Representing Scott Sets in Algebraic Settings*

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Abstract

We prove that for every Scott Set S there are S-saturated real closed fields and S-saturated models of Presburger arithmetic.

1 Introduction

Recall that $S \subseteq 2^\omega$ is called a *Scott set* if and only if:

i) S is a Turing ideal, i.e., if $x, y \in S$ and $z \leq_T x \oplus y$, then $z \in S$, where $x \oplus y$ is the disjoint union of x and y;

ii) If $T \subseteq 2^{<\omega}$ is an infinite tree computable in some element of S, then there is $f \in S$ an infinite path through T.

Scott sets first arose in the study of completions of Peano arithmetic (PA) and models of PA. Scott [9] shows that the countable Scott sets are exactly the families of sets "representable" in a completion of PA. If $\mathcal{M}$ is a nonstandard model of Peano arithmetic and $a \in \mathcal{M}$, let

$$r(a) = \{n \in \omega : \mathcal{M} \models p_n | a\}$$

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where $p_0, p_1, \ldots$ is an increasing enumeration of the standard primes. The *standard system* of $\mathcal{M}$

$$SS(\mathcal{M}) = \{r(a) : a \in \mathcal{M}\}$$

is a Scott set. A longstanding and vexing problem in the study of models of arithmetic is whether every Scott set arises as the standard system of a model of Peano arithmetic. The best result is from Knight and Nadel [6].

Proposition 1.1 *If S is a Scott set and $|S| \leq \aleph_1$, then there is a model of Peano Arithmetic with standard system S.*

Thus the Scott set problem has a positive solution if the Continuum Hypothesis is true, but the question remains open without additional assumptions. Later in this section, we sketch a proof of Proposition 1.1.

Scott sets also are important when studying recursively saturated structures. We assume that we are working in a computable language $\mathcal{L}$. We fix a Gödel coding of $\mathcal{L}$ and say that a set of $\mathcal{L}$-formulas is in S if the corresponding set of Gödel codes is in S.

Let T be a complete $\mathcal{L}$-theory. The following ideas were introduced in [5], [10] and [7].

Definition 1.2 Let $S \subseteq 2^\omega$. We say that a model $\mathcal{M}$ of T is *S-saturated* if:

i) every type $p \in S_n(\emptyset)$ realized in $\mathcal{M}$ is computable in some element of S;

ii) if $p(x, \overline{y}) \in S_{n+1}(\emptyset)$ is computable in some element of S , $\overline{a} \in M^n$ and $p(x, \overline{a})$ is finitely satisfiable in $\mathcal{M}$, then $p(x, \overline{a})$ is realized in $\mathcal{M}$.

If a model is S-saturated for some $S \subseteq 2^\omega$, then the model is certainly recursively saturated. At least for countable models, the converse is proved in [5] Theorem 2.10.

Proposition 1.3 *If $\mathcal{M} \models T$ is countable and recursively saturated, then $\mathcal{M}$ is S-saturated for some Scott set S.*

If the theory T has limited coding power, then we can say little about S. For example, an algebraically closed field of infinite transcendence degree will be S-saturated for every Scott set S. On the other hand, the associated Scott set is unique for many natural examples, such as Peano arithmetic, divisible ordered abelian groups, real closed fields, $\mathbb{Z}$-groups (models of $\mathrm{Th}(\mathbb{Z},+)$) and Presburger arithmetic (models of $\mathrm{Th}(\mathbb{Z},+,<)$).

Definition 1.4 We say that a theory T is *effectively perfect* if there is a tree $(\phi_\sigma : \sigma \in 2^{<\omega})$ of formulas in n-free variables computable in T such that:
i) $T + \exists \overline{v}\ \phi_\sigma(\overline{v})$ is consistent for all σ;
ii) if $\sigma \subset \tau$, then $T \models \phi_\tau(\overline{v}) \rightarrow \phi_\sigma(\overline{v})$;
iii) $\phi_{\sigma^\frown 0}(\overline{v}) \wedge \phi_{\sigma^\frown 1}(\overline{v})$ is inconsistent with T for all σ.

