CASTELNUOVO-MUMFORD REGULARITY AND BRIDGELAND STABILITY OF POINTS IN THE PROJECTIVE PLANE

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Abstract. In this paper, we study the relation between Castelnuovo-Mumford regularity and Bridgeland stability for the Hilbert scheme of n points on $\mathbb{P}^2$. For the largest $\lfloor \frac{n}{2} \rfloor$ Bridgeland walls, we show that the general ideal sheaf destabilized along a smaller Bridgeland wall has smaller regularity than one destabilized along a larger Bridgeland wall. We give a detailed analysis of the case of monomial schemes and obtain a precise relation between the regularity and the Bridgeland stability for the case of Borel fixed ideals.

1. Introduction

In this paper, we consider the relation between the Castelnuovo-Mumford regularity and the Bridgeland stability of zero-dimensional subschemes of $\mathbb{P}^2$. Our study is motivated by the following result which relates geometric invariant theory (GIT) stability and Castelnuovo-Mumford regularity.

Theorem. [HH13, Corollary 4.5] *Let $C \subset \mathbb{P}^{3g-4}$ be a c-semistable bicanonical curve. Then $\mathcal{O}_C$ is 2-regular.*

Note that *c-semistability* of curves [HH13, Definition 2.6] is a purely geometric notion concerning singularities and subcurves, whereas Castelnuovo-Mumford regularity is an algebraic notion regarding the syzygies of ideal sheaves.

For points in $\mathbb{P}^2$, a similar but weaker statement holds. A set of n points in $\mathbb{P}^2$ is GIT semistable if and only if at most $2n/3$ of the points are collinear, in which case the regularity is at most $2n/3$. However, the regularities of semistable points cover a broad spectrum. Our goal in this paper is to use Bridgeland stability to obtain a closer relationship between stability and regularity.

There is a distinguished half-plane $H = \{(s,t) | s > 0, t \in \mathbb{R}\}$ of Bridgeland stability conditions for $\mathbb{P}^2$. Let ξ be a Chern character. The half-plane H admits a wall-and-chamber decomposition, where in each chamber the set of Bridgeland semistable objects with Chern character ξ remains constant.

The Bridgeland walls where an ideal sheaf of points is destabilized consist of the vertical line $s = 0$ and a finite set of *nested* semicircular walls $\mathcal{W}_c$ centered along the s-axis at $s = -c - \frac{3}{2} < 0$ [ABCH13, Section 6]. Since the semicircular Bridgeland

2010 *Mathematics Subject Classification.* 14C05, 13D02, 14D20 primary, 13D99, 14D99, 14C99 secondary.

Key words and phrases. Castelnuovo-Mumford regularity, Hilbert schemes of points, Bridgeland stability, monomial schemes.

The first author was partially supported by the NSF CAREER grant DMS-0950951535 and the NSF grant DMS-1500031; the second author was supported by the following grants funded by the government of Korea: NRF grant 2011-0030044 (SRC-GAIA) and NRF grant NRF-2013R1A1A2010649.

walls are nested, we can order them by inclusion. If an ideal sheaf $\mathcal{I}_Z$ is destabilized along the wall $\mathcal{W}_c$, then $\mathcal{I}_Z$ is Bridgeland stable in the region bounded by $\mathcal{W}_c$ and $s = 0$. Let $\sigma \prec \sigma'$ if all σ'-semistable ideal sheaves with Chern character ξ are σ-semistable. Consequently, Bridgeland stability induces a stratification of $\mathbb{P}^{2[n]}$

$$\mathbb{P}^{2[n]} = \coprod_{\alpha} X^{\alpha},$$

where

$$X^{\alpha} = \{Z \in \mathbb{P}^{2[n]} \,|\, \mathcal{I}_Z \text{ is } \alpha\text{-semistable but } \beta\text{-unstable } \forall \alpha \prec \beta\}$$

and α runs over a Bridgeland stability condition in each chamber. We have $\overline{X^{\alpha}} = \bigcup_{\beta \preceq \alpha} X^{\beta}$ (see Section 2). By [ABCH13, Sections 9,10] and [LZ], this stratification coincides with the stratification of $\mathbb{P}^{2[n]}$ according to the stable base loci of linear systems. Recall that the effective cone of a variety has a wall and chamber decomposition such that in each chamber the stable base locus of the divisors remain constant.

Similarly, there is a stratification induced by Castelnuovo-Mumford regularity:

$$\mathbb{P}^{2[n]} = \coprod_{r \in \mathbb{Z}} X^{r\text{-reg}},$$

where $X^{r\text{-reg}}$ is the collection of ideals whose Castelnuovo-Mumford regularity is r. The regularity, being a cohomological invariant [Eis95, Proposition 20.16], is upper-semicontinuous and we have $\overline{X^{r\text{-reg}}} = \coprod_{r' \geq r} X^{r'\text{-reg}}$.

