# Multibody System Investigation of Contact Geometry: Application to Deformable and Variable Profile Rails 

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## THESIS

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This thesis is dedicated to my parents, Suzan Bertucci and Martin J. Hamper.

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## SUMMARY

Modeling the contact problem is a fundamental feature in a variety of multibody system (MBS) dynamics applications. It is of particular importance in the area of railroad vehicle dynamics. In this case, the interaction of the vehicle and track can significantly impact the kinematics and kinetics of the system. In many cases, the rail is treated as a rigid solid, however there are some physical phenomena, such as the effect of broken or missing ties, which cannot be appropriately captured under this assumption. It is for this reason that the development of an alternative method for modeling the flexibility of the track is one of the contributions of this thesis. There are, of course, many phenomena which can be captured under the assumption that the rail is a rigid solid. In a subset of these cases, the geometry can vary along the length of the rail as is evident in special trackwork, such as turnouts, frogs, and crossing diamonds. These cases differ from the case of stock rail in which the rail geometry has no appreciable variation along the length of the rail. To accurately model this geometry for the rigid body contact problem, there are certain criteria related to the position and spatial derivatives of the surface which must be taken into account for the contact evaluation method chosen. Many existing contact surface representations do not meet these criteria for certain contact evaluation methods. As such, it is one of the contributions of this thesis to develop a contact surface representation that satisfies these requirements.

The finite segment method (FSM) has been used in the literature to model the dynamics of a deformable body by using a set of rigid bodies connected by elastic force elements. This approach can be applied, as demonstrated in this thesis, to the simulation of some rail movements. In order to ensure that the rail geometry is not distorted as the result of the finite segment displacements, a new track model that consistently integrates the absolute nodal

## SUMMARY (continued)

coordinate formulation (ANCF) geometry and the FSM is developed. ANCF finite elements define the track geometry in the reference configuration as well as the change in the geometry due to the movement of the finite segments of the track. Using ANCF geometry and the FS kinematics, the location of the wheel/rail contact point is predicted on-line and used to update the creepage expressions due to the finite segment displacements and rotations.

The location of the wheel/rail contact point and the updated creepage expressions are used to evaluate the creep forces. The three-dimensional non-conformal elastic contact formulation - algebraic equations (ECF-A), which allows for wheel/rail separation, is used in this thesis to determine the point of contact between two rigid bodies. The rail displacement due to the applied loads is modeled by a set of rigid finite segments that are connected by a series of spring-damper elements. Each rail FS is assumed to have all six rigid body degrees of freedom. The equations of motion of the finite segments are integrated with the railroad vehicle system equations of motion in a sparse matrix formulation. The resulting dynamic equations are solved using an explicit predictor-corrector numerical integration scheme that has a variable order and variable step size. The finite segments may be used to model specific phenomena that occur in railroad vehicle applications, including rail rotations and gage widening. The procedure used in this thesis to implement the FSM in a general purpose MBS computer program is described. Two simple models are presented in order to demonstrate the implementation of the FSM in MBS algorithms.

In order to examine in detail the applicability of using the FSM for modeling track flexibility, three methods suited to this purpose are presented while conclusions about their implementations and features are made. The first method is based on the floating frame of

## SUMMARY (continued)

reference (FFR) formulation which allows for the use of a detailed finite element (FE) mesh with the component mode synthesis technique in order to obtain a reduced order model. The second is the FSM in which the flexible rail is modeled as a finite number of rigid elements that are connected by springs and dampers. This method requires the use of rigid MBS formulations only. In the third method, the FFR formulation is used to obtain a model that is equivalent to the FSM model by assuming that the rail segments are very stiff; thereby allowing the exclusion of the high frequency modes associated with the rail deformations. This FFR/FSM model demonstrates that some rail movement scenarios, such as gage widening, can be captured using the FE/FFR formulation. The three procedures FFR, FSM, and FFR/FSM will be compared in order to establish differences among them and analyze the specific application of the FSM to modeling track flexibility. Convergence of the methods is analyzed. The three methods proposed in this thesis for modeling the movement of three-dimensional tracks are used in conjunction with ECF-A which predicts contact points on-line and allows for updating the creepages to account for the rail deformations. Several conclusions will be drawn in view of the results obtained. The limitations of using the finite segments approach for modeling the track structure and rail flexibility are also discussed.

While the modeling of the flexibility of bodies in contact is important for many models, contact between two rigid bodies can also be a fundamental feature in a variety of models, including those focused on railroad vehicles. Many procedures have been proposed to solve the rigid body contact problem, most of which belong to one of two categories: off-line and on-line contact search methods. This thesis will present the development of a contact surface geometry model for the rigid body contact problem in the case where an on-line three-dimensional non-

## SUMMARY (continued)

conformal contact evaluation procedure, such as ECF-A, is employed. It is shown that the contact surface must have continuity in the second order spatial derivatives when used in conjunction with ECF-A. Many of the existing contact surface models rely on direct linear interpolation of profile curves which leads to first order spatial derivative discontinuities. While these procedures may be well suited for use in the off-line approach, these procedures lead to erroneous spikes in the prediction of contact forces when an on-line approach such as ECF-A is employed. To this end, an ANCF thin plate surface model is developed in order to ensure second order spatial derivative continuity which satisfies the requirements of the contact formulation. A simple example of a railroad vehicle negotiating a turnout, which includes a variable crosssection rail surface, is tested for the cases of the new ANCF thin plate element surface, an existing ANCF thin plate element surface with first order spatial derivative continuity, and the direct linear profile interpolation method. A comparison of the numerical results reveals the benefits of using the new ANCF surface geometry developed in this thesis.

## CHAPTER 1

## INTRODUCTION

In multibody system (MBS) dynamics, the contact between various bodies in the system is a fundamental feature for a variety of applications. One such application is the modeling of contact between the wheels of a vehicle and the rails in a railroad vehicle dynamics simulation. The interaction between the vehicle and the rails can have a significant impact on both the kinematics and kinetics of the system. One common assumption in these simulations is that the rails may be treated as rigid solids. In many cases this assumption causes no hindrance, however in some cases, such as that of broken or missing ties, this assumption leads to a poor model of the physical system. It is for this reason that an alternative method for modeling the flexibility of the track is one of the contributions of this thesis. As previously mentioned, there are many cases in which the flexibility of the track can be neglected without sacrificing the accuracy of the model. In a subset of these cases, the geometry of the rail can be of paramount importance for an accurate prediction of the systems dynamic response. In the majority of simulations, the vehicle is traveling along stock rail which has a constant surface profile along the length of the rail. However there are some cases, for example the case of special trackwork, such as turnouts, frogs, and crossing diamonds, in which the geometry of the rail varies along its longitudinal length. In this case the geometry of the track can have a significant impact on the motion of the vehicle, and as a consequence, the geometry must be accurately modeled in order to correctly simulate the motion of the vehicle. For certain contact evaluation procedures implemented in MBS programs, there are criteria related to the position and spatial derivatives of the surfaces which must be taken into account for an accurate prediction of the contact forces. Many of the existing contact surface representation techniques do not meet these criteria for certain contact
evaluation procedures. As such, it is one of the contributions of this thesis to develop a contact surface representation that satisfies these requirements.

### 1.1 Background

In MBS dynamics, the transient motion of a collection of bodies connected via elastic force elements and various joint types is analyzed. Analytically, this results in a set differential and algebraic equations (DAEs) which must be solved as is described in the literature (Huston and Liu, 2001; Roberson and Schwertassek, 1988; Shabana, 2005; Shabana, 2010; Shabana 2012). The non-linear algebraic equations generally result from the kinematic constraints imposed upon the motion of the bodies in the system. As is described in the literature (Huston and Liu, 2001; Roberson and Schwertassek, 1988; Shabana, 2005; Shabana, 2010), these equations can be used to represent a wide variety of joint types ranging from simple revolute joints to universal joints. Additionally, these equations can be used to define the conditions for contact between two bodies as is described by (Kalker, 1979; Kalker, 1990;Shabana et al., 2008).

The bodies contained in a MBS may be considered as either rigid or flexible as described by Shabana (Shabana, 2005; Shabana, 2012). In the physical world, an arbitrary body is considered to be flexible as it can undergo deformation provided sufficient force is applied. In the context of MBS applications, the amount of deformation many bodies undergo is too small to impact the bulk motion of the system and are therefore considered to be rigid and are numerically incapable of deformation. Other bodies, such as those considered to be flexible in the MBS context, are modeled with additional coordinates which represent the elastic characteristics of the bodies which allow them to undergo deformation. This is typically accomplished through either the floating frame of reference (FFR) formulation (Canavin and Likins, 1977; Shabana and Schwertassek, 1997; Shabana, 2005; Shabana, 2012) for the case
where the deformations are small or through the absolute nodal coordinate formulation (ANCF) (Shabana, 1997; Shabana, 1998; Shabana, 2005; Shabana, 2012) for the case where the deformations are large.

The finite element method (FEM) can be combined with the FFR formulation to develop a detailed FE mesh; the dimension of which can be significantly reduced through component mode synthesis techniques. In an alternative approach, called the finite segment method (FSM), flexible solids are modeled as a series of rigid bodies connected with spring-damper elements. The FSM was developed, as an alternative to the FEM, to model MBS applications including robotics and spatial systems. Several authors have worked on developing this method and have obtained high quality results for some applications. Different FSM variations and some examples can be found in the literature (Connelly and Huston, 1994a; Connelly and Huston, 1994b; Huston, 1981; Wang and Huston, 1994; Wittbrodt and Wojciech, 1995; Wittbrodt et al., 2006). In these studies, several variations of the FSM are proposed; for example, the number of degrees of freedom between finite segments (Wang and Huston, 1994; Wittbrodt et al., 2006), the choice of parameters for the spring-damper elements (Huston, 1981; Wittbrodt et al., 2006), as well as linearization in transformation matrices (Huston, 1981; Wittbrodt and Wojciech, 1995) have been the subject of discussion.

The set of DAEs governing multibody systems may be highly nonlinear due to the application of constraint equations and certain flexible body material models. Due to these nonlinearities, closed form solutions are often not available and the set of DAEs must be solved numerically. Typically a numerical integration scheme, such as the explicit Adams-Bashforth procedure (Shampine and Gordon, 1975), is employed. The result is an approximation to the transient evolution of the position, velocity, acceleration, and forces associated with the MBS. In
this thesis, the differential and algebraic equations of motion are solved in all numerical examples using the general purpose multibody code SAMS/2000 (Shabana, 2010).

An example of one such MBS is a railroad vehicle traveling along a track with variable geometry. The accuracy of the dynamic analysis of railroad vehicles depends on the accuracy of the wheel/rail contact model, the representation of the vehicle components and its suspension elements, the description of the track geometry, and the track elasticity model. This thesis focuses on methods which improve upon existing procedures for modeling the track elasticity and geometry. This is accomplished by introducing two new methods. In the first method, a simplified approach to modeling the flexibility of the track using the FSM is introduced. In the second method, a detailed approach to modeling the geometry of rails with variable profiles, such as turnouts, is presented. The later approach may also be extended to general rigid body contact problems.

### 1.2 FSM Rail Modeling

In this thesis, the FSM is used to model both track structure and rail flexibility. With this approach, the rail is modeled as a set of moveable rigid bodies connected by elastic force elements. Each moveable rail segment will be referred to as a finite segment (FS) which is assumed to have known length and inertia properties. A FS may represent a section of either tangent or curved track. In the case of unconstrained motion, the FS will have six degrees of freedom including three translations and three rotations. The motion of a FS is affected by the forces resulting from both the contact with the wheel and the constraints applied to the FS. For generality, the track geometry is rigidly attached to the corresponding FS at its boundaries as defined by the longitudinal rail arc length parameter. The track geometry is developed using a
pre-processor computer program that is described in the literature (Shabana et al., 2008). The pre-processor output provides the geometry data required to predict the location of the contact point between the wheel and the finite segment or rail on-line. The three-dimensional wheel/rail contact model used in this thesis requires the evaluation of the spatial derivatives at the point of contact with respect to the wheel and rail surface parameters. In order to accurately evaluate these derivatives, three-dimensional fully parameterized ANCF 3D beam elements are employed (Shabana, 2005; Shabana, 2012). The ANCF description provides a position vector field which can be used to accurately define the derivatives that are required in the formulation and solution of the nonlinear contact conditions.

Modeling the deformation of the structural elements of railroad tracks is necessary in order to be able to develop detailed models that address different issues related to railroad vehicles performance, dynamics, and stability. Studies related to passenger comfort, damages in the track, and/or noise generation, require nonlinear formulations of the vehicle/track system in which the deformation of the track components play a key role. This fact makes it fundamental to develop methods that describe such deformation. The effect of the track structure (sleepers, ballast, subgrade, etc.) was the focus of several previous investigations (Goicolea and Antolín, 2011; Wanming and Xiang, 2008; Zhai et al., 2004). Several approaches have been employed to model the track structure and rail flexibility. Some researchers proposed the use of a simplified beam on elastic foundation to model rail flexibility and track structure (Cai and Raymond, 1994; Duffy, 1990; Grassie and Cox, 1982; Ilias and Muller, 1994; Ishida et al., 1997). Recently, Shabana et al. proposed the use of FFR and FEM to model rail flexibility and track structure (Shabana et al., 2007). The integration of the FFR formulation and the FEM generally leads to a non-linear MBS approach in which component modes can be used to systematically reduce the
number of the deformation modes. The FFR formulation was also validated as a rail flexibility analysis approach (Rathod et al., 2009).

The use of the FSM to describe the deformation of the rail could offer an alternative to the use of the more general FEM in certain applications. In this thesis, the FSM and ANCF geometry are integrated in order to accurately describe the track geometry and deformation. In this model, the rigid finite segments are used to represent a flexible continuum material by connecting the segments with spring-damper elements similar to those proposed by Wittbrodt et al. (Wittbrodt et al., 2006). The method of Wittbrodt et al. leads to a system of rigid bodies, each of which has constant mass and inertia properties. One beneficial aspect of this procedure is that it may be contained entirely within a MBS program without the need for a third party FEM modeling package. In this thesis, the FFR formulation will serve as baseline for comparison with the proposed FSM rail model in order to demonstrate the positive aspects and limitations of these procedures.

### 1.3 Contact Surface Modeling

The later portion of this thesis is concerned with modeling the geometry of surfaces for use in the solution of the rigid body contact problem. As previously mentioned, contact between two bodies is a fundamental feature in a variety of applications in MBS dynamics. For example, in the study of the wheel/rail contact in a railroad vehicle dynamics model, an accurate geometric model and a robust contact formulation are prerequisites for a realistic prediction of the wheel/rail interaction forces. To this end, many methods have been introduced which rely on curve geometry to represent the contact surfaces such as the curve network representation (Shabana et al., 2008). Here, a profile curve of the rail surface is swept along a space curve
which represents the centerline of the rail in order to define the contact surface of the rail. It has been shown that this method is viable in the case where the profile of the rail is constant; however it is insufficient for rail sections which have geometry that varies along the rail space curve. It is one of the contributions of this thesis to develop a new method with which a surface with variable geometry may be modeled more accurately for the rigid body contact problem.

Many procedures have been proposed to solve the rigid body contact problem, most of which belong to one of two categories: off-line and on-line contact search methods. In off-line contact search methods, a significant amount of work is performed at a pre-processing stage to determine the location of a contact point between two surfaces under a limited range of scenarios. This data is then compiled in tabular form as a function of a reduced set of parameters which are interpolated at run-time to determine the location of the contact point as well as other parameters required to determine the contact forces. Due to the complexity of the contact problem, these tables are often formed under the assumption that one of the two bodies is fully constrained to the ground. An example is the method applied by Kassa et al. in which a contact table is formed to solve the wheel/rail contact problem for the case of variable profile rail sections (Kassa et al., 2006). A similar approach is presented by Alfi and Bruni (Alfi and Bruni, 2009). Sugiyama et al. presented a method in which the derivatives of the profile curve, which are described using a three-layer spline, are also computed via linear interpolation of the two adjacent profiles (Sugiyama et al., 2011; Sugiyama et al., 2012). Kassa et al. used the distance traveled along the rail and the lateral shift of the vehicle to define a set of tabular contact functions which contain the information necessary to define the location and forces associated with the wheel/rail contact. Linear interpolation between these contact functions is performed at run-time with some success as demonstrated by Kassa and Nielsen (Kassa and Nielsen, 2008)
where a series of tests were performed to compare measured and numerical results. It is important to note that many off-line tabular contact evaluation methods do not account for relative rotation between the two bodies. In many scenarios this approximation is sufficient, as is demonstrated by the validation presented in the literature (Kassa and Nielsen, 2008); however this assumption is only valid provided that the relative rotations remain small.

In on-line contact search methods, computational geometry (CG) methods are employed at run-time to determine the location of a contact point between two bodies. Additional parameters are often computed in order to determine the associated contact forces. To this end, the elastic contact formulation - algebraic equations (ECF-A), which requires the solution of a set of four nonlinear algebraic equations (Pombo and Ambrósio, 2003; Pombo et al, 2007; Shabana et al., 2008), is chosen to solve for the location contact points in this thesis. This formulation is based on four non-linear equations that are solved numerically using a NewtonRaphson algorithm to determine the four surface parameters which define the geometry of the wheel and rail surfaces. The wheel and rail surface parameters are used to define the location of the contact points. The geometry at the contact point, the material properties, and the creepages are used to calculate the normal and creep forces at the point of contact. Several methods can be used to calculate the creep forces between two bodies in rolling contact (De Pater, 1988; Garg and Dukkipati, 1984; Kalker, 1979; Kalker 1990). However, this thesis employs Kalker's USETAB program to determine these forces (Vollebregt, 2008).

Many authors have made use of curve geometry for the on-line contact problem with some success. For example, Schupp et al. presented a method in which linear interpolation between two adjacent profiles is used at run-time to determine an intermediate profile of the rail at the current location (Schupp et al., 2004). Wan et al. presented a method of reducing the three-
dimensional problem to a two dimensional case where only the distance along the trajectory curve and vertical height of the rail profile are considered (Wan et al., 2013). The common feature of these approaches is the use of curve geometry to represent three-dimensional surfaces. As with the tabular approach, this method is successful over a range of scenarios but cannot capture the full range of features available in surface-based geometric methods.

Since the contact problem is highly non-linear, the off-line approach has an advantage in computation time needed for a simulation; however, this method is limited to the range of scenarios calculated at the pre-processing stage, giving the on-line approach an advantage in robustness. There also exists an additional limitation in the direct profile curve interpolation method presented by Schupp et al. (Schupp et al., 2004) when employed in conjunction with either the three-dimensional non-conformal elastic or constraint contact approaches (Shabana et al., 2008). As was discussed by Sinokrot et al. (Sinokrot et al., 2008), the elastic contact approach requires second order spatial derivative continuity $\left(C^{2}\right)$ throughout the contact surface while the constraint contact approach requires third order spatial derivative continuity $\left(C^{3}\right)$. A surface created via linear interpolation using more than two curves will have a discontinuity in the first order spatial derivatives at the location of each interior profile, thus the surface has only position level continuity $\left(C^{0}\right)$. A direct consequence of this, as will be shown in the numerical results of a comparative example in this thesis, is a fictitious spike in the predicted contact forces at the location of the spatial derivative discontinuity. It is for this reason that a new technique is proposed in this thesis to model a variable profile surface without the need for direct interpolation between profile curves at run-time.

Rather than using the curve based approach previously discussed, the new method developed in this thesis relies on the construction of a surface at a pre-processing stage using a mesh of ANCF thin plate elements (Dmitrochenko and Pogorelov, 2003; Shabana, 2012). The most direct approach to generating a surface composed of ANCF thin plate elements begins with the construction of a non-uniform rational B-spline (NURBS) surface which may be easily converted into a collection of ANCF thin plate elements using a technique discussed in a later chapter of this thesis. Note that ANCF elements are preferred to NURBS surfaces in this context as ANCF surfaces may easily be adapted for use in the modeling of contact with flexible surfaces. ANCF solid elements with similar surface continuity conditions may be developed in the future to model contact between flexible solids such as a wheel and rail. Due to the fact that NURBS surfaces may employ a very fine discretization, the conversion from NURBS to ANCF may also be applied by direct evaluation of the position and spatial derivatives of the NURBS surface to generate the ANCF nodal coordinates. This provides the option for the user to select a smaller subset of the potential ANCF nodes thereby producing a more coarse mesh than is generated by direct transformation from NURBS to ANCF.

Many methods are available for the construction of NURBS surfaces. This task is commonly divided into two categories based on the type of input: surface data points with regular or irregular spacing. Piegl and Tiller presented a series of surface reconstruction techniques which rely on regularly spaced data as input (Piegl and Tiller, 1997). This is ideal in the case where a device which measures profile curves of a surface is used to generate the input data. Similar procedures are implemented in the open source NURBS package SISL (SINTEF ICT, 2005), which is used for NURBS surface reconstruction in this thesis. This category of methods is limited in accuracy by the spacing of the input profile curves as any surface features
between the profile curves are omitted from the reconstructed surface. More recently, threedimensional scanning technology has become commercially available and many authors have developed techniques for reconstructing NURBS surfaces from the scattered data produced by these devices. Gálvez and Iglesias presented an excellent survey of the current state of the art in scattered data surface reconstruction techniques as well as a new (and highly efficient) method for reconstructing NURBS surfaces using particle swarm optimization techniques (Gálvez and Iglesias, 2010). This method is an ideal choice for the generation of a NURBS surface as a highly accurate surface may be generated with a limited investment of time and resources.

Mikkola et al. presented an ANCF thin plate element which has first order derivative continuity $\left(C^{1}\right)$ at the element boundaries in a surface mesh (Mikkola et al., 2012). This thin plate element was tested for use as a rigid contact surface model and was found to provide a significant improvement over the direct linear interpolation method. It still suffers, however, from the issue of second order spatial derivative discontinuities and the associated fictitious spikes in the contact forces. Additionally, it is shown that with the application of constraints, as proposed by Lan and Shabana (Lan and Shabana, 2010), $C^{2}$ continuity cannot be achieved for this element without the destruction of the element's intended geometry. To this end, a new ANCF thin plate element which has $C^{2}$ continuity at the element boundaries in a surface mesh is developed and tested in this thesis. A comparison between the three aforementioned methods is presented.

### 1.4 Scope and Objectives of the Thesis

Chapter 2 was first published in Acta Mechanica (Hamper et al., 2012a) and is reproduced in this thesis with permission which is provided in Appendix A. In this chapter, a discussion of
finite segment kinematics and the ANCF discretization used to define the rail geometry is presented. Both the position and velocity functions, which are required for use in the contact formulation and creepage evaluation, are presented. It is shown that these discrete rail segments may be employed without distortion of the rail geometry in the case in which they are stationary. The contact conditions used in the on-line prediction of the wheel/rail contact points in the case where the FSM is employed are discussed. The equations of motion in which the FSM rail description is used are presented. Additionally, the geometric formulation chosen to represent curved track is discussed. Subsequently, two numerical examples are presented which demonstrate the ability of finite segments in modeling both stationary and moving rail sections.

Chapter 3 was first published in the ASME Journal of Computational and Nonlinear Dynamics (Hamper et al., 2012b) and is reproduced in this thesis with permission which is provided in Appendix B. In this chapter, the modeling of rail flexibility using FSM and FFR is discussed. The method with which the stiffness and damping properties which define the connectivity between finite segments is presented. A brief summary of the FFR formulation which is used to develop the detailed track model is provided. An alternative approach which explains how to use the FFR formulation with component mode synthesis methods to obtain a model that is equivalent to the FS model is developed. This is achieved by developing a FE mesh for the model in which the rail finite elements can experience only rigid body displacements. The on-line contact formulation for both the cases of FSM and FFR rail are presented. A simple railroad example that serves to validate the new methods applied and comment on the approaches used is provided. This railroad example allows for highlighting the differences between the methods.

