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SUBSETS OF SUPERSTABLE STRUCTURES ARE WEAKLY BENIGN

BEKTUR BAIZHANOV, JOHN T. BALDWIN, AND SAHARON SHELAH

Baizhanov and Baldwin [1] introduce the notions of benign and weakly benign sets to investigate the preservation of stability by naming arbitrary subsets of a stable structure. They connect the notion with work of Baldwin, Benedikt, Bouscaren, Casanovas, Poizat, and Ziegler. Stimulated by [1], we investigate here the existence of benign or weakly benign sets.

Definition 0.1. (1) *The set A is* benign *in M if for every α, $\beta \in M$ if $p = \mathrm{tp}(\alpha/A) = \mathrm{tp}(\beta/A)$ then $\mathrm{tp}_*(\alpha/A) = \mathrm{tp}_*(\beta/A)$ where the $*$-type is the type in the language L^* with a new predicate P denoting A.*

(2) *The set A is* weakly benign *in M if for every $\alpha, \beta \in M$ if $p = \mathrm{stp}(\alpha/A) = \mathrm{stp}(\beta/A)$ then $\mathrm{tp}_*(\alpha/A) = \mathrm{tp}_*(\beta/A)$ where the $*$-type is the type in language with a new predicate P denoting A.*

Conjecture 0.2 (too optimistic). *If M is a model of stable theory T and $A \subseteq M$ then A is benign.*

Shelah observed, after learning of the Baizhanov-Baldwin reductions of the problem to equivalence relations, the following counterexample.

Lemma 0.3. *There is an ω-stable rank 2 theory T with ndop which has a model M and set A such that A is not benign in M.*

Proof. The universe of M is partitioned into two sets denoted by Q and R. Let Q denote $\omega \times \omega$ and R denote $\{0, 1\}$. Define $E(x, y, 0)$ to hold if the first coordinates of x and y are the same and $E(x, y, 1)$ to hold if the second coordinates of x and y are the same. Let A consist of one element from each $E(x, y, 0)$-class and one element of all but one $E(x, y, 1)$-class such that no two members of A are equivalent for either equivalence relation. It is easy to check that letting α and β denote the two elements of R, we have a counterexample. In this case, the type p is algebraic. Algebraicity is a completely artificial restriction. Replace each α and β by an infinite set of points which behave exactly as α, β respectively. We still have a counterexample. In either case, α and β have different strong types. This leads to the following weakening of the conjecture. ⊣

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CONJECTURE 0.4 (Revised). *If M is a model of stable theory T and A is an arbitrary subset of M then A is weakly benign.*

We give here a proof of Conjecture 0.4 in the superstable case. There are two steps. In the first we show that if (M, A) is not (weakly) benign then there is a certain configuration within M. (This uses only T stable.) The second shows that this configuration is contradicted for superstable T. Note that if (M, A) is not weakly benign, neither is any L^*-elementary extension of (M, A) so we may assume any counterexample is sufficiently saturated.

§**1. Refining a counterexample.** In this section we choose a specific way in which a sufficiently saturated pair (M, A) where $\mathrm{Th}(M)$ is stable, fails to be weakly benign. Fix M, a κ^+-saturated model of a stable theory T where $\kappa = \kappa^{|T|}$ is regular.

We introduce some notation. Recall that A is *relatively κ-saturated* in M if every type over (a subset of A) whose domain has cardinality less than κ and which is realized in M, is also realized in A. First note that for any $c \in M - A$, there is a pair (M_1, A_1) such that A_1 is relatively κ-saturated in A; $A_1 \cup c \subseteq M_1$ and M_1 is independent from A over A_1; A_1 and M_1 have cardinality κ and M_1 is κ-saturated. For this, choose $A_0 \subset A$ with c independent from A over A_0 and $|A_0| < \kappa$ (which follows since $\kappa \geq |T|^+ \geq \kappa(T)$). Then extend A_0 to a subset A_1 of A with cardinality at most κ which is relatively κ-saturated in A. Finally, let $M_1 \prec M$ be κ-prime over $A_1 \cup c$. We have shown the following class $\boldsymbol{K}_c$ is not empty.

NOTATION 1.1. (1) *For any $c \in M$, let $\boldsymbol{K}_c$ be the class of pairs (M_1, A_1) with $c \in M_1 \prec M$ such that A_1 is relatively κ-saturated in A; $A_1 \cup c \subseteq M_1$ and M_1 is independent from A over A_1; A_1 and M_1 have cardinality κ and M_1 is κ-saturated with $|M_1| \leq \kappa$.*

(2) *For any a, b in M which realize the same type over A, let $\boldsymbol{K}^1_{a,b}$ be the set of tuples $\langle A_1, M_a, M_b, N_a, g\rangle$ such that (M_a, A_1) and (M_b, A_1) are in $\boldsymbol{K}_a, \boldsymbol{K}_b$ respectively, g is an isomorphism between M_a and M_b (subsets of M) over A_1 (taking a to b), N_a contains M_a and is saturated with cardinality κ, and N_a is independent from A over A_1.*

(3) *Let $\boldsymbol{K}^2_{a,b}$ be the set of tuples $\langle A_1, M_a, M_b, N_a, g\rangle \in \boldsymbol{K}^1_{a,b}$ such that g is an isomorphism between M_a^{eq} and M_b^{eq} over A_1^{eq} (taking a to b). a and b realize the same type over A_1^{eq}, so they realize the same strong type over A_1.*

