Effective Divisors on Kontsevich Moduli Spaces

by

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To David, without whom I would still be researching this thesis.

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SUMMARY

Let $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)$ denote the Kontsevich moduli space of n-pointed rational curves of degree d in projective r-space. The rational Picard group of this space was computed by Pandharipande in (1), and in the case $n = 0, r \geq d$ its cone of effective $\mathbb{Q}$-divisors was computed by Coskun, Harris and Starr in (2). In this thesis we extend the former result to certain Gromov-Witten varieties, and the latter result to the case $n = 1$.

CHAPTER 1

INTRODUCTION

In this thesis, we will describe the effective cone of the Kontsevich moduli space which compactifies the space of pointed rational normal curves in terms of Pandharipande's generators and an additional geometrically meaningful class, the divisor D_{deg} of degree d rational curves in $\mathbb{P}^d$ whose image is contained in a hyperplane. We will also show that if we fix the image of a marked point (thereby obtaining the simplest example of a so-called Gromov-Witten variety), we can use Pandharipande's generators to obtain a presentation for the rational Picard groups of the Gromov-Witten variety.

The Kontsevich moduli space $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)$ is the course moduli space of the stack $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ which represents the functor of families of stable maps, morphisms $f : (C, p_1, ..., p_n) \to \mathbb{P}^r$ such that

- C is a reduced, connected, at-worst-nodal curve of arithmetic genus 0.
- The points $p_1, ..., p_n \in C$ are distinct, smooth points.
- The line bundle $L := f^*\mathcal{O}_{\mathbb{P}^r}(1)$ has degree d on C (i.e. $\sum_i \deg(L|_{C_i}) = d$).

These spaces were introduced by Maxim Kontsevich in (3) for use in the study of mirror symmetry. Kontsevich employed them in an example to solve the problem of counting rational plane curves of degree d passing through $3d-1$ points, which he proved is determined via a recursive formula and the base case $d = 1$ (there is a unique line through any two distinct points). Foundational issues are thoroughly treated in (4). In the special case $d = 1$, we recover the Grassmannian of lines in $\mathbb{P}^r$ for $n = 0$, and the universal line over the Grassmannian for

$n = 1$. For larger n, we get a simple compactification of the space of lines with n distinct marked points; when two points come together, an additional copy of $\mathbb{P}^1$ "sprouts" from the point of collision on which f has degree 0, and this configuration is unique up to isomorphism. In (1) and (5), Rahul Pandharipande computed the rational Picard groups of these spaces and subsequently their canonical classes in terms of his generators.

Of particular interest in the case $n > 0$ are an n-tuple of natural maps

$$ev_i : \overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d) \to \mathbb{P}^r$$

given on closed points by $(C, p_1, ..., p_n, f) \mapsto f(p_i)$, which by pulling back $\mathcal{O}_{\mathbb{P}^r}(1)$ yield n natural line bundles on the moduli space. One can also use these maps to define Gromov-Witten varieties, which are obtained by taking preimages of linear spaces in $\mathbb{P}^r$ and intersecting them.

Let X be an algebraic variety; X may be singular, but no worse than $\mathbb{Q}$-factorial. The Picard group is one of the most fundamental invariants of X. It is the codimension-1 group of Chow cycles, hence it is closely related to the 2nd topological cohomology group in the classical topology over $\mathbb{C}$, for example (at least in the smooth case) by the exponential exact sequence

$$0 \to 2\pi\sqrt{-1}\mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0.$$

In certain situations it is more convenient to pass to the tensor product with $\mathbb{Q}$, as we will do here. In the simplest case of a Gromov-Witten variety, the preimage of a point, we will show that the Picard group can be presented as a quotient, with one of Pandharipande's generators spanning the kernel of the quotient map.

Main Theorem 1. *For a point $p \in \mathbb{P}^r$ and any $i = 1, ..., n$, the rational Picard group of $ev_i^{-1}(p)$ is a quotient of $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d))$; the kernel of the quotient map is generated by $\mathcal{L} := ev_i^* \mathcal{O}_{\mathbb{P}^r}(1)$.*

In general, the rational Picard vector space contains a number of cones of divisors which are in some sense "positive." We are interested here in effective divisors; more precisely, we will compute the closed cone of psuedoeffective divisors on the Kontsevich space when $n = 1$ and $d \geq r$, which encodes rational contractions of the space.

Main Theorem 2. *The extremal rays of the effective cone of $\overline{\mathrm{M}}_{0,1}(\mathbb{P}^r, d)$ for $r \geq d$ are*

1. *D_{deg}, the divisor of stable maps whose image does not span a hyperplane in $\mathbb{P}^d$.*
2. *$\mathcal{L}$, the divisor of stable maps sending the marked point p to a fixed hyperplane.*
3. *Those rays which pullback to $\mathfrak{S}_{d+1}$-invariant extremal rays under Kapranov's embedding*

$$K : \overline{\mathrm{M}}_{0,d+2} \hookrightarrow \overline{\mathrm{M}}_{0,1}(\mathbb{P}^d, d).$$

CHAPTER 2

PRELIMINARIES & BACKGROUND

We first fix some conventions and notation for the remainder of this document. We assume (for the most part tacitly) familiarity with the contents of Robin Hartshorne's textbook (6), and we follow his conventions unless stated otherwise. We work over the field of complex numbers $\mathbb{C}$. A scheme is Noetherian and separated. A variety is a reduced scheme of finite type over $\mathbb{C}$, but is not necessarily irreducible. Curves are connected (projective unless stated otherwise) varieties of dimension 1. We denote by $\mathfrak{S}_d$ the symmetric group on d symbols.

2.1 Effective Divisors

A more complete discussion of what follows in this section can be found in *Positivity in Algebraic Geometry I* by R.K. Lazarsfeld (7).

A scheme X is $\mathbb{Q}$-**factorial** if every Weil divisor on X is $\mathbb{Q}$-Cartier. Let X be a $\mathbb{Q}$-factorial projective variety of dimension n, and let $N^1(X)_{\mathbb{Q}}$ denote the Neron-Severi group of $\mathbb{Q}$-divisors on X modulo numerical equivalence. Recall that a Cartier divisor D is **effective** if it is given locally by a regular function; equivalently, D is effective if the associated line bundle $\mathcal{O}_X(D)$ admits a global section. A $\mathbb{Q}$-divisor class is said to be **effective** if it is a $\mathbb{Q}_{>0}$-linear combination of classes of effective Cartier divisors. We may thus define the **psuedoeffective cone**

$$\overline{\mathrm{Eff}}(X) \subset N_1(X)_{\mathbb{Q}}$$

to be the closure of the convex cone of effective divisor classes. A divisor whose class lies in this cone is called **psuedoeffective**.

Remark 2.1.1. For all examples considered in this thesis, we have an isomorphism $N^1_{\mathbb{Q}} \cong \mathrm{Pic}_{\mathbb{Q}}$. As such, we will often refer to the Neron-Severi space as the Picard group; implicit in this abuse of language is another abuse, namely that we will refer to the rational Picard vector space as the Picard group throughout the remainder of the document since integral Picard groups are never considered.

Remark 2.1.2. It is a useful convention in modern birational geometry to regard a curve on a variety not as a closed subscheme $C \subset X$ but as a map $f : C \to X$, which may not be injective or even immersive. This is a more natural and flexible setting for birational geometry which allows us to do things like replace an immersed curve with its normalization or pass to a finite cover without changing the type of object under consideration.

Definition 2.1.3. A curve $f : C \to X$ with irreducible domain is called **moving** if (images of) its deformations cover a Zariski-dense subset of X.

The following lemma is fundamental to the study of cones of effective divisors:

Lemma 2.1.4. *Let D be an irreducible effective divisor in X, and let $f : C \to X$ be a moving curve. Then the intersection number $C.D := \deg(f^*(D))$ is nonnegative.*

Proof. Since C is moving, it has a deformation C' whose image is not contained in D, so that $C'.D \geq 0$. Since the intersection number is invariant under flat deformations, we conclude that $C.D \geq 0$. □

2.1.1 Computing Effective Cones

The simple idea underlying all cone computations in Mori theory is that an equality is two inequalities, which in this case means that we need to produce cones contained in the effective cone as well as cones in which it is itself contained, and make these match up. In practice for effective cones, the former amounts to producing a list of effective divisors (which tautologically spans a subcone), which will be the appropriate "inner bound" if we list all extremal rays of the cone. Obtaining the "outer bound" typically requires the construction of moving curves, which by 2.1.4 cut out a hyperplane which cannot intersect the interior of the effective cone. A suitable collection of moving curves thus induces a cone containing the effective cone, which was the strategy employed by (2) in their investigation of the effective cone of the Kontsevich space.