Chapter 4 was first published in the ASME Journal of Computational and Nonlinear Dynamics (Hamper et al., 2014) and is reproduced in this thesis with permission which is provided in Appendix B. In this chapter, the development of a new contact surface geometry model is presented. First, the existing curve network geometry procedure is provided in order to shed light on the problems of discontinuities in the spatial derivatives and how this problem negatively impacts the contact force prediction. Following this, ANCF thin plate surface geometry is discussed beginning with the existing $C^{1}$ plate element available in the literature (Mikkola et al, 2012). The associated spatial derivative continuity constraints are also discussed. This leads to the development of a new ANCF thin plate element which has $C^{2}$ continuity. This element is presented along with the conditions necessary to maintain $C^{2}$ continuity. A set of guidelines which may be used in the construction of a surface mesh composed of ANCF thin plate elements is discussed. It is also shown that this new ANCF thin plate element may be converted to an equivalent Bezier patch and vice-versa. The three-dimensional non-conformal contact formulation for the case of generic rigid bodies is presented. A brief discussion of the equations of motion used in the case of rigid body contact is provided. Finally, a numerical example comparing the results of the three different methods employed to model contact surfaces is presented to illuminate the negative impact of using contact surface models with insufficient spatial derivative continuity in conjunction with on-line approaches such as ECF-A. It is shown that the new ANCF plate element presented alleviates these issues.

## CHAPTER 2

## FINITE SEGMENT RAIL MODELING

In the finite segment method (FSM), the dynamics of a deformable body is described using a set of rigid bodies that are connected by elastic force elements. This approach can be adapted for use, as demonstrated in this chapter, in the simulation of some rail movements. In order to ensure that the rail geometry is not distorted as the result of the finite segment (FS) displacements, a new track model that consistently integrates the absolute nodal coordinate formulation (ANCF) geometry and the FSM is developed. ANCF finite elements define the track geometry in the reference configuration as well as the change in the geometry due to the movement of the finite segments of the track. Using ANCF geometry and the finite segment kinematics, the location of the wheel/rail contact point is predicted on-line and used to update the creepage expressions due to the FS displacements and rotations. The location of the wheel/rail contact point and the updated creepage expressions are used to evaluate the creep forces. The three-dimensional nonconformal elastic contact formulation - algebraic equations (ECF-A) that allows for wheel/rail separation is used. The rail displacement due to the applied loads is modeled by a set of rigid finite segments that are connected by spring-damper elements. Each rail FS is assumed to have six rigid body degrees of freedom. The equations of motion of the finite segments are integrated with the railroad vehicle system equations of motion in a sparse matrix formulation. The resulting dynamic equations are solved using an explicit predictor-corrector numerical integration scheme that has a variable order and step size. The finite segments may be used to model specific phenomena that occur in railroad vehicle applications, including rail rotations and gage widening. The procedure used in this chapter to implement the finite segment method in a general purpose multibody system (MBS) computer program is described. Two simple models are
presented in order to demonstrate the implementation of the FSM in MBS algorithms. The limitations of using the finite segment approach for modeling the track structure and rail flexibility are also discussed.

### 2.1 Finite Segment Kinematics

As the rail finite segments move due to the wheel/rail interaction forces, the geometry of the rail surface should not be distorted; that is, the rail surface geometry should remain invariant under an arbitrary FS rigid body displacement. Therefore, it is necessary to use a method, such as the ANCF representation, that preserves the geometry in the case of an arbitrary rigid body displacement. This section explains how the ANCF representation is used in conjunction with the FS rigid body kinematics to ensure that the rail geometry is not distorted during the movement of finite segments. The FS geometry is defined using ANCF finite elements, while the FS kinematic equations are expressed in terms of absolute Cartesian and orientation coordinates. The ANCF nodal coordinates have to be consistently updated in order to avoid shape distortion and at the same time account for the change of the FS configuration resulting from the wheel/rail interaction forces.

In this chapter, the track geometry is described by nodes that define the track center line and rail space curve segments. The track pre-processor output defines the position of the nodes and the orientation of the profile frame at these nodes. The orientation of the profile frame at the nodes is defined using three Euler angles which can be utilized to define at each node three unit vectors which are required in the ANCF interpolation. The position of the nodes is defined with respect to a track coordinate system as shown in Fig. 1. The global position vector of an arbitrary track node is defined as follows:


Figure 1 Track Model

$$
\begin{equation*}
\mathbf{r}^{r}=\mathbf{R}^{r}+\mathbf{A}^{r} \overline{\mathbf{u}}_{k}^{r}, \quad k=1,2, \ldots, n \tag{2.1}
\end{equation*}
$$

where subscript $r$ refers to rail or track, $\mathbf{R}^{r}$ defines the position of the origin of the track frame, $\overline{\mathbf{u}}_{k}^{r}$ defines the position of node $k$ with respect to the track frame, $n$ is the total number of nodes in the track, and $\mathbf{A}^{r}$ is a three-dimensional transformation matrix which defines the orientation of the track frame with respect to the global frame. Using Eq. 2.1, the location of an arbitrary point on the rail space curve may be defined by interpolating between any two nodes using ANCF finite elements. If the track structure is assumed to be fixed, $\mathbf{R}^{r}$ is a constant vector and $\mathbf{A}^{r}$ is a constant transformation. The track structure can be fixed, while the rail finite segments can experience rigid body displacements with respect to the track coordinate system. Because the geometry is initially defined in the track coordinate system, the relative displacement of the finite segments with respect to the track coordinate system must be evaluated in order to define the change of coordinates necessary to avoid a distortion of the geometry.

In the FSM employed in this chapter, each segment is defined within a specified location along the rail space curve. In this case, the geometry of the rail surface must be defined with respect to the FS centroidal body coordinate system. In order to simplify the equations defining


Figure 2 Contact Point on a Finite Segment
the wheel/rail or wheel/FS contact, the rail profile coordinate system is introduced as shown in Fig. 2. Using this profile coordinate system, the global position vector at the point of contact between the wheel and the finite segment can be written as follows

$$
\begin{equation*}
\mathbf{r}^{f s}=\mathbf{R}^{f s}+\mathbf{A}^{f s}\left(\overline{\mathbf{R}}^{r p}+\mathbf{A}^{r p} \overline{\mathbf{u}}^{r p}\right) \tag{2.2}
\end{equation*}
$$

where $\mathbf{R}^{f s}$ defines the position of the finite segment body coordinate system with respect to the global coordinate system, $\mathbf{A}^{f s}$ defines the orientation of the finite segment with respect to the global coordinate system, $\overline{\mathbf{R}}^{r p}$ defines the location of the origin of the profile frame with respect to the finite segment, $\mathbf{A}^{r p}$ defines the orientation of the rail profile frame with respect to the finite segment, and $\overline{\overline{\mathbf{u}}}^{r p}$ defines the location of the contact point with respect to the profile frame.

The solution procedure used in this chapter for predicting the wheel/rail contact point online requires the evaluation of the tangent and normal vectors at the point of contact. ANCF geometry is used to determine a differentiable position vector field that can be used to determine the tangent and normal vectors as well as their spatial derivatives at the contact point on the rail surface. The location of the origin of the profile frame may be defined using three-dimensional fully parameterized ANCF beam elements as (Shabana, 2005; Shabana, 2012)

$$
\begin{equation*}
\overline{\mathbf{R}}^{r p}\left(x^{f s}, y^{f s}, z^{f s}, t\right)=\mathbf{S}^{f s}\left(x^{f s}, y^{f s}, z^{f s}\right) \mathbf{e}^{f s}(t) \tag{2.3}
\end{equation*}
$$

where $t$ is time; $x^{f s}, y^{f s}$, and $z^{f s}$ are the spatial coordinates; and $\mathbf{S}^{f s}$ is the shape function matrix defined as

$$
\mathbf{S}^{f s}=\left[\begin{array}{llllllll}
S_{1} \mathbf{I} & S_{2} \mathbf{I} & S_{3} \mathbf{I} & S_{4} \mathbf{I} & S_{5} \mathbf{I} & S_{6} \mathbf{I} & S_{7} \mathbf{I} & S_{8} \mathbf{I} \tag{2.4}
\end{array}\right]
$$

In this shape function matrix, $\mathbf{I}$ is a $3 \times 3$ identity matrix, and $S_{i}(i=1,2, \ldots, 8)$ are the shape functions defined as

$$
\left.\begin{array}{cccc}
S_{1}=1-3 \xi^{2}+2 \xi^{3}, & S_{2}=l\left(\xi-3 \xi^{2}+\xi^{3}\right), & S_{3}=l(\eta-\xi \eta), & S_{4}=l(\zeta-\xi \zeta) \\
S_{5}=3 \xi^{2}-2 \xi^{3}, & S_{6}=l\left(\xi^{3}-\xi^{2}\right), & S_{7}=l \xi \eta, & S_{8}=l \xi \zeta \tag{2.5}
\end{array}\right\}
$$

where $\xi=x^{f s} / l, \eta=y^{f s} / l$, and $\zeta=z^{f s} / l$, where $l$ is the length of the element. In Eq. 2.3, $\mathbf{e}^{f s}$ is the vector of time dependent nodal coordinates. This vector is assumed to depend on time since the finite segments are allowed to translate and rotate in order to be able to simulate the rail movements. If the contact point is assumed to lie between nodes $i$ and $i+1$, the vector of nodal coordinates $\mathbf{e}^{f s}$ may be defined as

$$
\mathbf{e}^{f s}=\left[\begin{array}{llllllll}
\overline{\mathbf{u}}_{i}^{f s^{T}} & \overline{\mathbf{e}}_{x, i}^{f s^{T}} & \overline{\mathbf{e}}_{y, i}^{f s^{T}} & \overline{\mathbf{e}}_{z, i}^{f s^{T}} & \overline{\mathbf{u}}_{i+1}^{f s^{T}} & \overline{\mathbf{e}}_{x, i+1}^{f s^{T}} & \overline{\mathbf{e}}_{y, i+1}^{f s^{T}} & \overline{\mathbf{e}}_{z, i+1}^{f s s^{T}} \tag{2.6}
\end{array}\right]
$$

where $\overline{\mathbf{u}}_{k}^{f s}$ defines the position of node $k$ with respect to the finite segment body coordinate system, and $\overline{\mathbf{e}}_{j, k}^{f s}$ is the gradient vector taken at node $k$ with respect to coordinate $j$ ( $j=$ $\left.x^{f s}, y^{f s}, z^{f s}\right)$. The Euler angles in the track pre-processor output define the orientation of the profile frames with respect to the track frame. In order to correctly define the geometry when the rail finite segments move, the vectors in Eq. 2.6 must be computed in the finite segment body system as shown in Fig. 3. This can be accomplished as follows

$$
\begin{equation*}
\overline{\mathbf{u}}_{k}^{f s}=\mathbf{A}^{f s^{T}}\left(\mathbf{R}^{r}+\mathbf{A}^{r} \overline{\mathbf{u}}_{k}^{r}-\mathbf{R}^{f s}\right), \quad \overline{\mathbf{e}}_{j, k}^{f s}=\mathbf{A}^{f s^{T}} \mathbf{A}^{r} \overline{\mathbf{e}}_{j, k}^{r}, \quad j=x, y, z \quad k=i, i+1 \tag{2.7}
\end{equation*}
$$

In this chapter, two parameters are used to describe the surface of the finite segment. The first is the longitudinal surface parameter $s_{1}^{f s}$ and the second is the lateral surface parameter


Figure 3 Reference, Body, and FS Coordinate Systems


Figure 4 Wheel and Rail Surface Parameters (Shabana et al., 2008)
$s_{2}^{f s}$ as shown in Fig. 4. It is convenient to define $\overline{\overline{\mathbf{u}}}^{r p}$ such that it lies entirely within the $Y^{r p} Z^{r p}$ plane. Assuming the rail has a constant predefined profile, $\overline{\mathbf{u}}^{r p}$ can be defined by the vector $\overline{\overline{\mathbf{u}}}^{r p}\left(s_{2}^{f s}\right)=\left[\begin{array}{lll}0 & s_{2}^{f s} & f\left(s_{2}^{f s}\right)\end{array}\right]^{T}$. Furthermore, the position of the origin of the profile frame will be function of only the longitudinal surface parameter, that is $\overline{\mathbf{R}}^{r p}\left(s_{1}^{f s}, t\right)=\mathbf{S}^{f s}\left(s_{1}^{f s}\right) \mathbf{e}^{f s}(t)$.

As will be discussed in the following section, the velocity creepage expressions used in this chapter are functions of the velocities of the wheel and rail finite segments at the point of contact. Because of the FS movement, the rail translational and angular velocities cannot be assumed to be zero when the creepages are calculated. The velocity of the finite segment at the contact point is defined as

$$
\begin{equation*}
\dot{\mathbf{r}}^{f s}=\dot{\mathbf{R}}^{f s}+\widetilde{\boldsymbol{\omega}}^{f s} \mathbf{A}^{f s}\left(\mathbf{S}^{f s} \mathbf{e}^{f s}+\mathbf{A}^{r p} \overline{\mathbf{u}}^{r p}\right)+\mathbf{A}^{f s} \mathbf{S}^{f s} \dot{\mathbf{e}}^{f s} \tag{2.8}
\end{equation*}
$$

where $\dot{\mathbf{R}}^{\text {fs }}$ is the absolute velocity of the finite segment center of mass (reference point), $\dot{\mathbf{e}}^{\text {fs }}$ is the time derivative of the vector of nodal coordinates defined as

$$
\dot{\mathbf{e}}^{f s}=\left[\begin{array}{llllllll}
\dot{\mathbf{u}}_{i}^{f s^{T}} & \dot{\dot{\mathbf{e}}}_{x, i}^{f s^{T}} & \dot{\mathbf{e}}_{y, i}^{f s^{T}} & \dot{\mathbf{e}}_{z, i}^{f s^{T}} & \dot{\mathbf{u}}_{i+1}^{f s^{T}} & \dot{\mathbf{e}}_{x, i+1}^{f s^{T}} & \dot{\dot{\mathbf{e}}}_{y, i+1}^{f s^{T}} & \dot{\mathbf{e}}_{z, i+1}^{f s^{T}} \tag{2.9}
\end{array}\right]
$$

where

$$
\begin{equation*}
\dot{\mathbf{u}}_{k}^{f s}=\mathbf{A}^{f s^{T}}\left(\widetilde{\boldsymbol{\omega}}^{f s}\left(\mathbf{R}^{r}+\mathbf{A}^{r} \overline{\mathbf{u}}_{k}^{r}-\mathbf{R}^{f s}\right)-\dot{\mathbf{R}}^{f s}\right) \quad k=i, i+1 \tag{2.10}
\end{equation*}
$$

and

$$
\begin{equation*}
\dot{\overline{\mathbf{e}}}_{j, k}^{f s}=\mathbf{A}^{f s^{T}}\left(\mathbf{A}^{r} \dot{\overline{\mathbf{e}}}_{j, k}^{r}-\widetilde{\boldsymbol{\omega}}^{f s} \mathbf{A}^{r} \overline{\mathbf{e}}_{j, k}^{r}\right) \quad j=x, y, z \quad k=i, i+1 \tag{2.11}
\end{equation*}
$$

and $\widetilde{\boldsymbol{\omega}}^{f s}$ is the skew symmetric matrix associated with the FS angular velocity vector $\boldsymbol{\omega}^{f s}$ which may be written in terms of Euler parameters $\boldsymbol{\theta}^{f s}$ as $\boldsymbol{\omega}^{f s}=2 \mathbf{E}^{f s} \dot{\boldsymbol{\theta}}^{f s}$, where

$$
\mathbf{E}^{f s}=\left[\begin{array}{cccc}
-\theta_{1}^{f s} & \theta_{0}^{f s} & -\theta_{3}^{f s} & \theta_{2}^{f s}  \tag{2.12}\\
-\theta_{2}^{f s} & \theta_{3}^{f s} & \theta_{0}^{f s} & -\theta_{1}^{f s} \\
-\theta_{3}^{f s} & -\theta_{2}^{f s} & \theta_{1}^{f s} & \theta_{0}^{f s}
\end{array}\right], \quad \boldsymbol{\theta}^{f s}=\left[\begin{array}{c}
\theta_{0}^{f s} \\
\theta_{1}^{f s} \\
\theta_{2}^{f s} \\
\theta_{3}^{f s}
\end{array}\right]
$$

In Eqs. 2.10 and 2.11, the coordinate system of the track is assumed to be fixed. Because the Euler parameter axis-angle representation is difficult to visualize, Euler parameters are often converted to a three angle representation, such as the Euler Angles. The necessary formulae to convert Euler parameters to Euler angles may be found in the literature (Shabana, 2005).

### 2.2 Finite Segment/Wheel Contact

The FS approach can be integrated with any on-line wheel/rail contact model. However, in the case of the wheel/rail contact model that employs the constraint approach (Shabana et al., 2008), a constraint addition and deletion procedure is needed to switch between finite segments that are used to define the rail. For this reason, the FS model presented in this chapter is used in conjunction with the three-dimensional non-conformal wheel/rail elastic contact formulation ECF-A. In ECF-A, the wheel is assumed to have 6 degrees of freedom with respect to the rail and small penetration and separation between the wheel and rail are allowed. The point of contact is found in terms of the four surface parameters contained in the vector $\mathbf{s}$

$$
\mathbf{s}=\left[\begin{array}{llll}
s_{1}^{w} & s_{2}^{w} & s_{1}^{f s} & s_{2}^{f s} \tag{2.13}
\end{array}\right]^{T}
$$

where the superscripts $w$ and $f s$ denote the wheel and finite segment bodies, and the subscripts 1 and 2 denote first and second surface parameter of each body as shown in Fig. 4. In the case of three-dimensional non-conformal wheel/rail elastic contact, two conditions must be satisfied at the contact point. The first condition requires that position vectors of the contact point on the two surfaces are the same. The second condition requires that the tangent planes of the two surfaces at this point be coplanar. These two conditions are satisfied by four non-linear algebraic equations $\mathbf{E}(\mathbf{s})=\mathbf{0}$ where the first two equations correspond to the first condition, and the later
two equations correspond to the second condition. The four equations can be written explicitly as follows (Shabana et al., 2008)

$$
\mathbf{E}(\mathbf{s})=\left[\begin{array}{llll}
\mathbf{t}_{1}^{f s} \cdot \mathbf{r}^{w f s} & \mathbf{t}_{2}^{f s} \cdot \mathbf{r}^{w f s} & \mathbf{t}_{1}^{w} \cdot \mathbf{n}^{f s} & \mathbf{t}_{2}^{w} \cdot \mathbf{n}^{f s} \tag{2.14}
\end{array}\right]^{T}=\mathbf{0}
$$

where the $\mathbf{t}$ and $\mathbf{n}$ are the tangent and normal vectors defined as (Kreysig, 1991)

$$
\begin{equation*}
\mathbf{n}^{f s}=\mathbf{t}_{1}^{f s} \times \mathbf{t}_{2}^{f s}, \quad \mathbf{t}_{i}^{f s}=\frac{\partial \mathbf{r}^{f s}}{\partial s_{i}}, \quad \mathbf{t}_{i}^{f s}=\frac{\partial \mathbf{r}^{w}}{\partial s_{i}}, \quad i=1,2 \tag{2.15}
\end{equation*}
$$

where $\mathbf{r}^{w}$ is the global position vector of the contact point on the wheel, $\mathbf{r}^{f s}$ is the global position vector of the FS point of contact as defined by Eq. 2.2, and the vector $\mathbf{r}^{w f s}$ is defined as $\mathbf{r}^{w f s}=\mathbf{r}^{w}-\mathbf{r}^{f s}$. It is important to note that Eq. 2.14 is applied only at the position level, and therefore, it is not considered as a set of kinematic constraint equations that introduce constraint forces and must be satisfied at the velocity and acceleration levels. The solution of the nonlinear algebraic equations contained in the vector $\mathbf{E}$ is found using Newton-Raphson algorithm.

In case of elastic contact formulation, the normal contact force is determined by using a compliant force model. In this case, Hertzian contact is assumed in addition to the material damping. The normal contact force can be written as $F^{N}=-k_{H} \delta^{3 / 2}-c \dot{\delta}|\delta|$ where $F^{N}$ is the normal force, $k_{H}$ is the Hertzian constant (Johnson, 1985), $c$ is a damping constant, and $\delta$ is the wheel/FS penetration defined as $\delta=\mathbf{r}^{w f s} \cdot \mathbf{n}^{f s}$. The penetration is used to determine if contact occurs at the point found by the numerical solution of Eq. 2.14. In the case that the penetration is greater than or equal to zero, the contact forces are not calculated. In the case where the penetration has a value less than zero, the normal contact force and subsequently the lateral and longitudinal creep forces as well as the creep spin moment at this point are evaluated. The creep forces are calculated using Kalker's USETAB (Vollebregt, 2008) as a function of the longitudinal, lateral, and spin creepages. USETAB is a table based program that works under the
assumption that the contact area is in the form of a Hertzian ellipse. The longitudinal, lateral, and spin creepages are defined for the case of finite segment/wheel contact by

$$
\begin{equation*}
\zeta_{x}=\frac{\left(\mathbf{v}^{w}-\mathbf{v}^{f s}\right)^{T} \hat{\mathbf{t}}_{1}^{f s}}{V}, \quad \zeta_{y}=\frac{\left(\mathbf{v}^{w}-\mathbf{v}^{f s}\right)^{T} \hat{\mathbf{t}}_{2}^{f s}}{V}, \quad \varphi=\frac{\left(\boldsymbol{\omega}^{w}-\boldsymbol{\omega}^{f s}\right)^{T} \widehat{\mathbf{n}}^{f s}}{V} \tag{2.16}
\end{equation*}
$$

where $\mathbf{v}^{w}$ and $\mathbf{v}^{f s}$ are the global velocities of the points of contact on the wheel and finite segment or rail, respectively; $\boldsymbol{\omega}^{w}$ and $\boldsymbol{\omega}^{f s}$ are the global absolute angular velocities of the wheel and FS or rail at the point of contact, $\hat{\mathbf{t}}_{1}^{f s}$ and $\hat{\mathbf{t}}_{2}^{f s}$ are the unit tangent vectors at the points of contact on the FS or rail, $\widehat{\mathbf{n}}^{f s}$ is the unit normal vector at the point of contact, and $V$ is defined as the forward velocity of the center of the wheel.

In the case of fully constrained rail, the rail has zero velocity and as a result the FS translational and angular velocity terms are zero in Eq. 2.16. However, when the point of contact lies within a moveable finite segment, the velocities at the point of contact need not be zero and the rail translational and angular velocity terms will be defined by Eqs. 2.8 and 2.12, respectively. Knowing the point of contact and the resulting contact forces and moments, the vector of generalized forces for the finite segment in the dynamic equations of motion can be determined.

### 2.3 Finite Segment Equations of Motion

In this section, a brief explanation is given to the equations of motion and the implementation in computational MBS algorithms that are designed to solve system of differential and algebraic equations (DAEs). The differential equations arise from the equations of motion, while the algebraic equations arise from the kinematic constraints imposed on the motion of the system when the method of Lagrange multipliers is employed. The algebraic constraint equations at the
acceleration level can be written as $\mathbf{C}_{\mathbf{q}} \ddot{\mathbf{q}}=\mathbf{Q}_{c}$ (Roberson and Schwertassek, 1988; Shabana, 2005), where $\mathbf{C}_{\mathbf{q}}$ is the constraint Jacobian matrix obtained by differentiating the system constraint equations $\mathbf{C}$ with respect to the system coordinates $\mathbf{q}$, and $\mathbf{Q}_{c}$ is the vector of terms resulting from this differentiation that are quadratic in the velocities. The principle of virtual work in dynamics for the constrained system can be written as $\delta \mathbf{q}^{T}\left(\mathbf{M} \ddot{\mathbf{q}}+\mathbf{C}_{\mathbf{q}}^{T} \lambda-\mathbf{Q}_{e}\right)=\mathbf{0}$, where $\mathbf{M}$ is the system mass matrix, $\delta \mathbf{q}$ is the virtual change in the system coordinates, $\boldsymbol{\lambda}$ is the vector of Lagrange multipliers, and $\mathbf{Q}_{e}$ is the vector of generalized external forces. The normal contact and creep forces between the wheel and the finite segment are contained within $\mathbf{Q}_{e}$. Using these definitions, the equations of motion are given in the augmented form as (Shabana, 2005)

$$
\left[\begin{array}{cc}
\mathbf{M} & \mathbf{C}_{\mathbf{q}}^{T}  \tag{2.17}\\
\mathbf{C}_{\mathbf{q}} & \mathbf{0}
\end{array}\right]\left[\begin{array}{l}
\ddot{\mathbf{q}} \\
\lambda
\end{array}\right]=\left[\begin{array}{l}
\mathbf{Q}_{e} \\
\mathbf{Q}_{c}
\end{array}\right]
$$

Assuming the initial coordinates and velocities are known, this set of algebraic equations is solved for the accelerations and the vector of Lagrange multipliers. Once the accelerations are obtained, the independent accelerations can be identified and integrated forward in time to determine the independent coordinates and velocities at the next time step. The explicit AdamsBashforth predictor-corrector numerical integration scheme is employed to find the independent coordinates and velocities (Shampine and Gordon, 1975). The dependent coordinates and velocities are then determined using the constraint equations at the position and velocity levels. With all the coordinates and velocities determined, Eq. 2.17 is solved again and this process is repeated until the specified end time of the simulation is reached.