This naturally raises the question of comparing the two stratifications. We will show that a general scheme destabilized at one of the $\lfloor \frac{n}{2} \rfloor$ largest Bridgeland walls has smaller regularity than the general scheme destabilized along the larger walls. Our main theorem will be proved in Section 5:

Theorem. *Let $\mathfrak{p}_i$ be the maximal ideal of the closed point $p_i \in \mathbb{P}^2$, $i = 1, \dots, s$. Let Z be the subscheme given by $\cap_{i=1}^{s} \mathfrak{p}_i^{m_i}$ and let n be its length. Define*

$$h := \max \left\{ \sum_{j=1}^{t} m_{i_j} \;\middle|\; p_{i_1}, \dots, p_{i_t} \text{ are collinear} \right\}.$$

If $n \leq 2h - 3$, then Z is destabilized at the wall $\mathcal{W}_{\mathrm{reg}(Z)-1}$. In particular, general points destabilized at $\mathcal{W}_{k+1}$ have higher regularity than those destabilized at $\mathcal{W}_k$, $\forall k \geq \frac{n}{2} - 1$.

For zero-dimensional subschemes cut out by monomials, we have a more precise connection between regularity and Bridgeland stability:

Proposition. *Let Z be a zero-dimensional monomial scheme in $\mathbb{P}^2$. If the ideal sheaf $\mathcal{I}_Z$ is destabilized at the wall $\mathcal{W}_{\mu(Z)}$ with center $x = -\mu(Z) - \frac{3}{2}$, then*

$$\frac{3}{4}(\mathrm{reg}(\mathcal{I}_Z) - 1) \leq \mu(Z) \leq \mathrm{reg}(\mathcal{I}_Z) - 1.$$

(1) *The left equality holds if and only if $\mathrm{reg}(\mathcal{I}_Z) + 1 = 2m$ is even and $\mathcal{I}_Z = \langle x^m, y^m \rangle$*
(2) *The right equality holds if and only if $\mathcal{I}_Z = \langle x^{a_1}, x^{a_2}y^{b_2}, \dots, y^{b_r} \rangle$ with $\max_{1 \leq i \leq r-1}(a_i + b_{i+1} - 1) \leq \max(a_1, b_r)$.*

In particular, for Borel fixed ideals, the regularity and the Bridgeland stability completely determine each other:

Corollary. *Let $Z \subset \mathbb{P}^2$ be a zero-dimensional monomial scheme whose ideal is Borel-fixed (which holds if it is a generic initial ideal, for instance). Then the ideal sheaf $\mathcal{I}_Z$ is destabilized at the wall $\mathcal{W}_{\mathrm{reg}(\mathcal{I}_Z)-1}$.*

In general, the relation between regularity and the Bridgeland slope is not monotonic. Let Z_1 and Z_2 be two schemes of length n destabilized along $\mathcal{W}_{\mu(Z_1)}$ and $\mathcal{W}_{\mu(Z_2)}$, respectively. It may happen that while $\mathrm{reg}(Z_1) > \mathrm{reg}(Z_2)$, we have $\mu(Z_1) < \mu(Z_2)$. We close the introduction with the following simple but illustrative example.

Example 1.1. Let Z_1 and Z_2 be the monomial scheme defined by $\langle x^4, y^4\rangle$ and $\langle x^6, x^5y, x^4y^2, xy^3, y^4\rangle$, respectively. Both are of length 16, and by the arguments of Section 3, we see that $\mathrm{reg}(\mathcal{I}_{Z_1}) = 7, \mathrm{reg}(\mathcal{I}_{Z_2}) = 6$ and $\mu(Z_1) = \frac{9}{2}, \mu(Z_2) = 5$.

We work over an algebraically closed field $\mathbb{K}$ of characteristic zero.

Acknowledgements. We would like to thank Aaron Bertram and Jack Huizenga for enlightening conversations.

2. Preliminaries on Bridgeland stability conditions

We briefly review the basics of Bridgeland stability conditions on $\mathbb{P}^2$. We refer the reader to [ABCH13] and [CH14] for more details. Let $\mathcal{D}^b(\mathbb{P}^2)$ be the bounded derived category of coherent sheaves on $\mathbb{P}^2$, and $K(\mathbb{P}^2)$ be the K-group of $\mathcal{D}^b(\mathbb{P}^2)$.

Definition 2.1. A *Bridgeland stability condition* on $\mathbb{P}^2$ consists of a pair $(\mathcal{A}, \mathcal{Z})$, where $\mathcal{A}$ is the heart of a t-structure on $\mathcal{D}^b(\mathbb{P}^2)$ and $\mathcal{Z} : K(\mathbb{P}^2) \to \mathbb{C}$ is a homomorphism (called the *central charge*) satisfying

- if $0 \neq E \in \mathcal{A}$, $\mathcal{Z}(E)$ lies in the semi-closed upper half-plane $\{re^{i\pi\theta} \mid r > 0, 0 < \theta \leq 1\}$.
- $(\mathcal{A}, \mathcal{Z})$ has the Harder-Narasimhan property, which will be defined below.