(4) *We will write K^i to denote either K^1 or K^2. Note the only difference between them is that K^2 has a more restrictive requirement on the isomorphism g.*

Note that the last clause of item 2 implies that N_a is independent from A over $N_a \cap A$ and that $N_a \cap A = A_1 = M_a \cap A$. Moreover, if $\langle A_1, M_a, M_b, N_a, g\rangle \in \boldsymbol{K}_{a,b}$ and $B \subseteq A$ with $|B| \leq \kappa$ then there is an $\langle A_1', M_a', M_b', N_a', g'\rangle \in \boldsymbol{K}_{a,b}$ with $A_1 \cup B \subseteq A_1'$. (Just include B when making the construction from the first paragraph of this section to show $\boldsymbol{K}_{a,b}$ is nonempty). We need a couple of other properties of $\boldsymbol{K}_{a,b}$. Note that $\boldsymbol{K}_{a,b}$ is naturally partially ordered by coordinate by coordinate inclusion.

LEMMA 1.2. *Every increasing chain from $\boldsymbol{K}^i_{a,b}$ of length δ a limit ordinal less than κ^+ has an upper bound in $\boldsymbol{K}^i_{a,b}$.*

PROOF. If the cofinality of the chain is at least κ, just take the union (in each coordinate). We check that N_a^δ, A are independent over A^δ: By induction, for every

$\alpha < \beta < \delta$, $tp(N_a^\alpha/A)$ does not fork over A_1^β (by monotonicity of nonforking). Hence if δ is a limit ordinal, $tp(N_a^\delta/A)$ does not fork over A_1^δ.

But if the cofinality is smaller the union may not preserve κ-saturation. In this case, let $\langle A_1', M_a', M_b', N_a', g'\rangle$ denote the union of the respective chains; each has cardinality κ. Choose $A_1 \subseteq A$ with $|A_1| = \kappa$ and such that A_1 is relatively κ-saturated in A and A_1 contains A_1'. Then let the bound be $\langle A_1, M_a, M_b, N_a, g\rangle$ where M_a is κ-prime over $M_a' \cup A_1$, M_b is κ-prime over $M_b' \cup A_1$, g is the induced isomorphism extending g' and N_a is any κ-saturated elementary extension of $M_a \cup N_a'$ in M with N_a independent from A over A_1. ⊣

Lemma 1.3. *If $t = \langle A_1, M_a, M_b, N_a, g\rangle \in \boldsymbol{K}^i_{a,b}$ and $p \in S(M_a)$ is non-algebraic, orthogonal to A and $p \not\perp \operatorname{tp}(N_a/M_a)$, then there is $t' = \langle A_1', M_a', M_b', N_a', g'\rangle \in \boldsymbol{K}^i_{a,b}$ with t' extending t and $\operatorname{tp}(N_a/M_a')$ forking over M_a.*

Proof. Since M is κ^+-saturated, we can find $d \in M$ realizing p such that $\operatorname{tp}(d/N_a)$ forks over M_a and $d' \in M$ realizing $g(p)$. Now, construct t' by letting $A_1' = A_1$, M_a' be κ-prime over $M_a \cup \{d\}$, M_b' be κ-prime over $M_b \cup \{d'\}$, g' be an extension of g taking d to d', and $N_a' \prec M$ any κ-saturated extension of $M_a' \cup N_a$. We need to show that M_a' and M_b' are independent from A over A_1'. For this, note that since $p \in S(M_a)$ is orthogonal to A (*a fortiori* to A_1) and A is independent from M_a over A_1, d is independent from A over M_a. Since M_a' is κ-prime over $M_a \cup \{d\}$, it follows that M_a' is independent from A over A_1'. An analogous argument shows M_b' is independent from A over A_1'. Since $d \in M_a'$, we have fulfilled the lemma. ⊣

For any ordinal μ and any sequence $\langle \mathbf{a}_i : i < \mu \rangle$ and any finite $w \subseteq \mu$, $\mathbf{a}_w$ denotes $\langle \mathbf{a}_i : i \in w \rangle$. We require one further technical notion.

Definition 1.4. *We say M_a is A-full in M if for any N κ-prime over M_aA and for any $C_0 \subseteq M_a$, $|C_0| \leq |T|$, $C_1 \subseteq A$ with $|C_1| \leq |T|$, and C_2 with $C_0 \subseteq C_2$, $C_1 \subseteq C_2 \subseteq N$, and $|C_2| \leq |T|$, there is an elementary map f taking C_1C_2 into M_a over C_0 with $f(C_1) \subseteq A$ and if C_2 is independent from A over C_1 then $f(C_2)$ is independent from A over $f(C_1)$.*

We prove a characterization of a weakly benign pair; a similar result for benign (using $\boldsymbol{K}^1$ instead of $\boldsymbol{K}^2$) also holds. In view of the counterexample in given in the introduction, weakly benign is the interesting case.