Figure 5 Relationship Between the Space Curve $s$ and its Projection $S$ (Shabana et al., 2008)

### 2.4 Curved Track

In order to solve the equations of motion presented in the preceding section, the initial coordinates and velocities of the FS bodies must be provided. The initial FS configuration must be accurately defined when these finite segments are used to model the movements of tangent and curved tracks. The algorithm used allows the finite segments to be located on a curved track section. In this case, the geometry of the curved track must be used in order to define the correct FS position and orientation. The position of the track nodes and the three Euler angles that define the orientation of the profile frame can also be used to define the FS position and orientation. In the case of curved track, the position of an arbitrary point, designated $\overline{\mathbf{u}}_{k}^{f s}$ in Section 2.1, can be written in terms of the rail longitudinal tangent as

$$
\overline{\mathbf{u}}_{k}^{f s}=\overline{\mathbf{u}}_{k-1}^{f s}+\int_{s_{0}}^{s}\left[\begin{array}{c}
\cos \psi \cos \theta  \tag{2.18}\\
\sin \psi \cos \theta \\
\sin \theta
\end{array}\right] d s
$$

The projection of the space curve arc length $s$ on the $X Y$ plane is referred to as $S$, as is shown in Fig. 5. Knowing that the horizontal curvature $C_{H}$ is equivalent to $1 / R_{H}$, where $R_{H}$ is the radius of
curvature, the relationship $d \psi=d s / R_{H}=C_{H} d S$ can be obtained. Upon integration one has the following

$$
\begin{equation*}
\psi_{k}(S)=\psi_{k-1}+\int_{s_{k-1}}^{s_{k}} C_{H_{k}}(S) d S \tag{2.19}
\end{equation*}
$$

The relationship between the projected arc length and the arc length is defined as $d S=\cos \theta d s$, which is integrated to yield the following expression at node $k$

$$
\begin{equation*}
S_{k}=S_{k-1}+\int_{S_{k-1}}^{S_{k}} \cos \theta_{k} d s \tag{2.20}
\end{equation*}
$$

Substituting Eq. 2.20 into Eq. 2.18 yields

$$
\overline{\mathbf{u}}_{k}^{f s}=\left[\begin{array}{c}
x_{k-1}^{f s}  \tag{2.21}\\
y_{k-1}^{f s} \\
z_{k-1}^{f s}
\end{array}\right]+\left[\begin{array}{c}
\int_{S_{k-1}}^{S_{k}} \cos \psi_{k}(S) d S \\
\int_{S_{k-1}}^{s_{k}} \sin \psi_{k}(S) d S \\
\int_{S_{k-1}}^{s_{k}} \sin \theta_{k}(s) d s
\end{array}\right]
$$

Equation 2.21 implies that in order to determine the position coordinates at each node, the orientation, arc length, and projected arc length at each node must also be calculated. Equation 2.21 can be used to determine the position of the nodes that define the FS longitudinal boundaries. The position vector in Eq. 2.21 can be evaluated if proper definitions of the angles $\psi$ and $\theta$ in terms of $S$ and $s$, respectively, are obtained.

One of the inputs to the track pre-processor is the horizontal curvature $C_{H}$. Within the track segments, the horizontal curvature is assumed to vary linearly. Knowing the horizontal curvature at the two ends of a track segment, the horizontal curvature can be written as a function of $S$ as

$$
\begin{equation*}
C_{H}(S)=\frac{C_{1}\left(S-S_{0}\right)-C_{0}\left(S-S_{1}\right)}{S_{1}-S_{0}} \tag{2.22}
\end{equation*}
$$

where $C_{0}, C_{1}, S_{0}$ and $S_{1}$ are the values of the curvature and the projected arc length at the two end points of the segment, respectively. Using this expression for $C_{H}$, the angle $\psi$ can be defined as a function of the projected arc length $S$ as

$$
\begin{equation*}
\psi=\psi_{0}+\frac{1}{S_{1}-S_{0}}\left[\frac{C_{1}}{2}\left(S-S_{0}\right)^{2}-\frac{C_{0}}{2}\left(S-S_{1}\right)^{2}\right]+\frac{C_{0}}{2}\left(S_{1}-S_{0}\right) \tag{2.23}
\end{equation*}
$$

Note that the first two elements of the vector $\overline{\mathbf{u}}_{k}^{f s}$ in Eq. 2.21 are expressed in terms of the angle $\psi$. Using the relationship $d \theta=C_{v} d s$ and assuming that the vertical curvature $C_{v}$ varies linearly with respect to $\theta$, one obtains

$$
\begin{equation*}
\theta(s)=\theta_{0}+C_{v}\left(s-s_{0}\right) \tag{2.24}
\end{equation*}
$$

A similar linearity assumption can be used for the bank angle $\varphi$, leading to

$$
\begin{equation*}
\varphi=\frac{\varphi_{1}\left(S-S_{0}\right)-\varphi_{0}\left(S-S_{1}\right)}{S_{1}-S_{0}} \tag{2.25}
\end{equation*}
$$

The projected arc length $S$ can also be expressed in terms of the actual arc length $s$ as follows (Shabana et al., 2008)

$$
S(s)=\left\{\begin{array}{cc}
S_{0}+\frac{1}{C_{v}} \sin (\theta(s))-\frac{1}{C_{v}} \sin \left(\theta_{0}\right) & \text { if } C_{v} \neq 0  \tag{2.26}\\
S_{0}+\left(s-s_{0}\right) \cos \theta & \text { if } C_{v}=0
\end{array}\right.
$$

Knowing the nodal positions and Euler angles at the nodes that define the boundaries of the FS, simple rigid body kinematic equations can be used to define the position of the FS center of mass of mass as well as the FS orientation.


Figure 6 Components of the Truck

### 2.5 Numerical Examples

In this section, two examples are provided to demonstrate the capabilities and limitations of the finite segment approach. The first example demonstrates the ability to switch the contact between a FS and rail online. The second model will demonstrate the ability of the FS model to transfer contact from one moving segment to another. All simulation results presented in this section were obtained using the general purpose MBS program SAMS/2000 (Shabana, 2010).

### 2.5.1 Curved Track

In this example, a single vehicle model based on a generic material handling car that uses a typical GSI truck shown in Fig. 6 is used. The geometric and dynamic properties of the model are given in Table 1, while the properties of the connections between different bodies are given in Tables 2 and 3. The track used in this example consists of the following sections: 60.96 m (200 $\mathrm{ft})$ tangent section; $152.4 \mathrm{~m}(500 \mathrm{ft})$ spiral section; $91.44 \mathrm{~m}(300 \mathrm{ft})$ 3-degree with 4-inch

Table 1. Vehicle Body Properties

| Body | Mass $(\mathbf{K g})$ | $I_{x x}\left(\mathbf{k g} \cdot \mathbf{m}^{\mathbf{2}}\right)$ | $I_{y y}\left(\mathbf{k g} \cdot \mathbf{m}^{\mathbf{2}}\right)$ | $I_{z z}\left(\mathbf{k g} \cdot \mathbf{m}^{\mathbf{2}}\right)$ |
| :---: | :---: | :---: | :---: | :---: |
| Wheelset | 2091 | 1098 | 191 | 1098 |
| Equalizer Beam | 469 | 6.46 | 255 | 252 |
| Truck Frame | 3214 | 1030 | 1054 | 2003 |
| Bolster | 1107 | 498 | 20.4 | 459 |
| Car Body | 24,170 | 30,000 | 687,231 | 687,231 |

Table 2. Individual Truck Connection Properties

| Description | Connected Bodies | Stiffness <br> $(\mathbf{N} / \mathbf{m})$ | Damping <br> $(\mathbf{N} \cdot \mathbf{s} / \mathbf{m})$ |
| :---: | :---: | :---: | :---: |
| Primary Suspension Vertical | Equalizer Beam \& Truck Frame | $1.68 \cdot 10^{6}$ | 0 |
| Primary Suspension Lateral | Equalizer Beam \& Truck Frame | $1.10 \cdot 10^{6}$ | 0 |
| Primary Suspension Longitudinal | Equalizer Beam \& Truck Frame | $1.10 \cdot 10^{6}$ | 0 |
| Bearing Friction Vertical/Lateral | Bearing Block \& Truck Frame | $8.75 \cdot 10^{6}$ | $8.75 \cdot 10^{3}$ |

Table 3. Vehicle Connection Properties

| Description | Connected Bodies | Stiffness <br> $(\mathbf{N} / \mathbf{m})$ | Damping <br> $(\mathbf{N} \cdot \mathbf{s} / \mathbf{m})$ |
| :---: | :---: | :---: | :---: |
| Secondary Suspension Vertical | Car Body \& Bolster | $1.97 \cdot 10^{6}$ | 0 |
| Secondary Suspension Lateral | Car Body \& Bolster | $1.28 \cdot 10^{6}$ | 0 |
| Secondary Suspension Longitudinal | Car Body \& Bolster | $1.10 \cdot 10^{6}$ | 0 |
| Lateral Body Damper | Car Body \& Bolster | 0 | $5.96 \cdot 10^{5}$ |
| Longitudinal Body Anchor | Car Body \& Bolster | $6.17 \cdot 10^{6}$ | $4.83 \cdot 10^{3}$ |
| Vertical Snubber | Car Body \& Bolster | 0 | Non- Linear |

super-elevation left hand curve; $304.8 \mathrm{~m}(1000 \mathrm{ft})$ spiral section; $91.44 \mathrm{~m}(300 \mathrm{ft})$ 3-degree with 4 inch super-elevation right hand curve; $152.4 \mathrm{~m}(500 \mathrm{ft})$ spiral section; and $121.92 \mathrm{~m}(400 \mathrm{ft})$ tangent section. The right rail is modeled as two bodies (finite segments). The first body is used to represent the right rail from the beginning to $199.79 \mathrm{~m}(655.48 \mathrm{ft})$, before the end of the first spiral section. The second segment is used from the end of the first segment to the end of the right rail as shown in Fig. 7. Both segments are assumed to be fully constrained. This model is presented only to demonstrate that the implementation allows for a smooth change of the contact between two segments without adversely affecting the quality of the results. As expected, Fig. 8 shows the predicted vertical forces of the right wheel of the axle of the lead truck. The predicted


Figure 7 Arrangement of the Segments of the Right Rail (----_ First Segment, $\qquad$ Second Segment)


Figure 8 Vertical Contact Force of the Right Wheel of the Lead Axle of the Lead Truck



Figure 9 Local Longitudinal Position of the Tread Contact of the Right Wheel of the Lead Axle of the Lead Truck
(-_-_-_ Tread Contact - Regular, ___Tread Contact - FS)
tread and flange contact forces, using the finite segment model, agree with the predicted forces using the right rail as a single body (referred to in the figure caption as regular rail). The results presented in Fig. 8 show that the method proposed in this thesis to switch the contact between bodies on-line does not negatively impact the quality of the results. Figure 9 shows a comparison between the predicted local longitudinal position of the tread contact using the regular rail and the rail composed of two segments. Clearly, when the right rail is modeled by using two segments, the local position of the contact starts from zero when the contact switches to the second segment, which is relative to the second segment reference point used in this model. Here, the reference point for each segment corresponds to the location at the beginning of the segment. While this location does not correspond to the physical center of gravity, it was chosen to more easily identify the point at which the transition between the rail and the finite segments

Table 4. Stiffness and Damping Values for the FS

| Translational |  | Rotational |  |
| :---: | :---: | :---: | :---: |
| $k_{x}\left(\mathrm{kN} / \mathrm{m} \cdot 10^{5}\right)$ | 2.71 | $k_{\theta}(\mathrm{kN} \cdot \mathrm{m} / \mathrm{rad})$ | 781.16 |
| $k_{y}\left(\mathrm{kN} / \mathrm{m} \cdot 10^{5}\right)$ | 1.04 | $k_{\varphi}(\mathrm{kN} \cdot \mathrm{m} / \mathrm{rad})$ | 656.10 |
| $k_{z}\left(\mathrm{kN} / \mathrm{m} \cdot 10^{5}\right)$ | 1.04 | $k_{\psi}(\mathrm{kN} \cdot \mathrm{m} / \mathrm{rad})$ | 103.03 |
| $c_{x}(\mathrm{kN} \cdot \mathrm{s} / \mathrm{m})$ | 91.29 | $c_{\theta}(\mathrm{kN} \cdot \mathrm{s} \mathrm{m} / \mathrm{rad})$ | 4.83 |
| $c_{y}(\mathrm{kN} \cdot \mathrm{s} / \mathrm{m})$ | 56.69 | $c_{\varphi}(\mathrm{kN} \cdot \mathrm{s} \mathrm{m} / \mathrm{rad})$ | 4.49 |
| $c_{z}(\mathrm{kN} \cdot \mathrm{s} / \mathrm{m})$ | 56.69 | $c_{\psi}(\mathrm{kN} \cdot \mathrm{s} \mathrm{m} / \mathrm{rad})$ | 1.78 |



Figure 10 Arrangement of the Right Rail for Single Wheelset Example
occur. As the finite segments in this example are fully constrained, the location of the reference point has no impact on the results. Meanwhile, if the regular model is used, the local longitudinal tread contact position continues to increase as the wheel travels along the rail.

### 2.5.2 Transition Between Moving Segments

In this second example, the transition of the location of the contact between two moving finite segments is demonstrated. A model of a single wheelset travelling on a tangent track with a speed equal to $4.4704 \mathrm{~m} / \mathrm{s}(10 \mathrm{mph})$ is used. The right rail is assumed to consist of three segments. The first segment of the right rail is fully constrained. The second and third segments are 5 m long. The second segment is connected to the first segment by a revolute joint that
allows a relative pitch rotation which is restricted using a torsional spring. Similarly, the second and third segments are connected by a revolute joint that allows a relative pitch rotation restricted by a torsional spring as shown in Fig. 10. The two ends of each finite segment are connected to the ground by vertical spring-damper element as shown in Fig. 10. The characteristics of the spring and damper coefficients are given in Table 4. The wheelset mass is 1568 kg and the mass moments of inertia are 656,168 and $656 \mathrm{~kg} \mathrm{~m}^{2}$ about its $X, Y$ and $Z$ axes, respectively. The wheelset has a static vertical applied load equal to 88964.43 N to simulate the vehicle weight. The mass of the second and third segments of the right rail is 341.78 kg , and their mass moments of inertia are $1.84,713.63$, and $712.29 \mathrm{~kg} \mathrm{~m}^{2}$ about the $X, Y$ and $Z$ axes, respectively.

Figure 11 shows the predicted vertical contact force of the right and left wheels. Clearly, as the wheelset entered the second segment of the right rail, a change in the vertical contact force is predicted. This is caused by the change in the characteristics of the rail support from a fully constrained segment to a segment that is supported by a spring-damper element. As the wheelset moves forward, the segments begin to respond to the applied vertical load. The static applied load led to a change in the vertical location and pitch rotation of the second and third segments as shown in Figs. 12 and 13. As the wheelset entered the third segment, it encountered a sudden change in the pitch orientation of the rail space curve. This caused the sudden change in the predicted vertical force shown in Fig. 11. The change in the local longitudinal contact location with respect to the segments is shown in Fig. 14.


Figure 12 Vertical Displacement of the Second and Third Right Rail Segments (___ Right Rail - Second Segment, $\qquad$ Right Rail - Third Segment)


Figure 13 Pitch Rotation of the Second and Third Right Rail Segments (___ Right Rail - Second Segment, _ _ _ _ Right Rail - Third Segment)


Figure 14 Local Longitudinal Contact Position of the Right Wheel

### 2.6 Concluding Remarks

This chapter proposes a computational method based on the FSM that can be used to model the track structure and rail movements in the MBS simulation of railroad vehicle dynamics. In order to avoid distortion of the geometry during the rail movements, ANCF finite elements are used to interpolate the rail space curve geometric properties. It is shown that a systematic method can be developed in which this FS model can be created with limited user input. It is clear from the presented numerical results that this approach is capable of modeling some simulation scenarios. The FS approach outlined in this chapter may be used to study the phenomena of gage widening and rail rollover. The FS approach can also be applied to more complex models pertaining to the study of broken rails, analysis of the effect of missing ties, and excitation due to bridge or seismic motion, provided the appropriate constraints and external forces are applied and an adequate number of finite segments are used.

Using the rigid body assumption to simulate the finite segment can introduce some discontinuities in the rail surface that cause spikes in the predicted contact force. This problem can be avoided if the finite element method is used in place of the FSM, as will be demonstrated in the following chapter. A second limitation of the proposed FS method is the failure to capture the change in the longitudinal rail arc length as the result of the FS longitudinal motion. A third limitation of the FS method is the need to use several small finite segments in order to capture the rail deformation. As will be shown in the following chapter, this can lead to a significant increase in the problem dimension.

## CHAPTER 3

## FSM AND FEM MODELING OF RAIL FLEXIBILITY

This chapter will present three methods suited for the study of flexible track models while conclusions about their implementations and features are made. The first method is based on the floating frame of reference (FFR) formulation which allows for the use of a detailed finite element (FE) mesh with the component mode synthesis technique in order to obtain a reduced order model. In the second method, which was introduced in the preceding chapter, the flexible body is modeled as a finite number of rigid elements that are connected by springs and dampers. This method, called the finite segment method (FSM) or rigid finite element method, requires the use of rigid multibody systems (MBS) formulations only. In the third method, the FFR formulation is used to obtain a model that is equivalent to the FSM model by assuming that the rail segments are very stiff; thereby allowing the exclusion of the high frequency modes associated with the rail deformations. This FFR/FSM model demonstrates that some rail movement scenarios such as gage widening can be captured using the finite element FFR formulation. The three procedures FFR, FSM, and FFR/FSM will be compared in order to establish differences among them and analyze the specific application of the FSM geometry developed in the previous chapter to modeling track flexibility. Convergence of the methods is analyzed. The three methods proposed in this chapter for modeling the movement of threedimensional tracks are used in conjunction with a three-dimensional elastic wheel/rail contact formulation that predicts contact points on-line and allows for updating the creepages to account for the rail deformations. Several conclusions will be drawn in view of the results obtained in this chapter.


Figure 15 Finite Segment Connectivity

### 3.1 Finite Segment Connectivity

The FSM has been used in the literature to model flexible bodies by assuming that the flexible body consists of a collection of rigid bodies, with each referred to as a finite segment (FS), connected by spring-damper elements that define the bodies' elastic and visco-elastic characteristics. The selection of the spring and damper coefficients is a major issue when the FSM is utilized to model flexible bodies. There exist in the literature several methods for defining the FS spring stiffness coefficients (Huston, 1981; Wittbrodt et al., 2006). Some methods consider coupling between stresses leading to a non-diagonal stiffness matrix, while others do not consider such a coupling leading to a diagonal stiffness matrix. The latter description will be used in this chapter as a special case of the bushing element implemented in many general purpose MBS codes. A bushing element is assumed to connect two bodies using $6 \times 6$ stiffness and damping matrices; this allows for adding stiffness and damping forces associated with the 6 degrees of freedom that represent the relative motion between the two bodies connected by the bushing element. In the special case of diagonal stiffness and damping matrices, a single bushing element may be used in place of 3 linear and 3 torsional springdamper elements.

In the FSM, the relative rotational and translational displacements at the ends of the finite segments define the deformation of the continuum. The stiffness of the continuum is described in this chapter by a diagonal stiffness matrix $\mathbf{k}$ that defines the connection forces between the finite segments as shown in Fig. 15. The elements of the diagonal stiffness matrix used in this chapter are defined as

$$
\left.\begin{array}{lll}
k_{x}=\frac{E A}{\Delta l}, & k_{y}=\frac{G A}{\Delta l}, & k_{z}=\frac{G A}{\Delta l}  \tag{3.1}\\
k_{\varphi}=\frac{G J}{\Delta l}, & k_{\theta}=\frac{E I_{y}}{\Delta l}, & k_{\psi}=\frac{E I_{z}}{\Delta l}
\end{array}\right\}
$$

where $E$ is the modulus of elasticity, $G$ is the shear modulus, $J$ is the polar moment of inertia, $I_{i}$ is the second moment of inertia along the axis $i=x, y, z, A$ is the cross-sectional area of the FS, $\Delta l$ is the length of the FS , and $\varphi, \theta$, and $\psi$ represent rotations about the $X, Y$, and $Z$ axes respectively. The stiffness matrix used in this chapter can also be modified to account more precisely for shear deformation (Hutchinson, 2001; MacNeal, 1978) and/or coupling between deformations.

A similar procedure can be used to define the coefficients of the damping matrix that specifies the visco-elastic characteristics of the bodies modeled with the finite segments. The diagonal damping matrix $\mathbf{c}$ can be defined using the following coefficients:

$$
\begin{equation*}
c_{i}=2 \xi \sqrt{k_{i} m}, \quad i=x, y, z ; \quad c_{j}=2 \xi \sqrt{k_{j} I_{j}}, \quad j=\varphi, \theta, \psi \tag{3.2}
\end{equation*}
$$

where $\xi$ is the damping factor, which is assumed in this chapter to be $3 \%, m$ is the part of the FS mass that corresponds to the node at which the finite segments are connected; and $I_{j}$ is the mass moment of inertia associated with a simple rotation described by angle $j$.

### 3.2 Floating Frame of Reference Formulation

The analysis presented in the preceding material clearly shows that the FSM suffers from three serious limitations, particularly when bodies with complex geometries are considered. First, there is no systematic way to choose the number and distribution of the rigid finite segments that are used to model the flexible body. Second, the determination of the stiffness and damping coefficients, number and points of application, and direction of the forces of the discrete springdamper elements can be a problem. Third, as the segments become small, the magnitude of the components of the stiffness matrix increase; this in turn may lead to a stiff system which can cause stability issues with explicit integration schemes such as the Adams-Bashforth approach (Shampine and Gordon, 1975). While the use of the finite element method (FEM) addresses these major issues, it is important when the FEM is used to correctly update the rail and track geometry as the result of the deformation.

A FEM for modeling railroad track flexibility has recently been developed and validated (Shabana et al., 2007; Rathod et al., 2009). The change in the geometry of the rails due to the elastic track deformation can be considered using the FFR formulation. This method accounts for the effect of the track flexibility with regard to the position of the contact points and creepages as well as the normal and creep forces. The points of contact between the wheels and rails are calculated on-line using a three-dimensional contact formulation that utilizes a two-dimensional parameterization of the wheel and rail surfaces. The FFR approach employed uses the component modes of the track to account for the deformation due to the dynamic loading conditions resulting from wheel/rail contact. The geometry of each rail in the undeformed or reference configuration is defined by a collection of three-dimensional absolute nodal coordinate formulation (ANCF) 3D beam elements. This reference geometry must be updated continuously
in order to account for the flexibility of the rail and allow for an accurate prediction of the contact point and the associated spatial derivatives required to evaluate the wheel/rail contact forces.