Definition 2.2. Writing $\mathcal{Z} = -d + ir$, the *slope* $\mu(E)$ of $0 \neq E \in \mathcal{A}$ is defined by $\mu(E) = d(E)/r(E)$ if $r(E) \neq 0$ and $\mu(E) = \infty$ otherwise.

Definition 2.3. An object $E \in \mathcal{A}$ is called *stable* (resp. *semistable*) if for every proper subobject $F \subset E$ in $\mathcal{A}$, $\mu(F) < \mu(E)$ (resp. $\mu(F) \leq \mu(E)$).

Definition 2.4. The pair $(\mathcal{A}, \mathcal{Z})$ has the *Harder-Narasimhan property* if any nonzero object $E \in \mathcal{A}$ admits a finite filtration

$$0 \subset E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that each *Harder-Narasimhan factor* $F_i = E_i/E_{i-1}$ is semistable and $\mu(F_1) > \mu(F_2) > \cdots > \mu(F_n)$.

Let L be the class of a line in $\mathbb{P}^2$.

Definition 2.5. Let E be a coherent sheaf on $\mathbb{P}^2$. The *Mumford slope* of E is defined by $\deg(E)/\mathrm{rank}(E)$, where $\deg(E) = \mathrm{ch}_1(E) \cdot L$ and $\mathrm{rank}(E) = \mathrm{ch}_0(E) \cdot L^2$ are the ordinary degree and rank.

Let $\mu_{\min}(E)$ (resp. $\mu_{\max}(E)$) denote the minimum (resp. maximum) slope of a Harder-Narasimhan factor of a coherent sheaf E with respect to the Mumford slope. For $s \in \mathbb{R}$, let $\mathcal{Q}_s$ and $\mathcal{F}_s$ be the full subcategory of $\mathrm{Coh}(\mathbb{P}^2)$ defined by

- $Q \in \mathcal{Q}_s$ if Q is torsion or $\mu_{\min}(Q) > s$.
- $F \in \mathcal{F}_s$ if F is torsion-free, and $\mu_{\max}(F) \leq s$.

Each pair $(\mathcal{F}_s, \mathcal{Q}_s)$ is a torsion pair [Bri08, Lemma 6.1], and induces a t-structure via tilting on $D^b(\mathbb{P}^2)$ with heart [HRS96]

$$\mathcal{A}_s = \{E \in \mathcal{D}^b(\mathbb{P}^2) \mid H^{-1}(E) \in \mathcal{F}_s, H^0(E) \in \mathcal{Q}_s, \text{ and } H^i(E) = 0 \text{ otherwise}\}.$$

Theorem. [Bri08, AB13, BM11] *For each $s \in \mathbb{R}$ and $t > 0$, define*

$$\mathcal{Z}_{s,t}(E) = -\int_{\mathbb{P}^2} e^{-(s+it)L} \mathrm{ch}(E).$$

Then the pair $(\mathcal{A}_s, \mathcal{Z}_{s,t})$ defines a Bridgeland stability condition on $\mathcal{D}^b(\mathbb{P}^2)$.

We thus obtain an upper half-plane H of Bridgeland stability conditions.

Fix a class ξ in the numerical Grothendieck group. If ξ has positive rank, define the *slope* and the *discriminant* by

$$\mu(\xi) = \frac{\mathrm{ch}_1(\xi)}{\mathrm{rank}\,(\xi)}, \quad \Delta = \frac{1}{2}\mu(\xi)^2 - \frac{\mathrm{ch}_2(\xi)}{\mathrm{rank}\,(\xi)}.$$

For an ideal sheaf $\mathcal{I}_Z$ of n points, we have $\mu = 0$ and $\Delta = n$. A sheaf E of positive rank is *Gieseker semistable* if for every proper subsheaf $0 \neq F \subset E$, $\mu(F) \leq \mu(E)$ and in case of equality $\Delta(F) \geq \Delta(E)$. The sheaf is called *Gieseker stable* if the second inequality is strict. The sheaf E is Gieseker semistable if and only if for some s, E is $\mathcal{Z}_{s,t}$-semistable for all $t \gg 0$ [ABCH13, Section 6]. Every ideal sheaf of points is Gieseker (in fact, slope) stable.

There exists a locally finite set of walls in the (s,t)-half plane depending on ξ such that the set of σ-(semi)stable objects of class ξ does not change as the σ varies in a chamber [Bri08, BM11, BM14]. These walls are called *Bridgeland walls*. For $\mathbb{P}^2$, the Bridgeland walls where a Gieseker semistable sheaf is destabilized consist of line $s = \mu(\xi)$ and a finite number of nested semicircles with center $(c, 0)$ with $c < \mu$ [ABCH13, Section 6]. The largest semicircular wall is called the *Gieseker wall* and the smallest semicircular wall is called the *collapsing wall*. If $\xi = (1, 0, -n)$, the Chern character of the ideal sheaf of a zero-dimensional subscheme of $\mathbb{P}^2$ of length n, then the wall with center $(c, 0)$ has radius $\sqrt{c^2 - 2n}$. Throughout the paper $\mathcal{W}_\mu = \mathcal{W}_\mu^n$ will denote the wall centered at $(-\mu - \frac{3}{2}, 0)$. An ideal sheaf destabilized along $\mathcal{W}_\mu$ is Bridgeland stable for all Bridgeland stability conditions outside $\mathcal{W}_\mu$ and not semistable for any Bridgeland stability condition contained in $\mathcal{W}_\mu$. All Bridgeland walls for $n \leq 9$ were explicitly computed in [ABCH13, Section 10].

