psi_1'(x)+\frac{1}{4}(3x+1)u\biggr]B_1\\&\quad=-\frac{1}{2}B_0-\frac{\sqrt{2}}{2}\sqrt{1-2x+5x^2}B_{0,x}\\&\qquad+\biggl[\biggl(2+\frac{\sqrt{2}(5x-1)}{2\sqrt{1-2x+5x^2}}\biggr)u+2\psi_1'(x)\biggr]B_{0,u}.\end{aligned} \tag{3.56}$$

In addition, (3.54) must satisfy the boundary condition (3.3), so that

$$B_0(x,0)=0,\qquad B_1(x,0)=0. \tag{3.57}$$

The PDEs in (3.55) and (3.56) are really ordinary differential equations in u, with x appearing only as a parameter in (3.55). The general solution to (3.55) that decays as $u\to+\infty$ is given by

$$B_0(x,u)=F(x)\,\mathrm{Ai}\biggl(\frac{(-3x-1)^{1/3}}{2}\biggl[u+\frac{2\sqrt{2}\sqrt{1-2x+5x^2}}{3x+1}\psi_1'(x)\biggr]\biggr), \tag{3.58}$$

where $\mathrm{Ai}(\cdot)$ is the Airy function, and $F(\cdot)$ is at this stage undetermined. In order to satisfy the first boundary condition in (3.57) at $u=0$ we must have

$$-\frac{\sqrt{2}\sqrt{1-2x+5x^2}}{(-3x-1)^{2/3}}\psi_1'(x)=r_k;\quad k=0,1,2,\ldots, \tag{3.59}$$

where the r_k are the roots of Airy function, thus $\mathrm{Ai}(r_k)=0$, and we order the roots as $0>r_0>r_1>\cdots$. We choose $k=0$, since the higher roots would lead to exponentially smaller terms in the ray expansion in (3.37). The necessity of taking $k=0$ will also follow from asymptotic matching considerations, between (3.37) and the corner scale, where $u = \mathrm{O}(1)$ and $x+1 = \mathrm{O}(\varepsilon^{1/3})$. Then setting $k=0$ in (3.59) we integrate the elementary ODE in (3.59), choosing $\psi_1(-1) = 0$ for convenience, and thus obtain (3.21). So, we have determined the subexponential factor in (3.37).

It remains to obtain $K_0(s)$ in (3.52) and $F(x)$ in (3.58). We next show that the analysis of (3.56) for the correction term B_1 in (3.54) will imply that $F(x)$ satisfies a certain linear differential equation. We note that changing variables from u to v, with

$$\begin{aligned} u &= \frac{2\sqrt{2}\sqrt{1-2x+5x^2}}{-3x-1}\psi_1'(x) + 2(-3x-1)^{-1/3}v \\ &= 2(-3x-1)^{-1/3}(v - r_0), \end{aligned} \tag{3.60}$$

(3.55) becomes $B_{0,vv} = vB_0$ and (3.56) becomes an inhomogeneous Airy equation, of the form

$$L[B_1] \equiv B_{1,vv} - vB_1 = \alpha(x)\,\mathrm{Ai}(v) + \beta(x)\,\mathrm{Ai}'(v) + \gamma(x)v\,\mathrm{Ai}'(v). \tag{3.61}$$

Here α, β and γ can be identified by using (3.58) in (3.56), and writing the result in terms of v instead of u. Since $\mathrm{Ai}''(v) = v\,\mathrm{Ai}(v)$ we have

$$\begin{aligned} &L\big[\mathrm{Ai}'(v)\big] = \mathrm{Ai}(v), \qquad L\left[\frac{1}{2}v\,\mathrm{Ai}(v)\right] = \mathrm{Ai}'(v), \\ &L\big[v^2\,\mathrm{Ai}(v) - 2\,\mathrm{Ai}'(v)\big] = 2v\,\mathrm{Ai}'(v). \end{aligned} \tag{3.62}$$

Using (3.62) we see that the general solution to (3.61) that decays as $v \to +\infty$ is given by

$$\begin{aligned} B_1 &= F_1(x)\,\mathrm{Ai}(v) + \alpha(x)\,\mathrm{Ai}'(v) + \frac{1}{2}\beta(x)v\,\mathrm{Ai}(v) \\ &\quad + \gamma(x)\left[\frac{1}{2}v^2\,\mathrm{Ai}(v) - \mathrm{Ai}'(v)\right]. \end{aligned} \tag{3.63}$$