When the FFR approach is employed to model rail flexibility, the global position of any point on a given finite element $j$ of rail $i$ can be written as follows (Shabana et al, 2007)

$$
\begin{equation*}
\mathbf{r}^{i j}=\mathbf{R}^{i}+\mathbf{A}^{i}\left(\overline{\mathbf{u}}_{0}^{i j}+\overline{\mathbf{u}}_{f}^{i j}\right), \quad j=1,2, \ldots, n_{e} \tag{3.3}
\end{equation*}
$$

where $\mathbf{R}^{i}$ is the global position vector which defines the location of the coordinate system of rail $i, \mathbf{A}^{i}$ is a transformation matrix which defines the orientation of the rail coordinate system with respect to the global coordinate system, $n_{e}$ is number of finite elements used to describe rail $i$, $\overline{\mathbf{u}}_{0}^{i j}$ is a local position vector of the point in the undeformed configuration, and $\overline{\mathbf{u}}_{f}^{i j}$ defines the elastic deformation of the point in the current or deformed configuration. The aforementioned vectors are defined as (Shabana et al, 2007)

$$
\begin{equation*}
\overline{\mathbf{u}}_{0}^{i j}=\mathbf{S}_{b}^{i j} \mathbf{B}_{c}^{i j} \mathbf{e}_{b o}^{i}, \quad \overline{\mathbf{u}}_{f}^{i j}=\mathbf{S}_{b}^{i j} \mathbf{B}_{c}^{i j} \mathbf{B}_{r}^{i} \mathbf{e}_{f}^{i} \tag{3.4}
\end{equation*}
$$

In these equations, $\mathbf{S}_{b}^{i j}$ is the matrix of shape functions which define the interpolation within element $j, \mathbf{B}_{c}^{i j}$ is a binary matrix which is used to describe the element connectivity conditions, $\mathbf{e}_{b o}^{i}$ is the vector of nodal coordinates which defines the shape of the undeformed configuration of rail $i, \mathbf{B}_{r}^{i}$ is a binary matrix which is used to enforce a set of reference conditions which allows for a unique displacement field to be defined, and $\mathbf{e}_{f}^{i}$ is the vector of nodal coordinates which defines the elastic deformation of rail $i$. This vector contains linearization of the angles that describe finite element nodal rotations. The track structural flexibility may be defined in terms of a selected set of the component modes of vibration. To this end, the nodal coordinates which define the elastic deformation may be written in terms of the modal coordinates corresponding to
the selected modes of vibration using the following relationship $\mathbf{e}_{f}^{i}=\mathbf{B}_{m}^{i} \mathbf{q}_{f}^{i}$, where $\mathbf{B}_{m}^{i}$ is a matrix in which each column defines one of the selected modes of vibration, and $\mathbf{q}_{f}^{i}$ is the vector of modal coordinates which define the elastic deformation. Substituting this relationship into Eq. 3.4 allows for the modal coordinates, rather than the nodal coordinates, to define the elastic deformation vector as:

$$
\begin{equation*}
\overline{\mathbf{u}}_{f}^{i j}=\mathbf{S}_{b}^{i j} \mathbf{B}_{c}^{i j} \mathbf{B}_{r}^{i} \mathbf{B}_{m}^{i} \mathbf{q}_{f}^{i} \tag{3.5}
\end{equation*}
$$

This relationship, along with Eqs. 3.4 and 3.5, allows for the global position of any point on a finite element to be written explicitly in terms of the modal, rather than nodal, coordinates. The modal coordinates are used as the system generalized coordinates thus greatly reducing the number of system degrees of freedom. Knowing these modal coordinates from the solution of the system nonlinear dynamic equations of motion, the deformation of the track geometry nodes can be determined. These deformations can be used to update the position coordinates and rotations at the track geometry nodes. The updated rotations are used to define the ANCF gradient vectors. The updated nodal position and gradient coordinates are used to define the position field required for the finite element ANCF interpolation of the rail space curves.

As previously mentioned, a systematic procedure can be used to update the position, tangent, and normal vectors as well as their derivatives once the deformations are determined using the FFR formulation. As an acceptable approximation, the shape function $\mathbf{S}_{b}^{r j}$ associated with element $j$ of a rail $r$ can be written in terms of the two surface parameters $s_{1}^{r}$ and $s_{2}^{r}$ as $\mathbf{S}_{b}^{r j}=\mathbf{S}_{b}^{r j}\left(s_{1}^{r j}, s_{2}^{r}, f\left(s_{2}^{r}\right)\right)$. In this equation, $s_{1}^{r j}$ is a parameter that defines the longitudinal distance of the contact point from the origin of the coordinate system of the finite element $j$. It is further assumed that $\partial \mathbf{S}_{b}^{r j} / \partial s_{1}^{r}=\partial \mathbf{S}_{b}^{r j} / \partial s_{1}^{r j}$. The third argument $f\left(s_{2}^{r}\right)$ of the shape function
$\mathbf{S}_{b}^{r j}$ can be determined using a cubic interpolation routine once the surface parameter $s_{2}^{r}$ that defines the location of the contact point on the profile is determined. The deformation predicted using the FFR formulation can be used to define the two tangent vectors as

$$
\begin{equation*}
\mathbf{t}_{1}^{r j}=\frac{\partial \mathbf{r}^{r j}}{\partial s_{1}^{r}}=\mathbf{A}^{r}\left(\overline{\mathbf{t}}_{1 o}^{r j}+\frac{\partial \overline{\mathbf{u}}_{f}^{r j}}{\partial s_{1}^{r}}\right), \quad \mathbf{t}_{2}^{r j}=\frac{\partial \mathbf{r}^{r j}}{\partial s_{2}^{r}}=\mathbf{A}^{r}\left(\overline{\mathbf{t}}_{2 o}^{r j}+\frac{\partial \overline{\mathbf{u}}_{f}^{r j}}{\partial s_{2}^{r}}\right) \tag{3.6}
\end{equation*}
$$

In this equation, $\overline{\mathbf{t}}_{1 o}^{r j}$ and $\overline{\mathbf{t}}_{2 o}^{r j}$ are the two tangent vectors in the undeformed state defined in the rail body coordinate system. Note also that $\partial \overline{\mathbf{u}}_{f}^{r j} / \partial s_{2}^{r}$ must include the effect of the derivative of the function $f\left(s_{2}^{r}\right)$ with respect to $s_{2}^{r}$. The preceding equation can be used to define the normal of the rail surface at the contact point as $\mathbf{n}^{r j}=\mathbf{t}_{1}^{r j} \times \mathbf{t}_{2}^{r j}$. This normal accounts for the effect of the deformation since this effect is included when the tangent vectors are evaluated. Higher order derivatives of the tangent and normal vectors can then be systematically evaluated and used in the wheel/rail contact formulations as previously described in the literature (Shabana et al., 2008).

The implementation of this geometry updating procedure requires the use of two data sets as previously described in the literature (Shabana et al., 2007). One set includes a description of the geometry in the undeformed state in terms of segments and nodes, while the other set has a description of the finite element model used in the FFR formulation. The rail segment used to define the geometry is referred to as the geometry segment, and each node used to define the geometry is referred to as the geometry node. The nodes used in the finite element FFR formulation are called the finite element nodes. In the contact formulations used in this thesis, the locations of the wheel/rail contact points are predicted on-line. These contact points can be arbitrary points on the rail surface. The location of the contact point on the rail surface can be defined since all contact formulations can be used to solve for $s_{1}^{r}$ and $s_{2}^{r}$. Using the location of
the contact point, one can identify the finite element $j$ to which the contact point corresponds. Knowing from the finite element data set the locations of the finite element nodes, one can determine the parameter $s_{1}^{r j}$. Having identified the finite element to which the contact point corresponds, the nodal deformation coordinates of the finite elements can be identified and used to evaluate the vector $\overline{\mathbf{u}}_{f}^{r j}$. One can differentiate $\overline{\mathbf{u}}_{f}^{r j}$ with respect to the surface parameters to determine the change in the tangent vectors due to the deformation. The tangent vectors can then be used to determine the normal vectors. Using the expressions for the tangent and normal vectors, higher order derivatives can be systematically evaluated.

### 3.3 FE/FS Approach

The FFR formulation described briefly in the preceding section can be used to develop a model that is equivalent to the FSM model previously discussed in this chapter. In this model, the rail sections are considered as rigid bodies by neglecting their modes of deformation. It has been demonstrated that the use of the FSM can lead to a good approximation of the dynamic behavior of a beam in some applications (Wang and Huston, 1994). Using this fact, the third FFR/FS approach is proposed in this chapter to model the track and rail displacements. In this approach, a FE mesh is developed for the track. This FE mesh is used to determine the deformation modes using a standard eigenvalue analysis. If sections or segments of the track are assumed to be rigid, the modes associated with the deformations of these sections are not included in the dynamic model; thereby allowing these sections to move as rigid bodies that are connected by discrete elastic elements. This FFR and component mode synthesis approach leads to a model similar to the FSM model previously discussed in this chapter. In developing this FSM model using the FFR formulation and component mode synthesis method, the mode shapes are extracted using a
model that employs diagonal stiffness and damping matrices based on the coefficients defined in Eqs. 3.1 and 3.2. In the actual implementation, the beam elements used to model the rigid segments are assumed to have very high modulus of elasticity. Using very high stiffness for these beam elements, the modes associated with their deformations correspond to very high frequencies. Such high frequency modes can be neglected, and therefore, the determination of these modes is not necessary, leading to a more efficient procedure for the solution of the eigenvalue problem.

The results obtained using this third approach (FFR/FS) will be compared with the results obtained using the FSM and the FFR formulation. The convergence of the modeling methods described in this chapter will be examined in order to shed light on the advantages and drawbacks of the FSM in the modeling of track flexibility.

### 3.4 Nonlinear MBS Constrained Equations of Motion

As previously mentioned, three different approaches are used in this chapter to account for the flexibility of the rail. If the FSM is used, the coordinates that define the flexible body deformation are the six degrees of freedom of each finite segment. The geometric updating of the flexible body is thus performed using the variables that represent the three translations and the three rotations of each rigid segment. In the FFR approach, modal coordinates are used to define the deformation of the body with respect to the body coordinate system. The rail geometry is updated on-line and the deformation and deformation rate of the flexible track are used in the definition of the wheel/rail contact and interaction forces. Therefore, in all the three methods that will be used in this chapter, wheel/rail contact parameters and forces are updated in order to account for the track movements.

In the case of the FSM, the differential equations of motion and the nonlinear algebraic constraint equations presented in Chapter 2 are solved to find the time domain response of the system. These equations can be written as (Shabana, 2005)

$$
\left[\begin{array}{cc}
\mathbf{M} & \mathbf{C}_{\mathbf{q}}^{T}  \tag{3.7}\\
\mathbf{C}_{\mathbf{q}} & \mathbf{0}
\end{array}\right]\left[\begin{array}{l}
\ddot{\mathbf{q}} \\
\lambda
\end{array}\right]=\left[\begin{array}{l}
\mathbf{Q}_{e} \\
\mathbf{Q}_{c}
\end{array}\right]
$$

where $\mathbf{M}$ is the system mass matrix, $\mathbf{C}_{\mathbf{q}}$ is the Jacobian constraint matrix obtained by differentiating the system constraint equations $(\mathbf{C}(\mathbf{q}, t)=\mathbf{0})$ with respect to the generalized coordinates, $\boldsymbol{\lambda}$ is the vector of Lagrange multipliers, $\ddot{\mathbf{q}}$ is the vector of the system absolute Cartesian accelerations, $\mathbf{Q}_{c}$ is the vector that contains quadratic velocity terms resulting from the differentiation of the constraint equations, and $\mathbf{Q}_{e}$ is the vector of generalized forces which include the forces of the bushing elements which connect the finite segments.

In the case of the FFR formulation and component mode synthesis method, the differential equations of motion are defined as (Shabana, 2005)

$$
\left[\begin{array}{cc}
\mathbf{m}_{r r} & \mathbf{m}_{r f}  \tag{3.8}\\
\mathbf{m}_{f r} & \mathbf{m}_{f f}
\end{array}\right]\left[\begin{array}{l}
\ddot{\mathbf{q}}_{r} \\
\ddot{\mathbf{q}}_{f}
\end{array}\right]=\left[\begin{array}{l}
\left(\mathbf{Q}_{e}\right)_{r} \\
\left(\mathbf{Q}_{e}\right)_{f}
\end{array}\right]+\left[\begin{array}{c}
\left(\mathbf{Q}_{v}\right)_{r} \\
\left(\mathbf{Q}_{v}\right)_{f}
\end{array}\right]-\left[\begin{array}{c}
\mathbf{C}_{\mathbf{q}_{r}}^{T} \\
\mathbf{C}_{\mathbf{q}_{f}}^{T}
\end{array}\right] \lambda-\left[\begin{array}{c}
\mathbf{0} \\
\mathbf{K}_{f f} \mathbf{q}_{f}
\end{array}\right]
$$

where $\mathbf{q}=\left[\begin{array}{ll}\mathbf{q}_{r}^{T} & \mathbf{q}_{f}^{T}\end{array}\right]^{T}$ is the vector of system generalized coordinates, $\mathbf{q}_{r}$ is the vector coordinates which define the rigid body motion of the track, $\mathbf{q}_{f}$ is the vector of modal coordinates which describes the elastic deformation of the track, $\mathbf{m}_{r r}$ is the inertia matrix related to the rigid body motion of the rails, $\mathbf{m}_{r f}$ and $\mathbf{m}_{f r}$ are the inertia matrices which define the coupling between the rigid and flexible motion of the body, $\mathbf{m}_{f f}$ is the inertia matrix related to the elastic deformation of the body, $\mathbf{C}_{\mathbf{q}_{r}}$ and $\mathbf{C}_{\mathbf{q}_{f}}$ are the matrices which define the Jacobian of the constraint equations with respect to the rigid and elastic coordinates, $\boldsymbol{\lambda}$ is the vector of Lagrange multipliers associated with the imposed constraints, $\left(\mathbf{Q}_{e}\right)_{r}$ and $\left(\mathbf{Q}_{e}\right)_{f}$ are the vectors of
generalized applied forces, $\left(\mathbf{Q}_{v}\right)_{r}$ and $\left(\mathbf{Q}_{v}\right)_{f}$ are the vectors which contain the contributions of the Coriolis and gyroscopic inertia forces, and $\mathbf{K}_{f f}$ is the stiffness matrix which defines the elasticity of the track.

### 3.5 Wheel/Rail Elastic Contact Formulation

In this chapter the three-dimensional elastic contact formulation that allows for wheel/rail penetration and separation, which was presented in Chapter 2, is reviewed for the case of wheel/rail, wheel/FS, and wheel/FFR rail contact. This formulation, called ECF-A (Shabana et al., 2008), does not reduce the number of degrees of freedom between a wheel/rail pair. The following four non-linear algebraic equations are solved on-line in order to determine the surface parameters which define the location of the contact point between a wheel/rail pair

$$
\mathbf{E}(\mathbf{s})=\left[\begin{array}{llll}
\mathbf{t}_{1}^{r} \cdot \mathbf{r}^{w r} & \mathbf{t}_{2}^{r} \cdot \mathbf{r}^{w r} & \mathbf{t}_{1}^{w} \cdot \mathbf{n}^{r} & \mathbf{t}_{2}^{w} \cdot \mathbf{n}^{r} \tag{3.9}
\end{array}\right]^{T}=\mathbf{0}
$$

where the subscript denotes to the surface parameter, the superscript $w$ denotes the wheel body, the superscript $r$ denotes the wheel rail body, while $\mathbf{t}$ and $\mathbf{n}$ are tangent and normal vectors to the surfaces at the contact point, $\mathbf{r}^{w r}$ is the vector that defines the relative position of the contact point on the wheel with respect to the point on the rail surface, and $\mathbf{s}$ is the vector of surface parameters defined as $\mathbf{s}=\left[\begin{array}{llll}s_{1}^{w} & s_{2}^{w} & s_{1}^{r} & s_{2}^{r}\end{array}\right]^{T}$. Following the solution for the location of the contact point, the normal contact force and normalized creepage velocities can be calculated. The normal contact force defined as $F^{N}$ may be calculated using the equation $F^{N}=$ $-k_{H} \delta^{3 / 2}-c \dot{\delta}|\delta|$, where $k_{H}$ is the Hertzian constant (Johnson, 1985), $c$ is the damping constant, and $\delta$ is the wheel/rail penetration defined as $\delta=\mathbf{r}^{w r} \cdot \mathbf{n}^{r}$, and $\dot{\delta}$ is the first time derivative of the wheel/rail penetration. The dimensions of the Hertzian contact ellipse may be computed once
the value of the penetration is known. The Hertzian contact ellipse is then used to compute the tangential creep forces and creep spin moment via Kalker's non-linear creep theory.

In the case where the rail is considered flexible, the effect of the elastic deformation must be included in the computation of the contact location and associated forces. To this end, the appropriate rail finite element in which the contact point resides must first be found. The appropriate element is found as a function of the longitudinal rail surface parameter. With the correct element found, the contact force is applied to the flexible track model and the rail geometry is updated to account for the predicted elastic deformation. The deformed geometry of the rail is used in the iterative process employed to determine the location and velocity of the contact point on the rail.

### 3.5.1 Creepage Definition

After determining the position and velocity of the contact points on the wheel and rail, the tangential and spin creepages are computed as follows (Shabana et al., 2008)

$$
\begin{equation*}
\zeta_{x}=\frac{\left(\mathbf{v}^{w}-\mathbf{v}^{r}\right)^{T} \hat{\mathbf{t}}_{1}^{r}}{V}, \quad \zeta_{y}=\frac{\left(\mathbf{v}^{w}-\mathbf{v}^{r}\right)^{T} \hat{\mathbf{t}}_{2}^{r}}{V}, \quad \varphi=\frac{\left(\boldsymbol{\omega}^{w}-\boldsymbol{\omega}^{r}\right)^{T} \widehat{\mathbf{n}}^{r}}{V} \tag{3.10}
\end{equation*}
$$

In these equations, $\mathbf{v}^{\boldsymbol{w}}$ and $\mathbf{v}^{r}$ are absolute velocity vectors of the wheel and rail, $\boldsymbol{\omega}^{\boldsymbol{w}}$ and $\boldsymbol{\omega}^{r}$ are absolute angular velocity vectors of the wheel and rail, $\hat{\mathbf{t}}_{1}^{r}$ and $\hat{\mathbf{t}}_{2}^{r}$ are the unit vectors which define the directions of the longitudinal and lateral tangents of the rail surface at the location of the contact point, $\widehat{\mathbf{n}}^{r}$ is the unit vector which defines the direction of the normal of the rail surface at the location of the contact point, and $V$ is defined as the forward velocity of the wheelset centroid. Following this, the longitudinal and lateral creep forces as well as the creep spin moment are computed using Kalker's non-linear creep theory.

### 3.5.2 Wheel/FS Contact

In the case of contact with a FS, the body coordinates and velocities of the FS can be used to determine the coordinates and velocities of the contact point on the rail surface via Eqs. 2.2 and 2.8. The FS orientation coordinates can also be used to determine the FS angular velocity via Eq. 2.12. Therefore, the absolute velocity of the contact point and the absolute angular velocity of the rail that enter into the formulation of the creepage expressions in Eq. 3.10 can easily be computed. The flexible links between finite segments allow for changes in the relative displacement and orientation along the three axes. Since each FS possesses its own dynamical state, the rail velocity at the interface between segments can experience a sudden change which can lead to some excitations in the predicted contact forces.

### 3.5.3 Wheel/FFR Rail Contact

In the case of contact when the FFR formulation and component mode synthesis are used to model the rail, the effect of the deformation of the rail on the rail tangent and normal vector is accounted for at run-time by iteratively updating the deformed rail geometry. The absolute velocity of the rail at the contact point, including the effect of the rail elastic deformation required in the computation of the creepages, is computed using the following equation

$$
\begin{equation*}
\mathbf{v}_{c}^{r}=\dot{\mathbf{R}}^{r}+\boldsymbol{\omega}_{c}^{r} \times \mathbf{A}^{r}\left(\overline{\mathbf{u}}_{c o}^{r}+\overline{\mathbf{u}}_{c f}^{r}\right)+\mathbf{A}^{r} \dot{\overline{\mathbf{u}}}_{c f}^{r} \tag{3.11}
\end{equation*}
$$

where $\overline{\mathbf{u}}_{c o}^{r}$ is defines the location of the contact point in the undeformed configuration of the rail and $\overline{\mathbf{u}}_{c f}^{r}$ is defines the elastic deformation of the rail at the contact point. This equation demonstrates that both the elastic deformation and the elastic deformation velocity vector contribute to the computation of the velocity of the rail at the contact point when the FFR formulation is employed. The definition of the rail angular velocity at the location of the contact


Figure 16 Suspended Wheelset Model
point must also be updated to account for the effect of the elastic deformation. The rail angular velocity vector at the contact point may be defined as:

$$
\begin{equation*}
\boldsymbol{\omega}_{c}^{r}=\boldsymbol{\omega}^{r}+\boldsymbol{\omega}_{c f}^{r} \tag{3.12}
\end{equation*}
$$

where $\boldsymbol{\omega}^{r}$ is defined as the angular velocity vector of the rail, and $\boldsymbol{\omega}_{c f}^{r}$ is defined as the angular velocity of the contact frame with respect to the rail due to the elastic deformation. In the case where rail is considered to be both rigid and stationary, the vectors $\dot{\mathbf{R}}^{r}$ and $\boldsymbol{\omega}^{r}$ in Eqs. 3.11 and 3.12 are identically zero.

### 3.6 Numerical Example

In this section, simple railroad models are used to compare between the three different modeling procedures described in this chapter (FS, FFR, and FFR/FS). This numerical comparative study will be used to illustrate the advantages and drawbacks of the FSM.

### 3.6.1 Wheelset Model

Figure 16 shows the suspended wheelset model, which is composed of a frame suspended over a wheelset by means of lateral and longitudinal spring-damper elements, that is used in this
example. The wheelset mass is assumed to be 1568 kg while the mass moments of inertia associated with roll, pitch, and yaw rotations are assumed to be 656,168 and $656 \mathrm{~kg} \cdot \mathrm{~m}^{2}$. The mass of the frame is assumed to be 3875 kg , its roll and pitch mass moments of inertia are assumed to be $1799 \mathrm{~kg} \cdot \mathrm{~m}^{2}$, and its yaw mass moment of inertia is assumed to be 2450 $\mathrm{kg} \cdot \mathrm{m}^{2}$. The stiffness and damping parameters of the spring-damper elements which connect the frame to the wheelset are assumed to be $c_{l 1}=c_{l 2}=1 \mathrm{kN} \cdot \mathrm{s} / \mathrm{m}, c_{t 1}=c_{t 2}=1 \mathrm{kN} \cdot \mathrm{s} / \mathrm{m}, k_{l 1}=$ $k_{l 2}=13.5 \mathrm{kN} / \mathrm{m}$, and $k_{t 1}=k_{t 2}=25 \mathrm{kN} / \mathrm{m}$. The frame is assumed to have a constant forward velocity of $15 \mathrm{~m} / \mathrm{s}$, while constraints are imposed to restrict lateral and vertical translations and all rotations. To account for the effect of the weight of the car body on the wheelset, a 38 kN load is applied to the wheelset in the vertical direction.

### 3.6.2 Track Model

In the track model used in this chapter, the left rail is assumed to be a rigid solid while the right rail is assumed to consist of three stretches. The stretch in the middle is assumed to be a flexible body modeled by a clamped-clamped beam, while the other two stretches are treated as rigid bodies. The first rigid stretch permits the algorithm to reach steady state values before the vehicle reaches the flexible section. Similarly, the clamped end conditions ensure a smooth transition between the rigid and the flexible parts. The dimensions and the coordinate system used are shown in Fig. 17 while the track model properties are presented in Table 5. In this model the effect of the rail weight (gravity force) is neglected. This track model has been implemented using the three procedures previously discussed in this thesis. The most relevant details of each procedure will be briefly discussed. In this section, the simulations have been carried out using the general purpose MBS program SAMS/2000 (Shabana, 2010) in which the FFR and FSM track models are implemented.

