Extremal Problems on Directed Hypergraphs and the Erdös-Gyárfás Ramsey Problem Variant for Graphs

BY

ALEXANDER CAMERON
B.A., Hendrix College, Conway, AR, 2006
J.D., Duke University, Durham, NC, 2011
M.S., University of Illinois at Chicago, Chicago, IL, 2017

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Defense Committee:
György Turán, Chair and Advisor
Dhruv Mubayi, Advisor
Shmuel Friedland
Lev Reyzin
Robert Sloan, Computer Science

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CONTRIBUTION OF AUTHORS

Chapters 1- 8 as well as Chapter 10 reproduce, with some minor changes, a paper published by the Electronic Journal of Combinatorics [12] for which I was the sole author. I retain the copyright to the publication. An early version of this paper was originally published on arXiv in two parts [8, 9]. Chapter 9 contains, with some minor changes, a paper published to arXiv [10] for which I was the sole author. Chapter 11 provides definitions and background on the (p, q)-coloring problem. Chapters 12 and 13 reproduce, with some minor changes, joint work with Emily Heath that has been reviewed and accepted for publication in Combinatorics, Probability & Computing [14]. It has yet to appear, but an early version has been published to arXiv [13]. Chapter 14 reproduces, with some minor changes, part of a paper published to arXiv [11] for which I am the sole author. Chapter 15 is a brief conclusion to the chapters about (p, q)-coloring. Appendix A is an algorithm used in Chapters 13 and 14 and is contained in both of the relevant papers.

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SUMMARY

This thesis has two parts. The first part, about Turán-type problems for directed hypergraphs, consists of Chapters 1 through 10. The second part, about a variation of the Ramsey problem, consists of Chapters 11 through 15. All of the material is related under the umbrella of extremal combinatorics.

The first part of this thesis primarily examines the extremal number of edges for directed hypergraphs given certain forbidden subgraphs. These sorts of questions are a big area of research for graphs and hypergraphs. Here the question is applied to the $2 \to 1$ directed hypergraph. Chapter 1 defines all of the relevant concepts. Chapters 2 through 8 give the extremal numbers for every $2 \to 1$ directed hypergraph with exactly two edges. Chapter 9 generalizes the concept of a directed hypergraph to include many different relational structures and extends some classical extremal results to this larger class of models. Chapter 10 concludes the first part of the thesis with a few stray results and open questions.

The second part of this thesis is about the (p, q)-coloring problem. A (p, q)-coloring is an edge-coloring of the complete graph K_n for which any p vertices must span at least q distinct colors. The goal is to find the minimum number of colors necessary for which such a coloring exists. Chapter 11 defines the necessary concepts and provides background. Chapter 12 provides a construction that will be used in subsequent chapters. Chapter 13 details a $(5, 5)$-coloring. Chapter 14 details a $(5, 6)$-coloring. Finally, Chapter 15 briefly explores some additional ideas for continued research.

CHAPTER 1

Introduction to Extremal Problems and Directed Hypergraphs

This chapter contains material from a paper published by the Electronic Journal of Combinatorics. [12]

We will begin this chapter with the basic definitions of uniform graphs and hypergraphs. For any set V and any positive integer r we let $\binom{V}{r}$ denote the set of r-subsets of V in what follows. Additionally, given a positive integer n, we let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

DEFINITION 1.1. *A **graph** is an ordered pair of sets $G = (V, E)$ where V, the set of **vertices**, is finite, and E, the set of **edges**, is some subset of $\binom{V}{2}$.*

Technically, this defines a simple graph in that it has no loops (an edge from a vertex to itself) or multiedges (two or more edges between the same two vertices). This definition generalizes to larger r.

DEFINITION 1.2. *For some integer $r \geq 2$, an r-uniform **hypergraph** is an ordered pair of sets $H = (V, E)$ where V, the set of **vertices**, is finite, and E, the set of **edges** (or **hyperedges**), is some subset of $\binom{V}{r}$. Often, we let $V(H)$ and $E(H)$ denote the vertex and edge sets of H when these sets have not been given explicitly.*

Graphs and hypergraphs are widely-studied combinatorial objects, and many questions have been asked (and sometimes answered) about them. Here, we are concerned primarily with extremal questions.

1. The forbidden subgraph problem

Turán-type extremal problems for uniform graphs and hypergraphs make up a large and well-known area of research in combinatorics. Of these problems, the forbidden subgraph problem is the most basic: "Given a family of forbidden r-uniform hypergraphs $\mathcal{F}$, what is the maximum number of edges an r-uniform hypergraph on n vertices can have without containing any member of $\mathcal{F}$ as a (not necessarily induced) subgraph?" Such problems were named after Paul Turán due to his important early results and conjectures concerning forbidden complete r-graphs [42, 43, 44]. We formalize this idea in the definitions that follow.

DEFINITION 1.3. *Given two r-uniform hypergraphs, H and G, we call a function $\phi : V(H) \to V(G)$ a **homomorphism** if it preserves the edges of H:*

$$v_1 v_2 \cdots v_r \in E(H) \implies \phi(v_1)\phi(v_2)\cdots\phi(v_r) \in E(G).$$

We will write $\phi : H \to G$ to indicate that ϕ is a homomorphism.

DEFINITION 1.4. *Given a family $\mathcal{F}$ of r-uniform hypergraphs, we say that a hypergraph H is* $\mathcal{F}$-free *if no injective homomorphism $\phi : F \to H$ exists for any $F \in \mathcal{F}$. If $\mathcal{F} = \{F\}$ we will simply write that G is F-free.*

DEFINITION 1.5. *Given a family $\mathcal{F}$ of r-uniform hypergraphs, let the* nth extremal number *$ex(n, \mathcal{F})$ denote the maximum number of edges that any $\mathcal{F}$-free hypergraph on n vertices can have.*

The extremal numbers of families of forbidden hypergraphs indicate the threshold number of edges at which any hypergraph, no matter how unstructured, is forced to have some local substructure.

For example, if a graph on n vertices has more than $n^2/4$ edges, then it must contain a 3-clique, three vertices that are all pairwise adjacent, no matter its structure. The complete bipartite graph with nearly equal parts (see Figure 1) demonstrates that we

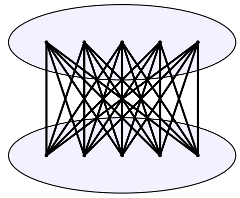


FIGURE 1. A triangle-free graph with many edges.

can have at least this many edges without a 3-clique. The fact that this is the best that we can do is called Mantel's Theorem, and we say that $n^2/4$ is the nth extremal number for the 3-clique K_3, $\mathrm{ex}(n, K_3) = n^2/4$. Turán's Theorem generalized this result for cliques of any size [42, 43, 44]. It states that the maximum number of edges that a graph can have before it is forced to contain a clique of k vertices is the same as the number of edges found in the complete balanced $(k-1)$-partite graph. These kinds of questions are difficult to answer in general for hypergraphs and other combinatorial structures. Even for 3-uniform hypergraphs, the extremal number of a 4-clique is unknown.

Often, it is easier to discuss these notions in terms of edge density rather than number of edges.

DEFINITION 1.6. *Let H be an r-uniform hypergraph with n vertices and $e(H)$ edges. Then the* ***edge density*** *of H is*

$$d_H = \frac{e(H)}{\binom{n}{r}}.$$

DEFINITION 1.7. *Given a forbidden family of r-uniform hypergraphs $\mathcal{F}$, the limit of the maximum edge densities of $\mathcal{F}$-free hypergraphs as the number of vertices goes to infinity is known as the* ***Turán density*** *of the family,*

$$\pi(\mathcal{F}) = \lim_{n\to\infty} \frac{ex(n, \mathcal{F})}{\binom{n}{r}}.$$

A simple averaging argument demonstrates that such a limit always exists. For graphs, it is well-known that the Turán density of any forbidden family is determined by the minimum chromatic number of the graphs in the family. This is the famous Erdős-Stone Theorem [23]. Loosely speaking, the chromatic number of a graph is the minimum number of parts that any partition of the vertices can have such that no edge is contained inside a part.

DEFINITION 1.8. *Let G be a graph and k be a positive integer. A* ***proper vertex coloring*** *of G with k colors is an assignment $f : V(G) \to [k]$ such that $f(x) \neq f(y)$ for any $xy \in E(G)$. The minimum k for which a proper vertex coloring of G exists is known as the chromatic number of G, $\chi(G)$. When $\chi(G) = 2$, we say that G is* ***bipartite****.*

For any forbidden family of graphs $\mathcal{F}$, Erdős and Stone [23] showed that $\pi(\mathcal{F}) = \frac{k-2}{k-1}$ where

$$k = \min_{F \in \mathcal{F}} \{\chi(F)\}.$$

In particular, a forbidden graph F has Turán density zero if and only if it is bipartite. This idea extends to hypergraphs as well.

DEFINITION 1.9. *Let H be an r-uniform hypergraph. We call H* ***degenerate*** *if $\pi(H) = 0$.*

DEFINITION 1.10. *Let H be an r-uniform hypergraph. We say that H is* r***-partite*** *if there exists a partition of the vertices of H into r parts such that every edge of H contains exactly one vertex from each part.*

Erdős [18] showed that an r-uniform hypergraph is degenerate if and only if it is r-partite. Moreover, $\pi(H) \geq \frac{r!}{r^r}$ for any r-uniform hypergraph H that is not r-partite. This can be seen by taking the sequence of r-partite hypergraphs with nearly equal parts and every possible edge. The hypergraphs in this sequence are all H-free and their edge densities tend towards $\frac{r!}{r^r}$ as the number of vertices increase.

However, there is no known generalization of the result relating chromatic number and Turán density to hypergraphs. Even for the complete 3-graph on 4 vertices, $K_4^{(3)}$, the Turán density is unknown.

2. Additional extremal concepts

The label "Turán-type problem" is applied to more than just the forbidden subgraph problem. A closely related result for hypergraphs known as **supersaturation** says any large hypergraph with an edge density slightly more than the Turán density of some forbidden hypergraph F will not only contain a copy of F, but many copies. That is, let F be a an r-uniform hypergraph and let $\epsilon > 0$. For sufficiently large $n \geq n_0(F, \epsilon)$, any r-uniform hypergraph H on n elements with density $d_H \geq \pi(F) + \epsilon$ will contain at least $c\binom{n}{r}$ copies of F for some constant $c = c(F, \epsilon)$. In fact, this supersaturation result is used to prove the characterization of degenerate hypergraphs mentioned above.

Another, closely related, extremal question for hypergraphs known as the "jumping constant conjecture" was proposed by Erdős [22, 23].

DEFINITION 1.11. *A real number $\alpha \in [0, 1)$ is called a **jump** for an integer $r \geq 2$ if there exists some positive constant c which depends only on α such that for any $\epsilon > 0$ and positive integer l there exists a positive integer N for which any r-uniform hypergraph on $n \geq N$ vertices which has edge density at least $\alpha + \epsilon$ contains a subgraph on l vertices with edge density at least $\alpha + c$.*

Informally, a jump is an edge density α for which any very large hypergraph with a slightly larger edge density must contain an arbitrarily large subgraph with edge density at least $\alpha + c(\alpha)$. That is, the density "jumps" by some fixed length $c(\alpha)$ when the overall edge density increases beyond α. The overall edge structure must get "clumpy."

It is well-known that when $r = 2$, every $\alpha \in [0, 1)$ is a jump [22, 23]. Moreover, every $\alpha \in \left[0, \frac{r!}{r^r}\right)$ is a jump for $r \geq 3$ [19]. The jumping constant conjecture asserted that every $\alpha \in [0, 1)$ is a jump for any r.

In 1984, Frankl and Rödl disproved the conjecture when they found the first instance of a nonjump for each $r \geq 3$ [27]. Since then many infinite sequences of nonjumps have been found, but the smallest known nonjump to date is $\frac{5r!}{2r^r}$ for each $r \geq 3$ determined by Frankl, Peng, Rödl, and Talbot [26]. The only additional jumps that have been found are all $\alpha \in [0.2299, 0.2316)$, $\left[0.2871, \frac{8}{27}\right)$ for $r = 3$ found by Baber and Talbot [4], using Razborov's flag algebra method [40].

3. Directed graphs and hypergraphs

Extremal problems like these have also been considered for directed graphs and multigraphs (with bounded multiplicity) [5, 6] and for the more general directed multi-hypergraphs [7].

Brown and Harary [6] determined the extremal numbers for several types of specific directed graphs including all tournaments - that is, a digraph with one edge in some orientation between every pair of vertices. Brown, Erdős, and Simonovits [5] determined the general structure of extremal sequences for every forbidden family of digraphs analogous to the Turán graphs for simple graphs.

The model of directed hypergraphs studied in [7] have r-uniform edges such that the vertices of each edge are given a linear ordering. However, there are many other ways that one could conceivably define a uniform directed hypergraph. The graph theoretic properties of a more general definition of a nonuniform directed hypergraph were studied by Gallo, Longo, Pallottino, and Nguyen [28]. They defined a directed hyperedge as some subset of vertices with a partition into head vertices and tail vertices.

Langlois, Mubayi, Sloan, and Gy. Turán [31, 32] studied extremal properties of certain small configurations in a directed hypergraph model. This model can be thought of as a $2 \to 1$ directed hypergraph where each edge has three vertices, two of which are "tails" and the third is a "head." They determined the extremal number for one such subgraph with two edges, and found the extremal number of a second configuration with two edges up to asymptotic equivalence. We will discuss their results in more detail in the following section. In Chapters 2 through 8, we determine the exact extremal numbers for every $2 \to 1$ directed hypergraph with exactly two edges.

The totally directed hypergraph model considered in [7] and the $r \to 1$ directed hypergraph model resulting from the study of Horn clauses both lead to the natural question of all possible ways to define a directed hypergraph. The definition in this paper of the class of general directed hypergraph models attempts to unify all of the possible "natural" ways one could define a directed hypergraph so that certain extremal questions can be answered about all of them at once. Adding to the motivation of considering more general structures is the recent interest in Razborov's flag algebra method which applies to all relational theories and not just undirected hypergraphs. The fact that the d-simplex model studied by Leader as well as many other somewhat geometric models come out of the class defined in Chapter 9 was a very interesting accident.

4. $2 \to 1$ directed hypergraphs

The combinatorial structure treated in Chapters 2- 8 is the $2 \to 1$ directed hypergraph defined as follows.

DEFINITION 1.12. *A* $2 \to 1$ directed hypergraph *is a pair* $H = (V, E)$ *where* V *is a finite set of* vertices *and the set of* edges E *is some subset of the set of all pointed* 3*-subsets of* V*. That is, each edge is three distinct elements of* V *with one marked as special. This special vertex can be thought of as the* head *vertex of the edge while the other two make up the* tail set *of the edge. If* H *is such that every* 3*-subset of*

V contains at most one edge of E, then we call H oriented*. For a given H we will typically write its vertex and edge sets as $V(H)$ and $E(H)$. We will write an edge as $ab \to c$ when the underlying 3-set is $\{a, b, c\}$ and the head vertex is c.*

For simplicity we will usually refer to $2 \to 1$ directed hypergraphs as *graphs* or sometimes as $(2 \to 1)$-*graphs* when needed to avoid confusion. This structure comes up as a particular instance of the model used to represent definite Horn formulas in the study of propositional logic and knowledge representation [1, 41]. Some combinatorial properties of this model were recently studied by Langlois, Mubayi, Sloan, and Turán [32, 31].

Before we can discuss their results we will need the following definitions which extend the concepts defined earlier in the chapter for graphs and hypergraphs to $2 \to 1$ directed hypergraphs.

DEFINITION 1.13. *Given two graphs H and G, we call a function $\phi : V(H) \to V(G)$ a homomorphism if it preserves the edges of H:*

$$ab \to c \in E(H) \implies \phi(a)\phi(b) \to \phi(c) \in E(G).$$

We will write $\phi : H \to G$ to indicate that ϕ is a homomorphism.

DEFINITION 1.14. *Given a family $\mathcal{F}$ of graphs, we say that a graph G is* $\mathcal{F}$-free *if no injective homomorphism $\phi : F \to G$ exists for any $F \in \mathcal{F}$. If $\mathcal{F} = \{F\}$ we will write that G is F-free.*

DEFINITION 1.15. *Given a family $\mathcal{F}$ of graphs, let the* nth extremal number *$ex(n, \mathcal{F})$ denote the maximum number of edges that any $\mathcal{F}$-free graph on n vertices can have. Similarly, let the* nth oriented extremal number *$ex_o(n, \mathcal{F})$ be the maximum number of edges that any $\mathcal{F}$-free oriented graph on n vertices can have. Sometimes we will call the extremal number the* standard *extremal number or refer to the problem of determining the extremal number as the* standard version *of the problem to* *distinguish these concepts from their oriented counterparts. As before, if* $\mathcal{F} = \{F\}$, *then we will write* $ex(n, F)$ *or* $ex_o(n, F)$ *for simplicity.*

In [32, 31], the authors studied the extremal numbers for two small $(2 \to 1)$-graphs. They refer to these two graphs as the 4-resolvent and the 3-resolvent configurations after their relevance in propositional logic. Here, we will denote these graphs as R_4 and R_3 respectively and define them formally as

$$V(R_4) = \{a, b, c, d, e\} \text{ and } E(R_4) = \{ab \to c, cd \to e\}$$

and

$$V(R_3) = \{a, b, c, d\} \text{ and } E(R_3) = \{ab \to c, bc \to d\}.$$

In [31], the authors determined $\mathrm{ex}(n, R_4)$ exactly for sufficiently large n, and in [32] they determined the sequence $\mathrm{ex}(n, R_3)$ up to asymptotic equivalence. In these papers, the authors discuss a third graph with two edges which they call an Escher configuration because it calls to mind the famous M.C. Escher piece in which two hands draw each other. This graph is on four vertices $\{a, b, c, d\}$ and has edge set $\{ab \to c, cd \to b\}$. In this paper, we will denote this graph by E. These three graphs turn out to be the only three graphs with exactly two edges and more than three vertices for which the extremal numbers are cubic in n. They are also the only three with two edges on more than three vertices that do not satisfy the following definition.

DEFINITION 1.16. *A graph H is* degenerate *if its vertices can be partitioned into three sets, $V(H) = T_1 \cup T_2 \cup K$ such that every edge of $E(H)$ is of the form $t_1t_2 \to k$ for some $t_1 \in T_1$, $t_2 \in T_2$, and $k \in K$.*

An immediate consequence of Theorem 9.4 shown in Chapter 9 is that the extremal numbers for a graph H are cubic in n if and only if H is not degenerate.

In this specific model of directed hypergraphs, there are nine different graphs with exactly two edges. Of these, four are not degenerate. One of these is the graph on three vertices with exactly two edges, $V = \{a, b, c\}$ and $E = \{ab \to c, ac \to b\}$. It is trivial to see that both the standard and oriented extremal numbers for this graph are $\binom{n}{3}$. The other three nondegenerate graphs are R_4, R_3, and E. We will determine both the standard and oriented extremal numbers for each of these graphs in Sections 2, 3, and 4 respectively.

Of the five degenerate graphs with exactly two edges, one has extremal numbers that are trivial to find. This is the graph with two independent edges, $V = \{a, b, c, d, e, f\}$ and $E = \{ab \to c, de \to f\}$. The extremal number for this graph comes directly from the known extremal number of the undirected 3-graph that consists of two independent edges - that is, the maximum number of edges in a 3-graph with edge intersection sizes never equal to zero. That extremal number is $\binom{n-1}{2}$ for sufficiently large n. Therefore, the oriented extremal number for two independent $2 \to 1$ edges is also $\binom{n-1}{2}$ and the standard extremal number is $3\binom{n-1}{2}$.

We will call the other four degenerate graphs with two edges I_0, I_1, H_1, and H_2 and define them as follows:

- $V(I_0) = \{a, b, c, d, x\}$ and $E(I_0) = \{ab \to x, cd \to x\}$
- $V(I_1) = \{a, b, c, d\}$ and $E(I_1) = \{ab \to c, ad \to c\}$
- $V(H_1) = \{a, b, c, d, x\}$ and $E(H_1) = \{ax \to b, cx \to d\}$
- $V(H_2) = \{a, b, c, d\}$ and $E(H_2) = \{ab \to c, ab \to d\}$

Here, the subscripts indicate the number of tail vertices common to both edges. The I graphs also share a head vertex while the H graphs do not. We will determine the oriented and extremal numbers for each of these graphs in Chapters 5- 8.

The proofs that follow rely heavily on the concept of a link graph. For undirected r-graphs, the link graph of a vertex is the $(r-1)$-graph induced on the remaining vertices such that each $(r-1)$-set is an $(r-1)$-edge if and only if that set together with the specified vertex makes an r-edge in the original r-graph [29]. In the directed hypergraph model here, there are a few ways that we could define the link graph of a vertex. We will need the following three definitions.

DEFINITION 1.17. *Let $x \in V(H)$ for some graph H. The* tail link graph *of x T_x is the simple undirected 2-graph on the other $n-1$ vertices of $V(H)$ with edge set defined by all pairs of vertices that exist as tails pointing to x in some edge of H. That is, $V(T_x) = V(H) \setminus \{x\}$ and*

$$E(T_x) = \{yz : yz \to x \in H\}.$$

The size of this set, $|T_x|$ will be called the tail degree *of x. The degree of a particular vertex y in the tail link graph of x will be denoted $d_x(y)$.*

Similarly, let D_x be the directed link graph *of x on the remaining $n-1$ vertices of $V(H)$. That is, let $V(D_x) = V(H) \setminus \{x\}$ and*

$$E(D_x) = \{y \to z : xy \to z \in E(H)\}.$$

Finally, let L_x denote the total link graph *of x on the remaining $n-1$ vertices. That is, $V(L_x) = V(H) \setminus \{x\}$ and*

$$E(L_x) = E(T_x) \cup E(D_x).$$

So L_x is a partially directed 2-graph.

The following notation will also be used when we want to count edges by tail sets.

DEFINITION 1.18. *For any pair of vertices $x, y \in V(H)$ for some graph H let $t(x,y)$ denote the number of edges with tail set $\{x,y\}$. That is*

$$t(x,y) = |\{v : xy \to v \in E(H)\}|.$$

CHAPTER 2

The 4-resolvent Graph R_4

This chapter contains material from a paper published by the Electronic Journal of Combinatorics. [12]

In [32], the authors gave a simple construction for an R_4-free graph. Partition the vertices into sets T and K and take all possible edges with tail sets in T and head vertex in K. When there are n vertices, this construction gives $\binom{t}{2}(n-t)$ edges where $t = |T|$. This is optimized when $t = \left\lceil \frac{2n}{3} \right\rceil$. In [31], the authors showed that this number of edges is maximum for R_4-free graphs for sufficiently large n and that the construction is the unique extremal R_4-free graph.

We now give an alternate shorter proof that $\left\lfloor \frac{n}{3} \right\rfloor \binom{\lceil \frac{2n}{3} \rceil}{2}$ is an upper bound on the extremal number for R_4 for sufficiently large n in both the standard and oriented versions of the problem. The proof also establishes the uniqueness of the construction.

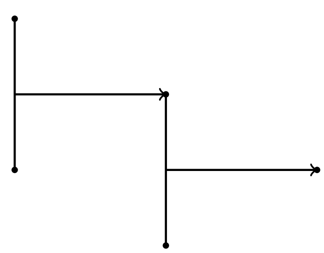


FIGURE 2. The 4-resolvent graph R_4.

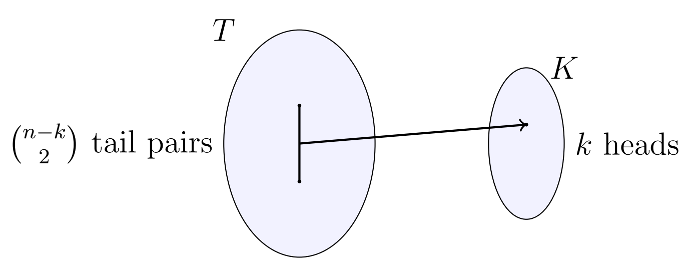


FIGURE 3. The lower bound construction for a graph with no R_4.

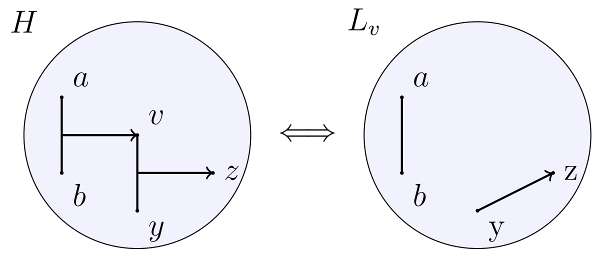


FIGURE 4. H contains a copy of R_4 if and only if the link graph of some vertex v contains a directed edge and an undirected edge that do not intersect.

THEOREM 2.1. *For all $n \geq 29$,*

$$ex_o(n, R_4) = \left\lfloor \frac{n}{3} \right\rfloor \binom{\lceil \frac{2n}{3} \rceil}{2}$$

and for all $n \geq 56$,

$$ex(n, R_4) = \left\lfloor \frac{n}{3} \right\rfloor \binom{\lceil \frac{2n}{3} \rceil}{2}.$$

Moreover, in each case there is one unique extremal construction up to isomorphism when $n \equiv 0, 1$ mod 3 and exactly two when $n \equiv 2$ mod 3.

PROOF. In either the standard or the oriented model, let H be an R_4-free graph on n vertices. Partition $V(H)$ into sets $T \cup K \cup B$ where T is the set of vertices that appear in tail sets of edges but never appear as the head of any edge, K is the set of vertices that do not belong to any tail set, and B is the set of vertices that appear as both heads and tails.

If B is empty, then H is a subgraph of some R_4-free graph with the same structure as the lower bound construction. Therefore, H is either isomorphic to this construction or has strictly fewer edges. So assume that there exists some $v \in B$. The link graph L_v must contain at least one undirected edge and at least one directed edge. If any undirected edge is independent from any directed edge in L_v, then v would be the intersection vertex for an R_4 in H. Therefore, every directed edge in L_v is incident to every undirected edge.

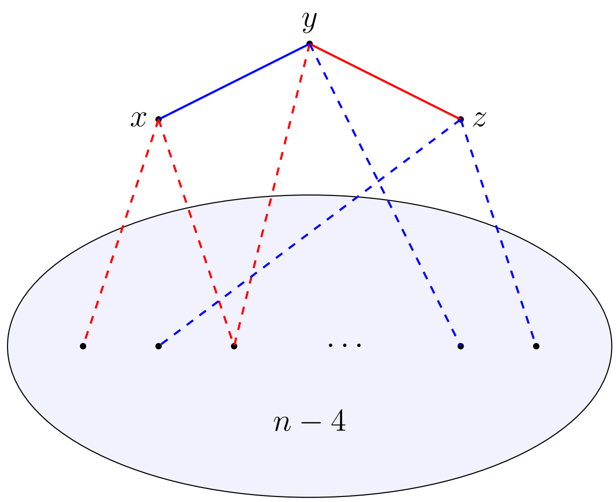


FIGURE 5. A simple graph on $n-1$ vertices with red and blue edges such that each red edge is incident to each blue edge and there is at least one blue edge, xy, and at least one red edge, yz, can have no edge contained in the remaining $n-4$ vertices. Moreover, only red edges can go from x to the remaining vertices and only blue edges can go from z to the remaining vertices.

We want to show that if $v \in B$, then $|E(L_v)| = O(n)$. Determining an upper bound on the number of edges in L_v is equivalent to determining an upper bound on the number of red and blue edges on $n-1$ vertices such that each red edge is incident to each blue edge and there is at least one edge of each color.

If we are working in the oriented model where multiple edges on the same triple are not allowed then no pair of vertices in L_v can hold more than one edge. If we are working in the standard model, then two vertices in this graph may have up to three edges between them, say two red and one blue.

First, we consider the oriented version. In this case we have at least one edge of each color and they must be incident. So let xy be blue and let yz be red. Then all other edges must be incident to x, y, or z. Moreover, any edge from x to the remaining $n-4$ vertices must be red since it is independent from yz and any edge from z to the remaining $n-4$ must be blue. Therefore, there are at most $2(n-4)$ edges from $\{x, y, z\}$ to the remaining $n-4$ vertices.

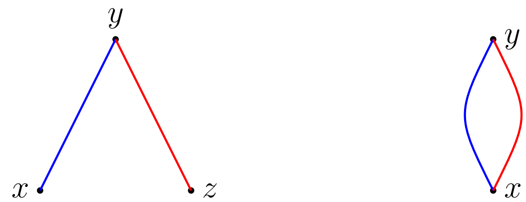


FIGURE 6. When two vertices are allowed to have up to two red edges and one blue edge, then an adjacent red and blue edge pair is either incident in one or two vertices.

In the standard case our initial two red and blue edges may either be incident as before with xy blue and yz red or they might be incident in two vertices so that xy holds both a red and a blue edge. If none of the first type of incidence exists, then there can be at most 3 edges, all on xy.

So assume that the first type of incidence exists - xy is a blue edge and yz is a red edge. As before, all other edges must be incident to these three vertices such that any edge from x to the remaining $n-4$ vertices must be red, and any edge from z to these vertices must be blue. Edges from y may be either color.

However, note that if any vertex of the $n-4$ has a red edge from x, then none of the other vertices can have a blue edge from y or z. Similarly, any vertex with a blue edge from z means that no other vertices can have red edges from x or y. Therefore, if x has more than one red neighbor among the $n-4$ vertices, then there are at most $4(n-4)$ edges between $\{x, y, z\}$ and the $n-4$ remaining vertices (since red edges have multiplicity up to 2). If z has more than one blue neighbor, then there are at most $2(n-4)$ edges between $\{x, y, z\}$ and the $n-4$ remaining vertices. Otherwise, x and z each have at most one neighbor among the $n-4$ vertices, and the best we can do is $3(n-4)$ edges, all from y. Therefore, there are at most $4(n-4)$ additional edges.

In either the standard or oriented versions of the problem, edges that do not contain vertices of B must have their tails in T and their heads in K. So there are at most

$$\left\lfloor \frac{n-b}{3} \right\rfloor \binom{\left\lceil \frac{2(n-b)}{3} \right\rceil}{2}$$

edges that do not intersect B where $b = |B|$. Hence,

$$|E(H)| < \left\lfloor \frac{n-b}{3} \right\rfloor \binom{\left\lceil \frac{2(n-b)}{3} \right\rceil}{2} + cnb$$

where $c = 2$ in the oriented case and $c = 5$ in the standard case.

This expression is maximum on $b \in [0, n]$ only at the endpoint $b = 0$ for all $n \geq 29$ when $c = 2$ and for all $n \geq 56$ when $c = 4$.

Therefore, we can never do better than the lower bound construction. Moreover, since B must be empty to reach this bound, then the construction is unique when $n \equiv 0, 1 \bmod 3$. When $n \equiv 2 \bmod 3$, then

$$\left\lfloor \frac{n}{3} \right\rfloor \binom{\left\lceil \frac{2n}{3} \right\rceil}{2} = \left\lceil \frac{n}{3} \right\rceil \binom{\left\lfloor \frac{2n}{3} \right\rfloor}{2}$$

so there are exactly two non-isomorphic extremal constructions in that case. ■

CHAPTER 3

The 3-resolvent Graph R_3

This chapter contains material from a paper published by the Electronic Journal of Combinatorics. [12]

In [32], the authors gave a simple construction for an R_3-free graph. Partition the vertices into sets A and B and take all possible edges with a tail set in A and head vertex in B plus all possible edges with a tail set in B and head vertex in A. When there are n vertices, this construction gives $(n-a)\binom{a}{2} + a\binom{n-a}{2}$ edges where $a = |A|$. This is optimized when $a = \left\lceil \frac{n}{2} \right\rceil$. The authors showed that this number of edges is asymptotically equivalent to the extremal numbers for R_3.

We show that in both the standard and the oriented versions of this problem that this construction is in fact the best that we can do. We will start with the oriented case since it is less technical.

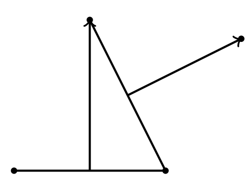


FIGURE 7. The 3-resolvent graph R_3.

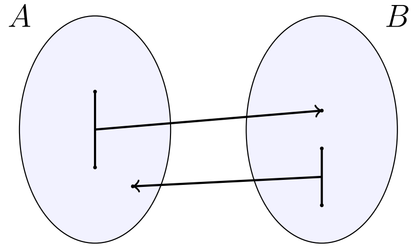


FIGURE 8. The unique R_3-free extremal construction.

FIGURE 9. Forbidden intersection types in L_x for any vertex x in an R_3-free graph.

1. The oriented version

THEOREM 3.1. *For all n,*

$$ex_o(n, R_3) = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \frac{n-2}{2}.$$

Moreover, there is one unique extremal R_3-free construction up to isomorphism for each n.

PROOF. Let H be an R_3-free oriented graph on n vertices. Consider the total link graph, L_x, for some $x \in V(H)$. If

$$yz, z \to t \in E(L_x)$$

or if

$$y \to z, z \to t \in E(L_x),$$

then H is not R_3-free (See Figure 9).

Let $U_x \subseteq V(L_x)$ be the set of vertices that appear as the tail vertex of some directed edge in L_x. Then no edges of L_x can be contained entirely inside U_x - it is an independent set with respect to both directed and undirected edges. Moreover, all undirected edges of L_x must appear entirely within the complement, $C_x := V(L_x) \backslash U_x$. Hence, if we let $u_x = |U_x|$, then

$$2|E(H)| = \sum_{x \in V(H)} |D_x| \leq \sum_{x \in V(H)} u_x(n - 1 - u_x).$$

Each term of this sum is maximized when $u_x \in \left\{ \left\lfloor \frac{n-1}{2} \right\rfloor, \left\lceil \frac{n-1}{2} \right\rceil \right\}$. Therefore, the result is immediate if n is even. The situation is slightly more complicated for odd n.

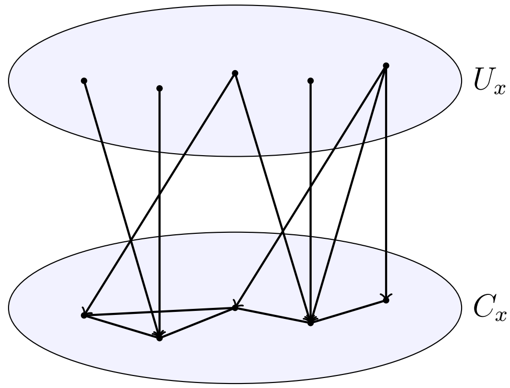


FIGURE 10. The structure of L_x for any x in an R_3-free graph

In this case,

$$u_x(n-1-u_x) \leq \left(\frac{n-1}{2}\right)^2$$

for each x. However, we need $u_x = \frac{n-1}{2}$ in order to attain this maximum value. This would mean that there are $\frac{n-1}{2}$ vertices in C_x, and so there are at most $\binom{\frac{n-1}{2}}{2}$ edges in T_x. Therefore, if $u_x = \frac{n-1}{2}$ for each $x \in V(H)$, then

$$|E(H)| = \sum_{x \in V(H)} |T_x| < \frac{(n-2)(n-1)(n+1)}{8} = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \frac{n-2}{2}.$$

Hence, we must assume that there exist some vertices for which $u_x \neq \frac{n-1}{2}$.

For each x let $i_x \in \{0, \ldots, \frac{n-1}{2}\}$ be the integer such that

$$u_x(n-1-u_x) = \left(\frac{n-1}{2} - i_x\right)\left(\frac{n-1}{2} + i_x\right).$$

Then,

$$|E(H)| \leq \frac{1}{2} \sum_{x \in V(H)} \left(\frac{n-1}{2} - i_x\right)\left(\frac{n-1}{2} + i_x\right) = \frac{n(n-1)^2}{8} - \frac{1}{2} \sum_{j=0}^{\frac{n-1}{2}} k_j j^2$$

where k_j is the number of vertices $x \in V(H)$ for which $i_x = j$.

