Examples of Isotrivial Elliptic Threefolds over $\mathbb{P}^2$ and Their Discriminants

by

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To my parents

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TABLE OF CONTENTS

CHAPTER **PAGE**

TABLE OF CONTENTS (Continued)

CHAPTER **PAGE**

LIST OF TABLES

LIST OF FIGURES

SUMMARY

Let E_0 and E_1 be the elliptic curves with j-invariant equal to 0 and 1728 respectively. We consider the μ_3-action on E_0 and the μ_4-action E_1. There are two elliptic fibrations $E_0^2/\mu_3 \to \mathbb{P}^1$ and $E_1^2/\mu_4 \to \mathbb{P}^1$. We found the relative minimal models S_0 and S_1 to each elliptic fibration respectively. S_0 and S_1 have a singular fiber configuration $\{IV, IV, IV\}$ and $\{III, III, I_0^*\}$ respectively. Moreover, we showed that a minimal elliptic surface with singular fiber configuration $\{IV, IV, IV\}$ or $\{III, III, I_0^*\}$ is unique up to isomorphisms between elliptic surfaces. We found pencils of cubics that induce S_0 and S_1. We also calculate the Mordell-Weil group of $S_0 \to \mathbb{P}^1$ by applying Shioda-Tate formula.

We further studied the elliptic fibrations $E_0^3/\mu_3 \to E_0^2/\mu_3$ and $E_1^3/\mu_4 \to E_1^2/\mu_4$. We constructed smooth birational models to each elliptic fibration, $X \to \mathbb{P}^2$ and $Y \to \mathbb{P}^2$. We identified their discriminant loci and studied their singular fibers. Based on the discriminant locus of $X \to \mathbb{P}^2$, we calculated the Mordell-Weil rank of the elliptic fibration $E_0^3/\mu_3 \to E_0^2/\mu_3$.

CHAPTER 1

INTRODUCTION

This paper mainly focuses on studying elliptic threefolds over rational surfaces, in particular over $\mathbb{P}^2$. The main approach is to study the discriminant locus of an elliptic fibration and find which geometric and topological invariants of an elliptic threefold can be determined from the geometric and topological properties of its discriminant locus.

Elliptic threefolds have been studied for decades. Some major results include the following: R.Miranda constructed a smooth and equidimensional model for any elliptic Weierstrass threefold (Miranda, 1983). If X and S are smooth and the discriminant locus $\Sigma_{X/S}$ is a simple normal crossing divisor, Kawamata showed that the elliptic modular function $J : S \to \mathbb{P}^1$ is a morphism and $\pi_*(K_{X/S})$ is an invertible sheaf. Kawamata also gave a formula of $\pi_*(K_{X/S})$ in terms of $\Sigma_{X/S}$ and J (Kawamata, 1983). Again assuming $\Sigma_{X/S}$ is a simple normal crossing divisor, Fujita had a formula for the canonical bundle K_X (Fujita, 1986). Also, A.Grassi discussed the notion of relative minimal model of an elliptic threefold (Grassi, 1991). Let $\pi : X \to S$ be an elliptic threefold which is not uniruled. Grassi showed that there is a birational equivalent fibration $\bar{\pi} : \bar{X} \to \bar{S}$ such that $\bar{X}$ has at worst terminal singularities and $\bar{S}$ has worst log-terminal singularities. Furthermore, the canonical bundle $K_{\bar{X}}$ is nef and a pullback from $\bar{S}$.

In our work, we don't make the assumption that the discriminant locus $\Sigma_{X/S}$ is a simple normal crossing divisor. In fact, the invariants of discriminant locus that we consider depend on the complexity of singularities of $\Sigma_{X/S}$.

We construct the following models. Let E_0 be the elliptic curve with j-invariant equal to 0. Consider the diagonal μ_3-action on E_0^3. The quotient space E_0^3/μ_3 is a Calabi-Yau threefold with terminal singularities. It has been studied in several aspects. In my research, I focus on the elliptic fibration of $E_0^3/\mu_3 \to E_0^2/\mu_3$. We construct a smooth model X of E_0^3/μ_3. There is an isotivial elliptic fibration $\pi : X \to \mathbb{P}^2$ with generic fiber isomorphic to E_0. We also show that the discriminant locus $\Sigma_{X/\mathbb{P}^2}$ is a dual Hesse arrangement in $\mathbb{P}^2$, which is the set of 9 lines and 12 multiple points of order 3. We study the singular fibers of $\pi : X \to \mathbb{P}^2$ over a smooth point and over a triple point of $\Sigma_{X/\mathbb{P}^2}$. The Mordell-Weil rank of an elliptic fibration over a rational surface is related to the Alexander polynomial of the discriminant locus by the work of Cogolludo and Libgober (Cogolludo-Agustín and Libgober, 2014). In this way, we can determine the Mordell-Weil rank of X.

For another model, let E_1 be the elliptic curve with j-invariant equal to 1728 and consider the diagonal μ_4-action on E_1^3, we find that the quotient space E_1^3/μ_4 has a smooth model Y that admits two isotrivial elliptic fibrations to the base $\mathbb{P}^2$. The discriminant loci of the two fibrations are the images of Cremona transform of each other. The two discriminant loci contain different types of singularities, such as tacodes and cusps. We will analyze the singular fibers over the singularities of the loci. Our main goal is to build connections between the invariants of elliptic threefolds to the invariants of their discriminant loci.

1.1 A Smooth Elliptic Threefold Birational to E_0^3/μ_3

Let E_0 be the elliptic curve with j-invariant equal to 0 and g be the automorphism of order 3 that generates a cyclic group μ_3. We consider the diagonal μ_3-action on the surface E_0^2. The projection

$\pi : E_0^2 \to E_0$ to the first component is equivariant with respect to the μ_3-actions. Therefore, we have the following diagram:

$$\begin{array}{ccc} E_0^2 & \xrightarrow{q_2} & E_0^2/\mu_3 \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow \\ E_0 & \xrightarrow{q_1} & E_0/\mu_3 \cong \mathbb{P}^1 \end{array}$$

where q_2 and q_1 are quotient maps with respect to the μ_3-actions. The morphism $E_0^2/\mu_3 \to \mathbb{P}^1$ has a general fiber isomorphic to E_0. To resolve the singularities of E_0^2/μ_3, we blow up the μ_3-fixed points in E_0^2:

$$\widetilde{E_0^2} \longrightarrow E_0^2.$$

Then the quotient space $\widetilde{E_0^2}/\mu_3$ is a smooth elliptic surface. After contracting all the (-1)-curves along the fibers $\widetilde{E_0^2}/\mu_3 \to S_0$, we have a relative minimal elliptic surface

$$S_0 \longrightarrow \mathbb{P}^1.$$

It is a rational elliptic surface and is isotrivial with the modular function $J \equiv 0$. Also S_0 has the singular fiber configuration $\{IV, IV, IV\}$.

A rational elliptic surface with a section can be obtained by blowing up the 9 points of a pencil of cubics in $\mathbb{P}^2$. In order to find such a pencil, we found 9 disjoint sections of S_0. Under the blowing down the 9 disjoint sections, the images of the singular fibers of S_0 form a dual Hesse arrangement. Then we found that there is a pencil of cubics containing a dual Hesse arrangement, which produces S_0 by blowing up its base points. We have the uniqueness of S_0 in the following sense:

Proposition 1.1. *The relative minimal model* S_0 *of the elliptic fibration* $E_0^2/\mu_3 \to \mathbb{P}^1$ *is the unique smooth elliptic surface that has the singular fiber configuration* $\{IV, IV, IV\}$*(in Kodaira's notion of singular fibers, see Table I). In particular,* S_0 *is isomorphic to the blowup of* $\mathbb{P}^2$ *at the base points of the pencil of cubics* $\lambda(x^3 - y^3) + \mu(x^3 - z^3)$.

Let's further consider the diagonal μ_3-action on E_0^3. The projection $p : E_0^3 \to E_0^2$ to the first two components is equivariant with respect to the μ_3-actions. Then we have the following diagram:

$$\begin{array}{ccc} E_0^3 & \xrightarrow{q_3} & E_0^3/\mu_3 \\ \downarrow{\scriptstyle p} & & \downarrow \\ E_0^2 & \xrightarrow{q_2} & E_0^2/\mu_3 \end{array}$$

where q_3 is the quotient map with respect to the μ_3-action on E_0^3.

We construct a smooth elliptic threefold over $\mathbb{P}^2$ that is birational to E_0^3/μ_3 as following. We first blow up E_0^3 along the fibers of $p : E_0^3 \to E_0^2$ over μ_3-fixed points in E_0^2,

$$\widetilde{E_0^3} \longrightarrow E_0^3.$$

There is a morphism $\widetilde{E_0^3} \to \widetilde{E_0^2}$. Then we blow up the curves in $\widetilde{E_0^3}$, on which μ_3 acts trivially,

$$Bl(\widetilde{E_0^3}) \longrightarrow \widetilde{E_0^3}.$$

Let $X = (Bl(\widetilde{E_0^3}))/\mu_3$ be the quotient space, and we have an elliptic fibration

$$X \longrightarrow \widetilde{E_0^2}/\mu_3.$$

Contracting all the (-1)-curves along the fibers of $\widetilde{E_0^2}/\mu_3 \to \mathbb{P}^1$, we have $X \to S_0$. After further contracting 9 disjoint sections of S_0, we obtain

$$f : X \longrightarrow \widetilde{E_0^2}/\mu_3 \longrightarrow S_0 \longrightarrow \mathbb{P}^2.$$

For this elliptic threefold we have the following result:

Theorem 1.2. *$f : X \to \mathbb{P}^2$ is an isotrivial elliptic fibration, with generic fiber isomorphic to E_0. The discriminant locus Σ is a dual Hesse arrangement. Moreover,*

- *The singular fiber over a smooth point of Σ is isomorphic to the union of 4 rational curves, one of which intersects the other three transversely, and the other three are disjoint, as shown in Figure 7.*
- *The singular fiber over a multiple point of Σ is isomorphic to the rational elliptic surface $\widetilde{E_0^2}/\mu_3$.*

The Alexander polynomial of a dual Hesse arrangement in $\mathbb{P}^2$ is $\Delta(t) = (t-1)^7(t^2+t+1)^2$ (Libgober, 2012). Based on Cogolludo and Libgober's work (Cogolludo-Agustín and Libgober, 2014), we will have that

$$rankMW(X) = deg(t^2+t+1)^2 = 4.$$

Theorem 1.3. *The elliptic fibration $X \to^2$ has Mordell-Weil rank equal to 4.*

1.2 Another Smooth Elliptic Threefold over $\mathbb{P}^2$

Let E_1 be the elliptic curve with j-invariant equal to 1728, and let g be an automorphism of E_1 of order 4. Then g generates the cyclic group μ_4 acting on E_1. Consider the diagonal μ_4-action on E_1^2, we have the following:

$$\begin{array}{ccc} E_1^2 & \xrightarrow{q_2} & E_1^2/\mu_4 \\ \downarrow p & & \downarrow \\ E_1 & \xrightarrow{q_1} & E_1/\mu_4 \cong \mathbb{P}^1 \end{array}$$

where p is the projection to the first component of E_1^2 and q_1, q_2 are quotient maps with respect to the μ_4-actions. Then we have a rational elliptic surface $E_1^2/\mu_4 \to \mathbb{P}^1$ with cyclic quotient singularities. Let $\widetilde{E_1^2/\mu_4}$ be the minimal resolution of E_1^2/μ_4. Then $\widetilde{E_1^2/\mu_4} \to \mathbb{P}^1$ is a smooth elliptic surface. It has the relative minimal model $S_1 \to \mathbb{P}^1$. We also found that S_1 has the singular fiber configuration $\{I_0^*, III, III\}$. Similar as the case of S_0, we contract several disjoint sections of $S_1 \to \mathbb{P}^1$ and several singular fiber components. Then we have a representation $\tau_1 : S_1 \to \mathbb{P}^2$ of S_1 as a blowup of $\mathbb{P}^2$ at base points of a pencil of cubics.

We found that such a pencil of cubics could contain 3 singular members:

$$l_1Q_1,\ \ l_2Q_2,\ \ lL^2,$$

where $l_1,\ l_2,\ l$ and L are lines, and $Q_1,\ Q_2$ are conics in $\mathbb{P}^2$. Furthermore, they satisfy the following configuration, as shown in Figure 12,

- l_1 is tangent to Q_1 and Q_2, and l_2 is tangent to Q_1 and Q_2;
- Q_1 and Q_2 intersect transversely at 2 points and are tangent at 1 tacnode,

- The points of tangency $l_1 \cap Q_2$, $l_2 \cap Q_1$, and the tacnode of $Q_1 \cap Q_2$ lie in the line L,
- The intersection point $l_1 \cap l_2$ and the two transversely intersection points of $Q_1 \cap Q_1$ lie in the line l.

We can choose different sections and singular fiber components to contract and have a different representation $\tau_2 : S_1 \to \mathbb{P}^2$, which is the blowup the base points of another pencil of cubics. Such a pencil of cubics contains 3 singular members:

$$C_1, \ \ C_2, \ \ L_1 L_2 L_3,$$

where C_1 and C_2 are cubic curves with a cusp and L_1, L_2 and L_3 are lines in $\mathbb{P}^2$. Furthermore they satisfy the following configuration, as shown in Figure 13,

- L_1 and C_1 are tangent, and the tangency point is the cusp of C_2;
- L_2 and C_2 are tangent, and the tangency point is the cusp of C_1;
- C_1 and C_2 intersect transversely at 2 points, and L_3 passes through the 2 points;
- C_1 and C_2 intersect at one point of multiplicity 3, and L_1, L_2, L_3 are concurrent at the same point.

We have the uniqueness of a pencil of cubics that contains such arrangements as following:

Proposition 1.4. A pencil of cubics in $\mathbb{P}^2$ that contains the singular members $\{l_1Q_1, l_2Q_2, lL^2\}$ or $\{C_1, C_2, L_1L_2L_3\}$ satisfying the conditions as above is unique up to projective transformations. Also, the pencil containing $\{l_1Q_1, l_2Q_2, lL^2\}$ and the pencil containing $\{C_1, C_2, L_1L_2L_3\}$ can be obtained by a Cremona transformation from each other.

Then we have the uniqueness of the rational elliptic surface S_1:

Proposition 1.5. *A minimal elliptic surface with a section that has singular fiber configuration* $\{I_0^*\ , III\ , III\}$ *is unique up to isomorphism. In particular, it is isomorphic to the relative minimal model of* $E_1^2/\mu_4 \to \mathbb{P}^1$.

Now we consider the diagonal μ_4-action on E_1^3. The projection $E_1^3 \to E_1^2$ to the first two components induces

$$\begin{array}{ccc} E_1^3 & \xrightarrow{q_3} & E_1^3/\mu_4 \\ \downarrow{\scriptstyle p} & & \downarrow \\ E_1^2 & \xrightarrow{q_2} & E_1^2/\mu_4 \end{array}$$

where q_3 is the quotient map with respect to the μ_4-action on E_1^3. A similar construction gives us a smooth birational model to E_1^3/μ_4,

$$Y \longrightarrow S_1,$$

whose general fiber is isomorphic to E_1. Since we have two representations of S_1 as 9-fold blowups of $\mathbb{P}^2$:

$$\tau_1, \tau_2 : S_1 \longrightarrow \mathbb{P}^2,$$

as described above. Then we have two elliptic fibrations,

$$f_1, f_2 : Y \longrightarrow \mathbb{P}^2.$$

Let Σ_i be the discriminant locus of f_i, $i = 1, 2$. We have following results about the singular fibers of f_1 and f_2.

Theorem 1.6. *The two elliptic fibrations $f_1, f_2 : X \longrightarrow \mathbb{P}^2$ are isotrivial with generic fiber isomorphic to E_1, and they have discriminant loci*

$$\Sigma_1 = l_1 \cup Q_1 \cup l_2 \cup Q_2 \cup l,$$

$$\Sigma_2 = C_1 \cup C_2 \cup L_1 \cup L_2 \cup L_3,$$

satisfying the configurations as above, as shown in Figure 12 and Figure 13. For the singular fibers of f_1, we have the following results,

- *Over a smooth point of $l \subset \Sigma_1$, the singular fiber is of Kodaira Type I_0^*(see Table I).*
- *Over a smooth point of l_1, l_2, Q_1 and Q_2, the singular fiber has 4 components of rational curves as shown in Figure 14.*
- *Over the transverse points of $l_1 \cap l_2$ and $Q_1 \cap Q_2$, the singular fibers are isomorphic to $\widetilde{E_1^2}/\mu_4$.*
- *The singular fibers over the tangency points $l_1 \cap Q_1$, $l_1 \cap Q_2$, $l_2 \cap Q_1$, $l_2 \cap Q_2$ and the tacnode of $Q_1 \cap Q_2$ consist of 6 components. One is isomorphic to $\widetilde{E_1^2}/\mu_4$, one is isomorphic to the Hirzebruch surface $\mathbb{F}_1$ and the other 4 components are isomorphic to the Hirzebruch surface $\mathbb{F}_0$. The 5 Hirzebruch surfaces intersect $\widetilde{E_1^2}/\mu_4$ along a singular fiber of type I_0^*, as shown in Figure 16 (The singular fibers of $\widetilde{E_1^2}/\mu_4$ is shown in Figure 9).*

For the singular fibers of f_2, we have the following,

- *Over a smooth point of* L_1, L_2 *and* L_3, *the singular fiber is of Kodaira Type* I_0^**(see Table I).*

- *Over a smooth point of* C_1 *and* C_2 *the singular fiber has 4 components of rational curves as shown in Figure 14.*

- *Over the two points where* C_1, C_2 *and* L_3 *intersect transversely with each other the singular fibers of* f_2 *are isomorphic to* $\widetilde{E_1^2}/\mu_4$.

- *The singular fibers of* f_2 *over the cusps of* C_1 *and* C_2 *have 4 components. One is isomorphic to* $\widetilde{E_1^2}/\mu_4$, *one is isomorphic to* $\mathbb{F}_0$ *and the other three are isomorphic to* $\mathbb{F}_1$. *The four Hirzebruch surfaces intersect* $\widetilde{E_1^2}/\mu_3$ *along one of its singular fibers that have 4 components, as shown in Figure 17.*

- *The singular fiber of* f_2 *over the concurrent point* $L_1 \cap L_2 \cap L_3$ *has 7 components. One is isomorphic to the K3 surface* $\widetilde{E_1^2}/\mu_2$, *one is isomorphic to* $\widetilde{E_1^2}/\mu_4$, *one is isomorphic to* $\mathbb{F}_1$ *and the other four are isomorphic to* $\mathbb{F}_1$. *The 5 Hirzebruch surfaces intersect* $\widetilde{E_1^2}/\mu_2$ *and* $\widetilde{E_1^2}/\mu_4$ *along their singular fibers of Kodaira Type* I_0^*, *as shown in Figure 18.*

CHAPTER 2

PRELIMINARIES AND BACKGROUND

In this chapter, we will go over some preliminary materials. This includes basics about elliptic curves and elliptic surfaces, which will be used in the following chapters. In the first section, we review the definition of elliptic curve and the group structure on an elliptic curves. In the second section, we review the definition of elliptic surface and relative minimal model. We also summarize some important results about minimal elliptic surfaces, such as Kodaira's classification of singular fibers and classifications of minimal models. At last we will introduce the Mordell-Weil group of an elliptic fibration and Shioda-Tate theorem.

2.1 Elliptic Curves

Let K be a field of characteristic zero and $\bar{K}$ be the algebraic closure of K. We will be particularly interested in the case $K = \mathbb{C}$ or a functional fields over $\mathbb{C}$, say $\mathbb{C}(x)$ or $\mathbb{C}(x, y)$. Let $\mathbb{P}^n$ denote the projective $n-$space over $\bar{K}$. We denote $\mathbb{P}^n(K)$ the set of K-rational points in $\mathbb{P}^n$.

Definition 2.1. An *elliptic curve* (E, O) over a field K is a smooth projective curve E of genus one defined over K, together with a chosen K-rational point O called its *base point*.

The following proposition describes Weierstrass form.

Proposition 2.2. Let (E, O) be an elliptic curve over K.

(a) There exist K-valued rational functions $x, y \in K(E)$ and the map

$$\phi : E \longrightarrow \mathbb{P}^2, \quad \phi = (x : y : 1),$$

such that ϕ is an ismorphism of E onto a plane curve defined by a Weierstrass equation

$$Y^2 = X^3 + a_4 X + a_6,$$

in the affine chart $(X : Y : 1)$, with coefficients $a_4, a_6 \in K$. Moreover, $\phi(O) = (0 : 1 : 0)$.

(b) Any two Weierstrass equations for an elliptic curve as in (a) are related by a linear change of coordinates:

$$X = u^2 X' \quad Y = u^3 Y',$$

with $u \in \bar{K}^*$.

(c) Conversely, every smooth Weierstrass plane cubic curve C (defined by a Weierstrass equation) is an elliptic curve defined over K with the base point $O = (0 : 1 : 0)$.

Proof. See (Silverman, 2009) chapter III Proposition 3.1. □

The *discriminant* of the Weierstrass equation above is

$$\Delta = (4a_4^3 + 27a_6^2),$$

The *j-invariant* of the corresponding elliptic curve is given by

$$j = 1728\frac{4a_4^3}{\Delta}.$$

The following theorem tells us how to determine an elliptic curve from its discriminant and $j-$invariant.

Theorem 2.3. (a) The elliptic curve E given by a Weierstrass equation $y^2 = x^3 + Ax + B$ has the following properties

(i) E is nonsingular if and only if $\Delta \neq 0$.

(ii) E has a node if and only if $\Delta = 0$ and $A \neq 0$.

(iii) E has a cusp if and only if $\Delta = A = 0$.

In the cases (ii) and (iii), E has only one singular point.

(b) Two elliptic curves are isomorphic over $\bar{K}$ (as algebraic curves) if and only if they have the same j-invariant.

(c) Let $j_0 \in \bar{K}$, there is an elliptic curve defined over $K(j_0)$ whose j-invariant equals to j_0.

Proof. See (Silverman, 2009) chapter III Proposition 1.4. □

There are two ways to realize an elliptic curve (E, O) as an abelian group. For geometric group law on an elliptic curve, we refer to (Silverman, 2009) Chapter III. The algebraic group law on (E, O) is given by the following proposition.

Proposition 2.4. Let (E, O) be an elliptic curve.

(a) For every degree-0 divisor $D \in Div^0(E)$ there exists a unique point $P \in E$ such that

$$D \sim (P) - (O).$$

Then we can define the map $\sigma : Div^0(E) \longrightarrow E$ sending D to the point P.

(b) σ induces a bijection of sets

$$\bar{\sigma} : Pic^0(E) \longrightarrow E.$$

The following theorem tells us about the automorphism group of an elliptic curve.

Theorem 2.5. Let (E, O) be an elliptic curve defined over a field K. Then its automorphism group $Aut(E)$ is a finite group. The order of $Aut(E)$ is given by the following table:

$\lvert Aut(E)\rvert$	$j(E)$
2	$j(E) \neq 0, 1728$
4	1728
6	0

Proof. See (Silverman, 2009) chapter III Theorem 10.1. □

In the case (E, O) is defined over $\mathbb{C}$, E is biholomorphic to a one-dimensional complex torus (Kirwan, 1992), i.e.

$$E \cong \mathbb{C}/\Lambda,$$

where $\Lambda \subset \mathbb{C}$ is a lattice of rank 2. The holomorhpic map above is also an isomorphism between abelian groups, where the group structure of $\mathbb{C}/\Lambda$ is induced from $\mathbb{C}$. Two complex tori are isomorphic if and only if the corresponding lattices are *homothetic* (Serre, 2012).

Example 2.6. Let $K = \mathbb{C}$. The Weierstrass equation $y^2 = x^3 + B$ with $B \neq 0$ defines an elliptic curve E_0 with $j = 0$. From Theorem 2.3, we can see that all the ellitpic curves defined by an Weierstrass equation of such form are isomorphic. E_0 has the analytic structure as

$$E_0 \cong \mathbb{C}/(\mathbb{Z} \oplus \omega\mathbb{Z}),$$

where ω is a primitive third root of unity. From Theorem 2.5 we have that

$$\mu_3 \subset Aut(E_0),$$

where μ_3 is the cyclic group of order 3. In fact, E_0 is the unique elliptic curve over $\mathbb{C}$ that admits an automorphism of order 3. The multiplication by ω on $\mathbb{C}$ induces such automorphism on the complex torus E_0.