Table 5. Track properties

| Description | Value |
| :---: | :---: |
| Length of the track $(\mathrm{m})$ | 40 |
| Beginning of the flexible part $(\mathrm{m})$ | 25 |
| Length of the flexible part $(\mathrm{m})$ | 5 |
| Rail area $\left(\mathrm{cm}^{2}\right)$ | 88.18 |
| Rail second moment of inertia, $I_{y y}\left(\mathrm{~cm}^{4}\right)$ | 4050 |
| Rail second moment of inertia, $I_{z z}\left(\mathrm{~cm}^{4}\right)$ | 636 |
| Polar moment of inertia, $J\left(\mathrm{~cm}^{4}\right)$ | 4686 |
| Rail density $\left(\mathrm{kg} / \mathrm{m}^{3}\right)$ | 7840 |
| Rail modulus of elasticity $E\left(\mathrm{~N} / \mathrm{m}^{2}\right)$ | 210 |
| Rail modulus of rigidity $G\left(\mathrm{~N} / \mathrm{m}^{2}\right)$ | 81 |



Figure 17 Flexible Track Model

### 3.6.3 FS Track Model

When the FSM is used, the flexible rail is comprised of a number of rigid finite segments of equal length and cross-section dimensions. Therefore, the coefficients used for the stiffness, damping, and inertia properties are the same for each segment. The inertia properties of the finite segments must be calculated by treating each segment as an independent rigid body. At the ends of the section modeled by finite segments; the stiffness coefficients are doubled since these ends
are assumed to be connected to a ground section that is stationary throughout the simulation (Wang and Huston, 1994). Five FS models with different numbers of finite segments are considered in the numerical study presented in this section. These models are used to analyze the rate of convergence of the FSM in the simulations as well as for comparison with FFR formulation. Due to the fact that six degrees of freedom between finite segments are considered, a discontinuity in the position is introduced between segments where shear and axial deformation occur. This discontinuity makes it difficult to obtain good results from the contact formulation that requires a higher degree of continuity. For this reason, comparisons among models with different number of elements will be made, and a new model in which only bending and torsional deformation are considered will be presented.

### 3.6.4 FFR/FS Track Model

A FS model can be developed using the FFR formulation and component mode synthesis technique (see Section 3.3). In this model, each finite segment is described by a beam element that is allowed to experience rigid body displacement only. As mentioned in Section 3.3, one way to achieve this is to assume a very high modulus of elasticity for the beam elements such that all their modes of deformation are associated with very high frequencies, and therefore, there is no need to evaluate these modes. The stiffness properties of such a model are concentrated at the interface of the finite elements, as discussed in Section 3.1. The nodes of two elements at the interface are assumed to have a physical separation of 1 mm . This is required due to a geometric limitation introduced in the FFR rail model used in this work which does not allow two different nodes to occupy the same point in space. A modal analysis can be carried out to obtain the mode shapes and the frequencies of the track. However, these modes will possess specific features that characterize this new FS model: the deformation is concentrated at the interface between the

Table 6. Natural Frequencies of the Mode Shapes in FSM (in Hz)

| Mode No. |  | $\mathbf{1}$ |  | 2 |  | 3 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | Error (\%) | Freq. | Error (\%) | Freq. | Error (\%) |  |
| 5 finite <br> seg. | Lateral | 17.045 | 13.71 | 40.010 | 26.3 | 64.070 | 39.6 |
|  | Vertical. | 42.716 | 13.46 | 100.03 | 25.4 | 160.57 | 37.8 |
| $\mathbf{1 6}$ finite <br> seg. | Lateral | 19.451 | 1.54 | 52.523 | 3.29 | 100.28 | 5.47 |
|  | Vertical. | 48.63 | 1.49 | 129.98 | 3.12 | 245.41 | 5.01 |
| 25 finite <br> seg. | Lateral | 19.634 | 0.61 | 53.589 | 1.32 | 103.73 | 2.22 |
|  | Vertical. | 49.069 | 0.60 | 132.48 | 1.26 | 253.12 | 2.03 |
| 50 finite <br> seg. | Lateral | 19.729 | 0.12 | 54.165 | 0.26 | 105.61 | 0.45 |
|  | Vertical. | 49.304 | 0.12 | 133.83 | 0.25 | 257.29 | 0.41 |
| $\mathbf{1 0 0}$ finite <br> seg. | Lateral | 19.754 | - | 54.307 | - | 106.09 | - |
|  | Vertical. | 49.364 | - | 134.17 | - | 258.36 | - |

Table 7. Natural Frequencies of the Mode Shapes in FEM (in Hz)

| Mode No. |  | $\mathbf{1}$ |  | $\mathbf{2}$ |  | 3 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | Error (\%) | Freq. | Error (\%) | Freq. | Error (\%) |  |
| 5 finite <br> el. | Lateral | 19.744 | 0.11 | 53.721 | 1.21 | 99.383 | 6.54 |
|  | Vertical. | 49.385 | 0.14 | 132.78 | 1.42 | 241.77 | 7.04 |
| 16 finite <br> el. | Lateral | 19.766 | 0.00 | 54.376 | 0.01 | 106.30 | 0.04 |
|  | Vertical. | 49.452 | 0.00 | 134.66 | 0.03 | 259.79 | 0.11 |
| 25 finite <br> el. | Lateral | 19.766 | 0.00 | 54.379 | 0.00 | 106.33 | 0.01 |
|  | Vertical. | 49.454 | 0.00 | 134.69 | 0.01 | 259.98 | 0.04 |
| 50 finite <br> el. | Lateral | 19.766 | 0.00 | 54.380 | 0.00 | 106.34 | 0.00 |
|  | Vertical. | 49.454 | 0.00 | 134.70 | 0.00 | 260.06 | 0.01 |
| 100 <br> finite <br> el. | Lateral | 19.765 | - | 54.380 | - | 106.34 | - |
|  | Vertical. | 49.454 | - | 134.70 | - | 260.08 | - |

finite segments and the finite segments behave as rigid bodies. Fourteen modes of vibration have been selected to describe the deformation of the rail. The first 14 modes are bending modes, 5 of which describe deformation in the $z$ direction; while the rest define lateral deformation. The first three natural frequencies for the lateral and vertical mode shapes using the FFR/FS model are presented in Table 6. The reference used to measure the error in Tables 6 and 7 are the results given by the 100 -element models.

### 3.6.5 FFR Track Model

A FE track model, that includes the deformation modes due to the distributed elasticity and inertia, can also be developed and integrated with the FFR formulation. The results obtained using this model will be compared with the FSM and FFR/FS models. The left rail and the first and third stretches of the right rail in the FE model used in this section are assumed to be rigid. This can be conveniently achieved in the FE model by using one clamped-clamped finite element that has six coordinates at each of its nodes to represent each rigid section. The clamped end conditions will eliminate all the element degrees of freedom. The flexible stretch is modeled using a number of beam elements that account for shear deformation. The flexible stretch in this method is modeled with clamped-clamped boundary conditions at the rigid/flexible track interface to ensure a smooth transition between the rigid and flexible stretches. The results obtained by performing a modal analysis on these models are shown in Table 7. It can be observed that the rate of convergence of the FEM is much higher than the FS model. Two models that have different numbers of modes are used in this chapter. The first model contains 14 mode shapes, 5 of which are vertical and 9 of which are lateral. The second model contains 30 modes shapes, 12 of which are vertical modes and 16 of which are lateral. In all cases when using FFR approach, 100 elements are used and modal damping has been selected as $3 \%$ for all modes.

### 3.6.6 Simulation Results

In this section, unless otherwise mentioned, 50 finite elements or finite segments are used in the comparative study. When using the FFR approach, 14 mode shapes will be used to describe the rail deformation. The first lateral mode shape is shown in Fig. 18. A good agreement between the models of the FFR/FS and FEM is obtained in frequencies, shown in Tables 6 and 7, and deformation shown in Fig. 18. However it is possible to see the negative effect of concentrating
the deformation at the interface between rigid segments and the linear interpolation of the geometry within these rigid segments. The convergence of the FSM model can be examined using the results presented in Figs. 19 and 20. Figure 19 shows better convergence results for the vertical displacement as compared to the lateral displacement shown in Fig. 20. This is attributed to the fact that the static load on the wheelset is largely responsible for the vertical force at the point of contact while the lateral forces are more heavily influenced by dynamic forces. This fact can be seen in Fig. 20 where a large number of finite segments is required to achieve convergence. The three methods discussed in this chapter are used to obtain the results of Figs. 21 and 22. A good agreement in vertical and lateral displacements may be observed in these two figures although several factors must be taken into consideration:

1) Inertia, elastic, and damping properties are differently distributed in each one of the models.
2) The geometry of the rail deformed centerline is also different among the three methods. This fact alters the dynamics of the entire system since such difference in geometry can significantly influence the calculation of the contact forces.
3) Mode shapes are only used in the two methods that employ the FFR formulation.

Although one advantage of the FSM is its simplicity and straightforward implementation, in some cases, this method leads to poor descriptions of the deformation, thereby limiting its application. When 5 vertical modes of vibrations are used, the two procedures involving the FFR formulation offer a similar shape which, if Fig. 23 is observed in detail, is different from the shape obtained by applying the FSM. However, if one increases the number of vertical modes of vibration to 12 , which corresponds to the model with 30 mode shapes, the FFR solution converges to the FSM solution. The displacement of the middle point of the beam as function of


Figure 18 First Lateral Mode Shape Using FSM and FEM Models (__FS,_--_ FE)


Figure 19 Vertical Displacement at Centroid of Each FS at 27.5m
(.. $\qquad$ 5 FS $\square$ 50 FS)


Figure 20 Lateral Displacement at Centroid of Each FS at 27.5m (................. 5 FS , .-..._ 16 FS,


Figure 21 Vertical Displacements at Contact Location 27.5m (___FS, _-__ FFR (FS), _._-_ FFR (FE))


Figure 22 Lateral Displacements at Contact Location 27.5m (___FS, ____ FFR (FS), _._._ FFR (FE))


Figure 23 Vertical Displacements at Contact Location 25.5 m
$\qquad$ FS, _--- _ FFR (FS), FFR (FE, 14 Modes), ....... . FFR (FE, 30 Modes))


Figure 24 Temporal Evolution of the Vertical Displacement at 27.5 m (___FS, __-_ FFR (FS), _._._ FFR (FE))
time is plotted in Fig. 24. In general terms, all models give similar results. However, FSM discretization leads to vibrations with different frequency contents.

The convergence of the FSM can be clearly examined using the results presented in Fig.
25. An increase in the number of finite segments makes the discontinuities in deformation between finite segments smaller. This fact is observed in the normal contact force: when more elements are used, the results are not as erratic. Furthermore, it can be easily observed how the model with 5 finite segments introduces an unrealistic oscillatory normal force. In Fig. 26, it is shown that the smooth results that the combined use of FEM and FFR generates cannot be achieved by using either the FSM or the FFR/FS method. The explanation comes from the fact that the mode shapes generated by applying the FSM are not smooth themselves. The smoothness of the rail surface is modified due to the concentration of the deformation at the


Figure 25 Normal Force Using the Three FS Models
$\qquad$ 5 FS, $\qquad$ 16 FS, $\qquad$ 50 FS)


Figure 26 Normal Force Using the Three Methods
$\qquad$ FS, $\qquad$ FFR (FS), $\qquad$ FFR (FE))
interface of the finite segments, in such a way that this method also introduces geometric irregularities. Despite these drawbacks, the average results of the contact forces of the procedures involving the FSM match with the smooth results from FFR formulation.

As expected, the lateral and longitudinal creepages experience jumps and oscillations in the case of the FSM that are not present when the FFR formulation is used. For clarity, 16 finite segments are used for comparison in the following discussion. In Figs. 27 and 28 it can be seen how the FSM introduces forces of high magnitude due to the geometric description of the rail. Figures 29 and 30 show the lateral and longitudinal contact force, respectively, on the right wheel. Again, the discontinuity at the interfaces between finite segments leads to unrealistic spikes in the contact forces at the interfaces of the finite segments. In a more complex model, these non-realistic forces may excite high frequency modes of the system, leading to results which diverge further from the results obtained using the FFR formulation or the FFR/FS approach even when the average of the FSM contact force is observed.

In view of the contact force results, one may suggest removing the relative translational degrees of freedom between the finite segments. Since the deformation of the beam is dominated by bending deformation, one can create FS models in which the finite segments are linked by 3 torsional springs and spherical joints. By using these models, with the same parameters as previously discussed, one circumvents the negative effect of the step between two finite segments even though discontinuities in the spatial derivatives will still be present in the surface due to the relative rotations. As will be discussed in Chapter 4, discontinuities in the spatial derivatives of contact surfaces can lead to inaccurate contact force predictions when ECF-A is chosen. Figures 31 and 32 present the results obtained using this simplified model. Smoother


Figure 27 Lateral Creepage in the Flexible Stretch ( $\quad$ FS (16 FS), _._._ FFR (FE))


Figure 28 Longitudinal Creepage in the Flexible Stretch
$\qquad$ FS (16 FS), _.-_FFR (FE))


Figure 29 Lateral Contact Force in the Flexible Stretch (_ FS (16 FS), _._._ FFR (FE))


Figure 30 Longitudinal Contact Force in the Flexible Stretch
$\qquad$ FS (16 FS), _._.FFR (FE))


Figure 31 Normal Contact Force Using Spherical Joints FS (16 FS), _._._ FS (16 FS, Spher.) ,__-_ _ FE)


Figure 32 Wheelset Velocity Using Spherical Joints ( $\qquad$ FS (16 FS), _._._ FS (16 FS, Spher.)
contact force results are achieved; however, oscillations in the aforementioned force lead to a rough prediction of the dynamical state of the wheelset, as can be seen in Fig. 32.

### 3.7 Concluding Remarks

This chapter is concerned with the evaluation of the FSM as a tool for modeling track flexibility. In order to evaluate the performance of the FSM, several procedures which aim to deal with flexible railroad tracks have been applied to a simple track model in which only one stretch in one rail is considered flexible. In addition to the FSM which does not employ component modes, two other procedure are used in the numerical comparative study presented in this chapter. The first is the FFR formulation and the second is the combined FFR/FS method. The latter allows for developing a track model which consists of rigid segments using a FE pre-processor. Remarkable agreement among the three methods used has been found in terms of deformation and the dynamic performance of the model.

The geometry description of the FSM represents the major drawback of this approach in contact applications. Removing the discontinuities between finite segments due to shear deformation is not enough to ensure smooth results in the contact forces as continuity of the spatial derivatives of contact surface is necessary for a correct application of the elastic contact formulation (Sinokrot et al., 2008), as will be discussed in the following chapter. In other words, concentrating the deformation of the beam as in the FSM jeopardizes the effectiveness of the contact method used. This issue is partially solved by using a large number of finite segments; however this may cause an additional shortcoming of the method by introducing fictitious excitations to the system that may negatively impact the accuracy of the results. Due to these limitations, it is recommended that the use of the FSM be limited to simple simulation scenarios
or to studies concerned with gage widening, rail segment movements, and deformation analysis as discussed in the precious chapter. This is due to the fact that the forces predicted by the FS method are somewhat lacking in accuracy.

## CHAPTER 4

## ANCF MODELING OF VARIABLE PROFILE SURFACES

In the analysis of multibody system (MBS) dynamics, contact between two rigid bodies is a fundamental feature in a variety of models. Many procedures have been proposed to solve the rigid body contact problem, most of which belong to one of two categories: off-line and on-line contact search methods. This chapter will focus on the development of a contact surface geometry model for the rigid body contact problem in the case where an on-line threedimensional non-conformal contact evaluation procedure, such as the elastic contact formulation - algebraic equations (ECF-A), is employed. It is shown that the contact surface must have continuity in the second order spatial derivatives when used in conjunction with ECF-A. Many of the existing surface models rely on direct linear interpolation of profile curves which leads to first order spatial derivative discontinuities. This, in turn, leads to erroneous spikes in the predicted contact forces when ECF-A is used. To this end, an absolute nodal coordinate formulation (ANCF) thin plate surface model is developed in order to ensure second order spatial derivative continuity which satisfies the requirements of the contact formulation. A simple example of a railroad vehicle negotiating a turnout, which includes variable cross-section rail, is tested for the cases of the new ANCF thin plate element surface, an existing ANCF thin plate element surface with first order spatial derivative continuity, and the direct linear profile interpolation method. A comparison of the numerical results reveals the benefits of using the ANCF surface geometry developed in this chapter.

### 4.1 Contributions and Scope of this Chapter

The objective of this chapter is to develop a new finite element-based procedure for representing surface geometry in MBS contact problems. This procedure ensures an appropriate degree of continuity at the element interface, thereby allowing for more accurate predictions of kinetic results that include the contact forces. Specifically, the main contributions of this chapter can be summarized as follows:

1. This chapter clearly identifies and explains the limitations of using curve representations in the description of surface geometry. It also identifies and explains the limitations of using low order interpolations with contact formulations that demand a higher degree of spatial derivative continuity. These two geometric approaches for modeling surfaces can lead to fundamental kinematic and kinetic problems that cannot be ignored in the analysis of important engineering applications such as railroad vehicle systems. To this end, this chapter provides in Section 4.3 a clear explanation of the potential loss of accuracy when continuity conditions are imposed in the case in which lower order interpolation is used.
2. This chapter proposes a new finite element-based surface geometry that ensures a higher degree of continuity at the element interface. The new geometry, which is based on ANCF finite element geometry, is proposed in order to address the fundamental problems associated with the use of the curve representation or the use of lower order interpolation. A bi-quintic interpolation is employed in order to address the kinetic problems that result from the use of lower order geometric descriptions.
3. This chapter presents a comparative analysis, both qualitative and quantitative, to demonstrate the value of using the proposed geometric approach. To this end, three different approaches are compared analytically and numerically. These three approaches


Figure 33 Swept Surface
are the curve network representation of the surface, the lower order surface interpolation, and the proposed higher order surface interpolation techniques. The results of this comparative analysis demonstrate that the use of higher order surface interpolation is feasible in many challenging problems.
4. Lastly, a numerical example of a rail vehicle negotiating a turnout is used to demonstrate the feasibility of using a rail computer-aided design (CAD) geometry model that can be systematically integrated with complex MBS models. The example presented in this chapter clearly demonstrates the need for the use of the new geometric approach to model important technological applications. The results also clearly demonstrate the limitations of other existing approaches.


Figure 34 Linear Interpolation Lofted Surface

### 4.2 Curve Network Representation

In the MBS analysis of contact problems, it is important to accurately describe the surfaces of the bodies in contact. The correct description of the surfaces is crucial for an accurate evaluation of the contact forces, which have a significant effect on the system dynamics. The assumption of a constant profile swept along a curve, as shown in Fig. 33, is implemented in many cases to simplify the geometric problem and, for many common scenarios, such as the idealized wheel/rail contact problem in railroad vehicle simulations, this swept surface description is a prudent choice. However, there are many scenarios for which this assumption does not provide an accurate model of the contact surface. For instance, a varying rail profile is a necessity in the cases of modeling turnouts, frogs, worn rails, etc. It is the goal of this section to describe a way in which the surface description proposed by (Rathod, et al., 2009) may be generalized to allow for the varying profiles of surfaces such as these by making use of a direct linear interpolation
scheme. The result is referred to as a lofted surface, such as that shown in Fig. 34. Additionally, it will be explained why this method is a poor choice when used in combination with the on-line elastic or constraint contact formulations outlined in Section 4.7.

It is convenient in numerical simulation to define the contact surfaces as unique functions of two surface parameters. The contact surface of body $i$ is defined in terms of the longitudinal arc length surface parameter, $s_{1}^{i}$, which follows the space curve the profile is swept along and the lateral surface parameter, $s_{2}^{i}$, which defines the profile of the surface as a function $f\left(s_{2}^{i}\right)$. As was described by Rathod et al., an arbitrary point on a swept surface may be described using the following equation (Rathod et al., 2009)

$$
\begin{equation*}
\mathbf{r}^{i}=\mathbf{R}^{i}+\mathbf{A}^{i} \overline{\mathbf{u}}^{i} \tag{4.1}
\end{equation*}
$$

where $\mathbf{r}^{i}$ is the global position of the arbitrary point on the surface of body $i, \mathbf{R}^{i}$ is the global position of the local coordinate system of body $i, \mathbf{A}^{i}$ is the transformation matrix which defines the orientation of the coordinate system of body $i$ with respect to the global coordinate system, and $\overline{\mathbf{u}}^{i}$ is the local position of an arbitrary point within the coordinate system of body $i$. In previous studies (Rathod et al., 2009), $\overline{\mathbf{u}}^{i}$ was defined for a constant profile using ANCF geometry to discretize the space curve and spline geometry to define the surface along the space curve as

$$
\begin{equation*}
\overline{\mathbf{u}}^{i}=\mathbf{S}\left(\xi\left(s_{1}^{i}\right), \eta\left(s_{2}^{i}\right), \zeta\left(f\left(s_{2}^{i}\right)\right)\right) \mathbf{e}^{m} \tag{4.2}
\end{equation*}
$$

where $\mathbf{S}$ is the matrix of shape functions for the three-dimensional beam element (Shabana, 2012), $\mathbf{e}^{m}$ is the vector of nodal coordinates including the position and gradient vectors at the two nodes of element $m$, and the local element coordinates are defined as $\xi=\left(s_{1}^{i}-x^{r}\right) / l$, $\eta=\left(s_{2}^{i}-y^{r}\right) / l$, and $\zeta=\left(f\left(s_{2}^{i}\right)-z^{r}\right) / l$ where the superscript $r$ designates the number of the
first node of element $m$, and $l$ is defined by the distance between nodes $r$ and $r+1$ in the parametric domain. Using these formulae, any point on the swept surface may be found using the ANCF geometric interpolation between the nodes of the space curve. Equation 4.1 may be generalized to allow for varying profiles of the surface by modifying the definition of the local coordinate $\zeta$ as (Sinokrot, 2009) $\zeta=\left(h\left(s_{1}^{i}, s_{2}^{i}\right)-z^{r}\right) / l$ which allows the height of the surface with respect to the space curve to vary as a function of both the lateral and longitudinal surface parameters. With this new definition for $\zeta$, Eq. 4.1 can be modified to define a lofted surface.

In a MBS code, a lofted surface may be created by providing profile curves which define the relationship between $s_{2}^{i}$ and $f\left(s_{2}^{i}\right)$. These profile curves may then be interpolated to define the surface of body $i$ in the interval between the two neighboring profile curves. Provided the intervals are small, a series of linear interpolations may be used to determine the value of $h\left(s_{1}^{i}, s_{2}^{i}\right)$ for any values of $s_{1}^{i}$ and $s_{2}^{i}$ on the surface. Nonetheless, according to (Sinokrot et al., 2008), continuity in the spatial derivatives of the surface up to the second order in both the lateral and longitudinal directions is required for the elastic contact formulation described in Section 4.7. Clearly the use of linear interpolation will result in discontinuities in some of these derivatives. To overcome this, one may use higher order direct interpolation methods on-line in a similar procedure as described above. According to (Shikin and Plis, 1995), this results in either an interpolating or smoothing surface. In the former, the surface passes through all of the data points provided but may result in fictitious oscillations in the surface geometry. In the latter, the surface need not pass through the provided data points but it is generally smooth and relatively free of the fictitious oscillations. With either choice, a sacrifice in accuracy or smoothness of the surface must be made. Neither choice is ideal for an on-line surface interpolation scheme as the user will be unable to determine the extent of the distortion in the surface geometry until the


Figure 35 ANCF Thin Plate Element in Parametric (left) and Physical (Right) Domains
simulation is complete. For this reason, an alternative surface description, which provides an acceptable level of both accuracy and smoothness, and which may be easily viewed prior to simulation, is developed using ANCF surface geometry as described in the following sections.