But $B_1(x,0) = 0$ implies that the right-hand side of (3.63) must vanish when $v = r_0$ (corresponding, by (3.60), to $u=0$) and since $\mathrm{Ai}'(r_0) \neq 0$ we must have

$$\alpha(x) = \gamma(x). \tag{3.64}$$

By using (3.58) to evaluate the right-hand side of (3.56) we find that

$$\alpha(x) = (-3x-1)^{-2/3}\sqrt{2}\sqrt{1-2x+5x^2}F'(x) - (-3x-1)^{-2/3}F(x) \tag{3.65}$$

and

$$\gamma(x) = (-3x-1)^{-2/3}\left\{\frac{\sqrt{2}\sqrt{1-2x+5x^2}}{-(3x+1)}F(x) + \left[4 + \frac{\sqrt{2}(5x-1)}{\sqrt{1-2x+5x^2}}\right]F(x)\right\}. \tag{3.66}$$

Using (3.64)–(3.66) leads to

$$\frac{F'(x)}{F(x)} = -\frac{5x-1}{1-2x+5x^2} + \frac{1}{3x+1} - \frac{5}{\sqrt{2}\sqrt{1-2x+5x^2}} \tag{3.67}$$

and thus

$$F(x) = F(-1)\frac{2^{7/6}(2\sqrt{10}-6)^{\sqrt{10}/2}(-3x-1)^{1/3}}{\sqrt{1-2x+5x^2}(5x-1+\sqrt{5}\sqrt{5x^2-2x+1})^{\sqrt{10}/2}}. \tag{3.68}$$

We have thus obtained the leading term in (3.55), as

$$B_0(x,u) = F(x)\,\mathrm{Ai}\left(\frac{1}{2}(-3x-1)^{1/3}u + r_0\right),$$

up to the constant $F(-1)$.

We proceed to match the expansion (3.54), as $u \to \infty$, to the ray expansion (3.37), having now determined $\psi_1(x,y) = \psi_1(s)$ in (3.49), in view of (3.59). Using the expansion of the Airy function in the form

$$\mathrm{Ai}(z+a) \sim \frac{1}{2\sqrt{\pi}} z^{-1/4} \mathrm{e}^{-2z^{3/2}/3} \mathrm{e}^{-a\sqrt{z}}; \quad z \to \infty, a = \mathrm{O}(1) \tag{3.69}$$

we find that

$$\begin{aligned} &\varepsilon^{\nu_1} B_0(x,u) \\ &\quad \sim \varepsilon^{\nu_1} \frac{1}{2\sqrt{\pi}} 2^{1/4}(-3x-1)^{-1/12} u^{-1/4} F(x) \\ &\quad\quad \times \exp\left[-\frac{(-3x-1)^{1/2}}{3\sqrt{2}} u^{3/2}\right] \exp\left[-\frac{(-3x-1)^{1/6}}{\sqrt{2}} r_0 \sqrt{u}\right]. \end{aligned} \tag{3.70}$$

Now, as we approach the boundary $x + y = 1$ we have $s \to x$ and $t \to 0$ in (3.20) (with (3.18)). Also, the Jacobian in (3.23) vanishes, with

$$\Delta_0(s,t) \sim (-3s-1)t \sim (-3x-1)t, \quad t \to 0. \tag{3.71}$$

From (3.17) and (3.19), after some calculation, we find that

$$\begin{aligned} &\psi(x,y) - \psi_0(x) - \psi_x(x, 1-x)(y+x-1) \\ &\quad \sim -\frac{(-3x-1)^{1/2}}{3\sqrt{2}}(1-x-y)^{3/2} = -\varepsilon\frac{(-3x-1)^{1/2}}{3\sqrt{2}} u^{3/2} \end{aligned} \tag{3.72}$$

and

$$\begin{aligned}\psi_1(s) - \psi_1(x) &\sim \psi_1'(x)(s-x) \sim \psi_1'(x)\frac{\sqrt{1-2s+5s^2}}{\sqrt{2}}t \\ &\sim -(-3x-1)^{2/3} r_0 \frac{t}{2} \\ &\sim -\varepsilon^{1/3}(-3x-1)^{1/6}\frac{r_0}{\sqrt{2}}\sqrt{u}.\end{aligned} \tag{3.73}$$