For effectively perfect theories there is a much tighter correspondence (see, for example, [7] Theorem 1.5).

Proposition 1.5 *If T is effectively perfect, then every recursively saturated model of T is S-saturated for a unique Scott set S.*

The theories we will be considering are all effectively perfect. For Peano arithmetic, Presburger arithmetic and $\mathbb{Z}$-groups we can use the formulas $p_n|v$ to find such a tree. For real closed fields we can use $q < v$ for $q \in \mathbb{Q}$ and for ordered divisible abelian groups we can use the binary formulas $mv < nw$ for $m, n \in \mathbb{Z}$.

We now sketch a proof of Proposition 1.1. We first note that for models $\mathcal{M}$ of Peano arithmetic, $\mathcal{M}$ is recursively saturated if and only if $\mathcal{M}$ is S-saturated where S is the standard system of $\mathcal{M}$. (For more details see [4]).

Lemma 1.6 *If S is a countable Scott set and $T \in S$ is a completion of Peano arithmetic, then there is an S-saturated model of T.*

Proof Sketch Build $\mathcal{M}$ by a Henkin construction. At any stage, we will have a finite tuple $\overline{a}$ and will be committed to $\mathrm{tp}(\overline{a})$ the complete type of $\overline{a}$, where $T \subseteq \mathrm{tp}(\overline{a}) \in S$. At alternating stages, we either witness an existential quantifier or realize a type $p(v, \overline{a}) \in S$, using the join property of Scott sets to compute $p(v, \overline{x}) \cup \mathrm{tp}(\overline{a})$, and using the tree property to find completions. $\square$

Lemma 1.7 *Suppose $S_0 \subset S_1$ are countable Scott sets, $T \in S_0$ is a completion of Peano arithmetic, and $\mathcal{M}_0$ and $\mathcal{M}_1$ are countable recursively saturated models of T, where S_i is the standard system of $\mathcal{M}_i$. Then there is an elementary embedding of $\mathcal{M}_0$ into $\mathcal{M}_1$.*

Proof Sketch Let $a_0, a_1, \ldots$ be a list of the elements of $\mathcal{M}_0$. Suppose we have a partial elementary map $(a_0, \ldots, a_n) \mapsto (b_0, \ldots, b_n)$. If $\mathrm{tp}(a_{n+1}, a_0, \ldots, a_n) = p(v, a_0, \ldots, a_n)$, there is $b \in \mathcal{M}_1$ realizing $p(v, b_0, \ldots, b_n)$, and we can extend the embedding. $\square$

We can now prove Proposition 1.1. Suppose $|S| = \aleph_1$ and S is the union of an ω_1-chain of countable Scott sets

$$S_0 \subseteq S_1 \subseteq \ldots \subseteq S_\alpha \subseteq \ldots$$

where $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$ when α is a limit ordinal. We can build an elementary chain $(\mathcal{M}_\alpha : \alpha < \omega_1)$ where $\mathcal{M}_\alpha$ is recursively saturated with standard system S_α. Then $\bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha$ is recursively saturated with standard system S.

While we have nothing new to say about the Scott set problem for Peano arithmetic, we show that the analgous problem for recursively saturated models has a positive solution in some related algebraic settings.

For divisible ordered abelian groups, this follows easily from an unpublished result of Harnik and Ressayre. Let $(G, +, <)$ be a divisible ordered abelian group. Define an equivalence relation on G by $g \equiv h$ if and only if there is a natural number n such that $|g| < n|h|$ and $|h| < n|g|$. Let $\Gamma = \{|g|/\equiv : g \in G\}$, the set of equivalence classes of positive elements. The ordering of G induces an ordering on Γ. Suppose S is a Scott set and k_S is the set of real numbers computable in some element of S (where we identify a real with its cut in the rationals). It is easy to see that k_S is a real closed field.