Since the construction gives $\frac{(n-2)(n-1)(n+1)}{8}$ for odd n, then we are only interested in beating this. So set

$$\frac{(n-2)(n-1)(n+1)}{8} \leq \frac{n(n-1)^2}{8} - \frac{1}{2}\sum_{j=0}^{\frac{n-1}{2}} k_j j^2.$$

This gives

$$\sum_{j=0}^{\frac{n-1}{2}} k_j j^2 \leq \frac{n-1}{2}. \tag{1}$$

Since we can also find $|E(H)|$ by counting the number of undirected edges over the L_x, then we can upper bound the number of these by assuming $u_x = \frac{n-1}{2} - i_x$ for each x since this increases the size of C_x. This gives

$$|E(H)| \leq \sum_{x \in V(H)} \binom{\frac{n-1}{2} + i_x}{2} = \frac{n^3 - 4n^2 + 3n}{8} + \frac{1}{2}\sum_{j=0}^{\frac{n-1}{2}} j(n+j-2)k_j.$$

We can also set this greater than or equal to the known lower bound:

$$\frac{(n-2)(n-1)(n+1)}{8} \leq \frac{n^3 - 4n^2 + 3n}{8} + \frac{1}{2}\sum_{j=0}^{\frac{n-1}{2}} j(n+j-2)k_j$$

to get

$$\frac{(n-1)^2}{2} \leq \sum_{j=0}^{\frac{n-1}{2}} k_j j^2 + (n-2)\sum_{j=0}^{\frac{n-1}{2}} k_j j. \tag{2}$$

Subtracting (1) from (2) gives

$$\frac{(n-1)(n-2)}{2} \leq (n-2)\sum_{j=0}^{\frac{n-1}{2}} k_j j.$$

Therefore,

$$\sum_{j=0}^{\frac{n-1}{2}} k_j j^2 \leq \frac{n-1}{2} \leq \sum_{j=0}^{\frac{n-1}{2}} k_j j,$$

and so

$$0 \leq \sum_{j=0}^{\frac{n-1}{2}} k_j(j - j^2).$$

Since $j - j^2 < 0$ for any $j \geq 2$ and $j - j^2 = 0$ when $j = 0, 1$, then $k_j = 0$ for all $j \geq 2$.

Moreover, once all these are set to zero we get that

$$k_1 \leq \frac{n-1}{2} \leq k_1.$$

Therefore, $k_1 = \frac{n-1}{2}$ and so $k_0 = \frac{n+1}{2}$ since $\sum k_j = n$. This gives the desired upper bound.

Now we can show that the lower bound construction is the unique extremal example up to isomorphism. Let H be an extremal example on n vertices, and define a relation, $\sim$, on the vertices such that $x \sim y$ if and only if either $x = y$ or $y \in U_x$. This defines an equivalence relation on $V(H)$. Reflexivity and symmetry are both immediate. For transitivity note that the proof of the upper bound requires that every possible directed edge be taken from U_x to C_x for each $x \in V(H)$. Therefore, if we assume towards a contradiction that $y \in U_x$ and $z \in U_y$ but $z \notin U_x$, then $z \in C_x$. So $xy \to z \in E(H)$ which means $z \in C_y$, a contradiction.

When n is even there must be exactly two equivalence classes each of size $\frac{n}{2}$. Similarly, when n is odd there must be two equivalence classes of sizes $\frac{n-1}{2}$ and $\frac{n+1}{2}$. Therefore, the lower bound construction must be unique. ∎

2. The standard version

THEOREM 3.2. *For all $n \geq 6$,*

$$ex(n, R_3) = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \frac{n-2}{2}.$$

Moreover, there is one unique extremal R_3-free construction up to isomorphism for each n.

PROOF. Let H be an R_3-free graph on n vertices. Let $x \in V(H)$, and call any pair of vertices in L_x a multiedge if they contain more than one edge. Let $V(L_x) = U_x \cup C_x \cup M_x$ where M_x is the set of vertices that are incident to multiedges (that is, the minimal subset of vertices that contains all multiedges) and U_x and C_x are defined on the rest of the vertices as in Theorem 3.1. The goal is to show that if M_x is nonempty for any vertex x, then H has strictly fewer than the number of edges in the unique oriented construction given in Theorem 3.1. Therefore, that construction must be the unique extremal R_3 example for the standard problem as well.

There are three possibilities for multiedges in M_x: two oppositely directed edges, one directed edge and one undirected edge, and one undirected edge with two oppositely directed edges. If $y, z \in M_x$ have two directed edges between them, then neither y nor z is incident to any other edge in L_x since any incidence would create one of the two forbidden edge incidences of L_x as discussed in the previous theorem.

If y and z have only one directed edge (assume it is $y \to z$) and one undirected edge between them, then y cannot be incident to any more edges for the same reason as before, but z can be incident to undirected edges as well as directed edges with z at the head. This means that z may be the vertex of intersection of a star of these types of multiedges within M_x. Between any two such stars, the vertices of intersection may have an undirected edge between them, but no directed edges.

Therefore, the structure of the internal directed edges of M_x looks like Figure 11 with only the vertices of intersection of the single directed edge stars able to accept more edges from the rest of L_x. Directed edges from the rest of the graph to M_x must originate in U_x. Therefore, if M_x consists of d double directed edge pairs of vertices and k single directed stars with the ith star containing s_i vertices, then the total number of directed edges incident to vertices of M_x is at most

$$2d + \sum_{i=1}^{k} (s_i - 1 + u)$$

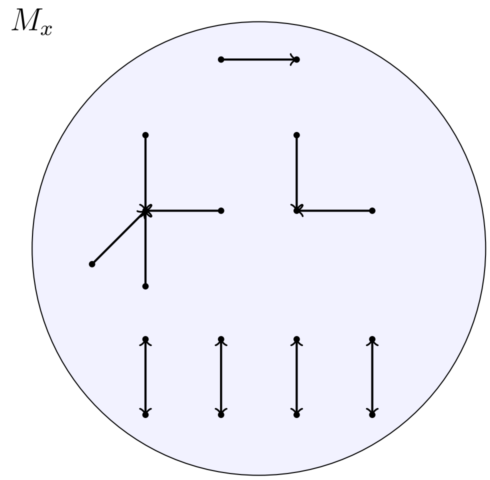


FIGURE 11. Example structure of M_x with 3 single directed edge stars and 4 double directed pairs.

where u is the number of vertices in U_x.

If we assume that M_x is nonempty, then $|M_x| = m \geq 2$. The number of directed edges incident to or inside of M_x is at most $m + k(u-1)$. Therefore, for $u \geq 2$, the number of directed edges incident to vertices of M_x is maximized when the number of single directed edge stars is maximized. This is $\lfloor \frac{m}{2} \rfloor$ stars. Therefore, there are at most

$$\frac{m}{2}(u+1)$$

directed edges incident to vertices of M_x. Thus, if $|C_x| = c$, then L_x can have at most $uc + \frac{m}{2}(u+1)$ directed edges. And since $u \geq 2$, then

$$uc + \frac{m}{2}(u+1) < u(c+m).$$

So L_x has strictly less directed edges than a complete bipartite graph on the same number of vertices would. In Theorem 3.1 every L_x needed to be a complete bipartite graph in terms of the directed edges in order for the maximum number of edges to be obtained, and only in the case of odd n could some of these bipartitions be less than equal or almost equal. In those cases the parts could only have $\frac{n-1}{2} - 1$ and

$\frac{n-1}{2}+1$ vertices. Therefore, the only way that $u(c+m)$ could have more than this is if $u=c+m$ and so $u=\frac{n-1}{2}$.

We assume that $m \geq 2$ and $u \geq 2$, but if both are equal to 2, then $c=u-m=0$ and $n=4$, a contradiction since n is odd. Therefore, one of them must be strictly greater. So

$$uc+\frac{m}{2}(u+1)<(u-1)(u+1)=\left(\frac{n-1}{2}-1\right)\left(\frac{n-1}{2}+1\right).$$

This leaves only the cases where $u=0$ and $u=1$ which are both trivial.

So every link graph of H that contains a multiedge has strictly fewer than $(\frac{n-1}{2})^2-1$ directed edges. This is enough to prove that an extremal R_3-free graph on an even number of vertices must be oriented. However, if there are an odd number of vertices it is possible that there could be enough directed link graphs with the maximum $\left(\frac{n-1}{2}\right)^2$ directed edges to make up the deficit for the directed link graphs with strictly less than $\left(\frac{(n-3)(n+1)}{4}\right)$ due to multiedges.

In this case there would need to be at least $\frac{n+3}{2}$ vertices with directed link graphs that are complete bipartite graphs with parts of size $\frac{n-1}{2}$ each. Let S be the set of these vertices. For any $x,y \in S$ define the relation $x \sim y$ if and only if $y \in U_x$. As in the proof of Theorem 3.1, this turns out to be an equivalence relation. By the definition of S one equivalence class can hold at most $\frac{n+1}{2}$ vertices. So there must be two nonempty classes. Let these classes be A and B.

Given some $x,y \in A$, suppose there is some $z \notin S$ such that $z \in U_x$ and $z \notin U_y$. Then it follows that $z \in C_y$ and therefore there is an edge $xy \to z$ and an edge $xz \to w$ for some $w \in C_x$. Together these make a copy of R_3, a contradiction. Therefore, any z that is in U_x for some $x \in A$ is in U_y for all $y \in A$.

Let C be the set of vertices that are in every U_x for $x \in A$ but not in A itself. Since A is nonempty, there is at least one vertex $x \in A$, and by definition $|U_x|=\frac{n-1}{2}$. Therefore, $|A|+|C|=\frac{n+1}{2}$. Similarly, let D be the set of vertices that are in every U_x

for $x \in B$ but are not in B itself. By the same reasoning we get that $|B|+|D| = \frac{n+1}{2}$. Hence, $|A|+|B|+|C|+|D| = n+1$. However, note that the sets A, B, C, and D are disjoint. So $|A|+|B|+|C|+|D| \leq n$, a contradiction. This is enough to show the result. $\blacksquare$

CHAPTER 4

The Escher Graph E

This chapter contains material from a paper published by the Electronic Journal of Combinatorics. [12]

In this chapter, we will prove the following result on the maximum number of edges of an E-free graph.

THEOREM 4.1. *For all n,*

$$ex(n, E) = \binom{n}{3} + 2$$

and there are exactly two extremal construction up to isomorphism for each $n \geq 4$.

But first we will prove the easier oriented version of the problem. This result will be needed to prove Theorem 4.1.

1. The oriented version

THEOREM 4.2. *For all n,*

$$ex_o(n, E) = \binom{n}{3}$$

and there is exactly one extremal construction up to isomorphism.

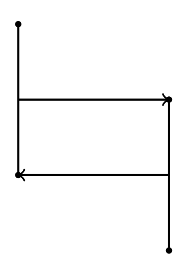


FIGURE 12. The Escher graph E.

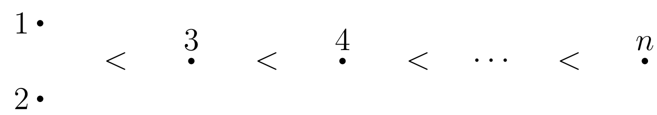


Figure 13. An "almost" linear ordering on the vertices of an E-free directed hypergraph.

Proof. The upper bound here is trivial so we need only come up with an E-free construction that uses $\binom{n}{3}$ edges. Let H be the directed hypergraph defined on vertex set $V(H) = [n]$ and edge set,

$$E(H) = \{ab \to c : a < b < c\}.$$

That is take some linear ordering on the n vertices and for each triple direct the edge to the largest vertex. Then every triple has an edge and H contains no copy of E.

Now we will show that this construction is unique. Let H be an E-free graph on n vertices and $\binom{n}{3}$ edges. Define a relation on the vertices, $\prec$, where $x \prec y$ if and only if there exists an edge in $E(H)$ with x in the tail and y as the head vertex. Then $\prec$ is a partial ordering of the vertices that is almost linear in that every pair of vertices are comparable except for the two smallest elements (see Figure 13). ■

We now shift our attention to the standard version of the problem where a triple of vertices can have more than one edge. Here, both of the lower bound constructions are similar to the unique extremal construction in the oriented version.

2. Two lower bound constructions for ex(n, E)

The first construction is the same as the extremal construction in the oriented case but with two additional edges placed on the "smallest" triple. That is, let $H_1 = ([n], E_1)$ where

$$E_1 = \{ab \to c : a < b < c\} \cup \{13 \to 2, 23 \to 1\}.$$

See Figure 14.

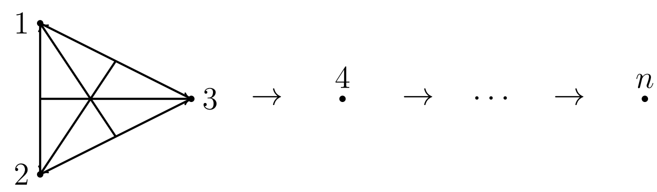


FIGURE 14. The first extremal construction, H_1, for an E-free directed hypergraph on n vertices.

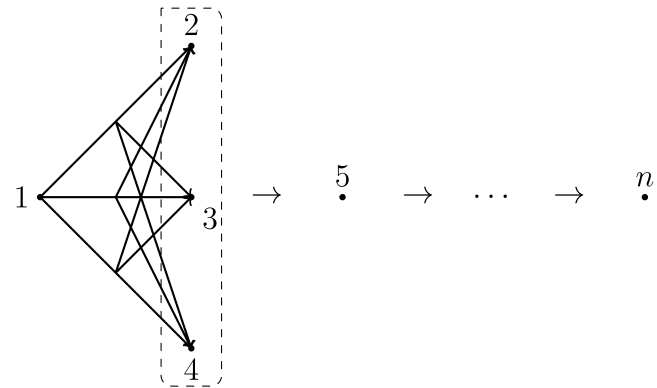


FIGURE 15. The second extremal construction, H_2, for an E-free graph on n vertices.

Moreover, it is important to note that if an E-free graph with $\binom{n}{3} + 2$ edges has at least one edge on every vertex triple, then it must be isomorphic to H_1. This is because we can remove two edges to get an E-free subgraph where each triple has exactly one edge. Therefore, this must be the unique extremal construction established in Theorem 4.2. The only way to add two edges to this construction and avoid creating an Escher graph is to add the additional edges to the smallest triple under the ordering.

The second construction is also based on the oriented extremal construction. Let $H_2 = ([n], E_2)$ where

$$E_2 = (E_1 \setminus \{23 \to 4, 23 \to 1\}) \cup \{14 \to 2, 14 \to 3\}.$$

See Figure 15.

For the rest of this section we will show that any E-free graph is either isomorphic to one of these two constructions or has fewer than $\binom{n}{3} + 2$ edges. Roughly speaking, the strategy is to take any E-free graph and show that we can add and remove edges to it so that we preserve E-freeness, remove most multiple edges from triples that had more than one, and never decrease the overall number of edges.

3. Add and remove edges

Let H be an E-free graph and represent its vertices as the disjoint union of three sets:

$$V(H) = D \cup R \cup T$$

where D (for 'Done') is the set of all vertices that have complete graphs on three or more vertices as tail link graphs, R (for 'Ready to change') is the set of vertices not in D that have at least three edges in their tail link graphs, and T is the set of all other vertices (those with 'Two or fewer edges in their tail link graphs').

The plan is now to remove and add edges in order make a new graph H' which is also E-free, has at least as many edges as H, and whose vertices make a disjoint union,

$$V(H') = D' \cup T'$$

where D' and T' are defined exactly the same as D and T except in terms of the vertices of H'.

That is, for each vertex $x \in R$, we will add all possible edges to complete T_x. This moves x from R to D. The edges removed will be all those that pointed from x to a vertex that points to x. This will destroy triples with more than one edge as we go. The following observation will ensure that this procedure only ever moves vertices from R to D, from R to T, from R to R, and from T to T. Since each step moves one vertex from R to D and ends when R is empty, then the procedure is finite. Here is the observation:

LEMMA 4.1. *Let H be an E-free graph, and let $x, y \in V(H)$. If $d_x(y), d_y(x) > 0$, then $d_x(y) = d_y(x) = 1$. In other words, for any two vertices, x and y, if $d_y(x) \geq 2$, then $d_x(y) = 0$.*

PROOF. Suppose not. Let $d_x(y), d_y(x) > 0$ and suppose $d_x(y) \geq 2$. Then there exist two distinct vertices, a and b such that

$$ay \to x, by \to x \in E(H).$$

There also exists a vertex c such that $xc \to y \in E(H)$. Since c must be distinct from either a or b if not both, then this yields an Escher graph. ■

Now, let us make the procedure slightly more formal: While there exist vertices in R, pick one, $x \in R$, and for each pair $a, b \in V(T_x)$, add the edge $ab \to x$ to $E(H)$ if it is not already an edge. Then, for each $a \in V(T_x)$, remove all edges of $E(H)$ of the form $xs \to a$ for any third vertex s.

Since there were at least three edges in T_x, then the added edges will move x from R to D. The removed edges, if any, will only affect vertices in R or in T since if xs is removed from T_a, then this implies that $a \in T_x$ and that $x \in T_a$ and so both had degree one in the other's tail link graph. Hence, $a \notin D$. Moreover, an affected vertex in R will either stay in R or move to T while an affected vertex in T will stay in T since it is only losing edges from its tail link graph.

At the end of this process D' will contain no triple of vertices with more than one edge. Therefore, the only such triples of vertices of H' will be entirely in T' or will consist of vertices from both T' and D'. We will show later that there cannot be too many of these triples. First, we need to show that after each step of this procedure, no Escher graph is created and at least as many edges are added to the graph as removed.

4. No copy of E is created and the number of edges can only increase

Fix a particular vertex $x \in R$ to move to D. Add and remove all of the designated edges. Suppose that we have created an Escher graph. Since the only added edges point to x, then the configuration must be of the form, $ab \to x, xc \to a$ for some distinct vertices, a, b, and c. Therefore, $a \in V(T_x)$ and so $xc \to a$ would have been removed in the process.

Now we will show that at least as many edges have been added to H as removed by induction on the number of independent edges in T_x. Start by assuming there are 0 independent edges in T_x and assume that there are k vertices in T_x that have degree one. Then at most k edges will be removed. If $k = 0$, then no edges are removed and there is a strict increase in the number of edges.

If $k = 1$, then let y_1 be the vertex with degree one and let y_2 be the vertex it is incident to. Since $d_x(y_2) \neq 1$ and $d_x(y_2) \geq 1$, then $d_x(y_2) \geq 2$. So there exists a third vertex, y_3, and similarly, $d_x(y_3) \geq 2$ but y_3 is not adjacent to y_1. Hence, there exists a fourth vertex, y_4. So at most one edge is removed and at least two edges are added, $y_1y_3 \to x$ and $y_1y_4 \to x$. Therefore, there is a strict increase in the number of edges.

If $k = 2$, then the fact that T_x has at least three edges means that there must be at least two additional vertices in T_x. Hence, at most two edges are removed but at least three are added. If $k \geq 3$, then at most k are removed but $\binom{k}{2}$ are added which nets

$$\binom{k}{2} - k = \frac{k(k-3)}{2} \geq 0$$

edges added.

Now, for the induction step, assume that T_x has $m > 0$ independent edges and that the process on a T_x with $m - 1$ independent edges adds just as many edges as it removes. Let yz be an independent edge in T_x and let A be the set of vertices of T_x that are not y or z. Since T_x has at least three edges, then A contains at least three vertices. Therefore, the number of added edges is at least 6 between A and $\{y, z\}$.

The number of edges removed from T_y and T_z together is at most 2. By assumption, the number of edges removed from the other tail link graphs of vertices in A is offset by the number of edges added inside A. Therefore, there is a strict increase in the number of edges.

To summarize, we have shown that H' is an E-free graph such that

$$|E(H)| \leq |E(H')|$$

and

$$V(H') = D' \cup T'$$

such that any triple of vertices of H' with more than one edge must intersect the set T'. We will now consider what is happening in T' by cases.

5. Case 1: $|T'| \geq 5$

Let $T' = \{x_1, x_2, \ldots, x_t\}$ for $t \geq 5$. For each x_i remove all edges of H' that have x_i as a head. By the definition of T' this will remove at most $2t$ edges from H'.

Next, add all edges to T' that follow the index ordering. That is, for each triple $\{x_i, x_j, x_k\}$ add the edge that points to the largest index, $x_i x_j \to x_k$ where $i < j < k$. This will add $\binom{t}{3}$ edges. The new graph has

$$\binom{t}{3} - 2t \geq 0$$

more edges than H'. Moreover, it is E-free and oriented. Therefore, $|E(H)| < \binom{n}{3}$.

6. Case 2: $|T'| \leq 4$ and some $x \in T'$ gives an T_x with 2 independent edges

Assume that some $x \in T'$ has a tail link graph T_x such that $ab, cd \in E(T_x)$ for four distinct vertices, $\{a, b, c, d\}$. If

$$d_a(x) = d_b(x) = d_c(x) = d_d(x) = 1,$$

then $a, b, c, d, x \in T'$, a contradiction of the assumption that $|T'| \leq 4$.

Therefore, we can add the edges

$$ac \to x, ad \to x, bc \to x, bd \to x$$

and remove any edges that point to a vertex from $\{a, b, c, d\}$ with x in the tail set. Because x has zero degree in at least one of those tail link graphs, then we have removed at most three edges and added four, a strict increase. We have also not created any triples of vertices with more than one edge or any Escher graphs.

We may now assume that $|T'| \leq 4$ and that the tail link graphs of vertices in T' are never two independent edges.

7. Case 3: $|T'| = 0, 1, 2$

First, note that if H' has a triple with more than one edge $\{x, y, z\}$ then at least two of its vertices must be in T' as a consequence of Lemma 4.1. Therefore, if $|T'| = 0, 1$, then H' is oriented and so

$$|E(H)| \leq |E(H')| \leq \binom{n}{3}.$$

Moreover, if $T' = \{x, y\}$ and H' is not oriented, then any vertex triple with more than one edge must have two edges of the form,

$$zx \to y, zy \to x$$

for some third vertex z. If there exist two such vertices $z_1 \neq z_2$ that satisfy this, then there would be an Escher graph. Hence, there is at most one vertex triple with more than one edge and it would have at most two edges. Therefore,

$$|E(H)| \leq |E(H')| \leq \binom{n}{3} + 1.$$

8. Case 4: $|T'| = 3$

First, suppose that there exists a triple $\{x, y, z\}$ with all three possible edges. Then $T' = \{x, y, z\}$. Since any triple with multiple edges must intersect T' in at least two vertices, then any additional such triple would make an Escher graph with one of the edges in T'. Therefore, H' has exactly one triple of vertices with all three edges on it and no others. So

$$|E(H) \leq |E(H')| \leq \binom{n}{3} + 2.$$

Moreover, to attain this number of edges, no triple of vertices can be empty of edges. In this case, H' must be isomorphic to the first construction H_1.

Next, assume that no triple of vertices has all three edges and let $T' = \{x, y, z\}$. Therefore, H' needs at least two triples of vertices that each hold two edges or else

$$|E(H)| \leq |E(H')| \leq \binom{n}{3} + 1$$

automatically. Suppose one of the multiedges is $\{x, y, z\}$ itself. Then without loss of generality let the edges be $xy \to z$ and $xz \to y$. The second triple with two edges must have its third vertex in D'. Call this vertex v. The vertex x cannot be in this second triple of vertices without creating an Escher graph. So the edges must be $vy \to z$ and $vz \to y$. But this also creates an Escher graph.

Therefore, neither of the two triples that hold two edges are contained entirely within T'. So without loss of generality they must be $vx \to y, vy \to x$ and $wy \to z, wz \to y$. If $v \neq w$, then $vx, wz \in T_y$, a contradiction to our assumption that T' contains no vertices with tail link graphs that are two independent edges. Hence, $v = w$.

Since $v \in D$, then T_v has at least three vertices. Moreover, since v is in the tail link graphs of each vertex of T', then none of these vertices can be in T_v. Remove all edges pointing to the vertices of T'. This is at most 6 edges. Add all possible edges with v as the head and a tail set among the set $V(T_v) \cup \{x, y, z\}$. This adds at least 12 new edges. The new graph is oriented and E-free. Therefore, $|E(H)| < \binom{n}{3}$.

9. Case 5: $|T'| = 4$

First, assume that there is some triple $\{x, y, z\}$ that contains all three possible edges. As before, there are no additional triples with more than one edge. So

$$|E(H)| \le |E(H')| \le \binom{n}{3} + 2.$$

The first construction H_1 is the unique extremal construction under this condition since all triples must be used at least once.

So assume that all triples with more than one edge have two edges each. Then we must have at least two. Assume that one of them is contained within $T' = \{a, b, c, d\}$. Without loss of generality let it be $ab \to c, ac \to b$. Since the second such triple intersects T' in at least two vertices, then it must intersect $\{a, b, c\}$ in at least one vertex.

If it intersects $\{a, b, c\}$ in two vertices, then without loss of generality (to avoid a copy of E) the second triple must be of the form $ab \to x, ax \to b$. Hence, $x \in T'$ so $x = d$.

But now there is no edge possible on $\{b, c, d\}$. Therefore, there must be a third such triple for H' to have $\binom{n}{3} + 2$ edges. This triple must be $ac \to d, ad \to c$. And the only way to actually make it to the maximum number of edges now must be to have an edge on every other triple.

Every triple of the form $\{b, c, s\}$ for $s \in D$ must have the edge $bc \to s$ since the other two options would create an Escher graph. Similarly, $bd \to s$ and $cd \to s$ are the only options for triples of the form $\{b, d, s\}$ and $\{c, d, s\}$ respectively. Next, any triple of the form $\{a, b, s\}$ must hold the edge $ab \to s$ since the other two edges create Escher graphs. Similarly, every triple of the forms $\{a, c, s\}$ and $\{a, d, s\}$ must hold the edges $ac \to s$ and $ad \to s$ respectively.

Since each triple contained in D holds exactly one edge, then the induced subgraph on D must be isomorphic to the oriented extremal example of an E-free graph on $n - 4$

vertices. Therefore, the entire graph H' must be isomorphic to the second extremal construction H_2 in order to attain $\binom{n}{3} + 2$ edges.

So assume that the second triple with two edges intersects $\{a, b, c\}$ in only one vertex. Then these edges must be $xa \to d, xd \to a$. This can be the only additional triple with two edges. So to make it to $\binom{n}{3} + 2$ edges we need each triple to have an edge. However, the edge for $\{a, b, d\}$ is forced to be $ad \to b$ and the edge for $\{b, c, d\}$ is forced to be $bc \to d$. This makes an Escher graph. So

$$|E(H)| \leq |E(H')| \leq \binom{n}{3} + 1.$$

Now assume that no vertex triple with multiple edges is contained entirely within T', but assume that there are at least two such triples in H'. The only way that two triples could have distinct vertices in D' is if they were of the forms (without loss of generality), $xa \to b, xb \to a$, and $yc \to d, yd \to c$. Otherwise, the pairs of the two triples that are in T' would intersect resulting in either a copy of E (if both triples use the same pair) or a vertex in T' with two independent edges as a tail link graph.

So there must be exactly two such triples. Therefore, all other triples of vertices must contain exactly one edge in order to reach $\binom{n}{3} + 2$ edges overall. To avoid the forbidden subgraph this edge must be $ab \to c$ for the triple $\{a, b, c\}$ and $cd \to a$ for the triple $\{a, c, d\}$. But this is an Escher graph. Hence, not all triples may be used and so

$$|E(H)| \leq |E(H')| \leq \binom{n}{3} + 1.$$

Therefore, we may now assume for each multiedge triple that the vertex from D' is always x. First, assume that there are only two such triples. As before, if we assume that the only two such triples are $xa \to b, xb \to a$ and $xc \to d, xd \to c$, then there can be not be an edge on both $\{a, b, c\}$ and $\{a, c, d\}$. Hence, there would be a suboptimal number of edges overall.

On the other hand, if the only two such triples are adjacent in T', then they are, without loss of generality, $xa \to b, xb \to a$ and $xb \to c, xc \to a$. In this case, no edge can go on the triple $\{a, b, c\}$ at all and so there are at most $\binom{n}{3} + 1$ edges overall.

Therefore, we must assume there are at least three such triples that meet at x. If these three triples make a triangle in T', then they are $xa \to b, xb \to a$, $xb \to c, xc \to b$, and $xc \to a, xa \to c$. Again, there can be no edges on the triple $\{a, b, c\}$. Hence, every other triple must hold an edge to attain $\binom{n}{3} + 2$ edges overall.

On the triple $\{a, b, d\}$ this edge must be $ab \to d$ to avoid making a copy of E. Similarly, we must have the edges $ac \to d$ and $bc \to d$. But this means that $d \notin T'$, a contradiction.

On the other hand, if there are three triples of vertices with more than one edge on each that do not make a triangle in T' or if there are four or more such triples, then x is in the tail link graphs for each vertex in T'. Hence, none of these vertices may be in the tail link graph, T_x. However, $x \in D'$ so its tail link graph has at least three vertices. Remove all edges pointing to vertices of T' (at most 8). Add all edges pointing to x with tail sets in T' (6 new edges) and between T' and $V(T_x)$ (at least 12 new edges). So this adds at least ten edges to H' to create H''. H'' is oriented so

$$|E(H)| < |E(H'')| \leq \binom{n}{3}.$$

This exhausts all of the cases and establishes that

$$\mathrm{ex}(n, E) = \binom{n}{3} + 2$$

with exactly two extremal examples up to isomorphism.

CHAPTER 5

The Graph I_0

This chapter contains material from a paper published by the Electronic Journal of Combinatorics. [12]

Let I_0 denote the forbidden graph where two edges intersect in exactly one vertex such that this vertex is the head of both edges. That is $V(I_0) = \{a, b, c, d, x\}$ and $E(I_0) = \{ab \to x, cd \to x\}$ (see Figure 16). In this chapter, we will prove the following result on the oriented extremal numbers of I_0.

THEOREM 5.1. *For all $n \geq 9$,*

$$ex_o(n, I_0) = \begin{cases} n(n-3) + \frac{n}{3} & n \equiv 0 \text{ } mod \text{ } 3 \\ n(n-3) + \frac{n-4}{3} & n \equiv 1 \text{ } mod \text{ } 3 \\ n(n-3) + \frac{n-5}{3} & n \equiv 2 \text{ } mod \text{ } 3 \end{cases}$$

with exactly one extremal example up to isomorphism when $3|n$, exactly 18 non-isomorphic extremal constructions when

$$n \equiv 1 \text{ } mod \text{ } 3,$$

and exactly 32 constructions when

$$n \equiv 2 \text{ } mod \text{ } 3.$$

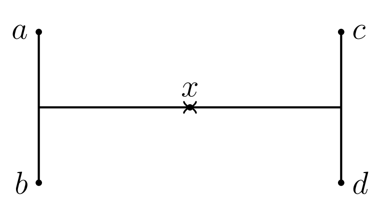


FIGURE 16. The graph I_0.

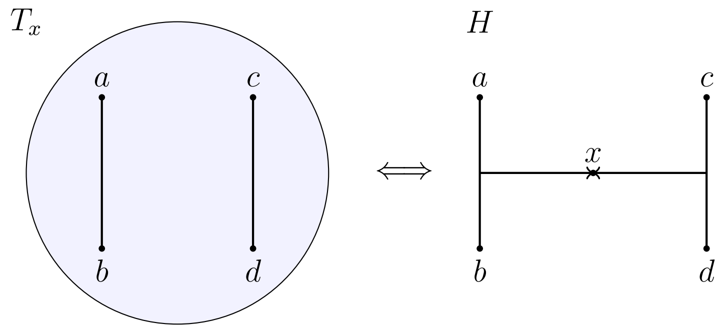


FIGURE 17. $ab, cd \in E(T_x)$ if and only if $ab \to x, cd \to x \in H$.

The proof for this is rather long. However, the standard version of the problem is much simpler so we will begin there.

THEOREM 5.2. *For each $n \geq 5$,*

$$ex(n, I_0) = n(n-2)$$

and for each $n \geq 6$, there are exactly $(n-1)^n$ different labeled I_0-free graphs that attain this maximum number of edges.

PROOF. Let H be I_0-free on $n \geq 5$ vertices. For any $x \in V(H)$, the tail link graph T_x cannot contain two independent edges (see Figure 17). Therefore, by the Erdős-Ko-Rado Theorem [21] the edge structure of T_x is either a triangle or a star with k edges all intersecting in a common vertex for some $0 \leq k \leq n-2$. So each vertex $x \in V(H)$ is at the head of at most $n-2$ edges. Hence,

$$|E(H)| = \sum_{x \in V(H)} |E(T_x)| \leq n(n-2).$$

On the other hand, many different extremal constructions exist that give $n(n-2)$ edges on n vertices without the forbidden intersection. Let

$$f : [n] \to [n]$$

be any function such that $f(x) \neq x$ for any $x \in [n]$. Define H_f as the graph with vertex set $V(H_f) = [n]$ and edge set

$$E(H_f) = \bigcup_{x\in[n]} \{f(x)y \to x : y \in [n] \setminus \{x, f(x)\}\}.$$

Certainly each vertex x is at the head of $n-2$ edges and each of its tail sets contains $f(x)$ which prevents the forbidden subgraph. So $|E(H_f)| = n(n-2)$, and H_f is I_0-free for any such function f.

Moreover, there are $(n-1)^n$ different functions f that will make such a construction on $[n]$. So this gives us $(n-1)^n$ labeled extremal I_0-free graphs. Conversely, since any I_0-free graph with the maximum number of edges must have $n-2$ edges in T_x for each vertex x, then all tail link graphs must be $(n-2)$-stars for all $n \geq 6$. Therefore, these constructions give all possible extremal examples. ■

The oriented version of this problem is less straight forward, but determining $\mathrm{ex}_o(n, I_0)$ also begins with the observation that every tail link graph of an I_0-free graph will either be a triangle, a star, or empty. Broadly speaking, as n gets large, it would make more sense for most, if not all, tail link graphs to be stars in order to fit as many edges into an I_0-free graph. This motivates the following auxiliary structure.

1. Gates

Let H be some I_0-free graph. For each $x \in V(H)$ for which T_x is a star (with at least one edge), let $g(x)$ denote the common vertex for the edges of T_x. We will refer to this vertex as the *gatekeeper* of x (in that it is the gatekeeper that any other vertex must pair with in order to "access" x). In the case where T_x contains only a single edge we may choose either of its vertices to serve as the gatekeeper. In this way, we have constructed a partial function, $g : V(H) \nrightarrow V(H)$.

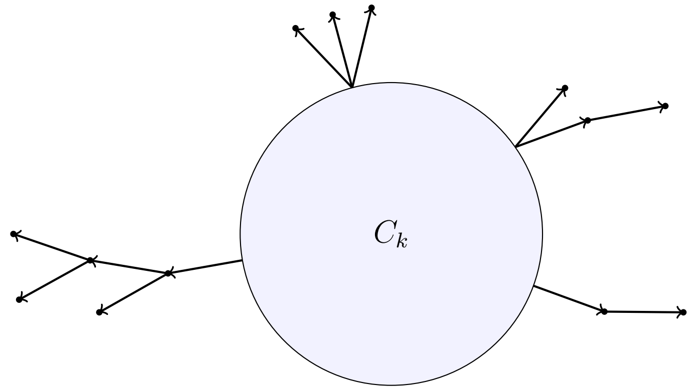


FIGURE 18. The structure of a connected component of the gate G.

Next, construct a directed 2-graph G on the vertex set $V(H)$ based on this partial function:

$$y \to x \in E(G) \iff y = g(x).$$

We will call this digraph the *gate* of H (or more properly, G is the gate of H *under* g since g is not necessarily unique).

The edge structure of any gate G is not difficult to determine. Since g is a partial function, then each vertex has in-degree at most one in G. Therefore, the structure of any connected component of G can be described as a directed cycle on k vertices, C_k, for $1 \leq k$ (where $k = 1$ implies a single vertex) unioned with k disjoint directed trees, each with its root vertex on this cycle (see Figure 18). We will refer to this kind of general structure as a *k-cycle with branches.*

Let

$$\mathcal{C} = \bigcup_{k=1}^{n} \mathcal{C}_k$$

be the set of maximal connected components of a gate of H where, for each k, $\mathcal{C}_k$ is the set of maximal connected components that are k-cycles with branches. Note that

$$|E(H)| = \sum_{x \in V(H)} |T_x| = \sum_{C \in \mathcal{C}} \left(\sum_{x \in V(C)} |T_x| \right) = \sum_{k=1}^{n} \left(\sum_{C \in \mathcal{C}_k} \left(\sum_{x \in V(C)} |T_x| \right) \right).$$

The next section determines for each k an upper bound on

$$\sum_{x\in V(C)} |T_x|$$

as a function of the number of vertices, $|V(C)|$, for any $C \in \mathcal{C}_k$.

2. Bounding $\sum_{x\in V(C)} |T_x|$ for any connected component C of the gate

Loosely speaking, each gatekeeper edge of a connected component C represents at most $n-2$ edges of H. We will arrive at an upper bound on the sum $\sum_{x\in V(C)} |T_x|$ by adding this maximum for each edge of C, and then subtracting the number of triples of vertices that such a count has included more than once. This will happen for any triple of vertices which contain two or three gatekeeper edges. We make this observation formal in the following definition and lemma.