Here we also used the asymptotic relations

$$\varepsilon^{2/3}u = 1 - x - y \sim \frac{-3s-1}{2}t^2 \sim \frac{-3x-1}{2}t^2, \quad t \to 0 \tag{3.74}$$

and

$$s - x \sim \frac{1}{\sqrt{2}}\sqrt{1-2x+5x^2}t, \quad t \to 0, \tag{3.75}$$

which follow by approximately inverting (3.20), with (3.18), for $t \to 0$. By using (3.70) in (3.54) and comparing the result to (3.37) (for $x + y \to 1$), we find that

$$\varepsilon^{\nu}(-3x-1)^{-1/2}t^{-1/2}K_0(x) \sim \varepsilon^{\nu_1}\frac{1}{2\sqrt{\pi}}F(x)\frac{2^{1/4}}{(-3x-1)^{1/12}u^{1/4}}. \tag{3.76}$$

But by (3.74) $t \sim \varepsilon^{1/3}\sqrt{u}\sqrt{2}(-3x-1)^{-1/2}$, so that the matching condition in (3.76) may be satisfied, provided that

$$\nu - \frac{1}{6} = \nu_1 \quad \text{and} \quad K_0(x) = \frac{(-3x-1)^{1/6}}{\sqrt{2\pi}}F(x). \tag{3.77}$$

We have thus determined the function $K_0(s)$ in (3.52), up to the multiplicative constant $F(-1)$. Also, the expansion in (3.54) is determined up to this constant, and the factor ε^{ν_1}.

Next we consider the "corner" region, where (x, y) is close to $(-1, 2)$. We set, as in item (iii) of Proposition 3,

$$1 - x - y = \varepsilon^{2/3}u, \qquad x + 1 = \frac{1}{2}\tau\varepsilon^{1/3} \tag{3.78}$$

and expand $T(x, y)$ on this scale as

$$T(x,y) = C(\varepsilon)\big[S(u,\tau) + o(1)\big]. \tag{3.79}$$

Using (3.78) and (3.79) in (3.32) leads to the limiting PDE

$$2S_{uu} + \frac{1}{2}\tau S_u + 4S_\tau = 0; \quad u > 0, -\infty < \tau < \infty. \tag{3.80}$$

The boundary condition is $S(0,\tau)=0$ for all τ, and we can also obtain asymptotic matching conditions for $S(u,\tau)$, as $\tau\to\pm\infty$. For $x>-1$ the boundary layer expansion in (3.11), noting that

$$\frac{1}{2}(x+1)\zeta=\frac{1}{2\varepsilon}(x+1)(1-x-y)=\frac{\tau u}{4},$$

leads to

$$S(u,\tau)\sim 1-\mathrm{e}^{-\tau u/4},\quad \tau\to\infty, u\to 0 \tag{3.81}$$

with $u\tau$ fixed. By expanding (3.54) as $x\to -1$, noting that

$$\psi(x,1-x)=\psi_0(x)\sim\frac{1}{48}(x+1)^3,$$
$$\psi_1(x)\sim\psi_1'(-1)(x+1)=-r_0 2^{-4/3}(x+1)$$

and

$$\psi_y(x,1-x)=B(x)\sim\frac{1}{4}(x+1),$$

we get a second matching condition:

$$\begin{aligned}S(u,\tau)&\sim\varepsilon^{\nu_1}F(-1)\mathrm{e}^{\tau^3/384}\mathrm{e}^{-u\tau/8}\\&\quad\times\exp\left[-\frac{r_0}{2^{7/3}}\tau\right]\mathrm{Ai}\big(2^{-2/3}u+r_0\big),\quad \tau\to-\infty.\end{aligned} \tag{3.82}$$