In Figure 1, we reproduce the example of $n = 5$. Along the Gieseker wall $\mathcal{W}_{-\frac{11}{2}}$ ideal sheaves of collinear points are destabilized. Along the wall $\mathcal{W}_{-\frac{9}{2}}$ ideal sheaves of schemes with a collinear subscheme of length four are destabilized. All ideal sheaves are destabilized along the collapsing wall $\mathcal{W}_{-\frac{7}{2}}$.

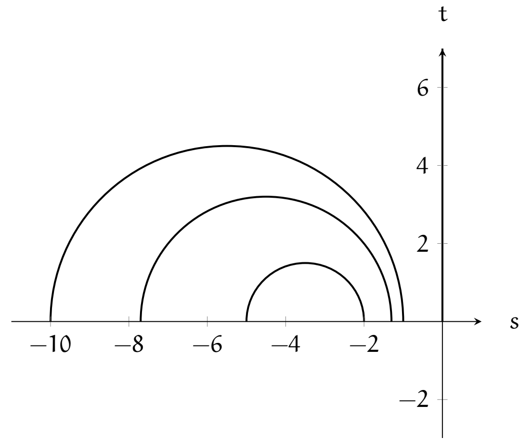

FIGURE 1. The Bridgeland walls for $\mathbb{P}^{2[5]}$.

3. Monomial schemes

A *monomial subscheme* of $\mathbb{P}^2$ is a subscheme whose ideal is generated by monomials. For these schemes, the relation between Castelnuovo-Mumford regularity and Bridgeland stability is clear because the regularity is easy to compute and the Bridgeland stability is explicitly described by [CH14]. To reveal the relation, we need to study the combinatorics.

Proposition 3.1. *Let Z be a zero-dimensional monomial scheme in $\mathbb{P}^2$. If the ideal sheaf $\mathcal{I}_Z$ is destabilized at the wall $\mathcal{W}_{\mu(Z)}$ with center $x = -\mu(Z) - \frac{3}{2}$, then*

$$\frac{3}{4}(\operatorname{reg}(\mathcal{I}_Z) - 1) \leq \mu(Z) \leq \operatorname{reg}(\mathcal{I}_Z) - 1.$$

(1) *The left equality holds if and only if* $\operatorname{reg}(\mathcal{I}_Z) + 1 = 2m$ *is even and* $\mathcal{I}_Z = \langle x^m, y^m \rangle$.
(2) *The right equality holds if and only if* $\mathcal{I}_Z = \langle x^{a_1}, x^{a_2}y^{b_2}, \ldots, y^{b_r} \rangle$ *satisfies* $\max_{1\leq i\leq r-1}(a_i + b_{i+1} - 1) \leq \max(a_1, b_r)$.

A zero-dimensional monomial subscheme Z in $\mathbb{P}^2$, in a suitable affine coordinate system, has defining ideal I_Z generated by a set of monomials

$$x^{a_1}, x^{a_2}y^{b_2}, \ldots, y^{b_r} \tag{†}$$

where $a_1 > \ldots > a_{r-1} > a_r = 0$ and $0 = b_1 < b_2 < \ldots < b_r$.

It is convenient to represent monomial subschemes by their block diagrams. The *block diagram* D for Z consists of b_r left-aligned rows of consecutive boxes such that the ith row counting from the bottom has a_j boxes if $b_j < i \leq b_{j+1}$. The lower left corner represents the monomial 1. The box to the right of (resp. above) x^iy^j represent $x^{i+1}y^j$ (resp. x^iy^{j+1}). With this interpretation, the box diagram D records the monomials in $\mathbb{K}[x, y]$ which are not in I_Z. The next figure shows an example.

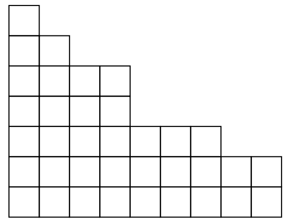

FIGURE 2. The block diagram for $\langle x^9, x^7y^2, x^4y^3, x^2y^5, xy^6, y^7\rangle$

We will always place the lower left corner of D at the origin and assume that the boxes in D are unit length.