DEFINITION 5.1. *Let G be some gate and let C be a maximal connected component of G. Let $P(C)$ be the set of* $2 \to 1$ possible edges *defined by*

$$P(C) = \bigcup_{a\to b\in E(C)} \{av \to b : v \in V(H) \setminus \{a,b\}\}.$$

LEMMA 5.2. *Let G be a gate, and let C be a maximal connected component of G. If a set of three distinct vertices $\{x,y,z\} \subseteq V(C)$ are spanned by two gatekeeper edges of G, then $P(C)$ contains at least two edges on these three vertices.*

PROOF. Without loss of generality, the two spanning edges on $\{x,y,z\}$ are either of the form

$$x \to y \to z \text{ or } x \leftarrow y \to z.$$

In the former case, $P(C)$ contains the edges $xz \to y$ and $yx \to z$. In the latter case, $P(C)$ contains the edges $yz \to x$ and $yx \to z$. ∎

Now comes the main counting lemma.

LEMMA 5.3. *Let H be an I_0-free graph on $n \geq 8$ vertices. Let G be a gate of H. Let C be a maximal connected component of G with m vertices. Then*

- $\sum_{x\in V(C)}|T_x| \leq m(n-3)$ *if $C \in \mathcal{C}_k$ for any $k \neq 3$ with equality possible only if $C = C_k$ for some $k \geq 4$,*
- $\sum_{x\in V(C)}|T_x| \leq m(n-3)+1$ *if $C = C_3$, and*
- $\sum_{x\in V(C)}|T_x| \leq m(n-3)$ *for all other $C \in \mathcal{C}_3$ with equality possible only if C is a 3-cycle with exactly one nonempty directed path coming off of it.*

PROOF. For convenience let

$$S = \sum_{x\in V(C)} |T_x|.$$

Note that for each $x \in V(C)$ with in-degree one, $ab \in T_x$ implies that $ab \to x \in P(C)$. Hence, if $C \notin \mathcal{C}_1$, then every edge counted in the sum S is in $P(C)$. Moreover, $|P(C)| = m(n-2)$.

If $C \in \mathcal{C}_k$ for $k \geq 4$, then by Lemma 5.2, each intersection of gatekeeper edges of C yields two edges on the same triple of vertices in $P(C)$. Conversely, since C contains no C_3, then each distinct triple of vertices contains at most two gatekeeper edges. Therefore, each triple contains at most two edges of $P(C)$. Hence,

$$S \leq m(n-2) - \sum_{x\in V(C)} \binom{d_G(x)}{2}$$

where $d_G(x)$ denotes the total number of vertices incident to x in the gate.

Since C has m edges, then $\sum_{x\in V(C)} d_G(x) = 2m$. So

$$S \leq m(n-2) - \sum_{x\in V(C)} \binom{d_G(x)}{2} \leq m(n-3)$$

by Jensen's Inequality. Moreover, equality happens if and only if $d_G(x) = d_G(y)$ for all $x, y \in V(C)$. Therefore, this inequality is strict for all $C \in \mathcal{C}_k$ unless $C = C_k$.

Similarly, if $C \in \mathcal{C}_2$, then $P(C)$ contains at least $\sum_{x \in V(C)} \binom{d_G(x)}{2}$ multiedges for the same reason as before. But here there are an additional $n-2$ edges counted for each triple containing the C_2. Also,

$$\sum_{x \in V(C)} d_G(x) = 2(m-1).$$

Hence,

$$S \leq m(n-2) - (n-2) - \sum_{x \in V(C)} \binom{d_G(x)}{2} \leq (m-1)(n-2) - m\binom{\frac{2(m-1)}{m}}{2}$$

by Jensen's Inequality. This is strictly less than $m(n-3)$.

In the acyclic case, Lemma 5.2 implies that the sum of all $|T_x|$ for each $x \in V(C)$ other than the root vertex is less than or equal to

$$(m-1)(n-2) - \sum_{x \in V(C)} \binom{d_G(x)}{2}.$$

The root vertex itself is the head vertex of at most 3 edges in H so Jensen's Inequality gives

$$S \leq (m-1)(n-2) - m\binom{\frac{2(m-1)}{m}}{2} + 3 < m(n-3)$$

for all $n \geq 8$.

Finally, if $C \in \mathcal{C}_3$, then each intersection of gatekeeper edges of C yields two edges on the same triple of vertices in $P(C)$. However, exactly one triple of vertices contains three gatekeeper edges and has three edges in $P(C)$. But the rest have at most two since there is only one triangle in C. Therefore, $\sum_{x \in V(C)} \binom{d_G(x)}{2}$ counts each triple of vertices that contain more than one gatekeeper edge exactly once except for the triple that makes up the C_3 which it counts three times. Since we must subtract off two edges in $P(C)$ on these three vertices to eliminate repeated triples, then we must subtract $\sum_{x\in V(C)} \binom{d_G(x)}{2} - 1$ from $|P(C)|$. Therefore,

$$S \leq m(n-2) - \sum_{x\in V(C)} \binom{d_G(x)}{2} + 1.$$

So by Jensen's Inequality,

$$S \leq m(n-3) + 1$$

with equality possible only if all of the degrees $d_G(x)$ are equal. This can only happen if $C = C_3$.

If we want to see for which $C \in \mathcal{C}_3$ the second best bound of $m(n-3)$ could be attained, then we need to set

$$\sum_{x\in V(C)} \binom{d_G(x)}{2} = m + 1.$$

Assume that the vertices are $x_1, \ldots, x_m$, and for each x_i let

$$d_i = d_G(x_i) - 2.$$

Then $\sum_{i=1}^m d_i = 0$ and a quick calculation shows that $\sum_{i=1}^m d_i^2 = 2$. Therefore, the only possibility is for some $d_i = 1$ and another to equal -1 and all the rest must be 0. This corresponds with one vertex degree equal to 3, another equal to 1, and all others equal to 2. The only way that this can happen in a C_3 with branches is to have exactly one branch, and that branch must be a directed path. ■

This shows that the best we can hope for in terms of the average number of edges per vertex over any connected component of the gate is $n - 3 + \frac{1}{3}$, and this could be attained only in the case where the component is a directed triangle with no branches. Otherwise, the average number of edges of a component is at most $n-3$, and this is attainable only if the component is a directed triangle with a single directed path coming off of one of its vertices or a directed k-cycle with no branches for some $k \geq 4$.

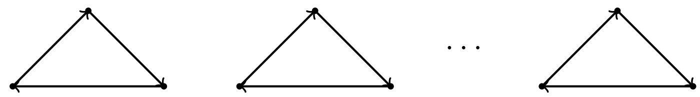


FIGURE 19. Structure of the gate for an extremal I_0-free graph when $n \equiv 0 \bmod 3$.

This is enough for us to establish the upper bound for $\mathrm{ex}_o(n, I_0)$ and to characterize the necessary structure of the gate for any graph attaining this upper bound.

3. Upper bound on $\mathbf{ex}_o(n, I_0)$

Let H be an I_0-free graph on $n \geq 9$ vertices. Let G be a gate of H. Let $\mathcal{C}$ be the set of maximal connected components of G and break $\mathcal{C}$ into three disjoint subsets based on the maximum average number of edges attainable for the components in each. That is, let

$$\mathcal{C} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$$

where $\mathcal{D}_1$ contains all components with maximum average number of edges per vertex strictly less than $n-3$: those components that are either acyclic, contain a C_2, contain a C_3 with nonempty branches that are more than just a single path, or contain a C_k for $k \geq 4$ with some nonempty branch; $\mathcal{D}_2$ is the set of all components with maximum number of edges per vertex of $n-3$: those that contain a directed C_3 and exactly one directed path or those that are a directed k-cycles for any $k \geq 4$ and no branches; and $\mathcal{D}_3$ is the set of components with a maximum average greater than $n-3$: the directed triangles.

For each i let d_i be the total number of vertices contained in the components of $\mathcal{D}_i$. Then

$$|E(H)| \leq d_3\left(n - 3 + \frac{1}{3}\right) + (n - d_3)(n - 3)$$

with equality possible only if $d_1 = 0$. This is enough to prove the following.

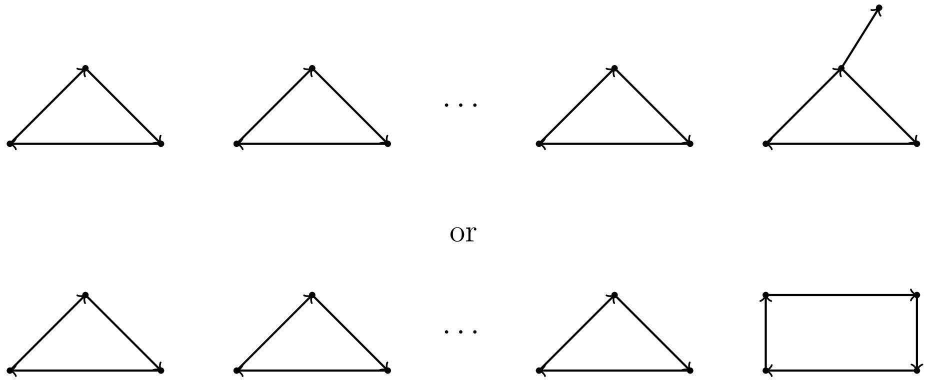


FIGURE 20. The only possible structures of the gate of an extremal I_0-free graph when $n \equiv 1 \bmod 3$.

LEMMA 5.4. *Let H be an I_0-free graph on $n \geq 9$ vertices such that $n \equiv 0 \bmod 3$, then*

$$|E(H)| \leq n(n-3) + \frac{n}{3}.$$

Moreover, the only way for H to attain this maximum number of edges is if the gate of H is a disjoint union of directed triangles.

The next two lemmas give the maximum number when $n \equiv 1, 2 \bmod 3$. There is only slightly more to consider in these cases.

LEMMA 5.5. *Let H be an I_0-free graph on $n \geq 9$ vertices such that $n \equiv 1 \bmod 3$, then*

$$|E(H)| \leq n(n-3) + \frac{n-4}{3}.$$

Moreover, the only way for H to attain this maximum number of edges is if the gate of H is a disjoint union of $\frac{n-4}{3}$ directed triangles together with either a directed C_4 or a 3-cycle with an extra edge.

PROOF. Since $n \equiv 1 \bmod 3$, then $d_3 \leq n-1$. If $d_3 = n-1$, then the gate consists of $\frac{n-1}{3}$ disjoint directed triangles and one isolated vertex which means that

$$|E(H)| \leq (n-1)\left(n - 3 + \frac{1}{3}\right) + 3.$$

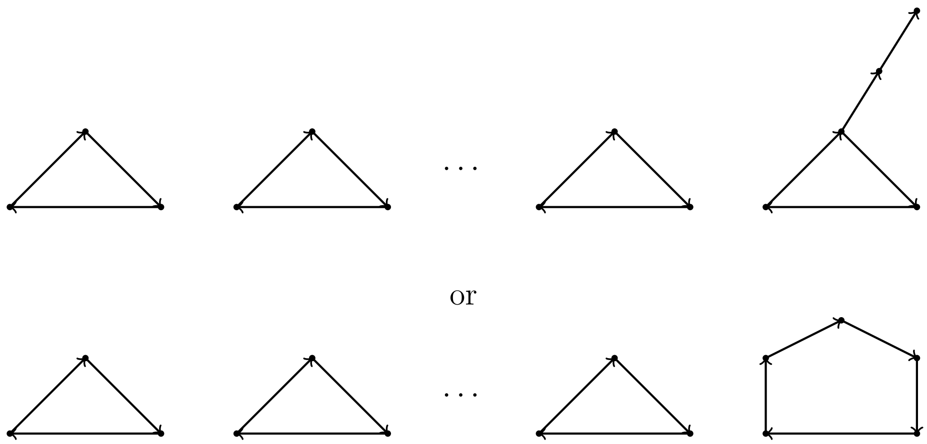


FIGURE 21. The only possible structures of the gate of an extremal I_0-free graph when $n \equiv 2 \bmod 3$.

If $d_3 \leq n-4$, then we can do better with

$$|E(H)| \leq (n-4)\left(n-3+\frac{1}{3}\right)+4(n-3)$$

only in the case of $\frac{n-4}{3}$ disjoint directed triangles and one component from $\mathcal{D}_2$ in the gate. Therefore,

$$|E(H)| \leq n(n-3)+\frac{n-4}{3}.$$

■

LEMMA 5.6. *Let H be an I_0-free graph on $n \geq 11$ vertices such that $n \equiv 2$ mod 3, then*

$$|E(H)| \leq n(n-3)+\frac{n-5}{3}.$$

Moreover, the only way for H to attain this maximum number of edges is if the gate of H is a disjoint union of $\frac{n-5}{3}$ directed triangles together with either a directed C_5 or a 3-cycle with a directed path of two edges.

PROOF. Since $n \equiv 2 \bmod 3$, then $d_3 \leq n-2$ and equality implies that G consists of $\frac{n-2}{3}$ disjoint directed triangles and two additional vertices that are either both isolated, contain one edge, or are a C_2 giving 6, $3+(n-2)$, or $n-2$ additional edges respectively. The best we can do when $d_3 = n-2$ is therefore,

$$|E(H)| \leq (n-2)\left(n-3+\frac{1}{3}\right)+(n+1).$$

Otherwise, $d_3 \leq n-5$ and the best we can do is

$$|E(H)| \leq (n-5)\left(n-3+\frac{1}{3}\right)+5(n-3).$$

This is better. Moreover, this will happen only when the five non-triangle vertices are in a component (or components) of G that give an average of $n-3$. So they must either make a C_5 or a directed triangle with one path. ■

4. Lower bound constructions

The structure of the gates necessary to attain the maximum number of edges for a I_0-free graph determined in the previous section are also sufficient. Of these gates, none of them have acyclic components. Therefore, any graph that produces one of these gates has only vertices with stars for tail link graphs. This immediately implies that there is no I_0 in any graph that has such a gate.

Moreover, if H is a graph with a gate G that is one of these configurations, then

$$E(H) \subseteq \bigcup_{C \in \mathcal{C}} P(C)$$

where $\mathcal{C}$ is the set of maximal connected components of G. All that is left to do in order to construct an extremal example is to pick which edges of each $P(C)$ to delete in order to eliminate triples of vertices with more than one edge.

LEMMA 5.7. *Let H be an I_0-free graph on $n \geq 9$ vertices such that $n \equiv 0$ mod 3, then*

$$|E(H)| \geq n(n-3)+\frac{n}{3}$$

and there is exactly one extremal construction up to isomorphism.

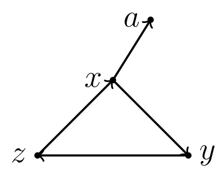


FIGURE 22. C_3 plus an edge.

PROOF. We know from Lemma 5.4 that the only way H can possibly attain

$$n(n-3) + \frac{n}{3}$$

edges is if its gate is the disjoint union of $\frac{n}{3}$ directed triangles. Therefore, each $P(C_3)$ contains exactly one vertex triple with all three possible edges. So two of these must be deleted for each component in order to arrive at an extremal construction. The three choices for this deletion on each component are all isomorphic to each other. Therefore, there is exactly one extremal construction up to isomorphism. ■

LEMMA 5.8. *Let H be an I_0-free graph on $n \geq 9$ vertices such that $n \equiv 1$ mod 3, then*

$$|E(H)| \geq n(n-3) + \frac{n-4}{3}$$

and there are exactly 18 extremal constructions up to isomorphism.

PROOF. We know from Lemma 5.5 that if H has $n(n-3) + \frac{n-4}{3}$ edges, then its gate is the disjoint union of $\frac{n-4}{3}$ directed triangles with either a directed C_4 or a C_3 plus an edge on the remaining 4 vertices. As in the previous proof, there is only one choice up to isomorphism for which edges to delete from each $P(C_3)$. However, this will not be true of the last component on the remaining four vertices.

First, let's consider the case where the last component is a C_3 plus one edge. Call the vertices $\{x, y, z, a\}$ where $x \to y \to z \to x$ is the C_3 and $x \to a$ is the additional edge. First, note that we have the following three mutually exclusive choices for edges with head vertices in this component:

(1) $xa \in T_y$ or $xy \in T_a$,

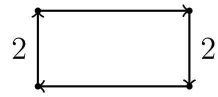


FIGURE 23. C_4 with 2 additional edges in opposite tail link graphs.

(2) $za \in T_x$ or $xz \in T_a$, and

(3) $zx \in T_y$, $yz \in T_x$, or $xy \in T_z$.

This gives 12 choices, and each choice is unique up to isomorphism.

Next consider the case of C_4. Each 3-subset of these four vertices holds two edges of $P(C)$ - one that points along the direction of the two gatekeeper edges and one that points the middle vertex of the two gatekeeper edges. For each triple one of these edges must be deleted to arrive at a legal oriented construction.

Each tail link graph must have at least $n-4$ edges, and combined they must contain four additional edges. Since each can have up to two more edges, then the distribution of these additional edges must be one of the following integer partitions of 4:

- 2, 2, 0, 0
- 2, 1, 1, 0
- 1, 1, 1, 1

There is only one choice up to isomorphism with a distribution of 2, 2, 0, 0. Each of the three ways to place 2, 1, 1, 0 around C_4 are possible but each distribution has only one way up to isomorphism. Finally, there are two ways up to isomorphism to put an extra edge into each tail link graph. So all together there are six nonisomorphic ways to distribute these extra edges to the C_4 tail link graphs. ■

LEMMA 5.9. *Let H be an I_0-free graph on $n \geq 9$ vertices such that $n \equiv 2$ mod 3, then*

$$|E(H)| \geq n(n-3) + \frac{n-5}{3}$$

and there are exactly 32 extremal constructions up to isomorphism.

Proof. We can do the same kind of analysis when $n = 3k + 2$ as in the previous proof. We know from Lemma 5.6 that the gate of any extremal construction must be all directed triangles together with either a directed C_5 or a directed triangle with a directed path of length two coming off of it (see Figure 21).

First, consider the C_5 case. Let the vertices be $\{x_0, \ldots, x_4\}$. For each gatekeeper edge, $x_i \to x_{i+1}$, every edge of the form $x_i v \to x_{i+1}$ must be an edge in H for any vertex

$$v \neq x_i, x_{i+1}, x_{i-1}, x_{i+2}.$$

Each gatekeeper edge can represent up to two additional edges of H, but again, every intersection of gatekeeper edges requires a mutually exclusive choice. Ultimately, we can add 5 additional edges so the extra edges must be distributed in one of the following ways:

- 2, 2, 1, 0, 0
- 2, 1, 1, 1, 0
- 1, 1, 1, 1, 1

There are 2 ways to get the first distribution up to isomorphism, 4 ways to get the second, and 2 ways to get the third. Therefore, there are 8 extremal constructions with this gate up to isomorphism.

Now consider the case of a directed triangle with a directed two path coming off of it. If we label the vertices as $\{x, y, z, a, b\}$ (see Figure 24), the mutually exclusive choices are

(1) $ax \to y$ or $yx \to a$,
(2) $az \to x$ or $zx \to a$,
(3) $zx \to y$, $yz \to x$, or $xy \to z$, and
(4) $xa \to b$ or $bx \to a$

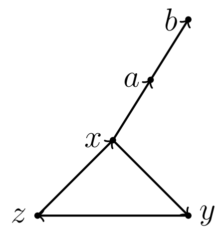


FIGURE 24. C_3 plus a 2-path.

This gives 24 ways of reaching the maximum, and each way is unique up to isomorphism. Therefore, there are 32 total distinct extremal graphs up to isomorphism. ■

This establishes the main result of this chapter.

CHAPTER 6

The Graph I_1

This chapter contains material from a paper published by the Electronic Journal of Combinatorics. [12]

Let I_1 denote the forbidden graph where two edges intersect in exactly two vertices such that one vertex is the head for both edges and the other is in the tail set of each edge. That is $V(I_1) = \{a, b, c, d\}$ and $E(I_1) = \{ab \to c, ad \to c\}$ (see Figure 40).

THEOREM 6.1. *For all* $n \geq 4$,

$$ex(n, I_1) = ex_o(n, I_1) = n \left\lfloor \frac{n-1}{2} \right\rfloor$$

and there are

$$\left(\frac{(n-1)!}{2^{\lfloor \frac{n-1}{2} \rfloor} \lfloor \frac{n-1}{2} \rfloor !} \right)^n$$

labeled graphs that attain this maximum in the standard case.

PROOF. Let H be an I_1-free graph on n vertices. For any $x \in V(H)$, T_x is a simple undirected 2-graph on $n-1$ vertices such that no two edges are adjacent (this is true for either version of the problem). Therefore, the edges of T_x are a matching on at most $n-1$ vertices. So there are at most $\lfloor \frac{n-1}{2} \rfloor$ edges in T_x for every $x \in V(H)$.

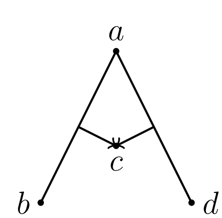


FIGURE 25. The graph I_1.

Thus,

$$|E(H)| = \sum_{x \in V(H)} |T_x| \leq n \left\lfloor \frac{n-1}{2} \right\rfloor.$$

This shows the upper bound for both versions.

Now we want to find lower bound constructions. In the standard version of the problem there are many extremal constructions since for each vertex x, we may pick any maximum matching on the remaining $n-1$ vertices to serve as the edges of T_x. So

$$\mathrm{ex}(n, I_1) = n \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Moreover, the number of labeled graphs that attain this maximum equals the number of ways to take a maximum matching to construct each tail link graph. For even k, the number of matchings on k vertices is

$$M_k = (k-1)M_{k-2}$$

since if we fix some vertex, then we can pick any of the remaining $k-1$ vertices to go with it and then take the number of matchings on the remaining $n-2$. Since $M_2 = 1$, then in general for even k,

$$M_k = \prod_{i=1}^{\frac{k}{2}} (2i-1).$$

If k is odd, then we can first select the vertex left out of the matching to get

$$M_k = kM_{k-1} = k \cdot \prod_{i=1}^{\frac{k-1}{2}} (2i-1) = \prod_{i=1}^{\frac{k+1}{2}} (2i-1).$$

Therefore, the number of labeled extremal I_1-free graphs on n vertices is

$$\left(\prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} (2i-1) \right)^n = \left(\frac{(n-1)!}{2^{\lfloor \frac{n-1}{2} \rfloor} \lfloor \frac{n-1}{2} \rfloor !} \right)^n.$$

In the oriented version of the problem we need to be more careful with the construction. First, assume that n is even and define a graph H with vertex set $V(H) = \mathbb{Z}_n$

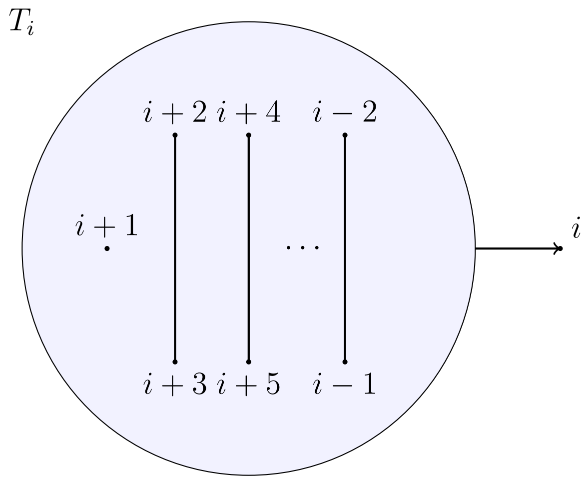


FIGURE 26. T_i in the oriented extremal construction for even n.

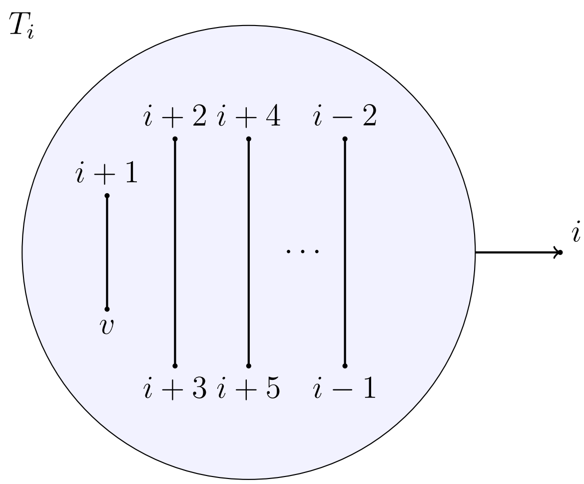


FIGURE 27. T_i in the oriented extremal construction on $n+1$ vertices for even n.

and edge set

$$E(H) = \bigcup_{i=0}^{n-1} \left\{ (i+2k)(i+2k+1) \to i : k = 1, 2, \dots, \frac{n-2}{2} \right\}.$$

This construction creates a maximum matching for each tail link graph (with $i+1$ as the odd vertex out for each T_i). So H has the extremal number of edges and contains no I_1. Therefore, all we need to show is that it has no triple with more than one edge.

If H does contain such a triple, then there exist three integers in $\mathbb{Z}_n$ that can be represented as both $\{a, a+2k, a+2k+1\}$ and $\{b, b+2i, b+2i+1\}$ with $a \neq b$. Without

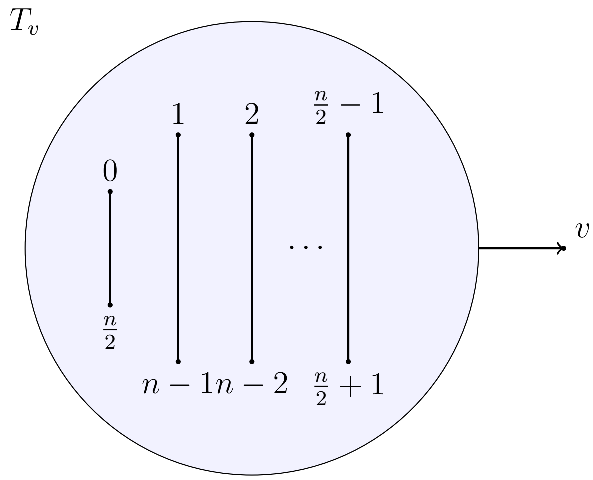


FIGURE 28. T_v in the oriented extremal construction on $n+1$ vertices for even n.

loss of generality we can assume that $b=0$. If $a+2k=0$, then $a+2k+1=1$, but 1 is not in any tail set that points to 0. Therefore, it must be the case that $a+2k+1=0$, but then $a+2k=n-1$. Therefore, the set is equal to $\{0,n-1,n-2\}$, and $a=n-2$, but $n-1$ does not point to $n-2$, a contradiction. Therefore, H can have no such triple.

Now, we consider odd $n+1$. Here, let $V(H)=\mathbb{Z}_n\cup\{v\}$ where v is a new vertex and use all of the edges from the even construction plus some new ones that all contain v. So $E(H)=E_{even}\cup E_{new}\cup E_v$ where

$$E_{even}=\bigcup_{i=0}^{n-1}\left\{(i+2k)(i+2k+1)\to i : k=1,2,\ldots,\frac{n-2}{2}\right\},$$

and

$$E_{new}=\{v(i+1)\to i : i=0,1,\ldots,n-1\}.$$

Certainly, the construction has so far avoided the forbidden subgraph and given each of the first n vertices the maximum number of tails. Now E_v can be constructed as any set of $\frac{n}{2}$ disjoint pairs of elements from $\mathbb{Z}_n$ all pointing at v so that no pair consists of two sequential numbers mod n. So any maximum matching of the n elements that observes this condition will do.

In particular, we can let

$$E_v = \left\{(i)(n-i) \to v : i = 1, \ldots, \frac{n}{2} - 1\right\} \cup \left\{(0)\left(\frac{n}{2}\right) \to v\right\}.$$

So

$$\mathrm{ex}_o(n, I_1) = n \left\lfloor \frac{n-1}{2} \right\rfloor.$$

$\blacksquare$

CHAPTER 7

The Graph H_1

This chapter contains material from a paper published by the Electronic Journal of Combinatorics. [12]

Let H_1 denote the forbidden graph where two edges intersect in exactly one vertex such that it is in the tail set of each edge. That is $V(H_1) = \{a, b, c, d, x\}$ and $E(H_1) = \{ax \to b, cx \to d\}$ (see Figure 29). First we will show the following result for the oriented version of the problem.

THEOREM 7.1. *For all $n \geq 6$,*

$$ex_o(n, H_1) = \left\lfloor \frac{n}{2} \right\rfloor (n-2).$$

We will use this result to solve the standard version of the problem.

THEOREM 7.2. *For all $n \geq 8$,*

$$ex(n, H_1) = \binom{n+1}{2} - 3.$$

Moreover, there is one unique extremal construction up to isomorphism for each n.

First, note that the proof of Theorem 7.1 is straightforward when n is even. To get a lower bound construction we can take a maximum matching on the n vertices and

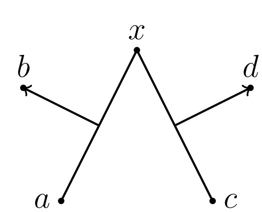


FIGURE 29. The graph H_1.

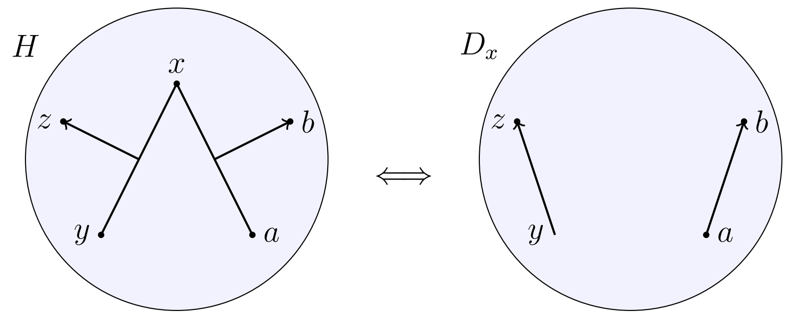


FIGURE 30. H has a copy of H_1 with intersection vertex x if and only if the directed link graph D_x has a pair of disjoint directed edges.

use each pair of this matching as the tail set to point at all $n-2$ other vertices. That is, let H be the graph with vertex set,

$$V(H) = \{x_1, \ldots, x_{\frac{n}{2}}, y_1, \ldots, y_{\frac{n}{2}}\}$$

and edge set,

$$E(H) = \bigcup_{i=1}^{\frac{n}{2}} \{x_i y_i \to z : z \in V(H) \setminus \{x_i, y_i\}\}.$$

To show that this is also an upper bound, let H be an H_1-free oriented graph on n vertices. Then for any $x \in V(H)$, the directed link graph D_x cannot have two independent edges (see Figure 30). Therefore, D_x is either empty, a triangle, or a star with at most $n-2$ edges. Since $n \geq 5$, then $|D_x| \leq n-2$ for each x. So

$$|E(H)| = \frac{1}{2} \sum_{x \in V(H)} |D_x| \leq \frac{1}{2} n(n-2).$$

Hence, we are finished for even n. However, this proof falls apart when n is odd. We will need a different strategy.

1. Counting edges by possible tail pairs

The basis of our strategy in getting an upper bound on $\mathrm{ex}_o(n, H_1)$ is to count the edges of an H_1-free graph H by its tail sets. That is,

$$|E(H)| = \sum_{\{x,y\} \in \binom{V(H)}{2}} t(x, y)$$

It is simple but important to note that if H is H_1-free, then any two pairs of vertices that each points to two or more other vertices must necessarily be disjoint.

LEMMA 7.1. *Let H be a H_1-free oriented graph. If $x_1, x_2, y_1, y_2 \in V(H)$ so that*

$$t(x_1, y_1), t(x_2, y_2) \geq 2$$

and $\{x_1, y_1\} \neq \{x_2, y_2\}$, then $\{x_1, y_1\} \cap \{x_2, y_2\} = \emptyset$.

PROOF. Suppose, towards a contradiction, that $x_1 = x_2 = x$ but $y_1 \neq y_2$. Since $t(x, y_1) \geq 2$, then there exists some vertex z_1 distinct from x, y_1, and y_2 such that

$$xy_1 \to z_1 \in E(H).$$

Similarly, since $t(x, y_2) \geq 2$, then there exists some vertex z_2 distinct from x, y_1, and y_2 such that

$$xy_2 \to z_2 \in E(H).$$

If $z_1 \neq z_2$, then this gives a copy of H_1.

So assume that they are the same vertex, $z_1 = z_2 = z$. Since $t(x, y_1) \geq 2$, then there is some second vertex that x and y_1 point to that is distinct from z. The only choice that would not create a copy of H_1 with the edge $xy_2 \to z$ is y_2. Similarly, since $t(x, y_2) \geq 2$, then there is some second vertex that x and y_2 point to that is distinct from z. The only choice that would not create a copy of H_1 with the edge $xy_1 \to z$ is y_1. So

$$xy_1 \to y_2, xy_2 \to y_1 \in E(H)$$

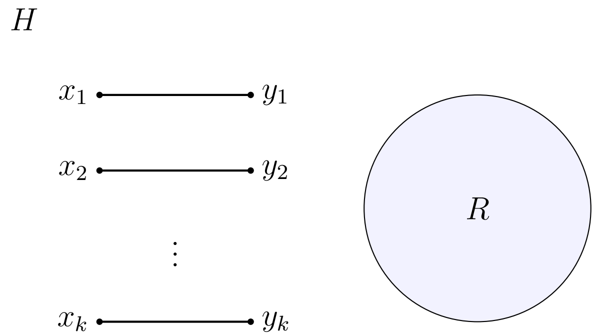


FIGURE 31. An H_1-free graph on n vertices breaks down into k disjoint pairs that each point to at least two other vertices plus a remainder set R with $n-2k$ vertices that belong to no such pair.

which contradicts the fact that H is oriented. ■

Therefore, if we assume that H is H_1-free on n vertices, then we can split its vertices up into k disjoint pairs such that each serves as a tail set to at least two edges of H plus a set of $n-2k$ vertices that belong to no such pair. That is,

$$V(H) = \{x_1, y_1, \ldots, x_k, y_k\} \cup R$$

so that $t(x_i, y_i) \geq 2$ for $i = 1, \ldots, k$ and $t(w, v) \leq 1$ for all other vertex pairs, $\{w, v\}$ (see Figure 31).

We now have two cases to consider. Either there are no such pairs ($k = 0$) or there is at least one ($k \geq 1$).

2. No pair points to more than one vertex

Assume that $k = 0$. Then $t(x, y) \leq 1$ for every pair $\{x, y\} \in \binom{V(H)}{2}$. If $|D_x| \leq n-3$ for all $x \in V(H)$, then

$$|E(H)| = \frac{1}{2} \sum_{x \in V(H)} |D_x| \leq \frac{1}{2} n(n-3) < \frac{1}{2}(n-1)(n-2)$$

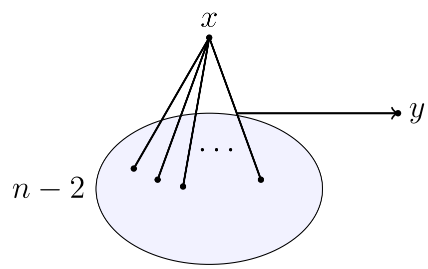


FIGURE 32. The special case configuration discussed in Lemma 7.2. Here, vertex x joins with every other element to point to vertex y.

and we are done. Otherwise, there exists some vertex x that belongs to $n-2$ tail sets. Therefore, D_x is a star of directed edges with some common vertex of intersection y. Either $t(x,y)=0$ or $t(x,y)=1$.

If $t(x,y)=0$, then all of the $n-2$ directed edges of D_x must point to y (see Figure 32). Such a configuration in H limits the number of edges to $\binom{n-1}{2}$ as proven in Lemma 7.2.

On the other hand, if $t(x,y)=1$, then $xy \to z \in E(H)$ for some vertex z, and $xv \to y$ for all other vertices $v \neq x, y, z$. Such a configuration in H will limit the number of edges to $\binom{n-1}{2}$ as proven in Lemma 7.3.

LEMMA 7.2. *Let H be an oriented graph on $n \geq 6$ vertices such that $t(x,y) \leq 1$ for each pair $\{x,y\} \in \binom{V(H)}{2}$. If H is H_1-free and contains vertices x and y such that $xv \to y \in E(H)$ for each $v \in V(H) \setminus \{x,y\}$, then*

$$|E(H)| \leq \binom{n-1}{2}.$$

See Figure 32.

PROOF. We want to show that there can be no more than $\binom{n-2}{2}$ additional edges in H other than the $n-2$ edges described in the statement of the lemma. This would give an upper bound on the total number of edges of

$$\binom{n-2}{2} + (n-2) = \binom{n-1}{2}.$$

First, note that every triple of the form $\{x, y, v\}$ already holds an edge. This implies that any additional edge cannot contain both x and y since H is oriented. On the other hand, if we were to add an edge, $vw \to u$, that excluded both x and y completely, then this new edge would create a copy of H_1 with the existing edge, $vx \to y$. Therefore, every additional edge must be on a triple of the form $\{v, w, x\}$ or $\{v, w, y\}$.