Example 2.7. Again let $K = \mathbb{C}$. The Weierstrass equation $y^2 = x^3 + Ax$ with $A \neq 0$ defines an elliptic curve E_1 with $j = 1728$. Its analytic structure is

$$E_1 \cong \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z}),$$

where i is the imaginary unit. The multiplication by i on $\mathbb{C}$ induces an automorphism of order 4 on E_1. Again from Theorem 2.5, we have that E_1 is the unique elliptic curve over $\mathbb{C}$ that admits an automorphism of order 4.

2.2 Elliptic Surfaces

We will consider the case $K = \mathbb{C}$ in this section.

Definition 2.8. Suppose X is a complex surface (smooth or singular). A *genus one fibration* is a proper, connected and holomorphis map $f : X \to C$, where C is a smooth complex curve, such that a general fiber is a smooth curve of genus one. If there is a section $s : C \to X$ of f, we say that $f : X \to C$ is an *elliptic fibration* (with a given section), and X is an *elliptic surface* over C. A *smooth* elliptic surface $f : X \to C$ is *relatively minimal*, if all fibers of f do not have $(-1)-$curve.

Given a smooth elliptic surface $X \to C$ over C, we can blow down $(-1)-$curves along fibers and have a relative minimal model of X, which is also smooth. We will see that a minimal model of a smooth elliptic surface is *unique* in a sense described below.

Definition 2.9. Let $f_1 : X_1 \to C$ and $f_2 : X_2 \to C$ be two elliptic surfaces over C. A *morphism of elliptic surfaces* (over C) is a morphism of complex surfaces $\psi : X_1 \to X_2$ such that $f_1 = f_2 \circ \psi$. If ψ is also an isomorphism of complex surfaces we say that ψ is an *isomorphism of elliptic surfaces* (over C) and X_1 and X_2 are *isomorphic as elliptic surfaces* (over C). Furthermore, X_1 and X_2 are *birational as elliptic surfaces* (over C) if there is a birational map $\phi : X_1 \to X_2$ such that $f_1 = f_2 \circ \phi$.

Proposition 2.10. Suppose X_1 and X_2 are minimal elliptic surfaces over C, which are birational as elliptic surfaces over C. Let $f : X_1 \to X_2$ be a birational map that is compatible with the two elliptic fibrations. Then f is an isomorphism of elliptic surfaces over C.

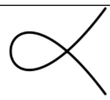
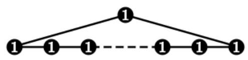
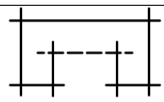
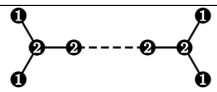
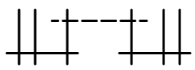
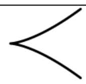
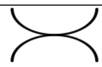
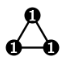
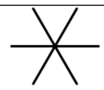
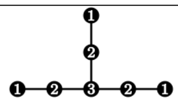
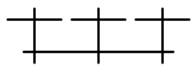
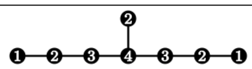
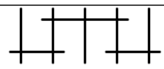
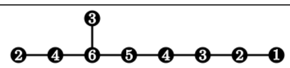
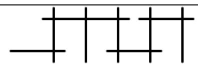
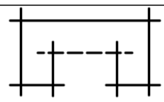
Proof. See (Miranda, 1989) (II.1.2) Proposition. □

Corollary 2.11. Suppose $X \to C$ is an elliptic surface over C, then there is a unique minimal elliptic surface $X' \to C$ that is birational to X as elliptic surfaces over C.

Kodaira classified all possible singular fibers of a minimal elliptic surface as shown in Table I (see (Kodaira, 1963)). The first column of the Table lists Kodaira's notations of singular fiber type. In the second column, a Dynkin diagram represents the intersection matrix of the irreducible components of a singular fiber. Each solid dot of a Dynkin diagram represents an irreducible component. The number in each solid dot denotes the multiplicity of corresponding component. There are three types of fibers that are irreducible: a smooth elliptic curve (I_0), a nodal rational curve (I_1) and a cuspidal rational curve (II). For a reducible singular fiber, all its irreducible components are smooth rational curve with self-intersection equal to -2.

Let $f : X \to C$ be a minimal elliptic surface over C with a section $s : C \to X$. The image of the section $S = s(C)$ is a divisor on X. Then for each singular fiber, S intersects exactly one of components with multiplicity 1. We introduce Weierstrass fibration as following (Miranda, 1981).

TABLE I Kodaira's Classification of Singular Fibers of A Minimal Elliptic Surface

Kodair	Dynkin Diagram	Fiber	Components
I_0			smooth elliptic curve
I_1			nodal rational curve
I_2			two smooth rational curves
$I_N, N \geq 3$			N smooth rational curves
$I_N^*, N \geq 3$			N+5 smooth rational curves
II			a cuspidal rational curve
III			two smooth rational curves
IV			three smooth rational curves
$IV*$			7 smooth rational curves
III*			8 smooth rational curves
II*			9 smooth rational curves
$I_{N,M}$			each component has multiplicity M.

Definition 2.12. Let X be a surface and C be a smooth curve. A *Weierstrass fibration* is a flat and proper map $f : X \to C$ such that every geometric fiber is either

1. a smooth genus one curve,
2. a rational curve with a node, or
3. a rational curve with a cusp,

and a general fiber is smooth. Moreover, there is a given section S that dose not pass through nodes or cusps of any fiber.

Suppose that $f : X \to C$ is a minimal elliptic surface with a section S. The matrices of intersection numbers for irreducible components of singular fibers of f can be represented by Dynkin diagrams in Table I and, in particular, all of them are negative semi-definite. Due to Grauert's contractibility criterion (Grauert, 1962) we can contract the union of all components of each singular fibers that do not intersect S. Such contraction gives a singular surface X' that admits a Weierstrass fibration,

$$X \xrightarrow{contract} X' \xrightarrow{Weierstrass\ fibration} C.$$

The singularities of X' are rational double points of the type denoted by the Dynkin diagrams for the corresponding singular fibers of X (Miranda, 1981). On the other hand, $X \to X'$ is the minimal resolution of the singularities of X'.

Let $f' : X' \to C$ be a Weierstrass fibration obtained from a minimal elliptic surface $f : X \to C$ with a section S. We still let S to be the corresponding section of X'. The normal bundle of $S \subset X'$ is denoted by $\mathcal{N}_{S/X'}$. Since $f'|_S$ is an isomorphism onto C, $f'_*\mathcal{N}_{S/X'}$ is a line bundle on C. We denote its dual bundle by

$$(f'_*\mathcal{N}_{S/X'})^{-1} = \mathbb{L}.$$

We call $\mathbb{L}$ the *fundamental line bundle* of the Weierstrass fibration $f' : X' \to C$. (See (Miranda, 1989))

A Weierstrass fibration $X' \to C$, with its fundamental line bundle $\mathbb{L}$ can be realized as a divisor inside a $\mathbb{P}^2-$bundle over C, which is $\mathbb{P}(\mathcal{O}_C \oplus \mathbb{L}^{-2} \oplus \mathbb{L}^{-3})$. Furthermore, X' has a global Weiertrass form:

$$y^2 = x^3 + Ax + B,$$

where x is a global section of $\mathbb{L}^2$, y is a global section of $\mathbb{L}^3$, A is a global section of $\mathbb{L}^4$ and B is a global section of $\mathbb{L}^6$. We say $(\mathbb{L}, A, B)$ is the *Weierstrass data* of X'. The *discriminant* of the fibration is the section $\Delta = 4A^3 + 27B^2$ of $\mathbb{L}^{12}$. The divisor (Δ) on C is called the *discriminant divisor* of the elliptic surface $X \to C$, where $X \to X'$ is the minimal resolution of X', therefore is a minimal elliptic surface over C. We also call $\mathbb{L}$ the *fundamental line bundle* of the elliptic surface $X \to C$.

Lemma 2.13. Let $(\mathbb{L}, A, B)$ be Weierstrass data over a projective curve C. Then $deg(\mathbb{L}) \geq 0$. Moreover, if $deg(\mathbb{L}) = 0$, $\mathbb{L}$ is torsion of order $1, 2, 3, 4$ or 6.

Example 2.14. Let f, g be two irreducible homogeneous polynomials of degree 3 in $k[x, y, z]$. And suppose that f and g define two smooth plane curves in $\mathbb{P}^2$: $C_1 = V(f)$ and $C_2 = V(g)$. Consider the pencil of cubics

$$\{sf + tg | (s : t) \in \mathbb{P}^1\}.$$

It has nine base points counting the multiplicities. Let $\epsilon : X = \widetilde{\mathbb{P}^2} \to \mathbb{P}^2$ be the successive 9-fold blowup at the base points of the pencil of cubics. Then the pullback of the pencil of cubics is base point free and induces a morphism $f : X \to \mathbb{P}^1$. A general fiber of f is the strict transform of a general member of the pencil of cubics, which is a smooth elliptic curve. Then $f : X \to \mathbb{P}^1$ is an elliptic surface with a section. The section S can be chosen to be the exceptional divisor of the last blowup of $X \to \mathbb{P}^2$. Since S is a (-1) curve in X, the fundamental line bundle of $X \to \mathbb{P}^1$ is $\mathbb{L} \cong \mathcal{O}_{\mathbb{P}^1}(1)$. We will see in Lemma 2.17 below that all minimal rational elliptic surfaces have $\mathbb{L} = \mathcal{O}_{\mathbb{P}^1}(1)$.

One has the *canonical bundle formula* for minimal elliptic surfaces.

Theorem 2.15. Let $f : X \to C$ be a minimal elliptic surface and $\mathbb{L}$ be the fundamental line bundle. Then the canonical bundle of X is a pullback bundle from the base curve,

$$K_X = f^*(K_C \otimes \mathbb{L}),$$

where K_C is the canonical bundle of C. Furthermore, $deg\mathbb{L} = \chi(X)$, where $\chi(X)$ is the Euler characteristic of X

Proof. Cf.(Miranda, 1989) Prop (III.1.1) or (Shioda and Schütt, 2010) Theorem 6.8. □

Recall *Noether's formula* (Beauville, 1996)

$$12\chi(X) = \omega_X^2 + e(X),$$

where ω_X is the canonical divisor of X and $e(X)$ is the topological Euler characteristic of X (equal to the 2nd chern number c_2), we have the following corollary,

Corollary 2.16. $e(X) = 12deg\mathbb{L}$.

Proof. Theorem 2.15 implies that ω_X is a multiple of fiber. Therefore $\omega_X^2 = 0$. Then apply Noether's formula and $deg\mathbb{L} = \chi(X)$. $\square$

We have a classifiaction of minimal elliptic surfaces based on the genus of base curve and the degree of fundamental line bunlde.

Lemma 2.17. Let $f : X \to C$ be a minimal elliptic surface with a section and $f' : X' \to C$ be the corresponding Weierstrass fibration with fundamental line bundle $\mathbb{L}$. Let $g = g(C)$ be the genus of C.

(a) If $g = 0$, then

- X is a product of an elliptic curve and $\mathbb{P}^1$ if $deg(\mathbb{L}) = 0$,
- X is a rational surface if $deg(\mathbb{L}) = 1$,
- X is a $K3$ surface if $deg(\mathbb{L}) = 2$,
- a properly elliptic surface if $deg(\mathbb{L}) \geq 3$.

(b) If $g = 1$, then X is

- an abelian surface (a product of two elliptic curves) if $\mathbb{L} \cong \mathcal{O}_C$,
- a hyperelliptic surface if $\mathbb{L}$ is torsion of order 2,3,4, or 6,
- a properly elliptic surface if $deg(\mathbb{L}) \geq 1$.

(c) If $g = 2$, then X is a properly elliptic surface.

Proof. See (Miranda, 1989) Lemma (III.4.6). □

Lemma 2.18. Let $f : X \to C$ be a minimal elliptic surface with a section and $\mathbb{L}$ be its fundamental line bundle. If X is not a product surface, then its Hodge numbers are

$$\begin{array}{ccccc} & & 1 & & \\ & g & & g & \\ g + deg\mathbb{L} - 1 & & 10deg\mathbb{L} + 2g & & g + deg\mathbb{L} - 1 \\ & g & & g & \\ & & 1. & & \end{array}$$

If X is a product surface, then its Hodge numbers are

$$\begin{array}{ccccc} & & 1 & & \\ & g & & g & \\ g + deg\mathbb{L} & & 10deg\mathbb{L} + 2g + 2 & & g + deg\mathbb{L} \\ & g & & g & \\ & & 1. & & \end{array}$$

Proof. See (Miranda, 1989) Lemma (IV.1.1). □

In Example 2.14, we have seen how to construct a rational elliptic surface from a pencil of plane cubics. The following lemma shows that all the rational elliptic surfaces can be constructed in this way.

Lemma 2.19. Let $f : X \to \mathbb{P}^1$ be a rational minimal elliptic surface. Then X is the 9-fold blowup of the plane $\mathbb{P}^2$ at the base points of a pencil of generically smooth cubic curves which induces the fibration f.

Proof. See (Miranda, 1989) Lemma (IV.1.2). □

2.3 Mordell-Weil Group of An Elliptic Surface

Definition 2.20. Let $X \to C$ be an elliptic surface with a chosen section s_0. Then the set of sections is an abelian group with the group addition defined fiber by fiber. This group is called *Mordell Weil group* of the elliptic surface, denoted by $MW(X)$. The chosen section s_0, which is the zero element of $MW(X)$ is called the *zero section*.

Let $f : X \to C$ be a minimal ellitpic surface with a section and $\mathbb{L}$ be its fundamental line bundle. We have the following lemmas.

Lemma 2.21.

1. $f^* : Pic(C) \to Pic(X)$ is injective.
2. If $f : X \to C$ is not a trivial fibration, i.e. X is not a product, then $f^* : Pic(C) \to Pic(X)$ is an isomorphism.
3. If $deg(\mathbb{L}) > 0$, then $Pic(X)/Pic^0(X)$ is torsion free.

Proof. See (Miranda, 1989) Lemma (VII.1.1) and Lemma (VII.1.2). □

Corollary 2.22. With the notations as above, assume that $deg\mathbb{L} > 0$, then Neron-Severi group $NS(X)$ of X is isomorphic to $Pic(X)/Pic^0(X)$. Furthermore, we have that

$$NS(X)/\mathbb{Z}[F] \cong Pic(X)/Pic^0(C),$$

where $\mathbb{Z}[F]$ is the free abelian group generated by the fiber class $[F] \in NS(X)$.

Now we introduce some notions of divisors. As we have seen, from a section s of $f : X \to C$ we have a divisor $S = s(C)$ in X. We call a divisor *horizontal* if it is the image of a section. On the other hand, we call a divisor *vertical*, if it is either a fiber or a component of a singular fiber. Let H and V be the groups generated by horizontal and vertical divisors respectively. We have that

Theorem 2.23. Let A be the subgroup of $NS(X)$ generated by the class of zero section S_0 and the classes of vertical divisors. We have an exact sequence

$$0 \to A \hookrightarrow NS(X) \xrightarrow{\beta} MW(X) \to 0.$$

Proof. See (Miranda, 1989) Theorem (VII.2.1). □

Recall that $(\Delta) \subset C$ is the discriminant divisor of $f : X \to C$. For $c \in (\Delta)$, the fiber over c denoted by $f^{-1}(c) = X_c$ is a singular fiber. Let n_c be the number of irreducible components of X_c. Then we have the Shioda-Tate's Formula:

Corollary 2.24.

$$rank\ NS(X) = 2 + \sum_{c \in (\Delta)} n_c + rank\ MW(X).$$

Proof. See (Miranda, 1989) Corollary (VII.2.4). □

CHAPTER 3

THE ELLIPTIC SURFACE E_0^2/μ_3

In this chapter we consider the case that the base field $K = \mathbb{C}$. We will see that there is a μ_3-action on the Abelian surface E_0^2 and the quotient space E_0^2/μ_3 is an elliptic surface with cyclic quotient singularities. We construct a relative minimal model S of the elliptic surface $E_0^2/\mu_3 \to \mathbb{P}^1$. We will analyze singular fibres of the relative minimal elliptic surface $S \to \mathbb{P}^1$ and compute its Mordell-Weil group $MW(S)$. Furthermore, we will give a projective model of S as a 9-fold blowup of $\mathbb{P}^2$ and show the uniqueness of S as a minimal elliptic surface with singular fiber configuration $\{IV, IV, IV\}$.

3.1 The μ_3-actions on E_0 and E_0^2

In Example 2.6, we have the elliptic curve E_0 with $j-$invariant equal to zero. Theorem 2.5 tells us that E_0 is the unique elliptic curve that admits an automorphism of order 3. Recall that E_0 is biholomorphic to the complex torus $E_0 \cong \mathbb{C}/(\mathbb{Z} \oplus \omega\mathbb{Z})$, where we choose $\omega = e^{\frac{2\pi i}{3}}$. Consider the map

$$\begin{aligned} g : \mathbb{C} &\longrightarrow \mathbb{C} \\ x &\longmapsto \omega x, \end{aligned}$$

it preserves the lattice $\Lambda = \mathbb{Z} \oplus \omega\mathbb{Z}$, so it induces an automorphism of E_0, which we also denotes by g. Then g generates the cyclic group μ_3 acting on E_0. Let P_0, P_1 and P_2 be the cosets of the lattice Λ of 0, $\frac{1}{\sqrt{3}}e^{\frac{\pi i}{6}}$, and $\frac{1}{\sqrt{3}}e^{\frac{\pi i}{2}}$. Then P_0, P_1 and P_2 are the only fixed points under the μ_3-action on E_0.

Now we consider the quotient map

$$q_1 : E_0 \longrightarrow E_0/\mu_3$$

with respect to μ_3-action on E_0, it is a holomorphic map of degree 3 and ramifies at the three points P_i with ramification indices 3. Then Riemann-Hurwitz formula gives

$$2-2g(E_0) = deg(q_1)(2-2g(E_0/\mu_3)) - \sum_{i=0,1,2}(e_{P_i}-1),$$

where $g(E_0) = 1$ is the genus of E_0. Then the genus of the quotient space is $g(E_0/\mu_3) = 0$ and we conclude that

$$E_0/\mu_3 \cong \mathbb{P}^1.$$

We denote the images of the three μ_3-fixed points under q_1 by

$$q_1(P_0) = [P_0],$$

$$q_1(P_1) = [P_1],$$

$$q_1(P_2) = [P_2].$$

The μ_3-action on the elliptic curve E_0 induces the diagonal action on the product surface $E_0^2 = E_0 \times E_0$ as following

$$g(P,Q) = (gP, gQ), \quad g \in \mu_3, \quad P, Q \in E_0.$$

There are 9 points in E_0^2 fixed by the μ_3-action

$$P_{ij} = (P_i, P_j), \quad i, j = 0, 1, 2.$$

The quotient map with respect to the diagonal μ_3-action

$$q_2 : E_0^2 \longrightarrow E_0^2/\mu_3$$

ramifies at the 9 μ_3-fixed points $P_{ij}, i, j = 0, 1, 2$. The algebraic surface E_0^2/μ_3 has 9 singularities, all of which are cyclic quotient singularities of type (3,1) (see (Lamotke, 2013) Chapter IV Section 5 and 6).

3.2 A Smooth Resolution of E_0^2/μ_3

In order to resolve the singularities of E_0^2/μ_3, it suffices to look into the local pictures. For a μ_3-fixed point, say $P_{ij} = (P_i, P_j) \in E_0^2$, we choose a local chart (U, ϕ) of P_{ij}, where U is an open neighborhood of P_{ij}, and $\phi : U \to D$ is a biholomorphic map to the unite disk $D \subset \mathbb{C}^2$. We can choose coordinates (x, y) of D such that $\phi(P_{ij}) = (0, 0)$ and $g \in \mu_3$ acts on D via ϕ as:

$$\phi \circ g \circ \phi^{-1}(x, y) = (\omega x, \omega y), \quad (x, y) \in D$$

where ω is a cubic root of unity. In the disk D all the lines passing through the origin will be preserved by the group action. We blow up the origin $\widetilde{D} \to D$. The group action extends to $\widetilde{D}$ analytically, as following. We have the coordinates on the blowup $\widetilde{D} = \{(x, y, (u : v) | xv = yu)\} \subset D \times \mathbb{P}^1$. The group action on $\widetilde{D}$ is:

$$g(x, y, (u : v)) = (\omega x, \omega y, (u : v)).$$

It is clear that g acts trivially on the exceptional line, which is defiend by $x = y = 0$. This also can be seen from the fact that each point in the exceptional line corresponds to a line through the origin in D, which is preserved by g.

We consider the quotient map $\widetilde{D} \to \widetilde{D}/\mu_3$. It is totally ramified along the exception line. Then $\widetilde{D}/\mu_3$ is smooth, see (Prill and others, 1967).

Now we blowup at all the 9 μ_3-fixed points: $\epsilon : \widetilde{E_0^2} \to E_0^2$. The μ_3-action extends analytically to the exceptional lines. As shown in the local discussion, all the 9 exceptional lines are fixed under the μ_3-action pointwisely. Then we consider the quotient map:

$$\tilde{q}_2 : \widetilde{E_0^2} \longrightarrow \widetilde{E_0^2}/\mu_3.$$

This quotient map is totally ramified along the 9 exceptional lines. From the local discussion above, we have that $\widetilde{E_0^2}/\mu_3$ is a smooth algebraic surface.

3.3 An Elliptic Fibration of $\widetilde{E_0^2}/\mu_3$

In this section we will show that $\widetilde{E_0^2}/\mu_3$ admits an elliptic fibration over $\mathbb{P}^1$ and in particular, it is a rational elliptic surface.

First, we have the following diagram,

$$\begin{array}{ccc}
\widetilde{E_0^2} & \xrightarrow{\tilde{q}_2} & \widetilde{E_0^2}/\mu_3 \\
\downarrow \epsilon & & \downarrow f_1 \\
E_0^2 & \xrightarrow{q_2} & E_0^2/\mu_3 \\
\downarrow \pi & & \downarrow f_2 \\
E_0 & \xrightarrow{q_1} & E_0/\mu_3.
\end{array}$$

where $\epsilon : \widetilde{E_0^2} \longrightarrow E_0^2$ is the blowup at the 9 μ_3-fixed points, $\pi : E_0^2 \longrightarrow E_0$ is the projection to the first component of E_0^2, and q_1, q_2 and $\tilde{q}_2$ are the quotient maps with respect to μ_3-actions. Both ϵ and π are equivariant with respect to μ_3-actions. Then we have two vertical morphisms f_1 and f_2 on the right side of the diagram above, where the morphism

$$f_1 : \widetilde{E_0^2}/\mu_3 \to E_0^2/\mu_3$$

is the smooth resolution of E_0^2/μ_3 constructed in previous section. The morphism

$$f_2 : E_0^2/\mu_3 \to E_0/\mu_3 \cong \mathbb{P}^1$$

has a genernal fibre isomorphic to E_0. Consider the composition

$$f = f_2 \circ f_1 : \widetilde{E_0^2}/\mu_3 \longrightarrow E_0/\mu_3 \cong \mathbb{P}^1,$$

it also has a general fiber isomorphic to E_0. Moreover, f has a section. We consider a section of $\pi : E_0^2 \to E_0$,

$$\begin{aligned} s_0 : E_0 &\longrightarrow E_0^2 \\ x &\mapsto (x, P_0) \end{aligned}$$

Notice that s_0 is equivariant with respect to the μ_3-actions. Then the images of s_0 under the quotient map $\tilde{q}_2$ is a section of $f : \widetilde{E_0^2}/\mu_3 \to \mathbb{P}^1$, which we still denote by s_0.