Proof of Proposition 3.1. We briefly recapitulate the computation of $\mu(Z)$ in [CH14]. Index the rows of a box diagram D from bottom to top, and the columns from left to right. Let h_j (resp. v_j) be the number of boxes in the jth row (resp. column). Let $r(D)$ and $c(D)$ be the number of rows and columns in D. Define the kth horizontal slope μ_k and the ith vertical slope μ'_i by

$$\mu_k = \frac{1}{k}\sum_{j=1}^{k}(h_j + j - 1) - 1, \quad \mu'_i = \frac{1}{i}\sum_{j=1}^{i}(v_j + j - 1) - 1.$$

Then the slope $\mu(Z)$ of Z is defined by

$$\mu(Z) = \max_{1\le k\le r(D), 1\le i\le c(D)}\{\mu_k, \mu'_i\}.$$

By [CH14, Theorem 1.6], the ideal sheaf $\mathcal{I}_Z$ is destabilized at the wall $\mathcal{W}_{\mu(Z)}$ with center $x = -\mu(Z) - \frac{3}{2}$.

On the other hand, the regularity of $\mathcal{I}_Z$ can be computed from its minimal free resolution given by

$$0 \to \bigoplus_{i=1}^{r-1}\mathcal{O}(-a_i - b_{i+1}) \xrightarrow{M} \bigoplus_{i=1}^{r}\mathcal{O}(-a_i - b_i) \to \mathcal{I}_Z \to 0,$$

where M is the $r \times (r-1)$ matrix with entries

$$m_{i,i} = y^{b_{i+1}-b_i}, \quad m_{i+1,i} = -x^{a_i - a_{i+1}}, \quad \text{and} \quad m_{i,j} = 0 \ \text{ otherwise.}$$

Since $a_i + b_{i+1} - 1 \ge a_i + b_i$ for $i = 1, \dots, r-1$ and $a_{r-1} + b_r - 1 \ge a_r + b_r$, the Castelnuovo-Mumford regularity $\operatorname{reg}(\mathcal{I}_Z)$ of $\mathcal{I}_Z$ is

$$\operatorname{reg}(\mathcal{I}_Z) = \max_{1\le i\le r-1}(a_i + b_{i+1} - 1).$$

If we place the block diagram D in the a-b plane with its lower left corner at the origin and set every box to be a unit square, then the points (a_i, b_{i+1}) are the vertices of D contained in the first quadrant. Hence, the block diagrams representing ideals with regularity l are precisely those which lie below and touch the line $a + b = l + 1$.

Fix the regularity to equal l. To maximize $\mu(Z)$ subject to $\operatorname{reg}(Z) = l$, we need to maximize μ_k and μ'_i under the condition that the box diagram lies below and touches the line $a + b = l + 1$. Since the box diagram of $I_Z = \langle x^l, x^{l-1}y, \dots, y^l\rangle$ contains every positive integral lattice point under the line $a + b = l + 1$, it follows that Z gives the the maximum μ-value, which is $l - 1$. Note that $\mu_k = l - 1$ if and only if $h_1 = l, h_2 = l-1, \ldots, h_k = l-(k-1)$. Hence, $\mu(Z) = l-1$ precisely when either $h_1 = l$ or $v_1 = l$. Equivalently, equality holds for $I_Z = \langle x^{a_1}, x^{a_2}y^{b_2}, \ldots, y^{b_r}\rangle$ if I_Z satisfies $\max_{1\leq i\leq r-1}(a_i + b_{i+1} - 1) \leq \max(a_1, b_r)$.

To minimize $\mu(Z)$ subject to $\text{reg}(Z) = l$, we use as few boxes as possible to minimize the slopes μ_k and μ_i'. A box diagram that touches the line $a+b = l+1$ at (a', b') contains the box diagram of the ideal $\langle x^{a'}, y^{b'}\rangle$. It follows that the ideal of Z should be of the form $\langle x^a, y^b\rangle$ with $a+b = l+1$. Then

$$\max_{1\leq k\leq r(D)}\{\mu_k\} = \mu_b = a + \frac{b-1}{2} - 1$$

and similarly

$$\max_{1\leq i\leq c(D)}\{\mu_i'\} = \mu_a' = b + \frac{a-1}{2} - 1$$

so that

$$\mu(Z) = \max\left(a + \frac{b-1}{2} - 1, b + \frac{a-1}{2} - 1\right).$$

Thus $\mu(Z)$ achieves the minimum when a and b are almost equal. If l is even, then $(a, b) = (\frac{l}{2}+1, \frac{l}{2})$ gives $\mu(Z) = \frac{3l}{4} - \frac{1}{2}$. If l is odd, then $(a, b) = (\frac{l+1}{2}, \frac{l+1}{2})$ gives $\mu(Z) = \frac{3l}{4} - \frac{3}{4}$. Furthermore, if $n > \frac{(l+1)^2}{4}$, then either the horizontal slope $\mu_{\frac{l+1}{2}}$ or the vertical slope $\mu'_{\frac{l+1}{2}}$ is strictly larger than $\frac{3l}{4} - \frac{3}{4}$. We conclude that $\frac{3l}{4} - \frac{3}{4} \leq \mu(Z)$ with equality only if Z is the monomial ideal $\langle x^{\frac{l+1}{2}}, y^{\frac{l+1}{2}}\rangle$. □

Recall that an ideal I generated by monomials in x and y is *Borel fixed* if $x^iy^j \in I$ for some $j > 0$ implies $x^{i+1}y^{j-1} \in I$. Borel fixedness is one of the most important combinatorial properties in the study of monomial ideals. For instance, *generic initial ideals* with respect to a monomial order are Borel fixed. See [Eis95, Theorem 15.20] for a detailed discussion. We obtain the following corollary.