Lemma 1.5. *Use the notation of* 1.1. *Suppose (M, A) is κ^+-saturated where $\kappa = \kappa^{|T|}$ is regular and $T = \operatorname{Th}(M)$ is stable. The following are equivalent.*

(1) *(M, A) is not weakly benign.*
(2) *There exist A_*, M_a, N_a, M_b, g contained in M with $a \in M_a$, $b \in M_b$ such that:*
   (a) *$\langle A_*, M_a, M_b, N_a, g\rangle \in \boldsymbol{K}^2_{a,b}$ and $M_a \neq N_a$.*
   (b) *M_a is A-full in M.*
   (c) *$\operatorname{tp}(N_a/M_a)$ is orthogonal to every nonalgebraic type in $S(M_a)$ which is orthogonal to A.*
   (d) *If $\mathbf{d} \in N_a - M_a$, there is no $\mathbf{d}' \in M$ which realizes $g(\operatorname{tp}(\mathbf{d}/M_a))$ and such that $\mathbf{d}'$ is independent from A over M_b.*

We can easily deduce from condition a) that N_a is independent from A over A_* and also that M_a and M_b are isomorphic over A_* by a map g taking a to b and preserving strong types over A, i.e., $g\restriction(A^*)^{\mathrm{eq}}$ is the identity. By general properties of orthogonality, we could rephrase item c) as: $\mathrm{tp}(N_a/M_a)$ is orthogonal to every nonalgebraic type in $S(M_a)$ which is orthogonal to A_*.

PROOF OF LEMMA 1.5. First we show that condition (2) implies condition (1). By condition (2a), there is an a' in $N_a - M_a$. Note that since A^* is relatively κ^+-saturated in A and M_a (M_b) is independent from A over A^*, $M_a \cap A = M_b \cap A = A^*$. It follows that $g \cup (\mathrm{id}\restriction \mathrm{acl}(A^{eq}))$ is an elementary map in L^{eq}. Let $\mathbf{a} = \langle a_i : i < \kappa \rangle$ enumerate $M_a - A$ with $a_0 = a$; denote $g(a_i)$ by b_i so $\mathbf{b} = \langle b_i : i < \kappa \rangle$ enumerates M_b. For any finite set of L-formulas Δ and finite subset w of κ, let $\phi_{\Delta,w}(\mathbf{x}; a', \mathbf{a}_w, \mathbf{b}_w)$ be the L^*-formula which asserts that $x\mathbf{b}_w$ and $a'\mathbf{a}_w$ realize the same Δ-type over A. For any finite w, $\mathbf{a}_w$ and $\mathbf{b}_w$ realize the same L-type over A.

Now, let $q = \{ \phi_{\Delta,w}(\mathbf{x}; a', \mathbf{a}_w, \mathbf{b}_w) : 0 \in w \subset_\omega \kappa, \Delta \subset_\omega L \}$. Putting $0 \in w$ guarantees a, b are in any relevant $\mathbf{a}_w, \mathbf{b}_w$. So q is a set of κ L^*-formulas with free variable x and parameters from $M_a \cup M_b \cup \{a'\}$. If q is finitely satisfied in (M, A), then q is realized in M by some b', since M is κ^+-saturated as an L^*-structure. But since a' is independent from A over M_a, b' realizes the unique nonforking extension of $g(\mathrm{tp}(a'/M_a))$ to $M_b \cup A$ contradicting condition d). If q is not finitely satisfiable, there is a formula $\phi_{\Delta,w}$ which demonstrates the L^* type of $\mathbf{a}_w$ and $\mathbf{b}_w$ over A are different.

We will use the following basic fact (compare Lemma I.1.12 of [2]):

FACT 1.6. (1) *If A_1 is relatively κ-saturated in A and C is independent from A over A_1, then CA_1 is relatively κ-saturated in CA.*

(2) *If A_1 is relatively κ-saturated in A and D is κ-atomic over A_1, D is independent from A over A_1.*

To show (1) implies (2) of Lemma 1.5, we suppose that $\mathbf{a}$ and $\mathbf{b}$ realize the same (strong)-type over A but that there is an a' such that there is no $b' \in M$ with $\mathbf{a}a' \equiv_{A,L} \mathbf{b}b'$. We fix $\langle a, b \rangle$ as the $\mathbf{a}$, $\mathbf{b}$ and analyze $\boldsymbol{K}^2_{a,b}$ below.

LEMMA 1.7. *There is a $t = \langle A_*, M_a, M_b, N_a, g \rangle \in \boldsymbol{K}^2_{a,b}$ such that*

(A) *$N_a \neq M_a$,*

(B) *$\mathrm{tp}(N_a/M_a)$ is orthogonal to every nonalgebraic type in $S(M_a)$, which is orthogonal to A.*

(C) *If $\mathbf{d} \in N_a - M_a$, there is no $\mathbf{d}' \in M$ which realizes $g(\mathrm{tp}(\mathbf{d}/M_a))$ and such that $\mathbf{d}'$ is independent from A over M_b.*

(D) *M_a is A-full.*

PROOF. Try to construct by induction a sequence $\langle t_\alpha : \alpha < \kappa^+ \rangle$ where $t_\alpha = \langle A^\alpha_*, M^\alpha_a, M^\alpha_b, N^\alpha_a, g^\alpha \rangle$ of elements of $\boldsymbol{K}^2_{a,b}$ which are increasing in the natural partial order, continuous at limit ordinals of cofinality greater than $\kappa_r(T)$; t_0 is any element of $\boldsymbol{K}^2_{a,b}$ with $a' \in N^0_a$.