Now we have that $\widetilde{E_0^2}/\mu_3$ is a smooth elliptic surface over $\mathbb{P}^1$ with a general fiber isomorphic to E_0 and a chosen section s_0. Moreover $f : \widetilde{E_0^2}/\mu_3 \to \mathbb{P}^1$ have 3 singular fibres with discriminant locus of f being:

$$\Delta(f) = \{[P_0], [P_1], [P_2]\}.$$

In the following sections we will discuss the singular fibers of f and construct a relative minimal model of $\widetilde{E_0^2}/\mu_3 \to \mathbb{P}^1$.

3.4 The Singular Fibres

The singular fibres of $f : \widetilde{E_0^2}/\mu_3 \to \mathbb{P}^1$ can be analyzed as following. Pick a point in the discriminant locus, say, $[P_i] \in \Delta(f)$, its preimage under q_1 is $q_1^{-1}([P_i]) = P_i,$ and P_i has preimage under $\pi : E_0^2 \to E_0$

$$\pi^{-1}(P_i) = \{(P_i, x) | x \in E_0\},$$

which is isomorphic to E_0. Notice that there are 3 out of the 9 μ_3-fixed points belonging to $\pi^{-1}(P_i)$, which are P_{i0}, P_{i1} and P_{i2}. We denote $C_i = \pi^{-1}(P_i)$. Consider the blowup $\epsilon : \widetilde{E_0^2} \to E_0^2$ at P_{ij}

and let E_{ij} be the exceptional divisor over P_{ij} for $i, j = 0, 1, 2$. The pullback of the divisor C_i under $\epsilon^* : Div(E_0^2) \to Div(\widetilde{E_0^2})$ is:

$$\epsilon^*(C_i) = \hat{C}_i + E_{i1} + E_{i2} + E_{i3},$$

where $\hat{C}_i$ is the strict transform of C_i. From the previous sections, we have seen that μ_3 acts trivially on E_{ij}, and it acts on $\hat{C}_i$ the same way as on E_0. Therefore, $\tilde{q}_2(\epsilon^*(C_i))$ has four components. We denote the four components of $\tilde{q}_2(\epsilon^*(C_i))$ by

$$D_i = \tilde{q}_2(\hat{C}_i)$$

$$D_{ij} = \tilde{q}_2(E_{ij}).$$

Then $D_i, D_{ij}, j = 0, 1, 2$ are the four components of the fibre $f^{-1}[P_i]$. Next we are going to determine the multiplicity of each component.

To compute the multiplicities of each component, we can look into the local picture. Let $p \in \hat{C}_i$ be a general point other than P_{ij}, $j = 0, 1, 2$. Choose an analytic open neighborhood U_p of p in $\widetilde{E_0^2}$, small enough such that $U_p \cap g(U_p) \cap g^2(U_p) = \emptyset$. Such choice implies that no μ_3-fixed points lie in U_p. Then, the restricted map $\tilde{q}_2|_{U_p} : U_P \to \tilde{q}_2(U_p)$ is biholomorphic and therefore $\tilde{q}_2(U_p)$ is an analytic neighborhood of $\tilde{q}_2(p) \in \widetilde{E_0^2}/\mu_3$. We choose a local coordinates (U_p, x, y) for U_p centred at p and a local coordinates $(\tilde{q}_2(U_p), x', y')$ for $\tilde{q}_2(U_p)$ centred at $\tilde{q}_2(p)$.

On the other hand, under the projection π, we have that $\pi(U_p)$ is an analytic open neighborhood of $\pi(p) \in E_0$. Since $p \in \hat{C}_i$, it is projected to $\pi(p) = P_i \in E_0$. We choose a local coordinate $(\pi(U_p), s)$

of $\pi(U_p)$ centred at P_i and a local coordinate (V, s') centred at $[P_0] \in E_0/\mu_3$. The following diagram shows the local maps,

$$\begin{array}{ccc} p \in (U_p, x, y) & \xrightarrow{\tilde{q_2}} & (\tilde{q_2}(U_p), x', y') \\ \big\downarrow \pi & & \big\downarrow f \\ P_0 \in (\pi(U_p), s) & \xrightarrow{q_1} & [P_0] \in (V, s') \end{array}$$

In the diagram above, $\tilde{q_2}|_{U_p}$ is biholomorphic and it induces an isomorphism between rings of holomorphic functions. So we can choose the local coordinates such that

$$\tilde{q_2}^*(x') = x$$

$$\tilde{q_2}^*(y') = y.$$

Here $\tilde{q_2}^*$ is the pullback homomorphism of rings of holomorphic functions. Since $\pi : E_0^2 \to E_0$ is the projection to the first component, by making a proper choice of coordinate $(\pi(U_p), s)$, we can have

$$\pi^*(s) = x.$$

The quotient map q_1 is totally ramified at P_i with ramification index 3. Also $P_i \in \pi(U_P)$ is locally defined by $s = 0$. We may choose the local coordinates (V, s') such that

$$q_1^*(s') = s^3.$$

Now consider the commutative diagram :

$$\begin{tikzcd}
\mathbb{C}[s'] \arrow[r, "q_1^*"] \arrow[d, "f^*"] & \mathbb{C}[s] \arrow[d, "\pi^*"] \\
\mathbb{C}[x', y'] \arrow[r, "\tilde{q}_2^*"] & \mathbb{C}[x, y]
\end{tikzcd}$$

Since $\tilde{q}_2{}^* \circ f^* = \pi^* \circ q_1^*$, then we have that:

$$f^*(s') = (x')^3.$$

$[P_i]$ is locally defined by $s' = 0$ in $V \subset E_0/\mu_3$ and D_i is locally defined by $x' = 0$ in U_p. The multiplicity of D_i as a component of the fibre $f^{-1}([P_i])$ is the vanishing order of $f^*(s')$ along D_i. Therefore the multiplicity of D_i is 3.

The multiplicity of D_{ij} is one, which can be determined by a similar local argument as above. Then the singular fibre over $[P_i]$ as a divisor in $\widetilde{E_0^2}/\mu_3$ is

$$f^{-1}([P_i]) = 3D_i + D_{i0} + D_{i1} + D_{i2}, \quad i = 0, 1, 2.$$

The singular fibres are shown in Figure 1.

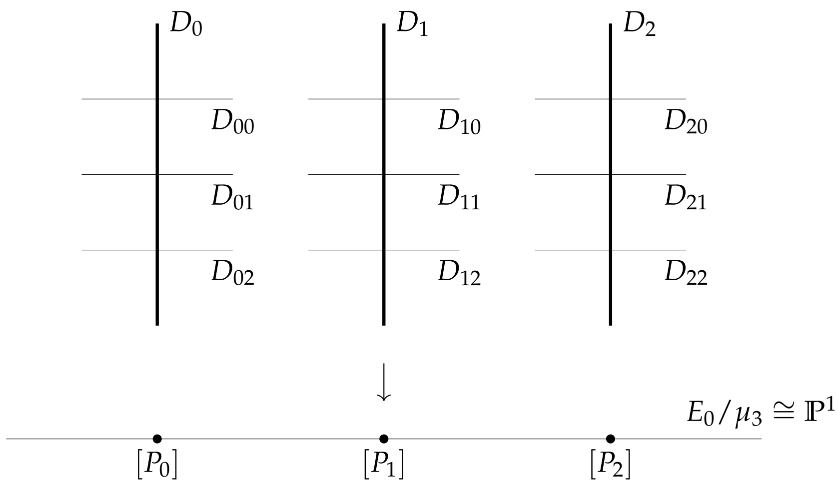


Figure 1: Singular Fibres of $\widetilde{E_0^2}/\mu_3 \longrightarrow \mathbb{P}^1$

All the three singular fibres are not in Kodaira's classification shown in Table I. We will see that all three singular fibers have a (-1)-component. Therefore, $\widetilde{E_0^2}/\mu_3 \to \mathbb{P}^1$ is not a minimal elliptic surface. In order to construct a relative minimal model of $\widetilde{E_0^2}/\mu_3 \to \mathbb{P}^1$, we are going to calculate the self-intersection numbers of the components of each singular fiber.

On the surface E_0^2 we have $C_i^2 = 0$, since C_i is the fiber over P_i of $\pi : E^2 \to E_0$. After blowing up at the 9 μ_3-fixed points, we have $\hat{C_i}^2 = -3$ on $\widetilde{E_0^2}$. The exceptional divisors are (-1)-cuvers $E_{ij}, i, j = 0, 1, 2$. Then we consider the quotient map: $\tilde{q}_2 : \widetilde{E_0^2} \to \widetilde{E_0^2}/\mu_3$. Recall our notations: $\tilde{q}_2(\hat{C}_i) = D_i$ and $\tilde{q}_2(E_{ij}) = D_{ij}$. In order to compute D_i^2 and D_{ij}^2 on $\widetilde{E_0^2}/\mu_3$, we need to borrow some intersection theory (Cf. (Eisenbud and Harris, 2016) and (Fulton, 2013)).

Now we apply the projection formula Theorem A.7 to the quotient map $\tilde{q}_2 : \widetilde{E_0^2} \to \widetilde{E_0^2}/\mu_3$. We have that

$$\tilde{q}_{2*}(\tilde{q}_2^*(D_i) \cdot \hat{C}_i) = D_i \cdot \tilde{q}_{2*}\hat{C}_i$$

Notice that $\tilde{q}_2|_{\hat{C}_i} : \hat{C}_i \to D_i$ is a degree 3 covering map. We can pull back the divisor D_i by Theorem A.6 and push-forward the divisor $\hat{C}_i$ by Definition A.3. Then we have that

$$\tilde{q}_{2*}\hat{C}_i = 3D_i,$$

$$\tilde{q}_2^*(D_i) = \hat{C}_i.$$

Plug the above two equations into the projection formula, we have that

$$\begin{aligned} \tilde{q}_{2_*}(\hat{C}_i \cdot \hat{C}_i) &= D_i \cdot 3D_i \\ -3 &= 3D_i^2 \\ -1 &= D_i^2 \end{aligned}$$

Since the μ_3-action restricted on the exceptional curves E_{ij} is a trivial action, we have that $\tilde{q}_2|_{E_{ij}} : E_{ij} \to D_{ij}$ is an isomorphism and

$$\tilde{q}_{2*}E_{ij} = D_{ij}.$$

Apply the projection formula Theorem A.7 to E_{ij} and D_i, we have

$$\begin{aligned}
\tilde{q}_{2*}(\tilde{q}_2^*(D_i) \cdot E_{ij}) &= D_i \cdot \tilde{q}_{2*} E_{ij} \\
\tilde{q}_{2*}(\hat{C}_i \cdot E_{ij}) &= D_i \cdot D_{ij} \\
1 &= D_i \cdot D_{ij}
\end{aligned}$$

Now we consider the singular fibre $f^{-1}([P_i]) = F_i = 3D_i + D_{i1} + D_{i2} + D_{i3}$. Since a fibre has zero self-intersection, we have

$$\begin{aligned}
0 &= F_i^2 \\
&= (3D_i + \sum_{j=0}^{2} D_{ij})^2 \\
&= 9D_i^2 + 6\sum_j D_i \cdot D_{ij} + \sum_j D_{ij}^2 \\
0 &= -9 + 18 + \sum_j D_{ij}^2 \\
-3 &= D_{ij}^2.
\end{aligned}$$

From the calculation above, we found that each singuler fiber F_i has one (-1)-component D_i.

3.5 The Relative Minimal Model of $\widetilde{E_0^2}/\mu_3$

We have found that the elliptic surface $\widetilde{E_0^2}/\mu_3 \longrightarrow \mathbb{P}^1$ has 3 singular fibres. Each singluar fibre has 4 irreducible components, all of which are rational curves, and there is one (-1)-curve in each singular fiber. Therefore $\widetilde{E_0^2}/\mu_3$ is not a minimal elliptic surface. To have a relavtive minimal model, we need to blow down the (-1)-component D_i in each singular fibre F_i,

$$\begin{tikzcd} \widetilde{E_0^2}/\mu_3 \arrow[r, "contract\ D_i"] \arrow[dr] & S \arrow[d, "f'"] \\ & \mathbb{P}^1 \end{tikzcd}$$

Let denote the image of D_{ij} under the blowdown by D'_{ij}.

The morphism $f' : S \to \mathbb{P}^1$ has 3 singular fibres $F'_i = f'^{-1}([P_i]), i = 0, 1, 2$, each of which consists of 3 concurrent (-2)-curves $D'_{ij}, j = 0, 1, 2$. All the singular fibers are of Kodaira Type IV (see Table I) as shown in Figure 2. Also, f' has a section. This is because the curves D_i we contract $\widetilde{E_0^2}/\mu_3 \to S$ are fiberal components of multiplicity 3. No sections of $f : \widetilde{E_0^2}/\mu_3 \to \mathbb{P}^1$ intersect D_i. The image of s_0 under the contraction map is a section of $f' : S \to \mathbb{P}^1$, which we still denote by s_0. Therefore, $f' : S \to \mathbb{P}^1$ is a minimal elliptic surface with a chosen section s_0.

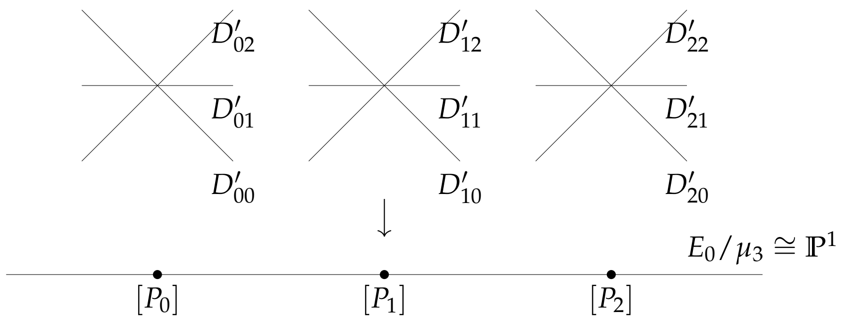


Figure 2: Singular Fibres of The Minimal Elliptic Surface $S \to \mathbb{P}^1$.

Furthermore, S is a rational elliptic surface, which can be seen as following. Due to Corollary 2.16 and Lemma 2.17, it suffices to calculate the topological Euler characteristic $e(S)$ of S.

Now let's calculate the topological Euler characteristic of S. Removing the discriminant locus $\{[P_0],[P_1],[P_2]\}$ from $\mathbb{P}^1$ and the 3 singular fibres from S, we have a trivial elliptic fibration with all fiber isomorphic to E_0,

$$S - \bigcup_{i=0,1,2} F_i \longrightarrow \mathbb{P}^1 - \{[P_0],[P_1],[P_2]\}$$

where F_i is the singular fibre over $[P_i]$. Then we have:

$$e(\mathbb{P}^1 - \{[P_0],[P_1],[P_2]\}) \cdot e(E_0) = e(S - \bigcup_i F_i).$$

E_0 is homeomorphic to a torus $\mathbb{T}^1$, so $e(E_0) = 0$. A singular fiber F_i of Kodaira type IV is homeomorphic to three 2-spheres S^2 intersecting at one point, so $e(F_i) = 4$. Then we have that

$$e(S) = 12.$$

Corollary 2.16 gives that $deg\mathbb{L} = 1$, and S is a rational elliptic surface by the classification Lemma 2.17.

Remark 3.1. The calculation above implies that the topological Euler characteristic of an elliptic surface $X \to C$ equals to the sum of the topological Euler characteristic of each singular fiber,

$$e(X) = \sum_{c \in (\Delta)} e(F_c)$$

where (Δ) is the discriminant divisor and F_c is the singular fiber over c.

Lemma 2.19 says that a minimal rational elliptic surface can be represented as a 9-fold blowup of $\mathbb{P}^2$ at the base points of a pencil of cubics. In the following sections we will give a pencil of cubics that induces the elliptic fibration $S \longrightarrow \mathbb{P}^1$ by blowing up its base points.

3.6 The Mordell-Weil Group of $E_0^2/\mu_3 \to \mathbb{P}^1$

In this section we will calculate the Mordell Weil group $MW(S)$ of the minimal elliptic surface $S \to \mathbb{P}^1$ we constructed in the previous section. Moreover, we will show that a choice of generators of $NS(S)$ determines a projective model of S.

Theorem 2.23 implies that it suffices to find $NS(S)$ and the subgroup A generated by vertical divisor classes and the zero section class. Lemma 2.19 says that S is a 9-fold blowup $\epsilon : \widetilde{\mathbb{P}^2} \to \mathbb{P}^2$. Therefore the class $[h] = \epsilon^*(H)$, which is the pull-back under the blowup of a general line $H \subset \mathbb{P}^2$ and the classes of the 9 exceptional divisors $[E_i]$ freely generate $NS(S)$ (Cf.(Beauville, 1996)) ,

$$NS(S) = [h]\mathbb{Z} \bigoplus_{i=1}^{9} [E_i]\mathbb{Z}.$$

Our strategy is as following. First, we find 9 *disjoint* sections of $S \to \mathbb{P}^1$. Then we show that these 9 sections are the exceptional curves E_i of a 9-fold blowup $S \to \mathbb{P}^2$. Then we will identify a set of generators of the subgroup A in terms H and E_i. Finally, the generators of A provide relations in the quotient $NS(S)/A$, which is isomorphic to $MW(S)$ by Theorem 2.23.

Remark 3.2. In fact there are infinitely many representations of S as a 9-fold blowup of $\mathbb{P}^2$ (Cf.(Miranda, 1989) Proposition (VIII.1.2)). We will see that $MW(S)$ is an infinite group and therefore $S \to \mathbb{P}^1$ has infinitely many disjoint sections. Any set of 9 disjoint sections gives a representations of S as a 9-fold blowup of $\mathbb{P}^2$.

First we claim the following lemma:

Lemma 3.3. Suppose X is a rational minimal elliptic surface. A curve in X is a section of the elliptic fibration $f : X \to \mathbb{P}^1$ if and only if it is a (-1)-curve.

Proof. Theorem 2.15 (Kodaira's formula of canonical divisor) gives

$$K_X = -F,$$

where F is the class of a fibre. For a curve $C \subset X$, by adjunction formula we have

$$K_X \cdot C + C^2 = 2g(C) - 2.$$

Suppose C is a section. We have $K_X \cdot C = -1$. Moreover, C is rational because the base curve is $\mathbb{P}^1$. Then adjunction formula gives $C^2 = -1$.

On the other hand, suppose C is a (-1)-rational curve. Since $g(C) = 0$ and $C^2 = -1$, we have $K_X \cdot C = -1$ and $F \cdot C = 1$. Therefore C is a section. □

Next we are going to find 9 sections of S such that they are disjoint with each other.

Let's look into the trivial elliptic fibration: $\pi : E_0^2 \to E_0$ projecting E_0^2 to its first component. We consider the following three sections of π:

$$s_0 = \{(x, P_0) | x \in E_0\}$$

$$s_1 = \{(x, P_1) | x \in E_0\}$$

$$s_2 = \{(x, P_2) | x \in E_0\}$$

We consider two automorphisms ϕ_1 and $\phi_{-\omega}$ on E_0^2. Explicitly, their actions on a general point $(x, y) \in E_0^2$ are

$$\begin{aligned} \phi_1(x,y) &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x+y \end{bmatrix} \\ \phi_{-\omega}(x,y) &= \begin{bmatrix} 1 & 0 \\ -\omega & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -\omega x + y \end{bmatrix} \end{aligned}$$

Apply ϕ_1 and $\phi_{-\omega}$ to s_i, $i = 0, 1, 2$, we have 9 curves in E_0^2 listed in the following table

curves	set description	through μ_3-fixed points
s_0	$\{(x, P_0)\|x \in E_0\}$	$(P_0, P_0), (P_1, P_0), (P_2, P_0)$
$\phi_1(s_0)$	$\{(x, x)\|x \in E_0\}$	$(P_0, P_0), (P_1, P_1), (P_2, P_2)$
$\phi_{-\omega}(s_0)$	$\{(x, -\omega x)\|x \in E_0\}$	$(P_0, P_0), (P_1, P_2), (P_2, P_1)$
s_1	$\{(x, P_1)\|x \in E_0\}$	$(P_0, P_1), (P_1, P_1), (P_2, P_1)$
$\phi_1(s_1)$	$\{(x, x + P_1)\|x \in E_0\}$	$(P_0, P_1), (P_1, P_2), (P_2, P_0)$
$\phi_{-\omega}(s_1)$	$\{(x, -\omega x + P_1)\|x \in E_0\}$	$(P_0, P_1), (P_1, P_0), (P_2, P_2)$
s_2	$\{(x, P_2)\|x \in E_0\}$	$(P_0, P_2), (P_1, P_2), (P_2, P_2)$
$\phi_1(s_2)$	$\{(x, x + P_2)\|x \in E_0\}$	$(P_0, P_2), (P_1, P_0), (P_2, P_1)$
$\phi_{-\omega}(s_2)$	$\{(x, -\omega x + P_2)\|x \in E_0\}$	$(P_0, P_2), (P_1, P_1), (P_2, P_0)$

where we also list the μ_3-fixed points that each curve passes through.

Lemma 3.4. The nine curves s_i, $\phi_1(s_i)$ and $\phi_{-\omega}(s_i)$,$i = 0, 1, 2$ are either disjoint or intersect each other at some μ_3-fixed points transversely. Furthermore, they all have zero self-intersection number.

Proof. For the second statement, since s_i are fibers of another fibration $E_0^2 \to E_0$, which projects E_0^2 to its second component, s_i have zero self-intersection. Since an automorphism preserves self-intersection, all the 9 curves listed above have zero self-intersection.

For the first statement, it is obvious that $s_i \cap s_j = \emptyset$. Therefore, $\phi_1(s_i) \cap \phi_1(s_j) = \phi_{-\omega}(s_i) \cap \phi_{-\omega}(s_j) = \emptyset$ for $i \neq j$.

We still need to consider $s_i \cap \phi_1(s_j)$. For a point $(x, y) \in E_0^2$, it belongs to $s_i \cap \phi_1(s_j)$ if and only if it satisfies the equation

$$x + P_j = P_i$$

Recall that $\{P_0, P_1, P_2\}$ is a subgroup of E_0 that is isomorphic the cyclic group $\mathbb{Z}_3$. We have $x = P_k$ for some $k \in \{0, 1, 2\}$.

Similarly, for $(x, y) \in s_i \cap \phi_{-\omega}(s_j)$, we need to look into the equation

$$-\omega x + P_j = P_i.$$

It has solution $x = -\omega^2(P_i - P_j)$, which is also a μ_3-fixed point.

For the case $(x, y) \in \phi_1(s_i) \cap \phi_{-\omega}(s_j)$, we need to solve the equation

$$x + P_i = -\omega x + P_j.$$

We can change of variable, by letting $t = x + P_k$ for some $k \in \{0, 1, 2\}$, then this equation turn into

$$t = -\omega t.$$

Then Lefschetz trace formula (Dold, 2012) implies that the automorphism $-\omega$ has only one fixed point, which is the zero element, i.e $t = P_0$.

Local presentations of the curves show the transversalities.

□

Let $\hat{s}_i, \widehat{\phi_1(s_i)}$ and $\widehat{\phi_{-\omega}(s_i)}$ denote the strict transforms of s_i, $\phi_1(s_i)$ and $\phi_{-\omega}(s_i)$ under the blowup at the nine μ_3-fixed points $\epsilon : \widetilde{E_0^2} \to E_0^2$. Now we consider the quotient map $\tilde{q}_2 : \widetilde{E_0^2} \longrightarrow \widetilde{E_0^2}/\mu_3$. We denote the images of the strict transforms by

$$\begin{aligned} s_{[i]} &= \tilde{q}_2(\hat{s}_i) \\ s^1_{[i]} &= \tilde{q}_2(\widehat{\phi_1(s_i)}) \\ s^\omega_{[i]} &= \tilde{q}_2(\widehat{\phi_{-\omega}(s_i)}) \end{aligned}$$

Lemma 3.5. The nine curves $s_{[i]}$, $s^1_{[i]}$ and $s^\omega_{[i]}$, $i = 0, 1, 2$ are (-1)-rational curves. Furthermore, they are disjoint with each other.

Proof. Each of the nine curves $s_i, \phi_1(s_i)$ and $\phi_{-\omega}(s_i)$ passes through 3 of the μ_3-fixed points (P_i, P_j) in E_0^2. After blowing up the nine μ_3-fixed points, the strict transforms $\hat{s}_i, \widehat{\phi_1(s_i)}$ and $\widehat{\phi_{-\omega}(s_i)}$ have self-intersection numbers equal to (-3) in $\widetilde{E_0^2}$. Lemma 3.4 implies that $\hat{s}_i, \widehat{\phi_1(s_i)}$ and $\widehat{\phi_{-\omega}(s_i)}$ are disjoint with each other for $i = 0, 1, 2$.