However, x is already in the maximum number of tails. So given any pair of non-$\{x, y\}$ vertices, $\{v, w\}$, the only possible additional edges are

$$vw \to x, vw \to y, yv \to w, \text{and } yw \to v.$$

The last three all appear on the triple, $\{v, w, y\}$, and are therefore mutually exclusive choices when it comes to adding them to the graph. The first two are also mutually exclusive choices since $t(v, w) \leq 1$.

So assume, towards a contradiction, that we could add $\binom{n-2}{2} + 1$ more edges to the existing configuration. Then some pair $\{v, w\}$ of non-$\{x, y\}$ vertices must be used twice. Without loss of generality, this means we must add the edges $vw \to x$ and $yv \to w$.

Now, let u be any of the remaining $n - 4$ vertices. The possible edge $uv \to y$ would create a copy of H_1 with $vw \to x$, and the possible edge $uv \to x$ would create a copy of H_1 with $vy \to w$. Therefore, the pair $\{v, u\}$ cannot be a tail set for any edge.

We can also view the potential additional edges as two different types: those that have a tail set of two non-$\{x, y\}$ vertices and those that have a tail set that includes y. There were originally at most $\binom{n-2}{2}$ of the first type that we are allowed to add in total, one edge for every distinct pair. However, v can now no longer be in a tail set with any of the other $n - 4$ vertices. So there are now at most $\binom{n-2}{2} - (n - 4)$ edges of this first type left possible to add. Therefore, in order to add $\binom{n-2}{2} + 1$ edges over all, we will need at least $n - 3$ of them to be of the second type - those that have y in the tail set.

Note that x must be an isolated vertex in the directed link graph D_y. Hence, there are at most $n-3$ tails containing y since otherwise the directed graph D_y would have $n-2$ edges among $n-2$ vertices. In this case, D_y would have two independent directed edges and so H would have a copy of H_1 with y as its intersection vertex. Moreover, D_y must be a star with a single vertex of intersection. Since $v \to w \in E(D_y)$, then this vertex of intersection must either be v or w.

Hence, in order to add $\binom{n-2}{2}+1$ edges, we will need to have $\binom{n-2}{2}-(n-4)$ edges that have non-$\{x, y\}$ tail sets. Since the tail set, $\{v, w\}$, already points to x, then this implies that all such edges must also point to x. Otherwise, we would have some edge of the form $ab \to y$. If $a = w$ or $b = w$, then this would create a copy of H_1 with $vw \to x$. If both elements are distinct from w, then we would still need to point the pair wa either to x or to y. Either choice would create a copy of H_1.

Let u be one of the remaining vertices. Then u must be adjacent to a directed edge of D_y for there to be $n-3$ edges added with y in the tail set. If v is the vertex of intersection of D_y, then this edge must either be $u \to v$ or $v \to u$. Either yields a copy of H_1. Similarly, if w is the vertex of intersection of D_y, then either $wy \to u \in E(H)$ or $uy \to w \in E(H)$. Again, either of these yields a copy of H_1. Therefore, $\binom{n-2}{2}+1$ edges cannot be added to the existing configuration. ■

LEMMA 7.3. *Let H be an oriented graph on $n \geq 6$ vertices such that for each pair $x, y \in V(H)$, $t(x, y) \leq 1$. If H is H_1-free and contains vertices x, y, and z such that $xy \to z \in E(H)$ and $xv \to y \in E(H)$ for each $v \in V(H) \setminus \{x, y, z\}$ (see Figure 33), then*

$$|E(H)| \leq \binom{n-1}{2}.$$

PROOF. Let $W = \{1, 2, \ldots, n-3\}$ be the set of non-$\{x, y, z\}$ vertices. Any additional edge to this graph must have a tail set of the form $\{i, j\}$, $\{i, y\}$, $\{i, z\}$, or $\{y, z\}$ for $i, j \in W$. An ij tail can only point to x or to y and there are $\binom{n-3}{2}$ pairs like this possible. An iy tail cannot point to x because H is oriented. It cannot point

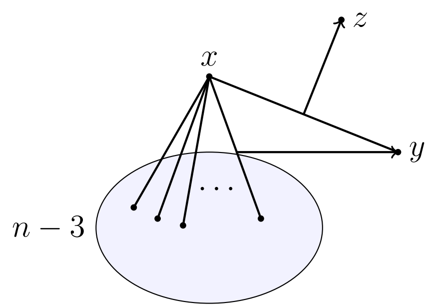


FIGURE 33. The special case configuration discussed in Lemma 7.3. Here, x joins with every vertex except z to point to y and then joins with y to point to z.

to j since that would create a copy of H_1 with $xy \to z$. Therefore, it could only point to z. An iz tail could not point to any j since this would create a copy of H_1 with the edge $ix \to y$. Therefore, it could only point to y or to x. And a yz tail could not point to x since H is oriented. Therefore, it could only point to some i.

Assume, towards a contradiction, that we can add

$$\binom{n-2}{2} + 1 = \binom{n-3}{2} + (n-3) + 1$$

edges to the existing configuration. Since we can add at most $\binom{n-3}{2}$ edges with tail sets made entirely of vertices from W, then we must have at least $n-2$ additional edges from the other possibilities.

For each $i \in W$ we could have

$$iy \to z, yz \to i, iz \to y, \text{and } iz \to x.$$

The first three of these are mutually exclusive choices since they are all on the same triple. Similarly, the last two are mutually exclusive choices since we are only allowing up to one edge per possible tail set.

Therefore, in order to add $n-2$ of these types of edges, two must use the same element of W. Given the mutually exclusive choices above this implies that there is some vertex $i \in W$ such that either $iz \to x, yi \to z \in E(H)$ or $iz \to x, yz \to i \in E(H)$.

In the first case, ij is no longer a possible tail for any edge for all $n-4$ remaining vertices $j \in W$. This is because $iz \to x$, $yi \to z$, and $ix \to y$ create a triangle in D_i. So any additional edge with i in the tail would give two independent edges in D_i and therefore a copy of H_1.

Hence, we can get at most $\binom{n-3}{2} - (n-4)$ edges with tails in W. This means that we will need $2(n-3)$ edges from the other possible edges to make up the difference if we want to add

$$\binom{n-3}{2} + (n-3) + 1$$

more edges.

Since each of the $n-3$ vertices from W can be in up to two of these additional edges, then $iz \to x$ would need to be an edge for every $i \in W$ and that $\{y, z, i\}$ also needs to hold one edge for every $i \in W$.

If $yz \to i$ is used once, then we get a copy of H_1 with $jz \to x$ for some other $j \in W$. Therefore, for all $i \in W$ we must have the edges $iy \to z$ and $iz \to x$. However, any pair $i, j \in W$ can now point to nothing since the only possibilities for such a tail were x or y to begin with and both of these options create copies of H_1. So in this case the most that we can add is

$$2(n-3) \leq \binom{n-3}{2} + (n-3)$$

for all $n \geq 6$.

In the other case we have added $iz \to x$ and $yz \to i$ for some i. Which means that $yz \to j$ is not allowed for any $j \neq i$ from W. Also, $jz \to y$ would make a copy of H_1 with $iz \to x$ and $jz \to x$ would make a copy of H_1 with $yz \to i$. Therefore, for all $j \neq i$ we can only add the edge $jy \to z$.

In order to add $\binom{n-3}{2} + n - 2$ edges, we will need all of these as well as all possible edges with tails in W. However, since $iz \to x$, all of the edges with tails completely in W must also point to x. Otherwise, some pair ab would point to y. If $a = i$ or

$b = i$, then this would make a copy of H_1 with $iz \to x$. If $i \neq a, b$, then consider where the pair ai points. It must either point to x or to y, but either of these would create a copy of H_1.

So all pairs of W must point to x and for all $j \in W$ not equal to i we must have the edge $jy \to z$. But $jy \to z$ and $ij \to x$ create a copy of H_1, a contradiction. Hence, it is not possible to add more than $\binom{n-3}{2} + (n-3)$ edges to the configuration. Since the configuration already has $n-2$ edges, then there can be no more than $\binom{n-1}{2}$ edges total. ■

Together these two lemmas take care of the cases where all pairs of vertices point to at most one vertex in H.

3. Some pair of vertices is the tail set to multiple edges of H

We return to our description of an H_1-free oriented graph as being made up of $k \geq 1$ vertex pairs that each serve as tail sets to strictly more than one edge plus a set R of the remaining $n - 2k$ vertices,

$$V(H) = \{x_1, y_1, \ldots, x_k, y_k\} \cup R$$

(see Figure 31). For each pair $\{x_i, y_i\}$ we want to prove the following upper bound,

$$t(x_i, y_i) + \sum_{v \neq x_i, y_i} (t(x_i, v) + t(y_i, v)) \leq n - 2.$$

That is, the total number of edges that include either x_i or y_i or both in the tail set is at most $n - 2$.

Now,

$$|E(H)| = \sum_{\{x,y\} \in \binom{V(H)}{2}} t(x, y) \leq \sum_{i=1}^{k} \left(t(x_i, y_i) + \sum_{v \neq x_i, y_i} (t(x_i, v) + t(y_i, v)) \right) + \sum_{\{x,y\} \in \binom{R}{2}} t(x, y).$$

Note that each pair of vertices in R act as a tail set at most once so

$$\sum_{\{x,y\}\in\binom{R}{2}} t(x,y) \le \binom{n-2k}{2}.$$

Therefore, proving the upper bound for each $\{x_i, y_i\}$ pair would imply that

$$|E(H)| \le k(n-2) + \binom{n-2k}{2}.$$

Since

$$k(n-2) + \binom{n-2k}{2} = 2k^2 - (n+1)k + \binom{n}{2}$$

is a quadratic polynomial with positive leading coefficient in terms of k, then it is maximized at the endpoints. Here, that means at $k = 1$ and at $k = \left\lfloor \frac{n}{2} \right\rfloor$.

When n is odd, both of these values for k give the upper bound,

$$|E(H)| \le \binom{n-1}{2}.$$

Only when n is even can we do better and get

$$|E(H)| \le \frac{n(n-2)}{2}$$

in the case where $k = \frac{n}{2}$. In either case this give an upper bound of

$$|E(H)| \le \left\lfloor \frac{n}{2} \right\rfloor (n-2).$$

So we need only prove that, in general,

$$t(x_i, y_i) + \sum_{v \neq x_i, y_i} (t(x_i, v) + t(y_i, v)) \le n - 2.$$

This is straightforward to show if $t(x_i, y_i) \ge 3$. However, when $t(x_i, y_i) = 2$ there is a case where it fails to hold. This is taken care of in the following lemma.

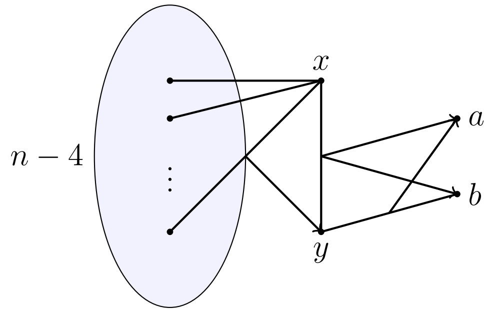


FIGURE 34. An H_1-free graph containing this configuration with have at most $\binom{n-1}{2}$ edges as shown in Lemma 7.4.

LEMMA 7.4. *Let H be an oriented graph on $n \geq 6$ vertices. If H is H_1-free and contains vertices x, y, a, and b such that $\{x, y\}$ is the tail set to exactly 2 edges with*

$$xy \to a, xy \to b, yb \to a \in E(H),$$

and for each $v \in V(H) \setminus \{x, y, a, b\}$, $xv \to y$ (see Figure 34), then

$$|E(H)| \leq \binom{n-1}{2}.$$

PROOF. First consider which pairs of vertices could possibly be a tail set to an edge in this graph. Let $W = \{1, \ldots, n-4\}$ be the set of vertices other than $\{x, y, a, b\}$. Then $\{i, j\}$ can be a tail set to $ij \to x$ and $ij \to y$ for any pair $i, j \in W$. Since $xy \to a$, then xi can point to nothing other than y. Similarly, xa and xb could only possibly point to b and a respectively, but either would create a copy of H_1 with $xi \to y$ for any $i \in W$. Also, by assumption xy can point to nothing else. Hence, x is in no additional tail sets.

Since $yb \to a$ and $xy \to a$, then ya cannot point to b or to x. It can also not point to any $i \in W$ since this would create a copy of H_1 with $xy \to b$. So y can be in no additional tails. The pair ab can point to anything aside from y since H is oriented, and ai can point to x or y for any $i \in W$ but not to b or another element of W since either would create a copy of H_1 with $xi \to y$. Similarly, bi can point to y for each

$i \in W$ but not to x or to a or to another element of W since these would create a copy of H_1 with either $yb \to a$ or $xi \to y$.

Leaving aside the edges with tail sets completely in W for the moment, this means there are $4(n-4)+1$ possible edges remaining. There are $n-4$ each of types $ai \to x$, $ai \to y$, $bi \to y$, and $ab \to i$ plus one extra edge which is $ab \to x$.

Suppose we are able to use at least $2(n-4)+1$ of these edges. First, if one of them is $ab \to x$, then there could be none of the types $ai \to y$ or $bi \to y$. So all of the ones of type $ab \to i$ and $ai \to x$ would need to be used. But since $n \geq 6$, there are at least two vertices in W. So there would exist edges $ai \to x$ and $ab \to j$ with $i \neq j$, a copy of H_1. Therefore, $ab \to x$ cannot be used if we want to get more than $2(n-4)$ of these edges.

Hence, we need to use at least three types of edges from the four possible types. Since any of the types $ai \to x$, $ai \to y$, and $bi \to y$ eliminate the possibility of using any edge $ab \to j$ where $j \neq i$, then we can use at most one of this last type of edge. But since $n \geq 6$, then $2(n-4)+1 \geq 5$ which means one of the other types gets used at least twice. Regardless of which one it is, there can be nothing used from the $ab \to i$ types of edges.

Therefore, we must use $2(n-4)+1$ edges from only the first three types. So there must be a vertex i from W that belongs to three of these edges, say

$$ai \to y, bi \to y, \text{and } ai \to x.$$

But the edges $bi \to y$ and $ai \to x$ form an H_1, a contradiction. Thus, at most $2(n-4)$ of these kinds of edges can be used over all.

Now let us look at the edges with tail sets contained in W. We have seen that each ij can point to x or to y, but nothing so far has kept the pair from pointing to both. However, if some pair does point to both, then no other tail could use either of these vertices since this would create a copy of H_1. Therefore, if there are $1 \leq l$ such pairs, then there are at most $2l + \binom{n-4-2l}{2}$ edges with tails from W. If $n = 6$, then this gives exactly one such pair and only 7 edges overall. If $n \geq 7$, then $l \leq \frac{n-4}{2}$ implies that

$$2l \leq n-4 \leq \binom{n-4}{2} - \binom{n-4-2l}{2}.$$

Hence,

$$2l + \binom{n-4-2l}{2} \leq \binom{n-4}{2}.$$

So there are at most $\binom{n-1}{2}$ edges in H. ■

4. The oriented extremal number

Now we can proceed with establishing the upper bound under the assumption that the configuration presented in Lemma 7.4 does not occur in our directed hypergraph. As we've seen, all that is necessary to show is that

$$t(x_i, y_i) + \sum_{v \neq x_i, y_i} (t(x_i, v) + t(y_i, v)) \leq n-2$$

for any pair of vertices $\{x_i, y_i\}$ that serves as the tail set to at least two edges.

So let $\{x, y\}$ be such a pair, and divide the rest of the vertices of H into two groups, those that are a head vertex to some edge with xy as the tail and those that are not. That is,

$$V(H) \setminus \{x, y\} = \{h_1, \dots, h_m\} \cup \{n_1, \dots, n_t\}$$

where for each $i = 1, \dots, m$, there exists an edge, $xy \to h_i \in E(H)$ and for each $j = 1, \dots, t$, $xy \to n_j \notin E(H)$ (note that $t(x, y) = m$ and that $m + t = n - 2$).

Now, consider an edge that contains either x or y in the tail but not both. Then the other tail vertex is either some h_i or some n_j. In the case of n_j, this edge must either be of the form $xn_j \to y$ or $yn_j \to x$ to avoid a copy of H_1 with both $xy \to h_1$ and

$xy \to h_2$. Moreover, since H is oriented, there can be at most one. Hence,

$$\sum_{j=1}^{t} (t(x, n_j) + t(y, n_j)) \leq t.$$

Now consider a tail set that includes either x or y and some h_i. Without loss of generality, assume that xh_1 is the tail to some edge. Since $t(x, y) \geq 2$, there is some other vertex h_2 such that $xy \to h_2 \in E(H)$. In order to avoid a copy of H_1 with this edge, xh_1 must either point to y or to h_2. However, $xh_1 \to y \notin E(H)$ since this would give the triple $\{x, y, h_1\}$ more than one edge.

Therefore, $xh_1 \to h_2$ is the only option. However, if $t(x, y) \geq 3$, then this will create a copy of H_1 with $xy \to h_3$. So xh_i and yh_i cannot be tails to any edge. So

$$\sum_{i=1}^{m} (t(x, h_i) + t(y, h_i)) = 0.$$

Therefore,

$$\begin{aligned}
&t(x, y) + \sum_{v \neq x,y} (t(x, v) + t(y, v)) \\
&= m + \sum_{j=1}^{t} (t(x, n_j) + t(y, n_j)) + \sum_{i=1}^{m} (t(x, h_i) + t(y, h_i)) \\
&\leq m + t \\
&= n - 2
\end{aligned}$$

when $t(x, y) \geq 3$.

The only other possibility is that $t(x, y) = 2$. So suppose this is the case and that the head vertices to xy are a and b. Without loss of generality, assume that $yb \to a \in E(H)$. Note that this precludes any edges of the form $yn_j \to x$. Similarly, if we added the edge $xa \to b$ or the edge $xb \to a$, then we could not add any edges of the form $xn_j \to y$ and so

$$\sum_{j=1}^{t} (t(x,n_j) + t(y,n_j)) = 0.$$

Moreover, $ya \to b$ would lead to more than one edge on the triple $\{y, a, b\}$. So

$$\sum_{i=1}^{m} (t(x,h_i) + t(y,h_i)) = 2$$

and in total we would have,

$$t(x,y) + \sum_{v \neq x,y} (t(x,v) + t(y,v)) = 4 \leq n-2.$$

On the other hand, if xa and xb are not tails to any edge, then the only way we could get a sum of more than $n-2$ is if $xn_j \to y \in E(H)$ for all $j = 1, \ldots, n-4$. But this is exactly the configuration described in Lemma 7.4 which we have excluded.

Therefore,

$$t(x,y) + \sum_{v \neq x,y} (t(x,v) + t(y,v)) \leq n-2$$

for any such pair, and this is enough to establish that

$$\mathrm{ex}_o(n, H_1) \leq \left\lfloor \frac{n}{2} \right\rfloor (n-2).$$

Conversely, we have already considered an extremal construction in the case where n is even, and this same construction will work when n is odd. That is, take a maximum matching of the vertices (leaving one out) and use each matched pair as the tail set for all $n-2$ possible edges.

Another construction that works for odd n that is not extremal for even n is to designate one vertex as the only head vertex and then make all $\binom{n-1}{2}$ pairs of the rest of the vertices tail sets.

Therefore,

$$\mathrm{ex}_o(n, H_1) = \left\lfloor \frac{n}{2} \right\rfloor (n-2).$$

Also, note that the only way that any construction could have more than $\binom{n-1}{2}$ edges is if n is even *and* the vertices are partitioned into $\frac{n}{2}$ pairs such that each points to at least two other vertices. This fact comes directly from the requirement that $k = \frac{n}{2}$ in the optimization of

$$k(n-2) + \binom{n-2k}{2}$$

in order for the expression to be more than $\binom{n-1}{2}$.

5. Intersections of multiedge triples in the standard version

Now, let H be an H_1-free graph on n vertices under the standard version of the problem so that any triple of vertices can now have up to all three possible directed edges. If we let t_H be the number of triples of vertices of H that hold at least one edge, and we let m_H be the number of triples that hold at least two, then we have the following simple observation:

$$|E(H)| \leq t_H + 2m_H.$$

We start our path towards an upper bound on $|E(H)|$ by finding an upper bound on the number of multiedge triples, m_H. We will need to prove some facts about the multiedge triples of H. First, any triple which holds two edges of H might as well hold three.

LEMMA 7.5. *Let H be an H_1-free graph such that some triple of vertices $\{x, y, z\}$ contains two edges. Define H' by $V(H') = V(H)$ and*

$$E(H') = E(H) \cup \{xy \to z, xz \to y, yz \to x\}.$$

Then H' is also H_1-free.

PROOF. Suppose H' is not H_1-free. Since H is H_1-free and the two graphs differ by at most one edge, then they must differ by exactly one edge. Without loss of generality, say

$$\{xy \to z\} = E(H') \setminus E(H).$$

This edge must be responsible for creating the copy of H_1 in H'. So it must intersect another edge in exactly one vertex that is in the tail set of both.

Therefore, without loss of generality, there is an edge $xt \to s \in H$ where $\{s,t\} \cap \{y,z\} = \emptyset$. However, since $\{x,y,z\}$ already contained two edges of H, then $xz \to y \in H$. Since $xt \to s$ and $xz \to y$ make a copy of H_1, then H cannot be H_1-free, a contradiction. ■

Next, we want to show that no two multiedge triples can intersect in exactly one vertex.

LEMMA 7.6. *Let H be a H_1-free graph. If two vertex triples $\{x,y,z\}$ and $\{s,t,r\}$ each contain two or more edges of H, then*

$$|\{x,y,z\} \cap \{s,t,r\}| \neq 1.$$

PROOF. Suppose

$$|\{x,y,z\} \cap \{s,t,r\}| = 1$$

By Lemma 7.5, since H is H_1-free, the graph created from H by adding all three possible edges on the triples $\{x,y,z\}$ and $\{s,t,r\}$ is also H_1-free. But if $x = r$ and x, y, z, s, and t are all distinct, then this graph contains $xy \to z$ and $xs \to t$ which is a copy of H_1, a contradiction. ■

Therefore, we can use an upper bound on the number of undirected 3-uniform hyperedges such that no two intersect in exactly one vertex as an upper bound on the number of multiedge triples. Moreover, the extremal examples are easy to describe which will be important for finding the upper bound for $\mathrm{ex}(n, H_1)$ as well as for establishing the uniqueness of the lower bound construction.

LEMMA 7.7. *Let H be a 3-uniform undirected hypergraph on n vertices such that no two edges intersect in exactly one vertex, then*

$$|E(H)| \leq \begin{cases} n & n \equiv 0 \ mod \ 4 \\ n-1 & n \equiv 1 \ mod \ 4 \\ n-2 & n \equiv 2,3 \ mod \ 4 \end{cases}$$

and H is the disjoint union of $K_4^{(3)}$s, $K_4^{(3)}$s minus an edge (K_4^-), and sets of edges that all share a common intersection of two vertices - a sunflower with a two vertex core.

PROOF. Two edges of H are either disjoint or they intersect in two vertices. So connected components of H that have 1 or 2 edges are both sunflowers. A third edge can be added to a two-edge sunflower by either using the two common vertices to overlap with both edges in two or by using one common vertex and the two petal vertices. So a connected component of H with 3 edges is either a sunflower or a K_4^-.

The only way to connect a fourth edge to the three-edge sunflower is to make a four-edge sunflower, and this is true for a k-edge sunflower to a $(k+1)$-edge sunflower for all $k \geq 3$. The only way to add a fourth edge to the K_4^- is to make a $K_4^{(3)}$ and then no new edges may be connected to a $K_4^{(3)}$ without intersecting two of its edges in exactly one vertex each. Therefore, these are the only possible connected components of H.

A sunflower with k edges uses $k+2$ vertices, and a $K_4^{(3)}$ has four edges on 4 vertices. Therefore, if $n \equiv 0$ mod 4 we can get at most n edges with a disjoint collection of $K_4^{(3)}$s. Similarly, the best we can do when $n \equiv 1$ mod 4 is $n-1$ edges with a disjoint collection of $K_4^{(3)}$s plus one isolated vertex since any sunflower will automatically limit the number of edges to $n-2$. And if $n \equiv 2$ mod 4 or $n \equiv 3$ mod 4, then $n-2$ is the best that we can do. ■

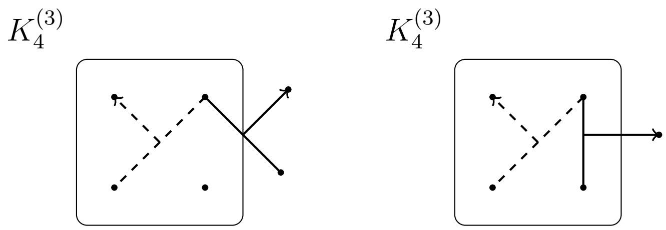


FIGURE 35. An edge that intersects a $K_4^{(3)}$ block of multiedge triples in one or two tail vertices will create a copy of H_1.

In general, the only way to actually have an H_1-free graph with n multiedge triples is if the multiedge triples form an undirected 3-uniform hypergraph of $\frac{n}{4}$ disjoint $K_4^{(3)}$ blocks when $n \equiv 0 \bmod 4$.

In this case there can be no additional directed edges in H since such an edge would either intersect one of these $K_4^{(3)}$s in one tail vertex which would create a copy of H_1 since this means it intersects three of the multiedge triples in exactly one tail vertex (we may assume that each multiedge has all three edges per Lemma 7.5) or it would intersect one of the $K_4^{(3)}$s in two tail vertices which means that it intersects two of the multiedge triples in exactly one tail vertex (see Figure 35).

So in this case, the number of total edges would be bound by

$$3n < \binom{n+1}{2} - 3$$

for all $n \geq 7$.

Next, the only way to have $n-1$ multiedge triples is to either have $\frac{n-1}{4}$ disjoint $K_4^{(3)}$ blocks when $n \equiv 1 \bmod 4$ or to have $\frac{n}{4} - 1$ disjoint $K_4^{(3)}$ blocks with one K_4^- when $n \equiv 0 \bmod 4$. In the first case any additional edge must have at least one and perhaps two of its tail vertices in a single $K_4^{(3)}$ block of multiedge triples which we have already seen will create a copy of H_1. So there are at most

$$3(n-1) < 3n < \binom{n+1}{2} - 3$$

total edges in this case.

In the second case, any additional edge that has no tail vertices in a $K_4^{(3)}$ block must have both tail vertices in the K_4^-. If the head to such an edge were outside of the K_4^-, then the edge must intersect one of the three multiedge triples of the block in exactly one tail vertex since there are two triples that it intersects in one tail vertex each, one of which must be a multiedge triple. On the other hand, it could have its head vertex inside the K_4^-. In this case, the additional edge must lie on the triple without multiple edges. This is the only edge that can be added. So there are at most

$$3(n-1)+1 < 3n < \binom{n+1}{2} - 3$$

total edges in this case.

6. An H_1-free graph with $n-2$ multiedge triples

Now, the only ways to have exactly $n-2$ multiedge triples is either to have $\frac{n}{4}-2$ of the $K_4^{(3)}$ blocks plus two K_4^- blocks of multiedge triples when $n \equiv 0 \bmod 4$ or to have k of the $K_4^{(3)}$ blocks of multiedge triples plus a sunflower with $n-4k-2$ petals. The first case is suboptimal for the same reasons already considered. So let us consider the second case.

First, assume that $k=0$ and that we have $n-2$ multiedge triples that make a sunflower (see Figure 36). How many edges can we add? This structure already has all possible edges with 2 vertices in the core (or so we may assume by Lemma 7.5). On the other hand, if an additional edge has no vertices in the core, then it would intersect two multiedge triples in one tail vertex each which would create a copy of H_1.

Therefore, any additional edge must include exactly one vertex from the core. If this vertex is in the tail set to the additional edge and the sunflower has at least three petals, then the additional edge intersects in exactly one tail vertex of the multiedge triples of the sunflower, a contradiction. Since we assume that $n \geq 6$, then the

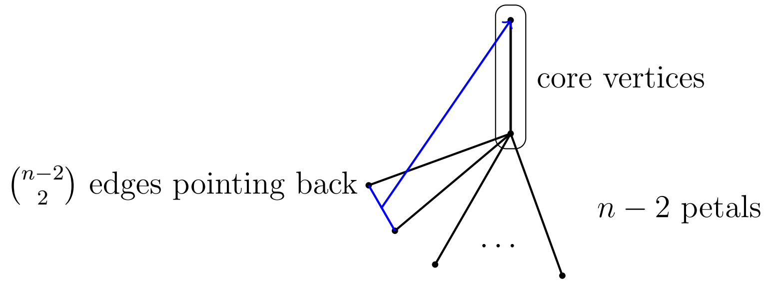


FIGURE 36. The unique extremal construction for an H_1-free graph has $\binom{n-2}{2}+3(n-2)$ edges.

sunflower has at least three petals. Hence, any additional edge must intersect the core in only its head vertex.

If any two additional edges have different core vertices as the head, then either the tail sets of these edges must be exactly the same or completely disjoint to avoid a copy of H_1. Hence, pairs of petal vertices that point to both core vertices must be independent of all other tail sets. And all other petal vertices fall into disjoint sets as to whether they are in additional edges that point to the first core vertex or the second. The number of additional edges will be maximized if every pair of petal vertices point to the same core vertex. Moreover, this will give a total of

$$3(n-2)+\binom{n-2}{2}=\binom{n+1}{2}-3$$

edges.

We will soon see that this is the best that we can do and that this construction, where the multiedge triples make a sunflower with $n-2$ petals with $\binom{n-2}{2}$ additional edges pointing from pairs of petal vertices to a single core vertex, is unique up to isomorphism.

First we will need to see that $k=0$ is the number of $K_4^{(3)}$ multiedge triple blocks that optimizes the total number of edges. So suppose there are k such blocks and that the other $n-4k$ vertices are in a sunflower. Then from prior considerations we know that any additional edge must have both tail vertices in this sunflower. If one of these tail vertices coincides with a petal vertex of the sunflower, then there will be a copy of H_1. Therefore, the tail vertices must coincide with the core and the only possibility for such an edge is to point out to a vertex in one of the k blocks.

Therefore, there are at most

$$3(4k)+3(n-4k-2)+\binom{n-4k-2}{2}+4k$$

edges in such a construction. Since this expression is quadratic in k with positive leading coefficient, then it must maximize at the endpoints, $k=0$ or $k=\frac{n}{4}$, and we already know that $k=\frac{n}{4}$ is suboptimal. Therefore, if there are exactly $n-2$ multiedge triples, then they must form a sunflower with a two-vertex core and from there the only way to maximize the total number of edges is to add every possible edge with tail set among the petal vertices all pointing to the same head vertex in the core.

7. Fewer than $n-2$ multiedge triples

Now suppose that H has fewer than $n-2$ multiedge triples. If $t_H \le \binom{n-1}{2}$, then

$$|E(H)| \le t_H + 2m_H < \binom{n-1}{2} + 2(n-2) = \binom{n+1}{2} - 3.$$

So we must assume that $t_H > \binom{n-1}{2}$. Also, if $m_H = 0$, then we know that

$$|E(H)| \le \mathrm{ex}_o(n, H_1) = \left\lfloor \frac{n}{2} \right\rfloor (n-2) < \binom{n+1}{2} - 3.$$

So assume that there is at least one multiedge triple, $\{x, y, z\}$. This triple has at least two edges. Assume without loss of generality that they are $xy \to z$ and $xz \to y$.

Let H' be an oriented graph arrived at by deleting edges from multiedge triples of H until each triple has at most one edge and every triple that had at least one edge in H still has at least one in H'. In other words, H' is any subgraph of H such that $t_{H'} = t_H$ and $m_{H'} = 0$. Without loss of generality, assume that

$$xy \to z \in E(H').$$

Since $t_{H'} > \binom{n-1}{2}$, then n must be even. Moreover, there is a matching on the vertices so that every matched pair $\{a, b\}$ points to at least two other vertices. That is, $t(a, b) \geq 2$.

Now consider the directed link graphs of the vertices. As stated before, these are either triangles or stars with a common vertex. However, if two or more of these link digraphs have three or fewer edges each (for instance, if they are triangles), then there are fewer edges than we are assuming since

$$|E(H')| = \frac{1}{2} \sum_{x \in V(H')} |D_x| \leq \frac{1}{2}\left(6 + (n-3)(n-2)\right) < \binom{n-1}{2}$$

for all $n \geq 8$. We will show that it must be the case that here at least two directed link graphs are restricted to at most three directed edges each, contradicting our assumptions about the number of edges in H.

First, note that $x \to z \in D_y$ and $y \to z \in D_x$. To avoid a contradiction, at least one of these two directed link graphs must have four or more edges. Without loss of generality, assume that it is D_y. Therefore, D_y is a star and not a triangle. So the additional three directed edges in D_y must either all be incident to z or to x.

If these directed edges are all incident to z, then y and z must be partners under the matching which means that x has another partner x' distinct from y and z. Since $t(x, x') \geq 2$ in H', then x' must point to two vertices in D_x. Since D_x already has $y \to z$ and no two edges may be independent in any directed link graph, then x' must point to y and to z, forming a triangle.

Next, consider $D_{x'}$. We know that

$$x \to y, x \to z \in D_{x'}.$$

If there is an additional edge in $D_{x'}$ that does not complete this triangle then it is either of the form $x \to t$ or $t \to x$. If $x \to t \in D_{x'}$ then $x' \to t, y \to z \in D_x$, a contradiction. If $t \to x \in D_{x'}$, then $x' \to x \in D_t$. But since t has its own matched vertex, then there exists a distinct t' such that

$$t' \to x, t' \to x' \in D_{t'}.$$

So either $|D_{x'}| \leq 3$ or $|D_{t'}| \leq 3$. Either way, this gives us two directed link graphs that have at most three edges each. So $t_{H'} < \binom{n-1}{2}$.

Therefore, we must assume that the three additional edges in D_y are incident to x and that y and x are partners under the matching. So z has some other partner under the matching z' distinct from x and y. Now, delete the edge $xy \to z$ from H' and add $xz \to y$ to get a new directed hypergraph H''. It follows that H'' has no multiedge triples and is H_1-free since we still have a subgraph of H.

In adding $xz \to y$ we have added $x \to y$ to D_z. Since z' must point to two vertices in D_z, then this addition means that D_z is a triangle under H''. Hence, $|D_z| = 2$ under H'.

Now, the same argument as above applies to $D_{z'}$. The only way for $|D_{z'}| > 3$ would mean either $z \to a \in D_{z'}$ or $a \to z \in D_{z'}$ for some a distinct from x, y, z, and z'. The first case would mean that two independent directed edges, $z' \to a$ and $x \to y$ are in D_z, a contradiction. The second case would mean that $z' \to z \in D_a$. Since a has its own partner under the matching that must point to two vertices in D_a, then in this case, D_a is a triangle.

Therefore, $t_H > \binom{n-1}{2}$ and $m_H \geq 1$ cannot both be true in any H_1-free graph. This is enough to complete the result,

$$\mathrm{ex}(n, H_1) = \binom{n+1}{2} - 3.$$

This also exhausts the remaining cases in order to demonstrate that the extremal construction is unique.

CHAPTER 8

The Graph H_2

This chapter contains material from a paper published by the Electronic Journal of Combinatorics. [12]

Let H_2 denote the forbidden graph where two edges intersect in exactly two vertices such that the set of intersection is the tail set to each edge. That is $V(H_2) = \{a, b, c, d\}$ and $E(H_2) = \{ab \to c, ab \to d\}$ (see Figure 37).

THEOREM 8.1. *For all $n \geq 5$,*

$$ex(n, H_2) = ex_o(n, H_2) = \binom{n}{2}.$$

Moreover, there are $(n-2)^{\binom{n}{2}}$ different labeled H_2-free graphs attaining this extremal number when in the standard version of the problem.

PROOF. Let H be H_2-free. Regardless of which version of the problem we are considering, each pair of vertices acts as the tail set to at most one directed edge. Therefore,

$$\mathrm{ex}(n, H_2), \mathrm{ex}_o(n, H_2) \leq \binom{n}{2}.$$

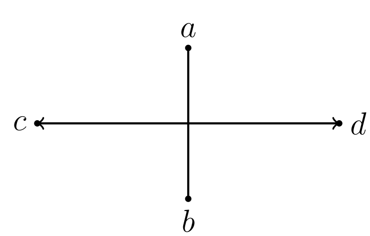


FIGURE 37. The graph H_2.

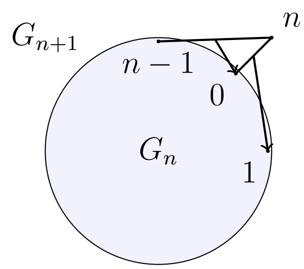


FIGURE 38. Inductive construction of H_2-free oriented graphs.