Corollary 3.2. *Let $Z \subset \mathbb{P}^2$ be a zero-dimensional monomial scheme whose ideal is Borel-fixed. Then the ideal sheaf $\mathcal{I}_Z$ is destabilized at the wall $\mathcal{W}_{\text{reg}(\mathcal{I}_Z)-1}$.*

Proof. A Borel-fixed ideal is of the form $\langle x^a, x^{a-1}y^{\lambda_{a-1}}, \ldots, y^{\lambda_0}\rangle$ with $\lambda_0 > \ldots > \lambda_{a-1} > 0$. Then $(i + \lambda_{i-1} - 1) \leq \lambda_0 = \max(a, \lambda_0)$ for $i = 1, \ldots, a$. The corollary follows from Proposition 3.1 (2). □

Every possible Betti diagram of a zero-dimensional scheme in $\mathbb{P}^2$ occurs as the Betti diagram of a monomial scheme [Eis05]. Let $\binom{k}{2} < n \leq \binom{k+1}{2}$ and let Z be a scheme of length n. Then the regularity of Z can be any integer between k and n. Given $k \leq l \leq n$, take a box diagram D with n boxes and at most l rows such that $h_1 = l$ and $h_i \leq l+1-i$ for $2 \leq i \leq l$. Since $n \leq \binom{l+1}{2}$ such diagrams D exist. Moreover, $\mu(Z) = l-1 = \text{reg}(\mathcal{I}_Z) - 1$, the maximum possible by Proposition 3.1.

We can also ask for the minimum possible $\mu(Z)$ given a scheme Z of length n and regularity l. If $0 < m \leq \frac{l}{2}$ and $m(l+1-m) \leq n < (m+1)(l-m)$, then the tallest rectangle with upper right vertex on the line $x+y = l+1$ is the $m \times (l-m+1)$ rectangle. Hence, $\mu(Z) \geq \text{reg}(\mathcal{I}_Z) - \frac{1}{2} - \frac{m}{2}$. Equality occurs, for instance, when $n = m(l+1-m)$. In case, l is even (resp. odd) and $n \geq \frac{l}{2}(\frac{l}{2}+1)$ (resp. $n \geq (\frac{l+1}{2})^2$), then $\mu(Z) \geq \frac{3}{4}\text{reg}(\mathcal{I}_Z) - \frac{1}{2}$ (resp. $\frac{3}{4}\text{reg}(\mathcal{I}_Z) - \frac{3}{4}$). In particular, we conclude that

$$1 \leq \text{reg}(\mathcal{I}_Z) - \mu(Z) \leq \frac{\sqrt{n}+1}{2}.$$

Equality is attained on the right hand side when $\mathrm{reg}(\mathcal{I}_Z)$ is odd and $n = \frac{(\mathrm{reg}(\mathcal{I}_Z)+1)^2}{4}$. We summarize this in the following proposition.

Proposition 3.3. *Let Z be a monomial scheme of length n and regularity l. If $0 < m \leq \frac{l}{2}$ and $m(l+1-m) \leq n < (m+1)(l-m)$, then*

$$1 \leq \mathrm{reg}(\mathcal{I}_Z) - \mu(Z) \leq \frac{m}{2} + \frac{1}{2}.$$

In general,

$$1 \leq \mathrm{reg}(\mathcal{I}_Z) - \mu(Z) \leq \frac{\sqrt{n}+1}{2}.$$

4. General points

In this section, we discuss the relation between Bridgeland stability and regularity for general points on $\mathbb{P}^2$.

Let $\binom{r}{2} < n \leq \binom{r+1}{2}$. Then, for a dense open set $U \in \mathbb{P}^{2[n]}$, the minimal free resolution of $\mathcal{I}_Z$ is the Gaeta resolution

$$0 \to \mathcal{O}^{\oplus a}(-r-1) \oplus \mathcal{O}^{\oplus \max(0,-b)}(-r) \to \mathcal{O}^{\oplus \max(0,b)}(-r) \oplus \mathcal{O}^{\oplus c}(-r+1) \to \mathcal{I}_Z \to 0,$$

where $a = n - \binom{r}{2} > 0$, $c = \binom{r+1}{2} - n \geq 0$ and $b = c - a + 1$ [Eis05]. The regularity of $\mathcal{I}_Z$ is r. Since regularity is upper-semicontinuous and $\mathbb{P}^{2[n]}$ is irreducible, there exists an open set U_1 containing U such that $\mathrm{reg}(\mathcal{I}_Z) = r$ for $Z \in U_1$.