### 4.3 Lower Order ANCF Surface Geometry

The $C^{1}$ continuous third order ANCF thin plate element was introduced by (Mikkola et al., 2012). This element is a four node quadrilateral that has four coordinate vectors at each node which results in a total of 48 degrees of freedom for the element. As with other ANCF thin plate elements presented in the literature (Dmitrochenko and Pogorelov, 2003; Shabana, 2005; Shabana, 2012), the position of an arbitrary point in the element may be computed by mapping an undeformed rectangular reference configuration to the deformed physical configuration, as shown in Fig. 35. A multiplicative decomposition of the spatial and temporal coordinates yields

$$
\begin{equation*}
\mathbf{r}=\mathbf{S}(\xi, \eta) \mathbf{e}^{m}(t) \tag{4.3}
\end{equation*}
$$



Figure 36 Two Element ANCF Thin Plate Mesh
where $\mathbf{S}$ is the matrix of shape functions, which is provided in Appendix C, $\xi$ and $\eta$ are the element coordinates defined in the natural coordinate system and are related to the reference configuration through the relationship $\xi=x / A$ and $\eta=y / B$ where $A$ and $B$ define the element width and height reference configuration, and $\mathbf{e}^{m}$ is the vector of nodal coordinates for element $m$ which are defined as $\mathbf{e}^{m}=\left[\begin{array}{llll}\mathbf{e}_{1}^{T} & \mathbf{e}_{2}^{T} & \mathbf{e}_{3}^{T} & \mathbf{e}_{4}^{T}\end{array}\right]^{T}$ where the coordinates at node $n$ are

$$
\mathbf{e}_{n}=\left[\begin{array}{llll}
\mathbf{r}_{n}^{T} & \frac{\partial \mathbf{r}_{n}^{T}}{\partial x} & \frac{\partial \mathbf{r}_{n}^{T}}{\partial y} & \frac{\partial^{2} \mathbf{r}_{n}^{T}}{\partial x \partial y} \tag{4.4}
\end{array}\right]^{T}
$$

where $\mathbf{r}_{n}$ is the position in the physical domain of node $n$. Note that in the case of a rigid ANCF surface, $\mathbf{e}^{m}$ has no temporal dependence as no deformation of the surface may occur.

It can be shown that this element has $C^{1}$ continuity by evaluating the first spatial derivatives of two elements at the element interface. For example, consider a two element mesh in which the elements share a boundary with a common $X$ coordinate as shown in Fig. 36. In this mesh, the element coordinate vectors may be defined as $\mathbf{e}^{1}=\left[\begin{array}{llll}\mathbf{e}_{1}^{T} & \mathbf{e}_{2}^{T} & \mathbf{e}_{3}^{T} & \mathbf{e}_{4}^{T}\end{array}\right]^{T}$ and $\mathbf{e}^{2}=$ $\left[\begin{array}{llll}\mathbf{e}_{2}^{T} & \mathbf{e}_{5}^{T} & \mathbf{e}_{6}^{T} & \mathbf{e}_{3}^{T}\end{array}\right]^{T}$. The $C^{1}$ continuity conditions at this interface may be defined as $(\partial \mathbf{S} / \partial x)(\xi=1, \eta) \mathbf{e}^{1}=(\partial \mathbf{S} / \partial x)(\xi=0, \eta) \mathbf{e}^{2}, \quad$ and $\quad(\partial \mathbf{S} / \partial y)(\xi=1, \eta) \mathbf{e}^{1}=(\partial \mathbf{S} / \partial y)(\xi=$ $0, \eta) \mathbf{e}^{2}$. These two conditions are satisfied in the case in which the two elements share a
common dimension $B$, which implies that the two elements have the same height in the reference configuration. Similarly, it can be shown that for a two-element mesh in which the elements share a common $Y$ boundary that the $C^{1}$ continuity conditions are satisfied for the case in which the two elements have a common dimension $A$. This implies that the two elements must have the same width. Therefore, in the case of a mesh which forms a rectangular grid in the undeformed configuration, the third order plate satisfies the $C^{1}$ continuity conditions throughout the entire surface.

Using a similar procedure, it can be shown that this element does not satisfy continuity in the second order spatial derivatives at the element interface. In order to enforce such conditions, one may extend the procedure presented by Lan and Shabana in which constraints are applied at the nodes on the element interface to increase the order of continuity between ANCF beam elements (Lan and Shabana, 2010). For the mesh shown in Fig. 36, it can be shown that the condition for the continuity in the second spatial derivative taken with respect to the $Y$ coordinate is satisfied provided the two elements have the same height. In other words, the condition $\left(\partial^{2} \mathbf{S} / \partial y^{2}\right)(\xi=1, \eta) \mathbf{e}^{1}=\left(\partial^{2} \mathbf{S} / \partial y^{2}\right)(\xi=0, \eta) \mathbf{e}^{2}$ is satisfied. However, a similar condition applied to the second spatial derivative taken with respect to the $X$ coordinate is not satisfied. In order to enforce continuity in the second spatial derivative taken with respect to the $X$ coordinate, the following two constraints must be applied to both nodes at the element interface

$$
\left.\begin{array}{c}
\frac{\partial^{2} \mathbf{S}}{\partial x^{2}}\left(\xi=\xi_{n}^{1}, \eta=\eta_{n}^{1}\right) \mathbf{e}^{1}=\frac{\partial^{2} \mathbf{S}}{\partial x^{2}}\left(\xi=\xi_{n}^{2}, \eta=\eta_{n}^{2}\right) \mathbf{e}^{2} \\
\frac{\partial^{3} \mathbf{S}}{\partial x^{2} \partial y}\left(\xi=\xi_{n}^{1}, \eta=\eta_{n}^{1}\right) \mathbf{e}^{1}=\frac{\partial^{3} \mathbf{S}}{\partial x^{2} \partial y}\left(\xi=\xi_{n}^{2}, \eta=\eta_{n}^{2}\right) \mathbf{e}^{2} \tag{4.5}
\end{array}\right\}
$$

where $\xi_{n}^{m}$ and $\eta_{n}^{m}$ are the natural coordinates of node $n$ defined in element $m$. These two equations are linear in the nodal coordinate vectors contained in $\mathbf{e}^{1}$ and $\mathbf{e}^{2}$ which allows for a
simple procedure in which two vectors of coordinates at both nodes on the element interface are constrained. This eliminates 12 degrees of freedom from the elements at the interface. A similar set of constraint equations can be applied to two elements which share a boundary with a common $Y$ coordinate. Consequently, for each interface an element shares with another element, 12 degrees of freedom must be constrained in order to ensure continuity in the second spatial derivative taken with respect to both the $X$ and $Y$ coordinates. Thus, for an element that does not lie on the boundary of the mesh, e.g. an element which has an interface with four other elements, 48 degrees of freedom would need to be constrained leaving the element with zero degrees of freedom. This implies that the constrained surface need not pass through the interior nodes the surface was originally constructed from. Clearly this does not represent an optimal choice for modeling a generic surface with second order spatial derivative continuity due to the substantial loss of geometric accuracy. For this reason, a higher order thin plate element with natural $C^{2}$ continuity was developed to maintain both higher order continuity and geometric accuracy.

### 4.4 Higher Order ANCF Surface Geometry

In order to ensure $C^{2}$ continuity of the new plate element, it is assumed that each element will employ bi-quintic interpolation and satisfy the nine conditions in terms of the position and spatial derivatives up to the second order in each direction at each internal node as prescribed by (Jones, 1988). These nine conditions are required at each node which is shared by up to four elements; this implies that a rectangular grid of elements must be chosen for these conditions to apply. Of course, one may desire to use a non-rectangular element, such as a triangular element, for ease of mesh refinement, however Jones demonstrated that this task is significantly more complicated as the nine aforementioned conditions are not sufficient in the case of non-rectangular elements.

These nine conditions can be systematically satisfied by using these conditions to form nine coordinate vectors for each node. With the chosen coordinates in mind, the shape functions for the quintic ANCF thin plate element were derived from the basis functions of a fifth order Bezier patch using the linear transformation that will be described in Section 4.6. The nodal coordinate vector for the new element is defined as

$$
\mathbf{e}_{n}=\left[\begin{array}{ccccccccc}
\mathbf{r}_{n}^{T} & \frac{\partial \mathbf{r}_{n}^{T}}{\partial x} & \frac{\partial \mathbf{r}_{n}^{T}}{\partial y} & \frac{\partial^{2} \mathbf{r}_{n}^{T}}{\partial x^{2}} & \frac{\partial^{2} \mathbf{r}_{n}^{T}}{\partial x \partial y} & \frac{\partial^{2} \mathbf{r}_{n}^{T}}{\partial y^{2}} & \frac{\partial^{3} \mathbf{r}_{n}^{T}}{\partial x^{2} \partial y} & \frac{\partial^{3} \mathbf{r}_{n}^{T}}{\partial x \partial y^{2}} & \frac{\partial^{4} \mathbf{r}_{n}^{T}}{\partial x^{2} \partial y^{2}} \tag{4.6}
\end{array}\right]^{T}
$$

which demonstrates that this element has 108 degrees of freedom, while the matrix of shape functions is provided in Appendix C.

As in Section 4.3, a set of conditions may be checked to ensure that continuity in the second spatial derivatives taken with respect to the $X$ and $Y$ coordinates at the element interfaces is guaranteed. For the configuration shown in Fig. 36, the following conditions can easily be verified in the case where the two elements share the same height $B$

$$
\begin{equation*}
\frac{\partial^{2} \mathbf{S}}{\partial x^{2}}(\xi=1, \eta) \mathbf{e}^{1}=\frac{\partial^{2} \mathbf{S}}{\partial x^{2}}(\xi=0, \eta) \mathbf{e}^{2}, \quad \frac{\partial^{2} \mathbf{S}}{\partial y^{2}}(\xi=1, \eta) \mathbf{e}^{1}=\frac{\partial^{2} \mathbf{S}}{\partial y^{2}}(\xi=0, \eta) \mathbf{e}^{2} \tag{4.7}
\end{equation*}
$$

It can be shown that similar conditions are satisfied for two elements which share a boundary with a common $Y$ coordinate provided that the two elements share a common width $A$. Thus, as explained by Jones, a rectangular grid of elements in the undeformed configuration is required in order to maintain the $C^{2}$ continuity conditions.

While this chapter is concerned with the case of a rigid contact surface, it is important to note that the new element presented in this section may be used to model a flexible surface using the same procedures employed to model other ANCF thin plate elements as flexible bodies in literature (Mikkola et al., 2012, Shabana, 2012). However, this new element has 108 degrees of freedom which can significantly penalize the computational time of the flexible body simulation
when compared to the 48 degree of freedom cubic plate element presented in Section 4.3. This large number of degrees of freedom per element may become prohibitive in the sequential computing paradigm. However, this large computational cost may be mitigated as demonstrated by (Melanz et al., 2012). The proposed method reported a speedup of 250x when compared to its serial programming counterpart by leveraging the parallel processing capabilities of modern computer hardware. A similar method could be employed in the case of the fifth order plate to reduce the prohibitive cost of the large number of degrees of freedom per element.

### 4.5 ANCF Thin Plate Element Meshing Schemes

In the design of an ANCF thin plate mesh which models an arbitrary surface, the choice of the reference configuration can represent a significant challenge. In the case of a rigid surface modeled with ANCF thin plate elements, the configuration of the body does not change with time and the body takes the so called deformed configuration at the initial time step. In this case, the reference configuration represents a parametric domain in which two surface parameters are defined to identify a specific point on the surface composed of a collection of elements which form the mesh. This lends itself easily to a contact formulation, such as that presented in Section 4.7, in which the entire surface must exist in a single continuous parametric domain. To this end, each node in the mesh is provided with an additional two coordinates which correspond to the surface parameters $s_{1}^{k}$ and $s_{2}^{k}$ where $k$ denotes the surface number. Consequently, one may use the equation $\mathbf{r}^{k}=\mathbf{S}\left(\xi\left(s_{1}^{k}\right), \eta\left(s_{2}^{k}\right)\right) \mathbf{e}^{m}$ to evaluate the position of a point $\left(s_{1}^{k}, s_{2}^{k}\right)$ on surface $k$. Recall that the reference configuration of an ANCF thin plate mesh using the elements presented


Figure 37 Right Hand Turnout Diagram (Shabana et al., 2008)
in Sections 4.3 and 4.4 must maintain a rectangular shape to ensure optimum continuity. The resulting rectangular mesh formed from this collection of elements is referred to as a rectangular grid. This does not, however, imply that the deformed configuration or the shape of the surface in the physical domain be defined as a rectangular shape. The aforementioned parameterization of the reference configuration allows for the following simple relationship between the surface parameters $s_{1}^{k}$ and $s_{2}^{k}$ and the element natural coordinates to be created: $\xi=\left(s_{1}^{k}-s_{1,1}^{k, m}\right) / A^{m}$ and $\eta=\left(s_{2}^{k}-s_{2,1}^{k, m}\right) / B^{m}$, where $s_{l, n}^{k, m}$ is the surface parameter $l$ stored for node $n$ of element $m$ on surface $k$, and the width and height of the element in the parametric configuration are defined as $A^{m}=s_{1,3}^{k, m}-s_{1,1}^{k, m}$ and $B^{m}=s_{2,3}^{k, m}-s_{2,1}^{k, m}$, respectively. Using these relationships, a large variety of surface shapes may easily be represented using ANCF thin plate elements of various orders of continuity.

The parameterization of the reference configuration presented in this section lends itself easily to the construction of surfaces which have continuous physical geometry. However, the physical geometry of a contact surface need not be continuous in the general case. Take for example a turnout in a railroad vehicle simulation. As is shown in Fig. 37, a turnout is composed of multiple rail segments which may physical discontinuities between them. These physical


Figure 38 ANCF Thin Plate Mesh with $C^{-1}$ Boundary (Left: Parametric Domain, Right: Physical Domain)
discontinuities are referred to as $C^{-1}$ continuity. Clearly it would be inappropriate to model the surface at the interface between two of these segments with $C^{2}$ continuity. To insert this $C^{-1}$ continuity into an otherwise $C^{2}$ mesh, the contact surface may be generated from two rectangular grids which are joined only in the parametric domain. In other words, the two rectangular grids would share no nodes or elements in common, however the parametric domains of the two grids would share a common boundary in either the longitudinal or lateral surface parameter directions thus allowing a continuous description in the parametric domain. An example of this configuration is provided in Fig. 38 in which two grids have been combined in a single mesh with a common boundary along the lateral direction. Using this approach, one may generate a variety of surfaces which are composed of multiple consecutive rectangular grids which have either $C^{-1}$ or $C^{2}$ continuity at the boundaries in the physical domain while retaining at least $C^{0}$ continuity at the boundaries in the parametric domain.

### 4.6 Mapping Between ANCF Elements and Bezier Patches

As has been shown in literature (Lan and Shabana, 2010; Mikkola et al., 2012), the geometric definition of ANCF curves and surfaces is compatible with that of Bezier curves and surfaces. In fact, a linear transformation may be used to convert certain ANCF curves and surfaces to equivalent Bezier curves or surfaces. This is particularly useful since, according to (Piegl and Tiller, 1997) any B-spline curve or surface may be converted to one or more Bezier curves or surfaces. In many CAD programs, B-spline geometry is used to represent many of the fundamental shapes. Thus, if one were to generate a complex geometric model of a surface in a B-spline based CAD system for use in a contact problem, it may be possible to convert this surface to an equivalent ANCF thin plate mesh without any geometric distortion.

Similar to ANCF thin plate elements, Bezier surface patches are generated via a multiplicative decomposition. The primary difference being that the coordinates of a Bezier surface patch, which are often referred to as control points, do not have clear physical meaning and some of which may not represent material points. On the other hand, the coordinates of an ANCF thin plate element have clear physical meaning such as the position or spatial derivative of the surface the element is used to represent. The surface of a Bezier patch of polynomial order $n$ in the $\xi$ direction and $m$ in the $\eta$ direction is defined as (Piegl and Tiller, 1997)

$$
\begin{equation*}
\mathbf{r}(\xi, \eta)=\sum_{i=0}^{n} \sum_{j=0}^{m} N_{\xi, i} N_{\eta, j} \mathbf{P}_{i, j} \tag{4.8}
\end{equation*}
$$

where $\mathbf{P}_{i, j}$ is the $i, j$ th control point of the Bezier patch, and $N_{\xi, i}$ and $N_{\eta, j}$ are the Bernstein polynomial basis functions of the Bezier patch defined as

$$
\begin{equation*}
N_{\chi, k}=\frac{l!}{k!(l-i)!} \chi^{k}(1-\chi)^{l-k} \tag{4.9}
\end{equation*}
$$

where $k=i$ and $l=n$ when $\chi=\xi$, and $k=j$ and $l=m$ when $\chi=\eta$.


Figure 39 Relationship Between Quintic Bezier Patch Control Points and ANCF Nodal Coordinates

Assuming that one desires to convert the Bezier patch to an equivalent four node ANCF thin plate element, it is convenient to restructure Eq. 4.8 as a series of dot products between the basis functions and control points which influence the Bezier patch at each corner. A similar restructuring may be applied to the ANCF thin plate shape functions and coordinates at each node as follows

$$
\begin{equation*}
\mathbf{r}(\xi, \eta)=\sum_{N=1}^{4} \mathbf{N}_{N} \mathbf{P}_{N}=\sum_{N=1}^{4} \mathbf{s}_{N} \mathbf{E}_{N} \tag{4.10}
\end{equation*}
$$

where $\mathbf{N}_{N}, \mathbf{P}_{N}, \mathbf{S}_{N}$, and $\mathbf{E}_{N}$ corresponding to a quintic Bezier patch and the related ANCF thin plate element are provided in Appendix C. Figure 39 shows graphically which control points have influence on Eq. 4.10 at the corners of the Bezier patch. It follows logically from this that these groups of control points, which are stored in $\mathbf{P}_{N}$, will be the only set which influence the ANCF coordinates at a given node. Note that the vectors of ANCF nodal coordinates $\mathbf{E}_{N}$ do not correspond exactly to the standard form of the vector of nodal coordinates provided for the quintic ANCF thin plate element presented in Section 4.4. This form was chosen such that the
same linear transformation may be used to determine the relationship between the ANCF nodal coordinates and the Bezier control points at all four nodes.

To determine the relationship between the control points and nodal coordinates, one may simply evaluate Eq. 4.10 and its derivatives in accordance with the chosen nodal coordinates at the corners of the Bezier patch. This leads to the construction of the linear transformation matrix B which relates the control points to the corresponding nodal coordinates through the following relationship

$$
\begin{equation*}
\mathbf{E}_{N}=\mathbf{B} \mathbf{P}_{N} \tag{4.11}
\end{equation*}
$$

where the matrix B corresponding to a quintic Bezier patch is provided in Appendix C. It is worth noting that the matrix $\mathbf{B}$ is non-singular which allows the converse relationship to be written by inverting the linear transformation matrix $\mathbf{B}$.

The shape functions for the fifth order ANCF thin plate element provided in Section 4.4 may be derived directly from the quintic Bezier patch. This is accomplished by substituting Eq. 4.11 into Eq. 4.10 at node $N$ and solving for the vector of shape functions which results in the following relationship

$$
\begin{equation*}
\mathbf{S}_{N}=\mathbf{N}_{N} \mathbf{B}^{-1} \tag{4.12}
\end{equation*}
$$

Following a similar procedure, the shape functions for a $C^{3}$ continuous ANCF thin plate element may be derived from a seventh order Bezier patch, as per the requirements of the constraint contact approach (Sinokrot et al., 2008). In this case, the vector of ANCF nodal coordinates would be composed of the sixteen vectors corresponding to the position and spatial derivatives of up to the third order in each parameter. Using this new vector of nodal coordinates, one could easily develop the linear transformation matrix B and use Eqs. 4.11 and 4.12 to convert the
control points and basis functions of the seventh order Bezier patch to the shape functions and nodal coordinates of the corresponding ANCF thin plate element.

### 4.7 Contact Formulations

In this section, two contact formulations are discussed. The first is the elastic contact formulation - algebraic equations (ECF-A), and the second is the augmented constraint contact formulation (ACCF). Although ECF-A and ACCF are both three-dimensional non-conformal contact formulations, there are some fundamental differences between the two. Note that both of the contact formulations employed in this chapter are solved on-line. The contact equations of ECFA are not treated as constraint equations and need not be satisfied by the MBS code at the velocity and acceleration levels. In ACCF, the contact patch is treated as a rigid solid such that small penetrations and separations are not allowed. As the name suggests, the contact equations of ACCF are treated as constraint equations which must be satisfied at the velocity and acceleration levels. Another important difference, as discussed by (Sinokrot et al., 2008), is the different requirements imposed on the spatial derivatives of the surfaces used for the two methods as will be discussed in the following sections.

### 4.7.1 Elastic Contact Formulation - Algebraic Equations (ECF-A)

As mentioned in previous chapters, ECF-A is a three-dimensional non-conformal contact formulation which assumes that the rigid bodies in contact are elastic in the contact patch. To find the contact point, a set of four algebraic equations in terms of the four surface parameters which describe the surfaces of bodies $i$ and $j$ are solved. The non-generalized surface parameters are stored in vector form as $\mathbf{s}=\left[\begin{array}{llll}s_{1}^{i} & s_{2}^{i} & s_{1}^{j} & s_{2}^{j}\end{array}\right]^{T}$. The vector of equations $\mathbf{E}(\mathbf{s})$ which is used to find the contact point is defined as

$$
\mathbf{E}(\mathbf{s})=\left[\begin{array}{llll}
\mathbf{t}_{1}^{j} \cdot \mathbf{r}^{i j} & \mathbf{t}_{2}^{j} \cdot \mathbf{r}^{i j} & \mathbf{t}_{1}^{i} \cdot \mathbf{n}^{j} & \mathbf{t}_{2}^{i} \cdot \mathbf{n}^{j} \tag{4.13}
\end{array}\right]^{T}=\mathbf{0}
$$

where $\mathbf{r}^{i j}$ is defined as $\mathbf{r}^{i}-\mathbf{r}^{j}, \mathbf{n}^{j}$ is the normal vector to the surface of body $j$ at the point of contact, and $\mathbf{t}_{l}^{p}$ is the tangent vector of the surface of body $p(p=i, j)$ taken with respect to surface parameter $l(l=1,2)$. The normal and tangent vectors are calculated, respectively, with the formulae (Kreyszig, 1991) $\mathbf{n}^{j}=\mathbf{t}_{1}^{j} \times \mathbf{t}_{2}^{j}$ and $\mathbf{t}_{l}^{p}=\partial \mathbf{r}^{p} / \partial s_{l}^{p}$. Equation 4.13 is solved using the iterative Newton-Raphson solution procedure to determine the location of the contact point in terms of the non-generalized surface parameters. When using ECF-A, the normal contact force is defined as a function of the penetration as has been discussed in literature (Shabana et al., 2008) and previous chapters. The penetration is defined as $\delta=\mathbf{r}^{i j} \cdot \mathbf{n}^{j}$, where $\mathbf{n}^{j}$ is defined as the normal vector at the contact point defined on body $j$.

As discussed by Sinokrot et al., the use of this elastic contact solution procedure requires continuity in the spatial derivatives up to the second order taken with respect to both the lateral and longitudinal surface parameters (Sinokrot et al. 2008). Consequently, the $C^{2}$ quintic ANCF thin plate element described in Section 4.4 is the ideal choice for modeling a surface when using this contact formulation. Conversely, if a surface does not guarantee $C^{2}$ continuity, such as the direct linear interpolation method described in Section 4.2 or the third order plate described in Section 4.3, significant numerical noise which has no physical meaning can be introduced into the solution. This is most pronounced in the normal contact force as will be shown in numerical example in Section 4.9.

### 4.7.2 Augmented Constraint Contact Formulation (ACCF)

In ACCF, five constraint equations (four of which were used in ECF-A) are imposed to guarantee the existence of a single contact point between each pair of bodies. Unlike ECF-A, no
penetration or separation is allowed as this would cause a violation in the imposed constraint equations. The five constraint equations are defined as follows (Shabana et al., 2008)

$$
\mathbf{C}\left(\mathbf{q}^{i}, \mathbf{q}^{i}, \mathbf{s}\right)=\left[\begin{array}{lllll}
\mathbf{t}_{1}^{j} \cdot \mathbf{r}^{i j} & \mathbf{t}_{2}^{j} \cdot \mathbf{r}^{i j} & \mathbf{n}^{j} \cdot \mathbf{r}^{i j} & \mathbf{t}_{1}^{i} \cdot \mathbf{n}^{j} & \mathbf{t}_{2}^{i} \cdot \mathbf{n}^{j} \tag{4.14}
\end{array}\right]^{T}=\mathbf{0}
$$

where $\mathbf{q}^{p}$ is the vector of generalized coordinates of body $p$, and the other parameters retain the same definition provided in the previous section. This equation is also solved using the iterative Newton-Raphson procedure at run-time. As will be shown in the following section, this solution is included in the augmented form of the equation of motion using the method of Lagrange multipliers. Due to the fact that there are five constraint equations and only four surface parameters, it can be shown that there is only one independent Lagrange multiplier (Shabana et al., 2008). This implies that there is only one independent constraint reaction force and it is this reaction force which is used to determine the normal contact force.