In some situations (such as ours) the construction of moving curves seems to be out of reach; however, there are other less systematic approaches that may still yield the necessary inequalities. For example: although the theory of positivity in higher codimension is still in development, a higher dimensional subvariety which is well-understood can be just as useful as a moving curve if the deformations needed to make transversality arguments work can be described explicitly. In the proof of one of our main theorems, we make use of an observation of Mikhail Kapranov that choosing $d+2$ points in general position in $\mathbb{P}^d$ induces a closed embedding $\overline{\mathrm{M}}_{0,d+2}/\mathfrak{S}_{d+2} \to \overline{\mathrm{M}}_{0,0}(\mathbb{P}^d, d)$ along which we may fruitfully pull back divisor classes. It was noted by Sean Keel that by allowing the points which define the embedding to vary, we obtain many deformations which lead to an alternative proof of the main theorem of (2). Our theorem concerns the case when $n = 1$; we make use of a theorem of William Rulla regarding the space of rational curves with a single distinguished marked point and many unlabelled marked points to show that the family of Kapranov embeddings behaves just as nicely if we slightly decrease the symmetry of the situation.

2.1.2 An Example

Obviously divisorial cones are only interesting on varieties with Picard rank greater than 1; we will begin with a simple such example:

Example 2.1.5. Let $\pi : \mathbb{F}_1 \to \mathbb{P}^2$ denote the blowup of $\mathbb{P}^2$ at a point. The Picard group is generated here by H, the pullback of the class of a line, and E, the class of the exceptional divisor. To determine the effective cone in the Picard rank 2 case we just need to determine the two extremal rays. To do this, we simply need to find a pair of effective divisors with a pair of orthogonal moving curves. Since H is a moving curve but $H.E = 0$, E is one extremal ray.

Now consider the class of a line through the center of the blowup, $H - E$. This is a moving curve, and $(H - E)^2 = H^2 + E^2 = 1 - 1 = 0$, so $H - E$ is the other extremal ray (associated to the Fano fibration $\mathbb{F}_1 \to \mathbb{P}^1$).

2.1.3 Other Notions of Positivity for Divisors

Along with effective divisors, the most familiar notion of positivity is ampleness; recall that a divisor is ample if a multiple is very ample, that is to say the pullback of $\mathcal{O}(1)$ under a projective embedding. The following lemma shows that ampleness is, unlike effectivity, a numerical property:

Lemma 2.1.6. *(Nakai-Moishezon Criterion) Let D be a divisor on a projective scheme X. Then D is ample if and only if $D^k.V > 0$ for all irreducible k-dimensional subvarieties $V \subset X$ of positive dimension.*

A theorem of Steve Kleiman says that a much weaker property, $C.D \geq 0$ for all irreducible curves, implies the weak versions of the intersection inequalities in the previous lemma, i.e. limits of ample classes can be detected on curves. This motivates the following definition:

Definition 2.1.7. A divisor D on a projective scheme X is **nef** if $C.D \geq 0$ for all irreducible curves $C \subset X$.

2.2 Moduli Spaces of Pointed Rational Curves

A smooth curve of genus 0 is **stable** if it is equipped with three or more marked points. A point on a nodal curve of arithmetic genus 0 with marked smooth points is **special** if it is marked or if it lies over a node in the normalization; such a curve is **stable** if every component of its normalization has at least three special points. Since the action of $PGL(2,\mathbb{C})$ on $\mathbb{P}^1$ is 3-transitive, there is a unique 3-pointed rational curve up to isomorphism (note that three marked points are insufficient to stabilize a nodal union of two copies of $\mathbb{P}^1$). The moduli space $\overline{\mathrm{M}}_{0,n}$ of stable n-pointed rational curves has dimension

$$\dim(\overline{\mathrm{M}}_{0,n}) = \dim((\mathbb{P}^1)^n) - \dim(PGL(2,\mathbb{C}) = n-3.$$

The boundary of this moduli space was described by Finn Knudsen in (8), who showed that a family of n-pointed curves (thought of as a codimension 1 fibration with n sections) in which the marked points collide can be uniquely transformed into a family of nodal curves with all marked points distinct by blowing up the loci along which sections intersect. What this amounts to is that if $p \to q$, one replaces q by a rational tail now marked with p and q at distinct points. Since this rational component has 3 special points, it is unique up to isomorphism.

The irreducible components of the boundary of $\overline{\mathrm{M}}_{0,n}$ are in bijection with partitions $[n] = S \sqcup T$ of the set $[n] := \{1, ..., n\}$ such that $|S|, |T| \geq 2$ (we do not distinguish between $S \sqcup T$ and $T \sqcup S$). The general point of the irreducible component which we will label $B_{S,T}$ parametrizes a nodal union of two copies of $\mathbb{P}^1$, one labeled with S and the other with T. Since the PGL-action on each component normalizes the location of the node and two of the markings, the dimension of this locus is

$$(|S| - 2) + (|T| - 2) = n - 4,$$

showing that we get a divisor in $\overline{\mathrm{M}}_{0,n}$.

It is classical (Castelnuovo's Lemma, see (9)) that through $(r + 3)$ general points in $\mathbb{P}^r$ there passes a unique rational normal curve, so through $(r + 2)$ general points there is an $(r - 1)$-dimensional family of rational normal curves. Taking $r + 3 = n$, we have an $(n - 3)$-dimensional family of rational curves with n-marked points. This is highly suggestive of the observation made by Kapranov (see (10)): the choice of $n - 1$ general points defines a rational map $\mathbb{P}^{n-3} \dashrightarrow \overline{\mathrm{M}}_{0,n}$ which can be resolved by a sequence of blowups along smooth centers. The resulting smooth variety $\widetilde{\mathbb{P}}^{n-3}$ comes with a morphism $f : \widetilde{\mathbb{P}}^{n-3} \to \overline{\mathrm{M}}_{0,n}$ which is a bijection on closed points (the exceptional loci essentially parametrizing all possible "choices" left ambiguous when the $n-1$ points are not in sufficiently general position). Since the moduli space is smooth (because the moduli stack is smooth, and n-pointed rational curves have no automorphisms for $n \geq 3$), it follows by Zariski's Main Theorem that f is an isomorphism.

Kapranov's initial construction was later simplified by Brendan Hassett (see (11)) to a beautiful description: in Hassett's version, one simply blows up the $n-1$ basepoints of the family of rational normal curves, then blows up the proper transforms of the $\binom{n-1}{2}$ lines spanned by pairs of base points, then the proper transforms of the $\binom{n-1}{3}$ planes spanned by triples of base points, and so on until one selects $n-3$ points and finds that one already has an effective Cartier divisor. The beauty and curse of this description is that under the map we call f, the boundary divisors pullback precisely to the exceptional divisors (which are naturally generators of the

Picard group) and the proper transforms of the various hyperplanes spanned by $(n-3)$-tuples of basepoints (which all have class of the form $H - \sum E_i$, the sum running over all exceptional divisors coming from the $n-3$ points). The beauty is that this description has reduced most questions about the geometry of $\overline{\mathrm{M}}_{0,n}$ to purely combinatorial questions; the curse is that the combinatorial complexity of these questions is immense.

Far better understood in many ways is the quotient moduli space $\widetilde{\mathrm{M}}_{0,n} := \overline{\mathrm{M}}_{0,n}/\mathfrak{S}_n$ which parametrizes stable rational curves with n unlabeled marked points. Although this variety is worse than $\overline{\mathrm{M}}_{0,n}$ in the sense that it possesses finite quotient singularities, it remains a fine moduli space and its birational geometry is far better understood. In particular we have the beautiful theorem of Keel and McKernan:

Theorem 2.2.1. *(12) The effective cone of* $\widetilde{\mathrm{M}}_{0,n}$ *is spanned by its* $\lfloor n/2 \rfloor$ *boundary divisors.*

Sketch of Proof. The boundary divisors here are sums of the $B_{S,T}$ with fixed $|S|$ and $|T|$, so we can just write them $B_2, ..., B_{\lfloor n/2 \rfloor}$. Any positive linear combination of B_i is effective, so we need to show that any effective D can be so expressed. To do so, we generalize the idea of 2.1.4: namely, a curve which is moving in an irreducible divisor (but not the entire space) still intersects all other irreducible effective divisors nonnegatively.