But then we must have $\nu_1=0$, and hence $\nu=1/6$ by (3.77). By making the substitution

$$S(u,\tau)=\mathrm{e}^{\tau^3/384}\mathrm{e}^{-u\tau/8}\widetilde{S}(u_1,\tau_1), \tag{3.83}$$

and also scaling $\tau=2^{7/3}\tau_1$, $u=2^{2/3}u_1$ the parabolic PDE in (3.80) becomes the separable PDE

$$\widetilde{S}_{u_1u_1}+\widetilde{S}_{\tau_1}-u_1\widetilde{S}=0;\quad u_1>0,-\infty<\tau_1<\infty. \tag{3.84}$$

The boundary condition is $\widetilde{S}(0,\tau_1)=0$ and the matching conditions in (3.82) and (3.81) translate to

$$\widetilde{S}\sim F(-1)\mathrm{e}^{-r_0\tau_1}\,\mathrm{Ai}(u_1+r_0),\quad \tau_1\to-\infty, \tag{3.85}$$
$$\widetilde{S}\sim\mathrm{e}^{-\tau_1^3/3}\big(\mathrm{e}^{u_1\tau_1}-\mathrm{e}^{-u_1\tau_1}\big),\quad \tau_1\to\infty, u_1\to 0. \tag{3.86}$$

We previously encountered very similar PDEs when analyzing Brownian particles moving in a time-dependent drift field [7], and problems in financial math dealing with option values evolving under the CEV (constant elasticity of variance) process [6]. The only difference was that in [6], τ_1 had the opposite sign in (3.84). The solution to (3.84), subject to the matching conditions in (3.85) and (3.86), and the boundary condition $\widetilde{S}(0,\tau_1) = 0$, is that in (3.13)–(3.15), and we have thus also determined $F(-1)$ as

$$F(-1) = \frac{1}{[\mathrm{Ai}'(r_0)]^2}. \tag{3.87}$$

This solution can also be obtained as a limiting case of some more general results in [7], where we analyzed problems similar to (3.84), but on semi-infinite τ_1 intervals. We have now completely determined the leading term in (3.37), where $\nu = 1/6$, and that in (3.54), where $\nu_1 = 0$.

It remains to analyze the vicinity of the transition curve $y = 2\sqrt{-x}$, and its analysis can be used to give an alternate derivation of (3.87). We first consider the scale $y - 2\sqrt{-x} = \varepsilon^{1/3}\eta$, with $\eta > 0$ and $x < -1$. On this scale we set

$$T(x,y) \sim \varepsilon^{1/6} C(\varepsilon) \mathrm{e}^{\psi(x,y)/\varepsilon} \widetilde{K}(x,y) G(x,y), \tag{3.88}$$

where ψ is as in (3.17) and

$$\widetilde{K}(x,y) = \frac{\mathrm{e}^{3t/2}}{|\Delta_0(s,t)|^{1/2}} \tag{3.89}$$

which corresponds to $K(x,y)$ in (3.37), divided by the factor $K_0(x)$. Then since ψ satisfies (3.38) and $\widetilde{K}$ satisfies (3.39), we find from (3.32) that

$$(2\psi_x - 2x)\frac{\partial G}{\partial x} + (2\psi_y - y)\frac{\partial G}{\partial y} = 0. \tag{3.90}$$

We rewrite (3.90) in terms of x and η, noting that as $s \to -1$ (which corresponds to $y \to 2\sqrt{-x}$) we have

$$\psi_x \sim A'(-1)(s+1)\mathrm{e}^{-2t} \sim \frac{s+1}{-4x}, \tag{3.91}$$

$$\psi_y \sim B'(-1)(s+1)\mathrm{e}^{-t} \sim \frac{s+1}{4\sqrt{-x}} \tag{3.92}$$

and

$$s + 1 \sim \varepsilon^{1/3} \frac{8(-x)^{3/2}}{(1-3x)(1-x)}\eta. \tag{3.93}$$

With (3.91)–(3.93), (3.90) is asymptotically equivalent to

$$-2x\frac{\partial G}{\partial x} + \left[-1 + \frac{4}{1-3x}\frac{1-x}{1+x}\right]\frac{\partial G}{\partial \eta} = 0, \tag{3.94}$$

whose general solution is

$$G(x,\eta) = G_0\left(\frac{\eta(-x)^{3/2}}{(1-3x)(-1-x)}\right). \tag{3.95}$$

We determine $G_0(\cdot)$ by asymptotically matching (3.88) to the corner layer.