In the standard version of the problem any function, $f : \binom{[n]}{2} \to [n]$, that sends each pair of vertices to a distinct third vertex, $f(\{a,b\}) \notin \{a,b\}$, has an associated H_2-free construction H_f with $\binom{n}{2}$ edges. That is, for any such function, f, let $V(H_f) = [n]$ and

$$E(H_f) = \left\{ a, b \to f(\{a,b\}) : \{a,b\} \in \binom{[n]}{2} \right\}.$$

Since each pair of vertices acts as the tail set to exactly one directed edge, then H_f is H_2-free and has $\binom{n}{2}$ edges. So

$$\mathrm{ex}(n, H_2) = \binom{n}{2}.$$

Moreover, there are $(n-2)^{\binom{n}{2}}$ distinct functions from $\binom{[n]}{2}$ to $[n]$ such that no pair is mapped to one of its members. Therefore, there are $(n-2)^{\binom{n}{2}}$ labeled graphs that are H_2-free with $\binom{n}{2}$ edges.

In the oriented version of the problem lower bound constructions can be defined inductively on n.

First, let $n = 5$ and define G_5 as the oriented graph with vertex set

$$V(G_5) = \{0, 1, 2, 3, 4\}$$

and the following edges: $0,1 \to 2$; $1,3 \to 0$; $0,4 \to 1$; $0,2 \to 3$; $2,4 \to 0$; $0,3 \to 4$; $2,3 \to 1$; $1,2 \to 4$; $1,4 \to 3$; and $3,4 \to 2$.

Each pair of vertices of G_5 are in exactly one tail set, and each triple of vertices appear together in exactly one edge. Therefore, this construction is H_2-free with $\binom{5}{2}$ edges.

Now, let $n \geq 5$, and define G_{n+1} by $V(G_{n+1}) = \{0, 1, \ldots, n\}$ and

$$E(G_{n+1}) = E(G_n) \cup \{ni \to (i+1) : i = 0, \ldots, n-1\}$$

where addition is taken modulo n.

Then G_{n+1} has n more edges than G_n. So $|E(G_{n+1})| = \binom{n+1}{2}$.

Any two new edges intersect in at most two vertices. Similarly, any new edge and any old edge also intersect in at most two vertices. Hence, at most one edge appears on a given triple of vertices. So G_{n+1} is oriented.

Moreover, all tail sets for the new edges are distinct from each other and from any tail sets for the edges of G_n. So G_{n+1} is H_2-free. Therefore,

$$\mathrm{ex}_o(n, H_2) = \binom{n}{2}.$$

■

CHAPTER 9

Generalized Directed Hypergraphs

In this chapter we will generalize the standard version of a $2 \to 1$ directed hypergraph to a class of combinatorial structures that can all be considered to be uniform directed hypergraphs. This chapter is organized as follows. In Section 1, we define the class of generalized directed hypergraphs (GDH) and extend the concepts of Turán density, blowups, and supersaturation to this setting. In Section 2, we define the idea of a jump for a given model of directed hypergraphs and prove several results about these jumps and how the jumps from one instance of the class relate to jumps in another. In Section 3, we adapt a couple of results proved in [7] for totally directed hypergraphs with multiplicity to any GDH.

1. Basic definitions and results

The following definition for a generalized directed hypergraph is intended to include most uniform models that could reasonably be called uniform directed hypergraphs. This includes models where the edges are r-sets each under some partition into k parts of fixed sizes $r_1, \ldots, r_k$ with some linear ordering on the k parts. The definition only includes structures where an r-set could include multiple edges up to the number of possible orientations allowed. That is, we do not consider the "oriented" versions of the models where only one edge is allowed per r-set. The definition is given in terms of logic and model theory for convenience only. No deep results from those subfields are used. The use of this notation also makes further generalizations like nonuniform directed hypergraphs or oriented directed hypergraphs easy.

DEFINITION 9.1. *Let $\mathcal{L} = \{E\}$, a language with one r-ary relation symbol E. Let T be an $\mathcal{L}$-theory that consists of a single sentence of the form*

$$\forall x_1 \cdots x_r E(x_1, \ldots, x_r) \implies \bigwedge_{i \neq j} x_i \neq x_j \wedge \bigwedge_{\pi \in J_T} E(x_{\pi(1)}, \ldots, x_{\pi(r)})$$

for some subgroup of the group of permutations on r elements, $J_T \subseteq S_r$. Call such a theory a generalized directed hypergraph theory *and any finite model of T is a* generalized directed hypergraph (GDH)*.*

Note that this definition includes graphs, hypergraphs, and $r \to 1$ directed hypergraphs. For example, the theory for a $2 \to 1$ directed hypergraph is

$$T = \{\forall xyzE(x, y, z) \implies x \neq y \wedge x \neq z \wedge y \neq z \wedge E(y, x, z)\}.$$

It is easy to see that when $r = 2$ we have only two GDH theories. The theory associated with the group S_2 is the theory of graphs, and the theory associated with the trivial group is the theory of directed graphs.

When $r = 3$ there are six subgroups of S_3. Three of these are all isomorphic to $\mathbb{Z}_2$ with each generated by a permutation that swaps two elements. The corresponding GDH theory for any of these can be thought of as having pointed 3-sets for edges or as being $(2 \to 1)$-graphs. Of the other subgroups, S_3 itself gives the theory of undirected 3-uniform hypergraphs, the trivial group gives totally directed 3-edges, and the subgroup generated by a three-cycle isomorphic to $\mathbb{Z}_3$ yields a GDH theory where the edges can be thought of as 3-sets that have some kind of cyclic orientation - either clockwise or counter-clockwise. Figure 39 summarizes the models of GDHs when $r = 3$. Note that in general, S_r always corresponds to the normal undirected r-graph model and the trivial group always corresponds to totally directed hypergraphs.

A fun thought experiment is to consider the kinds of edges that arise when $r = 4$. Many of them are geometric in nature. For instance, the alternating group A_4 gives a theory where edges can be thought of tetrahedrons (at least in an abstract sense).

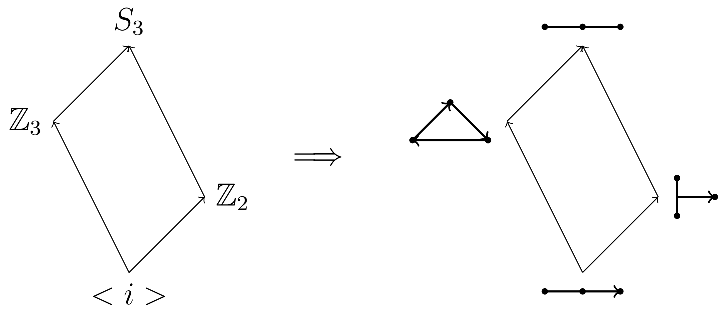


FIGURE 39. The subgroup lattice of S_3 and the corresponding lattice of directed hypergraphs.

In fact, Leader and Tan [33] study the "oriented" versions of the models that come from the alternating groups for any $r \geq 3$.

In this chapter, when the theory is not specified we are simply discussing GDHs that are all models of the same fixed theory. When discussing multiple theories we will often refer to T-graphs to mean models of a GDH theory T. Throughout the chapter, J_T will always stand for the subgroup $J_T \subseteq S_r$ that determines the GDH theory T and m_T will always be the order of this subgroup, $m_T = |J_T|$. Also, V_G and E_G will be used to denote the underlying set of elements of a model G and its relation set respectively.

The following basic propositions are given without proof. The first is a simple consequence that we are working in a relational language, and the second results from the fact that J_T is a group.

PROPOSITION 9.2. *For any GDH theory T and any nonnegative integer n, there exists a GDH $G \models T$ on n elements. Moreover, for any nonnegative integer $k < n$, the substructure of G induced on any k-subset of the elements of G is also a T-graph.*

PROPOSITION 9.3. *Given a GDH G with r-ary relation set E_G, there exists an equivalence relation $\sim$ on E_G defined by*

$$(a_1, \ldots, a_r) \sim (b_1, \ldots, b_r)$$

if and only if for each i, $b_i = a_{\pi(i)}$ for some $\pi \in J_T$.

We can now use these propositions to extend the concepts of extremal graph theory to GDHs in a natural way.

DEFINITION 9.4. *For any GDH G, an* edge *of G will always refer to an equivalence class of $[E_G]_\sim$.*

DEFINITION 9.5. *Given a GDH G on n elements, denote the number of edges of G by $e_T(G)$ and let the* edge density *of G be defined as*

$$d_T(G) := \frac{e_T(G)}{\frac{r!}{m_T}\binom{n}{r}}.$$

Note that since

$$e_T(G) = \frac{|E_G|}{m_T},$$

then the density is

$$d_T(G) = \frac{(n-r)!|E_G|}{n!}$$

and could have been defined this way while mostly avoiding talk of edges as equivalence classes of E_G. However, the above definition makes the following extremal concepts reduce to their standard definitions in the undirected case.

DEFINITION 9.6. *Given two GDHs G and H and a function $\psi : V_H \to V_G$, we say that ψ is a* homomorphism *if for all $(a_1, \ldots, a_r) \in E_H$, $(\psi(a_1), \ldots, \psi(a_r)) \in E_G$.*

We say that G contains a copy of H if there exists some injective homomorphism, $\psi : V_H \to V_G$. Otherwise, we say that G is H-free. Similarly, we would say that a GDH G is $\mathcal{F}$-free for some family $\mathcal{F}$ of GDHs if G is F-free for all $F \in \mathcal{F}$.

DEFINITION 9.7. *Given a family of GDHs $\mathcal{F}$ and a positive integer n, let the nth* extremal number*, $ex_T(n, \mathcal{F})$, be defined as the maximum number of edges over all $\mathcal{F}$-free GDHs on n elements,*

$$ex_T(n, \mathcal{F}) := \max_{\mathcal{F}\text{-free } G_n} \{e_T(G_n)\}.$$

The Turán density *of $\mathcal{F}$ is defined as*

$$\pi_T(\mathcal{F}) := \lim_{n\to\infty} \frac{ex_T(n,\mathcal{F})}{\frac{r!}{m_T}\binom{n}{r}}.$$

Our first main result is to show that these Turán densities exist for any GDH theory. The proof is the standard averaging argument used to show that these limiting densities exist for families of undirected hypergraphs [29].

THEOREM 9.1. *For any GDH family $\mathcal{F}$ the Turán density exists.*

PROOF. Let G be an $\mathcal{F}$-free GDH on n elements with $\mathrm{ex}_T(n,\mathcal{F})$ edges. For each $i = 1,\dots,n$ let G^i be the subGDH of G induced by removing the ith vertex. Each edge of G appears in exactly $n-r$ of these subGDHs. Therefore,

$$(n-r)e_T(G) = e_T(G^1) + \cdots e_T(G^n).$$

Moreover, $e_T(G) = \mathrm{ex}_T(n,\mathcal{F})$ and each G^i is also $\mathcal{F}$-free so $e_T(G^i) \le \mathrm{ex}_T(n-1,\mathcal{F})$. Therefore,

$$\mathrm{ex}_T(n,G) \le \frac{n}{n-r}\mathrm{ex}_T(n-1,\mathcal{F}).$$

So

$$\frac{\mathrm{ex}_T(n,G)}{\frac{r!}{m_T}\binom{n}{r}} \le \frac{n}{n-r}\frac{\mathrm{ex}_T(n-1,\mathcal{F})}{\frac{r!}{m_T}\binom{n}{r}} = \frac{\mathrm{ex}_T(n-1,\mathcal{F})}{\frac{r!}{m_T}\binom{n-1}{r}}.$$

Therefore, the sequence of these extremal densities is monotone decreasing as a function of n in the range $[0,1]$. Hence, the limit exists. ■

1.1. Blowups and blowup density. We'll now extend the concept of the blowup of uniform hypergraphs to the more general setting of GDHs and define the corresponding notion of the blowup density. As with hypergraphs, the blowup of a GDH can be thought of as the replacement of each vertex with many copies and taking all of the resulting edges. Formally,

DEFINITION 9.8. *Let G be a GDH with $V_G = \{x_1,\dots,x_n\}$, and let $t = (t_1,\dots,t_n)$ be a tuple of positive integers. Define the t*-blowup *of G to be the $\mathcal{L}$-structure $G(t)$*

where

$$V_{G(t)} = \{x_{11}, \ldots, x_{1t_1}, \ldots, x_{n1}, \ldots, x_{nt_n}\}$$

and

$$(x_{i_1 j_1}, \ldots, x_{i_r j_r}) \in E_{G(t)} \iff (x_{i_1}, \ldots, x_{i_r}) \in E_G.$$

PROPOSITION 9.9. *Let G be a GDH on n vertices, and let $t = (t_1, \ldots, t_n)$ be a tuple of positive integers. Then the t-blowup of G is also a GDH.*

PROOF. We need only show that the $\mathcal{L}$-structure $G(t)$ models T. So let

$$(x_{i_1 j_1}, \ldots, x_{i_r j_r}) \in E_{G(t)}.$$

Then $(x_{i_1}, \ldots, x_{i_r}) \in E_G$. Since $G \models T$ this implies that $i_a \neq i_b$ whenever $a \neq b$. Hence, the elements $x_{i_a j_a} \neq x_{i_b j_b}$ whenever $a \neq b$. It also implies that $(x_{i_{\pi(1)}}, \ldots, x_{i_{\pi(r)}}) \in E_G$ for any $\pi \in J_T$. Hence,

$$(x_{i_{\pi(1)} j_{\pi(1)}}, \ldots, x_{i_{\pi(r)} j_{\pi(r)}}) \in E_{G(t)}$$

for any $\pi \in J_T$. Therefore, $G(t) \models T$. ■

Next, we consider the edge density of a given blowup by defining the edge polynomial for a GDH.

DEFINITION 9.10. *Let G be a GDH on n vertices. For each r-set $R \in \binom{V_G}{r}$, let e_R be the number of edges of G in R. Then let the* edge polynomial *be*

$$p_G(x) := \sum_{R \in \binom{V_G}{r}} e_R \prod_{i \in R} x_i.$$

This polynomial is a simple generalization of the standard edge polynomial for undirected hypergraphs. To see this more easily note that for a given GDH G, the edges of G are in bijection with the monomials the sum p_G were we to write the sum out with no coefficients greater than one.

From this we see that the edge density of the $(t_1, \ldots, t_n)$-blowup of G is

$$\frac{p_G(t_1, \ldots, t_n)}{\frac{r!}{m_T}\binom{t}{r}} = m_T \frac{p_G(t_1, \ldots, t_n)}{t(t-1)\cdots(t-r+1)}$$

where $t = \sum t_i$. Let t increase to infinity and for each t pick a vector $(t_1, \ldots, t_n)$ that maximizes this edge density. Then this sequence of densities is asymptotically equivalent to the sequence of numbers

$$m_T p_G\left(\frac{t_1}{t}, \ldots, \frac{t_n}{t}\right).$$

This motivates the following definition.

DEFINITION 9.11. *Let G be a GDH on n vertices. Let*

$$S^n = \left\{(x_1, \ldots, x_n) | x_i \geq 0 \wedge \sum_{i=1}^{n} x_i = 1\right\},$$

the standard $(n-1)$-dimensional simplex. Define the blowup density *of G as*

$$b_T(G) = m_T \max_{x \in S^n}\{p_G(x)\}.$$

Since any $x \in S^n$ is the limit of some sequence $\left\{\left(\frac{t_1}{t}, \ldots, \frac{t_n}{t}\right)\right\}$ with positive t_i as $t \to \infty$, then the blowup density of a GDH G is the best limiting density of any sequence of blowups of G.

The remaining definition and basic result about blowups given in this subsection will be useful when extending results about jumps and nonjumps from undirected hypergraphs to GDHs generally in Section 3.

DEFINITION 9.12. *Let T' and T be GDH theories such that $J_{T'} \subseteq J_T \subseteq S_r$. For a T-graph F and a T'-graph F' we say that F* contains *F' if $V_F = V_{F'}$ and every edge of F' is contained in some edge of F (where the edges are considered under their equivalence class definition as subsets of E_F and $E_{F'}$). We say that F is the* minimum T-container *of F' if F has no edges that do not contain edges of F'.*

PROPOSITION 9.13. *Let T' and T be GDH theories such that $J_{T'} \subseteq J_T \subseteq S_r$. Let F' be a T'-graph and let F be the minimum T-container of F'. Then*

$$\frac{m_{T'}}{m_T} b_T(F) \leq b_{T'}(F') \leq b_T(F)$$

with equality on the left if F' has exactly one edge contained in each edge of F and equality on the right if each edge of F contains all $\frac{m_T}{m_{T'}}$ possible edges of F'.

Moreover, if F' has exactly k edges contained in each edge of F, then

$$b_{T'}(F') = \frac{k m_{T'}}{m_T} b_T(F).$$

PROOF. Let $|V_{F'}| = |V_F| = v$, then for any $x \in S^v$,

$$p_F(x) \leq p_{F'}(x) \leq \frac{m_T}{m_{T'}} p_F(x)$$

with equality on the left if F' has exactly one edge contained in each edge of F and equality on the right if each edge of F contains all $\frac{m_T}{m_{T'}}$ possible edges of F'. Hence,

$$\max_{x \in S^v} p_F(x) \leq \max_{x \in S^v} p_{F'}(x) \leq \max_{x \in S^v} \frac{m_T}{m_{T'}} p_F(x).$$

This implies that

$$\frac{m_{T'}}{m_T} b_T(F) \leq b_{T'}(F') \leq b_T(F).$$

In particular, if F' has exactly k edges contained in each edge of F, then for any $x \in S^v$,

$$p_{F'}(x) = k p_F(x)$$

which implies the result. ■

1.2. Supersaturation and related results. Supersaturation holds for GDHs as it does for undirected hypergraphs, and the proof of this result is the same as the one for hypergraphs found in [29] with only minor differences.

THEOREM 9.2 (Supersaturation). *Let F be a GDH on k elements. Let $\epsilon > 0$. For sufficiently large $n \geq n_0(F, \epsilon)$, any GDH G on n elements with density $d(G) \geq \pi_T(F) + \epsilon$ will contain at least $c\binom{n}{k}$ copies of F for some constant $c = c(F, \epsilon)$.*

PROOF. Fix some positive integer l so that

$$\mathrm{ex}_T(l, F) < \left(\pi_T(F) + \frac{\epsilon}{2}\right) \frac{r!}{m_T} \binom{l}{r}.$$

Let G be a GDH on $n > l$ elements with edge density $d_T(G) \geq \pi(F) + \epsilon$. Then G must contain more than $\frac{\epsilon}{2}\binom{n}{l}$ l-sets with density at least $\pi_T(F) + \frac{\epsilon}{2}$. Otherwise, at most $\frac{\epsilon}{2}\binom{n}{l}$ l-sets contain more than $\left(\pi_T(F) + \frac{\epsilon}{2}\right)\binom{l}{r}$ edges. Therefore, we can count the number of edges in G by l-sets and get an upper bound of

$$\binom{n-r}{l-r} e_T(G) \leq \frac{\epsilon}{2}\binom{n}{l}\binom{l}{r}\frac{r!}{m_T} + \left(1 - \frac{\epsilon}{2}\right)\binom{n}{l}\left(\pi_T(F) + \frac{\epsilon}{2}\right)\binom{l}{r}\frac{r!}{m_T}.$$

We can now replace $e_T(G)$ since

$$e_T(G) \geq (\pi_T(F) + \epsilon)\binom{n}{r}\frac{r!}{m_T}.$$

This is enough to get the contradiction.

Since G contains more than $\frac{\epsilon}{2}\binom{n}{l}$ l-sets with density at least $\pi_T(F)+\frac{\epsilon}{2}$, then it contains a copy of F in each. A given copy of F appears in $\binom{n-k}{l-k}$ l-sets of G. Therefore, there are more than

$$\frac{\epsilon}{2}\binom{n}{l}\binom{n-k}{l-k}^{-1} = c\binom{n}{k}$$

distinct copies of F in G where

$$c = \frac{\epsilon}{2}\binom{l}{k}^{-1}.$$

■

Similarly, the following theorem is an extension from the same result for undirected hypergraphs, and the proof is an adaptation of the one found in [29].

THEOREM 9.3. *Let F be a GDH on k vertices and let $t = (t_1, \ldots, t_k)$ be an k-tuple of positive integers. Then $\pi_T(F) = \pi_T(F(t))$.*

PROOF. That $\pi_T(F) \leq \pi_T(F(t))$ is trivial since $F(t)$ contains a copy of F so any F-free GDH is automatically $F(t)$-free.

Therefore, we only need to show that $\pi_T(F) \geq \pi_T(F(t))$. Suppose not, then for sufficiently large n there exists some $F(t)$-free GDH G on n elements with edge density strictly greater than $\pi_T(F)$. By supersaturation this implies that G contains $c\binom{n}{k}$ copies of F.

Define G^* to be the k-uniform hypergraph where $V_{G^*} = V_G$ and $\{a_1, \ldots, a_k\} \in E_{G^*}$ iff and only if $\{a_1, \ldots, a_k\}$ contains a copy of F in G. Since the edge density of G^* is $c > 0$, then for large enough n, G^* must contain an arbitrarily large complete k-partite subgraph.

For each edge F maps to the vertices in at least one out of $k!$ total possible ways to make an injective homomorphism in G. Therefore, by Ramsey Theory, if we take the parts of this complete k-partite subgraph large enough and color the edges by the finite number of non-isomorphic ways that F could possibly map to the k vertices, we will get an arbitrarily large monochromatic k-partite subgraph where each part has t vertices. This must have been a copy of $F(t)$ in G, a contradiction. ∎

The fact that the Turán density of a blowup equals the Turán density of the original GDH leads to the following nice characterization of degenerate families of GDH - those families with Turán density zero.

THEOREM 9.4 (Characterization of Degenerate GDH). *Let $\mathcal{F}$ be some family of GDHs, then $\pi_T(\mathcal{F}) = 0$ if and only if some member $F \in \mathcal{F}$ is a subGDH of the t-blowup of a single edge for some vector, $t = (t_1, \ldots, t_r)$, of positive integers. Otherwise, $\pi(\mathcal{F}) \geq \frac{m_T}{r^r}$.*

PROOF. Suppose that no member of $\mathcal{F}$ is such a blowup. Then no member is contained in the $(t, t, \ldots, t)$-blowup of S. Let $S(t)$ stand for this blowup, then the sequence of GDHs, $\{S(t)\}_{t=1}^{\infty}$, is an $\mathcal{F}$-free sequence. The density of any such $S(t)$ is

$$d_T(S(t)) = \frac{t^r}{\frac{r!}{m_T}\binom{tr}{r}} = \frac{m_T t^r (tr-r)!}{(tr)!}.$$

These densities tend to $\frac{m_T}{r^r}$ as t increases. Therefore,

$$\pi_T(\mathcal{F}) \geq \frac{m_T}{r^r} > 0.$$

Conversely, suppose some $F \in \mathcal{F}$ is a $(t_1, \ldots, t_r)$-blowup of a single edge. By Theorem 9.3, $\pi_T(F) = \pi_T(S) = 0$ since $\mathrm{ex}_T(n, S)$=0 for all n. Therefore, $\pi_T(\mathcal{F}) = 0$. ■

2. Jumps

Now we turn to the issue of finding jumps and nonjumps for GDH theories. The definition of a jump for undirected hypergraphs extends naturally to this setting as does the important connection between jumps and blowup densities.

DEFINITION 9.14. *Let T be a GDH theory, then $\alpha \in [0, 1)$ is a* jump *for T if there exists a $c > 0$ such that for any $\epsilon > 0$ and any positive integer l, there exists a positive integer $n_0(\alpha, \epsilon, l)$ such that any GDH G on $n \geq n_0$ elements that has at least $(\alpha + \epsilon)\frac{r!}{m_T}\binom{n}{r}$ edges contains a subGDH on l elements with at least $(\alpha + c)\frac{r!}{m_T}\binom{l}{r}$ edges.*

Note that by Theorem 9.4 every $\alpha \in \left[0, \frac{m_T}{r^r}\right)$ is a jump for any r-ary GDH theory T. This generalizes the well-known result of Erdős [18] that every $\alpha \in [0, \frac{r!}{r^r})$ is a jump for r-graphs. The following important theorem on jumps for GDH theories was originally shown by Frankl and Rödl [27] for undirected hypergraphs. Their proof works equally well in this setting so the differences here are in name only.

THEOREM 9.5. *The GDH theory T has a jump α if and only if there exists a finite family $\mathcal{F}$ of GDHs such that $\pi_T(\mathcal{F}) \leq \alpha$ and $b_T(F) > \alpha$ for each $F \in \mathcal{F}$.*

PROOF. Let α be a jump and let c be the supremum of all corresponding lengths c to the jump. Fix a positive integer k so that

$$\binom{k}{r}\left(\alpha + \frac{c}{2}\right) > \alpha \frac{k^r}{r!}.$$

Let $\mathcal{F}$ be the family of all GDHs on k elements with at least $\left(\alpha + \frac{c}{2}\right)\binom{k}{r}\frac{r!}{m_T}$ edges. Then $\pi_T(\mathcal{F}) \leq \alpha$ since any slightly larger density implies arbitrarily large subsets with density $\alpha + c$. This in turn would imply the existence of a k-subset with density at least $\alpha + c$. This k-subset would include some member of $\mathcal{F}$. On the other hand, a given $F \in \mathcal{F}$ will have blowup density

$$b_T(F) \geq m_T p_F\left(\frac{1}{k}, \ldots, \frac{1}{k}\right) > \alpha.$$

Conversely, suppose that such a finite family $\mathcal{F} = \{F_1, \ldots, F_k\}$ exists. Let $\epsilon > 0$ and let $\{G_n\}$ be an infinite sequence of GDHs with density that tends to $\alpha + \epsilon$. As in the proof of Theorem 9.2, for any positive integer l, G_n must contain at least $\frac{\epsilon}{2}\binom{n}{l}$ l-subsets with density at least $\alpha + \frac{\epsilon}{2}$.

Let l be large enough so that any GDH on l vertices with density at least $\alpha + \frac{\epsilon}{2}$ contains some F_i from $\mathcal{F}$. Therefore, any G_n with $n > l$ contains $\frac{\epsilon}{2}\binom{n}{l}$ l-sets each with some F_i. Since there are only k members of $\mathcal{F}$, then this implies that at least $\frac{\epsilon}{2k}\binom{n}{l}$ l-sets contain the same F_i.

Let $|V(F_i)| = v_i$. By the proof of Theorem 9.2 this implies that there is some positive constant b such that G_n contains at least $b\binom{n}{v_i}$ distinct copies of F_i. By the proof of Theorem 9.3 this shows that if n is large enough, then we get a copy of an arbitrarily large t-blowup of F_i.

Let $c = \min_{F_i \in \mathcal{F}} b_T(F_i)$. For some subset $\mathcal{F}' \subseteq \mathcal{F}$, each $F_i \in \mathcal{F}'$ yields an infinite subsequence of $\{G_n\}$ which contains arbitrarily large t-blowups of F_i. The densities of these blowups all tend to at least c. Therefore, for any positive integer m, there exists an m-set of each $\{G_n\}$ for sufficiently large n with density at least $\alpha+c$. Hence, α is a jump. ■

The following proposition is needed to compare jumps between different GDH theories.

PROPOSITION 9.15. *The GDH theory T has a jump α if and only if there exists some $c > 0$ such that for all families $\mathcal{F}$ of GDHs, either $\pi_T(\mathcal{F}) \leq \alpha$ or $\pi_T(\mathcal{F}) \geq \alpha+c$.*

PROOF. Let α be a jump for T and let $c > 0$ be some corresponding length to the jump. Suppose that $\mathcal{F}$ is a finite family of GDHs of type T for which $\alpha < \pi_T(\mathcal{F}) < \alpha + c$. Let $\{G_n\}$ be a sequence of extremal $\mathcal{F}$-free GDHs. For each positive integer k there exists some G_n that contains a k-subset with at least $(\alpha + c)\binom{k}{r}\frac{r!}{m_T}$ edges. Take the sequence of these subsets. They are all $\mathcal{F}$-free by assumption, and the limit of their densities is at least $\alpha + c$. Therefore, $\pi_T(\mathcal{F}) \geq \alpha + c$, a contradiction.

Conversely, assume that α is not a jump. Let $c > 0$, then for some $0 < \epsilon < c$ and some positive integer l, there exists an infinite sequence of GDHs, $\{G_n\}$ for which each GDH has density at least $\alpha+\epsilon$ and all l-sets have strictly less than $(\epsilon+c)\binom{l}{r}\frac{r!}{m_T}$ edges. Hence, $\{G_n\}$ is $\mathcal{F}$-free where $\mathcal{F}$ is the set of all l-GHDs with at least $(\alpha + c)\binom{l}{r}\frac{r!}{m_T}$ edges. So $\pi_T(\mathcal{F}) \geq \alpha + \epsilon$. Since any GDH with density at least $\alpha + c$ must have an l-set with density at least $\alpha + c$, then $\pi_T(\mathcal{F}) < \alpha + c$. ■

We will now look at how jumps are related between two different GDH theories for some fixed edge size r. We will see that in general jumps always "pass up" the subgroup lattice. That is, if $J_{T'} \subseteq J_T$ for GDH theories T' and T, then a jump for T' is a jump for T. The converse is not true in general. In fact, for any GDH theories T' and T with $J_{T'} \subseteq J_T$ such that the order of J_T is at least three times that of $J_{T'}$ we will show that the set of jumps for T' is not equal to the set of jumps for T. The case where $m_T = 2m_{T'}$ is open.

2.1. Jumps pass up the lattice. First, we will show that for GDH theories T and T' with $J_{T'} \subseteq J_T$ the set of Turán densities of forbidden families of T-graphs is a subset of the set of Turán densities for T'.

THEOREM 9.6. *Let T and T' be two GDH theories such that $J_{T'} \subseteq J_T$. Then for any family $\mathcal{F}$ of T-graphs there exists a family $\mathcal{F}'$ of T'-graphs for which $\pi_{T'}(\mathcal{F}') = \pi_T(\mathcal{F})$. Moreover, if $\mathcal{F}$ is a finite family, then $\mathcal{F}'$ is also finite.*

PROOF. For each $F \in \mathcal{F}$ let $F_{T'}$ be the set of all T'-graphs that have exactly one edge contained in every edge of F. That is, since $J_{T'} \subseteq J_T$, then there are $\frac{m_T}{m_{T'}}$ possible T' edges contained within one T edge. So $F_{T'}$ is a finite set with at most $\left(\frac{m_T}{m_{T'}}\right)^{e_T(F)}$ members. Let

$$\mathcal{F}' = \bigcup_{F \in \mathcal{F}} F_{T'}.$$

Then $\mathcal{F}'$ is a family of T'-graphs. Moreover, $\mathcal{F}'$ is finite if $\mathcal{F}$ is finite. We want to show that $\pi_{T'}(\mathcal{F}') = \pi_T(\mathcal{F})$.

First, let $\{G'_n\}$ be an extremal $\mathcal{F}'$-free sequence of T'-graphs. For each G'_n let G_n be the T-graph constructed by replacing each T'-edge of G'_n with its containing T-edge (multiple T'-edges could correspond to the same T-edge but each T-edge can only be added once).

The sequence $\{G_n\}$ is $\mathcal{F}$-free since otherwise some G_n contains some $F \in \mathcal{F}$ which means that G'_n must have contained at least one member of $F_{T'}$. Therefore,

$$\pi_T(\mathcal{F}) \geq \lim_{n\to\infty} d_T(G_n) \geq \lim_{n\to\infty} \frac{\frac{m_{T'}}{m_T} e_{T'}(G'_n)}{\frac{r!}{m_T}\binom{n}{r}} = \lim_{n\to\infty} \frac{ex_{T'}(n, \mathcal{F}')}{\frac{r!}{m_{T'}}\binom{n}{r}} = \pi_{T'}(\mathcal{F}').$$

Conversely, now let $\{G_n\}$ be an extremal $\mathcal{F}$-free sequence of T-graphs. For each G_n construct a T'-graph G'_n by replacing each T-edge with all $\frac{m_T}{m_{T'}}$ T'-edges contained in it. The sequence $\{G'_n\}$ is $\mathcal{F}'$-free with $\frac{m_T}{m_{T'}}\mathrm{ex}_T(n, \mathcal{F})$ edges. Therefore,

$$\pi_{T'}(\mathcal{F}') \geq \lim_{n\to\infty} \frac{\frac{m_T}{m_{T'}}\mathrm{ex}_T(n, \mathcal{F})}{\frac{r!}{m_{T'}}\binom{n}{r}} = \pi_T(\mathcal{F}).$$

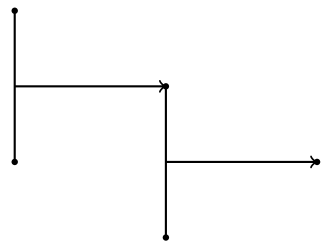


FIGURE 40. $\pi(F) = \frac{4}{27}$.

So $\pi_{T'}(\mathcal{F}') = \pi_T(\mathcal{F})$. ■

The converse of Theorem 9.6 is false in general. For example, the permutation subgroup for the theory T' of $(2 \to 1)$-uniform directed hypergraphs is a subgroup of the permutation group for the theory T of undirected 3-graphs, S_3. The extremal number for the directed hypergraph R_4 (see Chapter 2) is

$$ex_{T'}(n, R_4) = \left\lfloor \frac{n}{3} \right\rfloor \binom{\lceil \frac{2n}{3} \rceil}{2}.$$

Therefore, the Turán density is $\pi_{T'}(R_4) = \frac{4}{27}$. However, it is well-known that no Turán densities exist for 3-graphs in the interval $\left(0, \frac{6}{27}\right)$.

COROLLARY 9.16. *Let T and T' be two GDH theories such that $J_{T'} \subseteq J_T$. If α is a jump for T', then it is also a jump for T.*

PROOF. If α is not a jump for T, then for any $c > 0$ there exists by Proposition 9.15 a family $\mathcal{F}$ such that $\alpha < \pi_T(\mathcal{F}) < \alpha + c$. So by Theorem 9.6 there exists a family $\mathcal{F}'$ of T'-graphs with $\alpha < \pi_{T'}(\mathcal{F}') < \alpha + c$. So α is not a jump for T'. ■

Corollary 9.16 immediately implies that all nonjumps found for r-uniform undirected hypergraphs must also be non-jumps for any GDH with an r-ary relation. However, the converse is not true in general.

2.2. Jumps do not pass down the lattice. Roughly speaking, the current best method of demonstrating that a particular α is not a jump for r-uniform hypergraphs is to construct a sequence of hypergraphs each with blowup densities that are strictly larger than α but for which any relatively small subgraph has blowup density at most α. This method originated in [27] and generalizes to GDHs as the following definition and lemma demonstrate.

DEFINITION 9.17. *Let $\alpha \in [0, 1)$. Call α a* demonstrated nonjump *for a GDH theory T if there exists an infinite sequence of GDHs, $\{G_n\}$, such that $b_T(G_n) > \alpha$ for each G_n in the sequence and for any positive integer l there exists a positive integer n_0 such that whenever $n \geq n_0$ then any subGDH $H \subseteq G_n$ on l or fewer vertices has blowup density $b_T(H) \leq \alpha$.*

LEMMA 9.18. *Every demonstrated nonjump is a nonjump.*

PROOF. Suppose not. Assume that α is a demonstrated nonjump but is a jump. Then there exists a finite family of GDHs $\mathcal{F}$ such that $\pi_T(\mathcal{F}) \leq \alpha$ and $b_T(F) > \alpha$ for each $F \in \mathcal{F}$. Let l be the maximum number of vertices over the members of $\mathcal{F}$. Let n be large enough so that any subGDH on l or fewer vertices has blowup density at most α. Then some large enough blowup of G_n contains some $F \in \mathcal{F}$ as a subGDH since the blowup density of each G_n tends to something strictly greater than α. Let H be the minimal subGDH of G_n for which the corresponding blowup contains this copy of F. Since H has at most l vertices, then it has a blowup density at most α. Hence,

$$b_T(F) \leq b_T(H(t)) \leq b_T(H) \leq \alpha,$$

a contradiction. ■

We can now show that a demonstrated nonjump for a GDH theory T yields multiple nonjumps of equal and lesser values down the lattice to GDH theories T' for which $J_{T'} \subseteq J_T$.