It is simple to construct moving curves in the boundary divisors: consider a nodal union of two copies of $\mathbb{P}^1$, fix i points on one component and $n-i$ points on the other, and vary the point of attachment on the second component. This curve is moving in B_i, and intersects B_{i+1} in $n-i$ points when the varying point of attachment collides with one of the marked points. This allows one to inductively prove positivity of the coefficients in an expression $D = b_2 B_2 + \cdots b_{\lfloor n/2 \rfloor} B_{\lfloor n/2 \rfloor}$. $\square$

Although the effective cone of $\overline{\mathrm{M}}_{0,n}$ is still out of reach, some progress has been made on quotients by subgroups of $\mathfrak{S}_n$, i.e. moduli spaces of rational curves with a few labelled markings any many unlabelled. The following was proved by William Rulla in (13), and will be used in our main theorems:

Proposition 2.2.2. *The effective cone of* $\overline{\mathrm{M}}_{0,n}/\mathfrak{S}_{n-1}$ *is generated by boundary divisors.*

Sketch of Proof. The argument is a minor variation of that used by Keel and McKernan. Note that on this space, boundary divisors can be indexed by the total number of marked points on the component with the labelled point (hence there are now $n-3$ such divisors), so what are essentially the same moving curves can be used. The only new component of the argument is a curve arising as the fiber of the map $\overline{\mathrm{M}}_{0,n}/\mathfrak{S}_{n-1} \to \widetilde{\mathrm{M}}_{0,n-1}$ forgetting the labelled point, which is moving in the full moduli space and is used in establishing the base case of the induction. □

2.3 The Kontsevich Moduli Space

Recall that an abstract curve is **prestable** if its singularities are at worst ordinary nodes. The Kontsevich moduli space $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)$ is the coarse moduli space of the stack representing the functor of flat families of degree d morphisms to $\mathbb{P}^r$ from prestable curves of genus 0 with n marked smooth points.

The Kontsevich space was first introduced by Kontsevich in (3) in his investigation of the mathematical foundations of mirror symmetry in order to formulate a rigorous algebro-geometric definition of Gromov-Witten invariants. In this paper he investigated several examples, in particular the number of degree d rational curves in $\mathbb{P}^2$ through $3d-1$ points. Using the inductive structure of the boundary, he was able to show that these **characteristic numbers** are determined recursively from the fact that a unique line passes through any two distinct points. While Kontsevich's work is principally concerned with enumerative calculations and other applications, the basic properties of the moduli space are presented with details by Fulton-Pandharipande (4). In fact the existence of a projective coarse moduli space is carried out in greater generality than we need; maps from stable curves of arbitrary genus to arbitrary projective schemes are covered. The foundational theorems of (4) specialized to our situation read as follows:

Theorem 2.3.1. *The coarse moduli space $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)$ is a normal projective variety of pure dimension $r + d(r+1) + n - 3$ which is locally the quotient of a nonsingular variety by a finite group. The locus $\overline{\mathrm{M}}^*_{0,n}(\mathbb{P}^r, d)$ of maps without automorphisms is nonsingular and a fine moduli space with universal family. Up to finite group quotient, the boundary of $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)$ is a divisor with normal crossings.*

Although the spaces we are dealing with are nice varieties, the generalizations to the case of positive genus domain curves or the case of hypersurface targets (even smooth Fano hypersurfaces) often result in moduli spaces with multiple components of different dimensions.

Example 2.3.2. Consider $\overline{\mathrm{M}}_{1,0}(\mathbb{P}^2, 3)$. The most naive hope would be that the general point parametrizes a smooth plane cubic; unfortunately this is far from the case. We know that smooth plane cubics vary in a 9-dimensional family. On the other hand, the space of cubic maps $\mathbb{P}^1 \to \mathbb{P}^2$ is birational to the discriminant hypersurface in the $\mathbb{P}^9$ of plane cubics, and abstract curves of genus 1 have 1-dimensional moduli, so the space of maps from a curve with one genus 0 (on which the map is cubic) and one genus 1 component (which the map contracts to a point) has dimension $8 + 1 + 1 = 10$ (8 for the rational cubic, one for the point in $\overline{\mathrm{M}}_{1,1}$, one for the attaching point on the rational curve).

2.3.1 Divisors on Kontsevich Moduli Spaces

The study of the Kontsevich spaces as geometric objects in their own right was continued by Rahul Pandharipande who computed the rational Picard group (here isomorphic to the Neron-Severi group, in terms of which we phrased our previous discussion of effective divisors) in terms of geometrically natural classes, and then computed the canonical class in terms of his generators. Besides the irreducible components of the boundary, two of the simplest divisor classes one can define are $\mathcal{H}$, the class of maps intersecting a fixed codimension-2 linear space in $\mathbb{P}^r$, and the n classes $\mathcal{L}_i$, $i = 1, ..., n$, which are obtained by pulling back $\mathcal{O}_{\mathbb{P}^r}(1)$ along the canonical evaluation morphisms

$$ev_i : \overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d) \to \mathbb{P}^r.$$

Theorem 2.3.3. *(Pandharipande) Suppose $d > 0$ and $r > 1$, i.e. suppose the general point of $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)$ parametrizes a generically injective map. Then $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d))$ is generated by the irreducible components of the boundary together with*

- *$\mathcal{H}$ when $n \geq 3$ or $n = 0$,*
- *$\mathcal{L}_1, \mathcal{L}_2$ when $n = 2$,*
- *$\mathcal{L}_1, \mathcal{H}$ when $n = 1$.*

All other relations among these generators come from pull back along the forgetful map

$$\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d) \to \overline{\mathrm{M}}_{0,n}.$$

Remark 2.3.4. At this point we need to introduce an indexing scheme for the components of the boundary. We will let $\Delta_{i(S)}$ denote the class of the divisor whose general point parametrizes a stable map whose domain has two components, one with $|S|$ marked points labeled by $S \subset \{1, ..., n\}$ and on which the map has degree i, and one with marked points labeled by the complement of S and on which the map has degree $d - i$. Following Pandharipande, we will let

$$D_{i,j} = \sum_{|S|=j} \Delta_{i(S)}.$$

Theorem 2.3.5. *(Pandharipande) Assume $d > 0, r > 1$. The canonical class of $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d)$ is*

$$-\frac{(d+1)(r+1)}{2d}\mathcal{H} + \sum_{i=1}^{\lfloor d/2 \rfloor} \left(\frac{(r+1)(d-i)i}{2d} - 2 \right) D_{i,0}.$$

For $n > 0$, the canonical class of $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)$ is

$$-\frac{(d+1)(r+1)d - 2n}{2d^2}\mathcal{H} - \frac{2}{d}\sum_{i=1}^{n} \mathcal{L}_i + \sum_{i=1}^{\lfloor d/2 \rfloor} \sum_{j=0}^{n} \left(\frac{(r+1)(d-i)di + 2d^2 j - 4dij + 2ni^2}{2d^2} - 2 \right) D_{i,j}.$$

The proofs of both theorems rely on intersections with test curves, which in this situation amounts to studying linear systems on blowups of ruled surfaces.

2.3.2 The Stability of the Effective Cone

It was shown in (2) that for $n = 0$ and fixed d, the effective cones are "nested" in a precise sense as r increases, and for $r \geq d$ the effective cone stabilizes: observe that for fixed d and n, the Picard rank of $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)$ is constant as $r \geq 2$ varies; in fact Pandharipande's generators for these spaces all have the same names. As such, let $P_{d,n}$ denote the abstract $\mathbb{Q}$-vector space whose basis is the set of symbols that label Pandharipande's generators for any $r \geq 2$. The lemma of (2) is:

Lemma 2.3.6. *For $n = 0$ there is a $\mathbb{Q}$-linear isomorphism*

$$u_{d,r,n} : P_{d,n} \to Pic(\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d))$$

which yields an inclusion

$$u^{-1}_{d,r,0}(\mathrm{Eff}) \subset u^{-1}_{d,r+1,0}(\mathrm{Eff})$$

stabilizing for $r \geq d$.

This intuitively makes sense because a rational curve of degree d in $\mathbb{P}^r$ for $r > d$ is degenerate. The proof does not appear to depend on the assumption $n = 0$, so it seems that the same result holds for any number of marked points. However we do not use the pointed version of the lemma in any examples or theorems (and its proof is best phrased in the language of stacks which would interrupt the flow of the paper), so we omit the proof, which would constitute an exercise in synonym substitution.

2.3.3 Gromov-Witten Varieties

As noted above, moduli spaces of stable maps were introduced to rigorously define Gromov-Witten invariants, which can be tautologically defined as intersection numbers of **Gromov-Witten varieties**. The simplest Gromov-Witten varieties are obtained as preimages of projective varieties (usually single reduced points) under the evaluation morphisms, which one then intersects to obtain more complicated varieties. Roughly, the intersection of Gromov-Witten varieties with codimensions summing to 0 should count the number of stable maps mapping the appropriate marked points to the corresponding subvarieties.

In this thesis, we observe that much of the birational geometry of a Kontsevich space is contained in the fibers of the evaluation maps, i.e. the 1-point Gromov-Witten varieties. This is not terribly shocking since the evaluation maps are isotrivial fibrations over projective space; nevertheless, we prove that the rational Picard group of these spaces is a quotient of $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d))$ with kernel generated by $\mathcal{L}_i$, and that the effective cone restricts in a similar manner.