First we have $\psi(x,y)=\psi(x,2\sqrt{-x}+\varepsilon^{1/3}\eta)$ so on the η-scale we can expand $\psi(x,y)$ in (3.88) about $y=2\sqrt{-x}$, retaining through cubic terms and noting that $B(-1)=0$, and thus $\psi_y(x,2\sqrt{-x})=0$. Also, in the transition layer $t\sim\log(\sqrt{-x})$ and

$$\widetilde{K}(x,y)\sim\widetilde{K}(x,2\sqrt{-x})\sim\frac{\mathrm{e}^{3t/2}}{|\Delta_0(-1,t)|^{1/2}}\sim\frac{2(-x)}{\sqrt{-1-x}\sqrt{1-3x}}. \tag{3.96}$$

Expanding the corner layer (cf. (3.12) with (3.13)) for $u_1\to+\infty$ and $\tau_1\to-\infty$ with $\tau_1+\sqrt{u_1}=\mathrm{O}(1)$, we obtain, using (3.69) to approximate the Airy functions,

$$C(\varepsilon)\mathrm{e}^{\tau^3/384}\mathrm{e}^{-u\tau/8}\mathrm{e}^{-2u_1^{3/2}/3}\frac{1}{2\sqrt{\pi}u_1^{1/4}}\sum_{k=0}^{\infty}\frac{\mathrm{e}^{(\tau_1+\sqrt{u_1})|r_k|}}{[\mathrm{Ai}'(r_k)]^2}, \tag{3.97}$$

where the series converges for $\tau_1+\sqrt{u_1}<0$. By comparing (3.97) to (3.88), after expanding $\widetilde{K}$ and ψ about $y=2\sqrt{-x}$, we find that the exponential factors agree automatically and matching of the algebraic factors leads to

$$\begin{aligned}&\varepsilon^{1/2}\frac{2(-x)}{\sqrt{-1-x}\sqrt{1-3x}}G_0\left(\frac{\eta(-x)^{3/2}}{(1-3x)(-1-x)}\right)\Bigg|_{x\to-1}\\&\quad\sim\frac{1}{2\sqrt{\pi}u_1^{1/4}}\sum_{k=0}^{\infty}\frac{\mathrm{e}^{(\tau_1+\sqrt{u_1})|r_k|}}{[\mathrm{Ai}'(r_k)]^2}.\end{aligned} \tag{3.98}$$

Now

$$-1-x=-\varepsilon^{1/3}\tau/2=-\varepsilon^{1/3}\cdot 2^{4/3}\tau_1\sim\varepsilon^{1/3}\cdot 2^{4/3}\sqrt{u_1}$$

and

$$\begin{aligned}\frac{\eta(-x)^{3/2}}{(1-3x)(-1-x)}&\sim\frac{y-2\sqrt{-x}}{4(-1-x)\varepsilon^{1/3}}\\&=\frac{y+x-1+(1-\sqrt{-x})^2}{-2\tau\varepsilon^{2/3}}\\&\sim\left[\varepsilon^{2/3}u-\frac{1}{16}\varepsilon^{2/3}\tau^2\right]\frac{1}{2\tau\varepsilon^{2/3}}\\&=\frac{1}{32\tau}\left[16u-\tau^2\right]=\frac{(\sqrt{u_1}-\tau_1)(\sqrt{u_1}+\tau_1)}{2^{8/3}\tau_1}\\&\sim-2^{-5/3}(\sqrt{u_1}+\tau_1).\end{aligned} \tag{3.99}$$

Using (3.95) we conclude that

$$G_0\left(-\frac{\sqrt{u_1}+\tau_1}{2^{5/3}}\right)=\frac{2^{1/6}}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\frac{\mathrm{e}^{(\sqrt{u_1}+\tau_1)|r_k|}}{[\mathrm{Ai}'(r_k)]^2}. \tag{3.100}$$

This determines the functional form of $G_0(\cdot)$, and using (3.100) in (3.95) and (3.88) leads to the expression in (3.27). We discuss in more detail the calculation of ψ_{yy} and ψ_{yyy} along $y = 2\sqrt{-x}$ in the Appendix.