THEOREM 9.7. *Let T and T' be GDH theories such that $J_{T'} \subseteq J_T$. Let α be a demonstrated nonjump for T. Then $\frac{km_{T'}}{m_T}\alpha$ is a demonstrated nonjump for T' for $k = 1, \ldots, \frac{m_T}{m_{T'}}$.*

PROOF. Let α be a demonstrated nonjump for T. Let $\{G_n\}$ be the corresponding infinite sequence of GDHs. Fix some $k \in \{1, \ldots, \frac{m_T}{m_{T'}}\}$. For each n let G'_n be a T'-graph constructed from G_n by replacing each T-edge with k T'-edges in any orientation. Then by Proposition 9.13 we know that

$$b_{T'}(G'_n) = \frac{km_{T'}}{m_T} b_T(G_n)$$

and any $H' \subseteq G'_n$ corresponding to $H \subseteq G_n$ also gives:

$$b_{T'}(H') = \frac{km_{T'}}{m_T} b_T(H).$$

Therefore, $b_{T'}(G'_n) > \frac{km_{T'}}{m_T}\alpha$ for each n and for any positive integer l, there exists a n_0 such that $b_{T'}(H) \leq \alpha$ for any subGDH $H \subseteq G_n$ for all $n \geq n_0$. ∎

Constructions of sequences of undirected r-graphs which show that $\frac{5r!}{2r^r}$ is a demonstrated nonjump for each $r \geq 3$ were given in [26]. This gives the following corollary.

COROLLARY 9.19. *Let T be an r-ary GDH theory for $r \geq 3$. Then $\frac{5m_T k}{2r^r}$ is a nonjump for T for $k = 1, \ldots, \frac{r!}{m_T}$.*

This in turn shows that the set of jumps for a theory T' is a proper subset of the set of jumps for T for any T such that $J_{T'} \subseteq J_T$ and $m_T \geq 3m_{T'}$.

COROLLARY 9.20. *Let T and T' be r-ary GDH theories such that $J_{T'} \subseteq J_T$ and $m_T \geq 3m_{T'}$. Then there exists an α that is a nonjump for T' and a jump for T.*

PROOF. Take $k = 1$, then $\frac{5m_{T'}}{2r^r}$ is a nonjump for T'. Since $m_T \geq 3m_{T'}$, then $m_T > 2.5m_{T'}$. So

$$\frac{5m_{T'}}{2r^r} < \frac{m_T}{r^r}.$$

Therefore, $\frac{5m_{T'}}{2r^r}$ is a jump for T since every $\alpha \in \left[0, \frac{m_T}{r^r}\right)$ is a jump for T. ∎

3. Continuity and approximation

The following two results are direct adaptations of two theorems from [7]. They are both general extremal results related to everything discussed in this chapter but did not fit nicely into the other sections. The first result on continuity relates extremal numbers of any infinite family of GDHs to the extremal numbers of its finite subfamilies. The second result on approximation discusses structural aspects of (nearly) extremal sequences for any forbidden family.

THEOREM 9.8 (Continuity). *Let $\mathcal{F}$ be an infinite family of T-graphs. For each $\epsilon > 0$ there exists a finite subfamily $\mathcal{F}_\epsilon \subset \mathcal{F}$ such that*

$$ex_T(n, \mathcal{F}) \leq ex_T(n, \mathcal{F}_\epsilon) < ex_T(n, \mathcal{F}) + \epsilon n^r$$

for sufficiently large n.

PROOF. Let $\mathcal{F}$ be the infinite family of GDHs. For each positive integer k let $\mathcal{F}_k$ be the subfamily of $\mathcal{F}$ where each member has at most k vertices. Let

$$\gamma_k = \lim_{n\to\infty} \frac{\mathrm{ex}_T(n, \mathcal{F}_k)}{\frac{r!}{m_T}\binom{n}{r}}$$

and let

$$\gamma = \lim_{n\to\infty} \frac{\mathrm{ex}_T(n, \mathcal{F})}{\frac{r!}{m_T}\binom{n}{r}}.$$

Since $\mathcal{F}_k \subset \mathcal{F}$, then $\{\gamma_k\}_{k=1}^{\infty}$ is a monotone decreasing sequence and $\gamma_k \geq \gamma$ for all k. Assume for some $\epsilon > 0$ that $\gamma_k > \gamma + \epsilon$ for all k. Note that

$$\frac{\mathrm{ex}_T(n, \mathcal{F}_k)}{\frac{r!}{m_T}\binom{n}{r}} \geq \gamma_k$$

is true for all n. In particular, when $n = k$ there is an $\mathcal{F}_n$-free GDH on n vertices with strictly more than $(\gamma + \epsilon)\frac{r!}{m_T}\binom{n}{r}$ edges. Since an $\mathcal{F}_n$-free GDH on n vertices is also necessarily $\mathcal{F}$-free, then this implies that

$$\mathrm{ex}_T\,(n, \mathcal{F}) > (\gamma + \epsilon)\,\frac{r!}{m_T}\binom{n}{r},$$

a contradiction. ■

Theorem 6 in [7] is the Approximation Theorem for totally directed r-uniform hypergraphs with bounded multiplicity. We will use the following equivalent statement (in the case of multiplicity one) written in terms of Turán densities as a lemma to prove that this approximation result holds for all GDHs.

LEMMA 9.21. *Let $\mathcal{F}'$ be a family of forbidden totally directed r-graphs (r-GDHs under the trivial group), and let $\epsilon > 0$. Then there exists some totally directed r-graph G' such that every blowup of G' is $\mathcal{F}'$-free and*

$$\pi(\mathcal{F}') \geq b(G) > \pi(\mathcal{F}') - \epsilon.$$

THEOREM 9.9 (Approximation). *Let $\mathcal{F}$ be a family of forbidden T-graphs, and let $\epsilon > 0$, then there exists some T-graph G for which all blowups of G are $\mathcal{F}$-free and*

$$\pi_T(\mathcal{F}) \geq b_T(G) > \pi_T(\mathcal{F}) - \epsilon.$$

PROOF. Let $\mathcal{F}'$ be the family of totally directed r-graphs as defined in the proof of Theorem 9.6. That is, the family of directed hypergraphs for which we know that $\pi(\mathcal{F}') = \pi_T(\mathcal{F})$. We know from the proof of that theorem that any T-graph that is the minimal container for an $\mathcal{F}'$-free graph is $\mathcal{F}$-free. By Lemma 9.21 there exists some totally directed $\mathcal{F}'$-free r-graph , G', such that

$$\pi(\mathcal{F}') \geq b(G) > \pi(\mathcal{F}') - \epsilon.$$

By Proposition 9.13 we know that if G' is the minimal containing T-graph of G, then $b_T(G) \geq b(G')$. Hence,

$$\pi_T(\mathcal{F}) \geq b_T(G) \geq b(G') > \pi(\mathcal{F}') - \epsilon = \pi_T(\mathcal{F}) - \epsilon.$$

■

CHAPTER 10

Additional Questions about Directed Hypergraphs

This chapter contains material from a paper published by the Electronic Journal of Combinatorics. [12]

There are many additional questions that we can ask about $2 \to 1$ directed hypergraphs and about GDHs in general. In this chapter, we will briefly review several open questions that come up naturally in this work.

1. Extremal numbers for tournaments

Brown and Harary [6] started studying extremal problems for directed 2-graphs by determining the extremal numbers for many "small" digraphs and for some more general types of digraphs such as tournaments - a digraph where every pair of vertices has exactly one directed edge. We could follow their plan of attack in studying the $2 \to 1$ model and look for the extremal numbers of tournaments. Here, a tournament could be defined as a graph with exactly one directed edge on every three vertices. In particular, a transitive tournament might be an interesting place to begin. A transitive tournament is a tournament where the direction of each edge is based on an underlying linear ordering of the vertices as in the oriented lower bound construction of Theorem 4.2.

Denote the $2 \to 1$ transitive tournament on k vertices by TT_k. Since the "winning" vertex of the tournament will have a complete K_{k-1} as its tail link graph, then any H on n vertices for which each T_x is K_{k-1}-free must be TT_k-free. Therefore,

$$n\left(\frac{n-1}{k-2}\right)^2\binom{k-2}{2} \leq \mathrm{ex}(n, TT_k), \mathrm{ex}_o(n, TT_k).$$

This also immediately shows that the transitive tournament on four vertices with the "bottom" edge removed has this extremal number exactly.

THEOREM 10.1. *Let* TT_4^- *denote the graph with vertex set* $V(TT_4^-) = \{a, b, c, d\}$ *and edge set*

$$E(TT_4^-) = \{ab \to d, bc \to d, ac \to d\}.$$

Then

$$ex(n, TT_4^-) = n \left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil.$$

Is it still true if we add an edge to $\{a, b, c\}$?

CONJECTURE 10.1. *Let* TT_4 *denote the graph with vertex set* $V(TT_4) = \{a, b, c, d\}$ *and edge set*

$$E(TT_4) = \{ab \to d, bc \to d, ac \to d, ab \to c\}.$$

Then

$$ex(n, TT_4) = n \left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil.$$

2. GDHs with $r \to 1$ edges

The $2 \to 1$ directed hypergraph originally came to the author's attention as a way to model definite Horn clauses in propositional logic. Definite Horn clauses are more generally modeled by $r \to 1$ edges for any r. Therefore, it seems natural to ask about the extremal numbers for graphs with two $(r \to 1)$-edges. If we look at every $(r \to 1)$-graph with exactly two edges, then we see that these fall into four main types of graph. Let i be the number of vertices that belong to the tail set of both edges. Then let $I_r(i)$ denote the graph where both edges point to the same head vertex, let $H_r(i)$ denote the graph where the edges point to different head vertices neither of which are in the tail set of the other, let $R_r(i)$ denote the graph where the first edge points to a head vertex in the tail set of the second edge and the second edge points to a head not in the tail set of the first edge, and let $E_r(i)$ denote the graph where both edges point to heads in the tail sets of each other.

This extends the notation used in this paper. The degenerate cases are generalized to $I_r(i)$ and $H_r(i)$, and the nondegenerate cases generalize to $R_r(i)$ and $E_r(i)$. For example, the 3-resolvent R_3 is $R_2(1)$. The split between degenerate and nondegenerate is maintained in this way as well as shown in Theorem 9.4.

To what extent do the proofs presented in this paper extend to these graphs? Some translate immediately. For example, in the standard version of the problem it can easily be seen that

$$\mathrm{ex}(n, I_r(0)) = n\binom{n-2}{r-1}$$

using Erdős-Ko-Rado [21] for the upper bound and the same basic construction for the lower bound that we used in proving the same result for I_0. More generally, we can get an upper bound of

$$\mathrm{ex}(n, I_r(i)) \leq n\binom{n-1}{r-1}$$

by applying the uniform Ray-Chaudhuri - Wilson Theorem [39] to the tail link graph of each vertex of an $I_r(i)$-free graph. We can get a general lower bound of

$$n\binom{n-i-2}{r-i-1} \leq \mathrm{ex}(n, I_r(i))$$

by constructing an $I_r(i)$-free graph in the following way: for each vertex x fix a set of $i+1$ vertices not including x, C_x, and then add every possible edge with x at the head and C_x in the tail set.

An easy lower bound construction for an $H_r(i)$-free graph is to fix a vertex x and take all possible edges that point to it giving

$$\binom{n-1}{r} \leq \mathrm{ex}(n, H_r(i)).$$

To get an upper bound also on the order of n^r note that we can extend the concept of the directed link graph to apply to more than one vertices. For instance, here let the directed link graph of a set of vertices A of cardinality i be the $(r-i) \to 1$

directed hypergraph on $n-i$ vertices, $V \setminus A$, for which every edge becomes an edge of the original $(r \to 1)$-graph when A is added to the tail set. In this case, no directed tail link graph for any set of i vertices can contain two independent directed edges. Therefore,

$$\mathrm{ex}(n, H_r(i)) \leq \frac{(r-i+1)\binom{n-i-1}{r-i}\binom{n}{i}}{\binom{r}{i}} = \frac{n(r-i+1)}{n-i}\binom{n-1}{r}.$$

It is easy to see that any $r \to 1$ transitive tournament on n vertices would be $E_r(i)$-free. This immediately solves the oriented version and gives a lower bound for the standard version:

$$\mathrm{ex}_o(n, E_r(i)) = \binom{n}{r+1}.$$

As in the first lower bound construction for E we can add r edges to the smallest $r+1$ vertices in the linear order given by the transitive tournament to get a few more edges in the standard case. Is this the best that we can do?

CONJECTURE 10.2.

$$ex(n, E_r(i)) = \binom{n}{r+1} + r.$$

For the generalized resolvent configurations, the lower bound constructions for R_3 and R_4 both generalize to the $r \to 1$ setting. When $i \geq 1$, then the construction that worked for R_3 gives the better lower bound. Split the vertices into two equal or almost equal parts and take all edges that point from an r-set in one to a vertex in the other. This gives

$$n\binom{\frac{n}{2}}{r} \leq \mathrm{ex}(n, R_r(i))$$

for $i \geq 1$. When $i = 0$, the same generalization of the construction for R_4 will produce an $R_r(0)$-free graph.

3. Differences between oriented and standard extremal numbers

It is interesting to look at the differences between the oriented and standard extremal problems for a given forbidden graph not only in their values but in the difficulty level of their proofs. For instance, the proof of the standard case of I_0 is quite easy while the proof of the oriented case took a lot of effort. For the Escher graph E the situation was reversed. What about the character of these two graphs determines that one version of the problem should be easy and the other difficult, and what is the difference between the two that swaps which version is which?

A more exact request is to ask for a characterization that determines the difference in the value. For instance, H_2, I_1, R_3, R_4, and the case of two completely overlapping edges each have oriented and standard numbers that are exactly the same while H_1 and I_0 each have differences that are linear in n, the Escher graph E has a constant difference, and the graph made up of two independent edges has a quadratic difference.

Of course, we get an immediate easy bound by observing that every non-oriented F-free graph contains an oriented F-free graph that can be arrived at by removing edges from each triple of vertices until only one remains. So

$$\mathrm{ex}(n, F) \leq 3\mathrm{ex}_o(n, F) \leq 3\mathrm{ex}(n, F)$$

for any forbidden graph F. The cases in this paper where the difference between the two numbers is zero shows that the upper bound is tight while the case of two independent edges shows that the lower bound is also tight.

But what causes the difference? Perhaps, it would be good to begin answering this question by narrowing the focus to nondegenerate graphs since in this paper almost every nondegenerate case had no difference in the values, and the only one that did had only constant difference. Will the difference always be at most constant or at least $o(n^3)$? No, any graph F that contains a triple with all three possible edges is certainly not degenerate, and the standard extremal number of F is at least twice as much as the oriented extremal number.

But what if we restrict ourselves further and only consider oriented nondegenerate forbidden graphs, then is

$$\mathrm{ex}(n, F) - \mathrm{ex}_o(n, F) = o(n^3)$$

for every oriented nondegenerate F? Dániel Gerbner and Balázs Keszegh produced an interesting counterexample to this claim as well. At this point it is unclear to the author what might be an appropriate characterization for graphs with small differences between extremal numbers.

4. General structural results

On a more general level we can ask about the structure of extremal $(2 \to 1)$-graphs. For instance, it was already shown in [32] that the 4-resolvent configuration R_4 has a stability result. Roughly speaking, R_4-free graphs with many edges differ only slightly from the given extremal construction. While we have shown that several of the extremal constructions in this paper are unique, we have not shown that any are stable.

Another avenue of research is to ask for canonical extremal structures. That is, for a forbidden graph F can we fix some constant r such that we can construct an F-free graph on n vertices such that the n vertices are partitioned into r parts and whether $xy \to z$ is an edge or not depends entirely on which parts x, y, and z are in? If we have a general r-part structure like this that is F-free for every n and the limit of the ratio of the number of edges given by the structure over $\mathrm{ex}(n, F)$ is one, then we call this a canonical F-free extremal structure. For instance, the Turán graphs are canonical extremal structures with respect to 2-graphs. Applying this idea to hypergraphs is already a major area of research (see [38]) so it seems likely that the question of whether every $(2 \to 1)$-graph has such a canonical extremal structure would be even more difficult.

5. GDH questions

It would be nice to show that the set of jumps for some GDH theory T' is a proper subset of the set of jumps of any theory T' up the lattice including those for which $m_T = 2m_{T'}$. Or on the other hand it would be very interesting to learn that this is not true in certain cases for $r \geq 3$!

CONJECTURE 10.3. *Let T' and T be r-ary GDH theories for $r \geq 3$ such that $J_{T'} \subseteq J_T$ and $m_T = 2m_{T'}$. Then there exists some $\alpha \in [0,1)$ for which α is a jump for T but not for T'.*

It is known by a result in [7] that every $\alpha \in [0,1)$ is a jump for digraphs. Therefore, the conjecture is not true when $r = 2$. On a related note, is it always true that when $J_{T'} \subset J_T$, there always exists a family $\mathcal{F}'$ of T'-graphs such that $\pi_{T'}(\mathcal{F}')$ is not contained in the set of Turán densities for T?

CONJECTURE 10.4. *Let T' and T be theories such that $J_{T'} \subseteq J_T$. Then there exists some family $\mathcal{F}'$ of T'-graphs such that $\pi_{T'}(\mathcal{F}')$ is not contained in the set of Turán densities for T.*

Finally, it would be nice to generalize the definition of a GDH to include other combinatorial structures. For instance we could easily change the current formulation to include multiple relations in order to capture nonuniform GDHs and those with edges that have bounded multiplicity like the structures studied in [7]. We could even allow these theories to contain general statements that relate the different relations. An example of this might be the theory of some kind of GDH with an edge-coloring that behaves in a certain way (at least locally). In another direction we could take away the requirement that all vertices of an edge be distinct to allow for kinds of generalized loops or add a condition that the existence of certain edges preclude the existence of others such as in the oriented cases studied here and in [33].

CHAPTER 11

Introduction to the (p, q)-coloring problem.

We will use the standard asymptotic notation in the following chapters. That is, for two functions, $f(n)$ and $g(n)$, we write $f = O(g)$ if there exists some constant c and some integer N such that $f(n) \leq cg(n)$ for all $n \geq N$. We write $f = o(g)$ if $f/g \to 0$ as $n \to \infty$. We write $f = \Omega(g)$ if $g = O(f)$ and $f = \omega(g)$ if $g = o(f)$. Finally, we write $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$.

Given two integers $s, t \geq 2$, the central question in classical Ramsey theory for graphs asks for the minimum number of vertices N for which any 2-coloring, say red and blue, of the edges of K_N must yield a red K_s or a blue K_t. We say that $N = R(s, t)$, the Ramsey number for s, t. This question generalizes to more than 2 colors in a natural way. That is, we let $R(s_1, \ldots, s_k)$ denote the minimum number of vertices N for which a coloring of the edges of K_N with k colors results in either an s_1-clique in the first color, or an s_2-clique in the second color, etc.

A variation of the Ramsey problem is given by the following definition.

DEFINITION 11.1. *Let n, p, and q be positive integers such that $q \leq \binom{p}{2}$. A (p, q)-**coloring** of the complete graph on n vertices, K_n, is an edge coloring,*

$$c : E(K_n) \to [k],$$

for which every subset of p vertices of $V(K_n)$ span at least q distinct edge colors. Let $f(n, p, q)$ denote the minimum number of colors k for which a (p, q)-coloring of K_n exists.

If we let $q = 2$ in the above definition, then determining an upper bound for the function $f(n, p, 2) \leq k$ is equivalent to giving a lower bound, $n + 1 \leq R(p, \ldots, p)$, for the Ramsey number with k colors. Similarly, giving a lower bound $k \leq f(n, p, q)$ is equivalent to giving the upper bound $R(p, \dots, p) \leq n$ on the Ramsey number for $k - 1$ colors.

Erdős and Shelah [20, 24] introduced the function $f(n, p, q)$ in 1975, but it was not studied in depth until 1997 when Erdős and Gyárfás [25] looked at the growth rate of $f(n, p, q)$ as $n \to \infty$ for fixed values of p and q. They used the local lemma to give a general upper bound for the function,

$$f(n, p, q) \leq cn^{\frac{p-2}{\binom{p}{2} - q + 1}}.$$

Other than this, they looked for threshold values for q in terms of p for which $f(n, p, q)$ jumps in order of magnitude. For instance, they showed that when

$$q = \binom{p}{2} - p + 3,$$

$f(n, p, q) = \Theta(n)$ and $f(n, p, q - 1) = o(n)$. So the function becomes linear in n at this particular value of q. Similarly, they determined that

$$q = \binom{p}{2} - \left\lfloor \frac{p}{2} \right\rfloor + 2$$

is the first value at which $f(n, p, q)$ is quadratic in n and

$$q = \binom{p}{2} - \left\lfloor \frac{p}{4} \right\rfloor + 1$$

is the first value for which

$$\binom{n}{2} - c \leq f(n, p, q)$$

where c is some constant depending only on p.

Left as an open question was determining the threshold value of q for which $f(n, p, q)$ first becomes polynomial in n. They showed that

$$n^{\frac{1}{p-2}} - 1 \leq f(n, p, p).$$

So therefore, any $q \geq p$ gives a function $f(n,p,q)$ that is polynomial in n. However, it was unclear what the order of magnitude of $f(n,p,p-1)$ is in general. To this end they considered some small cases. When $p=3$, they pointed out that since determining $f(n,3,2)$ is equivalent to solving the multicolor Ramsey problem for 3-cliques, then

$$c_1 \frac{\log n}{\log \log n} \leq f(n,3,2) \leq c_2 \log n$$

for constants c_1, c_2. However, for $p=4$, they could not beat the probabilistic upper bound

$$f(n,4,3) = O(n^{1/2}).$$

For this reason, they called this the "most annoying" case.

In 1998, Mubayi [34] gave an explicit $(4,3)$-coloring using a subpolynomial number of colors. Specifically, he showed that

$$f(n,4,3) \leq e^{O(\sqrt{\log n})}.$$

In 2000, Mubayi and Eichhorn [17] demonstrated that for $p \geq 5$, this construction is in general a (p,q)-coloring for $q = 2\lceil \log_2 p \rceil - 2$. In 2015, Conlon, Fox, Lee, and Sudakov [15] finally proved that $f(n,p,p-1)$ is subpolynomial for all $p \geq 3$. We will discuss the construction they came up with to demonstrate this in Chapter 12.

In addition to their general results, Erdős and Gyárfás looked at several cases for small values of p. They found that

$$\frac{5}{6}(n-1) \leq f(n,4,5) \leq n$$

and that

$$f(n,9,34) = \binom{n}{2} - o(n^2).$$

Moreover, they singled out the cases of $(4,4)$ and $(5,9)$-colorings as being particularly interesting to look at. In 2000, Axenovich [2] gave a construction showing that $f(n,5,9) \leq n^{1+o(1)}$. Since Erdős and Gyárfás showed that $f(n,5,8) = \Theta(n)$, then this reduced the difference between the known upper and lower bounds for $f(n,5,9)$ to a subpolynomial factor. In 2013, E. Krop and I. Krop [30] improved the lower bound to

$$\frac{7}{4}n - 3 \leq f(n,5,9).$$

In 2004, Mubayi gave an explicit $(4,4)$-coloring which reduced the upper bound to

$$f(n,4,4) \leq n^{1/2+o(1)},$$

a subpolynomial factor away from the best known lower bound given by Erdős and Gyárfás of $n^{1/2} - 1$. We will discuss his construction in more detail in Chapter 14.

In Chapter 13 we will give an explicit $(5,5)$-coloring that uses only $n^{1/3+o(1)}$ colors, a subpolynomial factor away from the best known lower bound of $n^{1/3} - 1$. Similarly, in Chapter 14 we will give an explicit $(5,5)$-coloring that uses only $n^{1/2+o(1)}$ colors, a subpolynomial factor away from the best known lower bound of $\Omega(n^{1/2})$. In both cases, the constructions will be combinations of a modified version of the construction given in [15] which we will define in Chapter 12 and certain "algebraic" colorings that extend the idea behind Mubayi's $(4,4)$-coloring [35].

CHAPTER 12

The Modified CFLS coloring.

This chapter contains material from a paper to be published in Combinatorics, Probability & Computing. [14]

In this chapter, we will define a particular instance of the general $(p, p-1)$-coloring of Conlon, Fox, Lee, and Sudakov [15] which we will refer to as the CFLS coloring, and we will show that this coloring avoids certain configurations. These properties will be useful in later chapters. Then we will modify the coloring and give some additional useful properties. We will not define the CFLS coloring in full generality since only a simple case is needed. We borrow part of the notation used in [15], but change it somewhat for clarity in this particular instance.

Let $n = 2^{\beta^2}$ for some positive integer β. Associate each vertex of K_n with a unique binary string of length β^2. That is, we may assume that our vertex set is

$$V = \{0,1\}^{\beta^2}.$$

For any vertex $v \in V$, let $v^{(i)}$ denote the ith block of bits of length β in v so that

$$v = (v^{(1)}, \ldots, v^{(\beta)})$$

where each $v^{(i)} \in \{0,1\}^{\beta}$.

Between two vertices $x, y \in V$, the CFLS coloring is defined by

$$\varphi_1(x,y) = \left(\left(i, \{x^{(i)}, y^{(i)}\}\right), i_1, \ldots, i_\beta\right)$$

where i is the first index for which $x^{(i)} \neq y^{(i)}$, and for each $k = 1, \ldots, \beta$, $i_k = 0$ if $x^{(k)} = y^{(k)}$ and otherwise is the first index at which a bit of $x^{(k)}$ differs from the corresponding bit in $y^{(k)}$.

For convenience, when discussing any edge color α, we will let α_0 denote the first coordinate of the color (of the form $(i, \{x^{(i)}, y^{(i)}\})$) and let α_k denote the index of the first bit difference of the kth block for $k = 1, \ldots, \beta$. Furthermore, throughout this section, we will say that two vertices x and y *agree at* i if $x^{(i)} = y^{(i)}$ and that x and y *differ at* i if $x^{(i)} \neq y^{(i)}$.

1. Avoided configurations

We will show through the following series of lemmas that the CFLS coloring avoids certain specified arrangements of edge colors.

LEMMA 12.1. *The CFLS coloring forbids monochromatic odd cycles.*

PROOF. Suppose there exists a sequence of distinct vertices, $v_1, \ldots, v_k$, for which k is odd and

$$\varphi_1(v_1, v_2) = \varphi_1(v_2, v_3) = \cdots = \varphi_1(v_{k-1}, v_k) = \varphi_1(v_k, v_1) = \alpha.$$

Let $\alpha_0 = (i, \{x, y\})$. Without loss of generality we may assume that $v_1^{(i)} = x$ and $v_2^{(i)} = y$. It follows that

$$y = v_2^{(i)} = v_4^{(i)} = \cdots = v_{k-1}^{(i)} = v_1^{(i)} = x,$$

a contradiction. ∎

LEMMA 12.2. *The CFLS coloring forbids four distinct vertices $a, b, c, d \in V$ for which $\varphi_1(a, b) = \varphi_1(c, d)$ and $\varphi_1(a, c) = \varphi_1(a, d)$ (see Figure 50a).*

PROOF. Assume towards a contradiction that $\varphi_1(a, b) = \varphi_1(c, d) = \alpha$ and $\varphi_1(a, c) = \varphi_1(a, d) = \gamma$. Let $\alpha_0 = (i, \{x, y\})$. Without loss of generality, $a^{(i)} = c^{(i)} = x$ and

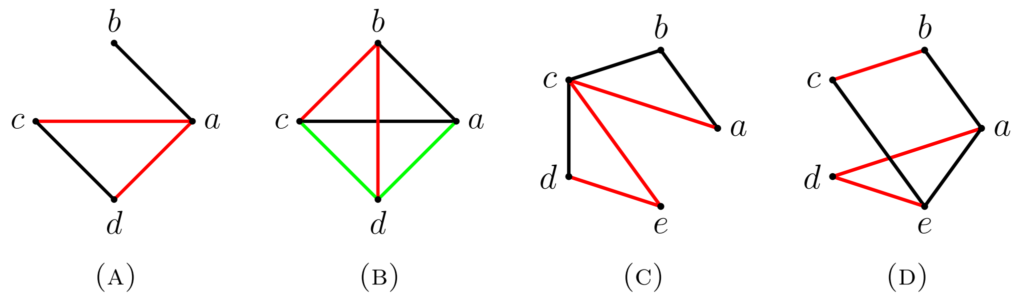


FIGURE 41. Four configurations avoided by the CFLS coloring.

$b^{(i)} = d^{(i)} = y$. Then $\gamma_i = 0$ since a and c agree at i, but $\gamma_i \neq 0$ as a and d differ at i, a contradiction. ∎

LEMMA 12.3. *The CFLS coloring forbids four distinct vertices* $a, b, c, d \in V$ *for which* $\varphi_1(a, b) = \varphi_1(a, c)$, $\varphi_1(b, d) = \varphi_1(b, c)$, *and* $\varphi_1(a, d) = \varphi_1(c, d)$ *(see Figure 41b).*

PROOF. Assume towards a contradiction that we have $\varphi_1(a, b) = \varphi_1(a, c) = \alpha$, $\varphi_1(b, d) = \varphi_1(b, c) = \gamma$, and $\varphi_1(a, d) = \varphi_1(c, d) = \pi$. Let $\alpha_0 = (i, \{x, y\})$, $\gamma_0 = (j, \{s, t\})$, and $\pi_0 = (k, \{w, v\})$. Without loss of generality we may assume that $a^{(i)} = x$ and $b^{(i)} = c^{(i)} = y$. Since b and c differ at j, then $i \neq j$. Without loss of generality we may assume that $b^{(j)} = s$ and $c^{(j)} = d^{(j)} = t$. So $\pi_j = 0$, and hence, $a^{(j)} = t$ since $\varphi_1(a, d) = \pi$. Therefore, $\alpha_j = 0$, which implies that $b^{(j)} = t$, a contradiction since $s \neq t$. ∎

LEMMA 12.4. *The CFLS coloring forbids five distinct vertices* $a, b, c, d, e \in V$ *that contain two monochromatic paths of three edges each that share endpoints:* $\varphi_1(a, b) = \varphi_1(b, c) = \varphi_1(c, d)$ *and* $\varphi_1(a, c) = \varphi_1(c, e) = \varphi_1(e, d)$ *(see Figure 41c).*

PROOF. Assume towards a contradiction that

$$\varphi_1(a, b) = \varphi_1(b, c) = \varphi_1(c, d) = \alpha$$

and

$$\varphi_1(a, c) = \varphi_1(c, e) = \varphi_1(e, d) = \gamma.$$

Let $\alpha_0 = (i, \{x, y\})$ and $\gamma_0 = (j, \{s, t\})$. Without loss of generality we may assume that $a^{(i)} = c^{(i)} = x$ and $b^{(i)} = d^{(i)} = y$. Note that $\varphi_1(a, c) = \gamma$ implies $\gamma_i = 0$. Then $e^{(i)} = d^{(i)} = y$ and $e^{(i)} = c^{(i)} = x$. So $x = y$, a contradiction. ■

LEMMA 12.5. *The CFLS coloring forbids five distinct vertices $a, b, c, d, e \in V$ for which $\varphi_1(a, b) = \varphi_1(a, e) = \varphi_1(e, c)$ and $\varphi_1(a, d) = \varphi_1(d, e) = \varphi_1(b, c)$ (see Figure 41d).*

PROOF. Assume towards a contradiction that $\varphi_1(a, b) = \varphi_1(a, e) = \varphi_1(e, c) = \alpha$ and $\varphi_1(a, d) = \varphi_1(d, e) = \varphi_1(b, c) = \gamma$. Let $\alpha_0 = (i, \{x, y\})$. We may assume without a loss of generality that $b^{(i)} = e^{(i)} = x$ and $a^{(i)} = c^{(i)} = y$. We also know that $b^{(k)} = a^{(k)} = e^{(k)} = c^{(k)}$ for all $k < i$. Since $\varphi_1(b, c) = \gamma$, then $\gamma_0 = (i, \{x, y\})$. So either $d^{(i)} = x$ or $d^{(i)} = y$. Therefore, d must agree with either a or e at i, a contradiction. ■

2. Modified CFLS

We will now add to the CFLS coloring to avoid the striped K_4, an edge-coloring of four distinct vertices a, b, c, d such that every pair of non-incident edges have the same color (see Figure 42). The CFLS coloring alone will not avoid such arrangements, but the product of φ_1 with another small edge-coloring, φ_2, will.

We will define the coloring φ_2 on the same set of vertices as the CFLS coloring, $V = \{0, 1\}^{\beta^2}$. However, we will also need to consider the vertices as an ordered set. Consider each vertex to be an integer represented in binary. Then order the vertices by the standard ordering of the integers. That is, $x < y$ if and only if the first bit at which x and y differ is zero in x and one in y. This ordering plays a large role in a recent construction by Mubayi [36] for a small case of the hypergraph version of the (p, q)-coloring problem. Note that each β-block is a binary representation of an integer from 0 to $2^\beta - 1$, so these blocks can be considered ordered in the same way.

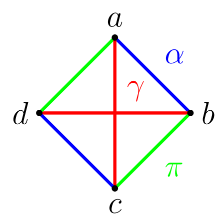


FIGURE 42. A striped K_4.

Moreover, note that if $x < y$ and if the first β-block at which x and y differ is i, then it must be the case that $x^{(i)} < y^{(i)}$.

Let $x, y \in V$ such that $x < y$. We define the second coloring as

$$\varphi_2(x, y) = (\delta_1(x, y), \ldots, \delta_\beta(x, y))$$

where for each i,

$$\delta_i(x, y) = \begin{cases} -1 & x^{(i)} > y^{(i)} \\ +1 & x^{(i)} \leq y^{(i)} \end{cases}$$

This construction uses 2^β colors. Therefore, the modified CFLS coloring, $\varphi = \varphi_1 \times \varphi_2$, uses

$$\beta^{\beta+1} 2^{3\beta} = \sqrt{\log n}^{\sqrt{\log n}+1} 2^{3\sqrt{\log n}} = 2^{O\left(\sqrt{\log n} \log \log n\right)}$$

colors.

LEMMA 12.6. *The modified CFLS coloring φ forbids four distinct vertices $a, b, c, d \in V$ with $\varphi(a, b) = \varphi(c, d)$, $\varphi(a, c) = \varphi(b, d)$, and $\varphi(a, d) = \varphi(b, c)$ (see Figure 42).*

PROOF. Assume towards a contradiction that a striped K_4 can occur. Then, $\varphi_1(a, b) = \varphi_1(c, d) = \alpha$, $\varphi_1(a, c) = \varphi_1(b, d) = \gamma$, and $\varphi_1(a, d) = \varphi_1(b, c) = \pi$. Let $\alpha_0 = (i, \{x, y\})$, $\gamma_0 = (j, \{s, t\})$, and $\pi_0 = (k, \{v, w\})$. Without loss of generality, assume that $i = \min\{i, j, k\}$. Since $\varphi_1(a, b) = \varphi_1(c, d)$, exactly one of $d^{(i)}$ and $c^{(i)}$ equals $a^{(i)}$. Say $d^{(i)} = a^{(i)}$ without loss of generality. Then, by the minimality of i, it must be the case that $j = i$ and that $i < k$.

Let $a^{(i)} = d^{(i)} = x$, $b^{(i)} = c^{(i)} = y$, $a^{(k)} = b^{(k)} = v$, and $c^{(k)} = d^{(k)} = w$. Without loss of generality we may assume that $x < y$. This implies that $a, d < b, c$ in the ordering of V as integers represented in binary. If $v < w$, then $\delta_k(a, c) = +1$ and $\delta_k(d, b) = -1$. Therefore, $\varphi_2(a, c) \neq \varphi_2(b, d)$, a contradiction. So, it must be the case that $w < v$. But then $\delta_k(a, c) = -1$ and $\delta_k(b, d) = +1$, which yields the same contradiction. ■

Note that to eliminate the striped K_4 configuration we needed just

$$\beta 2^{3\beta} = \sqrt{\log n}\, 2^{3\sqrt{\log n}} = 2^{O\left(\sqrt{\log n}\right)}$$

colors since only the first coordinate of the CFLS coloring was needed in the proof. Mubayi used on the order of $n^{1/2}$ colors to eliminate it while defining his $(4, 4)$-construction [35].

Before we move on from discussing the modified CFLS coloring, we need to point out one nice fact that will be used in Chapter 13.

LEMMA 12.7. *If $a < b < c$, then $\varphi(a, b) \neq \varphi(b, c)$.*

PROOF. Suppose $\varphi_1(a, b) = \varphi_1(b, c) = \alpha$ and that $\alpha_0 = (i, \{x, y\})$ for $x < y$. Then $a^{(i)} = x$ and $b^{(i)} = y$. But then $c^{(i)} = x$. Therefore, $c < b$, a contradiction. ■

CHAPTER 13

A $(5,5)$-coloring construction.

This chapter contains material from a paper to be published in Combinatorics, Probability & Computing. [14]

As previously stated, we know in general that $f(n,p,p) \geq \Omega\left(n^{1/(p-2)}\right)$. However, the local lemma gives the best general upper bound,

$$f(n,p,p) \leq O\left(n^{2/(p-1)}\right).$$

Only for $p = 3,4$ do we know of a better upper bound.