CHAPTER 3

WORKING WITH MODULI SPACES

If one wishes to understand a coarse moduli space, a common first step is to study the corresponding fine moduli stack, which has a canonical projection to the coarse space which is well understood in good situations. Studying the stack amounts to studying families of whatever objects the moduli space classifies.

3.1 Stable Reduction

One example is the separatedness and properness of the Deligne-Mumford compactification of the moduli space of curves of genus g. This follows from the corresponding properties of the stack, which when unwound amount to the existence of a unique central fiber (which is a **stable curve**, i.e. a prestable curve with finite automorphism group) completing any flat family of curves over a punctured disc (more precisely, the punctured spectrum of a discrete valuation ring), possibly after making a finite base change; this is known as the Stable Reduction Theorem. The incredible thing is that there is an algorithmic way to compute this fiber which is frequently effective, even when working by hand. The key observation is that if any necessary base changes are factored into base changes of prime degree, the induced map on the total space of the family can be described topologically and combinatorially (i.e. computations can be done without more subtle scheme-theoretic bookkeeping). A thorough exposition of these computations can be found in (14), while we will simply summarize the steps of the algorithm:

1. Start with a 1-parameter family of stable curves whose central fiber is not stable.

2. Blow up the excessively singular points of the central fiber until the supports of the irreducible components all intersect transversely.

3. Make a series of base changes of prime degree, normalizing the resulting surface after each step, until all components of the central fiber are reduced; the effect of a degree p base change and normalization on the total space is the taking of a degree p cover whose branch divisor is computed by viewing the central fiber as a divisor and reducing its coefficients $\bmod p$.

4. Once the central fiber is reduced, blow down any (-1)-curves and any (-2)-curves.

5. We have arrived at the stable model of the central fiber. The total space may have A_n singularities. If a smooth total space is desired, one can blow up these singularities (bringing back the (-2)-curves) and instead work with the semistable model (a prestable curve is **semistable** if its automorphism group is at most 1-dimensional).

Deployment of this algorithm also allows one to compute the degrees of divisor classes on $\overline{\mathcal{M}}_g$ in many cases, since the total space of a family of curves is frequently birational to a familiar variety. Suppose $\pi : X \to B$ is a 1-parameter family of stable curves with relative dualizing sheaf ω_π. The basic divisor classes are

- δ_0, which counts non-separating nodes (in the transverse case),

- δ_i, $i = 1, ... \lfloor g/2 \rfloor$, which counts nodes that separate a stable curve into connected components of genus i and $g - i$ (in the transverse case).

- $\delta := \delta_0 + \delta_1 + \cdots + \delta_{\lfloor g/2 \rfloor}$, which measures the total singularity of the family,

- λ, which is the degree of $c_1(\pi_*(\omega_\pi))$,

- $\kappa = \kappa_1$, which is the degree of $\pi_*(c_1(\omega_\pi)^2)$.

One may notice that the last two classes are defined by starting with the same sheaf and applying the direct image and Chern class functors in different orders. The failure of these functors to commute is measured by the Grothendieck-Riemann-Roch formula, and indeed Mumford established the relation $12\lambda = \delta + \kappa$ thereby in (15).

Example 3.1.1. Here we will compute the divisor class δ on $\overline{\mathcal{M}}_3$ on a general pencil of plane quartics specializing to a tacnode (a singular point with local equation $y^2 = x^4$).

The local description is as follows: blowing up $\mathbb{P}^2$ at the tacnode, we obtain a triple point consisting of the proper transforms of the branches meeting the exceptional divisor E_1 at the same point (it is also worth noting here that as a fiber of the family treated as a divisor, E_1 has multiplicity 2). Blowing up this point, we obtain another exceptional divisor E_2 through which E_1 and the branches each pass at distinct points (and which occurs with multiplicity 4).

At this point, the components of the central fiber intersect transversely, but we have nonreduced components. Denoting by $\tilde{C}$ the proper transform of the initial tacnodal quartic $\tilde{C}$, we can write this fiber as $F = \tilde{C} + 2E_1 + 4E_2$. E_2 meets E_1 once and $\tilde{C}$ twice. By the algorithm above, we now take a double cover of the total space branched along $\tilde{C}$, which means that we will take a double cover of E_2 branched at two points, so the inverse image of E_2 is also isomorphic to $\mathbb{P}^1$. This curve again meets $\tilde{C}$ twice, but there are now two disjoint copies of E_1 (call them E_1' and E_1'') meeting what we will continue to call E_2.

We now take another double cover, but this time the branch divisor is $\tilde{C} + E_1' + E_1''$, so over E_2 we get an elliptic curve E. The central fiber is now semistable, but to obtain the stable model we observe that any component meeting the rest of the central fiber in a single point is a (-1) curve. In our case these are the rational curves E_1', E_1'', so we blow them down to obtain the stable model: the normalization of the tacnodal curve, with an elliptic bridge connecting the points lying over the tacnode.

Having analyzed the local picture, we can now carry out the necessary bookkeeping to do the global calculation. First, let $X \subset \mathbb{P}^2 \times \mathbb{P}^1$ denote the blowup of $\mathbb{P}^2$ at the 16 basepoints of the pencil (we will call the exceptional divisors and their proper transforms F_i). The topological Euler characteristic is $\chi(X) = 3 + 16 = 19$. Blowing up the tacnode twice, we obtain a surface $X' \to X$ with $\chi(X') = 21$.

The global base change is obtained as the fiber product of $\pi : X' \to \mathbb{P}^1$ with the double cover $\mathbb{P}^1 \to \mathbb{P}^1$ branched over 0 and ∞, so the double cover of the surface is branched along $\tilde{C}$, a subcurve of the central fiber, and the fiber over ∞ which is a general smooth quartic. After normalizing, the new surface $f : Y \to X'$ has $\chi(Y) = 2(21 - (-4)) + (-4) = 46$.

The second base change is branched over $\tilde{C}+E_1'+E_1''$ in the central fiber and the general fiber at ∞. Normalizing again, the new surface has $\chi(Y') = 2(46) = 92$ (the branch locus happens to have Euler characteristic 0 here). Blowing down the two (-1)-curves we get $\chi(Y'') = 90$.

Finally, note that a topological fiber bundle of genus 3 surfaces over a 2-sphere has $\chi = (-4)(2) = -8$, so there are $90 - (-8) = 98$ nodes in the fibers of our pencil. Dividing by the total degree of base change, we get $\delta = 24\frac{1}{2}$. Since the non-central singular fibers are general, their nodes are of type δ_0, and the two nodes in central fiber with the elliptic bridge are as well (since smoothing either node results in an irreducible curve). Thus in fact $\delta_0 = 24\frac{1}{2}$ and $\delta_1 = 0$.

3.2 Computing Divisor Classes

Beyond the divisors from Pandharipande's description of the Picard group, the key to the description of the effective cone of the Kontsevich space is the divisor of stable maps whose images are set-theoretically degenerate; that is, fail to span a hyperplane. One may imagine a conic degenerating to a double line or a twisted cubic degenerating to a nodal plane cubic as basic examples of curves intersecting this divisor. This divisor is denoted D_{deg} and its class was computed by Coskun-Harris-Starr in (2):

Lemma 3.2.1. *The class of D_{deg} in $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^d, d)$ is*

$$\frac{1}{2d}\left[(d+1)\mathcal{H} - \sum_{k=1}^{\lfloor d/2 \rfloor} k(d-k)\Delta_{k,d-k}\right].$$

Sketch of Proof. We write D_{deg} as an arbitrary linear combination of $\mathcal{H}$ and the boundary classes, then compute its restriction to well-chosen "test families" to obtain relations among the coefficients until we have enough to deduce the class.

A classic test family (used at length in (14) and ever since) is obtained by taking the union of curves with genera totaling g, and varying the isomorphism class of one of these curves. The first test family we use here is a variation on this idea: fix a rational normal curve R of degree $d-i-1$ in a linear subspace of $\mathbb{P}^d$ and a rational surface scroll S of degree i (lying in a complimentary subspace) meeting it in one point p. Since the scroll is the image of either $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$, the blowup of $\mathbb{P}^2$ at a point (specifically the domain surface is $\mathbb{F}_{i \bmod 2}$), we can treat the divisor theory uniformly. Let f denote the fiber class on either surface and e the exceptional class or the other fiber class. If we take a general pencil of rational degree $i+1$ curves through p in the linear system $|e + \lfloor \frac{i+3}{2} \rfloor f|$, the union of these curves with R induces a curve C_i in the Kontsevich space.