In order to compute these self-intersection numbers of these nine curves, we apply the Projection Formula Theorem A.7. Note that $\tilde{q}_2|_{\hat{s}_i} : \hat{s}_i \to s_{[i]}$ is a degree 3 covering map, we have that $\tilde{q}_{2*}(\hat{s}_i) = 3s_{[i]}$. Then we have

$$
\begin{aligned}
\tilde{q}_{2*}(\hat{s}_i \cdot \tilde{q}_2^*(s_{[i]})) &= \tilde{q}_{2*}(\hat{s}_i) \cdot s_{[i]} \\
\tilde{q}_{2*}(\hat{s}_i \cdot \hat{s}_i) &= 3s_{[i]} \cdot s_{[i]} \\
-3 &= 3s_{[i]} \cdot s_{[i]} \\
-1 &= s_{[i]} \cdot s_{[i]}
\end{aligned}
$$

Similar calculations give

$$
\begin{aligned}
&s^1_{[i]} \cdot s^1_{[i]} = -1 \\
&s^\omega_{[i]} \cdot s^\omega_{[i]} = -1
\end{aligned}
$$

Applying the Projection formula again, it is easy to see that the disjointness among $s_{[i]}, s^1_{[i]}$ and $s^\omega_{[i]}$ is directly from the disjointness among $\hat{s}_i, \widehat{\phi_1(s_i)}$ and $\widehat{\phi_{-\omega}(s_i)}$ for $i = 0, 1, 2$.

□

Recall that $\widetilde{E_0^2}/\mu_3$ is not a relative minimal model, i.e. each singular fibre has a component that is a (-1)-rational curve D_i. It is not hard to see that $s_{[i]}$, $s^1_{[i]}$ and $s^\omega_{[i]}$ are disjoint from the (-1)-component in each singular fibre. This is because the (-1)-components D_i (see Figure 1) are the quotients of the fibres

C_i over P_i in the fibration $E_0^2 \longrightarrow E_0$. The nine curves $s_i, \phi_1(s_i)$ and $\phi_{-\omega}(s_i)$ in E_0^2 intersect C_i only at some μ_3-fixed points. The blowup at the nine μ_3-fixed points separates $\hat{C}_i$ and $s_i, \phi_1(s_i)$ and $\phi_{-\omega}(s_i)$ in $\widetilde{E_0^2}$. Therefore, $s_{[i]}$, $s^1_{[i]}$ and $s^{\omega}_{[i]}$ are disjoint from $D_i = \tilde{q}_2(\hat{C}_i)$.

In order to have a minimal elliptic surface, we contract D_i for $i = 0, 1, 2$, $\widetilde{E_0^2}/\mu_3 \longrightarrow S$. We use the same notation $s_{[i]}$, $s^1_{[i]}$ and $s^{\omega}_{[i]}$ to denote their images in S under the blowing down. Since they are disjoint from D_i, their images have the same self-intersections in S. We conclude that the nine curves $s_{[i]}$, $s^1_{[i]}$ and $s^{\omega}_{[i]}, i = 0, 1, 2$ are (-1)-rational curves and therefore they are sections of $S \longrightarrow \mathbb{P}^1$ due to Lemma 3.3. In particular, these 9 sections are disjoint from each other.

Recall that we denote D'_{ij} as the image of D_{ij} under the contraction $\widetilde{E_0^2} \longrightarrow S$. The singular fibre of $S \longrightarrow \mathbb{P}^1$ over $[P_i] \in \mathbb{P}^1$ is denoted by F_i and

$$F_i = D'_{i0} + D'_{i1} + D'_{i2}$$

as shown in Figure 2. In the Table below, we list the singular fibres components of $S \to \mathbb{P}^1$, which each of the nine sections intersects. This is a directly result from which μ_3-fixed points the nine curves s_i, $\phi_1(s_i)$ and $\phi_{-\omega}(s_i)$ pass through in E_0^2.

TABLE II Nine Disjoint Sections of The Relative Minimal Model to $E_0^2/\mu_3 \to \mathbb{P}^1$ and The Singular Fiber Components They Intersect

Section	Intersects Singular Fibers Components
$s_{[0]}$	$D'_{00}, D'_{10}, D'_{20}$
$s_{[1]}$	$D'_{01}, D'_{11}, D'_{21}$
$s_{[2]}$	$D'_{02}, D'_{12}, D'_{22}$
$s^1_{[0]}$	$D'_{00}, D'_{11}, D'_{22}$
$s^1_{[1]}$	$D'_{01}, D'_{12}, D'_{20}$
$s^1_{[2]}$	$D'_{02}, D'_{10}, D'_{21}$
$s^\omega_{[0]}$	$D'_{00}, D'_{12}, D'_{21}$
$s^\omega_{[1]}$	$D'_{01}, D'_{10}, D'_{22}$
$s^\omega_{[2]}$	$D'_{02}, D'_{11}, D'_{20}$

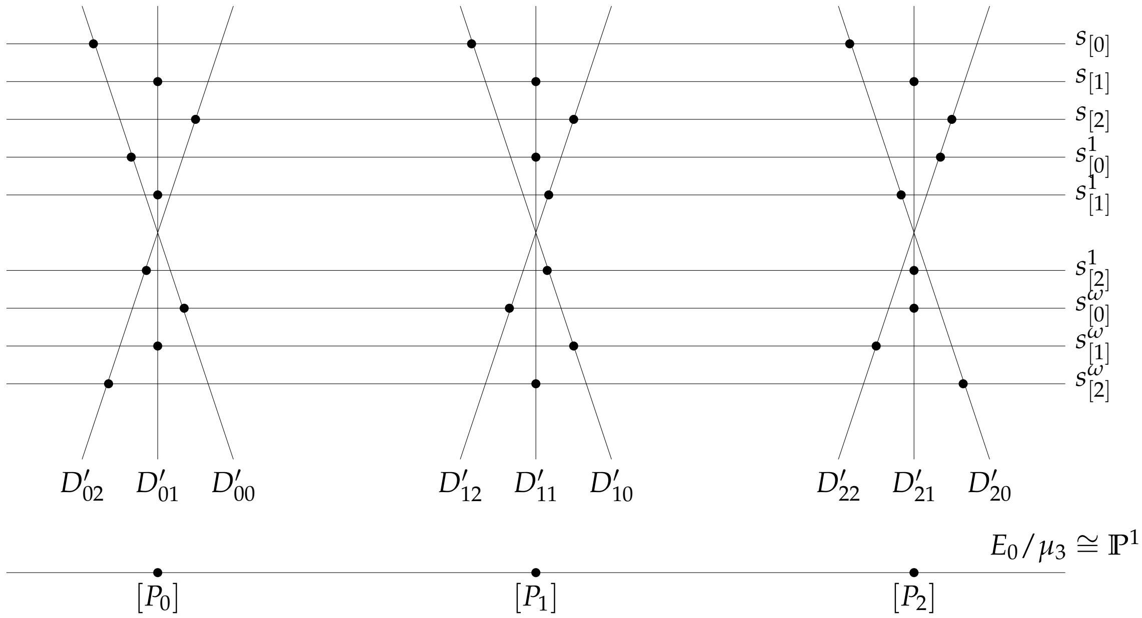


Figure 3: The 9 Sections of $S \to \mathbb{P}^1$.

Now we blow down the 9 disjoint sections of $f : S \to \mathbb{P}^1$ listed above,

$$\epsilon : S \longrightarrow \mathbb{S}'$$

We claim that the image S' of the contraction map ϵ is $\mathbb{P}^2$. This can be seen as following. We had that S is rational and $e(S) = 12$. Then the Hodge number $h^{1,1}(S) = 10$. After contracting the 9 disjoint sections, which are (-1)-rational curves, we have that $h^{1,1}(S') = 1$. Since $\mathbb{P}^2$ is the only smooth rational surface with $h^{1,1} = 1$, we conclude that $S' \cong \mathbb{P}^2$.

We denote the images of D'_{ij} under ϵ by

$$\epsilon(D'_{ij}) = L_{ij}, \quad i, j = 0, 1, 2.$$

One can see that L_{ij} is the strict transform of D'_{ij} under the blowup $\epsilon : S \longrightarrow \mathbb{P}^2$. One notices that each component D'_{ij} of singular fibers intersects three of the nine sections. Since $D'^2_{ij} = -2$ in S, its image L_{ij} in $\mathbb{P}^2$ is a smooth rational curve with self-intersection number equal to 1. We conclude that $L_{ij}, i, j = 0, 1, 2$ are lines in $\mathbb{P}^2$.

Since the singular fibers of $f : S \to \mathbb{P}^1$ are of Kodaira type IV, which is a triple of concurrent rational curves (see Figure 2), the image of each singular fiber is a triple of concurrent lines in $\mathbb{P}^2$. Furthermore, Lemma 2.19 implies that all the 3 triples of concurrent lines belongs to one pencil of cubics in $\mathbb{P}^2$, and $\epsilon : S \to \mathbb{P}^2$ is the blowup at the 9 base points of this pencil of cubics. We will find such pencil of cubics in the next section, from which we will have a projective model of S.

As the discussion above $\{L_{00}, L_{01}, L_{02}\}$,$\{L_{10}, L_{11}, L_{12}\}$ and $\{L_{20}, L_{21}, L_{22}\}$ are 3 triples of concurrent lines belonging to a pencil of cubics in $\mathbb{P}^2$. Denote the 9 base points of the pencil of cubics by $Q_1, ..., Q_9$. To be specific:

$$Q_1 = L_{00} \cap L_{10} \cap L_{20},$$

$$Q_2 = L_{00} \cap L_{11} \cap L_{22},$$

$$Q_3 = L_{00} \cap L_{12} \cap L_{21},$$

$$Q_4 = L_{02} \cap L_{10} \cap L_{21},$$

$$Q_5 = L_{02} \cap L_{11} \cap L_{20},$$

$$Q_6 = L_{02} \cap L_{12} \cap L_{22},$$

$$Q_7 = L_{01} \cap L_{10} \cap L_{22},$$

$$Q_8 = L_{01} \cap L_{11} \cap L_{21},$$

$$Q_9 = L_{01} \cap L_{12} \cap L_{20}.$$

We also denote the common point of each triple of concurrent lines by

$$R_1 = L_{00} \cap L_{01} \cap L_{01},$$

$$R_2 = L_{10} \cap L_{11} \cap L_{12},$$

$$R_3 = L_{20} \cap L_{21} \cap L_{22},$$

as shown in Figure 4

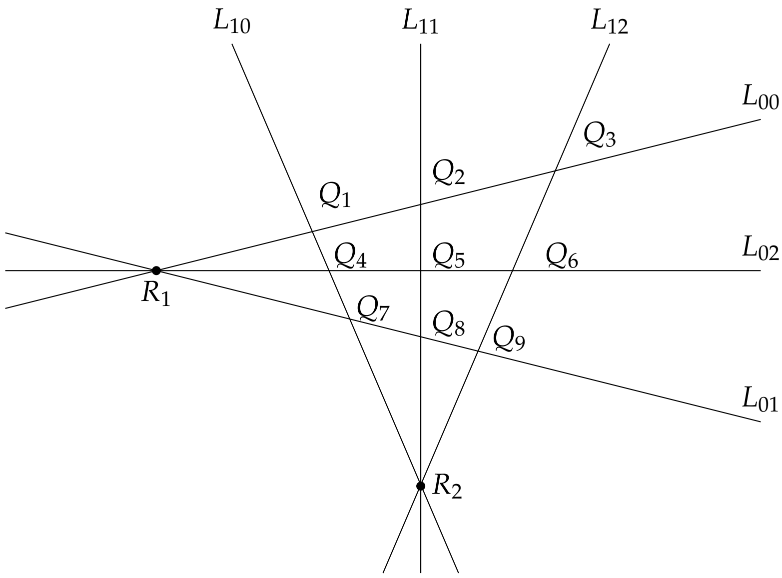


Figure 4: 6 out of 9 Lines of A Dual Hesse Arrangement

Under such notations, $\epsilon : S \to \mathbb{P}^2$ is the blowup at $Q_1, ..., Q_9$. Let E_i be the exceptional curve over Q_i for $i = 1, ..., 9$. In fact $E_i, i = 1, ..., 9$ are the 9 disjoint sections of $f : S \to \mathbb{P}^1$. Consider the intersections of the 9 sections with singular fibre components D'_{ij}, we can identify the 9 sections with the exceptional curves respectively. For example, the exceptional curve E_1 over Q_1 intersect the strict transforms of the three lines that are concurrent at Q_1, whihc are L_{00}, L_{10} and L_{20}. So E_1 is identified as the section intersecting D'_{00}, D'_{10} and D'_{20}, which is $s_{[0]}$ (see Table II). Similarly, we can identify all the 9 section as following

$$E_1 = s_{[0]}$$

$$E_2 = s^1_{[0]}$$

$$E_3 = s^{\omega}_{[0]}$$

$$E_4 = s^1_{[2]}$$

$$E_5 = s^{\omega}_{[2]}$$

$$E_6 = s_{[2]}$$

$$E_7 = s^{\omega}_{[1]}$$

$$E_8 = s_{[1]}$$

$$E_9 = s^1_{[1]}$$

Now we consider the Néron-Severi group $NS(S)$ of S. It is a free abelian group of rank 10 and generated by $[h]$, and $[E_i]$, $i = 1, ..., 9$, where $h = \epsilon^*(H)$ is the the pull-back of a general line H in $\mathbb{P}^2$, and $[E_i]$ is the class of the exceptional divisor E_i.

$$NS(S) = [h]\mathbb{Z} \oplus \bigoplus_{i=1}^{9} [E_i]\mathbb{Z}$$

Recall that $A \subset NS(S)$ is the subgroup generated by the class of a fiber $[F]$, the class of the zero section $[s_{[0]}] = [E_1]$ and the classes of all the singular fiber components $[D'_{ij}], i, j = 0, 1, 2$. We are going to express all the vertical classes, $[F]$ and $[D'_{ij}], i, j = 0, 1, 2$, in terms of the free generators $[h]$ and $[E_i], i = 1, ..., 9$.

In $\mathbb{P}^2$, L_{00} is the line passing through Q_1,Q_2 and Q_3. Therefore, the total transform of L_{00} under the blowup $\epsilon : S \to \mathbb{P}^2$ is

$$[h] = \epsilon^*(L_{00}) = [D'_{00}] + [E_1] + [E_2] + [E_3].$$

Also, the expression of $\epsilon^*(L_{ij}), i,j = 0,1,2$ gives a relation between $[D'_{ij}]$, $[h]$ and $[E_i], i = 1,...,9$. Then we have the relations of classes:

$$[D'_{00}] = [h] - [E_1] - [E_2] - [E_3]$$

$$[D'_{01}] = [h] - [E_7] - [E_8] - [E_9]$$

$$[D'_{02}] = [h] - [E_4] - [E_5] - [E_6]$$

$$[D'_{10}] = [h] - [E_1] - [E_4] - [E_7]$$

$$[D'_{11}] = [h] - [E_2] - [E_5] - [E_8]$$

$$[D'_{12}] = [h] - [E_3] - [E_6] - [E_9]$$

$$[D'_{20}] = [h] - [E_1] - [E_5] - [E_9]$$

$$[D'_{21}] = [h] - [E_3] - [E_8] - [E_4]$$

$$[D'_{22}] = [h] - [E_2] - [E_7] - [E_6]$$

Also we can have the class of a fibre

$$\begin{aligned} [F] &= [D'_{00}] + [D'_{01}] + [D'_{02}] \\ &= 3[h] - \sum_{i=1}^{9} [E_i]. \end{aligned}$$

Due to Theorem 2.23, Mordell-Weil group $MW(S)$ is the quotient group

$$MW(S) \cong NS(S)/A$$

The discussion above gives a set of relations of this quotient group in terms of the free generators of $NS(S)$,

$$\begin{aligned}
[E_1] &= 0 \\
[h] &= [E_1] + [E_2] + [E_3] \\
&= [E_7] + [E_8] + [E_9] \\
&= [E_4] + [E_5] + [E_6] \\
&= [E_1] + [E_4] + [E_7] \\
&= [E_2] + [E_5] + [E_8] \\
&= [E_3] + [E_6] + [E_9] \\
&= [E_1] + [E_5] + [E_9] \\
&= [E_3] + [E_8] + [E_4] \\
&= [E_2] + [E_7] + [E_6]
\end{aligned}$$

Some calculations based on the relations show that $[E_2], [E_3]$ and $[E_6]$ already generate $MW(S)$. To be explicit,

$$\begin{aligned}
[h] &= [E_2] + [E_3], \\
[E_4] &= [E_2] + [E_6], \\
[E_5] &= [E_3] + [E_6], \\
3[E_6] &= 0, \\
[E_8] &= -[E_6], \\
[E_7] &= [E_3] - [E_6], \\
[E_9] &= [E_2] - [E_6].
\end{aligned}$$

Then we find that $MW(S)$ can be generated by two free generators $[E_2]$ and $[E_3]$ and one torsion generator $[E_6]$ of order 3, i.e.

$$MW(S) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/3\mathbb{Z}.$$

3.7 A Pencil of Cubics Inducing The Relative Minimal Model of E_0^2/μ_3

In the previous section, we have found that S is the blowup at the 9 base points $\{Q_1, ..., Q_9\}$ of a pencil of cubics in $\mathbb{P}^2$, which contains three triples of concurrent lines $\{L_{00}, L_{01}, L_{02}\}$, $\{L_{10}, L_{11}, L_{12}\}$ and $\{L_{20}, L_{21}, L_{22}\}$. Let R_1, R_2, and R_3 be the three concurrent points of the three triples of concurrent lines. The nine lines L_{ij}, $i, j = 0, 1, 2$ intersect triply at the 12 points $Q_1, ..., Q_9$, R_1, R_2 and R_3. Note that each line passes through 4 triple points. Such arrangement of 9 lines and 12 points in $\mathbb{P}^2$ is called a *dual Hesse Arrangement*. We recall the definition of dual Hesse arrangement (Artebani and Dolgachev, 2006)

Definition 3.6. In the projective plane $\mathbb{P}^2$, a *dual Hesse arrangement* is a collection of 9 distinct reduced lines such that the 9 lines intersect triply at 12 points. In the dual projective plane $\mathbb{P}^{2*}$, the set of lines, which are dual to the 12 triple points of a dual Hesse arrangement is called a **Hesse arrangement**.

Remark 3.7. In a dual Hesse arrangement each line contains 4 triple points. This can be seen as follows. Let l be one of the 9 lines in a dual Hesse arrangement, and V be the union of the other 8 lines. Then l intersects with V at 8 points counting multiplicity. Also l intersects V only at triple points of the dual Hesse arrangement. For each intersecting point p, the intersection index is $(l, V)_p = 2$. So $l \cap V$ consists of 4 of the 12 triple points of the dual Hesse arrangement.

Therefore, in a Hesse arrangement each multiple point is a quadruple point and each line contains 3 quadruple points. A Hesse arrangement has 9 quadruple points.

Remark 3.8. A dual Hesse arrangement does not exist in the real projective plane. This is due to Motzkin's theorem, see (Motzkin, 1951), which says that a real arrangement of lines has a double point.

We will show that a dual Hesse arrangement is unique up to automorphism of $\mathbb{P}^2$.

Lemma 3.9. The 9 quadruple points of a Hesse arrangement can be realized by the 9 inflection points of a nonsingular plane cubic curve.

Proof. The following claim is well-known and we prove for completeness

Claim 3.10. If there is a nonsingular plane cubic curve C containing the 9 quadruple points of a Hesse arrangement, then the 9 quadruple points are the inflection points of C.

Proof. Let $\{P_1, ..., P_9\}$ be the 9 quadruple points of the Hesse arrangement. Recall that there are 12 triple points in a dual Hesse arrangement, so there are 12 lines in a Hesse arrangement, each of which passes through 3 quadruple points. Therefore, there are 12 linear relations among the 9 quadruple points, which becomes 12 relations among the elements of the Abelian group C,

$$P_1 + P_2 + P_3 = 0$$

$$P_1 + P_4 + P_7 = 0$$

$$P_1 + P_5 + P_9 = 0$$

$$P_1 + P_6 + P_8 = 0$$

$$P_2 + P_4 + P_9 = 0$$

$$P_2 + P_5 + P_8 = 0$$

$$P_2 + P_6 + P_7 = 0$$

$$P_3 + P_4 + P_8 = 0$$

$$P_3 + P_5 + P_7 = 0$$

$$P_3 + P_6 + P_9 = 0$$

$$P_4 + P_5 + P_6 = 0$$

$$P_7 + P_8 + P_9 = 0$$

Sum the first 3 equations we have

$$\begin{aligned} 3P_1 + (P_2 + P_4 + P_9) + (P_3 + P_5 + P_7) &= 0 \\ 3P_1 &= 0 \end{aligned}$$

Similarly, all the 9 points are of order 3:

$$3P_i = 0, \quad i = 1, ..., 9.$$

Therefore, the 9 quadruple points are inflection points of C. This finishes the proof of the claim.

$\square$

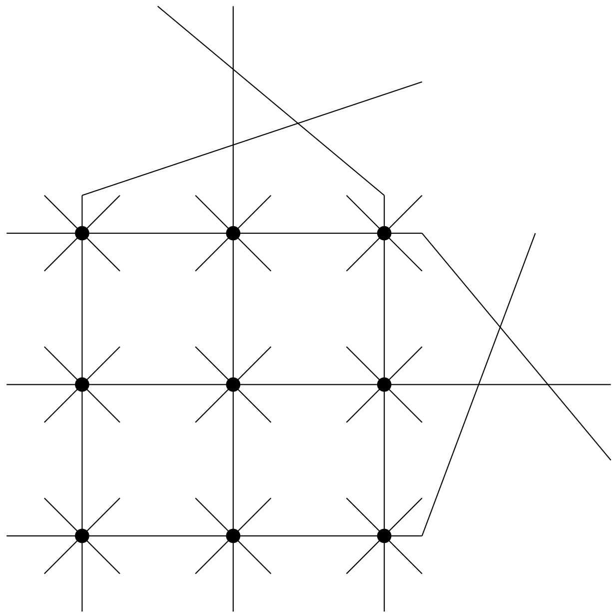


Figure 5: The 9 Quadruple Points And 2 out of 4 Triangles of A Hesse Arrangement

The dimension of plane cubic curves is $dim\mathbb{P}(H^0(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(3))) = 9$. Therefore given any 9 points in $\mathbb{P}^2$, there is at least one plane cubic curve contains them. Next, we need to show that there is a *nonsingular* plane cubic curve that contains the 9 quadruple points of a given Hesse arrangement.

By definition of a Hesse arrangement, the 12 lines in a Hesse arrangement consist of 4 triangles (non-concurrent triples of lines). Each triangle contains all 9 quadruple points. We can choose 2 of the 4 triangles (see Figure 5) to generate a pencil of cubics, whose base points are the 9 quadruple points of the Hesse arrangement. By Bertini's Theorem a general member of the pencil of cubics is smooth away from the base points. We need to argue that a general member of the pencil of cubics is also smooth at each base point.

To see this, pick one of the 9 base points, say p. In a local chart of p, let f_1 and f_2 be the local equations of the two generators of the pencil of cubics. Note that the two generators, which are two triangles, intersect at the nine base points each having multiplicity one. Therefore f_1 and f_2 both have vanishing order 1 at p. A general member of the pencil of cubics has local equation $\mu f_1 + \lambda f_2$ at p for a general point $(\mu : \lambda) \in \mathbb{P}^1$. Hence a general member of the pencil of cubics has multiplicity one at each base point and therefore it is smooth at the base points.

We conclude that there exists a smooth plane cubic curve C that contains the 9 quadruple points of a Hesse arrangement. Claim 3.10 implies that the 9 quadruple points are the inflection points of C. This finishes the proof of the lemma. □

Lemma 3.11. For any smooth plane cubic curve C there exists a coordinate system, in which it is defined by an equation

$$x^3 + y^3 + z^3 + \lambda xyz = 0$$

for some $\lambda \in \mathbb{C}$, which is called the Hesse canonical form of C.