Now (3.27) applies only for $\eta > 0$ as then the series converges, since $r_k < 0$. This expansion cannot match to the constant $T(x, y) \sim C(\varepsilon)$ that applies for $y < 2\sqrt{-x}$. We thus consider the scaling $y - 2\sqrt{-x} = \sqrt{\varepsilon}\Delta$ and note that $\Delta = \varepsilon^{-1/6}\eta$. Then setting

$$T(x, y) = C(\varepsilon)\big[H(x, \Delta) + o(1)\big] \tag{3.101}$$

we rewrite (3.3) on the (x, Δ) scale and obtain the limiting parabolic PDE

$$\frac{x-1}{x}H_{\Delta\Delta} - 2xH_x - \Delta H_\Delta = 0; \quad x < -1, -\infty < \Delta < \infty. \tag{3.102}$$

In order for H to match to the range $y < 2\sqrt{-x}$ we must have

$$H(x, \Delta) \to 1, \quad \Delta \to -\infty. \tag{3.103}$$

We can also derive a matching condition for H as $\Delta \to +\infty$ by requiring (3.101) to match to (3.27) as $\eta \to 0^+$. We can estimate the series in (3.27) for $\eta \to 0$ by noting that the Airy roots r_k satisfy $r_k \sim -(\frac{3}{2}k\pi)^{2/3}$ for $k \to \infty$ and then $\mathrm{Ai}'(r_k) \sim \pm(\frac{3}{2}k)^{1/6}\pi^{-1/3}$. We thus have, for $a \to 0$,

$$\begin{aligned}
&\sum_{k=0}^{\infty} \mathrm{e}^{-a|r_k|}\big[\mathrm{Ai}'(r_k)\big]^{-2} \\
&\quad\sim \sum_k \pi^{2/3}\left(\frac{2}{3}\right)^{1/3} k^{-1/3} \exp\left[-a\left(\frac{3}{2}\pi\right)^{2/3} k^{2/3}\right] \\
&\quad\sim a^{-1}\int_0^\infty \pi^{2/3}\left(\frac{2}{3}\right)^{1/3} u^{-1/3} \exp\left[-\left(\frac{3}{2}\pi\right)^{2/3} u^{2/3}\right] \mathrm{d}u \\
&\quad= a^{-1}.
\end{aligned} \tag{3.104}$$

Here we approximated a sum by an integral via the Euler–MacLaurin formula. We use (3.104) with $a = \eta(-x)^{3/2}2^{5/3}/[(-1-x)(1-3x)]$ and note that $\eta = o(1)$ and $\eta^2\varepsilon^{-1/3} = \Delta = O(1)$. Thus we get the matching condition

$$H(x, \Delta) \sim \frac{1}{\Delta}\frac{\sqrt{(-1-x)(1-3x)}}{2\sqrt{\pi}\sqrt{-x}} \exp\left[\frac{\Delta^2}{2}\psi_{yy}(x, 2\sqrt{-x})\right], \quad \Delta \to +\infty. \tag{3.105}$$

The solution to (3.102) that satisfies (3.103) and (3.104) can be obtained by using the similarity variable $\omega = \Delta/f(x)$. Setting $H(x, \Delta) = H_0(\Delta/f(x))$ we obtain

$$H_0''(\omega) + \frac{x\omega}{x-1}\big[2xf(x)f'(x) - f^2(x)\big]H_0'(\omega) = 0. \tag{3.106}$$

Choosing $f(\cdot)$ to satisfy the ODE

$$2xff' - f^2 = 2 - \frac{2}{x}, \quad f(-1) = 0 \tag{3.107}$$

we find that

$$f(x) = \sqrt{\frac{(3x-1)(1+x)}{-x}} \tag{3.108}$$

and then (3.106) becomes $H_0''(\omega) + 2\omega H_0'(\omega) = 0$ with $H_0(-\infty) = 1$ so that

$$H_0(\omega) = \frac{1}{\sqrt{\pi}} \int_\omega^\infty \mathrm{e}^{-v^2}\, \mathrm{d}v. \tag{3.109}$$

Then as $\Delta \to +\infty$ (thus $\omega \to +\infty$)

$$H(x, \Delta) \sim \frac{1}{2\sqrt{\pi}\omega} \mathrm{e}^{-\omega^2} = \frac{f(x)}{2\sqrt{\pi}\Delta} \exp\left[-\frac{\Delta^2}{f^2(x)}\right] \tag{3.110}$$

and this agrees with (3.105), as $\psi_{yy}(x, 2\sqrt{-x}) = -2/f^2(x)$. We have thus derived (3.30) and this completes the derivation of Proposition 3.