Since the curves in C_i by definition sweep out a scroll of degree i, we have $C_i.\mathcal{H} = i$. In (16), Coskun computed the cohomology of line bundles associated to effective divisors of type $e+mf$, from which one concludes that on a scroll of degree $i \neq 1$, a pencil of rational curves of degree $i+1$ is determined by imposing $i+2$ general basepoints (except for plane conics, which require $i+3=4$), including p. The other $i+1$ points together with R span $\mathbb{P}^d$, so all C_i are disjoint from D_{deg}.

For $2 \leq i \leq \lfloor d/2 \rfloor$, the pencil becomes reducible $i+1$ times, once for each fiber that contains a basepoint. The fiber is one component of the reducible curve, so we have $C_i.\Delta_{i,d-i} = 1$ (from the fiber through p) and $C_i.\Delta_{1,d-1} = i+1$ (for the other fibers). The last nonzero intersection is $C_i.\Delta_{i+1,d-i-1} = -1$, coming from the normal bundle to the exceptional divisor lying over p when one constructs the total space of this family. Also, note that we obtain contradictory answers for $C_1.\Delta_{1,d-1}$ from these two formulae; in fact we need to sum these numbers to get the correct answer of 3 (this "collapsing" of the calculations seems to correspond to $i=1$ being the lone case in which the Hirzebruch surface does not embed; rather the linear system $|e+2f|$ blows $\mathbb{F}_1$ down to a $\mathbb{P}^2 \subset \mathbb{P}^d$).

The next test family is obtained by fixing $d+2$ general points and moving one more point on a general line, and considering the induced family B_1 of rational curves. By construction, B_1 is disjoint from D_{deg} and only intersects the boundary divisor $\Delta_{1,d-1}$: any partition of the $d+2$ points into $i+1$ and $d-i+1$ points corresponds to a pair of linear spaces of dimensions i and $d-i$ in $\mathbb{P}^d$. But a general line can be chosen to avoid both of these linear spaces unless $i=1$. Similarly, when the curve breaks the degree 1 component must correspond to the size $i+1=2$ step of the partition, while the degree $d-1$ component is determined by the d remaining points and the intersection points of the $(d-1)$-plane with the fixed line and the degree 1 component. Concluding this line of thought, we have

$$
\begin{aligned}
B_1.\Delta_{1,d-1} &= \#\left\{(d-1) - \text{planes spanned by subsets of fixed points}\right\} \\
&= \binom{d+2}{d} \\
&= \frac{(d+2)(d+1)}{2}.
\end{aligned}
$$

The final intersection number here is $B_1.\mathcal{H}$, which is the number of rational curves of degree d through $d+2$ general points, meeting a general line, and meeting a general $(d-2)$-plane. To compute this, we specialize the $(d-2)$-plane to Λ, the hyperplane spanned by d of the fixed points. By the above analysis, any reducible curves have a component of degree 1 or $d-1$ lying in Λ, which we will refer to in the sequel as the Λ-component.

If the Λ-component is a line, there are $\frac{d(d-1)}{2}$ possible choices therefor; this line also necessarily handles the intersection with the $\mathbb{P}^{d-2}$. We have remaining d points through which must pass the degree $d-1$ component D, and in fact D must lie in the hyperplane spanned by the d points. Since D must also intersect the general line and the Λ-component, each of which meet the hyperplane in a point, D must pass through a total of $d+1$ general points in the hyperplane. Thus D is uniquely determined.

If the Λ-component has degree $d-1$, the complementary curve is the line between the other two points. This line intersects Λ in the point at which it must meet the Λ-component, and the general line intersects Λ at a $(d+2)$-nd point, proving uniqueness of the Λ-component. It intersects the $\mathbb{P}^{d-2}$ in $(d-1)$ points hence counts with that multiplicity. Thus the intersection number is

$$B_1.\mathcal{H} = \frac{d(d-1)}{2} + (d-1) = \frac{d^2+d-2}{2}.$$

Finally, fix a degree $d-1$ curve and a general pencil of lines with basepoint on the curve. It is straightforward that $C.\mathcal{H} = 1, C.D_{deg} = 1$, and $C.\Delta_{1,d-1} = -1$ are the only nonzero intersection numbers, which is enough to conclude the calculation. $\square$

Remark 3.2.2. Since the definition is purely in terms of the underlying curves, one can pullback D_{deg} along the forgetful map $\overline{\mathrm{M}}_{0,1}(\mathbb{P}^d, d) \to \overline{\mathrm{M}}_{0,0}(\mathbb{P}^d, d)$ and get almost the same class expression:

$$\frac{1}{2d}\left[(d+1)\mathcal{H} - \sum_{k=1}^{\lfloor d/2 \rfloor} 2k(d-k)\Delta_{k,d-k}\right],$$

if d is odd, and

$$\frac{1}{2d}\left[(d+1)\mathcal{H} - \frac{d^2-1}{4}\Delta_{\frac{d+2}{2}} - \sum_{k=1}^{d/2} 2k(d-k)\Delta_{k,d-k}\right],$$

if d is even.

3.3 Examples of Main Theorem for small values of r, d, n

Our Main Theorem 2 is about the effective cone of $\overline{\mathrm{M}}_{0,1}(\mathbb{P}^d, d)$. Nevertheless, it is instructive to look at a number of small cases which do not fall under this exact heading, if only to get a sense of the difficulties that begin to arise. We begin with the case $d = n = 1$ while allowing $r > 0$ to be arbitrary. It is also useful to consider the case where we allow a second marked point, which illustrates the kind of techniques needed, which do not carry through to the case of degree $d > 1$ (largely due to tangencies that do not happen when working with lines).

Example 3.3.1. Since no degenerations can happen, it is clear that $\overline{\mathrm{M}}_{0,1}(\mathbb{P}^r, 1)$ is the universal line over the Grassmannian, also known as the two-step flag variety $\mathbb{F}(1, 2, r+1)$. From Pandharipande's results, the Picard group of this space is spanned by $\mathcal{H}$ and $\mathcal{L}$, which are of course the Schubert divisors in the homogeneous space interpretation. From that interpretation we know that the psuedoeffective cone is equal to the nef cone and is in fact spanned by $\mathcal{H}$ and $\mathcal{L}$. For a direct proof, note that a pencil of lines through a fixed point in $\mathbb{P}^r$ (which is taken as the marked point) induces a moving curve on the moduli space whose intersection with $\mathcal{L}$ is 0, so $\mathcal{L}$ is an extremal ray of the effective cone (corresponding to the fiber contraction

$$ev : \overline{\mathrm{M}}_{0,1}(\mathbb{P}^r, 1) \to \mathbb{P}^r).$$

For the other extremal ray, consider a fixed line in $\mathbb{P}^r$ with a marked point moving thereupon. This induces a moving curve which, since the line can be chosen to be disjoint from any fixed codimension 2 linear space, has 0 intersection with $\mathcal{H}$ (this corresponds to the forgetful morphism

$$\overline{\mathrm{M}}_{0,1}(\mathbb{P}^r, 1) \to \overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, 1)).$$

Remark 3.3.2. Although we will not pursue this line of inquiry any further in this thesis, it follows from the general theory of homogeneous varieties that the cone of effective codimension k cycles on $\overline{\mathrm{M}}_{0,1}(\mathbb{P}^r, 1)$ is spanned by Schubert classes for all k.

Example 3.3.3. On $\overline{\mathrm{M}}_{0,2}(\mathbb{P}^r, 1)$, we begin to see degenerations occur when points collide. On this space, the Picard group is generated by $\mathcal{H}$, $\mathcal{L}_1$, and $\mathcal{L}_2$, and we have the relation $\Delta = \mathcal{L}_1 + \mathcal{L}_2 - \mathcal{H}$ for the only irreducible component of the boundary, which parametrizes degree 1 maps from a transverse union of two copies of $\mathbb{P}^1$ with both marked points on the degree 0 component. We claim that the extremal rays are $\Delta, \mathcal{H}, \mathcal{L}_1$, and $\mathcal{L}_2$. To prove this, we will find moving curves which are null on each face of the cone.

Take a general line L and a general point p in $\mathbb{P}^r$, and consider the pencil of lines meeting both. We can take the first marked point to be the intersection with L, and the second to be p. This induces a moving curve in the moduli space which we call C_1. Switching the marked points gives another curve C_2. Both are disjoint from the boundary and intersect $\mathcal{H}$ once, and we clearly have $C_i\mathcal{L}_j = \delta_{ij}$, where here δ_{ij} is the Kronecker delta symbol. Thus C_i cuts out the face spanned by Δ and $\mathcal{L}_i$.