Proof. Cf. (Artebani and Dolgachev, 2006) Lemma 1. □

Lemma 3.12. All the smooth plane cubic curves in Hesse canonical form share the same 9 inflection points.

Proof. The plane curve E defined by a Hesse form $x^3 + y^3 + z^3 + \lambda xyz = 0$ has its Hessian matrix formed by the second derivatives of its defining equation. The determinant of its Hessian matrix defines $He(E)$ the *Hessian curve* of E,

$$He(E): \quad -6\lambda^2(x^3 + y^3 + z^3) + (6^3 + 2\lambda^3)xyz = 0$$

The inflection points are the intersection of E with its Hessian curve $He(E)$. Combine the two equations we have

$$(6^3 + 8\lambda^3)xyz = 0$$

Notice that the curve E is nonsingular if and only if $6^3 + 8\lambda^3 \neq 0$. So we have,

$$\begin{aligned} xyz &= 0, \\ x^3 + y^3 + z^3 &= 0. \end{aligned}$$

The solutions consist of 9 distinct points independent of λ, which are

$$(1:-1:0), (1:-\omega:0), (1:-\omega^2:0),$$

$$(1:0:-1), (1:0:-\omega), (1:0:-\omega^2),$$

$$(0:1:-1), (0:1:-\omega), (0:1:-\omega^2).$$

□

The three lemmas above implies that

Proposition 3.13. In the projective plane $\mathbb{P}^2$, all dual Hesse arrangements are projectively equivalent.

Up to an automorphism of $\mathbb{P}^2$, we may assume that the three triples of concurrent lines are defined by

$$L_{00} \cup L_{01} \cup_{02} : \quad x^3 - y^3 = 0$$

$$L_{10} \cup L_{11} \cup_{12} : \quad x^3 - z^3 = 0$$

$$L_{20} \cup L_{21} \cup_{22} : \quad z^3 - y^3 = 0$$

It is easy to check that the 9 lines above intersect at 12 points of multiplicity 3, and they belong to the pencil of cubics

$$\lambda(x^3 - y^3) + \mu(x^3 - z^3) = 0.$$

Now we can identify the minimal elliptic surface S as the blowup at the 9 base points of the pencil of cubics $\lambda(x^3 - y^3) + \mu(x^3 - z^3) = 0$ in $\mathbb{P}^2$.

Proposition 3.14. The relative minimal model $S \to \mathbb{P}^1$ of the elliptic fibration $E_0^2/\mu_3 \to \mathbb{P}^1$ is isomorphic to the elliptic surface obtained by blowing up at the base points of the pencil of cubics $\lambda(x^3 - y^3) + \mu(x^3 - z^3) = 0$.

.

The minimal rational elliptic surfaces are classified by their configurations of singular fibres (Persson, 1990). Now we can show that a minimal rational elliptic surface with configuration $\{IV\ IV\ IV\}$ of singular fibres has a trivial moduli space.

Corollary 3.15. A minimal rational elliptic surface X with a section that has singular fibre configuration $\{IV\ IV\ IV\}$ is unique up to isomorphism. In particular, X is isomorphic to the relative minimal model of $E_0^2/\mu_3 \to E_0/\mu_3$.

Proof. By Lemma 2.19, X is biregular to the blowup at the 9 base points of a pencil of cubics in $\mathbb{P}^2$,

$$f : X \longrightarrow \mathbb{P}^2.$$

In such a model of X, the 9 exceptional curves of the blowup becomes 9 sections of the elliptic surface X.

Recall that Kodaira Type IV singular fibre consists of a triple concurrent (-2)-rational curves. Notice that a fibre of the elliptic surface X is the strict transform of a plane cubic curve, which is a member of the pencil of cubics in $\mathbb{P}^2$. Therefore a singular fibre of Kodaira Type IV is the strict transform of a plane cubic containing three rational components intersecting at a single point. Such a plane cubic has to be a concurrent triple of lines. Since each component of a singular fibre is a (-2)-curve, it intersects 3 exceptional curves of the blowup.

Suppose the three singular fibres have components $\{E_{i1}, E_{i2}, E_{i3}\}$, $i = 1, 2, 3$ for each fibre. Let $L_{ij} = f(E_{ij})$ be the image of blowing down. We have argued that L_{ij} are lines in $\mathbb{P}^2$. Let E be an exceptional curve of the blowup f. Since E is a section of the elliptic surface X, it intersects one of the

3 components of each singular fibre, say E_{1i}, E_{2j} and, E_{3k} for some $1 \leq i, j, k \leq 3$. If we blow down the exceptional curve E, the image of E_{1i}, E_{2j} and, E_{3k} will intersect triply at a single point. If we blow down the 9 exceptional curves, the image of $\{E_{ij}\}_{i,j=1,2,3}$ under blowing down, which is $\{L_{ij}\}_{i,j=1,2,3}$ will intersect triply at the 9 base points.

Together with the 3 concurrency points of $\{E_{ij}\}$ the three singular fibres of Kodaira Type IV, the image $\{L_{ij}\}$ of $\{E_{ij}\}$ under blowing down the 9 exceptional curves consists of 9 lines and 12 triple points, which form a dual Hesse arrangement by Def.3.6. By the uniqueness of dual Hesse arrangement in Prop.3.13, there is a unique pencil of cubics contains the dual Hesse arrangement up to projective equivalence.

Two projectively equivalent pencils of cubics induce two isomorphic elliptic surfaces by blowing up their base points. Therefore X is unique up to isomorphism. The relative minimal model S of E_0^2/μ_3 has the same configuration of singular fibres as X. Then X is isomorphic to S as elliptic surfaces by uniqueness. In particular, a general fibre of S is isomorphic to E_0, so X is isotrivial and has the modular function $\mathbb{J} \equiv 0$.

$\square$

CHAPTER 4

A SMOOTH BIRATIONAL MODEL OF E_0^3/μ_3

In the previous chapter we constructed and studied the relative minimal model of the elliptic surface $\widetilde{E_0^2}/\mu_3 \to \mathbb{P}^1$. In this chapter we will look into an elliptic fibration $E_0^3/\mu_3 \to E_0^2/\mu_3$. We will construct a smooth elliptic threefold over $\mathbb{P}^2$ that is birational to the elliptic fibration $E_0^3/\mu_3 \to E_0^2/\mu_3$.

4.1 A μ_3-action on E_0^3

First we look at a μ_3-action on E_0^3. The μ_3-action on E_0 induces the diagonal action on the threefold E_0^3 as following,

$$g(P,Q,R) = (gP, gQ, gR), \quad g \in \mu_3, \quad P,Q,R \in E_0.$$

This diagonal μ_3-action has 27 fixed points, which are

$$P_{ijk} = (P_i, P_j, P_k) \in E_0^3, \quad i,j,k = 0,1,2,$$

where $P_i, i = 0,1,2$ are the μ_3-fixed points of E_0. We consider the quotient map with respect to this group action:

$$q_3 : E_0^3 \longrightarrow E_0^3/\mu_3.$$

The quotient map q_3 ramifies at the 27 μ_3-fixed points. The quotient space E_0^3/μ_3 has 27 cyclic quotient singularities, which are locally biholomorphic to $\mathbb{C}^3/\Gamma$. Here Γ is the subgroup of $SL(3,\mathbb{C})$ generated by $\begin{bmatrix} \omega & & \\ & \omega & \\ & & \omega \end{bmatrix}$, where ω is a primitive third root of unity.

The threefold E_0^3/μ_3 admits an elliptic fibration. First we consider the projection map

$$\begin{array}{rcl} \pi : E_0^3 & \longrightarrow & E_0^2 \\ (x,y,z) & \mapsto & (x,y) \end{array}$$

It is easy to see that π is equivariant with respect to the μ_3-actions on E_0^2 and E_0^3, so we have the diagram

$$\begin{array}{ccc} E_0^3 & \xrightarrow{q_3} & E_0^3/\mu_3 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle f_3} \\ E_0^2 & \xrightarrow{q_2} & E_0^2/\mu_3. \end{array}$$

The general fiber of f_3 is isomorphic to E_0.

Both the threefold E_0^3/μ_3 and the base surface E_0^2/μ_3 are singular. We have constructed a smooth resolution $\widetilde{E_0^2/\mu_3} \to E_0^2/\mu_3$ in the previous chapter. We are going to construct a smooth resolution of E_0^3/μ_3, which has an elliptic fibration over $\widetilde{E_0^2/\mu_3}$.

4.2 Resolution of $\mathbb{C}^3/\Gamma$

We start with a local construction as following. Consider the diagonal μ_3-action on $\mathbb{C}^2$ and $\mathbb{C}^3$,

$$\begin{aligned}(x,y) &\mapsto (\omega x, \omega y),\\ (x,y,z) &\mapsto (\omega x, \omega y, \omega z).\end{aligned}$$

where ω is a third root of unity. Let $\pi : \mathbb{C}^3 \longrightarrow \mathbb{C}^2$ be the projection onto the first two components. It is equivariant with respect to the μ_3-actions on $\mathbb{C}^3$ and $\mathbb{C}^2$, then we have the following diagram:

$$\begin{tikzcd} \mathbb{C}^3 \arrow[r] \arrow[d, "\pi"] & \mathbb{C}^3/\mu_3 \arrow[d, "f_3"] \\ \mathbb{C}^2 \arrow[r] & \mathbb{C}^2/\mu_3 \end{tikzcd}$$

Both $\mathbb{C}^2/\mu_3$ and $\mathbb{C}^3/\mu_3$ are singular and have cyclic quotient singularities. We are going to construct smooth resolutions of $\mathbb{C}^2/\mu_3$ and $\mathbb{C}^3/\mu_3$ and a map $\tilde{f}_3$ between them, which is an extension of f_3.

First we blow up the origin $O \in \mathbb{C}^2$ and the line $L = \{(x,y,z)|x = y = 0\} \subset \mathbb{C}^3$. We have

$$\begin{aligned}Bl_O\mathbb{C}^2 = \widetilde{\mathbb{C}^2} &= \{(x,y,u:v)|xv = yu\}\\ Bl_L\mathbb{C}^3 = \widetilde{\mathbb{C}^3} &= \{(x,y,z,s:t)|xt = ys\}.\end{aligned}$$

The projection π can be extend to :

$$\begin{aligned}\tilde{\pi} : \widetilde{\mathbb{C}^3} &\longrightarrow \widetilde{\mathbb{C}^2}\\ (x,y,z,s:t) &\mapsto (x,y,s:t).\end{aligned}$$

Also the μ_3-actions extend holomorphically to the blowups:

$$\begin{aligned} on \quad \widetilde{\mathbb{C}^2} \quad &: (x, y, u : v) \longmapsto (\omega x, \omega y, u : v) \\ on \quad \widetilde{\mathbb{C}^3} \quad &: (x, y, z, r : s) \longmapsto (\omega x, \omega y, \omega z, r : s). \end{aligned}$$

The morphism $\tilde{\pi}$ is equivariant with respect to the extended μ_3-actions. We have the following diagram

$$\begin{array}{ccc} \widetilde{\mathbb{C}^3} & \xrightarrow{\tilde{q}_3} & \widetilde{\mathbb{C}^3}/\mu_3 \\ \Big\downarrow{\scriptstyle \tilde{\pi}} & & \Big\downarrow{\scriptstyle \tilde{f}_3} \\ \widetilde{\mathbb{C}^2} & \xrightarrow{\tilde{q}_2} & \widetilde{\mathbb{C}^2}/\mu_3 \end{array}$$

From the local discussion in previous chapter, $\widetilde{\mathbb{C}^2}/\mu_3$ is nonsingular. However, the quotient space $\widetilde{\mathbb{C}^3}/\mu_3$ is still singular. This is because the μ_3-action fixes the curve $C = \{(0, 0, 0, r : s)\}$ in $\widetilde{\mathbb{C}^3}$. We further blow up $\widetilde{C^3}$ along the curve C

$$Bl_C\widetilde{\mathbb{C}^3} \longrightarrow \widetilde{\mathbb{C}^3}.$$

The μ_3-action extends to $Bl_C\widetilde{\mathbb{C}^3}$. The extended μ_3-action on $Bl_C\widetilde{\mathbb{C}^3}$ acts trivially on the exceptional divisor over C of the second blowup. Then we conclude that $(Bl_C\widetilde{\mathbb{C}^3})/\mu_3$ is smooth.

Since the composed map $Bl_C\widetilde{\mathbb{C}^3} \longrightarrow \widetilde{\mathbb{C}^3} \longrightarrow \widetilde{\mathbb{C}^2}$ is equivariant with respect to the μ_3-actions, it induces a morphism

$$\left(Bl_C\widetilde{\mathbb{C}^3}\right)/\mu_3 \longrightarrow \widetilde{\mathbb{C}^2}/\mu_3,$$

whose general fiber is isomorphic to E_0.

4.3 A Nonsingular Elliptic Threefold over $\mathbb{P}^2$

Now we return to the construction of a nonsingular elliptic threefold over $\widetilde{E_0^2}/\mu_3$. As above we consider the isotrivial elliptic fibration $f_3 : E_0^3/\mu_3 \to E_0^2/\mu_3$.

Let $L_{ij} = \pi^{-1}(P_{ij})$, where π is the projection and P_{ij} is the μ_3-fixed points in E_0^2. We let

$$\widetilde{E_0^3} = Bl_{L_{ij}} E_0^3 \longrightarrow E_0^3.$$

denote the blowup along the 9 curves $L_{ij}, i, j = 0, 1, 2$ in E_0^3.

Recall that $\widetilde{E_0^2}$ is the blowup of E_0^2 at the 9 μ_3-fixed points. We also notice that the two blowups

$$\begin{aligned} \widetilde{E_0^3} &\longrightarrow E_0^3, \\ \widetilde{E_0^2} &\longrightarrow E_0^2, \end{aligned}$$

in a neighborhood of the μ_3-fixed points $P_{ijk} \in E_0$ and in a neighborhood of $P_{ij} \in E_0^2$ are locally biholomorphic to the blowups,

$$\begin{aligned} Bl_L\mathbb{C}^3 = \widetilde{\mathbb{C}^3} &\longrightarrow \mathbb{C}^3, \\ Bl_O\mathbb{C}^2 = \widetilde{\mathbb{C}^2} &\longrightarrow \mathbb{C}^2, \end{aligned}$$

where $L = \{(x, y, z) | x = y = 0\} \in \mathbb{C}^3(x, y, z)$, and $O = (0, 0) \in \mathbb{C}^2(x, y)$.

From the local construction in the previous section, we have a morphism:

$$\widetilde{E_0^3} \longrightarrow \widetilde{E_0^2}.$$

Also, the μ_3-action extends to $\widetilde{E_0^3}$ and $\widetilde{E_0^2}$ in the same way as the μ_3-action extends to $\widetilde{\mathbb{C}^3}$ and $\widetilde{\mathbb{C}^2}$. From the discussion of the local construction, we can see that the morphism $\widetilde{E_0^3} \to \widetilde{E_0^2}$ is equivariant with respect to the μ_3-actions, so we have the following diagram,

$$\begin{array}{ccc} \widetilde{E_0^3} & \longrightarrow & \widetilde{E_0^3}/\mu_3 \\ \Big\downarrow & & \Big\downarrow \\ \widetilde{E_0^2} & \longrightarrow & \widetilde{E_0^2}/\mu_3. \end{array}$$

We notice that $\widetilde{E_0^3}/\mu_3$ is not smooth. The local picture shows that the μ_3-action on $\widetilde{E_0^3}$ fixes 27 curves, for which we denote by $C_{ijk}, \;\; i,j,k = 0,1,2$. Therefore, $\widetilde{E_0^3}/\mu_3$ has a singular locus containing 27 curves, each of which has a neighborhood biholomophic to a neighborhood of the singular locus of $\widetilde{\mathbb{C}^3}/\mu_3$ as described in the local construction in the previous section.

In order to have a smooth resolution of $\widetilde{E_0^3}/\mu_3$, we blow up $\widetilde{E_0^3}$ along the 27 curves C_{ijk}, for $i,j,k = 0,1,2$,

$$Bl_{C_{ijk}}\widetilde{E_0^3} \longrightarrow \widetilde{E_0^3}$$

This blowup is locally biholomorphic to the blowup $Bl_C\widetilde{\mathbb{C}^3} \longrightarrow \widetilde{\mathbb{C}^3}$ in a neighborhood of $C_{ijk} \subset \widetilde{E_0^3}$ and in a neighborhood of $C \subset \widetilde{\mathbb{C}^3}$.

We can extend the μ_3-action to $Bl_{C_{ijk}}\widetilde{E_0^3}$ as we extended the μ_3-action from $\widetilde{\mathbb{C}^3}$ to $Bl_C\widetilde{\mathbb{C}^3}$ in the local picture in the previous section. The local discussion shows that $(Bl_C\widetilde{A^3})/\mu_3$ is smooth, therefore $\left(Bl_{C_{ijk}}\widetilde{E_0^3}\right)/\mu_3$ is also smooth. Since the blowup $Bl_{C_{ijk}}\widetilde{E_0^3} \to \widetilde{E_0^3}$ is equivariant with respect to the μ_3-actions, we have the morphism between quotient spaces

$$\left(Bl_{C_{ijk}}\widetilde{E_0^3}\right)/\mu_3 \longrightarrow \widetilde{E_0^3}/\mu_3.$$

Composing with $\widetilde{E_0^3}/\mu_3 \to \widetilde{E_0^2}/\mu_3$, we have a morphism

$$f' : \left(Bl_{C_{ijk}}\widetilde{E_0^3}\right)/\mu_3 \longrightarrow \widetilde{E_0^2}/\mu_3,$$

which is an elliptic fibration with a general fibre isomorphic to E_0.

In the previous chapter we have identify $c : \widetilde{E_0^2}/\mu_3 \longrightarrow \mathbb{P}^2$ as a blowup at the 12 triple points of a dual Hesse arrangement in $\mathbb{P}^2$. Then we have an isotrivial elliptic fibration,

$$f = c \circ f' : \left(Bl_{C_{ijk}}\widetilde{E_0^3}\right)/\mu_3 \longrightarrow \mathbb{P}^2.$$

4.4 The Singular Fibers of The Elliptic Threefold

In this section we are going to analyze the discriminant locus $\Delta(f)$ and the singular fibers of f. We will prove Theorem 1.2.

We first analyze the discriminant locus $\Delta(f')$ and singular fibers of f'. We have the following proposition,

Proposition 4.1. The elliptic fibration $f' : \left(Bl_{C_{ijk}} \widetilde{E_0^3} \right) / \mu_3 \longrightarrow \widetilde{E_0^2} / \mu_3$ has discriminant locus $\Delta(f')$ a disjoint union of the 9 exceptional curves $D_{i,j}, i, j = 0, 1, 2$ as shown in Figure 1. All its singular fibers have 4 components of rational curves as shown in Figure 7.

Proof. Recall Section 3.4 that

$$\widetilde{E_0^2} / \mu_3 \to E_0 / \mu_3$$

is an elliptic surface and its singular fiber over $[P_i] \in E_0 / \mu_3$ is $3D_i + D_{i0} + D_{i1} + D_{i2}$ for $i = 0, 1, 2$ (see Figure 1). One can check directly that for $P \notin D_{ij}, i, j = 0, 1, 2$, its fiber $f'^{-1}(P)$ is isomorphic to E_0.

Now we study the fibers over points of the exceptional curves D_{ij}. We will show that the discriminant locus of f' is the union of D_{ij} for $i, j = 0, 1, 2$.

We need to recall the construction of $\left(Bl_{C_{ijk}} \widetilde{E_0^3} \right) / \mu_3$ and trace the fibers step by step. First we recall that $\epsilon_3 : \widetilde{E_0^3} \to E_0^3$ is the blowup along the 9 curves $L_{ij}, i, j = 0, 1, 2$, where $L_{ij} = \{(P_i, P_j, z) | z \in E_0\} \subset E_0^3$ is the fiber of the projection $\pi : E_0^3 \to E_0^2$ over the μ_3-fixed point $P_{ij} \in E_0^2$. Let's denote the exceptional divisor of ϵ_3 over L_{ij} by $B_{ij} \subset \widetilde{E_0^3}$. Let $\epsilon_2 : \widetilde{E_0^2} \to E_0^2$ be the blowup at the 9 points $P_{ij} \in E_0^2$. We denoted the exceptional curve over P_{ij} by E_{ij}. Since the normal bundle of L_{ij} in E_0^3 is a trivial bundle, we have that

$$B_{ij} \cong L_{ij} \times \mathbb{P}^1.$$

The restriction of $\tilde{\pi} : \widetilde{E_0^3} \to \widetilde{E_0^2}$ to B_{ij} is a trivial elliptic fibration

$$\tilde{\pi}|_{B_{ij}} : L_{ij} \times \mathbb{P}^1 \cong B_{ij} \longrightarrow E_{ij} \cong \mathbb{P}^1.$$

For a point $P \in E_{ij}$, its fiber $\tilde{\pi}^{-1}(P)$ is isomorphic to E_0.

Next we blowup $\widetilde{E_0^3}$ along the 27 curves $C_{ijk}, i, j, k = 0, 1, 2$:

$$\tau : Bl_{C_{ijk}}\widetilde{E_0^3} \to \widetilde{E_0^3}.$$

We notice that B_{ij} contains three of the 27 curves, i.e. C_{ij0}, C_{ij1} and C_{ij2}. Let denote the exceptional divisor of τ over C_{ijk} by B_{ijk} and denote the strict transform of B_{ij} by $\hat{B}_{ij}$. Then the morphism

$$\pi' = \tilde{\pi} \circ \tau : Bl_{C_{ijk}}\widetilde{E_0^3} \to \widetilde{E_0^2}$$

restricts to

$$\pi' : \hat{B}_{ij} \cup B_{ij0} \cup B_{ij1} \cup B_{ij2} \longrightarrow E_{ij}.$$

For a point $P \in E_{ij}$, we denote its fiber of $\tilde{\pi}$ by $L_P = \tilde{\pi}^{-1}(P)$. One notices that L_P intersects C_{ijk} transversely in $\widetilde{E_0^3}$ for $k = 0, 1, 2$. We denote the fiber of π' over $P \in E_{ij}$ by $L'_P = \pi'^{-1}(P)$. Then L'_P consists of four components. One component of L'_P is the strict transform of L_P under τ, which we denote by $\hat{L_P}$. We denote $Q_{ijk} = L_P \cap C_{ijk}$ for $k = 0, 1, 2$. Then $\tau^{-1}(Q_{ijk})$ are the other three components of L'_P for $k = 0, 1, 2$. In Claim 4.2 below, We will show that B_{ijk} is a rational ruled surface over C_{ijk}. We have th $\tau^{-1}(Q_{ijk})$ is a fiber of the ruled surface B_{ijk}. Then the singular fiber of π' over $P \in E_{ij}$ consists of 4 components

$$\pi'^{-1}(P) = L'_p = \hat{L_P} \cup \tau^{-1}(Q_{ij0}) \cup \tau^{-1}(Q_{ij1}) \cup \tau^{-1}(Q_{ij2}),$$

as shown in Figure 6.

$$\begin{tikzcd}
E_0^3 & \widetilde{E_0^3} \arrow[l, "\epsilon_3"'] \arrow[d, "\tilde{\pi}"] & Bl_{C_{ijk}}\widetilde{E_0^3} \arrow[l, "\tau"'] \arrow[dl, "\pi'"] \\
E_0^2 \arrow[u, leftarrow, "\pi"] & \widetilde{E_0^2} \arrow[l, "\epsilon_2"'] &
\end{tikzcd}$$

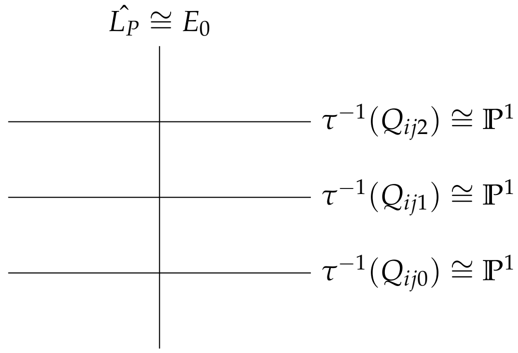


Figure 6: A Singular Fiber of $\pi' : Bl_{C_{ijk}}\widetilde{E_0^3} \to \widetilde{E_0^2}$.