(1) If α is an even ordinal there are several cases.
  (a) Suppose condition (B) fails, i.e. for some $d \in N_a$, $p = \mathrm{tp}(d/M_a)$ is nonorthogonal to some stationary type $q \in S(M_a)$ which is orthogonal to A. Then by Lemma 1.3, there is $t' = \langle A'_*, M'_a, M'_b, N'_a, g' \rangle \in \boldsymbol{K}^2_{a,b}$ with t' extending t and $\mathrm{tp}(N_a/M'_a)$ forks over M_a.
  (b) Suppose condition (B) holds.
    (i) If α is a limit ordinal of cofinality κ, stop.

(ii) If α is a limit ordinal of cofinality $< \kappa$ or α is a successor ordinal, let $t_{\alpha+1} = t_\alpha$.

(2) α is an odd successor ordinal. Choose an auxiliary $\hat{M}^\alpha_a$ κ-prime over $M^\alpha_a A$. Choose $A^{\alpha+1}_*, M^{\alpha+1}_a, M^{\alpha+1}_b$ such that $A^\alpha_* \subseteq A^{\alpha+1}_* \subseteq A$, $|A^{\alpha+1}_*| = \kappa$ and so that

$$(M^{\alpha+1}_a, A^{\alpha+1}_*) \prec_{L_{(|T|^+,|T|^+)}} (\hat{M}^\alpha_a, A)$$

and $M^{\alpha+1}_a$ is κ-prime over $M^\alpha_a A^{\alpha+1}_*$. This is possible since $\kappa = \kappa^{|T|}$. In particular, $M^{\alpha+1}_a$ is independent from A over $A^{\alpha+1}_*$. The κ-primeness allows us to easily construct $M^{\alpha+1}_b$ and $g_{\alpha+1}$. Now choose $N^{\alpha+1}_a$ to be a κ-saturated extension of $M^{\alpha+1}_a$ that is independent from A over $A^{\alpha+1}_*$

(3) If α is a limit ordinal choose t_α by Lemma 1.2.

We cannot carry out this construction for κ^+ steps. If we did, by clause (1) of the construction at each limit α with $\mathrm{cf}(\alpha) = \kappa$, clause (B) fails. Thus, $M^{\alpha+1}_a$ depends on N^α_a over M^α_a for all such α, which contradicts stability. (If we were dealing with finite sequences, the bound would be $\kappa(T)$; since we deal with sets of cardinality κ, the bound is κ^+.)

Fix α where the construction stops. We have constructed $t_\alpha = \langle A^\alpha_*, M^\alpha_a, M^\alpha_b, N^\alpha_a, g^\alpha\rangle$ but for any choice of $t_{\alpha+1} \in \boldsymbol{K}^2_{a,b}$, $M^{\alpha+1}_a$ is independent from N^α_a over M^α_a. Note that each member of $t_\alpha = \langle A^\alpha_*, M^\alpha_a, M^\alpha_b, N^\alpha_a, g^\alpha\rangle$ is the union of the respective member of t_β over $\beta < \alpha$. We claim this t_α is a t satisfying the conditions of the lemma.

For clause (A) note $N^\alpha_a \neq M^\alpha_a$ since $a' \in N^\alpha_a$ and a' cannot be in the domain of g^α by the original choice of a'. Since the construction stopped clause (B), holds.

For clause (C), we must show that if $\mathbf{d} \in N_a - M_a$, there is no $\mathbf{d}' \in M$ which realizes $g(\mathrm{tp}(\mathbf{d}/M_a))$ and such that $\mathbf{d}'$ is independent from A over M_b. Fix $\mathbf{d} \in N_a - M_a$; if such a $\mathbf{d}'$ exists, choose $M^{\alpha+1}_a, M^{\alpha+1}_b$ contained in M prime over $M^\alpha_a\mathbf{d}$ and $M^\alpha_b\mathbf{d}'$ respectively. We easily extend g^α to $g^{\alpha+1}$ mapping $M^{\alpha+1}_a$ to $M^{\alpha+1}_b$. By the construction, A^α_* is relatively κ-saturated in A. So, $M^\alpha_a \cup \{\mathbf{d}\}$ and A are independent over A^α_* by monotonicity, as N^α_a is independent from A over A^α_*. Now by Fact 1.6 (1), $M^\alpha_a \cup \{\mathbf{d}\}$ is relatively κ-saturated inside $M^\alpha_a \cup \{\mathbf{d}\} \cup A$. Whence, by Fact 1.6 (2) $M^{\alpha+1}_a$ and A are independent over $M^\alpha_a \cup \{\mathbf{d}\}$. By transitivity of nonforking, $M^{\alpha+1}_a$ and A are independent over A^α_*. Similarly, since $\mathbf{d}'$ is independent from A over M_b, $M^{\alpha+1}_b$ is independent from A over A^α_*. But now, N^α_a depends on $M^{\alpha+1}_a$ over M^α_a because $d \in (M^{\alpha+1}_a \cap N^\alpha_a) - M^\alpha_a$ and we have violated the choice of α.