A $(3,3)$-coloring is equivalent to a proper edge coloring, one in which no two incident edges can have the same color. Therefore, it is well known that

$$f(n,3,3) = \begin{cases} n & n \text{ is odd} \\ n-1 & n \text{ is even} \end{cases}$$

In 2004, Mubayi [35] provided an explicit $(4,4)$-coloring of K_n with only $n^{1/2}e^{O(\sqrt{\log n})}$ colors. This closed the gap for $p = 4$ to

$$n^{1/2} - 1 \leq f(n,4,4) \leq n^{1/2+o(1)}.$$

His construction was the product of two colorings. The first was his earlier $(4,3)$-coloring which used $n^{o(1)}$ colors. The second was an "algebraic" coloring that assigned to each vertex a vector from a two-dimensional vector space over a finite field, and then colored each edge with an element from the base field, giving $n^{1/2}$ colors. The algebraic part of his construction will be detailed further in Chapter 14.

The $(5,5)$-coloring defined in this chapter extends Mubayi's idea of combining a small $(p,p-1)$-coloring with an algebraic coloring to obtain the following result.

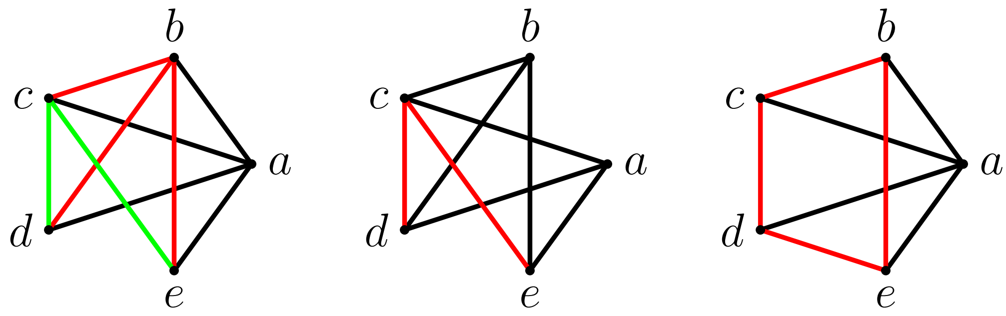
THEOREM 13.1. *As* $n \to \infty$,

$$f(n,5,5) \leq n^{1/3} 2^{O\left(\sqrt{\log n}\log\log n\right)}.$$

Before defining the algebraic part of our construction, we can systematically look at all edge-colorings of a K_5 up to isomorphism that use no more than four colors and do not contain any of the configurations eliminated in Chapter 12 to get a list of possible "bad" colorings of a K_5 that could survive the modified CFLS coloring. A careful mathematician with a free day could work through these cases by hand. A simple computer program like the inelegant one detailed in Appendix A is easier to verify. However this process is executed, we end up with three possible bad colorings of K_5 (see Figure 43). Avoiding these will require both the modified CFLS coloring and the MIP coloring defined in Section 1.

In Section 1, we define the first part of an algebraic coloring which we call the Modified Inner Product (MIP) coloring. Under this construction, each vertex is associated with a vector in a three-dimensional space over a finite field. As in Mubayi's construction [35] each edge is colored with a specific element in the base field. Some slight modifications are needed for special cases, but these will only split each color a constant number of times, ultimately giving $O\left(n^{1/3}\right)$ colors used in the MIP construction.

In Section 2, we will take the product of the modified CFLS coloring defined in Chapter 12 and the first part of the MIP to get a construction that uses $n^{1/3+o(1)}$ colors and eliminates the first two of the three remaining bad configurations. Finally, in Section 3 we define the rest of the MIP coloring to eliminate the third configuration.



(A) (B) (C)

FIGURE 43. Three configurations not avoided by the modified CFLS coloring.

1. The Modified Inner Product coloring

Let q be some odd prime power, and let $\mathbb{F}_q^*$ denote the nonzero elements of the finite field with q elements. The vertices of our graph will be the three-dimensional vectors over this set,

$$V = \left(\mathbb{F}_q^*\right)^3.$$

All algebraic operations used in defining the MIP coloring are the standard ones from the underlying field, and $\cdot$ will denote the standard inner product of two vectors,

$$x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Additionally, let $<$ be any linear order on the elements of $\mathbb{F}_q$, and extend this to a linear order on the vectors so that

$$x < y \iff x_i < y_i$$

where $i \in \{1, 2, 3\}$ is the first position at which $x_i \neq y_i$.

The MIP coloring will be broken up into two parts, $\chi = \chi_1 \times \chi_2$. The first part χ_1 uses at most $12n^{1/3}$ colors. The second part χ_2 uses only four colors and is used to split up colors from χ_1 in order to avoid one particularly difficult configuration. In this section, we will first define χ_1. Then, after a brief review of the necessary linear

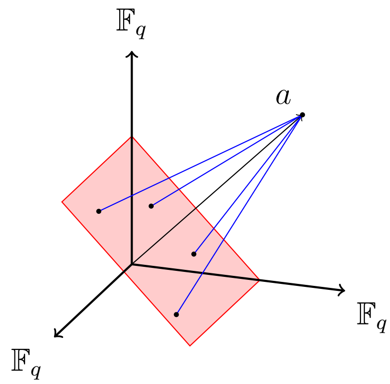


FIGURE 44. The geometric visualization can be misleading since we are working over a finite field, but monochromatic neighborhoods are contained in affine planes.

algebra concepts, we will prove some key properties of χ_1. The second part χ_2 will be defined in Section 5.

1.1. Motivation. The MIP coloring should be viewed as coloring each edge with the inner product of the two vectors with some adjustments for special cases. As motivation for this coloring, note that each of the three configurations with four colors that survive the modified CFLS coloring (see Figure 43) contains at least one pair of vertices in the intersection of monochromatic neighborhoods of the other three vertices.

For instance, vertices d and e in Figure 50b are in the same monochromatic neighborhood with respect to vertices a, b, and c. Under the MIP coloring, the monochromatic neighborhood of any vertex is contained in an affine plane of $\mathbb{F}_q^3$. So, if a, b, and c are linearly independent, then these planes intersect in one point, not two. Therefore, the 50b configuration could only happen under an inner product coloring if the span of a, b, and c has dimension at most 2.

This same idea applies to the other two configurations, and the adjustments to coloring with the inner product are all dedicated to handling the cases for which the five vectors are not in general position. In these cases we frequently end up with three vectors that must all lie on the same affine line, and the offending configurations

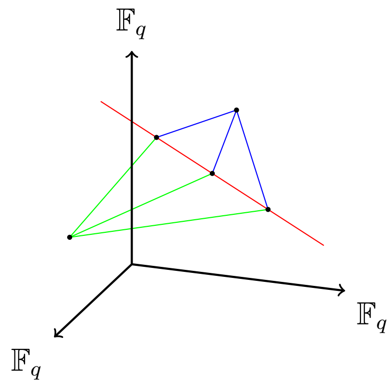


FIGURE 45. The intersection of two monochromatic neighborhoods usually lies on an affine line.

would be destroyed if the coloring could be modified to give a proper coloring on every affine line.

1.2. The coloring χ_1. In Lemma 13.11 we will show that χ_1 induces a proper edge coloring on every line, not just one-dimensional linear spaces but affine lines as well. This will be one of the key lemmas in showing that our construction avoids the remaining configurations. By itself, the inner product almost accomplishes this goal. However, a problem arises when one vector on a given line is orthogonal to the direction of the line. In this case, that particular vector has the same inner product with all other vectors on the line, so we must give these edges new colors. We accomplish this by replacing the inner product with another function.

The first part of χ_1 labels the type of edge-coloring we will have. For two distinct vectors, $x, y \in V$, let $T(x,y)$ be a function defined by

$$T(x,y) = \begin{cases} \mathrm{UP}_1 & x \cdot y = x \cdot x \text{ and } x_1 < y_1 \\ \mathrm{UP}_2 & x \cdot y = x \cdot x, x_1 = y_1, \text{ and } x < y \\ \mathrm{DOWN}_1 & x \cdot y \neq x \cdot x, x \cdot y = y \cdot y, \text{ and } x_1 < y_1 \\ \mathrm{DOWN}_2 & x \cdot y \neq x \cdot x, x \cdot y = y \cdot y, x_1 = y_1, \text{ and } x < y \\ \mathrm{ZERO} & x \cdot y \notin \{x \cdot x, y \cdot y\} \text{ and } x \cdot y = 0 \\ \mathrm{DOT} & \text{otherwise} \end{cases}$$

Here, the categories UP_i and DOWN_i let us know that at least one of the two vectors is orthogonal to the direction of the line between the two, and therefore this edge will need to receive something other than the inner product in the next part of the color. The words UP and DOWN describe the edge from the perspective of the "special" vertex. For instance, if x is orthogonal to the direction of the line it makes with y and $x < y$, then x looks up the edge to y. The need for different categories when $x_1 = y_1$ is a technical point. The category DOT stands for the inner product (or the "dot" product), and ZERO is the special case where the inner product is zero. The need to split the colors with zero inner product is also a technical point.

Let $f_T(x,y) : \mathbb{F}_q^3 \to \mathbb{F}_q$ be a function defined by

$$f_T(x,y) = \begin{cases} x_1 + y_1 & T \in \{\mathrm{UP}_1, \mathrm{DOWN}_1, \mathrm{ZERO}\} \\ x_2 + y_2 & T \in \{\mathrm{UP}_2, \mathrm{DOWN}_2\} \\ x \cdot y & T = \mathrm{DOT} \end{cases}$$

One final technical point is to differentiate colors based on whether the two vectors are linearly dependent or independent. Let

$$\delta(x,y) = \begin{cases} 0 & \{x,y\} \text{ is linearly dependent} \\ 1 & \{x,y\} \text{ is linearly independent} \end{cases}$$

This is enough to define the coloring. For vertices $x < y$, let $T = T(x,y)$, and set

$$\chi_1(x,y) = (T, f_T(x,y), \delta(x,y)).$$

1.3. Algebraic definitions and facts. We assume that the reader has some familiarity with basic linear algebra notions such as dimension, linear independence, linear combination, and span. The following definitions and facts are perhaps less familiar. All are reproduced from definitions and propositions in Chapter 2 of the great *Linear Algebra Methods in Combinatorics* book by László Babai and Péter Frankl [3].

DEFINITION 13.1. *Let $\mathbb{F}^n$ be a vector space, and let $S \subseteq \mathbb{F}^n$ be a set of vectors. The* rank *of S is the dimension of the linear space spanned by S.*

FACT 13.2. *Let $\mathbb{F}$ be a field, and let A be a $k \times n$ matrix over $\mathbb{F}$. Then the rank of the set of column vectors as vectors in $\mathbb{F}^k$ is equal to the rank of the set of row vectors as vectors in $\mathbb{F}^n$. We know this value as the rank of the matrix A, $rk(A)$.* ■

DEFINITION 13.3. *Let $\mathbb{F}^n$ be a vector space. An* affine combination *of vectors $v_1, \dots, v_k \in \mathbb{F}^n$ is a linear combination $\lambda_1 v_1 + \cdots + \lambda_k v_k$ for $\lambda_1, \dots, \lambda_k \in \mathbb{F}$ such that $\lambda_1 + \cdots + \lambda_k = 1$. An* affine subspace *is a subset of vectors that is closed under affine combinations.*

FACT 13.4. *Any affine subspace U is either empty or the translation of some linear subspace V. That is, each vector $u \in U$ can be written in the form $u = v + t$ where v is some vector in V and t is a fixed translation vector.* ■

DEFINITION 13.5. *The* dimension *$dim(U)$ of an affine subspace U is the dimension of the unique linear subspace of which U is a translate.*

DEFINITION 13.6. *Let $\mathbb{F}^n$ be a vector space. Let $v_1, \dots, v_k \in \mathbb{F}^n$. We say that these vectors are* affine independent *if*

$$\lambda_1 v_1 + \cdots + \lambda_k v_k = 0$$

implies that

$$\lambda_1 = \cdots = \lambda_k = 0$$

for any $\lambda_1, \dots, \lambda_k \in \mathbb{F}$ for which $\lambda_1 + \cdots + \lambda_k = 0$. Otherwise, these vectors are affine dependent. *We say that a set of vectors S is a* basis *for an affine subspace if they are affine independent and every vector in the subspace is an affine combination of vectors in S.*

FACT 13.7. *A basis of an affine subspace U contains exactly $dim(U) + 1$ elements.* ■

FACT 13.8. *Let $\mathbb{F}^n$ be some vector space. Let A be a $k \times n$ matrix over $\mathbb{F}$ and $b \in \mathbb{F}^n$. Then the solution set to $Ax = b$ is an affine subspace of dimension $n - rk(A)$.* ■

DEFINITION 13.9. *A vector $x \in \mathbb{F}^n$ is* isotropic *if $x \cdot x = 0$. A linear subspace $U \subseteq \mathbb{F}^n$ is* totally isotropic *if $x, y \in U$ implies that $x \cdot y = 0$.*

FACT 13.10. *For any nonzero vector $x \in \mathbb{F}^n$, the set of vectors $\{y : x \cdot y = 0\}$ is a linear subspace of $\mathbb{F}^n$ with dimension $n - 1$.* ■

1.4. Properties of χ_1.

LEMMA 13.11. *The coloring χ_1 induces a proper edge coloring on every one-dimensional affine subspace.*

PROOF. Let $a, b, c \in \mathbb{F}_q^3$ be three distinct vectors in a one-dimensional affine subspace. Then there exists some $\lambda \in \mathbb{F}_q$ such that $c = \lambda a + (1 - \lambda)b$. Suppose towards a contradiction that $\chi_1(a, b) = \chi_1(a, c)$, and let $T = T(a, b) = T(a, c)$. If $T \in \{\text{ZERO}, \text{DOT}\}$, then

$$a \cdot b = a \cdot (\lambda a + (1 - \lambda)b).$$

So $\lambda a \cdot (a - b) = 0$. Since $c \neq b$, then $\lambda \neq 0$. Therefore, $a \cdot (a - b) = 0$. But this contradicts the assumption that $T \in \{\text{ZERO}, \text{DOT}\}$.

If $T \in \{\text{UP}_1, \text{DOWN}_1\}$, then $f_T(a, b) = f_T(a, c)$ gives

$$a_1 + b_1 = a_1 + \lambda a_1 + (1 - \lambda)b_1.$$

So $b_1 = a_1$, a contradiction since $T \in \{\text{UP}_1, \text{DOWN}_1\}$ implies that $a_1 \neq b_1$. Similarly, if $T \in \{\text{UP}_2, \text{DOWN}_2\}$, then $a_2 = b_2$ by the same argument and $a_1 = b_1$ by definition. But then either $a_3(a_3 - b_3) = 0$ or $b_3(b_3 - a_3) = 0$. Both cases imply that $a_3 = b_3$. So $a = b$, a contradiction. ■

DEFINITION 13.12. *Given a vertex $a \in V$ and an edge-color A, let*

$$N_A(a) = \{x : \chi_1(a, x) = A\}$$

be the A-neighborhood of a

OBSERVATION 13.13. *Given a vector $a \in V$ and a color $A = (T, \alpha, i)$, the vectors in $N_A(a)$ all belong to the two-dimensional affine subspace defined by $\{x : f_T(a, x) = \alpha\}$. In particular, this plane can be defined as the solution space to either $a \cdot x = \alpha$ when $T = DOT$, $(1, 0, 0) \cdot x = \alpha - a_1$ when $T \in \{ZERO, UP_1, DOWN_1\}$, and $(0, 1, 0) \cdot x = \alpha - a_2$ when $T \in \{UP_2, DOWN_2\}$.*

In certain cases, we can actually say something a little stronger. First, note that if $T(a, x) \in \{\mathrm{UP}_1, \mathrm{DOWN}_1\}$, then we will have $a_1 < x_1$ if, and only if, $a < x$. Therefore, if $f_T(a, x) = \alpha$, since $f_T(a, x) = a_1 + x_1$, we will have $a_1 < \alpha = a_1$ if, and only if, $a < x$.

LEMMA 13.14. *Given a vector $a \in V$, and a color $A = (T, \alpha, i)$, the vectors of $N_A(a)$ all belong to a one-dimensional affine subspace if one of the following three cases holds for all $x \in N_A(a)$:*

(1) $T \in \{ZERO, UP_2, DOWN_2\}$;
(2) $T = UP_1$ and $a < x$;
(3) $T = DOWN_1$ and $a > x$.

PROOF. In the first case, if $T = \mathrm{ZERO}$, then every $x \in N_A(a)$ must satisfy the system of linear equations

$$a \cdot x = 0$$

$$(1, 0, 0) \cdot x = \alpha - a_1.$$

Since a contains no zero components, then the rank of $\{a, (1, 0, 0)\}$ is two. Therefore, the solution space must be a one-dimensional affine subspace. If $T \in \{\mathrm{UP}_2, \mathrm{DOWN}_2\}$,

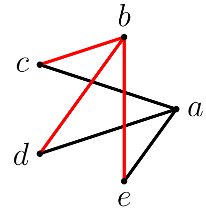


FIGURE 46. The intersection of two monochromatic neighborhoods.

then every $x \in N_A(a)$ must satisfy the system

$$(1,0,0) \cdot x = a_1$$
$$(0,1,0) \cdot x = \alpha - a_2.$$

Since $(1,0,0)$ and $(0,1,0)$ are linearly independent, then, as before, the set of solutions is a one-dimensional affine subspace.

In each of the other two cases, we see that every $x \in N_A(a)$ must satisfy the system

$$a \cdot x = a \cdot a$$
$$(1,0,0) \cdot x = \alpha - a_1.$$

As before, the solution space must be a one-dimensional affine subspace. ■

Therefore, we immediately get the following corollary by Lemma 13.11.

COROLLARY 13.15. *Let $a, b, c, d \in V$ be four distinct vertices such that*

$$\chi_1(a,b) = \chi_1(a,c) = \chi_1(a,d) = (T, \alpha, i).$$

The set of vertices $\{b, c, d\}$ span three distinct edge colors under χ_1 if any of the following are true:

1. *$T \in \{ZERO, UP_2, DOWN_2\}$;*
2. *$T = UP_1$ and $a < b, c, d$;*
3. *$T = DOWN_1$ and $a > b, c, d$.*

LEMMA 13.16. *Let $a, b, c, d, e \in V$ be vectors such that $\{a, b\}$ is linearly independent, $\chi_1(a, c) = \chi_1(a, d) = \chi_1(a, e)$, and $\chi_1(b, c) = \chi_1(b, d) = \chi_1(b, e)$ (see Figure 46). Then the set $\{c, d, e\}$ spans three distinct edge colors.*

PROOF. Let $\chi_1(a, c) = \chi_1(a, d) = \chi_1(a, e) = A$ and $\chi_1(b, c) = \chi_1(b, d) = \chi_1(b, e) = B$. The result is immediate if either pair (a, A) or (b, B) satisfies the conditions listed in Corollary 13.15. So assume not. If $A = (T_a, \alpha, i)$, then by Observation 13.13 we know that c, d, and e must either satisfy $a \cdot x = \alpha$ or $(1, 0, 0) \cdot x = \alpha - a_1$. Similarly, if $B = (T_b, \beta, j)$, then c, d, and e must either satisfy $b \cdot x = \beta$ or $(1, 0, 0) \cdot x = \beta - b_1$. Since the sets $\{a, b\}$, $\{a, (1, 0, 0)\}$, and $\{(1, 0, 0), b\}$ are all linearly independent, then every case gives us the result immediately except when $T_a, T_b \in \{\mathrm{UP}_1, \mathrm{DOWN}_1\}$. Since we assume that none of the cases from Corollary 13.15 hold, then this can only happen when $x \cdot (x - a) = x \cdot (x - b) = 0$ for $x = c, d, e$. In this case, c, d, and e all satisfy the two linear equations,

$$(a - b) \cdot x = 0$$

$$(1, 0, 0) \cdot x = \alpha - a_1.$$

Hence, c, d, and e are affine independent, and the result follows from Lemma 13.11 unless

$$a_2 - b_2 = a_3 - b_3 = 0.$$

But if this is true, then $c \cdot (c - a) = c \cdot (c - b)$ implies that $a = b$, a contradiction. ■

2. Combining the colorings

Let $n = (q - 1)^3$ where q is an odd prime power. To each $\alpha \in \mathbb{F}_q$ we associate the unique element $\alpha' \in \{0, 1\}^{\lceil \log q \rceil}$ which represents in binary the rank of α under the linear order given to the elements of $\mathbb{F}_q$ in Section 1. Let β be the minimum positive integer for which

$$3 \lceil \log q \rceil \leq \beta^2.$$

We associate each of the n vertices of K_n with a unique vector in $\left(\mathbb{F}_q^*\right)^3$ as in Section 1. To each vertex (x_1, x_2, x_3), we associate $(x_1', x_2', x_3', 0) \in \{0,1\}^{\beta^2}$ as well, where for each i, x_i' is the binary representation of the rank of x_i, and 0 denotes a string of $\beta^2 - 3\lceil \log q \rceil$ zeros. Let

$$C = \varphi \times \chi_1.$$

Since

$$\beta = \Theta\left(\sqrt{3\log q}\right) = \Theta\left(\sqrt{\log n}\right),$$

it follows that the number of colors used in this combined coloring is at most

$$12q\beta 2^{3\beta} = n^{1/3} 2^{O\left(\sqrt{\log n}\log\log n\right)}$$

colors. This bound on the number of colors generalizes to all n by the standard density of primes argument [37].

2.1. The first two configurations.

LEMMA 13.17. *Any distinct vertices $a, b, c, d, e \in V$ for which $C(a, c) = C(a, d) = C(a, e)$, $C(b, c) = C(b, d) = C(b, e)$, and $C(a, c) \neq C(b, c)$ (see Figure 46) span at least five distinct edge colors.*

PROOF. Lemma 12.1 implies that neither color between $\{a, b\}$ and $\{c, d, e\}$ can be repeated on the edges spanned by $\{c, d, e\}$. Therefore, if $\{a, b\}$ is linearly independent it follows from Lemma 14.1 that $\{a, b, c, d, e\}$ span at least 5 colors.

Otherwise, $b = \lambda a$ for some $\lambda \in \mathbb{F}_q$. If $C(a, b)$ repeats one of the colors from the edges spanned by $\{c, d, e\}$, then this gives us the configuration forbidden by Lemma 12.2. If $C(a, b) = C(a, c)$ or $C(a, b) = C(b, c)$, then all five vectors belong to a one-dimensional linear subspace spanned by a which must be properly edge-colored by Lemma 13.11. Therefore, the set of vertices $\{a, b, c, d, e\}$ spans at least 5 colors. ■

This immediately shows that the first configuration will not appear under the combined coloring.

COROLLARY 13.18. *Let $a, b, c, d, e \in V$ be five distinct vertices. It cannot be the case that $C(a, b) = C(a, c) = C(a, d) = C(a, e)$, $C(b, c) = C(b, d) = C(b, e)$, and $C(c, d) = C(c, e)$ as in Figure 50b.*

The second configuration also will not appear under the combined coloring.

LEMMA 13.19. *Let $a, b, c, d, e \in V$ be five distinct vertices. It cannot be the case that*

$$C(a, c) = C(a, d) = C(a, e) = C(b, c) = C(b, d) = C(b, e),$$

and $C(c, d) = C(c, e)$ as in Figure 43b.

PROOF. By Lemma 14.1, this can happen only if there exists some $\lambda \in \mathbb{F}_q$ such that $b = \lambda a$. In this case,

$$\chi_1(a, c) = \chi_1(a, d) = \chi_1(a, e) = \chi_1(\lambda a, c) = \chi_1(\lambda a, d) = \chi_1(\lambda a, e).$$

If this color is in DOT, then $c \cdot a = c \cdot \lambda a$. So either $\lambda = 1$, a contradiction, or $c \cdot a = 0$, a contradiction that the color is in DOT. If the color is not in DOT, then it must be the case that $a_1 = \lambda a_1$. Since $a_1 \neq 0$, then this forces $\lambda = 1$, a contradiction. ■

3. Splitting the coloring

Now we will split the colors of C to make a new coloring $C' = C \times \chi_2$, where χ_2 is the second part of the MIP coloring.

Let $U \subseteq \mathbb{F}_q^3$ be a two-dimensional linear subspace. Let G_U be an auxiliary graph where $V(G_U)$ is the set of non-isotropic vectors in U, and

$$xy \in E(G_U) \iff x \cdot y = 0.$$

We wish to show that G_U is bipartite. Note that $x \cdot y = 0$ implies that $\alpha x \cdot \beta y = 0$ for any $\alpha, \beta \in \mathbb{F}_q$. Suppose that $x \cdot z = 0$ for some $z \in V(G_U)$ such that $z \neq \beta y$ for any $\beta \in \mathbb{F}_q$. Then the intersection between U and the two-dimensional linear subspace orthogonal to x must also be a two-dimensional linear subspace. Therefore, x is contained in its own orthogonal linear subspace. So x is isotropic, a contradiction. Hence, G_U is comprised of disjoint complete bipartite graphs and so is itself bipartite.

For each two-dimensional linear subspace U, we label the vertices of G_U with A_U and B_U depending on their part in the bipartition, and then label all isotropic vectors in U with A_U as well.

For any two-dimensional linear subspace $U \subseteq \mathbb{F}_q^3$ and any $x \in U$ we define

$$S(x, U) = \begin{cases} A & x \in A_U \\ B & x \in B_U \end{cases}$$

For a given vector $a \in V$, and a given color type T, define

$$a_T = \begin{cases} a & T = \mathrm{DOT} \\ (1,0,0) & T \in \{\mathrm{UP}_1, \mathrm{DOWN}_1, \mathrm{ZERO}\} \\ (0,1,0) & T \in \{\mathrm{UP}_2, \mathrm{DOWN}_2\} \end{cases}$$

and let

$$U_{a,T} = \{x : a_T \cdot x = 0\}.$$

For convenience, let

$$a_b = \begin{cases} 0 & a_T \cdot a_T = 0 \\ (a_T \cdot b)(a_T \cdot a_T)^{-1} a_T & a_T \cdot a_T \neq 0 \end{cases}$$

for any vectors a and b where $T = T(a, b)$.

Now we can define the second part of the MIP coloring. For any two vectors, $a < b$ with $T = T(a, b)$, let

$$\chi_2(a, b) = (S(a - b_a, U_{b,T}), S(b - a_b, U_{a,T}))\,.$$

3.1. The third configuration. Let $a, b, c, d, e \in V$ be five distinct vertices such that

$$C'(a, b) = C'(a, c) = C'(a, d) = C'(a, e) = \text{Black},$$

and let

$$C'(b, c) = C'(c, d) = C'(d, e) = C'(e, b) = \text{Red}$$

as shown in Figure 43c. By Lemma 12.7 we know that either $b, d < c, e$ or $c, e < b, d$. Similarly, we know that either $a < b, c, d, e$ or $b, c, d, e < a$. So without loss of generality, we can say that either $a < b, d < c, e$ or $b, d < c, e < a$. In either case,

$$S(b - a_b, U_{a,T}) = S(c - a_c, U_{a,T})$$

where $T = T(a, b) = T(a, c)$.

By Corollary 13.15 we know that one of the following three cases must be true:

(1) Black $\in$ DOT,

(2) Black $\in$ UP$_1$ such that $b, d < c, e < a$, or

(3) Black $\in$ DOWN$_1$ such that $a < b, d < c, e$.

This abuses our notation slightly, but the meaning is hopefully clear. For example, Black $\in$ DOT means that the first component of the χ_1 part of the color Black is DOT.

We will show that none of these cases are possible through the following series of lemmas.

LEMMA 13.20. *If Black $\in$ DOT and RED $\in$ DOT, then the configuration in Figure 43c is not possible under the coloring C'.*

Proof. Let the inner product part of color Black be α and the inner product part of Red be β. Note that if either Black or Red encodes linear independence, then b, c, d, and e would all belong to the same one-dimensional linear subspace, a contradiction of Lemma 13.11. Also, since $c - e$ satisfies the three linear equations, $a \cdot x = 0$, $b \cdot x = 0$, and $d \cdot x = 0$, then $\{a, b, d\}$ cannot be linearly independent since then $c = e$, a contradiction. So there exist nonzero $\lambda_1, \lambda_2 \in \mathbb{F}_q$ such that $d = \lambda_1 a + \lambda_2 b$.

Note that $a \cdot a \neq 0$ since otherwise

$$a \cdot d = a \cdot (\lambda_1 a + \lambda_2 b)$$

$$\alpha = \lambda_2 \alpha$$

implies that $\lambda_2 = 1$ since $\alpha \neq 0$. If $\lambda_2 = 1$, then we would reach a contradiction by taking the inner product of both sides of $d = \lambda_1 a + b$ with c to get that $\beta = \lambda_1 \alpha + \beta$, a contradiction since $d \neq b$.

So $a_b = a_c = \alpha (a \cdot a)^{-1} a$, then

$$d = \lambda_1' a_b + \lambda_2 b$$

where $\lambda_1' = \lambda_1 \alpha^{-1} (a \cdot a)$. Taking the inner product of both side of this with a gives that

$$\alpha = (\lambda_1' + \lambda_2) \alpha.$$

So it follows that a_b, b, and d are affine dependent. By the same arguments we can conclude that a_b, c, and e are also affine dependent.

Note that $(b - d) \cdot (c - e) = 0$. Therefore, $(b - a_b) \cdot (c - a_c) = 0$. Since

$$S\left(b - a_b, U_{a,\mathrm{DOT}}\right) = S\left(c - a_c, U_{a,\mathrm{DOT}}\right),$$

then $b - a_b$ and $c - a_c$ are contained in the same part of the bipartition of the auxiliary graph on $U_{a,\mathrm{DOT}}$. Therefore, either $b - a_b$ or $c - a_c$ must be isotropic since otherwise the fact that they are orthogonal would have made them adjacent in the auxiliary graph.

Assume without loss of generality that $b - a_b$ is isotropic. Since

$$(b - a_b) \cdot (b - a_b) = 0,$$

then $b \cdot b = \alpha^2 (a \cdot a)^{-1}$. Since $(b - a_b) \cdot (c - a_c) = 0$, then $\beta = \alpha^2 (a \cdot a)^{-1}$. Therefore, $b \cdot b = \beta$. Hence,

$$b \cdot (b - c) = 0.$$

This contradicts our assumption that Red $\in$ DOT. ■

Note in what follows that if Red $\notin$ DOT, then $b_1 = d_1$ and $c_1 = e_1$.

LEMMA 13.21. *If Black $\in$ DOT and RED $\in$ ZERO, then the configuration in Figure 43c is not possible under the coloring C'.*

PROOF. If Red $\in$ ZERO, then

$$b \cdot (c - e) = d \cdot (c - e) = 0.$$

Also, recall that

$$a \cdot (c - e) = 0.$$

If a, b, d are linearly independent, then $c = e$, a contradiction. So we must assume that a, b, d are linearly dependent.

If either b or d depends on a, then $\delta(a, x) = 0$ for $x = b, c, d, e$ which implies that all five vectors belong to a one-dimensional linear subspace spanned by a, contradicting Lemma 13.11. If $d = \lambda b$ for some $\lambda \in \mathbb{F}_q$, then $b_1 = d_1 = \lambda b_1$. So either $b_1 = 0$ or $\lambda = 1$, both contradictions. So we must assume that $d = \lambda_1 a + \lambda_2 b$ for nonzero $\lambda_1, \lambda_2 \in \mathbb{F}_q$. But then

$$\begin{aligned} d \cdot c &= \lambda_1 (a \cdot c) + \lambda_2 (b \cdot c) \\ 0 &= \lambda_1 (a \cdot c) \end{aligned}$$

Since $\lambda_1 \neq 0$, then $a \cdot c = 0$ which implies that Black $\notin$ DOT, a contradiction. ■

LEMMA 13.22. *If Black $\in$ DOT and RED $\notin \{DOT, ZERO\}$, then the configuration in Figure 43c is not possible under the coloring C'.*

PROOF. If Red $\in \mathrm{UP}_2 \cup \mathrm{DOWN}_2$, then $b_1 = c_1 = d_1 = e_1$. So all four vectors b, c, d, e satisfy the linear equations $a \cdot x = \alpha$ and $(1,0,0) \cdot x = b_1$. Therefore, b, c, d, and e all belong to a one-dimensional affine subspace, a contradiction of Lemma 13.11.

If Red $\in \mathrm{UP}_1$, then

$$b \cdot (b - c) = b \cdot (b - e) = d \cdot (d - c) = d \cdot (d - e) = 0$$

since we assume that $b, d < c, e$. Therefore,

$$b \cdot (c - e) = b \cdot c - b \cdot e = b \cdot b - b \cdot b = 0.$$

Similarly, $d \cdot (c - e) = 0$. Since $c_1 = e_1$, then it follows that

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ b_1 & d_2 & d_3 \end{pmatrix} \begin{pmatrix} 0 \\ c_2 - e_2 \\ c_3 - e_3 \end{pmatrix} = 0.$$

Therefore, if any two of (a_2, a_3), (b_2, b_3), and (d_2, d_3) are linearly independent as vectors in $\mathbb{F}_q^2$, then $c = e$, a contradiction. Hence, there must exist $\lambda_1, \lambda_2 \in \mathbb{F}_q$ such that $(a_2, a_3) = \lambda_1(b_2, b_3)$ and $(d_2, d_3) = \lambda_2(b_2, b_3)$.

From the equations of the form $x \cdot (x - c) = 0$ for $x = b, d$ we get

$$b_1(b_1 - c_1) + b_2^2 + b_3^2 - b_2c_2 - b_3c_3 = 0$$
$$b_1(b_1 - c_1) + \lambda_2^2 b_2^2 + \lambda_2^2 b_3^2 - \lambda_2 b_2 c_2 - \lambda_2 b_3 c_3 = 0$$

So it follows that

$$c_3 = b_3^{-1}\left(b_1(b_1 - c_1) + b_2^2 + b_3^2 - b_2c_2\right)$$

which in turn gives that

$$(1-\lambda_2)b_1(b_1-c_1) = \lambda_2(1-\lambda_2)(b_2^2+b_3^2).$$

Since $\lambda_2 \neq 1$, then

$$b_1(b_1-c_1) = \lambda_2(b_2^2+b_3^2).$$

Since $b_1 \neq 0$ and $b_1 \neq c_1$, then $b_2^2 + b_3^2 \neq 0$.

Now, since Black $\in$ DOT, we get

$$a \cdot b = a \cdot d$$

$$a_1b_1 + \lambda_1 b_2^2 + \lambda_1 b_3^2 = a_1b_1 + \lambda_1\lambda_2 b_2^2 + \lambda_1\lambda_2 b_3^2$$

$$\lambda_1(1-\lambda_2)(b_2^2+b_3^2) = 0$$

But this is a contradiction, since none of these three terms are zero.

If Red $\in$ DOWN$_1$, then we swap b, d and e, c in the previous argument to obtain the same contradiction. ■

LEMMA 13.23. *If Black $\notin$ DOT, then the configuration in Figure 43c is not possible under the coloring C'.*

PROOF. In this case, either $b, d < c, e < a$ and Black $\in$ UP$_1$, or $c, e > b, d > a$ and Black $\in$ DOWN$_1$. In both cases,

$$b \cdot (b-a) = c \cdot (c-a) = d \cdot (d-a) = e \cdot (e-a) = 0.$$

Moreover, $b_1 = c_1 = d_1 = e_1$, which implies that

$$\text{Red} \in \text{ZERO} \cup \text{UP}_2 \cup \text{DOWN}_2 \cup \text{DOT}.$$

If Red $\in$ ZERO, then

$$b \cdot (c-e) = d \cdot (c-e) = 0.$$

Therefore,

$$\begin{pmatrix} b_2 & b_3 \\ d_2 & d_3 \end{pmatrix} \begin{pmatrix} c_2 - e_2 \\ c_3 - e_3 \end{pmatrix} = 0.$$

So $(d_2, d_3) = \gamma(b_2, b_3)$ for some $\gamma \in \mathbb{F}_q$, and $b \cdot c = d \cdot c$ gives

$$b_1^2 + c_2 b_2 + c_3 b_3 = b_1^2 + \gamma c_2 b_2 + \gamma c_3 b_3.$$

Thus, $(1 - \gamma)(c_2 b_2 + c_3 b_3) = 0$. Therefore, either $\gamma = 1$, a contradiction since $b \neq d$, or $c_2 b_2 + c_3 b_3 = 0$, also a contradiction since this implies that $b \cdot c = b_1^2 \neq 0$.

If Red $\in \mathrm{UP}_2 \cup \mathrm{DOWN}_2$, then we have $b_2 = d_2$, $c_2 = e_2$, and either $b \cdot (b - c) = b \cdot (b - e) = 0$ or $c \cdot (c - b) = c \cdot (c - d) = 0$. In the first case, $b \cdot (e - c) = 0$ so $b_3(e_3 - c_3) = 0$. So either $b_3 = 0$ or $e_3 = c_3$, both contradictions. Similarly, in the second case, $c \cdot (b - d) = 0$ means that $c_3(b_3 - d_3) = 0$, which gives the same contradictions.