Since π' is equivariant with respect to the μ_3-actions, it induces the morphism f'. We denote the image of $\hat{B}_{ij}$ under $q_3 : Bl_{C_{ijk}}\widetilde{E_0^3} \to \left(Bl_{C_{ijk}}\widetilde{E_0^3}\right)/\mu_3$ by

$$q_3(\hat{B}_{ij}) = G_{ij}.$$

We notice that $\hat{B}_{ij} \cong E_0 \times \mathbb{P}^1$ and the μ_3-action on $\hat{B}_{ij}$ is trivial on its second product component. Then we have that

$$G_{ij} \cong (E_0 \times \mathbb{P}^1)/\mu_3 \cong (E_0/\mu_3 \times)\mathbb{P}^1 \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

We denote the image of B_{ijk} under q_3 by

$$q_3(B_{ijk}) = G_{ijk}.$$

Since μ_3 acts on B_{ijk} trivially, we have that

$$G_{ijk} \cong B_{ijk}.$$

We have the following claim

Claim 4.2. G_{ijk} is isomorphic to the Hirzebruch surface $\mathbb{F}_1$.

Proof. The blowup $\tau : Bl_{C_{ijk}}\widetilde{E_0^3} \to \widetilde{E_0^3}$ restricted to the exceptional divisor B_{ijk} is the projectivization of the normal bundle $\mathcal{N}_{C_{ijk}|\widetilde{E_0^3}}$ of $C_{ijk} \subset \widetilde{E_0^3}$,

$$\tau|_{B_{ijk}} : B_{ijk} \cong \mathbb{P}\left(\mathcal{N}_{C_{ijk}|\widetilde{E_0^3}}\right) \longrightarrow C_{ijk}$$

Notice that $C_{ijk} \subset B_{ij} \subset \widetilde{E_0^3}$, we have the exact sequence:

$$0 \longrightarrow \mathcal{N}_{C_{ijk}|B_{ij}} \longrightarrow \mathcal{N}_{C_{ijk}|\widetilde{E_0^3}} \longrightarrow \left(\mathcal{N}_{B_{ij}|\widetilde{E_0^3}}\right)|_{C_{ijk}} \longrightarrow 0$$

This short exact sequence splits. It is because that the curve C_{ijk} is a complete intersection in $\widetilde{E_0^3}$. Consider the surface

$$S_k = \{(x, y, P_k) | x, y \in E_0\} \subset E_0^3,$$

it intersects L_{ij} at the μ_3-fixed point $P_{ijk} = (P_i, P_j, P_k)$ transversely. When we blowup E_0^3 along L_{ij} we denote the strict transform of S_k by $\widetilde{S_k}$, which is the blowup S_k at P_{ijk}. And we have that

$$C_{ijk} = \widetilde{S_k} \cap B_{ij},$$

is a complete intersection in $\widetilde{E_0^3}$. In particular, C_{ijk} is the exceptional divisor of $\widetilde{S_k} \to S_k$. Then we have that

$$\begin{aligned} \mathcal{N}_{C_{ijk}|\widetilde{E_0^3}} &= \mathcal{N}_{C_{ijk}|B_{ij}} \oplus \mathcal{N}_{C_{ijk}|\widetilde{S_k}} \\ &= \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1). \end{aligned}$$

Therefore, $G_{ijk} \cong B_{ijk} \cong \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\right) \cong \mathbb{F}_1$. $\square$

Recall our notation $\tilde{q}_2(E_{ij}) = D_{ij} \subset \widetilde{E_0^2}/\mu_3$. And $\tilde{q}_2$ restricted on E_{ij} is an isomorphism onto D_{ij}. Consider a point $Q \in D_{ij}$ and $P = \tilde{q}_2^{-1}(Q) \in E_{ij}$, the fiber $f'^{-1}(Q)$ of Q has four components, which are the images of the four components of $L'_P = \pi'^{-1}(P)$ under the quotient map q_3,

$$f'^{-1}(Q) = q_3(\hat{L_P}) \cup q_3(\tau^{-1}(Q_{ij0})) \cup q_3(\tau^{-1}(Q_{ij1})) \cup q_3(\tau^{-1}(Q_{ij2})).$$

Recall that μ_3 acts trivially on B_{ijk}, we have that

$$
\begin{aligned}
q_3(\hat{L_P}) &\cong \mathbb{P}^1, \\
q_3(\tau^{-1}(Q_{ijk})) &\cong \mathbb{P}^1.
\end{aligned}
$$

We conclude that the discriminant locus $\Delta(f')$ of f' is the union of $D_{ij}, i, j = 0, 1, 2$, which are 9 disjoint curves. All the singular fibers consist of four components as shown in Figure 7 $\square$

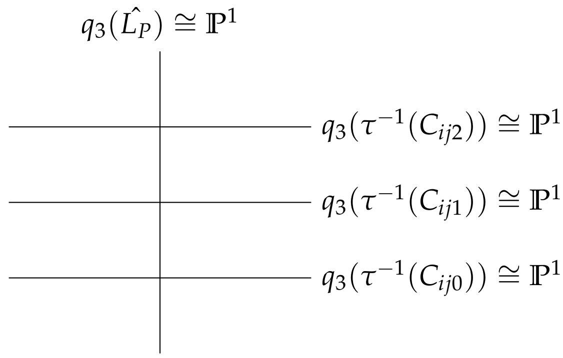


Figure 7: A Singular Fiber of $f' : \left(Bl_{C_{ijk}}\widetilde{E_0^3}\right)/\mu_3 \to \widetilde{E_0^2}/\mu_3$.

Now we are going to analyze the discriminant locus $\Delta(f)$ and singular fibers of $f : \left(Bl_{C_{ijk}}\widetilde{E_0^3}\right)/\mu_3 \to \mathbb{P}^2$, which is the composition of f' and $c : \widetilde{E_0^2}/\mu_3 \to \mathbb{P}^2$, where c is a contraction of 12 $(-1)-$curves. We first blow down 3 fibral components $\widetilde{E_0^2}/\mu_3 \to S$ to get the relative minimal elliptic surface $S \to \mathbb{P}^1$.

Then we further contract 9 disjoint sections $S \to \mathbb{P}^2$. As we have seen in the previous chapter, the images of $D_{ij}, i, j = 0, 1, 2$ under the contraction c is a dual Hesse arrangement. Therefore, we conclude that the discriminant locus $\Delta(f)$ of f is a dual Hesse arrangement in $\mathbb{P}^2$. The singular fiber of f over a smooth point P of $\Delta(f)$ is isomorphic to the singular of f' as shown in Figure 7.

Recall that a dual Hesse arrangement contains 9 lines and 12 triple points. We need to look into the singular fibers over triply points of $\Delta(f)$. We have denote the 12 triple points by $Q_1, ..., Q_9$,R_1, R_2, and R_3, see Figure 4. The contraction map $c : \widetilde{E_0^2}/\mu_3 \to \mathbb{P}^2$ is the blowup at the 12 points. First we need to identify the exceptional curves over each of the 12 points. Secondly, we will look into the preimage of the exceptional curves under the map f'. Then we can identify the fiber of each triple point as

$$f^{-1}(Q_i) = f'^{-1}(c^{-1}(Q_i)), \quad i = 1, ..., 9,$$

$$f^{-1}(R_i) = f'^{-1}(c^{-1}(R_i)), \quad i = 1, 2, 3.$$

Recall that we denote $C_i = \{(P_i, x) | x \in E_0\}$ is the fiber of $E_0^2 \to E_0$ over a μ_3-fixed point P_i. We denote its strict transform under the blowup $\widetilde{E_0^2} \to E_0^2$ by $\hat{C}_i$. And we also let $\tilde{q}_2(\hat{C}_i) = D_i$. Then the exceptional curve over $R_i \in \mathbb{P}^2$ is $D_i \subset \widetilde{E_0^2}/\mu_3$, i.e.

$$c^{-1}(R_i) = D_i \quad for \;\; i = 0, 1, 2.$$

We consider the projection $\pi : E_0^3 \to E_0^2$, we have that

$$\pi^{-1}(C_i) = \{(P_i, x, y) | x, y \in E_0\}.$$

Then we denote the strict transform of $\pi^{-1}(C_i)$ under the blowup $\epsilon_3 : \widetilde{E_0^3} \to E_0^3$ by $\widehat{\pi^{-1}(C_i)}$. One can check that $\widehat{\pi^{-1}(C_i)} \cong E_0^2$ and it intersects B_{ij} transversely for $j = 0, 1, 2$. Then we blow up $\widetilde{E_0^3}$ along C_{ijk}, $\tau : Bl_{C_{ijk}}\widetilde{E_0^3} \to \widetilde{E_0^3}$. Let's denote the strict transform of $\widehat{\pi^{-1}(C_i)}$ under τ by $\widehat{\widehat{\pi^{-1}(C_i)}}$. Then we have that

$$\widehat{\widehat{\pi^{-1}(C_i)}} \cong \widetilde{E_0^2},$$

where $\widetilde{E_0^2}$ is the blowup E_0^2 at the 9 μ_3-fixed points. The morphism $\pi' : Bl_{C_{ijk}}\widetilde{E_0^3} \to \widetilde{E_0^2}$ restricts to $\widehat{\widehat{\pi^{-1}(C_i)}}$ we have a morphism

$$\widehat{\widehat{\pi^{-1}(C_i)}} \longrightarrow \hat{C}_i,$$

which is equivariant with respect to the μ_3-actions. It induces a morphism between the quotient spaces,

$$\widehat{\widehat{\pi^{-1}(C_i)}}/\mu_3 \longrightarrow \hat{C}_i/\mu_3 = \tilde{q}_2(\hat{C}_i) = D_i,$$

which is the restriction of f' on $q_3(\widehat{\widehat{\pi^{-1}(C_i)}}) \subset \left(Bl_{C_{ijk}}\widetilde{E_0^3}\right)/\mu_3$. Then we have that

$$f'^{-1}(D_i) \cong \widetilde{E_0^2}/\mu_3.$$

Therefore, we conclude that the singular fiber over R_i

$$f^{-1}(R_i) \cong \widetilde{E_0^2}/\mu_3, \quad i = 1,2,3.$$

Similar discussion gives that

$$f^{-1}(Q_i) \cong \widetilde{E_0^2}/\mu_3, \quad i = 1,...,9.$$

We summarize the discussion above and have the following theorem

Theorem 4.3. *The elliptic threefold* $f : \left(Bl_{C_{ijk}}\widetilde{E_0^3}\right)/\mu_3 \to \mathbb{P}^2$ *constructed in the previous section has a general fiber isomorphic to* E_0*. Its discriminant locus* $\Delta(f)$ *is a dual Hesse arrangement. Furthermore,*

- *The singular fiber over a smooth point of* $\Delta(f)$ *has four components as shown in Figure 7.*
- *The singular fiber over a triple point of* $\Delta(f)$ *is isomorphic to the rational surface* $\widetilde{E_0^2}/\mu_3$*.*

4.5 The Hodge Structure of The Elliptic Threefold

In the previous section we constructed a smooth elliptic threefold $f : \left(Bl_{C_{ijk}}\widetilde{E_0^3}\right)/\mu_3 \to \mathbb{P}^2$. We are going to calculate its Hodge numbers in this section.

The Künneth formula and Hodge decompotisition give us, (See(Voisin, 2002))

$$H^{p,q}(E_0^3) = \bigoplus_{\substack{a_1+a_2+a_3=p \\ b_1+b_2+b_3=q}} H^{a_1,b_1}(E_0) \otimes H^{a_2,b_2}(E_0) \otimes H^{a_3,b_3}(E_0)$$

Since $H^{p,q}(E_0) = 1$ for $i, j = 0, 1$, we have the Hodge diamond for E_0^3,

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 3 & & 3 & & \\
 & 3 & & 9 & & 3 & \\
1 & & 9 & & 9 & & 1 \\
 & 3 & & 9 & & 3 & \\
 & & 3 & & 3 & & \\
 & & & 1 & & &
\end{array}$$

Now we are going to calculate the Hodge numbers of $\widetilde{E_0^3}$, which is the the blowup of E_0^3 along the nine disjoint curves $L_{ij} = \{(P_i, P_j, x) | x \in E_0\}, i, j = 0, 1, 2$. For the integral cohomology of a blownup space, we have the following theorem, (See (Voisin, 2002) Theorem 7.31)

Theorem 4.4. Let X be a Kähler manifold and let Z be a submanifold with codimension r. Consider the blowup $\tau : \widetilde{X}_Z \longrightarrow X$ of X along Z. Let $E = \tau^{-1}(Z)$ be the exceptional diviosr. When τ restricts on E, $\tau|_E : E \to Z$, it is a projective bundle of rank r-1 over Z. Let j be the embedding $j : E \to \widetilde{X}_Z$. We also denote the first Chern class of $\mathcal{O}_E(1)$ by $h = c_1(\mathcal{O}_E(1)) \in H^2(E, \mathbb{Z})$. Then we have an isomorphism of Hodge structures:

$$\tau^* \oplus \sum_i j_* \circ h^i \circ (\tau|_E)^* : H^k(X, \mathbb{Z}) \left(\bigoplus_{i=0}^{r-2} H^{k-2i-2}(Z, \mathbb{Z}) \right) \longrightarrow H^k(\widetilde{X}_Z, \mathbb{Z}).$$

Here h^i is given by taking the cup-product with $h^i \in H^{2i}(E, \mathbb{Z})$ and j_* is the Gysin morphism induced by j.

Now we can apply the theorem to $\widetilde{E_0^3} \to E_0^3$. The submanifold L_{ij} is of codimension $r = 2$ and isomorphic to E_0. Apply the theorem above and we have the Hodge diamond of $\widetilde{E_0^3}$:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 3 & & 3 & & \\ & 3 & & 18 & & 3 & \\ 1 & & 18 & & 18 & & 1 \\ & 3 & & 18 & & 3 & \\ & & 3 & & 3 & & \\ & & & 1. & & & \end{array}$$

Next we are going to calculate Hodge numbers of $Bl_{C_{ijk}}\widetilde{E_0^3}$, which is the blowup of $\widetilde{E_0^3}$ along 27 disjoint rational curves C_{ijk}, for $i, j, k = 0, 1, 2$. Recall that $C_{ijk} \cong \mathbb{P}^1$. Apply the theorem above, we can see that only the Hodge groups $H^{1,1}$ and $H^{2,2}$ get additional contributions from the second blowup. We have the Hodge numbers of $Bl_{C_{ijk}}\widetilde{E_0^3}$,

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 3 & & 3 & & \\
 & 3 & & 45 & & 3 & \\
1 & & 18 & & 18 & & 1 \\
 & 3 & & 45 & & 3 & \\
 & & 3 & & 3 & & \\
 & & & 1. & & &
\end{array}
$$

Now we are going to calculate the Hodge numbers of the quotient space $\left(Bl_{C_{ijk}}\widetilde{E_0^3}\right)/\mu_3$. In order to do this, we need to study the μ_3-action on the $(p,q)-$forms of $Bl_{C_{ijk}}\widetilde{E_0^3}$.

Claim 4.5. For a complex manifold X and a group G acting on X, We have that ,

$$H^{p,q}(X/G) = H^{p,q}(X)^G,$$

where $H^{p,q}(X)^G$ is the group of $G-$invariant $(p,q)-$forms on X.

Proof. (Smith, 1983).

□

From Theorem 4.4 we see that the two blowups $\tau \circ \epsilon_3 : Bl_{C_{ijk}}\widetilde{E_0^3} \to E_0^3$ induce isomorphisms

$$H^{p,q}(E_0^3) \cong H^{p,q}\left(Bl_{C_{ijk}}\widetilde{E_0^3}\right)$$

for $(p,q) = (0,0),(0,1),(1,0),(0,2),(2,0),(0,3)$ and $(0,3)$. Since both blowups are equivariant with respect to the μ_3-actions, it suffices to look into the μ_3-actions on the $(p,q)-$forms of E_0^3 for (p,q) listed above.

The space of $(0,0)-$forms on E_0^3 is generated by a non-zero constant function, which is obviously μ_3-invariant.

The space of $(0,1)-$forms on E_0^3 is generated by the forms $d\bar{z}_i$, for $i = 1,2,3$, where z_i is the holomorphic coordinates of the i-th component of E_0^3. The the μ_3-action transforms $d\bar{z}_i$ to $\bar{\omega} d\bar{z}_i$. Therefore, there are no μ_3-invariant $(0,1)$-forms. The space $(0,2)-$forms are generated by $d\bar{z}_i \wedge d\bar{z}_j$ for $i,j = 1,2,3$. The μ_3-action transforms $d\bar{z}_i \wedge d\bar{z}_j$ to $\bar{\omega}^2 d\bar{z}_i \wedge d\bar{z}_j$. Therefore, there are no μ_3-invariant (0,2)-fomrs. The space of (0,3)-forms of E_0^3 are generated by $d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3$, which is translated by μ_3 to $\bar{\omega}^3 \bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 = d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3$. Therefore all the $(0,3)$-forms are μ_3-invariant. Similar discussions can be applied to the $(p,0)$-forms. And we have the Hodge numbers

$$h^{0,0}\left(\left(Bl_{C_{ijk}}\widetilde{E_0^3}\right)/\mu_3\right) = h^{0,3} = h^{3,0} = 1,$$
$$h^{0,1}\left(\left(Bl_{C_{ijk}}\widetilde{E_0^3}\right)/\mu_3\right) = h^{0,2} = h^{1,0} = h^{2,0} = 0$$

For the $(2,1)-$forms on E_0^3, we have

$$\tau^* : H^{2,1}(\widetilde{E_0^3}) \cong H^{2,1}(Bl_{C_{ijk}}\widetilde{E_0^3}),$$

where the isomorphism is the pull-back induced by the blowup $\tau : Bl_{C_{ijk}}\widetilde{E_0^3} \to \widetilde{E_0^3}$. The (2,1)-forms of $Bl_{C_{ijk}}\widetilde{E_0^3}$ can be identified with the $(2,1)-$forms of $\widetilde{E_0^3}$, which come from two parts:

$$H^{2,1}(\widetilde{E_0^3}) \cong \epsilon_3^*(H^{2,1}(E_0^3)) \oplus j_* \circ (\epsilon|_{E_{ij}})^*(H^{1,0}(L_{ij})),$$

where $\epsilon_3 : \widetilde{E_0^3} \to E_0^3$ is the blowup along L_{ij} and j is the embedding $j : B_{ij} \to \widetilde{E_0^3}$ and B_{ij} is the exceptional divisor over L_{ij}.

The space of the (2,1)-forms of E_0^3 are generated by $dz_i \wedge dz_j \wedge d\bar{z}_k$, which is transformed by μ_3 to $\omega dz_i \wedge dz_j \wedge d\bar{z}_k$. Therefore $\epsilon_3^*(H^{2,1}(E_0^3, \mathbb{Z}))$ has no μ_3-invariant forms.

For the second part, the μ_3-action of E_0^3 restricted on L_{ij} is the same action as μ_3 acting on E_0. Therefore there is no μ_3-invariant (1,0)-form on L_{ij}. Also the blowup ϵ is equivariant with respect to μ_3-actions, and the Gysin map j_* is a composition of the pull back j^* and Poincaré duality, which are both equivariant with respect to the μ_3-actions. Therefore there is no μ_3-invariant (2,1)-forms of $\widetilde{E_0^3}$ that belong to the second part.

Then we have that $H^{2,1}(\widetilde{E_0^3})$ has no μ_3-invariant forms. Since the blowup $\tau : Bl_{C_{ijk}}\widetilde{E_0^3} \longrightarrow \widetilde{E_0^3}$ is equivariant with respect to the μ_3-acitons, it induces an equivariant isomorphism between the groups of the (2,1)-forms. We conclude that there is no non-zero μ_3-invariant (2,1)-forms on $Bl_{C_{ijk}}\widetilde{E_0^3}$. Similar discussions apply to the (1,2)-forms. We conclude that

$$h^{2,1}\left(\left(Bl_{C_{ijk}}\widetilde{E_0^3}\right)/\mu_3\right) = h^{1,2} = 0.$$

At last, we consider the $(1,1)-$forms of $Bl_{C_{ijk}}\widetilde{E_0^3}$, which come from 3 parts: $H^{1,1}(E_0^3)$, $H^{0,0}(L_{ij})$ and $H^{0,0}(C_{ijk})$. Recall our notations, $\epsilon_3 : \widetilde{E_0^3} \to E_0^3$ is the blowup along L_{ij} and $\tau : Bl_{C_{ijk}}\widetilde{E_0^3} \to \widetilde{E_0^3}$ is the blowup along C_{ijk}. We denote the exceptional divisor over L_{ij} by B_{ij} and the exceptional divisor over C_{ijk} by B_{ijk}. Let $j_1 : B_{ij} \to \widetilde{E_0^3}$ and $j_2 : B_{ijk} \to Bl_{C_{ijk}}\widetilde{E_0^3}$ be the embeddings. Then by Theorem 4.4 we have

$$H^{1,1}\left(Bl_{C_{ijk}}\widetilde{E_0^3}\right) \cong \tau^* \circ \epsilon_3^*(H^{1,1}(E_0^3)) \oplus \tau^* \circ j_{1*} \circ \epsilon_3|_{B_{ij}}^*(H^{0,0}(L_{ij})) \oplus j_{2*} \circ \tau|_{B_{ijk}}^*(H^{0,0}(C_{ijk})).$$

In the first part, $H^{1,1}(E_0^3)$ is generated by the forms $dz_i \wedge d\bar{z}_j$, which are translated by μ_3 to $\omega\bar{\omega}dz_i \wedge d\bar{z}_j = dz_i \wedge d\bar{z}_j$. Therefore $H^{1,1}(E_0^3)$ is μ_3-invariant. For the second part, $H^{0,0}(L_{ij})$ is generated by a non-zero constant function, which is obviously invariant under the μ_3-action. By the same argument, $H^{0,0}(C_{ijk})$ is invariant under the μ_3-action for $i, j, k = 0, 1, 2$. Since all the blowups and embeddings are equivariant with respect to the μ_3-actions, we have that

$$h^{1,1}\left(\left(Bl_{C_{ijk}}\widetilde{E_0^3}\right)/\mu_3\right) = h^{1,1}\left(Bl_{C_{ijk}}\widetilde{E_0^3}\right) = 45.$$

To conclude, we have the Hodge numbers of $\left(Bl_{C_{ijk}}\widetilde{E_0^3}\right)/\mu_3$ as following,

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 45 & & 0 & \\ 1 & & 0 & & 0 & & 1 \\ & 0 & & 45 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1. & & & \end{array}$$

4.6 The Mordell-Weil Rank of The Elliptic Threefold

Cogolludo and Libgober built a relation between the Mordell-Weil rank of an elliptic threefold over a rational surface to the Alexander polynomial of the discriminant locus (Cogolludo-Agustín and Libgober, 2014).

It follows from Theorem 4.3 that the discriminant locus $\Delta(f)$ of $f : \left(Bl_{C_{ijk}}\widetilde{E_0^3}\right)/\mu_3 \to \mathbb{P}^2$ is a dual Hesse arrangement. By the uniqueness of dual Hesse arrangement Prop 3.13, we may let $\Delta(f)$ be the locus defined by the equation $(x^3 - y^3)(x^3 - 1)(y^3 - 1) = 0$. Moreover, the monodromy

$$\pi_1(\mathbb{P}^2 - \Delta(f)) \to AutH_1(E_0, \mathbb{Z})$$

of the elliptic fibration f sends each meridian of any component of $\Delta(f)$ to the same element of μ_3. Also the elliptic fibration f is isotrivial and its modular function $J \equiv 0$. We have that $f : \left(Bl_{C_{ijk}} \widetilde{E_0^3}\right) / \mu_3 \rightarrow \mathbb{P}^2$ is birational to the elliptic threefold defined by $u^2 + v^3 = (x^3 - y^3)(x^3 - 1)(y^3 - 1)$ in $\mathbb{C}^4$ and the elliptic fibration induced by the projection $\mathbb{C}^4(u, v, x, y) \rightarrow \mathbb{C}^2(x, y)$ (Libgober, 2012).