Finally we verify clause (D): M_a is A-full. Choose N, which is κ-prime over AM_a. Then N can be embedded over AM_a into $\hat{M}^\alpha_a = \cup_{i<\alpha}\hat{M}^i_a$. By the Tarski union of chains theorem (using clause (2) of the construction), $(M^\alpha_a, A \cap M^\alpha_a) \prec_{L_{|T|^+,|T|^+}} (\hat{M}^\alpha_a, A)$. Let C_0, C_1, $C_2 \subseteq N$ satisfy the hypotheses of the definition of A-full. The elementary submodel condition easily allows us to define the required function f. ⊣

And thus, we have proved Lemma 1.5. ⊣

§2. The superstable case. The aim of this section is to prove that if M is a model of a superstable theory and $A \subset M$, then (M, A) is weakly benign. This is a generalization of a result of Bouscaren [3], who showed, in our terminology that every *submodel* of a superstable structure is benign.

THEOREM 2.1. *If M is a model of a superstable theory and $A \subset M$, then (M, A) is weakly benign.*

PROOF. We work in $\mathscr{M}^{\mathrm{eq}}$. Without loss of generality, assume (M, A) is κ^+-saturated for a regular κ satisfying $\kappa^{|T|} = \kappa$. By Lemma 1.5 if (M, A) is not weakly benign, there exist A_*, M_a, N_a, M_b, g contained in M satisfying the conditions of Lemma 1.5 and with $\langle A_*, M_a, M_b, N_a, g\rangle \in \boldsymbol{K}^2_{a,b}$.

Since M_a is properly contained in N_a, we can choose $\mathbf{c} \in M_a$ and $\phi(x, \mathbf{c})$ to have minimal D-rank among all formulas with $\phi(N_a, \mathbf{c}) \neq \phi(M_a, \mathbf{c})$. Then for any $d^* \in \phi(N_a, \mathbf{c})\backslash\phi(M_a, \mathbf{c})$, $p^* = \mathrm{tp}(d^*/M_a)$ is regular. Without loss of generality again, we can fix d^*, which does not fork over $\mathbf{c}$ and so that p^* has the same D-rank as $\phi(x, \mathbf{c})$ and $\mathrm{tp}(d^*/\mathbf{c})$ is stationary. By clause (c) of Lemma 1.5, p^* is not orthogonal to A_*. So, there is a $q' \in S(M_a)$ which does not fork over A_* and is nonorthogonal and so non-weakly orthogonal to p^*. Fix $C \subseteq A_*$ with $|C| \leq |T|$ and $\mathbf{c}$ is independent from A_* over C. Without loss of generality $\mathrm{tp}(\mathbf{d}^*/A_*\mathbf{c}) \not\perp^w q\restriction(A_*\mathbf{c})$ and $\mathrm{tp}(\mathbf{d}^*/C\mathbf{c}) \not\perp^w q\restriction(C\mathbf{c})$. Let $\mathscr{P} = \{\, p : p$ is regular, stationary, and nonorthogonal to $p^* \,\}$. $\mathscr{P}$ is based on $B = \mathrm{acl}^{\mathrm{eq}}(C)$, i.e. every automorphism of $\mathscr{M}$ fixing B maps $\mathscr{P}$ to itself.

If $\mathbf{c}'' \in M$ realizes $\mathrm{tp}(\mathbf{c}/A^{\mathrm{eq}})$ and $d''\mathbf{c}''$ realizes $r = \mathrm{tp}(d^*\mathbf{c}/B)$, then $\mathrm{tp}(d''/\mathbf{c}'')$ is regular and nonorthogonal to p^*. We can find $\langle \mathbf{c}_i : i < \omega\rangle$ in M_a with $\mathbf{c}_0 = \mathbf{c}$ which are indiscernible over B and which are based on B. The $r(\mathbf{x}, \mathbf{c}_i)$ are regular, pairwise nonorthogonal, and all nonorthogonal to $\mathscr{P}$ and each $r(\mathbf{x}, \mathbf{c}_i)$ is not weakly orthogonal to $q'\restriction(B\mathbf{c}_i)$. Note $r(\mathbf{x}, \mathbf{c}_i) \subset p^*$. Let $r_i \in S(M)$ denote the nonforking extension of $r(\mathbf{x}, \mathbf{c}_i)$ to $S(M)$. By Section V.4 of [4], there is a $q \in S(B)$, which is $\mathscr{P}$-simple and $k < \omega$ such that $w_{\mathscr{P}}(q) > 0$ and $q(\mathscr{M}) \subseteq \mathrm{acl}(B \cup \bigcup_{i<k} \mathbf{c}_i \cup \bigcup_{i<k} r(\mathscr{M}, \mathbf{c}_i))$. (This q is actually q'/E for an appropriate definable (over B) equivalence relation; compare V.4.17 (8) of [4].)

Let q^+ denote the unique nonforking extension of q to $S(M)$, p_a^+ denote the unique nonforking extension of p^* to $S(M)$, and p_b^+ denote the unique nonforking extension of $g(p^*)$ to $S(M)$. Clearly, $p_a^+\restriction(M_a \cup A)$ is a nonforking extension of the stationary type p^* and is realized by $\mathbf{d}^*$; so it is equivalent to $p_a^+\restriction\mathrm{acl}(M_a \cup A)$.

REMARK 2.2. *Note $(g \cup \mathrm{id}_A)(p_a^+\restriction(M_a \cup A) = p_b^+\restriction(M_b \cup A) \sim p_b^+\restriction\mathrm{acl}(M_b \cup A)$ is omitted in M.*

We use the next lemma several times.