Take a general line in $\mathbb{P}^r$, fix a point thereupon, and let another point move. When the points coincide, the stable limit is a map of the type described in the definition of Δ, wherein the degree 0 component is collapsed to the fixed point. Let C_3 denote the moving curve in the moduli space induced by letting the fixed point be the second marking, and let C_4 be the curve with the marked points switched. Each of these curve intersects Δ once and is disjoint from $\mathcal{H}$, and we clearly have $C_i\mathcal{L}_j = \delta_{i,j+2}$. Thus C_i cuts out the face spanned by $\mathcal{H}$ and $\mathcal{L}_{i-2}$. Since we have shown that the region connecting each of our extremal rays is in fact a face of the effective cone, we have completed the computation.

Next, we will forget about marked points and consider incomplete linear systems, i.e. curves whose degree exceeds the dimension of the projective space.

3.3.1 $\underline{\mathrm{Eff}(\overline{\mathrm{M}}_{0,0}(\mathbb{P}^2, 3))}$

The Picard rank here is 2, with standard generators $\mathcal{H}$ and $\Delta = \Delta_1$ (here the boundary is irreducible, which can be seen directly as it is clearly a blowup of $\mathbb{P}^5 \times \mathbb{P}^{2\vee}$). We also need the divisor class NI, which is defined as follows: since all smooth cubics in $\mathbb{P}^2$ have genus 1, any map parametrized by this moduli space has singular image, so the general map paramatrized by $\overline{\mathrm{M}}$ (which has smooth domain) is not an isomorphism onto its image. NI is the class of maps who “fail to be an isomorphism” along a fixed line. The basic strategy is to prove that the extremal rays of the effective cone are Δ and NI by exhibiting orthogonal moving curves. We will also, for convenience, use the following lemma whose proof can be found in (17):

Lemma 3.3.4. *Suppose that C is a moving curve in an irreducible effective divisor D satisfying $C.D < 0$. Then D is extremal and rigid.*

3.3.1.1 The curve C_1

Let C_1 denote the curve obtained by taking a pencil of lines in $\mathbb{P}^2$ together with a fixed conic passing through the basepoint. If we fix a general reference line to establish a representative of the class NI, its two points of intersection with the conic indicate the two members of the pencil which contribute, hence $C_1.NI = 2$. On the other hand, every map in the pencil has reducible domain. The only contribution to Δ comes from the exceptional divisor over the basepoint, so $C_1.\Delta = -1$. By the lemma, Δ is extremal.

3.3.1.2 The curve C_2

Let C_2 denote the curve obtained as follows: fix general points $p, q_1, ..., q_5 \in \mathbb{P}^2$, and consider the pencil of cubic curves which are double at p and pass through $q_1, ..., q_5$. The general member of this pencil is irreducible, while the singular members are the union of a smooth conic and a line. Since the stable maps in the associated pencil fail to be an isomorphism only at p, by calculating relative to a fixed line not through p we see $C_2.NI = 0$. To compute the other intersection number, observe that the line and the conic both need to pass through p, so we get a Δ contribution for each of the 5 points q_i (which determines a line $\overline{pq_i}$, and a conic through p and the remaining 4 base points), hence $C_2.\Delta = 5$. Since C_2 is a moving curve, we conclude that NI is an extremal ray of Eff. Since the Picard rank is 2, this completes the calculation.

3.3.2 $\mathrm{Eff}(\overline{\mathrm{M}}_{0,0}(\mathbb{P}^3, 4))$

The Picard rank here is 3, with standard generators $\mathcal{H}$, Δ_1, and Δ_2. We also again need the divisor class NI, here and in higher dimensions more simply defined as the locus of maps not isomorphic onto their images.

3.3.2.1 The curve C_1

Let C_1 denote the pencil of maps induced by a general pencil of $(1,3)$ curves on a smooth quadric surface $Q \subset \mathbb{P}^3$. The topological Euler characteristic of the total space is

$$\chi(\mathbb{P}^1 \times \mathbb{P}^1) + (1,3).(1,3) = 4 + 6 = 10,$$

while the total space of an abstract pencil of smooth rational curves has $\chi = 4$, so there must be 6 singular fibers. Since a $(1,3)$ curve cannot break into a pair of conics (i.e. a pair of $(1,1)$ curves), we conclude that $C_1.\Delta_1 = 6$ and $C_1.\Delta_2 = 0$, while $C_1.NI = 0$ since every curve in the (flat) family on Q has arithmetic genus 0.

3.3.2.2 The curve C_2

Consider the Veronese surface $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$, its general projection the quartic surface $V \subset \mathbb{P}^4$, and *its* general projection $S \subset \mathbb{P}^3$, a Steiner surface. While $\mathbb{P}^2 \cong v_2(\mathbb{P}^2) \cong V$ abstractly, S acquires three double lines, which meet at a triple point. In terms of the original plane, the singular locus of S is the image of a "triangle" of three lines in $\mathbb{P}^2$; each of these double covers one of the double lines with branching over the triple point, which is the image of the three vertices of the triangle.

Observe now that a pencil of conics in $\mathbb{P}^2$ induces a pencil of quartic curves on $v_2(\mathbb{P}^2)$ hence a pencil of quartics on V. We claim that this pencil can be chosen so that the resulting pencil of curves on S never acquires a node. Obviously this can only happen along one of the double lines, specifically if the underlying conic in $\mathbb{P}^2$ meets the underlying line in two points p, q of a fiber of $f : \mathbb{P}^2 \to S \subset \mathbb{P}^3$. This happens precisely if every conic through p also passes through q, so obtaining a pencil of smooth quartics in $\mathbb{P}^3$ from this construction is a simple matter of choosing a pencil of conics whose four base points avoid the triangle $f^{-1}(S_{sing})$. So $C_2.NI = 0$.

To compute the intersection with the boundary, observe that the three singular conics in the underlying plane pencil are responsible for the only contribution. Since the singular conics are transversely intersecting line pairs, they map to pairs of transversely (on S) intersecting conics, so $C_2.\Delta_2 = 3$ and $C_2.\Delta_1 = 0$.

3.3.2.3 Conclusion

The preceding two test curves are both moving curves in in the moduli space, hence identify faces $\langle NI, \Delta_2 \rangle$ and $\langle NI, \Delta_1 \rangle$ of the effective cone. By the Stability Lemma and the Theorem of (2) on $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^d, d)$, $\langle \Delta_1, \Delta_2 \rangle$ is also a face, so these three divisors constitute the three extremal rays of $\mathrm{Eff}_{4,3,0}$.

3.3.3 $\mathrm{Eff}(\overline{\mathrm{M}}_{0,1}(\mathbb{P}^2, 2))$

The Picard rank here is 3, with standard generators $\mathcal{H}$, $\Delta = \Delta_{1(1)}$, and $\mathcal{L} = \mathcal{L}_1$. We claim that the extremal rays of Eff are D_{deg}, Δ and $\mathcal{L}$.

3.3.3.1 The curve C_1

Fix a smooth conic in $\mathbb{P}^2$ and vary a marked point thereon. Evidently

$$C_1.\Delta = C_1.D_{deg} = 0.$$

A general reference line intersects the conic twice, so the marked point lies on that line in two fibers, hence $C_1.\mathcal{L} = 2$. Since C_1 is a moving curve in the moduli space, we conclude that $\langle D_{deg}, \Delta \rangle$ is an extremal face.

3.3.3.2 The curve C_2

Consider a general pencil of conics marked at a base point. Since the pencil has four noncollinear base points, $C_2.D_{deg} = 0$, and since the marked point is fixed, $C_2.\mathcal{L} = 0$. Finally, since the pencil has three reducible members, $C_2.\Delta = 3$. Since C_3 moves in the moduli space, we conclude that $\langle D_{deg}, \mathcal{L} \rangle$ is an extremal face.

3.3.3.3 The curve C_3

Fix a line ℓ and a point p in $\mathbb{P}^2$, and consider the pencil of lines Λ_t through p. We can construct a pencil of stable maps by taking the double cover of Λ_t branched at p and $\Lambda_t \cap \ell$, with marked point p. Since the branch points never collide, the domain of every stable map in the pencil is $\mathbb{P}^1$ and $C_3.\Delta = 0$. Since the marked point does not move, $C_3.\mathcal{L} = 0$. Finally, we recall the class

$$D_{deg} = \tfrac{1}{4}(3\mathcal{H} - 2\Delta),$$

so

$$C_4.D_{deg} = \tfrac{3}{4}(C_4.\mathcal{H}) = \tfrac{3}{4}.$$

As such, $\langle \Delta, \mathcal{L} \rangle$ is the final extremal face (or rather, any effective divisor whose base locus does not contain D_{deg} is in the span of the three indicated).

3.3.4 $\mathrm{Eff}(\overline{\mathrm{M}}_{0,1}(\mathbb{P}^3, 3))$

Proposition 3.3.5. *The extremal rays of* $\mathrm{Eff}_{3,3,1}$ *are* D_{deg}, $\Delta_{1(0)}$, $\Delta_{1(1)}$, *and* $\mathcal{L}$.