Theorem 1.8 (Harnik–Ressayre) *A divisible ordered abelian group G is S-saturated if and only if Γ is a dense linear order without endpoints and each equivalence class under $\equiv$ is isomorphic to the ordered additive group of k_S.*

A complete proof is given in [3].

Corollary 1.9 *For any Scott set S, there is an S-saturated divisible ordered abelian group.*

Proof Let G be the set of functions $f : \mathbb{Q} \to k_S$ such that $\{q \in \mathbb{Q} : f(q) \neq 0\}$ is finite. We add elements of G coordinatewise and order G lexicographically. By the Harnik–Ressayre Theorem, G is S-saturated. $\square$

2 Real Closed Fields

In [2] D'Aquino, Knight and Starchenko show that if $\mathcal{M}$ is a nonstandard model of Peano arithmetic with standard system S, then the real closure of the fraction field of $\mathcal{M}$ is an S-saturated real closed field. Thus, it is natural to ask whether we can find an S-saturated real closed field for every Scott set S.

Theorem 2.1 *For any Scott set S, there is an S-saturated real closed field.*

The value group of an S-saturated real closed field will be an S-saturated divisible ordered abelian group. Thus, Corollary 1.9 will also follow from Theorem 2.1.

Theorem 2.1 is a simple induction using the following Lemma.

Lemma 2.2 *Let S be a Scott set. Let K be a real closed field such that every type realized in K is in S. Suppose $p(v, \overline{w})$ is a set of formulas in S, $\overline{a} \in K$, and $p(v, \overline{a})$ is finitely satisfiable in K. Then we can realize $p(v, \overline{a})$ by a (possibly new) element b such that every type realized in $K(b)^{\mathrm{rcl}}$ is in S.*

Proof The set of formulas $p(v, \overline{a}) \cup \mathrm{tp}(\overline{a})$ is a consistent partial type in S, and, hence, has a completion in S. Thus, without loss of generality, we may assume $p(v, \overline{a})$ is a complete type. If $p(v, \overline{a})$ is realized in K, then there is nothing to do. If $p(v, \overline{a})$ is not realized in K then it determines a cut in the ordering of $\mathbb{Q}(\overline{a})^{\mathrm{rcl}}$ that is not realized in K, and, hence, by o-minimality, it determines a unique type over K. Let b realize $p(v, \overline{a})$ and let $\overline{c} \in K$. We need to show that $\mathrm{tp}(b, \overline{c})$ is in S.

How do we decide whether $K(b)^{\mathrm{rcl}} \models \phi(b, \overline{c})$? By o-minimality, $\phi(v, \overline{c})$ defines a finite union of points and intervals with endpoints in $\mathbb{Q}(\overline{c})^{\mathrm{rcl}}$. Since $b \notin K$, b is neither one of the distinguished points nor an end point of of one of the intervals. There are $\emptyset$-definable Skolem functions f and g such that:

i) $f(\overline{a}) < v < g(\overline{a}) \in \mathrm{tp}(b/\overline{a})$;

ii) $f(\overline{a}) < v < g(\overline{a}) \rightarrow \phi(v, \overline{c})$ or $f(\overline{a}) < v < g(\overline{a}) \rightarrow \neg\phi(v, \overline{c})$;

Given $\mathrm{tp}(b/\overline{a})$ and $\mathrm{tp}(\overline{a}, \overline{c})$ we can computably search and find the decomposition of $\phi(v, \overline{c})$ and f and g as above. We can then decide whether $\phi(b, \overline{c})$ holds. Since S is closed under join and Turing reducibility, $\mathrm{tp}(b, \overline{c})$ is in S. $\square$

The above argument works for any o-minimal theory $T \in S$.