4. Discussion

We have thus studied some first passage problems, in the small diffusion limit, and where the domain of interest can be divided into two parts. In one part D_- the process flows towards the exit boundary and in the other part D_+ away from it, toward a stable equilibrium point. Our analysis of the simple two-dimensional example showed that the asymptotic structure of the problem can be quite intricate, necessitating the analysis of several different regions of the domain Ω. The exit boundary $\partial\Omega$ can also be divided into two parts, according to where D_+ (resp., D_-) meets $\partial\Omega$, and then the normal component of the flow points away from (resp., into) $\partial\Omega$. The key element of the analysis is to consider a vicinity of the point on $\partial\Omega$ where the flow becomes tangent to the boundary. This analysis involves a complicated combination of Airy functions, and shows that as we move away from the tangency point toward where D_- meets $\partial\Omega$, this part of the boundary becomes caustic, and this also indicates how the first passage time T behaves as we cross the curve $\mathcal{C}$ that separates D_+ from D_-. Upon crossing into D_- we enter a range D_* where the caustic ray expansion dominates and the mean first passage times are exponentially large but depend on the starting point. It is not until we cross the curve Γ, which lies entirely within D_- and hits the boundary $\partial\Omega$, that the mean first passage times become O(1) as $\varepsilon \to 0$, and then the deterministic or fluid approximation becomes valid asymptotically.

Even though we chose a particular example, we believe this type of analysis should apply for any two-dimensional problem for which Ω can be divided into two parts, and where the flow has a unique stable equilibrium and a unique subcharacteristic that becomes tangent to $\partial\Omega$, thus causing the division of Ω into D_+ and D_-. The solution to the problem where the subcharacteristic is tangent to the boundary (cf. item (iii) in Proposition 3) should be "canonical" for these types of problems, as should be the transition scales/expansions that apply near the curve $\mathcal{C}$ that separates D_+ from D_-. Of course, if, for example, the boundary had some curvature then the caustic boundary rays may themselves form secondary caustics or cusps which would lead to additional asymptotic ranges/expansions. Also, if the boundary $\partial\Omega$, or a part of it, was itself a characteristic curve, or the problem had multiple equilibrium points, then it is likely that the asymptotic structure would be quite different from that here. For this reason we did not attempt to treat two-dimensional problems in any generality.

Appendix

Here we discuss in more detail the caustic rays, as given by (3.20), and the behavior of $\psi(x,y)$ along the critical parabola $y = 2\sqrt{-x}$.

By using (3.18) in (3.20) and eliminating the radicals we obtain, after some calculation,

$$\begin{aligned} &-2(3s-1)y^4 - (3s-1)^2(s+1)\big(x^2+y^2\big) \\ &\quad + 8xy^2\big(2s^2-s+1\big) + 2x(3s-1)(s+1)\big(2s^2-s+1\big) \\ &\quad - (s+1)\big(2s^2-s+1\big)^2 = 0, \quad s < -1. \end{aligned} \tag{A.1}$$

If $s = -1$, (A.1) collapses to $8(y^4 + 4xy^2) = 0$ which contains the curve $y = 2\sqrt{-x}$. For $s < -1$, (A.1) applies in the range where $x < -1$ and $y > 2$. The equation in (A.1) defines an algebraic curve that is quadratic in x, quartic in y (and quadratic in y^2), and also quintic in s, which is the parameter that indexes the rays.