Since these constraint equations must be differentiated twice when coupled with the equations of motion as discussed by Sinokrot et al., a $C^{3}$ continuous surface is required in order to obtain a result free of fictitious numerical excitation (Sinokrot et al., 2008). Considering the fact that ECF-A has a continuity requirement one order lower than ACCF and allows for separation between the two bodies in contact, ECF-A is chosen in place of ACCF for the numerical investigation presented in Section 4.9. As discussed in Section 4.6, a higher order plate element with $C^{3}$ continuity could be developed following the procedure used to generate the fifth order plate element. This would, of course, require a much larger number of coordinates than the fifth order plate element. Consequently, ECF-A is preferred to ACCF in the case of contact surfaces with variable geometry.

### 4.7.3 Evaluation of the Contact Forces

Once the normal contact force is determined, the tangential creep forces and spin moment must be evaluated. These forces are calculated using the same method for both contact formulations presented. First, the longitudinal, lateral and spin creepages are calculated as follows (Shabana et al., 2008)

$$
\begin{equation*}
\zeta_{x}=\frac{\left(\mathbf{v}^{i}-\mathbf{v}^{j}\right)^{T} \hat{\mathbf{t}}_{1}^{j}}{V}, \quad \zeta_{y}=\frac{\left(\mathbf{v}^{i}-\mathbf{v}^{j}\right)^{T} \hat{\mathbf{t}}_{2}^{j}}{V}, \quad \varphi=\frac{\left(\boldsymbol{\omega}^{i}-\boldsymbol{\omega}^{j}\right)^{T} \widehat{\mathbf{n}}^{j}}{V} \tag{4.15}
\end{equation*}
$$

where $\mathbf{v}^{i}$ and $\mathbf{v}^{j}$ are, respectively, the absolute velocity vectors of bodies $i$ and $j$ at the contact point, $\hat{\mathbf{t}}_{1}^{j}$ and $\hat{\mathbf{t}}_{2}^{j}$ are the longitudinal and lateral unit tangent vectors of body $j$ at the contact point, $V$ is defined as the forward velocity of the centroid of body $i, \boldsymbol{\omega}^{i}$ and $\boldsymbol{\omega}^{j}$ are the absolute angular velocity vectors of bodies $i$ and $j$ respectively, and $\widehat{\mathbf{n}}^{j}$ is defined as the unit normal vector of body $j$ at the point of contact. With the creepages found, the forces associated with them can be calculated using Kalker's non-linear creep theory with the use of Kalker's USETAB (Vollebregt, 2008). This program takes the creepages and other pertinent quantities as input and provides the longitudinal and lateral creep forces as well as the creep spin moment as output.

### 4.8 MBS Dynamic Equations

In this section, two versions of the augmented form of the equations of motion are presented. When the elastic contact formulation is used, the non-generalized surface parameters are not included in the equation of motion while the generalized coordinates are included. However, when the constraint contact method is chosen, the non-generalized surface parameters must appear in the augmented form of the equations of motion or systematically eliminated using four of the contact constraint equations. Using the principle of virtual work, the following formula may be obtained for the elastic contact formulation (Shabana et al., 2008) $\delta \mathbf{q}^{T}\left(\mathbf{M} \ddot{\mathbf{q}}+\mathbf{C}_{\mathbf{q}}^{T} \lambda-\right.$
$\left.\mathbf{Q}_{e}\right)=0$, while for the constraint contact method, this equation is written as $\delta \mathbf{q}^{T}\left(\mathbf{M} \ddot{\mathbf{q}}+\mathbf{C}_{\mathbf{q}}^{T} \boldsymbol{\lambda}-\right.$ $\left.\mathbf{Q}_{e}\right)+\delta \mathbf{s}^{T} \mathbf{C}_{\mathbf{s}}^{T} \boldsymbol{\lambda}=0$, where $\mathbf{M}$ is the system mass matrix, $\mathbf{q}$ is the vector of generalized coordinates, $\mathbf{s}$ is the vector of non-generalized surface parameters, $\mathbf{C}_{\mathbf{q}}$ and $\mathbf{C}_{\mathbf{s}}$ are the Jacobian constraint matrices formed by the differentiation of the constraint equations with respect to $\mathbf{q}$ and $\mathbf{s}$ respectively, $\mathbf{Q}_{e}$ is the generalized force vector, and $\boldsymbol{\lambda}$ is the vector of Lagrange multipliers. The differentiation of the constraint equations twice with respect to time yields $\mathbf{C}_{\mathbf{q}} \ddot{\mathbf{q}}=\mathbf{Q}_{c}$ for the elastic contact formulation, and $\mathbf{C}_{\mathbf{q}} \ddot{\mathbf{q}}+\mathbf{C}_{\mathbf{s}} \ddot{\mathbf{s}}=\mathbf{Q}_{c}$ for the constraint contact method where $\mathbf{Q}_{c}$ is a quadratic velocity vector which results from this differentiation. Using the previous equations, the augmented form of the equation of motion for the elastic contact formulation may be written as (Roberson and Schwertassek, 1988, Shabana et al., 2008)

$$
\left[\begin{array}{cc}
\mathbf{M} & \mathbf{C}_{\mathbf{q}}^{T}  \tag{4.16}\\
\mathbf{C}_{\mathbf{q}} & \mathbf{0}
\end{array}\right]\left[\begin{array}{l}
\ddot{\mathbf{q}} \\
\lambda
\end{array}\right]=\left[\begin{array}{l}
\mathbf{Q}_{e} \\
\mathbf{Q}_{c}
\end{array}\right]
$$

while written for the constraint contact formulation as

$$
\left[\begin{array}{ccc}
\mathbf{M} & \mathbf{0} & \mathbf{C}_{\mathbf{q}}^{T}  \tag{4.17}\\
\mathbf{0} & \mathbf{0} & \mathbf{C}_{\mathbf{s}}^{T} \\
\mathbf{C}_{\mathbf{q}} & \mathbf{C}_{\mathbf{s}} & \mathbf{0}
\end{array}\right]\left[\begin{array}{c}
\ddot{\mathbf{q}} \\
\ddot{\mathbf{s}} \\
\lambda
\end{array}\right]=\left[\begin{array}{c}
\mathbf{Q}_{e} \\
\mathbf{0} \\
\mathbf{Q}_{c}
\end{array}\right]
$$

These augmented forms of the equation of motion are solved for the generalized and, where applicable, the non-generalized accelerations as well as the Lagrange multipliers. In this chapter, the explicit Adams-Bashforth predictor-corrector numerical integration scheme (Shampine and Gordon, 1975) is used to find the independent coordinates and velocities. The dependent coordinates and velocities are computed using the Newton-Raphson algorithm.


Figure 40 Suspended Wheelset Model

### 4.9 Numerical Example

In this section, a simple example is presented to demonstrate the surface modeling techniques discussed in this chapter. A suspended wheelset traveling at a constant velocity over a partial left hand turnout is chosen as an idealized scenario in which a railroad vehicle may encounter a rail with a variable profile. The numerical simulations are carried out for three different scenarios that correspond to the three surface types (curve network representation, low order interpolation, and high order interpolation) discussed in throughout chapter. The results obtained using the three different surface types are compared. The simulations are carried out using the on-line nonconformal elastic contact formulation (ECF-A) implemented in the general purpose multibody package SAMS/2000 (Shabana, 2010).

### 4.9.1 Simulation Parameters

The suspended wheelset used in this example is composed of a single wheelset and a frame connected by linear spring-damper elements as is shown in Fig. 40. The stiffness and damping of the suspension components are defined as $k_{l 1}=k_{l 2}=13,499.36 \mathrm{~N} / \mathrm{m}(925 \mathrm{lbf} / \mathrm{ft}), k_{t 1}=k_{t 2}=$
$24,999.36 \mathrm{~N} / \mathrm{m}(1713 \mathrm{lbf} / \mathrm{ft}), c_{l 1}=c_{l 2}=999.68 \mathrm{~N} \cdot \mathrm{~s} / \mathrm{m}(68.5 \mathrm{lbf} \cdot \mathrm{s} / \mathrm{ft})$, and $c_{t 1}=c_{t 2}=999.68$ $\mathrm{N} \cdot \mathrm{s} / \mathrm{m}(68.5 \mathrm{lbf} \cdot \mathrm{s} / \mathrm{ft})$. A constant velocity constraint is applied to the frame with a value of 4.4704 $\mathrm{m} / \mathrm{s}(10 \mathrm{mph})$ to simulate the vehicle motion while a load of $97,860.88 \mathrm{~N}(22,000 \mathrm{lbs})$ is applied to the wheelset to simulate the vehicle weight. The frame has a mass of $9,999.74 \mathrm{~kg}$ ( 685.2 slug), with roll, pitch, and yaw mass moments of inertia defined as $1,799.03,1,799.03$, and 2,499.96 $\mathrm{kg} \cdot \mathrm{m}^{2}\left(1,326.9,1,326.9\right.$, and $1,807.0$ slug. $\left.\cdot \mathrm{ft}^{2}\right)$ respectively; while the wheelset has a mass of $1,567.39 \mathrm{~kg}$ ( 107.4 slug), with roll, pitch, and yaw mass moments of inertia defined as 655.94 , 167.99, and $655.94 \mathrm{~kg} \cdot \mathrm{~m}^{2}\left(483.8,123.9\right.$, and $\left.483.8 \mathrm{slug} \cdot \mathrm{ft}^{2}\right)$ respectively. The wheel profile used in the example is the AAR-1B-WF which is positioned such that a flange clearance of 7.391 mm ( 0.291 inches) is maintained in the equilibrium position.

A partial turnout is considered for the track model; the components included are the left and right stock rails, the left tongue rail, and lead rail. For simplicity of the analysis, the guard, frog, and right tongue rail sections are not included in the model. Each of the three geometric models is created from the same set of rail profiles. The left rail is modeled using 36 profiles for the stock rail and 28 profiles for the tongue and lead rails while the right rail is modeled using 36 profiles. These profiles, which are formed using between 400 and 500 discrete points, were developed based on CAD drawings provided by Cleveland Track Materials (CTM). The stock profile of the rail used in this model is the 136 RE , while the tongue and lead rails belong to a No. 9 left hand turnout typically used in yards with a maximum speed rating of $6.67056 \mathrm{~m} / \mathrm{s}$ ( 15 $\mathrm{mph})$. The rails are positioned such that the track gage is 1.4351 m ( 56.5 inches) and the tongue rail is located at 6.858 m ( 270 inches) along the left stock rail.

The linear interpolation surface is generated by direct interpolation of the aforementioned profiles. It is important to note that the profiles of the left stock and tongue rails are combined


Figure 41 ANCF Quintic Thin Plate Turnout
into a single curve due to the inability of the direct linear interpolation method to capture the $C^{-1}$ boundary between the stock and tongue rails. The ANCF thin plate meshes were generated by extracting the nodal coordinates from a B-spline surface created by the NURBS package SISL (SINTEF ICT, 2005). A total of 12,500 thin plate elements are used to model the left rail, while 10,000 are used to model the right rail. Note that fewer discrete nodal points are used in the ANCF models when compared with the direct interpolation method although more coordinates are used in the ANCF model considering that 8 spatial derivative vectors are required at each discrete nodal point in addition to the position vector. Each rail is modeled as a separate surface with a unique parametric domain. Additionally, the method presented in Section 4.5 was used to insert a $C^{-1}$ boundary into the mesh of the left rail between the stock and tongue rails for both types of ANCF thin plate meshes. Figure 41 shows the geometry of the turnout as produced by the quintic ANCF thin plate mesh.

### 4.9.2 Numerical Results

Among the three models, the fastest is the linear interpolation method which requires 5 minutes and 8 seconds of CPU time on a personal computer using serial computations, while the cubic ANCF method required 5 minutes and 27 seconds, and the quintic ANCF method required 6 minutes and 22 seconds. While the linear interpolation method is faster, the improved accuracy of the quintic ANCF method far overweighs the additional CPU time it requires, as will be demonstrated by the numerical results. The best agreement is found in the location of the contact point. In Fig. 42, it is shown that the difference in the computed lateral position of the contact point is negligible between the cubic and quintic ANCF thin plate models while the discrepancies are more pronounced when compared to the linear profile interpolation method. Note that the large shift in the location of the contact point at 7.9502 m (313 inches) corresponds to the time at which the contact point switches from the stock rail to the tongue rail. A similar phenomenon can be seen in the plot of the vertical position of the contact point shown in Fig. 43. As the contact point transitions from the stock rail to the tongue rail there is a small vertical shift downward. Following this, the contact point shifts vertically by $6.35 \mathrm{E}-3 \mathrm{~m}(0.25 \mathrm{in})$ as per the design of the tongue rail which includes this elevation increase. It is also important to note here the linear nature of the change in the vertical position of the contact point in the case of the direct profile interpolation method. Recall that linear interpolation is used in the longitudinal interpolation between any 2 profiles, consequently this leads to a linear change in the height of the profile along the rail space curve. This phenomenon is less pronounced in the lateral shift as the individual profiles are described with cubic interpolation. Figure 44 shows the trace of the contact points along the left rail in the proximity of the tongue rail. Here the cause for the lateral


Figure 42 Y Coordinate of Contact Point on Left Rail
(___ Linear Interpolation, _ _ _ Cubic Plate, ___ Quintic Plate)


Figure 43 Z Coordinate of Contact Point on Left Rail
(_-_ Linear Interpolation, _ _ _ Cubic Plate, $\qquad$ Quintic Plate)


Figure 44 Trace of Contact Point Along Quintic ANCF Thin Plate Turnout shift is more pronounced: the contact point shifts laterally to follow the stock rail until such a time that the primary contact point transitions from the stock rail to the tongue rail.

The difference between the three examples is more pronounced when the normal contact forces are compared. In Fig. 45, a comparison is shown for the normal contact force at the left wheel/rail interface between the direct linear interpolation method and the quintic ANCF thin plate mesh. Here it can be seen that the linear interpolation method produces fictitious spikes in the forces which is certainly an undesirable and unrealistic feature. However, the trend line of the linear interpolation method follows the same path as the quintic plate between these fictitious spikes. The contact forces are far more similar when the cubic and quintic plates are compared as is shown in Fig. 46. However, it is clear that some small fictitious force spikes are still predicted in the case of the cubic ANCF thin plate. Note the abrupt change in the normal force at 7.9052 m (313 in), this is the location at which the contact shifts from the stock rail to the switch point.


Figure 45 Normal Force at Left Contact: Linear Interpolation Vs. ANCF Quintic Plate (_- - Linear Interpolation, $\qquad$ Quintic Plate)


Figure 46 Normal Force at Left Contact: ANCF Cubic Plate Vs. ANCF Quintic Plate (___ Cubic Plate, $\qquad$ Quintic Plate)

This transition does cause some physical disturbance in the wheel/rail interaction forces. Note that this change in the forces is small due to the idealized nature of the suspended wheelset model.

### 4.10 Concluding Remarks

In this chapter, three different methods that define variable profile surface geometry are presented. In the first method, a linear interpolation between two adjacent profiles is used to define the surface between the profiles. As a consequence, fictitious spikes in the contact forces are produced due to both first and second order spatial derivative discontinuities which are unavoidable with this method. In the second method, a surface mesh is produced using a collection of cubic ANCF thin plate elements. This method shows marked improvement over the direct profile interpolation method, however some small fictitious spikes in the contact forces are predicted due to second order spatial derivative discontinuities at the element boundaries. In the third method, a surface mesh is produced using a series of the newly developed quintic ANCF thin plate elements. This element has natural $C^{2}$ continuity and as a result does not produce the fictitious spikes in the contact forces that result from spatial derivative discontinuities when used in combination with the on-line ECF-A approach. It was shown that a linear transformation may be used to convert this quintic plate element to a quintic Bezier patch. This allows for a simple conversion from CAD geometry to the surface geometry used in the contact evaluation procedure. Since this quintic plate element does not rely on geometry lofted along a curve, a surface of arbitrary shape may easily be created using this element type. This would allow, for example, irregular terrain geometry for use in the simulation of contact for various types of vehicles to be easily developed.

With regard to the simulation of railroad vehicles on variable profile rails, it was shown that the linear interpolation method produces reasonable accuracy in predicting the location of the contact point when combined with ECF-A. For such analyses that are not highly concerned with the contact forces, this method is ideal due to the simplicity of model creation. The cubic ANCF thin plate model produces nearly identical results at the position level when compared with the quintic ANCF thin plate model; the only discrepancy is related to some small fictitious spikes in the normal contact forces. Taking into consideration that model construction and implementation is nearly identical for the two types of ANCF thin plate elements and the small difference in the CPU time, it is advisable to choose the quintic plate in place of the cubic plate as the increased accuracy in the force prediction outweighs the slight increase in computational time required when the quintic ANCF thin plate surface is chosen.

## CHAPTER 5

## CONCLUSIONS

The main contributions of this thesis are focused on the development of alternative methods for modeling track elasticity and geometry. This was accomplished by introducing two new methods. In the first method, a simplified approach for modeling the flexibility of the track and moveable rail segments in railroad vehicle dynamics simulations was introduced. This simplified approach is not, however, without limitations as was discussed in Chapters 2 and 3. The second method introduced is a detailed approach to modeling the geometry of contact surfaces. This method was applied to the simulation of vehicle/track interaction during the negotiation of a simplified model of a turnout. While requiring more CPU time than existing methods, the improved accuracy in the prediction of the contact forces demonstrates that this method is a viable alternative to the existing procedures as was demonstrated in Chapter 4.

In Chapter 2 a computational method, based on the finite segment (FS) approach, which can be used to model track structure and rail movements in railroad vehicle dynamics was introduced. In order to avoid distortion of the geometry during the rail movements, absolute nodal coordinate formulation (ANCF) finite elements are used to interpolate the rail space curve geometric properties. It was shown that a systematic method can be developed in which this FS model can be created with limited user input. It is clear from the presented numerical results that this approach is capable of modeling some useful simulation scenarios such as gage widening and rail rollover. The FS approach can also be applied to more complex models pertaining to the study of broken rails, analysis of the effect of missing ties, and excitation due to bridge or seismic motion, provided the appropriate constraints and external forces are applied and an adequate number of finite segments are used.

The use of the finite segment rail model can introduce some discontinuities in the rail surface that cause erroneous spikes in the predicted contact forces. This problem can be avoided if the finite element method (FEM) is used in place of the FS method, as was demonstrated in Chapter 3. A second limitation of the proposed FS method is the failure to capture the change in the longitudinal rail arc length as the result of the FS longitudinal motion.

In Chapter 3, the FS approach presented in Chapter 2 was adapted to model track flexibility. In order to evaluate the performance of the finite segment method (FSM) in this context, several procedures which aim to deal with flexible railroad tracks have been applied to a simple track model in which only one stretch in one rail is considered flexible. In addition to the FSM which does not employ component modes, two other procedures are used in the comparative numerical study presented Chapter 3. The first is the floating frame of reference (FFR) formulation and the second is the combined FFR/FS method, the latter allows for developing a track model that consists of rigid segments using a finite element (FE) preprocessor. Remarkable agreement among the three methods used has been found in terms of deformation and the dynamic performance of the models.

The geometric description of the FSM represents the major drawback of this approach in contact applications. Removing the spatial discontinuities between finite segments due to shear deformation is not enough to ensure smooth results in contact forces. Continuity in all second order spatial derivatives of the contact surface is necessary for a correct application of the on-line elastic contact formulation, was discussed in Chapter 4. In other words, concentrating the deformation as in the FSM jeopardizes the effectiveness of the contact method used. This issue is partially solved by using a large number of finite segments; however this may cause an additional shortcoming of the method by introducing fictitious excitations to the system that may
negatively impact the accuracy of the results. Due to these limitations, it is recommended that the use of the FSM be limited to simple simulation scenarios or to studies concerned with gage widening, rail segment movements, and deformation analysis. This is largely due to the fact that the forces predicted by the FS method are somewhat lacking in accuracy.

In Chapter 4, three different methods that define variable profile surface geometry for the three-dimensional non-conformal elastic contact formulation were presented. In the first method, a linear interpolation between two profiles is used to define the surface between the profiles. As a consequence, fictitious spikes in the contact forces are produced due to both first and second order spatial derivative discontinuities which are unavoidable with this method. In the second method, a FE mesh is produced to model the surface using a collection of cubic ANCF thin plate elements. This method shows marked improvement over the direct profile interpolation method, however some small fictitious spikes in the contact forces are predicted due to second order spatial derivative discontinuities at the element boundaries. In the third method, a FE mesh is produced to model the surface using a series of the newly developed quintic ANCF thin plate elements. This element has second order spatial derivative continuity and as a result does not produce fictitious spikes in the contact forces when used in combination with the elastic contact formulation - algebraic equations (ECF-A). It was shown that a linear transformation may be used to convert this quintic plate element to a quintic Bezier patch. This allows for a simple conversion from computer-aided design (CAD) geometry to the surface geometry used in the contact evaluation procedure. Since this quintic plate element does not rely on geometry lofted along a curve, a surface of arbitrary shape may easily be created using this element type. This would allow, for example, irregular terrain geometry for use in the simulation of contact for various types of vehicles to be easily developed.

With regard to the simulation of railroad vehicles in contact with variable profile rails, it was shown that the linear interpolation method produces reasonable accuracy in the prediction of the location of the contact point when combined with the on-line ECF-A approach. For such analyses that are not highly concerned with the contact forces, this method is ideal due to the simplicity of model creation and overall efficiency of the method. The cubic ANCF thin plate model produces nearly identical results at the position level when compared with the quintic ANCF thin plate model; the only discrepancy is related to some small fictitious spikes in the normal contact forces with the cubic ANCF thin plate model. Taking into consideration that model construction and implementation is nearly identical for the two types of ANCF thin plate elements, and the small difference in the computational efficiency, it is advisable to choose the quintic ANCF thin plate in place of the cubic ANCF thin plate as the increased accuracy in the force prediction outweighs the slight increase in the computational time required when the quintic ANCF thin plate surface is chosen.

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## APPENDIX C

In this appendix, the shape functions of the ANCF finite plate elements considered in this thesis are presented. Also the ANCF/Bezier mapping matrices introduced in Section 4.6 are presented.