Based on Cogolludo and Libgober's work (Cogolludo-Agustín and Libgober, 2014), we have that

Proposition 4.6. Let C be the curve defined by $F(x, y)$ in $\mathbb{C}^2$. Suppose that C intersects the line at the infinity transversely and 3 divides $degF(x, y)$. Consider the elliptic threefold defined by $u^2 + v^3 = F(x, y)$ in $\mathbb{C}^4$ and the elliptic fibration induced by the projection $\mathbb{C}^4(u, v, x, y) \rightarrow \mathbb{C}^2(x, y)$, if the Alexander polynomial of the complement of C in $\mathbb{P}^2$ is $(t^2 + t + 1)^s(t - 1)^k$, then the Mordell-Weil rank of the elliptic threefold is $2s$.

The Alexander polynomial of a dual Hesse arrangement in $\mathbb{P}^2$ is

$$\Delta(t) = (t - 1)^7(t^2 + t + 1)^2$$

see(Libgober, 1982) and (Libgober, 2012) Remark 4.1. Then we have that

Corollary 4.7. *The elliptic fibration* $f : \left(Bl_{C_{ijk}} \widetilde{E_0^3}\right) / \mu_3 \rightarrow \mathbb{P}^2$ *has Mordell-Weil rank equal to 4.*

CHAPTER 5

THE ELLIPTIC SURFACE E_1^2/μ_4

5.1 A μ_4-Action on The Elliptic Curve with j-Invariant 1728

Recall Example 2.7 the elliptic curve E_1 has j-invariant equal to 1728 and is isomorphic to $\mathbb{C}/\Lambda$ as abelian groups, where the lattice $\Lambda = \mathbb{Z} \oplus i\mathbb{Z}$. It has an automorphism g of order 4, which generates the cyclic group $\mu_4 = \{1, g, g^2, g^3\}$ acting on E_1. Consider the rotation of $\mathbb{C}$ by $\frac{\pi}{2}$ around the origin, we notice that such rotation preserves the lattice Λ. Then we let g be the automorphism induced from the rotation of $\mathbb{C}$.

There are 4 points of E_1 that have nontrivial stabilizers with respect the μ_4-action. To see this, we consider the four point $P_1 = 0$, $P_2 = \frac{1}{2} + \frac{i}{2}$, $Q_1 = \frac{1}{2}$ and $Q_2 = \frac{i}{2}$ in the fundamental domain $\{x + iy \mid 0 \leq x, y \leq 1\}$ of the lattice. The rotation acts on the four points as following:

$$\begin{aligned}
P_1 = 0 \quad &\longmapsto \quad 0 = P_1 \\
P_2 = \frac{1}{2} + \frac{i}{2} \quad &\longmapsto \quad -\frac{1}{2} + \frac{i}{2} \equiv P_2 \ (mod\ \Lambda) \\
Q_1 = \frac{1}{2} \quad &\longmapsto \quad \frac{i}{2} = Q_2 \\
Q_2 = \frac{i}{2} \quad &\longmapsto \quad -\frac{1}{2} \equiv Q_1 \ (mod\ \Lambda).
\end{aligned}$$

Taking the quotient of complex plane $\mathbb{C}$ with respect to the lattice Λ, we use the same notation for the images of the four points P_1, P_2, Q_1 and Q_2 in $\mathbb{C}/\Lambda \cong E_1$. Therefore $P_1, P_2 \in E_1$ are fixed by the μ_4-action, and $Q_1, Q_2 \in E_2$ have a stabilizer $\{1, g^2\} \subset \mu_4$ since g permutes Q_1 and Q_2.

We consider the quotient map with respect to the μ_4-action:

$$q_1 : E_1 \longrightarrow E_1/\mu_4,$$

it ramifies at P_1, P_2 of ramification index 4 and at Q_1, Q_2 of index 2. Then Riemann-Hurwitz Formula (see (Hartshorne, 1977) Chapter IV) says:

$$2g(E_1) - 2 = deg(q_1)(2g(E_1/\mu_4) - 2) + \sum_{p \in \{P_1, P_2, Q_1, Q_2\}} (e_p - 1).$$

Since $deg(q_1) = 4$ and the genus $g(E_1) = 1$, we have that $g(E_1/\mu_4) = 0$ and therefore E_1/μ_4 is isomorphic to $\mathbb{P}^1$.

5.2 The Elliptic Surface E_1^2/μ_4 and A Smooth Resolution

The μ_4-action on E_1 induces the diagonal action on the product surface $E_1^2 = E_1 \times E_1$. To be explicit, for $g \in \mu_4$, we have

$$g(x, y) = (gx, gx), \;\; for \;\; x, y \in E_1.$$

There are 16 points of E_1^2 that have a nontrivial stabilizer with respect to the μ_4-action. The four points $(P_i, P_j), i, j = 1, 2$ are fixed by the μ_4-action, and the other 12 points $(P_i, Q_j), (Q_i, P_j)$ and $(Q_i, Q_j), i, j = 1, 2$ have a stabilizer $\{1, g^2\}$.

Consider the quotient map with respect to the μ_4-action on E_1^2,

$$q_2 : E_1^2 \longrightarrow E_1^2/\mu_4,$$

it ramifies at (P_i, P_j) with ramification index 4 and ramifies at (P_i, Q_j), (Q_i, P_j) and (Q_i, Q_j) with ramification index 2. Therefore the quotient space E_1^2/μ_4 has 10 cyclic quotient singularities as following:

$$\begin{aligned} q_2(P_i, P_j), i, j = 1, 2, &\quad Type(4, 1) \\ q_2(P_i, Q_1) = q_2(P_i, Q_2), i = 1, 2 &\quad Type(2, 1) \\ q_2(Q_1, P_i) = q_2(Q_2, P_i), i = 1, 2 &\quad Type(2, 1) \\ q_2(Q_1, Q_1) = q_2(Q_2, Q_2), &\quad Type(2, 1) \\ q_2(Q_1, Q_2) = q_2(Q_2, Q_1), &\quad Type(2, 1) \end{aligned}$$

Remark 5.1. Here we use the notations (n, k) from (Lamotke, 2013) for the types of cyclic quotient singularities defined as following. Consider $\mathbb{C}^2$ and the finite group

$$G = < \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^k \end{bmatrix} >,$$

where $\alpha = exp\{\frac{2\pi i}{n}\}$. We say that $\mathbb{C}^2/G$ has a *cyclic quotient singularity of type* (n,k).

In order to have a smooth resolution of E_1^2/μ_4, we blow up the 16 points of E_1^2 where q_2 ramifies,

$$\epsilon : \widetilde{E_1^2} \longrightarrow E_1^2.$$

The μ_4-action extends to $\widetilde{E_1^2}$ continuously. From a similar discussion as in the section 3.2, we can see that μ_4 acts trivially on the exceptional curves over the μ_4-fixed points $(P_i, P_j), i, j = 1, 2$. Then the generator g permutes the exceptional curves over the points that it permutes on E_1^2, and g^2 fixes all the exceptional curves pointwisely. We consider quotient map

$$\tilde{q}_2 : \widetilde{E_1^2} \longrightarrow \widetilde{E_1^2}/\mu_4.$$

Since all the cyclic quotient singularities of E_1^2/μ_4 are of type either $(4,1)$ or $(2,1)$, the quotient space $\widetilde{E_1^2}/\mu_4$ is a smooth resolution of E_1^2/μ_4. For a detailed discussion about resolution of cyclic singularities of complex surfaces, please see (Lamotke, 2013).

We denote $E_{P_iP_j}$, $E_{P_iQ_j}$, $E_{Q_iP_j}$ and $E_{Q_iQ_j}$ to be the exceptional divisors over (P_i, P_j), (P_i, Q_j), (Q_i, P_j) and (Q_i, Q_j) respectively. The quotient map $\tilde{q}_2$ is totally ramified along the 16 exceptional curves with ramification index 4 along $E_{P_iP_j}$ and index 2 along the others. The images of the exceptional curves under the quotient map are denoted by

$$\begin{aligned}
\tilde{q}_2(E_{P_iP_j}) &= D_{P_iP_j} \\
\tilde{q}_2(E_{P_iQ_1}) = \tilde{q}_2(E_{P_iQ_2}) &= D_{P_iQ} \\
\tilde{q}_2(E_{Q_1P_i}) = \tilde{q}_2(E_{Q_2P_i}) &= D_{QP_i} \\
\tilde{q}_2(E_{Q_1Q_2}) = \tilde{q}_2(E_{Q_2Q_1}) &= D_{QQ'} \\
\tilde{q}_2(E_{Q_1Q_1}) = \tilde{q}_2(E_{Q_2Q_2}) &= D_{QQ}.
\end{aligned}$$

Applying the projection formula, we have that,

$$\begin{aligned}
D^2_{P_iP_j} &= -4, \\
D^2_{P_iQ} &= -2, \\
D^2_{QP_j} &= -2, \\
D^2_{QQ'} &= -2, \\
D_{QQ} &= -2.
\end{aligned}$$

The smooth surface $\widetilde{E_1^2}/\mu_4$ has an elliptic fibration. We consider the composition of the projection of E_1^2 to its first component $\pi : E_1^2 \to E_1$ and the blowup $\epsilon : \widetilde{E_1^2} \to E_1^2$,

$$\tilde{\pi} = \pi \circ \epsilon : \widetilde{E_1^2} \longrightarrow E_1.$$

The general fiber of $\tilde{\pi}$ is isomorphic to E_1. There are 4 singular fibers of $\tilde{\pi}$, which are the fibers over P_1, P_2, Q_1 and Q_2. We let

$$C_{P_i} = \pi^{-1}(P_i) = \{(P_i, x) | x \in E_1\}$$

$$C_{Q_i} = \pi^{-1}(Q_i) = \{(Q_i, x) | x \in E_1\}, \ \ for \ \ i = 1, 2.$$

Let's denote the strict transforms of C_{P_i} and C_{Q_i} under the blowup ϵ by $\widehat{C}_{P_i}$ and $\widehat{C}_{Q_i}$. Then the singular fibers of $\tilde{\pi}$ as divisors in $\widetilde{E_1^2}$ are

$$\begin{aligned}
\tilde{\pi}^{-1}(P_1) &= \widehat{C}_{P_1} + E_{P_1Q_2} + E_{P_1Q_1} + E_{P_1P_2} + E_{P_1P_1} \\
\tilde{\pi}^{-1}(P_2) &= \widehat{C}_{P_2} + E_{P_2Q_2} + E_{P_2Q_1} + E_{P_2P_2} + E_{P_2P_1} \\
\tilde{\pi}^{-1}(Q_1) &= \widehat{C}_{Q_1} + E_{Q_1Q_2} + E_{Q_1Q_1} + E_{Q_1P_2} + E_{Q_1P_1} \\
\tilde{\pi}^{-1}(Q_2) &= \widehat{C}_{Q_2} + E_{Q_2Q_2} + E_{Q_2Q_1} + E_{Q_2P_2} + E_{Q_2P_1},
\end{aligned}$$

as shown in Figure 8.

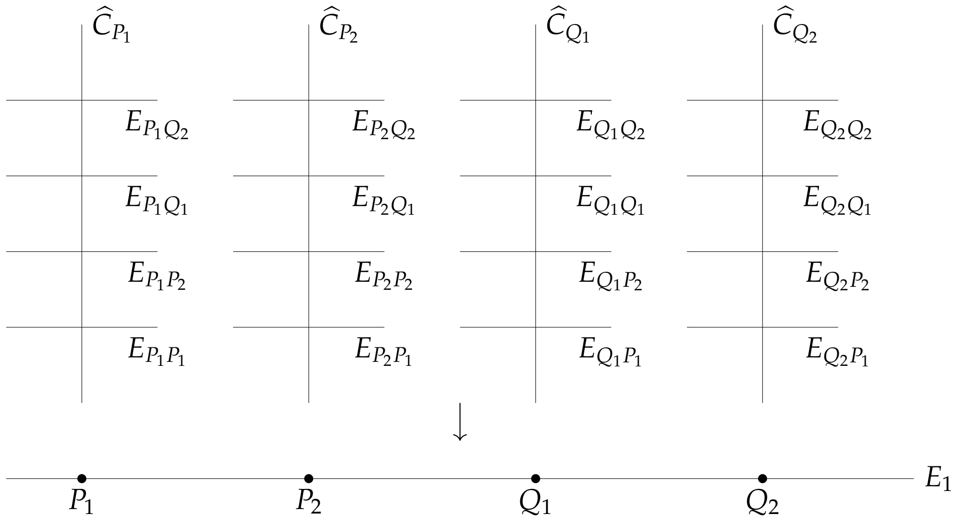


Figure 8: Singular Fibers of $\tilde{\pi} : \widetilde{E_1^2} \to E_1$

Since both π and ϵ are equivariant with respect to the μ_4-actions, so is $\tilde{\pi}$. Therefore we have the following diagram,

$$\begin{array}{ccc} \widetilde{E_1^2} & \xrightarrow{\tilde{q}_2} & \widetilde{E_1^2}/\mu_4 \\ \Big\downarrow{\scriptstyle \tilde{\pi}} & & \Big\downarrow{\scriptstyle f} \\ E_1 & \xrightarrow{q_1} & E_1/\mu_4 \end{array}$$

where f is induced by $\tilde{\pi}$ and it is an elliptic fibration over $\mathbb{P}^1$ with a general fiber isomorphic to E_0.

5.3 The Singular Fibers of $\widetilde{E_1^2}/\mu_4 \to \mathbb{P}^1$

Now we are going to study the elliptic surface $\widetilde{E_1^2}/\mu_4 \to \mathbb{P}^1$. First we will study its singular fibers, then we will construct a relative minimal model. The singular fibers of the relative minimal model will be classified in Kodaira's notations (see Table I). We will see that $\widetilde{E_1^2}/\mu_4$ is a rational elliptic surface. We will give a pencil of cubics inducing the relative minimal model in the following sections.

There are three singular fibers of $\widetilde{E_1^2}/\mu_4 \to \mathbb{P}^1$ over the 3 branched points of $q_1 : E_1 \to E_1/\mu_4$, which we denote by

$$\begin{aligned} [P_1] &= q_1(P_1), \\ [P_2] &= q_1(P_2), \\ [Q] &= q_1(Q_1) = q_1(Q_2). \end{aligned}$$

For $[P_1] \in E_1/\mu_4$, its preimage in E_1 is $q_1^{-1}([P_1]) = P_1$ and $\tilde{\pi}^{-1}(P_1)$ has 5 components in $\widetilde{E_1^2}$ as shown in Figure 8. Then we take the quotient with respect to the μ_4−action $\tilde{q}_2 : \widetilde{E_1^2} \to \widetilde{E_1^2}/\mu_4$. Let's denote the images of $\widehat{C}_{P_i}$ by

$$\begin{aligned} B_{P_i} &= \tilde{q}_2(\widehat{C}_{P_i}), \quad i = 1, 2, \\ B_Q &= \tilde{q}_2(\widehat{C}_{Q_1}) = \tilde{q}_2(\widehat{C}_{Q_2}). \end{aligned}$$

Now we can see that $\tilde{q}_2(\tilde{\pi}^{-1}(P_1))$ has 4 components, which are

$$B_{P_1}, \ D_{P_1P_1}, \ D_{P_1P_2}, \ D_{P_1Q}.$$

They are the components of the singular fiber $f^{-1}([P_1])$. To find the multiplicity of each components, we can apply a local calculation as we did in analyzing the singular fibers of $\widetilde{E_0^2}/\mu_3 \to E_0/\mu_3$ (see Section 3.4). Then we have the singular fiber of f over $[P_1]$ as a divisor in $\widetilde{E_1^2}/\mu_4$,

$$f^{-1}([P_1]) = 4B_{P_1} + D_{P_1P_1} + D_{P_1P_2} + 2D_{P_1Q}.$$

Similarly we have the singular fibers of f over $[P_2]$ and $[Q]$,

$$\begin{aligned} f^{-1}([P_2]) &= 4B_{P_2} + D_{P_2P_1} + D_{P_2P_2} + 2D_{P_2Q}, \\ f^{-1}([Q]) &= 2B_Q + D_{QP_1} + D_{QP_2} + D_{QQ'} + D_{QQ}. \end{aligned}$$

The singular fibers of $f : \widetilde{E_1^2}/\mu_4 \longrightarrow \mathbb{P}^1$ is shown in Figure 9.

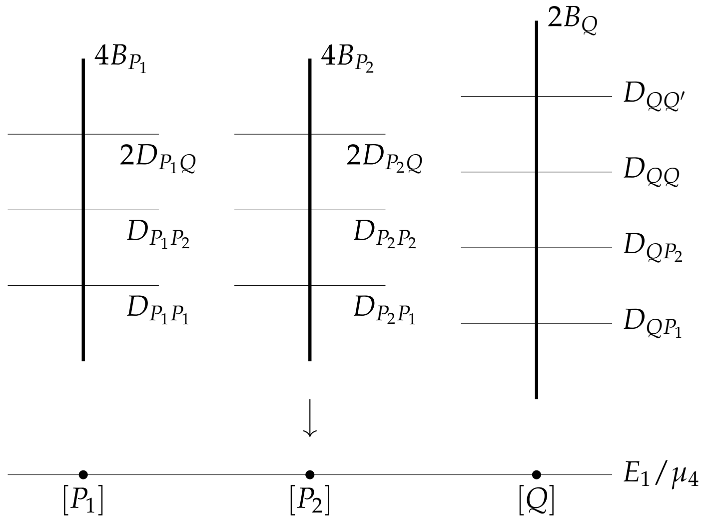


Figure 9: Singular Fibers of $f : \widetilde{E_1^2}/\mu_4 \longrightarrow \mathbb{P}^1$

5.4 The Relative Minimal Model of $\widetilde{E_1^2}/\mu_4$

In the previous chapter we calculated the self-intersection number of each component of all the singular fibers of $\widetilde{E_0^2}/\mu_3 \to \mathbb{P}^1$. Now we do the same calculation for the elliptic surface $f : \widetilde{E_1^2}/\mu_4 \to \mathbb{P}^1$. Applying Projection Formula Theorem A.7 to the quotient map $\tilde{q}_2$,

$$\begin{aligned}
\tilde{q}_{2*}(\tilde{q}_2^*(B_{P_i}) \cdot \hat{C}_{P_i}) &= B_{P_i} \cdot \tilde{q}_{2*}(\hat{C}_{P_i}) \\
\tilde{q}_{2*}(\hat{C}_{P_i} \cdot \hat{C}_{P_i}) &= 4B_{P_i} \cdot B_{P_i} \\
-4 &= 4B_{P_i}^2 \\
-1 &= B_{P_i}^2.
\end{aligned}$$

Here we notice that $\tilde{q}_2$ restricted to $\widehat{C}_{P_i}$ is a 4 to 1 covering of B_{P_i}, therefore $\tilde{q}_{2*}(\hat{C}_{P_i}) = 4B_{P_i}$. Also notice that $\tilde{q}_2$ restricted to $\widehat{C}_{Q_1}$ is a 2 to 1 covering of B_Q, therefore $\tilde{q}_{2*}(\hat{C}_{Q_1}) = 2B_Q$, and we have

$$\begin{aligned}
\tilde{q}_{2*}(\tilde{q}_2^*(B_Q)\cdot\widehat{C}_{Q_1}) &= B_Q\cdot\tilde{q}_{2*}(\widehat{C}_{Q_1}) \\
\tilde{q}_{2*}((\widehat{C}_{Q_1}+\widehat{C}_{Q_2})\cdot\widehat{C}_{Q_1}) &= 2B_Q\cdot B_Q \\
-4+0 &= 2B_Q^2 \\
-2 &= B_Q^2.
\end{aligned}$$

Similarly, we have the self-intersections of the other components of singular fibers

$$\begin{aligned}
&D^2_{P_1Q} = D^2_{P_2Q} = -2, \\
&D^2_{P_1P_2} = D^2_{P_2P_2} = -4, \\
&D^2_{P_1P_1} = D^2_{P_2P_1} = -4, \\
&D^2_{QP_1} = D^2_{QP_2} = D^2_{QQ} = D^2_{QQ'} = -2.
\end{aligned}$$

The singular fiber $f^{-1}([Q])$ is of Kodaira Type I_0^* (See Table I). And the singular fibers $f^{-1}([P_1])$ and $f^{-1}([P_2])$ have a component B_{P_1} and B_{P_2} of self-intersection (-1). In order to have a relative minimal model, we first contract B_1 and B_2. Then the image of $f^{-1}([P_1])$ and $f^{-1}([P_2])$ also have a (-1)-component, which is the image of D_{P_1Q} and D_{P_2Q}. We further contract the two (-1)-components. Then we have a relative minimal model of the elliptic surface $\widetilde{E_1^2}/\mu_4 \to \mathbb{P}^1$,

$$\begin{tikzcd}
\widetilde{E_1^2}/\mu_4 \arrow[r, "c"] \arrow[dr, "f"'] & S \arrow[d, "f'"] \\
& E_1/\mu_4 \cong \mathbb{P}^1
\end{tikzcd}$$

where $c : \widetilde{E_1^2}/\mu_4 \to S$ is the successive contraction of the fibral components B_{P_1},B_{P_2}, D_{P_1Q} and D_{P_2Q}.

Following the discussion above, the minimal elliptic surface $f' : S \to \mathbb{P}^1$ has three singular fibers. The singular fiber $f'^{-1}([Q])$ over $[Q]$ is of Kodaira Type I_0^* as stated above. And the singular fiber $f'^{-1}([P_i])$ over $[P_i]$ has two components $\widehat{D}_{P_iP_1}$ and $\widehat{D}_{P_iP_2}$, which are the images of $D_{P_iP_1}$ and $D_{P_iP_2}$ under the contraction map c. The two components are both (-2)-rational curves and they intersect at a double point. Therefore the singular fiber over $[P_i]$ is of Kodaria Type III for $i = 1, 2$ (See Table I). The singular fibers of $f' : S \to \mathbb{P}^1$ is shown in Figure 10.

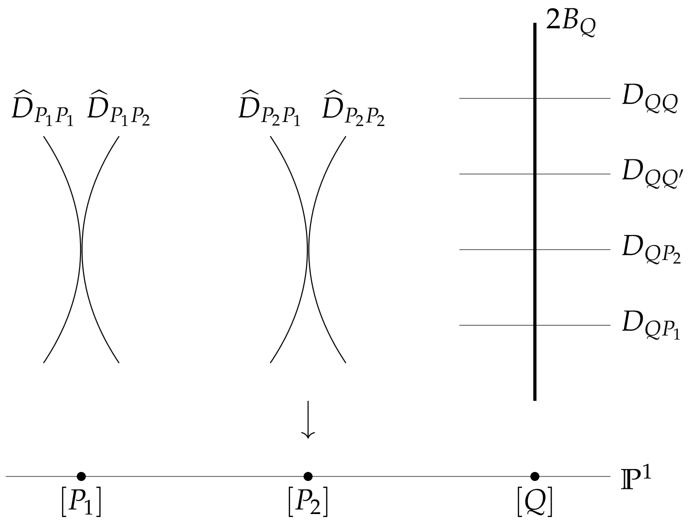


Figure 10: Singular Fibers of The Minimal Elliptic Surface $f' : S \to \mathbb{P}^1$

Due to Remark 3.1, the topological Euler characteristic of S is the sum of the topological Euler characteristic of its singular fibers and we have

$$\begin{aligned} e(S) &= e(III) + e(III) + e(I_0^*) \\ &= 3 + 3 + 6 = 12. \end{aligned}$$

Then Lemma 2.17 and Corollary 2.16 tell us that S is a rational elliptic surface. Furthermore, due to Lemma 2.19, S can be represented as a blowup of $\mathbb{P}^2$ at the base points of a pencil of cubics. We are going to determine a pencil of cubics that induces the rational elliptic surface $S \to \mathbb{P}^1$ in the next section.