On the other hand, there exists an open dense set $U_2 \in \mathbb{P}^{2[n]}$ such that for $Z \in U_2$ the ideal sheaf $\mathcal{I}_Z$ is destabilized at the collapsing wall $\mathcal{W}_{\mu_n}$ with center $(-\mu_n - \frac{3}{2}, 0)$. By a general point of $\mathbb{P}^{2[n]}$, we will mean a point $Z \in U_1 \cap U_2$. For such Z, there exists a precise relation between the regularity k and the Bridgeland slope μ_n. Huizenga computed μ_n for all n [Hui, Theorem 7.2]. The slope μ_n is the smallest positive slope of a stable vector bundle on the parabola $\mu^2 + 3\mu + 2 - 2n = 2\Delta$, where μ is the slope and Δ is the discriminant. The computation of μ_n, while easy for any given n, depends on a fractal curve. Consequently, it is hard to write a closed formula.

Luckily, there are good bounds for μ_n. Let

$$\mathcal{S} = \left\{ \frac{0}{1}, \frac{1}{2}, \frac{3}{5}, \frac{8}{13}, \dots \right\} \cup \left\{ \alpha > \phi^{-1} = \frac{\sqrt{5}-1}{2} \right\}$$

consisting of consecutive ratios of Fibonacci numbers and numbers larger than the inverse of the golden ratio. Let $n = \binom{k}{2} + s$ with $0 \leq s < k$. By [ABCH13, Theorem 4.5], we have

$$\mu_n = \begin{cases} k - 2 + \frac{s}{k-1} & \text{if } \frac{s}{k-1} \in \mathcal{S} \\ k - 1 - \frac{k-s}{k+1} & \text{if } 1 - \frac{s+1}{k+1} \in \mathcal{S}. \end{cases}$$

Furthermore, by [ABCH13, Lemma 4.1, Corollary 4.9], the inequalities

$$\mu_{n-1} \leq \mu_n \leq \begin{cases} k - 2 + \frac{s}{k-1} & \text{if } \frac{s}{k-1} \geq \frac{1}{2} \\ k - 1 - \frac{k-s}{k+1} & \text{if } \frac{s}{k-1} \leq \frac{1}{2} \end{cases}$$

hold. When k is odd and $s = \frac{k-1}{2}$, then $\frac{s}{k-1} = \frac{1}{2} \in \mathcal{S}$ and $\mu_n = k - \frac{3}{2}$. When k is even and $n = \binom{k}{2} + \frac{k}{2} + 1$, then the positive root x_p of $\frac{1}{2}(\mu^2 + 3\mu + 2) - n = \frac{1}{2}$ satisfies $x_p > k - \frac{3}{2}$. By [Hui, Theorem 7.2], we conclude that $\mu_n > k - \frac{3}{2}$. Combining these inequalities we deduce the following proposition.

Proposition 4.1. *Let Z be a general point of $\mathbb{P}^{2[n]}$. Let $\mathcal{W}_{\mu_n}$ be the collapsing wall.*

(1) *If $n = \binom{k}{2}$, then $\mu_n = \operatorname{reg}(\mathcal{I}_Z) - 1$.*
(2) *If $n = \binom{k}{2} + s$ with $\frac{1}{2} \geq \frac{s}{k-1} > 0$, then*
$$\operatorname{reg}(\mathcal{I}_Z) - 1 - \frac{\max(k-s, \lceil \phi^{-1}(k+1) \rceil)}{k+1} \leq \mu_n \leq \operatorname{reg}(\mathcal{I}_Z) - 1 - \frac{k-s}{k+1}$$
and the right inequality is an equality if $1 - \frac{s+1}{k+1} \in \mathcal{S}$.
(3) *If $n = \binom{k}{2} + s$ with $\frac{s}{k-1} \geq \frac{1}{2}$, then*
$$\operatorname{reg}(\mathcal{I}_Z) - \frac{3}{2} \leq \mu_n \leq \operatorname{reg}(\mathcal{I}_Z) - 2 + \frac{s}{k-1}$$
and the right inequality is an equality if $\frac{s}{k-1} \in \mathcal{S}$.

In particular, $\operatorname{reg}(\mathcal{I}_Z) - 2 < \mu_n \leq \operatorname{reg}(\mathcal{I}_Z) - 1$ for a general Z.

We point out that the sets $U_1 - U_2$ and $U_2 - U_1$ are both nonempty in general.

Example 4.2. The minimum regularity for a scheme Z of length 7 is 4 and $\mu_7 = \frac{12}{5}$ [Hui, Table 1]. Consider the monomial scheme generated with defining ideal $\langle x^4, xy, y^4 \rangle$. The regularity of this scheme is 4 but it is destabilized along the wall $\mathcal{W}_3$. Hence, this monomial scheme is a point of U_1 which is not in U_2.