The shape function matrix of the third order ANCF plate element presented in Section 4.3 is defined as (Mikkola et al., 2012)

$$
\begin{align*}
& \mathbf{S}=\left[\begin{array}{llllllll}
\mathrm{S}_{1} \mathbf{I} & \mathrm{~S}_{2} \mathbf{I} & \mathrm{~S}_{3} \mathbf{I} & \mathrm{~S}_{4} \mathbf{I} & \mathrm{~S}_{5} \mathbf{I} & \mathrm{~S}_{6} \mathbf{I} & \mathrm{~S}_{7} \mathbf{I} & \mathrm{~S}_{8} \mathbf{I}
\end{array}\right. \\
& \begin{array}{llllllll}
\mathrm{S}_{9} \mathbf{I} & \mathrm{~S}_{10} \mathbf{I} & \mathrm{~S}_{11} \mathbf{I} & \mathrm{~S}_{12} \mathbf{I} & \mathrm{~S}_{13} \mathbf{I} & \mathrm{~S}_{14} \mathbf{I} & \mathrm{~S}_{15} \mathbf{I} & \left.\mathrm{~S}_{16} \mathbf{I}\right]
\end{array} \tag{A.1}
\end{align*}
$$

where $\mathbf{I}$ is a $3 \times 3$ identity matrix,

$$
\left.\begin{array}{cccc}
\mathrm{S}_{1}=S_{\xi 1} S_{\eta 1} & \mathrm{~S}_{2}=S_{\xi 2} S_{\eta 1} & \mathrm{~S}_{3}=S_{\xi 1} S_{\eta 2} & \mathrm{~S}_{4}=S_{\xi 2} S_{\eta 2}  \tag{A.2}\\
\mathrm{~S}_{5}=S_{\xi 3} S_{\eta 1} & \mathrm{~S}_{6}=S_{\xi 4} S_{\eta 1} & \mathrm{~S}_{7}=S_{\xi 3} S_{\eta 2} & \mathrm{~S}_{8}=S_{\xi 4} S_{\eta 2} \\
\mathrm{~S}_{9}=S_{\xi 3} S_{\eta 3} & \mathrm{~S}_{10}=S_{\xi 4} S_{\eta 3} & \mathrm{~S}_{11}=S_{\xi 3} S_{\eta 4} & \mathrm{~S}_{12}=S_{\xi 4} S_{\eta 4} \\
\mathrm{~S}_{13}=S_{\xi 1} S_{\eta 3} & \mathrm{~S}_{14}=S_{\xi 2} S_{\eta 3} & \mathrm{~S}_{15}=S_{\xi 1} S_{\eta 4} & \mathrm{~S}_{16}=S_{\xi 2} S_{\eta 4}
\end{array}\right\}
$$

and

$$
\left.\begin{array}{cc}
\mathrm{S}_{\chi 1}=2 \chi^{3}-3 \chi^{2}+1 & \mathrm{~S}_{\chi 2}=\alpha\left(\chi^{3}-2 \chi^{2}+\chi\right)  \tag{A.3}\\
\mathrm{S}_{\chi 3}=-2 \chi^{3}+3 \chi^{2} & \mathrm{~S}_{\chi 4}=\alpha\left(\chi^{3}-\chi^{2}\right) \\
\alpha=A \text { for } \chi=\xi & \alpha=B \text { for } \chi=\eta
\end{array}\right\}
$$

The shape function of the fifth order ANCF plate element presented in Section 4.4 is defined as

$$
\mathbf{S}=\begin{array}{cccccccccccc}
{\left[\begin{array}{c}
\mathrm{S}_{1} \mathbf{I} \\
\mathrm{~S}_{13} \mathbf{I}
\end{array}\right.} & \mathrm{S}_{2} \mathbf{I} & \mathrm{~S}_{3} \mathbf{I} & \mathrm{~S}_{4} \mathbf{I} & \mathrm{~S}_{5} \mathbf{I} & \mathrm{~S}_{6} \mathbf{I} & \mathrm{~S}_{7} \mathbf{I} & \mathrm{~S}_{8} \mathbf{I} & \mathrm{~S}_{9} \mathbf{I} & \mathrm{~S}_{10} \mathbf{I} & \mathrm{~S}_{11} \mathbf{I} & \mathrm{~S}_{12} \mathbf{I} \\
\mathrm{~S}_{17} \mathbf{I} & \mathrm{~S}_{18} \mathbf{I} & \mathrm{~S}_{19} \mathbf{I} & \mathrm{~S}_{20} \mathbf{I} & \mathrm{~S}_{21} \mathbf{I} & \mathrm{~S}_{22} \mathbf{I} & \mathrm{~S}_{23} \mathbf{I} & \mathrm{~S}_{24} \mathbf{I}  \tag{A.4}\\
\mathrm{~S}_{25} \mathbf{I} & \mathrm{~S}_{26} \mathbf{I} & \mathrm{~S}_{27} \mathbf{I} & \mathrm{~S}_{28} \mathbf{I} & \mathrm{~S}_{29} \mathbf{I} & \mathrm{~S}_{30} \mathbf{I} & \mathrm{~S}_{31} \mathbf{I} & \mathrm{~S}_{32} \mathbf{I} & \mathrm{~S}_{33} \mathbf{I} & \mathrm{~S}_{34} \mathbf{I} & \mathrm{~S}_{35} \mathbf{I} & \left.\mathrm{~S}_{36} \mathbf{I}\right]
\end{array}
$$

where

$$
\begin{align*}
& \mathrm{S}_{1}=S_{\xi 1} S_{\eta 1} \quad \mathrm{~S}_{2}=S_{\xi 2} S_{\eta 1} \quad \mathrm{~S}_{3}=S_{\xi 1} S_{\eta 2} \quad \mathrm{~S}_{4}=S_{\xi 3} S_{\eta 1} \quad \mathrm{~S}_{5}=S_{\xi 2} S_{\eta 2} \quad \mathrm{~S}_{6}=S_{\xi 1} S_{\eta 3} \\
& \mathrm{~S}_{7}=S_{\xi 3} S_{\eta 2} \quad \mathrm{~S}_{8}=S_{\xi 2} S_{\eta 3} \quad \mathrm{~S}_{9}=S_{\xi 3} S_{\eta 3} \quad \mathrm{~S}_{10}=S_{\xi 4} S_{\eta 1} \quad \mathrm{~S}_{11}=S_{\xi 5} S_{\eta 1} \quad \mathrm{~S}_{12}=S_{\xi 4} S_{\eta 2} \\
& \mathrm{~S}_{13}=S_{\xi 6} S_{\eta 1} \quad \mathrm{~S}_{14}=S_{\xi 5} S_{\eta 2} \quad \mathrm{~S}_{15}=S_{\xi 4} S_{\eta 3} \quad \mathrm{~S}_{16}=S_{\xi 6} S_{\eta 2} \quad \mathrm{~S}_{17}=S_{\xi 5} S_{\eta 3} \quad \mathrm{~S}_{18}=S_{\xi 6} S_{\eta 3} \\
& \mathrm{~S}_{19}=S_{\xi 4} S_{\eta 4} \quad \mathrm{~S}_{20}=S_{\xi 5} S_{\eta 4} \quad \mathrm{~S}_{21}=S_{\xi 4} S_{\eta 5} \quad \mathrm{~S}_{22}=S_{\xi 6} S_{\eta 4} \quad \mathrm{~S}_{23}=S_{\xi 5} S_{\eta 5} \quad \mathrm{~S}_{24}=S_{\xi 4} S_{\eta 6}  \tag{A.5}\\
& \mathrm{~S}_{25}=S_{\xi 6} S_{\eta 5} \quad \mathrm{~S}_{26}=S_{\xi 5} S_{\eta 6} \quad \mathrm{~S}_{27}=S_{\xi 6} S_{\eta 6} \quad \mathrm{~S}_{28}=S_{\xi 1} S_{\eta 4} \quad \mathrm{~S}_{29}=S_{\xi 2} S_{\eta 4} \quad \mathrm{~S}_{30}=S_{\xi 1} S_{\eta 5} \\
& \mathrm{~S}_{31}=S_{\xi 3} S_{\eta 4} \quad \mathrm{~S}_{32}=S_{\xi 2} S_{\eta 5} \quad \mathrm{~S}_{33}=S_{\xi 1} S_{\eta 6} \quad \mathrm{~S}_{34}=S_{\xi 3} S_{\eta 5} \quad \mathrm{~S}_{35}=S_{\xi 2} S_{\eta 6} \quad \mathrm{~S}_{36}=S_{\xi 3} S_{\eta 6}
\end{align*}
$$

## APPENDIX C (continued)

and

$$
\left.\begin{array}{ccc}
S_{\chi^{1}}=(\chi-1)^{3}\left(6 \chi^{2}+3 \chi+1\right) & S_{\chi^{2}}=\alpha \chi(\chi-1)^{3}(3 \chi+1) & S_{\chi^{3}}=\frac{\alpha^{2}}{2} \chi^{2}(\chi-1)^{3} \\
S_{\chi 4}=-\chi^{3}\left(6 \chi^{2}-15 \chi+10\right) & S_{\chi^{5}}=\alpha \chi^{3}\left(3 \chi^{2}-7 \chi+4\right) & S_{\chi^{6}}=-\frac{\alpha^{2}}{2} \chi^{3}(\chi-1)^{2}  \tag{A.6}\\
\alpha=A \text { for } \chi=\xi & \alpha=B \text { for } \chi=\eta
\end{array}\right\}
$$

The ANCF/Bezier mapping matrices of Section 4.6 are

$$
\begin{align*}
& \left.\mathbf{P}_{1}=\left[\begin{array}{lllllllll}
\mathbf{P}_{0,0} & \mathbf{P}_{0,1} & \mathbf{P}_{0,2} & \mathbf{P}_{1,0} & \mathbf{P}_{1,1} & \mathbf{P}_{1,2} & \mathbf{P}_{2,0} & \mathbf{P}_{2,1} & \mathbf{P}_{2,2}
\end{array}\right]^{\mathbf{T}}\right) \\
& \mathbf{P}_{2}=\left[\begin{array}{lllllllll}
\mathbf{P}_{5,0} & \mathbf{P}_{5,1} & \mathbf{P}_{5,2} & \mathbf{P}_{4,0} & \mathbf{P}_{4,1} & \mathbf{P}_{4,2} & \mathbf{P}_{3,0} & \mathbf{P}_{3,1} & \mathbf{P}_{3,2}
\end{array}\right]^{\mathrm{T}} \\
& \left.\mathbf{P}_{3}=\left[\begin{array}{lllllllll}
\mathbf{P}_{5,5} & \mathbf{P}_{5,4} & \mathbf{P}_{5,3} & \mathbf{P}_{4,5} & \mathbf{P}_{4,4} & \mathbf{P}_{4,3} & \mathbf{P}_{3,5} & \mathbf{P}_{3,4} & \mathbf{P}_{3,3}
\end{array}\right]^{\mathrm{T}}\right\}  \tag{A.7}\\
& \left.\mathbf{P}_{4}=\left[\begin{array}{lllllllll}
\mathbf{P}_{0,5} & \mathbf{P}_{0,4} & \mathbf{P}_{0,3} & \mathbf{P}_{1,5} & \mathbf{P}_{1,4} & \mathbf{P}_{1,3} & \mathbf{P}_{2,5} & \mathbf{P}_{2,4} & \mathbf{P}_{2,3}
\end{array}\right]^{\mathrm{T}}\right] \\
& \mathbf{N}_{1}=\left[\begin{array}{lllll}
\mathbf{N}_{\xi, 0} \mathbf{N}_{\eta, 0} & \mathbf{N}_{\xi, 0} \mathbf{N}_{\eta, 1} & \mathbf{N}_{\xi, 0} \mathbf{N}_{\eta, 2} & \mathbf{N}_{\xi, 1} \mathbf{N}_{\eta, 0} & \mathbf{N}_{\xi, 1} \mathbf{N}_{\eta, 1}
\end{array}\right. \\
& \left.\mathbf{N}_{\xi, 1} \mathbf{N}_{\eta, 2} \quad \mathbf{N}_{\xi, 2} \mathbf{N}_{\eta, 0} \quad \mathbf{N}_{\xi, 2} \mathbf{N}_{\eta, 1} \quad \mathbf{N}_{\xi, 2} \mathbf{N}_{\eta, 2}\right] \\
& \mathbf{N}_{2}=\left[\begin{array}{llllll}
\mathbf{N}_{\xi, 5} \mathbf{N}_{\eta, 0} & \mathbf{N}_{\xi, 5} \mathbf{N}_{\eta, 1} & \mathbf{N}_{\xi, 5} \mathbf{N}_{\eta, 2} & \mathbf{N}_{\xi, 4} \mathbf{N}_{\eta, 0} & \mathbf{N}_{\xi, 4} \mathbf{N}_{\eta, 1}
\end{array}\right. \\
& \left.\begin{array}{lllll}
\mathbf{N}_{\xi, 4} \mathbf{N}_{\eta, 2} & \mathbf{N}_{\xi, 3} \mathbf{N}_{\eta, 0} & \mathbf{N}_{\xi, 3} \mathbf{N}_{\eta, 1} & \mathbf{N}_{\xi, 3} \mathbf{N}_{\eta, 2}
\end{array}\right] \\
& \mathbf{N}_{3}=\left[\begin{array}{llllll}
\mathbf{N}_{\xi, 5} \mathbf{N}_{\eta, 5} & \mathbf{N}_{\xi, 5} \mathbf{N}_{\eta, 4} & \mathbf{N}_{\xi, 5} \mathbf{N}_{\eta, 3} & \mathbf{N}_{\xi, 4} \mathbf{N}_{\eta, 5} & \mathbf{N}_{\xi, 4} \mathbf{N}_{\eta, 4}
\end{array}\right.  \tag{A.8}\\
& \left.\mathbf{N}_{\xi, 4} \mathbf{N}_{\eta, 3} \quad \mathbf{N}_{\xi, 3} \mathbf{N}_{\eta, 5} \quad \mathbf{N}_{\xi, 3} \mathbf{N}_{\eta, 4} \quad \mathbf{N}_{\xi, 3} \mathbf{N}_{\eta, 3}\right] \\
& \mathbf{N}_{4}=\left[\begin{array}{llllll}
\mathbf{N}_{\xi, 0} \mathbf{N}_{\eta, 5} & \mathbf{N}_{\xi, 0} \mathbf{N}_{\eta, 4} & \mathbf{N}_{\xi, 0} \mathbf{N}_{\eta, 3} & \mathbf{N}_{\xi, 1} \mathbf{N}_{\eta, 5} & \mathbf{N}_{\xi, 1} \mathbf{N}_{\eta, 4}
\end{array}\right. \\
& \left.\mathbf{N}_{\xi, 1} \mathbf{N}_{\eta, 3} \quad \mathbf{N}_{\xi, 2} \mathbf{N}_{\eta, 5} \quad \mathbf{N}_{\xi, 2} \mathbf{N}_{\eta, 4} \quad \mathbf{N}_{\xi, 2} \mathbf{N}_{\eta, 3}\right] \\
& \begin{array}{l}
\mathbf{E}_{1}=\left[\begin{array}{lcccccccc}
\mathbf{r}_{1} & \mathbf{r}_{1, \mathrm{x}} & \mathbf{r}_{1, \mathrm{y}} & \mathbf{r}_{1, \mathrm{xx}} & \mathbf{r}_{1, \mathrm{xy}} & \mathbf{r}_{1, \mathrm{yy}} & \mathbf{r}_{1, \mathrm{xxy}} & \mathbf{r}_{1, \mathrm{xyy}} & \mathbf{r}_{1, \mathrm{xxyy}}
\end{array}\right]^{\mathrm{T}} \\
\mathbf{E}_{2}=\left[\begin{array}{llll}
\mathbf{r}_{2} & -\mathbf{r}_{2, \mathrm{x}} & \mathbf{r}_{2, \mathrm{y}} & \mathbf{r}_{2, \mathrm{xx}} \\
-\mathbf{r}_{2, \mathrm{xy}} & \mathbf{r}_{2, \mathrm{yy}} & \mathbf{r}_{2, \mathrm{xxy}} & -\mathbf{r}_{2, \mathrm{xyy}} \\
\mathbf{r}_{2, \mathrm{xxyy}}
\end{array}\right]^{\mathrm{T}} \\
\mathbf{E}_{3}=\left[\begin{array}{lll}
\mathbf{r}_{3} & -\mathbf{r}_{3, \mathrm{x}} & -\mathbf{r}_{3, \mathrm{y}} \\
\mathbf{r}_{3, \mathrm{xx}} & \mathbf{r}_{3, \mathrm{xy}} & \mathbf{r}_{3, \mathrm{yy}} \\
-\mathbf{r}_{3, \mathrm{xxy}} & -\mathbf{r}_{3, \mathrm{xyy}} & \mathbf{r}_{3, \mathrm{xxyy}}
\end{array}\right]^{\mathrm{T}} \\
\mathbf{E}_{4}=\left[\begin{array}{lll}
\mathbf{r}_{4} & \mathbf{r}_{4, \mathrm{x}} & -\mathbf{r}_{4, \mathrm{y}} \\
\mathbf{r}_{4, \mathrm{xx}} & -\mathbf{r}_{4, \mathrm{xy}} & \mathbf{r}_{4, \mathrm{yy}} \\
-\mathbf{r}_{4, \mathrm{xxy}} & \mathbf{r}_{4, \mathrm{xyy}} & \mathbf{r}_{4, \mathrm{xxyy}}
\end{array}\right]^{\mathrm{T}}
\end{array}  \tag{A.9}\\
& \left.\begin{array}{l}
\mathbf{S}_{1}=\left[\begin{array}{lcccccccc}
S_{1} & S_{2} & S_{3} & S_{4} & S_{5} & S_{6} & S_{7} & S_{8} & S_{9}
\end{array}\right] \\
\mathbf{S}_{2}=\left[\begin{array}{llllll}
S_{10} & -S_{11} & S_{12} & S_{13} & -S_{14} & S_{15} \\
S_{16} & -S_{17} & S_{18}
\end{array}\right] \\
\mathbf{S}_{3}=\left[\begin{array}{llllll}
S_{19} & -S_{20} & -S_{21} & S_{22} & S_{23} & S_{24} \\
-S_{25} & -S_{26} & S_{27}
\end{array}\right] \\
\mathbf{S}_{4}=\left[\begin{array}{llll}
S_{28} & S_{29} & -S_{30} & S_{31} \\
-S_{32} & S_{33} & -S_{34} & S_{35} \\
S_{36}
\end{array}\right]
\end{array}\right\} \\
& \text { APPENDIX C (continued) }
\end{align*}
$$

$$
\mathbf{B}=\left[\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \tag{A.11}\\
\frac{-5}{A} & 0 & 0 & \frac{5}{A} & 0 & 0 & 0 & 0 & 0 \\
\frac{-5}{B} & \frac{5}{B} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{20}{A^{2}} & 0 & 0 & \frac{-40}{A^{2}} & 0 & 0 & \frac{20}{A^{2}} & 0 & 0 \\
\frac{25}{A B} & \frac{-25}{A B} & 0 & \frac{-25}{A B} & \frac{25}{A B} & 0 & 0 & 0 & 0 \\
\frac{20}{B^{2}} & \frac{-40}{B^{2}} & \frac{20}{B^{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-100}{A^{2} B} & \frac{100}{A^{2} B} & 0 & \frac{200}{A^{2} B} & \frac{-200}{A^{2} B} & 0 & \frac{-100}{A^{2} B} & \frac{100}{A^{2} B} & 0 \\
\frac{-100}{A B^{2}} & \frac{200}{A B^{2}} & \frac{-100}{A B^{2}} & \frac{100}{A B^{2}} & \frac{-200}{A B^{2}} & \frac{100}{A B^{2}} & 0 & 0 & 0 \\
\frac{400}{A^{2} B^{2}} & \frac{-800}{A^{2} B^{2}} & \frac{400}{A^{2} B^{2}} & \frac{-800}{A^{2} B^{2}} & \frac{1600}{A^{2} B^{2}} & \frac{-800}{A^{2} B^{2}} & \frac{400}{A^{2} B^{2}} & \frac{-800}{A^{2} B^{2}} & \frac{400}{A^{2} B^{2}}
\end{array}\right]
$$

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## PROFESSIONAL EXPERIENCE

Teaching Assistant I Dept. of Mech. Engr., UIC
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- Perform tasks including grading of homework assignments and examinations, validation of the solutions to these assignments provided by faculty, and holding office hours which students attend voluntarily to supplement the material presented in lecture.


## Research Assistant | Dynamic Simulation Laboratory, UIC

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- Participate in the development of the general purpose multi-body dynamics software SAMS/2000 \& railroad industry specific module SAMS/RAIL; formulation of mathematical models of multi-body systems of interconnected rigid and deformable bodies. Knowledge of FORTRAN used extensively in the development and implementation of rail flexibility and wheel/rail contact modules. Provide analysis of the accuracy and effectiveness of software implementations in the SAMS/2000 \& SAMS/RAIL software packages.


## JOURNAL PUBLICATIONS

- Hamper, M. B., Wei, C., Shabana, A. A., "Use of ANCF Surface Geometry in Rigid Body Contact Problems", Submitted to the Journal of Computational and Nonlinear Dynamics, 4 October 2013.
- El-Ghandour, A. I., Hamper, M. B., Foster, C. D.,"Coupled Finite Element and MultiBody Systems Modeling of a 3D Railroad System", Submitted to the Proceedings of the Institution of Mechanical Engineers, Part F: Journal of Rail and Rapid Transit, 10 September 2013.
- Shabana, A. A., Hamper, M. B., O'Shea, J. J., "Rolling Condition and Gyroscopic Moments in Curve Negotiations", Journal of Computational and Nonlinear Dynamics, 8(1), 2013, 011015.
- Hamper, M. B., Zaazaa, K. E., Shabana, A. A., 2012. "Modeling Railroad Track Structures Using the Finite Segment Method", Acta Mechanica, 223(8), 2012.
- Hamper, M. B., Recuero, A. M., Escalona, J. L., Shabana, A. A., "Use of Finite Element and Finite Segment Methods in Modeling Rail Flexibility: A Comparative Study", Journal of Computational and Nonlinear Dynamics, 7(4) 2012, 041007.


## CONFERENCE PUBLICATIONS \& TECHNICAL REPORTS

- Hamper, M. B., Ruppert, C., Wei, C., Shabana, A. A.,"A Spatial Geometry Approach to Variable Cross-section Rail Modeling in Multi-Body Dynamic Simulations", ASME/ASCE/IEEE 2014 Joint Rail Conference, Colorado Springs, CO, USA, 2-4 April 2013.
- Hamper, M. B., Wei, C., Shabana, A. A., "Use of ANCF Surface Geometry in Rigid Body Contact Problems", Technical Report MBS2013-4-UIC, Department of Mechanical Engineering, University of Illinois at Chicago, 2013.
- Shabana, A. A., Hamper, M. B., O'Shea, J. J., "Rolling Condition and Gyroscopic Moments in Curve Negotiations" Proceedings of the ASME 2012 International Design Engineering Technical Conferences \& Computers and Information in Engineering Conference, Chicago, IL, USA, 12-15 August 2012.
- Hamper, M. B., Zaazaa, K. E., Shabana, A. A., "Development of the Finite Segment Method for Modeling Railroad Track Structures" ASME/ASCE/IEEE 2011 Joint Rail Conference, Pueblo, CO, USA, 16-18 March 2011.
- Hamper, M. B., Recuero, A. M., Escalona, J. L., Shabana, A. A., 2011. "Modeling Rail Flexibility Using Finite Element and Finite Segment Methods", ASME/ASCE/IEEE 2011 Joint Rail Conference, Pueblo, CO, USA, 16-18 March 2011.


## CONFERENCE \& SEMINAR PRESENTATION

- Hamper, M. B., 2013. "Variable Cross-Section Rail Modeling In SAMS/2000", NURail/AOE Rail Research Monthly Review, University of Illinois at Chicago, November 5.
- Hamper, M. B., 2013. "Geometry and Wheel Rail Contact During Curve Negotiation", UIC Workshop on Multi-Body System Dynamics, University of Illinois at Chicago, August 10.
- Hamper, M. B., 2102. "Integration of Vehicle/Track Models", NURail Workshop on Vehicle/Track Interaction, University of Illinois at Chicago, September 16.
- Hamper, M. B., 2011. "Development of the Finite Segment Method for Modeling Railroad Track Structures", ASME/ASCE/IEEE 2011 Joint Rail Conference, Pueblo, CO, USA, March 16-18.


## PROFESSIONAL SERVICE

Journal Reviewer:

- Proceedings of the Institution of Mechanical Engineers, Part F: Journal of Rail and Rapid, March 2013-Present
- ASME Journal of Computational and Nonlinear Dynamics March 2014-Present Conference Reviewer:
- ASME IDETC/CIE, January 2014, August 2013, February 2012
- Conference Reviewer, ASME IMECE, June 2012, June 2011

Conference Session Moderator:

- ASME/ASCE/IEEE Joint Rail Conference, April 2014


## COLLABORATIONS

Variable Cross-section Rail Modeling Apr. 2013-May 2014
Conrad Ruppert, UIUC, Urbana, IL
Finite Element Modeling of Railroad Track Structures
Feb. 2012 - May 2014
Craig Foster, UIC, Chicago, IL
Virtual Reality Visualization of Multi-body Dynamics Models Feb. 2012 - Jun. 2013
Jason Leigh, UIC, Chicago, IL
Analysis of Deformable Rail Models
Aug. 2010 - Jan. 2012
José Escalona, University of Seville, Seville, Spain
ACADEMIC HONORS \& AW ARDS

ASME RTD Graduate Student Conference Scholarship Mar. 2014
NURail Student of the Year 2013
ASME RTD Graduate Student Conference Scholarship
Pi Tau Sigma Honor Society, Member
Mar. 2011
Nov. 2008 - Present

## PROFESSIONAL AFFILIATIONS

American Society of Mechanical Engineering (ASME)
RESEARCH INTERESTS
Multi-Body Dynamics
Flexible Body Modeling
Contact Modeling
Surface Reconstruction Techniques
Engineering Graphics and Visualization