LEMMA 2.3. *If $A^{\mathrm{eq}} \subseteq N_1 \subseteq N_2 \subseteq M$ and N_1, N_2 are $|T|^+$-saturated then*

$$w_{\mathscr{P}}(q(N_2), N_1) = w_{\mathscr{P}}(q(N_2), q(N_1)A^{\mathrm{eq}}).$$

PROOF. Fix $\mathbf{b} \in N_1$ and choose $D \subseteq q(N_1)A^{\mathrm{eq}}$ with $|D| \leq |T|$ such that $\mathrm{tp}(\mathbf{b}/q(N_1)A^{\mathrm{eq}})$ does not fork over D. If $\mathrm{tp}(\mathbf{b}/q(N_2)A^{\mathrm{eq}})$ forks over D, there are finite $\mathbf{d}_1 \subseteq q(N_2)$ and $\mathbf{d}_2 \subseteq A^{\mathrm{eq}}$ such that $\mathrm{tp}(\mathbf{b}/BD\mathbf{d}_1\mathbf{d}_2)$ forks over D. But there is a $\mathbf{d}' \in q(N_1)$ realizing $\mathrm{stp}(\mathbf{d}_1/D\mathbf{b}\mathbf{d}_2)$, which contradicts $\mathrm{tp}(\mathbf{b}/q(N_1)A^{\mathrm{eq}})$ does not fork over D.

So $\mathrm{tp}(\mathbf{b}/q(N_2)A^{\mathrm{eq}})$ does not fork over $q(N_1)A^{\mathrm{eq}}$. Since $\mathbf{b}$ was arbitrary in N_1, $\mathrm{tp}(N_1/q(N_2)A^{\mathrm{eq}})$ does not fork over $q(N_1)A^{\mathrm{eq}}$. By symmetry of forking, $\mathrm{tp}(q(N_2)/N_1A^{\mathrm{eq}})$ does not fork over $q(N_1)A^{\mathrm{eq}}$. Since $A^{\mathrm{eq}} \subseteq N_1$ we finish. $\dashv$

The proof now proceeds by a series of claims. The key idea is that $w_{\mathscr{P}}(q(M), A^{\mathrm{eq}})$ can be calculated either as $w_{\mathscr{P}}(q(M), q(M_b) \cup A^{\mathrm{eq}}) + w_{\mathscr{P}}(q(M_b), A^{\mathrm{eq}})$ or $w_{\mathscr{P}}(q(M), q(M_a) \cup A^{\mathrm{eq}}) + w_{\mathscr{P}}(q(M_a), A^{\mathrm{eq}})$. We will calculate both ways to obtain a contradiction. We begin with the M_a side.

Claim 2.4. *If* $\dim(r_0 \restriction A_* \mathbf{c}_0, M_a)$ *is finite, then* $w_{\mathscr{P}}(q(M_a), A_* \cup \bigcup_{i<k} \mathbf{c}_i)$ *is finite.*

Proof. If u is a finite subset of ω, since the r_i are regular, it is easy to show that for each i, $\dim(r_i \restriction (A_* \mathbf{c}_i), M_a)$ is finite iff $\dim(r_i \restriction (A_* \cup \mathbf{c}_i \cup_{j \in u} \mathbf{c}_j), M_a)$ is finite. Since the $r_i \restriction (A_* \mathbf{c}_i \mathbf{c}_j)$ are regular and pairwise not weakly orthogonal

$$\dim(r_i \restriction A_* \mathbf{c}_i \mathbf{c}_j, M_a) = \dim(r_j \restriction A_* \mathbf{c}_i \mathbf{c}_j, M_a).$$

The previous two sentences imply: $\dim(r_i \restriction A_* \mathbf{c}_i, M_a)$ is finite iff $\dim(r_j \restriction A_* \mathbf{c}_j, M_a)$ is finite. So if $\dim(r_0 \restriction A_* \mathbf{c}_0, M_a)$ is finite then $w_{\mathscr{P}}(\bigcup_{i<k} r_i(M_a, \mathbf{c}_i), A_* \cup \bigcup_{i<k} \mathbf{c}_i)$ is finite; whence $w_{\mathscr{P}}(q(M_a), A_* \cup \bigcup_{i<k} \mathbf{c}_i)$ is finite. ⊣

Now we drop the $\bigcup_{i<k} \mathbf{c}_i$ in the conclusion.

Claim 2.5. $\dim(r_0 \restriction A_* \mathbf{c}_0, M_a)$ *is finite implies* $w_{\mathscr{P}}(q(M_a), A_*)$ *is finite.*