Proof. We analyze a sequence of test curves, using the expression of an arbitrary effective divisor as

$$a'\mathcal{H} + b'\Delta_{1(0)} + c'\Delta_{1(1)} + d\mathcal{L}$$

to study the claimed expression as a nonnegative combination

$$aD_{deg} + b\Delta_{1(0)} + c\Delta_{1(1)} + d\mathcal{L}.$$

3.3.4.1 Plan

Modulo the construction and analysis of test curves, the proof is basic linear algebra. However, the logical implications can easily get lost in the analysis, so we provide a roadmap: suppose $a < 0$, then substituting in the expression for D_{deg} we get $a' < 0$. Therefore, to prove what we want, namely $a \geq 0$, it suffices to show $a' \geq 0$, which we do below. Then, since the coefficient d is the same in both expressions, we simply show that $d \geq 0$. Finally, we find curves vanishing on D_{deg}, $\mathcal{L}$, and each of the boundary divisors in turn, verifying that $b, c \geq 0$.

3.3.4.2 The curve C_1

We claim that if an effective divisor has class $D = a\mathcal{H} + b\Delta_{1(0)} + c\Delta_{1(1)} + d\mathcal{L}$, then $a \geq 0$. Embed $v_3 : \mathbb{P}^2 \hookrightarrow \mathbb{P}^9$ via $|\mathcal{O}(3)|$ and project generally to $\mathbb{P}^3$. The proper transform of a pencil of lines in $\mathbb{P}^2$ under this construction is a pencil of mostly smooth twisted cubics with a basepoint (which we mark). This is a moving curve in the moduli space with $C_1.\mathcal{H} = \deg(v_2(\mathbb{P}^2)) = 9$, and

$$C_1.\Delta_{1(0)} = C_1.\Delta_{1(1)} = C_1.\mathcal{L} = 0,$$

proving the claim.

3.3.4.3 The curve C_2

Next, we claim that $d \geq 0$. Fix a smooth twisted cubic in $\mathbb{P}^3$ and vary a marked point on it. Then $C_2.\mathcal{L} = 3$ and

$$C_2.\Delta_{1(0)} = C_2.\Delta_{1(1)} = C_2.\mathcal{H} = 0.$$

Since C_2 is moving in the moduli space, we conclude that $d \geq 0$.

3.3.4.4 The curve C_3

Consider the complete linear system of

$$D_{cub} = 3H - \sum_{i=1}^{6} E_i$$

on S, the blowup of $\mathbb{P}^2$ in 6 general points. We obtain a moving curve $C_3 \subset \overline{\mathrm{M}}_{0,1}(\mathbb{P}^3, 3)$ by taking the proper transform under $|D_{cub}|$ of a pencil of lines in $\mathbb{P}^2$, with the basepoint of the pencil marked so $C_3.\mathcal{L} = 0$. The lines are mapped to curves of degree $D_{cub}.H = 3$ on the embedded image of the cubic surface $S \subset \mathbb{P}^3$. The six singular curves all have the marked point on the conic component since $D_{cub}.E_i = -E_i^2 = 1$, so we have the following intersection numbers:

.	$\mathcal{H}$	$\Delta_{1(0)}$	$\Delta_{1(1)}$	$\mathcal{L}$
C_3	3	6	0	0.

Thus $C_3.D_{deg} = 0$, proving that $b \geq 0$.

3.3.4.5 The curve C_4

Fix a line $\ell \subset \mathbb{P}^3$ and a transverse plane $\Lambda \subset \mathbb{P}^3$. Let $p = \ell \cap \Lambda$, and consider the pencil of conics B_t through p and three other general points $q_1, q_2, q_3 \in \Lambda$. Then $C_t := B_t \cup \ell$, with q_1 as marked point, induces a moving curve $C_4 \subset \Delta_{1(0)}$. We have $C_4.\mathcal{H} = 1$ and $C_4.\mathcal{L} = 0$, and the basepoint at p contributes -1 to $\Delta_{1(0)}$. Meanwhile, the three singular fibers of B_t give rise to three 3-component fibers of C_t. The "middle" component and one of the "edge" components come from B_t, with one of the q_i on the middle in each case and the other two on the edge. Thus the node separating the edge from the rest of the curve has type $\Delta_{1(1)}$ when q_1 lies on the edge component, and type $\Delta_{1(0)}$ when q_1 lies on the middle component, so we conclude

$$C_4.\Delta_{1(0)} = 0, C_4.\Delta_{1(1)} = 2.$$

We thus have $C_4.D_{deg} = 0$. Suppose now that D (as expressed throughout this proof) is irreducible, effective, and not equal to $\Delta_{1(0)}$. Then $D \cap \Delta_{1(0)}$ has codimension ≥ 2 in $\overline{\mathrm{M}}_{0,1}(\mathbb{P}^3, 3)$, and the curve class C_4 (being moving in $\Delta_{1(0)}$) has $C_4.D \geq 0$. We conclude that $c \geq 0$, finishing the proof. $\square$

CHAPTER 4

PROOFS OF MAIN THEOREMS

4.1 The First Theorem

Recall the statement of our first theorem on the rational Picard groups of one-point Gromov-Witten varieties:

Main Theorem 1. For a point $p \in \mathbb{P}^r$ and any $i = 1, ..., n$, the rational Picard group of $\overline{\mathrm{M}}_p := ev_i^{-1}(p)$ is a quotient of $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d))$; the kernel of the quotient map is generated by $\mathcal{L}_i = ev_i^* \mathcal{O}_{\mathbb{P}^r}(1)$.

The proof essentially boils down to the fact that the evaluation maps are locally trivial fibrations.

Proof. During this proof, we will abbreviate $\overline{\mathrm{M}}_{0,n}(\mathbb{P}^r, d)$ to $\overline{\mathrm{M}}$. Furthermore, we will be working with a single evaluation map, so we will drop the index and simply write ev and $\mathcal{L}$. Consider the closed subvariety of $\overline{\mathrm{M}}$ obtained by taking the inverse image of a hyperplane in $\mathbb{P}^r$. After twisting by $\mathbb{Q}$, the excision sequence of Hartshorne II.6.5(c) evaluates to

$$0 \to \mathbb{Q} \to Pic(\overline{\mathrm{M}}) \to Pic(\overline{\mathrm{M}} \setminus ev^{-1}(H)) \to 0,$$

where the first map sends 1 to $\mathcal{L}$. We claim that

$$\overline{\mathrm{M}} \setminus ev^{-1}(H) \cong \overline{\mathrm{M}}_p \times \mathbb{A}^r,$$

which implies the result. Set-theoretically,

$$\overline{\mathrm{M}} \setminus ev^{-1}(H) = \{(C, x, f) \,|\, f(x) \in \mathbb{P}^r \setminus H\} \,,$$

while

$$\overline{\mathrm{M}}_p \times \mathbb{A}^r = \{(C, x, f, q) \,|\, f(x) = p\} \,.$$

We wish to give an isomorphism; we have maps

$$\begin{aligned} \overline{\mathrm{M}} \setminus ev^{-1}(H) &\to \overline{\mathrm{M}}_p \\ (C, x, f) &\mapsto (C, x, \alpha_{f(x)} \circ f), \end{aligned}$$

where α is a family of automorphisms of $\mathbb{P}^r$ constant along fibers of ev which takes $f(x) \mapsto p$ for all elements of the domain. If we choose coordinates $(x_0 : \cdots : x_r) \in \mathbb{P}^r$ such that H is given by the vanishing of x_0, $p = (1 : 0 : \cdots : 0)$, and $f(x) = (a_0 : a_1 : \cdots : a_n)$, the family is given by the matrix

$$\begin{pmatrix} 1/a_0 & 0 & 0 & 0 & \cdots & 0 \\ -a_1/a_0 & 1 & 0 & 0 & \cdots & 0 \\ -a_2/a_0 & 0 & 1 & 0 & \cdots & 0 \\ -a_3/a_0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_r/a_0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Furthermore, restricting the evaluation morphism yields

$$\overline{\mathrm{M}} \setminus ev^{-1}(H) \to \mathbb{A}^r$$

$$(C, x, f) \mapsto f(x).$$

Since we have an inverse isomorphism

$$\overline{\mathrm{M}}_p \times \mathbb{A}^r \to \overline{\mathrm{M}} \setminus ev^{-1}(H)$$

$$[(C, x, f), q] \mapsto (C, x, \alpha_q^{-1} \circ f),$$

we are done. □

Remark 4.1.1. Although the case of two-point Gromov-Witten varieties should be similarly straightforward due to the $(r+2)$-transitivity of the PGL action on closed points, the necessity of cutting away a hyperplane makes constructing another compatible local trivialization highly nontrivial (perhaps impossible). As such, more advanced techniques may be necessary.