Every real closed field K has a natural valuation for which the valuation ring is

$$\mathcal{O} = \{x : |x| < n \text{ for some } n \in \mathbb{N}\}.$$

If K is recursively saturated, then the value group is a recursively saturated divisible ordered abelian group. It is natural to ask if every recursively saturated divisible ordered abelian group arises this way.

D'Aquino, Kuhlmann and Lange [3] gave a valuation-theoretic characterization of recursively saturated real closed fields. In the following argument we assume familiarity with their results.[1]

Proposition 2.3 *Let G be a recursively saturated divisible ordered abelian group. There is a recursively saturated real closed field with value group G.*

Proof Sketch Let S be the Scott set of G. Start with the field

$$k_S(t^g : g \in G)^{\mathrm{rcl}}.$$

This is a real closed field with residue field k_S, value group G and all types recursive in S. Given K a real closed field K with value group G and all types recursive in S and suppose we have $\overline{a} \in K$ and $(f_0, f_1, \ldots)$ a sequence of Skolem functions recursive in S such that $(f_0(\overline{a}), f_1(\overline{a}), \ldots)$ is pseudo-Cauchy. If the sequence has no pseudo-limit in K it determines a unique type over K. Adding a realization b does not change the value group or residue field. As above, every type realized in $K(b)^{\mathrm{rcl}}$ is in S. We can iterate this construction to build the desired real closed field. $\square$

3 Presburger Arithmetic

In [6] Knight and Nadel proved that for every Scott set S there is an S-saturated $\mathbb{Z}$-group, i.e., an S-saturated model of $\mathrm{Th}(\mathbb{Z}, +)$. They asked whether the same is true for the theory of $(\mathbb{Z}, +, <)$. This is Presburger arithmetic, which we denote Pr. We answer this question in the affirmative.

Theorem 3.1 *For every Scott set S, there is an S-saturated model of Presburger arithmetic.*

[1] This is essentially our original proof of Theorem 2.1.

We will consider Presburger arithmetic in the language that includes constants for 0 and 1 and unary predicates $P_n(v)$ for $n = 2, 3, \ldots$ that hold if n divides v. We can eliminate quantifiers in this language and the resulting structure is quasi-o-minimal; i.e., any formula $\phi(v, \overline{a})$ defines a finite Boolean combination of $\emptyset$-definable sets and intervals with endpoints in $\mathrm{dcl}(\overline{a}) \cup \{\pm\infty\}$. We will use this in the following form. (See, for example [8] §3.1 for quantifier elimination and [1] for quasi-o-minimality.)

Lemma 3.2 *i) Any formula $\phi(v, \overline{a})$ is equivalent over $\mathrm{tp}(\overline{a})$ to a Boolean combination of formulas of the form $v \equiv m \bmod n$, $v = \alpha$, $v < \beta$ where α, β are in the definable closure of $\overline{a}$ and $m, n \in \mathcal{N}$.*
ii) $\mathrm{tp}(b, \overline{a})$ is determined by:

- $\mathrm{tp}(\overline{a})$*;*
- *the sequence $b \bmod 2, b \bmod 3, b \bmod 4, \ldots$;*
- *the cut of b in the definable closure of $\overline{a}$.*

Another useful fact about Presburger arithmetic is the existence of definable Skolem functions. Indeed, the definable closure of a set A is contained in the $\mathbb{Q}$-linear span of $A \cup \{1\}$.

We obtain Theorem 3.1 by an iterated construction using the following lemma.

Lemma 3.3 *Let S be a Scott set. Let $G \models \mathrm{Pr}$ such that every type realized in G is computable in S. Suppose $\overline{a} \in G$ and $p(v, \overline{w})$ is a complete type in S such that $p(v, \overline{a})$ is finitely satisfiable. Then there is $H \supseteq G$ such that $H \models \mathrm{Pr}$, such that $p(v, \overline{a})$ is realized in H and every type realized in H is in S.*

Proof If $p(v, \overline{a})$ is realized in G, then there is nothing to do, so we assume $p(v, \overline{a})$ is not realized in G. Let $p^-(v, \overline{a})$ be the partial type describing the cut of v over the definable closure of $\overline{a}$, i.e., p^- consists of all formulas of the form $mv < \sum n_i a_i$ or $mv > \sum n_i a_i$ that are in p where $m, n_i \in \mathbb{Z}$.