Solving (A.1) explicitly we then obtain

$$x = \frac{\big(2s^2-s+1\big)\big[4y^2+(3s-1)(s+1)\big] + y(s-3)\sqrt{5s^2-2s+1}\sqrt{2y^2+(3s-1)(s+1)}}{(s+1)(3s-1)^2}. \tag{A.2}$$

By implicit differentiation of (A.1) with respect to y we find that

$$s_y\big|_{s=-1} = \frac{-8(-x)^{3/2}}{(x+1)(3x-1)} < 0 \tag{A.3}$$

and

$$s_{yy}\big|_{s=-1} = \frac{-4x(11x^4+36x^3-30x^2+4x-5)}{(3x^2+2x-1)^3}. \tag{A.4}$$

Using (3.20) to solve for t in terms of y and s we obtain

$$\psi_y = \frac{1}{2}\left[y - \sqrt{y^2 + \frac{1}{2}(s+1)(3s-1)}\right]$$

and thus

$$4\left[\psi_y^2 - y\psi_y\right] = \frac{1}{2}\left(3s^2 + 2s - 1\right). \tag{A.5}$$

Again by implicit differentiation we have

$$8\psi_y\psi_{yy} - 4\psi_y - 4y\psi_{yy} = (3s + 1)s_y \tag{A.6}$$

so that along the ray $s = -1$ we have $\psi_{yy} = s_y/(2y) = s_y/(4\sqrt{-x})$ and, using also (A.3), this leads to (3.28). Differentiating (A.6) again with respect to y and setting $s = -1$ leads to

$$\psi_{yyy} = \frac{1}{2y}s_{yy} - \frac{1}{y^2}s_y - \frac{3y^2 - 2}{4y^3}s_y^2, \quad s = -1, y = 2\sqrt{-x}, \tag{A.7}$$

and then using (A.3) and (A.4) leads to the expression in (3.29).

References

[1] S. Chandrasekhar, Stochastic problems in physics and astronomy, in: *Noise and Stochastic Processes*, N. Wax, ed., Dover, New York, 1954.
[2] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*, Jones and Bartlett, Boston, 1993.
[3] E.B. Dyhkin, *Markov Processes I, II*, Springer-Verlag, New York, 1965.
[4] A. Einstein, *Investigations on the Theory of the Brownian Movement*, Dover, New York, 1956.
[5] M.A. Friedlin and A.D. Wentzell, *Random Perturbations of Dynamical Systems*, Springer-Verlag, New York, 1984.
[6] F. Hu and C. Knessl, Asymptotics of barrier option pricing under the CEV process, *Applied Mathematical Finance* **17** (2010), 261–300.
[7] C. Knessl and Y. Yang, Analysis of a Brownian particle moving in a time-dependent drift field, *Asymptotic Analysis* **27** (2001), 281–319.
[8] H.A. Kramers, Brownian motion in a field of force and the diffusion model of chemical reactions, *Physica* **7** (1940), 284–304.
[9] M. Mangel and D. Ludwig, Probability of extinction in a stochastic competition, *SIAM J. Appl. Math.* **33** (1977), 256–266.
[10] B. Matkowsky and Z. Schuss, The exit problem for randomly perturbed dynamical systems, *SIAM J. Appl. Math.* **33** (1977), 365–382.
[11] B. Matkowsky and Z. Schuss, The exit problem: a new approach to diffusion across potential barriers, *SIAM J. Appl. Math.* **36** (1979), 604–623.
[12] B. Matkowsky, Z. Schuss and C. Tier, Diffusion across characteristic boundaries with critical points, *SIAM J. Appl. Math.* **43** (1983), 673–695.
[13] L.S. Ornstein and G.E. Uhlenbeck, On the theory of Brownian motion, *Phys. Rev.* **30** (1930), 823–841.
[14] S. Redner, *A Guide to First-Passage Processes*, Cambridge Univ. Press, Cambridge, 2001.
[15] Z. Schuss, *Theory and Applications of Stochastic Differential Equations*, Wiley, New York, 1980.
[16] Z. Schuss, *Theory and Applications of Stochastic Processes: An Analytical Approach*, Springer, New York, 2010.
[17] M. Smoluchowski, Drei Vorträge über Diffusion, Brownsche Molekularbewegung und Koagulation von Kolloidteilchen, *Phys. Z.* **17** (1916), 557–571.
[18] M. Smoluchowski, Drei Vorträge über Diffusion, Brownsche Molekularbewegung und Koagulation von Kolloidteilchen, *Phys. Z.* **17** (1916), 585–599.
[19] S.R.S. Varadhan, *Large Deviations and Applications*, CBMS–NSF Regional Conference Series in Applied Mathematics, Vol. 46, SIAM, Philadelphia, 1984.
[20] A.J. Viterbi, *Principles of Coherent Communications*, McGraw-Hill, New York, 1966.