4.2 The Second Theorem

Our primary result is

Main Theorem 2. The extremal rays of the effective cone of $\overline{\mathrm{M}}_{0,1}(\mathbb{P}^r, d)$ for $r \geq d$ are

1. D_{deg}, the divisor of stable maps whose image does not span a hyperplane in $\mathbb{P}^d$.
2. $\mathcal{L}$, the divisor of stable maps sending the marked point p to a fixed hyperplane.
3. Those rays which pullback to $\mathfrak{S}_{d+1}$-invariant extremal rays under Kapranov's embedding

$$K : \overline{\mathrm{M}}_{0,d+2} \hookrightarrow \overline{\mathrm{M}}_{0,1}(\mathbb{P}^d, d).$$

This is a generalization of a theorem of Coskun-Harris-Starr:

Theorem 4.2.1. *(2) A $\mathbb{Q}$-divisor class on $\overline{\mathrm{M}}_{0,0}(\mathbb{P}^r, d)$, where $r \geq d$, is psuedoeffective if and only if it lies in the $\mathbb{Q}_{>0}$-linear span of D_{deg} and the irreducible components of the boundary. When $r < d$, every effective divisor lies in this span (in this case, by D_{deg} we mean the divisor with class*

$$\frac{1}{2d}\left[(d+1)\mathcal{H} - \sum_{k=1}^{\lfloor d/2 \rfloor} k(d-k)\Delta_{k,d-k}\right],$$

which is no longer effective).

The proof of the above cited theorem proceeds by the construction of moving curves which cut out the faces of the effective cone. If the Harbourne-Hirschowitz conjecture is proved, Coskun-Harris-Starr have constructed curves which will then cut out the faces exactly. In the meantime, they have also constructed infinite sequences of curves which cut out the cone in the limit, which is sufficient to prove the statement.

In our investigations of the pointed case, several things became apparent. First, a curve which lies entirely in the boundary introduces complications into any systematic attempt to study the cone. Second, everything would work out smoothly if we could find lots of families of rational normal curves which contained basepoints, since the basepoint provides a natural choice of marked point which has the advantage of making the divisor class $\mathcal{L}$ vanish. Thirdly, since families with basepoints are not sufficiently abundant, one must mark sections, but this is not conducive to constructing useful moving curves.

Instead, we have found that an argument due to Keel which does not construct moving curves (instead proving positivity of coefficients by a different manner) can be slightly generalized to our situation:

Proof. By the stability lemma, we need only concentrate on the case $r = d$. Suppose D is an effective divisor class on $\overline{\mathrm{M}}_{0,1}(\mathbb{P}^d, d)$. The claim is that we can write

$$D \equiv aD_{deg} + \ell\mathcal{L} + \sum_{k=1}^{d-1} d_k\Delta_k, \qquad a, \ell, d_k \geq 0,$$

where Δ_k parametrizes maps from reducible curves with degrees k and $d-k$ on the components, and the marked point on the degree k component. Positivity of a, as in the unpointed case, follows from taking the image of a pencil of lines in $\mathbb{P}^2$ under the d-th Veronese embedding which is then projected to $\mathbb{P}^d$ from a general linear space of the appropriate dimension, yielding a moving curve C_a. We take the basepoint of the pencil to be our marked point, so this curve has intersection 0 with $\mathcal{L}$. Furthermore, observe that all of these maps have irreducible domain so that the curve meets each Δ_k trivially. Finally, the images of a finite number of these maps meet a general codimension 2 linear space. Since C_a is a moving curve and

$$C_a.D_{deg} = \frac{d+1}{2d}C_a.\mathcal{H} = \frac{d(d+1)}{2} \geq 0,$$

we conclude that $a \geq 0$.

Next, consider C_ℓ, a moving curve obtained by fixing a rational normal curve in $\mathbb{P}^d$ and varying a marked point thereupon. Again, no maps in this family have reducible domain so $C_\ell.\Delta_k = 0$ for all k, and all of the images are the fixed rational normal curve so $C_\ell.D_{deg} = 0$. If we fix a hyperplane in $\mathbb{P}^d$, the moving marked point will intersect it d times, so $C_\ell.\mathcal{L} = d$. Since C_ℓ is a moving curve, we conclude that $\ell \geq 0$.

Finally, we use Kapranov's embedding: fix $p_1, ..., p_{d+2} \in \mathbb{P}^d$ linearly general, and consider the subscheme of $\overline{\mathrm{M}}_{0,1}(\mathbb{P}^d, d)$ parametrizing stable maps which meet the $d+2$ points and take the marked point to p_1. This subscheme is isomorphic to $\overline{\mathrm{M}}_{0,d+2}$, so we have an embedding $K : \overline{\mathrm{M}}_{0,d+2} \hookrightarrow \overline{\mathrm{M}}_{0,1}(\mathbb{P}^d, d)$ and the divisor Δ_k restricts to the boundary divisor parametrizing two-component rational curves with one component containing the labelled marked point p_1 and k additional unlabelled (but still marked) points, i.e.

$$K^*(\Delta_k) = \sum_{1 \in S, |S| = k+1} B_{S,T}.$$

In the unpointed case, it was observed by Keel that since the embedding is $\mathfrak{S}_{d+2}$-invariant, the boundary divisors of the Kontsevich space restrict precisely to the boundary divisors of $\overline{\mathrm{M}}_{0,d+2}/\mathfrak{S}_{d+2}$, which are precisely the extremal rays of that space by a theorem of Keel and McKernan.

We observe that, since the embedding is also tautologically invariant under subgroups of $\mathfrak{S}_{d+2}$, in particular the subgroup isomorphic to $\mathfrak{S}_{d+1}$ which fixes p_1, we obtain the analogous restrictions in the one-pointed case: the display in the last paragraph computes $K^*(\Delta_k) = B_{k+1}$, where we are as in Chapter 2 using B_i to denote the divisor on $\overline{\mathrm{M}}_{0,d+2}/\mathfrak{S}_{d+1}$ which partitions the markings into sets of i (containing the labelled point) and $d - i + 2$ points. Note also that by construction (all maps pass through a collection of $d + 2$ general points) the image of K is disjoint from D_{deg}. Furthermore, we can choose the reference hyperplane for computing $K^*(\mathcal{L})$ to be disjoint from $p_1, ..., p_{d+2}$, so that line bundle is trivial. Finally, given an arbitrary effective divisor D on the Kontsevich space, we can choose the points defining K to avoid the images of the maps parametrized by D so that D pulls back to an effective divisor $D_K \subset \overline{\mathrm{M}}_{0,n}$ and we can apply results on the effective cone of $\overline{\mathrm{M}}_{0,d+2}$, i.e.

$$\begin{aligned} D_K &= K^*(D) \\ &= K^*\left(aD_{deg} + \ell\mathcal{L} + \sum_{k=1}^{d-1} d_k\Delta_k\right) \\ &= \sum_{k=1}^{d-1} d_k B_{k+1}. \end{aligned}$$

By effectivity of D_K and the proposition 2.2.2 (the effective cone of $\overline{\mathrm{M}}_{0,d+2}/\mathfrak{S}_{d+1}$ is generated by boundary divisors), we conclude that $d_k \geq 0$ for all $k = 1, ..., d-1$, hence we have completely characterized the effective cone of $\overline{\mathrm{M}}_{0,1}(\mathbb{P}^d, d)$. □

Corollary 4.2.2. *The extremal rays of the effective cone of the Gromov-Witten variety* $\overline{\mathrm{M}}_p \subset \overline{\mathrm{M}}_{0,1}(\mathbb{P}^d, d)$ *are*

1. *D_{deg}, the divisor of stable maps whose image does not span a hyperplane in $\mathbb{P}^d$.*

2. *Those rays which pullback to $\mathfrak{S}_{d+1}$-invariant extremal rays under Kapranov's embedding*

$$K : \overline{\mathrm{M}}_{0,d+2} \hookrightarrow \overline{\mathrm{M}}_{0,1}(\mathbb{P}^d, d).$$

Proof. Observe that since it is defined in terms of fixed points which are taken as the markings, the Kapranov Embedding factors through $\overline{\mathrm{M}}_{p_1}$ (recalling the notation from the first main theorem):

$$\begin{tikzcd} \overline{\mathrm{M}}_{0,d+2} \arrow[rr, hook, "K"] \arrow[dr, hook, "\tilde{K}"] & & \overline{\mathrm{M}}_{0,1}(\mathbb{P}^d, d). \\ & \overline{\mathrm{M}}_{p_1} \arrow[ur, hook] & \end{tikzcd}$$

Furthermore, the curve C_a (the d-th Veronese twist of a pencil of lines in the plane) also takes a basepoint as the marked point. As such, the argument from the previous theorem goes through *mutatis mutandis* (simply dropping all mentions of the divisor $\mathcal{L}$). $\square$

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