Proof. Find $\mathbf{d} \subseteq q(M)$ such that $\bigcup_{i<k} \mathbf{c}_i$ is independent from $A_* \cup q(M)$ over $A_* \cup \mathbf{d}$. Now, as $\mathrm{tp}(\mathbf{d}/A_*)$ is $\mathscr{P}$-simple, $w_{\mathscr{P}}(q(M_a), A_*) = w_{\mathscr{P}}(q(M_a), A_* \mathbf{d}) + w_{\mathscr{P}}(\mathbf{d}, A_*)$. The second term is finite and $w_{\mathscr{P}}(q(M_a), A_* \mathbf{d}) = w_{\mathscr{P}}(q(M_a), A_* \mathbf{d} \cup \bigcup_{i<k} \mathbf{c}_i)$ by the independence. But, $w_{\mathscr{P}}(q(M_a), A_* \mathbf{d} \cup \bigcup_{i<k} \mathbf{c}_i) = w_{\mathscr{P}}(q(M_a), A_* \cup \bigcup_{i<k} \mathbf{c}_i) - w_{\mathscr{P}}(\mathbf{d}, A_* \cup \bigcup_{i<k} \mathbf{c}_i)$. Now the first of the last two terms is finite by Claim 2.4 (since $\dim(r_0 \restriction A_* \mathbf{c}_0, M_a)$ is finite) and the second by the finiteness of $\mathbf{d}$ so $w_{\mathscr{P}}(q(M_a), A_*)$ is finite. ⊣

Claim 2.6. $\dim(r_0, M_a)$ *is finite.*

Proof. Note that $p_a^+ \restriction (B\mathbf{c}_0) = r_0 \restriction (B\mathbf{c}_0)$. Choose by induction $\mathbf{a}_\alpha \in M_a$ so that $\mathbf{a}_\alpha$ realizes $p_a^+ \restriction A_*^{\mathrm{eq}} \cup g(\mathbf{c}_0) \cup \{\mathbf{a}_\beta : \beta < \alpha\}$ for as long as possible to construct: $\mathbf{I} = \langle \mathbf{a}_\alpha : \alpha < \alpha^* \rangle$. Clearly $\alpha^* < |M_a|^+$, but in fact α^* is finite. As, since M_a is independent from A over A_*, $\boldsymbol{I}$ is a set of indiscernibles over A. Since M is κ^+-saturated, if $\boldsymbol{I}$ is infinite $\langle g(\mathbf{a}_\alpha) : \alpha < \alpha^* \rangle$ can be extended to a set $\mathbf{J}$ of indiscernibles over A contained in M_b with cardinality κ^+. Then all but at most κ members of $\mathbf{J}$ realize $p_b^+ \restriction (M_b \cup A)$ contradicting Remark 2.2 that $p_b^+ \restriction (M_b \cup A)$ is omitted in M. ⊣

Now, easily we have

Claim 2.7. $w_{\mathscr{P}}(q(M_a), A_*) = w_{\mathscr{P}}(q(M_a), A^{\mathrm{eq}})$ *is finite.*

Proof. The equality holds by the independence of M_a and A over A_*. The finiteness follows from Claim 2.6 and Claim 2.5. ⊣

The next claim involves both M_a and M_b.

Claim 2.8. *Suppose* $w_{\mathscr{P}}(q(M_a), A_*)$ *is finite and* $N \prec M$ *is* κ*-prime over* $M_b A$. *Then* $w_{\mathscr{P}}(q(N), q(M_b)A) = 0$.

Proof. Since $w_{\mathscr{P}}(q(M_a), A_*)$ is finite, and A, M_a are independent over A_*,we can choose finite $D \subseteq q(M_a)$ with $w_{\mathscr{P}}(q(M_a), A_*) = w_{\mathscr{P}}(q(M_a), A) = w_{\mathscr{P}}(D, A_*) = w_{\mathscr{P}}(D, A)$.

Now assume for contradiction that $w_{\mathscr{P}}(q(N), q(M_b)A) > 0$. Let $N' \prec M$ be κ-prime over $M_a \cup A$, so there is $g^+ \supseteq g \cup \mathrm{id}_A$ which is an isomorphism from N' onto N. Then there is a finite $D_2 \subseteq q(N')$ with $w_{\mathscr{P}}(D_2, M_a A) > 0$. Choose $C_0 \subseteq M_a$, $|C_0| \leq |T|$ with $DB \subseteq C_0$ and $C_1 \subseteq A$ with $|C_1| \leq |T|$ so that D_2

is independent from $M_a A$ over C_0C_1 and is the unique nonforking extension of $\mathrm{tp}(D_2/C_0C_1)$ to $S(M_aA)$ which is realized in M. Recall that M_a is A-full and apply the Definition 1.4 of A-full with $C_0C_1D_2$ playing the role of C_2 to obtain an embedding f. Then, $f(D_2) \subseteq q(M_a)$ and $f(D_2)$ is independent from C_0A over $C_0 f(C_1)$. Thus,

$$w_{\mathscr{P}}(f(D_2), AD) = w_{\mathscr{P}}(D_2, AD) \geq w_{\mathscr{P}}(D_2, q(M_a)A) > 0.$$

This implies $w_{\mathscr{P}}(q(M_a), A) \geq w_{\mathscr{P}}(Df(D_2), A) = w_{\mathscr{P}}(D, A) + w_{\mathscr{P}}(f(D_2), AD) > w_{\mathscr{P}}(D, A)$, which contradicts our original choice of D. ⊣

Claim 2.9. $w_{\mathscr{P}}(q(M), q(M_b)A) = 0$.