Finally, if Red $\in$ DOT, then $b \cdot (c - e) = 0$ and $d \cdot (c - e) = 0$. So

$$\begin{pmatrix} b_2 & b_3 \\ d_2 & d_3 \end{pmatrix} \begin{pmatrix} c_2 - e_2 \\ c_3 - e_3 \end{pmatrix} = 0.$$

Therefore, either $c = e$, a contradiction, or $(d_2, d_3) = \lambda(b_2, b_3)$ for some nonzero $\lambda \in \mathbb{F}_q$.

If β is the inner product represented by Red, then we get that

$$\beta = b_1^2 + b_2 c_2 + b_3 c_3$$

$$\beta = b_1^2 + \lambda b_2 c_2 + \lambda b_3 c_3$$

So,

$$(1 - \lambda)(b_2 c_2 + b_3 c_3) = 0.$$

Therefore, either $\lambda = 1$ or $b_2c_2 + b_3c_3 = 0$. If $\lambda = 1$, then $b = d$, a contradiction. So we must assume that $b_2c_2 + b_3c_3 = 0$. But then

$$(b - (b_1, 0, 0)) \cdot (c - (b_1, 0, 0)) = 0.$$

Since $a_b = a_c = (b_1, 0, 0)$ we know that

$$(b - a_b) \cdot (c - a_c) = 0.$$

Therefore, since

$$S(b - a_b, U_{a,T}) = S(c - a_c, U_{a,T}),$$

it must be that either $(0, b_2, b_3)$ or $(0, c_2, c_3)$ is isotropic.

First, assume that $(0, b_2, b_3)$ and $(0, c_2, c_3)$ are linearly independent. Then they must span the linear subspace $U_{a,T}$. Without loss of generality assume that $(0, b_2, b_3)$ is isotropic. Therefore, it is orthogonal to every vector in the subspace $U_{a,T}$. Since this space is defined to be orthogonal to $(1, 0, 0)$, then this means that $(0, b_2, b_3)$ is linearly dependent on $(1, 0, 0)$, a contradiction.

So we must assume that $(0, b_2, b_3)$ and $(0, c_2, c_3)$ are linearly dependent. Since at least one of them is isotropic, then they belong to a totally isotropic one-dimensional linear subspace. So $b_2^2 + b_3^2 = 0$ and $c = (b_1, \lambda b_2, \lambda b_3)$ for some $\lambda \in \mathbb{F}_q$. But then $b \cdot (b - c) = 0$, a contradiction of the assumption that Red $\in$ DOT. ∎

Since these lemmas show that the third and final configuration does not appear, the coloring C' is a $(5, 5)$-coloring of K_n.

CHAPTER 14

A $(5,6)$-coloring construction.

In this chapter, we improve the probabilistic upper bound of $f(n,5,6) \le cn^{3/5}$ by giving an explicit $(5,6)$-coloring of K_n that uses few colors. The new upper bound comes close to matching the best known lower bound in order of magnitude.

THEOREM 14.1. *As $n \to \infty$,*

$$\left(\frac{5}{6}n - \frac{95}{144}\right)^{1/2} \le f(n,5,6) \le n^{1/2} 2^{O\left(\sqrt{\log n}\log\log n\right)}.$$

The lower bound comes from the following lemma, a generalization of an argument used by Erdős and Gyárfás [25], and stated explicitly as equation 11 in [15].

LEMMA 14.1. *Let $t = f(n,p,q)$, then*

$$f\left(\left\lceil \frac{n-1}{t} \right\rceil, p-1, q-1\right) \le t.$$

PROOF. Suppose we have a (p,q)-coloring of K_n with t colors. Fix some vertex x, then at least $\lceil \frac{n-1}{t} \rceil$ vertices must appear in a monochromatic neighborhood of x. The number of colors t must be enough to give a $(p-1,q-1)$-coloring on this set. ∎

Erdős and Gyárfás showed that $\frac{5}{6}(n-1) \le f(n,4,5)$ [25]. This, combined with the lemma, gives the stated lower bound in Theorem 14.1.

The construction providing the upper bound combines two existing constructions with some modification. The first is the modified CFLS construction given in Chapter 12. The second construction is the "algebraic" part of the $(4,4)$-coloring given by Mubayi in [35].

After defining the construction in the next section, we will demonstrate that it avoids many different configurations of colored edges on five or fewer vertices. By ruling these cases out, the algorithm in Appendix A is used to show that no copy of K_5 can span fewer than six distinct colors.

1. The algebraic construction

We will now define the algebraic part of the construction, $\zeta = \zeta_1 \times \zeta_2$. The first part of this construction, ζ_1, is exactly the algebraic part of the $(4, 4)$-coloring given by Mubayi [35]. The second part, ζ_2, is a modification original to this chapter but based on a similar modification used to alter the algebraic portion of the $(5, 5)$-coloring in Chapter 13.

Let $n = q^2$ where q is some odd prime power. Associate each vertex of K_n with a unique vector in the space $\mathbb{F}_q^2$ over the finite field with q elements. Between any two vectors $x = (x_1, x_2)$ and $y = (y_1, y_2)$, we define the color ζ_1 of the edge between them as

$$\zeta_1(xy) = (x_1y_1 - x_2 - y_2, \delta(x_1, y_1))$$

where

$$\delta(x_1, y_1) = \begin{cases} 0 & x_1 = y_1 \\ 1 & x_1 \neq y_1 \end{cases}.$$

Here, all algebraic operations are taken to be the standard ones defined by the finite field.

The modification to this coloring, ζ_2, requires that we give the elements of $\mathbb{F}_q^2$ some linear order. When we combine the algebraic part of the coloring with the modified CFLS coloring, this order will agree with the order put on the binary strings, but for now we just assume that there is some linear order.

For each element $\alpha \in \mathbb{F}_q$ let G_α be the graph with vertex set $\mathbb{F}_q \setminus \{\alpha\}$ such that

$$xy \in E(G) \iff x + y = 2\alpha.$$

It is straightforward to show that the edges of G_α form a complete matching of the vertices. The vertices can therefore be partitioned into two sets, S_α and T_α, such that no edge lies inside either set.

For two distinct elements, $\alpha, \beta \in \mathbb{F}_q$, define the function

$$f_\alpha(\beta) = \begin{cases} S & \beta \in S_\alpha \\ T & \beta \in T_\alpha \end{cases}.$$

Now we can define ζ_2 for two vectors, $x < y$, as

$$\zeta_2(xy) = (f_{x_1}(y_1), f_{y_1}(x_1)).$$

The coloring, ζ_1, gives at most $2q$ colors on q^2 vertices, and the modification, ζ_2, gives four colors. So overall the modified algebraic coloring ζ uses at most $8q = 8\sqrt{n}$ colors.

2. Combining the constructions

Begin with $n = q^2$ for some odd prime power q, and associate each vertex with a distinct vector of $\mathbb{F}_q^2$ as in the previous section. Give some linear order for the elements of the base field, $\mathbb{F}_q$. To each $\alpha \in \mathbb{F}_q$ we associate the unique element $\alpha' \in \{0,1\}^{\lceil \log q \rceil}$ which represents in binary the rank of α under the this linear order.

Let β be the minimum positive integer for which

$$2 \lceil \log q \rceil \leq \beta^2.$$

To a vertex of K_n associated with vector $(x_1, x_2) \in \mathbb{F}_q^2$, we also associate the binary string $(x_1', x_2', 0) \in \{0,1\}^{\beta^2}$ where for each i, x_i' is the binary representation of the rank of x_i, and 0 denotes a string of $\beta^2 - 2\lceil \log q \rceil$ zeros.

The edge-coloring $\varphi \times \zeta$ is then given by applying ζ to the vectors and φ to the binary strings. Since

$$\beta = \Theta\left(\sqrt{2\log q}\right) = \Theta\left(\sqrt{\log n}\right),$$

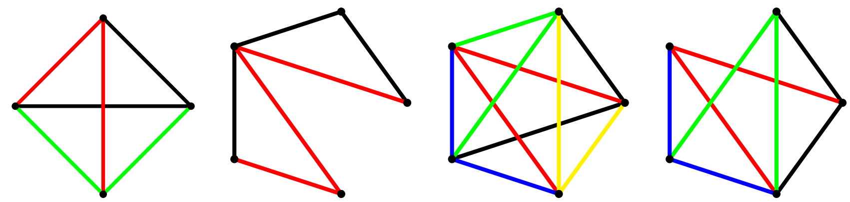


FIGURE 47. Four configurations that each contain a forbidden "color cycle."

it follows that the number of colors used in this combined coloring is at most

$$8q\beta 2^{2\beta} = n^{1/2} 2^{O\left(\sqrt{\log n} \log\log n\right)}$$

colors. This upper bound on the number of colors generalizes to all n by the standard density of primes argument [35, 37].

3. Configurations avoided by CFLS

In Chapter 12, we showed that the modified CFLS coloring, φ, avoids several possible configurations of edge colors on small cliques. Several of these cases, including monochromatic odd cycles, are covered by Lemma 14.2.

3.1. General "color cycle" configurations. Let p and q be positive integers. Assume that we have a copy of K_p under an edge-coloring

$$c : E(K_p) \to \{C_1, \ldots, C_q\}.$$

Define an auxiliary digraph D on the set of edge colors, $V(D) = \{C_1, C_2, \ldots, C_q\}$, such that $C_i \to C_j \in E(D)$ if and only if there exist vertices $v_1, \ldots, v_k \in V(K_p)$ for $k \geq 3$ such that

$$c(v_1v_2) = c(v_2v_3) = \cdots = c(v_{k-1}v_k) = C_i$$

and $c(v_kv_1) = C_j$.

Now, color the directed edges of D "Odd" or "Even" depending on the parity of the number of vertices k that gives the directed edge. Note that multiedges with different parities are possible in D.

LEMMA 14.2. *The CFLS coloring avoids any edge-colored copy of K_p for which the auxiliary digraph on the colors spanned by that clique contains a directed cycle with at least one Odd edge.*

PROOF. Suppose that colors $C_1, \ldots, C_m$ make such a directed cycle:

$$C_1 \to C_2, C_2 \to C_3, \ldots, C_m \to C_1 \in E(D)$$

such that (without loss of generality) $C_1 \to C_2$ is colored Odd. Then there exist vertices $v_1, \ldots, v_k \in V(K_p)$ for some odd integer $k \geq 3$ for which $\varphi_1(v_i v_{i+1}) = C_1$ for $i = 1, \ldots, k-1$ and $\varphi_1(v_k v_1) = C_2$. Let the zero coordinate of the color C_1 be $(i, \{x, y\})$. Without loss, assume that $v_1^{(i)} = x$. It follows that $v_k^{(i)} = x$ as well. Therefore, the ith coordinate of C_2 is zero.

Now, each subsequent directed edge $C_j \to C_{j+1}$ for $j = 2, \ldots, m$, regardless of color, forces the ith coordinate of the "head" color to be zero as well. To see this assume that C_j is zero in its ith coordinate. A monochromatic path in color C_j, $u_1 u_2 \cdots u_l$, implies that u_1 agrees with u_l at i. Therefore, the ith coordinate of C_{j+1} must also be zero. The same must be true for C_1 since this is a directed cycle, a contradiction. ■

For the $(5, 6)$-coloring we use Lemma 14.2 to eliminate the configurations shown in Figure 47 as well as monochromatic odd cycles.

3.2. Configurations containing a monochromatic P_3. Assume that we have an edge-colored K_5 that contains a monochromatic P_3 on vertices *abcd* in color Black. The edges *ac* and *bd* cannot be Black since color classes are bipartite in CFLS. So we color edge *ac* Red. Let $\text{Black}_0 = (i, \{x, y\})$, then $\text{Red}_i = 0$. Therefore, any Red edge from the Black P_3 to vertex e fixes the value of $e^{(i)}$ as either x or y. CFLS would then forbid any third color from having an edge between an x and a y as well as one between two vertices that agree at i.

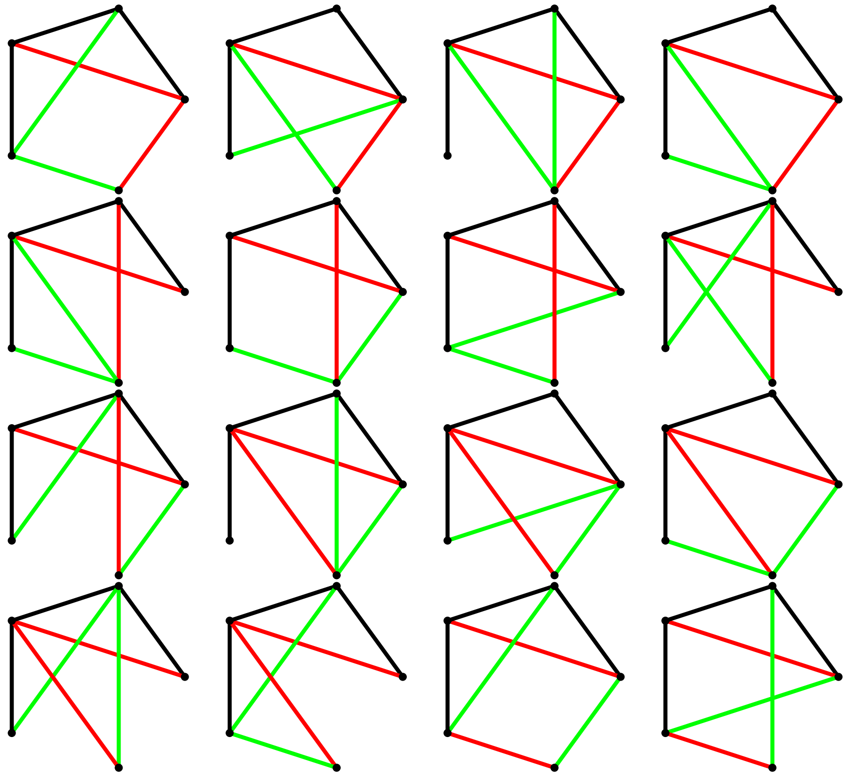


FIGURE 48. CFLS forbids these 16 configurations, each contain a monochromatic P_3.

There are 16 possible configurations that fit this description when we do not consider those in which the third color never touches vertex e (these cases are summarized separately). Figure 48 presents these 16 configurations.

3.3. Configurations containing an alternating C_4. Next, consider configurations that contain a 2-colored C_4 with alternating colors. If the configuration also has two same-colored edges adjacent at the fifth vertex so that the other two endpoints are each incident to either endpoint of an edge of the C_4 (as shown by the edge-colored cliques in Figure 49), then we can say that under CFLS, the color from the fifth vertex must be distinct from any color spanned by the other four vertices. Moreover, none of these spanned colors can be incident with the fifth vertex.

LEMMA 14.3. *Let a, b, c, d, e be distinct vertices such that $\varphi(ab) = \varphi(cd) = \alpha$, $\varphi(bc) = \varphi(ad) = \beta$, $\varphi(ae) = \varphi(de) = \gamma$, $\varphi(ac) = \pi_1$, $\varphi(bd) = \pi_2$, $\varphi(be) = \lambda_1$, and*

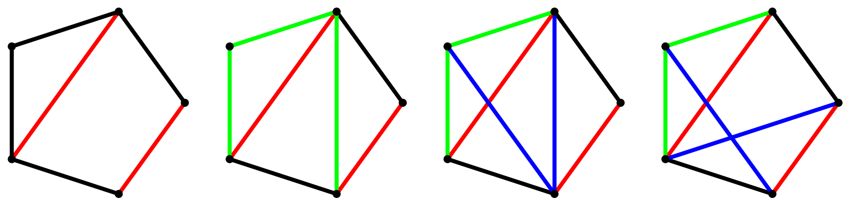


FIGURE 49. Configurations containing an alternating C_4.

$\varphi(ce) = \lambda_2$. *Then*

$$\gamma, \lambda_1, \lambda_2 \notin \{\alpha, \beta, \pi_1, \pi_2\}.$$

PROOF. Let $\beta_0 = (i, \{x, y\})$. Without loss of generality, we may assume that $a^{(i)} = x$ and $d^{(i)} = y$. If $e^{(i)} = x$ or $e^{(i)} = y$, then we get that $\gamma_i = 0$ and $\gamma_i \neq 0$, a contradiction. Hence,

$$e^{(i)} = z \notin \{x, y\}.$$

This alone shows that $\gamma \neq \beta$.

If $\gamma = \alpha$, then $\alpha_i \neq 0$ so it must be the case that $b^{(i)} = y$ and $c^{(i)} = x$. Therefore, three distinct binary strings, x, y, z, pairwise have the same first index of difference α_i. This is impossible since two must either both be zero or both be one.

If $\gamma = \pi_1$, then $c^{(i)} = y$ and so $b^{(i)} = x$. Hence, the distinct binary strings x, y, z again pairwise have the same first index of difference, γ_i, a contradiction. The same argument applies if $\gamma = \pi_2$.

Next, assume that $\lambda_1 = \pi_1$. Since $e^{(i)} = z$ and $b^{(i)} \in \{x, y\}$, then λ_1 is nonzero at i. Since $\varphi(ac) = \lambda_1$ and $a^{(i)} = x$, then $c^{(i)} = y$ and so $b^{(i)} = x$. Hence, colors γ and λ_1 must agree in coordinate i. This again gives us that x, y, z all pairwise differ at the same first index, a contradiction. The same argument applies if $\lambda_2 = \pi_2$.

If $\lambda_1 = \pi_2$, then $b^{(i)} = x$ and so $c^{(i)} = y$. So again γ and λ_1 agree at coordinate i. This again forces the contradiction with x, y, z. The same argument applies if $\lambda_2 = \pi_1$.

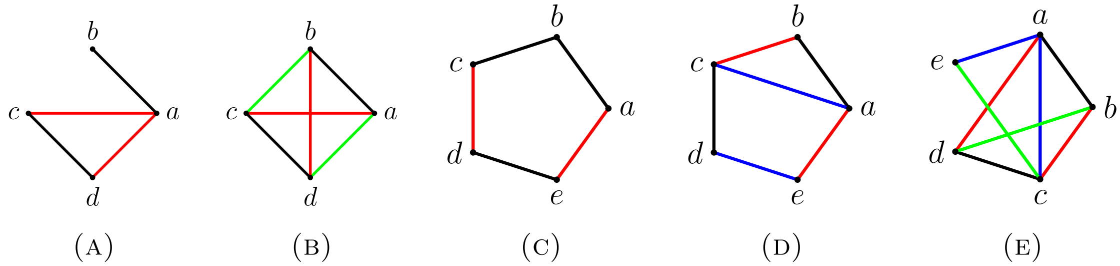


FIGURE 50. Five additional configurations avoided by φ.

Finally, note that $\lambda_1, \lambda_2 \neq \beta$ since $e^{(i)} = z$. If $\lambda_1 = \alpha$, then $b^{(i)} = y$ and so γ and α agree at coordinate i which gives the same contradiction as before. The same reasoning applies if $\lambda_2 = \alpha$. ■

3.4. Additional configurations. We finish this section by showing that the modified CFLS coloring eliminates the additional cases shown in Figure 50. We already showed in Chapter 12 that this coloring eliminates Figures 50a and 50b.

LEMMA 14.4. *Let a, b, c, d, e be distinct vertices. The CFLS coloring forbids $\varphi(ab) = \varphi(bc) = \varphi(de) = \alpha$ and $\varphi(cd) = \varphi(ae) = \beta$ (see Figure 50c).*

PROOF. Assume that this can happen and that $\alpha_0 = (i, \{x, y\})$. Without loss of generality, assume that $b^{(i)} = x$ and $a^{(i)} = c^{(i)} = y$. Moreover, without loss we can assume that $d^{(i)} = x$ and $e^{(i)} = y$. Then $a^{(i)} = e^{(i)}$ implies that $\beta_i = 0$. But $c^{(i)} \neq d^{(i)}$ implies that $\beta_i = 0$, a contradiction. ■

LEMMA 14.5. *Let a, b, c, d, e be distinct vertices. The CFLS coloring forbids $\varphi(ab) = \varphi(cd) = \alpha$, $\varphi(ae) = \varphi(bc) = \beta$, and $\varphi(ac) = \varphi(de) = \gamma$ (see Figure 50d).*

PROOF. Let $\gamma_0 = (i, \{x, y\})$. Without loss of generality we may assume that $a^{(i)} = x$ and $c^{(i)} = y$. If $d^{(i)} = y$ and $e^{(i)} = x$, then $\alpha_i = \beta_i = 0$ and so $b^{(i)} = x$ and $b^{(i)} = y$, a contradiction. Hence, $d^{(i)} = x$ and $e^{(i)} = y$. So $b^{(i)} = z \notin \{x, y\}$. Moreover, $\alpha_i = \beta_i$ both go between x and y. Therefore, the three distinct binary strings x, y, z are all pairwise different at the same first index, α_i, a contradiction. ■

LEMMA 14.6. *It cannot be the case that a, b, c, d, e are distinct vertices such that $\varphi(ab) = \varphi(cd) = \alpha$, $\varphi(bc) = \varphi(ad) = \beta$, $\varphi(ae) = \varphi(ac) = \gamma$, and $\varphi(bd) = \varphi(bc) = \pi$ (see Figure 50e).*

PROOF. Let $\pi_0 = (i, \{x, y\})$. Without loss of generality we may assume that $e^{(i)} = x$ and $c^{(i)} = y$. It follows from color γ that $a^{(i)} \notin \{x, y\}$. Since $d^{(i)} \in \{x, y\}$, then it follows that $\beta_i \neq 0$. Hence, $b^{(i)} = x$ and so $d^{(i)} = y$. So $\varphi(cd) = \alpha$ implies that $\alpha_i = 0$, but $\varphi(ab) = \alpha$ implies that $\alpha_i \neq 0$, a contradiction. ■

4. Configurations avoided by the algebraic coloring

We begin with two basic lemmas about the algebraic construction ζ.

LEMMA 14.7. *Let a, b, c be three distinct vertices such that $\zeta(ab) = \zeta(ac)$, then $b_1 \neq c_1$.*

PROOF. Let $\zeta(ab) = \zeta(ac)$. Then

$$a_1 b_1 - a_2 - b_2 = a_1 c_1 - a_2 - c_2$$

$$a_1(b_1 - c_1) = b_2 - c_2.$$

If $b_1 = c_1$, then $b_2 = c_2$ as well. Hence, $b = c$, a contradiction. ■

LEMMA 14.8. *Let a, b, c, d be four distinct vertices such that $\zeta(ab) = \zeta(ac)$ and $\zeta(db) = \zeta(dc)$, then $a_1 = d_1$.*

PROOF. Since $b_1 \neq c_1$ by Lemma 14.7, then we know that

$$a_1 = \frac{b_2 - c_2}{b_1 - c_1} = d_1.$$

■

As shown in [35], the algebraic construction ζ_1 avoids monochromatic C_4s (see Figure 51a) as well as the configuration shown in Figure 51b. We provide these two results for completeness.

LEMMA 14.9. *Let a, b, c, d be distinct vertices. The algebraic coloring ζ forbids $\zeta(ab) = \zeta(bc) = \zeta(cd) = \zeta(da)$ (see Figure 51a).*

PROOF. By Lemma 14.7, $b_1 \neq d_1$. But $b_1 = d_1$ by Lemma 14.8, a contradiction. ■

LEMMA 14.10. *Let a, b, c, d be distinct vertices. The algebraic coloring ζ forbids $\zeta(ab) = \zeta(ac) = \zeta(ad)$ and $\zeta(bc) = \zeta(bd)$ (see Figure 51b).*

PROOF. By Lemma 14.8, $a_1 = b_1$. Therefore, $\delta(a_1, b_1) = 0$. So $c_1 = a_1 = d_1$. But $c_1 \neq d_1$ by Lemma 14.7. ■

Now we will take care of a few additional configurations. We will use the following technical lemma.

LEMMA 14.11. *Let a, b, c, d be four distinct vertices such that $\zeta(ab) = \zeta(cd)$ and $\zeta(bc) = \zeta(ad)$. Then*

$$(a_1 + c_1)(b_1 - d_1) = 2(b_2 - d_2).$$

PROOF. The two colors give us the following relations:

$$a_1b_1 - a_2 - b_2 = c_1d_1 - c_2 - d_2$$

$$a_1d_1 - a_2 - d_2 = c_1b_1 - b_2 - b_2.$$

We subtract the second equation from the first to get the desired equation. ■

LEMMA 14.12. *Let a, b, c, d, e be distinct vertices. Then $\mathcal{C}(ab) = \mathcal{C}(cd)$, $\mathcal{C}(bc) = \mathcal{C}(ad)$, $\mathcal{C}(ae) = \mathcal{C}(ce)$, and $\mathcal{C}(be) = \mathcal{C}(de)$ (see Figure 51c) is forbidden by the coloring $\mathcal{C} = \varphi \times \zeta$.*

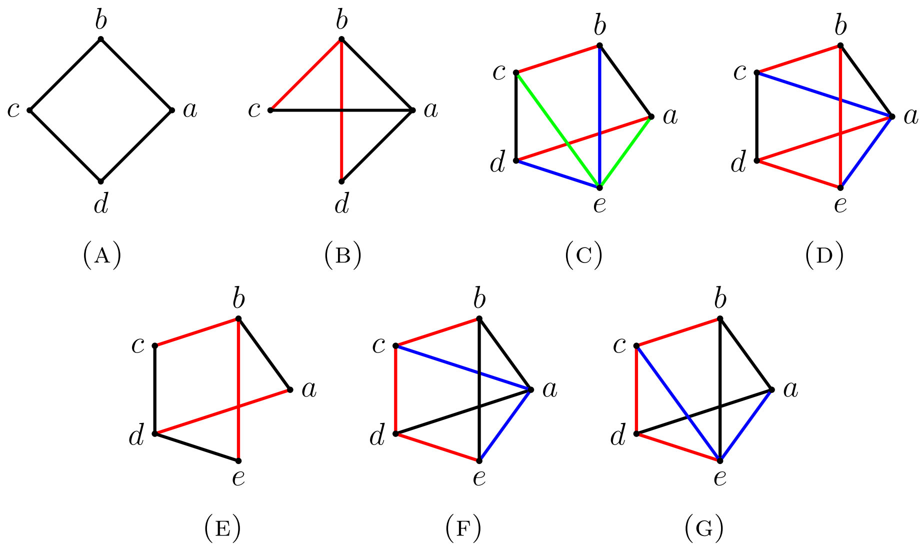


FIGURE 51. Configurations eliminated by the modified algebraic coloring.

PROOF. By Lemma 14.11, we know that

$$(a_1 + c_1)(b_1 - d_1) = 2(b_2 - d_2),$$

and from $\zeta(be) = \zeta(de)$ we get that

$$e_1(b_1 - d_1) = b_2 - d_2.$$

Therefore,

$$(a_1 + c_1)(b_1 - d_1) = 2e_1(b_1 - d_1)$$

$$a_1 + c_1 = 2e_1$$

since $b_1 \neq d_1$ by Lemma 14.7. So $f_{e_1}(a_1) \neq f_{e_1}(c_1)$. By Lemma 12.7, $\varphi(ae) = \varphi(ce)$ implies that either $a, c < e$ or $e < a, c$. In either case, $\zeta_2(ae) \neq \zeta_2(ce)$, a contradiction. ∎

LEMMA 14.13. *Let a, b, c, d, e be distinct vertices. Then $\mathcal{C}(ab) = \mathcal{C}(cd)$, $\mathcal{C}(bc) = \mathcal{C}(ad) = \mathcal{C}(be) = \mathcal{C}(de)$, and $\mathcal{C}(ac) = \mathcal{C}(ae)$ (see Figure 51d) is forbidden by the coloring $\mathcal{C} = \varphi \times \zeta$.*

PROOF. As in the previous proof we get that $a_1 + c_1 = 2e_1$. By Lemma 14.8 we know that $b_1 = a_1$. Therefore, it follows from the second part of ζ_1 that $c_1 = d_1$. Therefore, $b_1 + d_1 = 2e_1$ and so $f_{e_1}(b_1) \neq f_{e_1}(d_1)$. As before, this fact along with Lemma 12.7 forces $\zeta_2(be) \neq \zeta_2(de)$, a contradiction. ∎

LEMMA 14.14. *Let a, b, c, d, e be distinct vertices. Then $\zeta(ab) = \zeta(cd) = \zeta(de)$ and $\zeta(bc) = \zeta(ad) = \zeta(be)$ (see Figure 51e) is forbidden by the algebraic coloring ζ.*

PROOF. By Lemma 14.11 we know that

$$(a_1 + c_1)(b_1 - d_1) = 2(b_2 - d_2),$$

and by Lemma 14.8 we know that $b_1 = d_1$. Hence, $b_2 - d_2 = 0$ and so $b = d$, a contradiction. ∎

LEMMA 14.15. *Let a, b, c, d, e be distinct vertices. Then $\zeta(bc) = \zeta(cd) = \zeta(de)$, $\zeta(eb) = \zeta(ba) = \zeta(ad)$, and $\zeta(ac) = \zeta(ae)$ (see Figure 51f) is forbidden by the algebraic coloring ζ.*

PROOF. By Lemma 14.7 we get that $b_1 \neq d_1$. By Lemma 14.8 we get that $a_1 = d_1$. Therefore, since the color encodes equality in the first coordinate we see that $b_1 = d_1$, a contradiction. ∎

LEMMA 14.16. *Let a, b, c, d, e be distinct vertices. Then $\zeta(bc) = \zeta(cd) = \zeta(de)$, $\zeta(eb) = \zeta(ba) = \zeta(ad)$, and $\zeta(ec) = \zeta(ae)$ (see Figure 51g) is forbidden by the algebraic coloring ζ.*

PROOF. By Lemma 14.8 we get that $a_1 = c_1$. By Lemma 14.7, we get that $a_1 \neq c_1$, a contradiction. ∎

CHAPTER 15

Additional questions about (p, q)-colorings.

1. General (p, p)-coloring with $n^{1/(p-2)+o(1)}$ colors

In general, the method of combining a variation of the CFLS coloring with a general algebraic construction using vectors from a space of dimension $p-2$ has the potential to show that

$$n^{1/(p-2)} \leq f(n, p, p) = n^{1/(p-2)+o(1)}$$

for $p \geq 6$. Once the difficulty of case analysis is circumvented, then properties of vector spaces could hopefully be used to eliminate p-cliques which span only $p-1$ colors.

2. A better bound for $f(n, 5, 7)$

The $(5, 5)$ and $(5, 6)$-colorings have left $q = 7$ as the only remaining value for which a polynomial gap (in the order) between the known upper and lower bounds exists when $p = 5$. In this case we know that there are positive constants c_1 and c_2 such that

$$c_1 n^{2/3} \leq f(n, 5, 7) \leq c_2 n^{3/4}.$$

Is it possible to lower the upper bound to $n^{2/3+o(1)}$ using methods similar to those in Chapters 13 and 14?

3. The hypergraph version

Let $f_k(n, p, q)$ denote the minimum number of colors needed to color the edges of the complete k-uniform hypergraph on n vertices in such a way so that every p vertices span at least q colors. To date, little work has been done on this hypergraph version of the problem. There appear to be only two papers published on the topic, one by Conlon, Fox, Lee, and Sudakov [16] and one by Mubayi [36].

The main problem in the area is to determine for fixed p the threshold values for q at which there are large jumps in the order of the $f(n, p, q)$. For $p > k \geq 3$ and $0 < i < k$, Conlon, Fox, Lee, and Sudakov [16] showed that there exists a constant c dependent on k, p, and i for which

$$f_k\left(n, p, \binom{p-i}{k-i} + 1\right) = \Omega\left(log_{(i-1)}(n)^c\right)$$

where we define $log_0(x) = x$ and $\log_i(x) = \log\left(\log_{i-1}(x)\right)$. They conjecture that this value of q is such a jump in the order. Is it true that

$$f_k\left(n, p, \binom{p-i}{k-i}\right) = \left(\log_{(i-1)} n\right)^{o(1)}?$$

Perhaps algebraic constructions have a place in this area as well.

APPENDIX A

Algorithm for reducing cases

This chapter contains material from a paper to be published in Combinatorics, Probability & Computing. [14]

The following algorithm is not difficult to verify, so we present it here without proof. The specific implementation we rely on is a Python script that can be found (with comments) at http://homepages.math.uic.edu/~acamer4/EdgeColors56.py. This particular script forbids monochromatic odd cycles and the edge-colorings shown in Figures 47, 48, 49, 50, as well as Figures 51a and 51b. The output is seven edge-colored copies of K_5 that each contain one of the remaining configurations in Figure 51.

Suppose we want to find every edge-coloring, up to isomorphism, of K_n that uses at most m colors and does not contain a copy of any $F \in \mathcal{F}$, a list of edge-colored complete graphs on n or fewer vertices. The algorithm takes $\mathcal{F}$, n, and m as input and returns a list $\mathcal{R}$ of edge-colorings of K_n satisfying these requirements.

For each $k = 3, \ldots, n$, the algorithm creates a list L_k of acceptable edge-colorings of K_k by adding a new vertex to each K_{k-1} listed in L_{k-1} (where L_2 is the list of exactly one K_2 with its single edge given color 1), and then coloring the $k-1$ new edges in all possible ways from the color set $[m]$. For each graph in L_{k-1} and each way to color the new edges, we test the resulting graph to see if it contains any of the forbidden edge-colorings. If it does, then we move on. If not, then we test it against the new list L_k to see if it is isomorphic to any of the colorings of K_k already on the list. If it is, then we move on. Otherwise, we add it to the list L_k. The algorithm terminates when it has tested all colorings of K_n.

Algorithm 1: List all edge-colorings with no forbidden subcoloring

---

Data: number of vertices n; maximum number of colors m; list of forbidden colorings $\mathcal{F}$

initialize L_2 as list containing one K_2 with its edges colored 1;

for $k = 3, \ldots, n$ **do**

  initialize empty list L_k;

  for $H \in L_{k-1}$ **do**

    for *each function* $f : [k-1] \to [m]$ **do**

      let G be K_k with edge-colors same as H on the first $k-1$ vertices and color $f(i)$ on edge ki for $i = 1, \ldots, k-1$;

      if G *contains no element of* $\mathcal{F}$ *and is isomorphic to no element of* L_k **then**

        add G to the list L_k

      end

    end

  end

end

return L_n

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VITA

NAME:	Alex Cameron
EDUCATION:	B.A., Mathematics and Physics, Hendrix College, Conway, Arkansas, 2006
	J.D., Law, Duke University, Durham, North Carolina, 2011
	M.S., Mathematics, University of Illinois, Chicago, Illinois, 2017
	Ph.D., Mathematics, University of Illinois, Chicago, Illinois, 2018
TEACHING:	University of Illinois at Chicago, 2011-2018
PUBLICATIONS:	Alex Cameron. Extremal Numbers for Directed Hypergraphs with Two Edges. The Electronic Journal of Combinatorics 25(1), P1.56 (2018).
	Alex Cameron and Emily Heath. A $(5,5)$-coloring of K_n with few colors. To appear in Combinatorics, Probability and Computing, Preprint posted February 2017.
	Alex Cameron. Counting Ring Homomorphisms. In PME Journal, Vol. 13, pages 65-67, 2010.
POSTERS & TALKS	AMS Special Session on Research from GRWC Joint Mathematics Meetings, San Diego, CA Talk: A $(5,5)$-coloring of K_n with few colors, 2018.
	29th Cumberland Conference on Combinatorics, Graph Theory and Computing Vanderbilt University, Nashville, TN Talk: A $(5,5)$-coloring of K_n with few colors, 2017.
	Combinatorics Seminar Rényi Institute, Budapest, Hungary Talk: Extremal Problems on Directed Hypergraphs, 2016.

Combinatorics Seminar
University of Szeged, Szeged, Hungary
Talk: Extremal Problems on Directed Hypergraphs, 2016.

EXCILL III: Extremal Combinatorics at Illinois
IIT, Chicago, IL
Poster: Extremal Problems on Generalized Directed Hypergraphs, 2016.

Graduate Research Workshop in Combinatorics
University of Wyoming, Laramie, WY
Talk: (p, q)-colorings, 2016.

Graduate Students Combinatorics Conference
Clemson University, Clemson, SC
Talk: Extremal Problems on Directed Hypergraphs: Forbidden subgraphs with two edges, 2016.

2006 Oklahoma-Arkansas MAA meeting
University of Arkansas, Fayetteville, AR
Talk: Counting Ring Homomorphisms, 2006.

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EMILY HEATH
Department of Mathematics
University of Illinois at Urbana-Champaign
Urbana, IL 61801 USA

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