Case 1: Suppose p^- is omitted in G.

Let b be any realization of p, and let H be the definable closure of $G \cup \{b\}$. It is enough to show that if $\overline{c} \in G$, then $\mathrm{tp}(b, \overline{c})$ is in S. We will show that $\mathrm{tp}(b, \overline{c})$ is recursive in $\mathrm{tp}(b, \overline{a})$ and $\mathrm{tp}(\overline{a}, \overline{c})$. Using only $\mathrm{tp}(b, \overline{a})$, we can determine $b \bmod n$ for all n. Thus, we only need to consider formulas of the form $\alpha < v < \beta$ where $\alpha, \beta \in \mathrm{dcl}(\overline{c})$. Since $p^-(v, \overline{a})$ is omitted, we can, as in the case of real closed fields, search to find $\gamma, \delta \in \mathrm{dcl}(\overline{a})$ such that $\gamma < b < \delta$ and either $\alpha \le \gamma < \delta \le \beta$, $\delta < \alpha$ or $\gamma > \beta$. This can be done recursively in $\mathrm{tp}(b, \overline{a})$ and $\mathrm{tp}(\overline{a}, \overline{c})$. Thus, every type realized in H is in S.

Case 2: Suppose $b \in G$ realizes p^-.

Let $\widehat{b}$ be a realization of $p(v, \overline{a})$ and let $q_0(v)$ be the divisibility type of $\widehat{b} - b$, i.e., if $\widehat{b} \equiv m \bmod n$ and $b \equiv l \bmod n$ then "$v \equiv m - l \bmod n$" $\in q_0$ for $l, m \in \mathbb{Z}$ and $n > 1$.

Let $q(v) \in S_1(G)$ be the unique type containing

- $q_0(v)$;
- $n < v$ for all $n \in \mathbb{Z}$;
- $v < g$ for all $g \in G$ such that $\mathbb{Z} < g$.

Let ϵ realize q and let H be the definable closure of $G \cup \{\epsilon\}$. Suppose $\alpha \in \mathrm{dcl}(\overline{a})$ and $b < \alpha$. Since $G \models \mathrm{Pr}$, we have $b + n < \alpha$ for all $n \in \mathbb{Z}$. Thus, $\epsilon < \alpha - b$ and $b + \epsilon < \alpha$. Similarly, if $\alpha < b$, then $\alpha < b + \epsilon$ and, thus, $b + \epsilon$ realizes $p^-(v, \overline{a})$. By the choice of q_0, $b + \epsilon$ realizes $p(v, \overline{a})$.

Suppose $\overline{c} \in G$. It suffices to show that $\mathrm{tp}(\epsilon, \overline{c})$ is in S. Without loss of generality, we may assume that $\overline{c} = (c_1, \ldots, c_n)$ where all of the c_i are positive infinite and $1, c_1, \ldots, c_n$ are linearly independent over $\mathbb{Q}$. We need to decide the signs of expressions of the form

$$r + s\epsilon + \sum_{i=1}^{n} t_i c_i$$

where $r, s, t_i \in \mathbb{Z}$. Such an expression is positive if and only if

- $\sum t_i c_i > 0$, or
- $\sum t_i c_i = 0$ and $s > 0$, or
- $\sum t_i c_i = s = 0$ and $r > 0$

This can be computed using $\mathrm{tp}(\overline{c})$. Thus $\mathrm{tp}(\epsilon, \overline{c})$ is recursive in q_0 and $\mathrm{tp}(\overline{c})$. Hence, every type realized in H is in S. $\square$

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