Example 4.3. The minimum regularity for a scheme Z of length 9 is 4. For a complete intersection scheme of type $(3, 3)$, the minimal resolution is
$$0 \to \mathcal{O}(-6) \to \mathcal{O}(-3) \oplus \mathcal{O}(-3) \to \mathcal{I}_Z \to 0.$$
Hence, the regularity is 5. On the other hand, the general scheme and a complete intersection scheme both have $\mu = 3$ [ABCH13], [CH14, Theorem 5.1]. Hence, the complete intersection scheme is in U_2 but not in U_1.

5. Outer walls of the Bridgeland manifold

In general, it is hard to test whether a specific ideal sheaf $\mathcal{I}_Z$ is destabilized along a given wall $\mathcal{W}_\mu$. However, for the largest $\lfloor \frac{n}{2} \rfloor$ semicircular Bridgeland walls, one can give a concrete characterization of the ideal sheaves destabilized along the wall. This characterization allows us to compute the regularity.

Let Y_μ^n denote the locally closed subset of $\mathbb{P}^{2[n]}$ parameterizing subschemes Z destabilized along $\mathcal{W}_\mu$. By the one-to-one correspondence between the Bridgeland walls and Mori walls [ABCH13], we may rephrase [ABCH13, Proposition 4.16] as follows.

Proposition 5.1. *Let $n \leq k(k+3)/2$. Let $\mathcal{W}_k$ be the wall with center $x = -k - \frac{3}{2}$.*

(a) *If $n \leq 2k+1$, then Y_k^n parameterizes Z that have a linear subscheme of length $k+2$ but no linear subscheme of length greater than $k+2$;*
(b) *If $n = 2k+2$, then Y_k^n parameterizes Z that are contained in a conic or have a linear subscheme of length $k+2$ but does not have a linear subscheme of length greater than $k+2$.*

Fatabbi's theorem [Fat94] allows us to say more about the regularity of the schemes destabilized along $\mathcal{W}_k$.

Proposition 5.2. *(Fat points) Let* $\mathfrak{p}_i$ *be the maximal ideals of distinct closed points* $p_i \in \mathbb{P}^2$, $i = 1, \ldots, s$. *Let* Z *be the subscheme given by* $\cap_{i=1}^s \mathfrak{p}_i^{m_i}$ *and suppose that* Z *is of length* n. *Define*

$$h := \max \left\{ \sum_{j=1}^t m_{i_j} \,\middle|\, p_{i_1}, \ldots, p_{i_t} \text{ are collinear} \right\}.$$

If $n \leq 2h-3$, *then* Z *is destabilized at the wall* $\mathcal{W}_{\mathrm{reg}(Z)-1}$. *In particular, a general member of* Y^n_{k+1} *has a higher regularity than a general member of* Y^n_k, $\forall k \geq \frac{n}{2} - 1$.

Proof. The assumption $n \leq 2h - 3$ allow us to apply [Fat94, Theorem 3.3] and conclude that the regularity of Z equals h. We shall prove that Z has no linear subschemes of length $h+1$. Let L be a linear subcheme of Z supported on $p_{i_1}, \ldots, p_{i_t}$. Let f be a linear form vanishing on $p_{i_1}, \ldots, p_{i_t}$. Then $\mathfrak{p}_{i_j} = \langle f, g_{i_j} \rangle$ for some linear form g_{i_j} and f and $\mathfrak{p}_{i_j}^{m_{i_j}}$, $j = 1, \ldots, t$ are contained in the ideal I_L of L.

For the length of L to be as large as possible, we take the smallest possible ideal that contains $f + \sum_{j=1}^t \mathfrak{p}_{i_j}$. Since $\mathfrak{p}_{i_j}^{m_{i_j}} = \langle f^{m_{i_j}}, f^{m_{i_j}-1} g_{i_j}, \ldots, g_{i_j}^{m_{i_j}} \rangle$, any ideal containing $f + \sum_{j=1}^t \mathfrak{p}_{i_j}$ must also contain $g_{i_j}^{m_{i_j}}$. It follows that $\langle f, g_{i_1}^{m_{i_1}} \rangle \cap \ldots \cap \langle f, g_{i_t}^{m_{i_t}} \rangle$ defines a linear subscheme of Z of maximal length $\sum_{j=1}^t m_{i_j}$ supported on the cycle $\sum_{j=1}^t m_{i_j} p_{i_j}$. Since the regularity h is the maximum that the degree $\sum_{j=1}^t m_{i_j}$ can achieve, it is the maximum length of a linear subscheme of Z. Now, since $n \leq 2(h-2)+1$ by assumption, we may apply Proposition 5.1 and obtain the first assertion.

General points Z of Y^n_k, $k \geq \frac{n}{2} - 1$, have no multiplicities i.e. $m_i = 1$, $\forall i$; have $k+2$ collinear points; and the rest are in general position. This corresponds to the case $h = k+2 \geq \frac{n}{2} + 1 > \left[\frac{n}{2}\right]$, so Fatabbi's theorem applies and $\mathrm{reg}(\mathcal{I}_Z) = h = k+2$. □

We emphasize again that, as we have noted in the introduction, the relation between regularity and the Bridgeland slope in general is not monotonic (Example 1.1).

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