Proof. Let $N \prec M$ be κ-prime over $M_b \cup A$, so $p_b^+ \restriction (M_b \cup A)$ has a unique extension in $S(N)$. If $w_{\mathscr{P}}(q(M), N) > 0$ then for some $\mathbf{b} \in q(M)$, $w_{\mathscr{P}}(\mathbf{b}, N) > 0$ so $\mathrm{tp}(\mathbf{b}/N) \not\perp p_b^+$; recall p_b^+ is parallel to $p_b^+ \restriction N$. So $p_b^+ \restriction N$ is realized in M_b contradicting Remark 2.2. Now $0 = w_{\mathscr{P}}(q(M), N)$ which equals $w_{\mathscr{P}}(q(M), q(N)A^{\mathrm{eq}})$ by Lemma 2.3. Since $A^{\mathrm{eq}} \subseteq N_b \subseteq N \subseteq M$,

$$\begin{aligned} w_{\mathscr{P}}(q(M), q(M_b)A^{\mathrm{eq}}) &= w_{\mathscr{P}}(q(M), q(N)A^{\mathrm{eq}}) + w_{\mathscr{P}}(q(N), q(M_b)A^{\mathrm{eq}}) \\ &= 0 + 0 = 0. \end{aligned}$$

The first 0 was noted in the previous sentence and the second is Claim 2.8. ⊣

Now calculating with respect to M_b, we have:

Claim 2.10. $w_{\mathscr{P}}(q(M), A^{\mathrm{eq}}) = w_{\mathscr{P}}(q(M_b), A^{\mathrm{eq}})$ *is finite.*

Proof.

$$\begin{aligned} w_{\mathscr{P}}(q(M), A^{\mathrm{eq}}) &= w_{\mathscr{P}}(q(M), q(M_b)A^{\mathrm{eq}}) + w_{\mathscr{P}}(q(M_b), A^{\mathrm{eq}}) \\ &= 0 + w_{\mathscr{P}}(q(M_b), A^{\mathrm{eq}}) < \omega. \end{aligned}$$

The first equality holds by additivity [4] and Lemma 2.3, the second by Claim 2.9, and the third by the last observation. ⊣

Now we analyze using M_a.

Claim 2.11. $w_{\mathscr{P}}((q(M), q(M_a) \cup A) \geq 1$.

Proof. $w_{\mathscr{P}}(\mathbf{d}^*, M_a \cup A) \geq 1$ since $\mathbf{d}^*$ is independent from A over M_a. Let N be κ-prime over M_aA^{eq}. As $\mathrm{tp}(\mathbf{d}^*/M_aA^{\mathrm{eq}})$ has all its restrictions to set of size less than κ realized in M_aA^{eq}, $\mathrm{tp}(\mathbf{d}^*/N)$ does not fork over M_aA^{eq}. Thus, $\mathbf{d}^*$ realizes $p_a^+ \restriction N$. Since $p_a^+ \restriction N$ is not orthogonal to $q^+ \restriction N$, there is $\mathbf{b} \in q^+(M)$ which depends on $\mathbf{b}$ over N. So $w_{\mathscr{P}}(\mathbf{b}, N) > 0$ whence $w_{\mathscr{P}}(q(M), N) > 0$. By monotonicity, $w_{\mathscr{P}}((q(M), q(M_a) \cup A_*) \geq w_{\mathscr{P}}(q(M), q(N)A^{\mathrm{eq}})$. But, by Lemma 2.3, $w_{\mathscr{P}}(q(M), q(N)A^{\mathrm{eq}}) = w_{\mathscr{P}}(q(M), N) > 0$. ⊣

Now we have

$$\begin{aligned} &(1) \qquad w_{\mathscr{P}}(q(M), A^{\mathrm{eq}}) = w_{\mathscr{P}}(q(M), q(M_a)A^{\mathrm{eq}}) + w_{\mathscr{P}}(q(M_a), A^{\mathrm{eq}}) \\ &(2) \qquad \geq 1 + w_{\mathscr{P}}(q(M_a), A^{\mathrm{eq}}) < \omega. \end{aligned}$$

Here, the first equality is by [4] and Lemma 2.3 and the second by Claim 2.11. The finiteness comes from Claim 2.7. Since $g \cup \mathrm{id}_{A^{\mathrm{eq}}}$ is an elementary map, $w_{\mathscr{P}}(q(M_a), A^{\mathrm{eq}}) = w_{\mathscr{P}}(q(M_b), A_*)$. We substitute in Equation 1, using Claim 2.10:

$$w_{\mathscr{P}}(q(M_a), A^{\mathrm{eq}}) = w_{\mathscr{P}}(q(M), A^{\mathrm{eq}}) = w_{\mathscr{P}}(q(M_a), A^{\mathrm{eq}}) + 1,$$

or subtracting, $0 = 1$ so we finish. ⊣

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INSTITUTE FOR PROBLEMS OF INFORMATICS AND CONTROL
PUSHKIN STR. 125
ALMATY 480100, KAZAKHSTAN
E-mail: baizhanov@ipic.kz

DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE
UNIVERSITY OF ILLINOIS AT CHICAGO
851 S. MORGAN STREET CHICAGO, IL 60607, USA
E-mail: jbaldwin@math.uic.edu

INSTITUTE OF MATHEMATICS
THE HEBREW UNIVERSITY
JERUSALEM, ISRAEL
and
RUTGERS UNIVERSITY
MATHEMATICS DEPARTMENT
NEW BRUNSWICK, NJ, USA
E-mail: shelah@math.huji.ac.il