5.5 A Pencil of Cubics Inducing The Relative Minimal Model of E_1^2/μ_4.

In this section, we will find a pencil of cubics in $\mathbb{P}^2$, which induces a representation of the relative minimal elliptic surface S as a 9-fold blowup of $\mathbb{P}^2$. We first find several sections of $S \to \mathbb{P}^1$. Then we contract 9 selected curves, sections or singular fiber components, in a chosen order. We will show that the strict transforms of singular fiber components form 3 cubic curves in $\mathbb{P}^2$. We further show that there is a pencil of cubics in $\mathbb{P}^2$ that contains the 3 cubic curves as its singular members. Resolving the base points of the pencil of cubics is the 9-fold blowup that induces $S \to \mathbb{P}^1$.

We look at the two sections of $\pi : E_1^2 \longrightarrow E_1$:

$$s_1 = \{(x, P_1) | x \in E_1\},$$

$$s_2 = \{(x, P_2) | x \in E_1\}.$$

Let $\hat{s}_1$ and $\hat{s}_2$ be their strict transforms in $\widetilde{E_1^2}$. It is easy to see that $\hat{s}_i^2 = -4$, since s_i passes through the 4 points (P_1, P_i), (P_2, P_i), (Q_1, P_i) and (Q_2, P_i). Then we denote their images under the composition of the quotient map $\tilde{q}_2 : \widetilde{E_1^2} \to \widetilde{E_1^2}/\mu_4$ and the contraction map $c : \widetilde{E_1^2}/\mu_4 \to S$ by

$$s_{[i]} = c \circ \tilde{q}_2(\hat{s}_i).$$

By the projection formula, we have

$$s_{[i]}^2 = -1.$$

Due to Lemma 3.3, we have that $s_{[i]}$ is a section of $S \to \mathbb{P}^1$ for $i = 1, 2$.

We can find more sections of $S \to \mathbb{P}^1$ as we did in Section 3.6. Let g be a generator of μ_4 acting on E_1. We consider the following three automorphisms ϕ_1, ϕ_{-1} and ϕ_i of E_1^2, which act on a point $(x, y) \in E_1^2$ as following

$$\begin{aligned}
\phi_1 &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x+y \end{bmatrix} \\
\phi_{-1} &= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -x+y \end{bmatrix} \\
\phi_i &= \begin{bmatrix} 1 & 0 \\ g & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ gx+y \end{bmatrix}.
\end{aligned}$$

Here 1 is regarded as the identity map on E_1 and -1 is the involution of E_1. These automorphisms of E_1^2 act on s_1 and s_2, then we have the following curves:

$$
\begin{aligned}
\phi_1(s_1) &= \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ P_1 \end{bmatrix} | x \in E_1 \right\} = \{(x, x+P_1) | x \in E_1\} = d_1, \\
\phi_1(s_2) &= \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ P_2 \end{bmatrix} | x \in E_1 \right\} = \{(x, x+P_2) | x \in E_1\} = d_2, \\
\phi_i(s_1) &= \left\{ \begin{bmatrix} 1 & 0 \\ g & 1 \end{bmatrix} \begin{bmatrix} x \\ P_1 \end{bmatrix} | x \in E_1 \right\} = \{(x, gx+P_1) | x \in E_1\} = b_1, \\
\phi_i(s_2) &= \left\{ \begin{bmatrix} 1 & 0 \\ g & 1 \end{bmatrix} \begin{bmatrix} x \\ P_2 \end{bmatrix} | x \in E_1 \right\} = \{(x, gx+P_2) | x \in E_1\} = b_2, \\
\phi_{-1}(s_1) &= \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ P_1 \end{bmatrix} | x \in E_1 \right\} = \{(x, -x+P_1) | x \in E_1\} = c_1, \\
\phi_{-1}(s_2) &= \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ P_2 \end{bmatrix} | x \in E_1 \right\} = \{(x, -x+P_2) | x \in E_1\} = c_2.
\end{aligned}
$$

Let $\hat{b}_i$, $\hat{c}_i$ and $\hat{d}_i$ be the strict transforms of b_i, c_i and d_i in $\widetilde{E_1^2}$ and let $b_{[i]}$, $c_{[i]}$ and $d_{[i]}$ be the images of $\hat{b}_i$, $\hat{c}_i$ and $\hat{d}_i$ under the map $c \circ \tilde{q}_2 : \widetilde{E_1^2} \to S$. One can check that $b_{[i]}$, $c_{[i]}$ and $d_{[i]}$ are sections of $S \to \mathbb{P}^1$ for $i = 1, 2$.

Now we have 8 sections of $S \to \mathbb{P}^1$, which are

$$s_{[1]},\ s_{[2]},\ b_{[1]},\ b_{[2]},\ c_{[1]},\ c_{[2]},\ d_{[1]},\ d_{[2]}.$$

First we notice that all the 8 sections are disjoint. This is because that $s_1, s_2, b_1, b_2, c_1, c_2, d_1$ and d_2 intersect each other transversely only at the points, at which we blow up E_1^2. In order to find which fiber components that each section intersects, we can apply a similar discussion as we did in Section 3.6.

We list the singular fiber components that the 8 sections intersect in Table III, as shown in Figure 11,

TABLE III Eight Disjoint Sections of The Relative Minimal Model to $E_1^2/\mu_4 \to \mathbb{P}^1$ and The Singular Fiber Components They Intersect

Section	Intersects Singular Fibers Components
$s_{[1]}$	$\widehat{D}_{P_1P_1}, \widehat{D}_{P_2P_1}, D_{QP_1}$
$s_{[2]}$	$\widehat{D}_{P_1P_2}, \widehat{D}_{P_2P_2}, D_{QP_2}$
$b_{[1]}$	$\widehat{D}_{P_1P_1}, \widehat{D}_{P_2P_2}, D_{QQ'}$
$b_{[2]}$	$\widehat{D}_{P_1P_2}, \widehat{D}_{P_2P_1}, D_{QQ}$
$c_{[1]}$	$\widehat{D}_{P_1P_1}, \widehat{D}_{P_2P_2}, D_{QQ}$
$c_{[2]}$	$\widehat{D}_{P_1P_2}, \widehat{D}_{P_2P_1}, D_{QQ'}$
$d_{[1]}$	$\widehat{D}_{P_1P_1}, \widehat{D}_{P_2P_2}, D_{QQ}$
$d_{[2]}$	$\widehat{D}_{P_1P_2}, \widehat{D}_{P_2P_1}, D_{QQ'}$

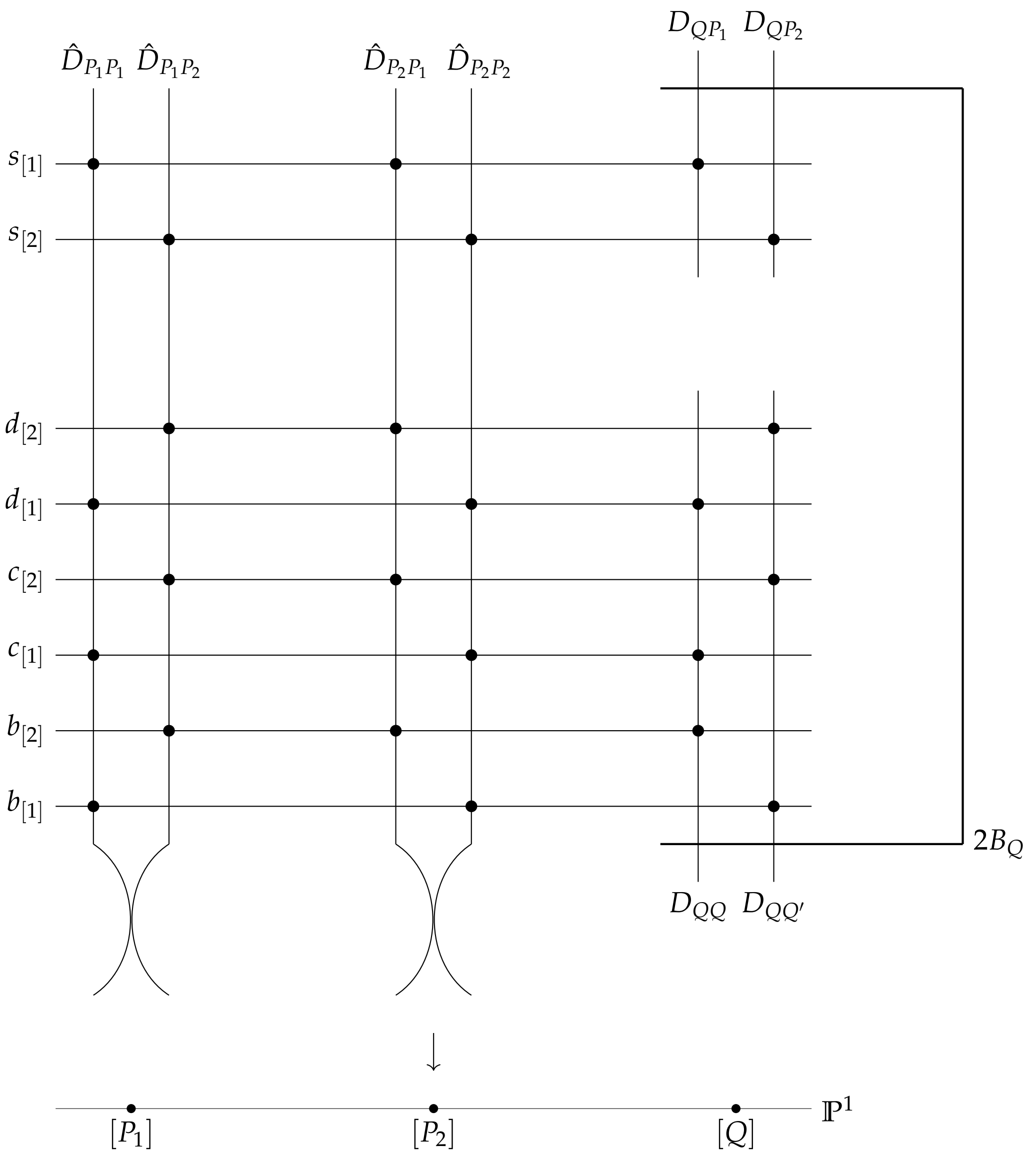


Figure 11: The 8 Sections of $f' : S \longrightarrow \mathbb{P}^1$

Now we contract 6 sections in the following order

$$s_{[1]},\ s_{[2]},\ b_{[1]},\ b_{[2]},\ c_{[2]},\ d_{[2]}.$$

One notices that after we contract the chosen 6 sections, the images of the following three curves are (-1)-curves:

$$D_{QP_1},\ D_{QP_2},\ D_{QQ}.$$

Then we further contract the three curves above. Let's denote the contraction of all the 9 curves by

$$\tau_1 : S \longrightarrow \mathbb{P}^2.$$

Here we notice that since S is a rational elliptic surface, the second Betti number $b_2(S) = 10$ due to Lemma 2.18. τ_1 is the successive contraction of 9 (-1)-curves, then $b_2(\tau_1(S)) = 1$. Since $\tau_1(S)$ is smooth and rational surface with $b_2 = 1$, we have that $\tau_1(S) \cong \mathbb{P}^2$. We will see that τ_1 is a representation of S as the blowup of $\mathbb{P}^2$ at the base points of a pencil of cubics.

Let us look at the images of the other singular fiber components that are not contracted by τ_1. We denote them by

$$\begin{aligned}
\tau_1(\widehat{D}_{P_1P_1}) &= l_1 \\
\tau_1(\widehat{D}_{P_1P_2}) &= Q_1 \\
\tau_1(\widehat{D}_{P_2P_1}) &= Q_2 \\
\tau_1(\widehat{D}_{P_2P_2}) &= l_2 \\
\tau_1(B_Q) &= L \\
\tau_1(D_{QQ'}) &= l.
\end{aligned}$$

We have the following claim:

Claim 5.2. $l_1,\ l_2,\ l$ and L are lines in $\mathbb{P}^2$ and $Q_1,\ Q_2$ are conics in $\mathbb{P}^2$. Furthermore, the lines and conics satisfy the following configuration, which we denote by $(\dagger)$, as shown in Figure 12

- l_1 is tangent to Q_1 and Q_2, and l_2 is tangent to Q_1 and Q_2;
- Q_1 and Q_2 intersect transversely at 2 points and are tangent at 1 tacnode,
- The points of tangency $l_1 \cap Q_2$, $l_2 \cap Q_1$, and the tacnode of $Q_1 \cap Q_2$ lie in the line L,
- $l_1 \cap l_2$ and the two transversely intersecting points of Q_1 and Q_1 lie in the line l.

Proof. Let's look at $\widehat{D}_{P_1P_2}$. Among the 6 sections we contract, $\widehat{D}_{P_1P_2}$ intersects $s_{[2]}$, $b_{[2]}$, $c_{[2]}$ and $d_{[2]}$. After the contraction of the 6 sections, the image of $\widehat{D}_{P_1P_2}$ intersects D_{QP_2} and D_{QQ}, which will be further contracted. So by the end the self-intersection of $\widehat{D}_{P_1P_2}$ increase from (-2) by 6. We conclude that $Q_1^2 = 4$ and Q_1 is a conic curve. Similar argument can be applied to the other curves.

When we contract $b_{[2]}$, the images of $\widehat{D}_{P_1P_2}$, $\widehat{D}_{P_2P_1}$ and D_{QQ} are concurrent at a point. Since we will further contract D_{QQ}, there is a tacnode of $Q_1 \cap Q_2$. Similar argument can be applied to verify the other conditions in $(\dagger)$. $\square$

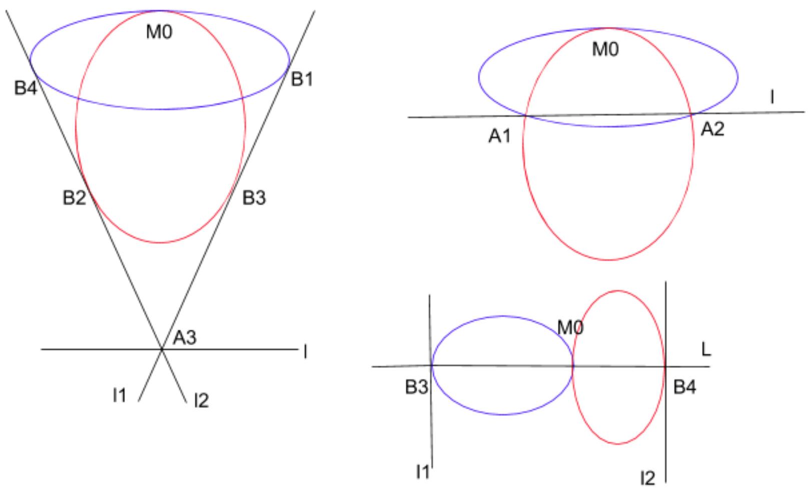


Figure 12: Q_1(Blue) and Q_2(Red) are conics, l_1, l_2, l and L are lines. B_1, B_2, B_3, and B_4 are tangency points. M_0 is a tacnode.

We have the following proposition:

Proposition 5.3. Let l_1, l_2, l and L be lines in $\mathbb{P}^2$ and Q_1 and Q_2 be conics in $\mathbb{P}^2$ satisfying $(\dagger)$. There is a pencil of cubics in $\mathbb{P}^2$ containg l_1Q_1, l_2Q_2, and lL^2 as its singular members, with 6 base points A_1, A_2, A_3, B_3, B_4 and M_0 as shown in Figure 12. Moreover, such a pencil of cubics is unique up to automorphisms of $\mathbb{P}^2$.

Proof. Without loss of generality, we can assume $l_1 = \{(x,y,z)|x=0\}$, $l_2 = \{(x,y,z)|y=0\}$ and $L = \{(x,y,z)|z=0\}$. We can also assume the line tangent to both Q_1 and Q_2 at the the tacnode to be $l' = \{(x,y,z)|x+y+z=0\}$. Notice that $Aut(\mathbb{P}^2)$ allows us to make such assumptions.

Now the conic Q_1 is tangent to $l_2 = \{(x,y,z)|y=0\}$ and $l' = \{(x,y,z)|x+y+z=0\}$ and the two tangency points lie on $L = \{(x,y,z)|z=0\}$. It follows that the equation of Q_1 is

$$Q_1 : z^2 + \alpha y(x+y+z) = 0.$$

for some $\alpha \in \mathbb{C}$. Similarly, we can write the equation of Q_2 as

$$Q_2 : z^2 + \beta x(x+y+z) = 0.$$

for some $\beta \in \mathbb{C}$.

Next we need to determine possible values for α and β such that the conditions of $(\dagger)$ satisfied. Notice that Q_1 is also tangent to $l_1 = \{(x,z,y)|x=0\}$. Plug $x=0$ into Q_1, we have

$$z^2 + \alpha yz + \alpha y^2 = 0,$$

which is supposed to have a double root. Then we have that

$$\alpha = 4.$$

Similarly, Q_2 is also tangent to $l_2 = \{(x, y, z)|y = 0\}$, and by the same argument we have that

$$\beta = 4.$$

Now we have that

$$\begin{aligned} l_1Q_1 &= xz^2 + 4xy(x + y + z), \\ l_2Q_2 &= yz^2 + 4xy(x + y + z). \end{aligned}$$

And

$$l_1Q_1 - l_2Q_2 = z^2(x - y) = lL^2,$$

where $l = \{(x, y, z)|x - y = 0\}$. Then the pencil of cubics generated by l_1Q_1 and l_2Q_2 have three singular members: l_1Q_1, l_2Q_2 and lL^2, which satisfy the conditions of configuration $(\dagger)$. In particular, one can see from the proof that such a pencil of cubics is unique up to projective automorphisms. $\square$

Such a pencil of cubics gives a representation of the rational minimal elliptic surface $S \to \mathbb{P}^1$. We have that $\tau_1 : S \to \mathbb{P}^2$ is the 9-fold blowup at the base points of the pencil of cubics

$$sx\left[z^2 + 4y(x+y+z)\right] + ty\left[z^2 + 4x(x+y+z)\right], \quad (s:t) \in \mathbb{P}^1,$$

upto automorphisms.

5.6 Another Pencil of Cubics.

In this section, we will give another representation of S as a 9-fold blouwup of $\mathbb{P}^2$. In fact there are infinitely many representations of S, since an automorphism of $\mathbb{P}^2$ will give another pencil of cubics that induces the same elliptic surface. However, we will find another pencil of cubics that is not in the orbit of the one we constructed in the previous section with respect to the action of $Aut(\mathbb{P}^2)$.

We find that there is another way to successively contract 9 (-1)-curves in S,

$$\tau_2 : S \longrightarrow \mathbb{P}^2$$

such that S is a 9-fold blowup at the base points of another pencil of cubics. To be explicit, let τ_2 be the successive contraction of 9 curves in the following order (see Figure 11),

$$b_{[2]}, \ D_{QQ}, \ B_Q, \ c_{[2]}, \ d_{[2]}, \ s_{[2]}, \ \hat{D}_{P_2P_2}, \ s_{[1]}, \ \hat{D}_{P_1P_1}.$$

We denote the images of the other singular fiber components that are not contracted by:

$$
\begin{aligned}
\tau_2(\widehat{D}_{P_2P_1}) &= C_1,\\
\tau_2(\widehat{D}_{P_1P_2}) &= C_2,\\
\tau_2(D_{QP_1}) &= L_1,\\
\tau_2(D_{QP_2}) &= L_2,\\
\tau_2(D_{Q_1Q_2}) &= L_3.
\end{aligned}
$$

We have the following claim:

Claim 5.4. C_1 and C_2 are cubic curves with a cusp and L_1, L_2 and L_3 are lines in $\mathbb{P}^2$. Furthermore they satisfy the following configuration, which we denote by $(\dagger\dagger)$, as shown in Figure 13,

- L_1 and C_1 are tangent, and the tangency point is the cusp of C_2,
- L_2 and C_2 are tangent, and the tangency point is the cusp of C_1,
- C_1 and C_2 intersect transversely at 2 points, and L_3 passes through the 2 points.
- C_1 and C_2 intersect at another point with index 3, and L_1, L_2 and L_3 are concurrent at the same point.

Proof. Let's look at $\widehat{D}_{P_2P_1}$. After we contract $b_{[2]}$, the image of $\widehat{D}_{P_2P_1}$ intersects D_{QQ}. Then we contract D_{QQ} and the image of $\widehat{D}_{P_2P_1}$ intersects B_Q. We further contract B_Q, $c_{[2]}$, $d_{[2]}$ and $s_{[2]}$, all of which intersect the image of $\widehat{D}_{P_2P_1}$ transversely. The self-intersection of the image of $\widehat{D}_{P_2P_1}$ increases from -2 by 5. Then we contract $\widehat{D}_{P_2P_2}$, which is tangent to $\widehat{D}_{P_2P_1}$, and the self-intersection of the image of

$\widehat{D}_{P_2P_1}$ increases by 4. Further, we contract $s_{[1]}$ then the image of $\widehat{D}_{P_2P_1}$ intersect $\widehat{D}_{P_1P_1}$, which is the last curve we contract. By the end the self-intersection of $\widehat{D}_{P_2P_1}$ increases from -2 by 11. We conclude that $C_1^2 = 9$ and C_1 is cubic curve in $\mathbb{P}^2$. Also when we contract $\widehat{D}_{P_2P_2}$, the tangency point of $\widehat{D}_{P_2P_1}$ and $\widehat{D}_{P_2P_2}$ becomes a cusp of the image of $\widehat{D}_{P_2P_1}$.

Similar argument can be applied to C_2 and L_is and to verify the other conditions in $(\dagger\dagger)$. $\square$

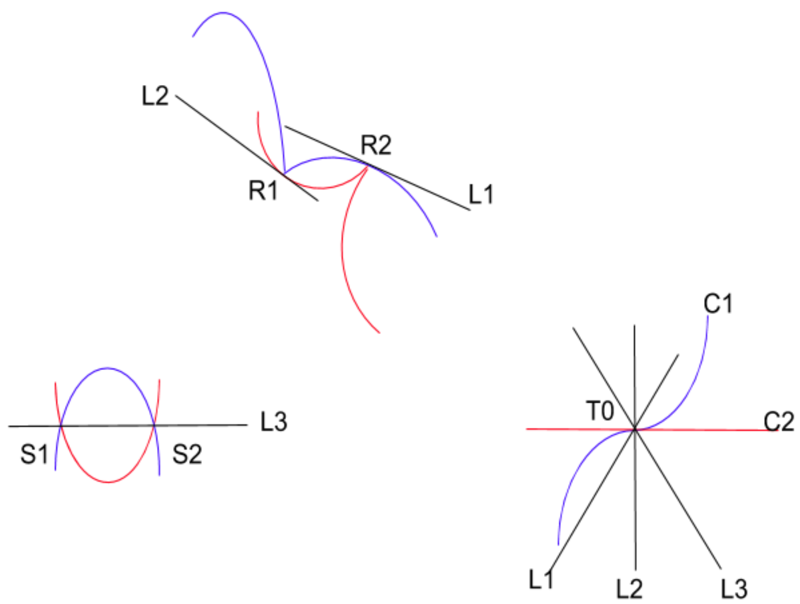


Figure 13: C_1(Blue) and C_2(Red) are cuspidal curves, L_1, L_2 and L_3 are lines. R_1 and R_2 are cusps of C_1 and C_2. T_0 is the concurrent point of L_is.

Remark 5.5. Given an arrangement $\{l_1, l_2, l, L, Q_1, Q_2\}$ satisfying $(\dagger)$, we can make a Cremona transform of $\mathbb{P}^2$ such that the images of the given arrangement form an arrangement satisfying $(\dagger\dagger)$.

We have the following proposition:

Proposition 5.6. Let L_1, L_2 and L_3 be lines and C_1 and C_2 be cuspidal curves in $\mathbb{P}^2$ satisfying $(\dagger\dagger)$. There is a pencil of cubics in $\mathbb{P}^2$ containg C_1, C_2, and $L_1L_2L_3$ as its singular members, with 5 base points R_1, R_2 S_1, S_2 and T_0 as shown in Figure 13. In particular, such a pencil of cubics is unique up to automorphisms of $\mathbb{P}^2$.

Proof. It follows from Remark 5.5 and Prop 5.3.

□

Corollary 5.7. A minimal rational elliptic surface with singular fibers configuration $\{III, III, I_0^*\}$(see Table I) is unique up to isomorphism.

Proof. Suppose that X is such a rational elliptic surface and $\tau_1 : X \to \mathbb{P}^2$ is a represestation of X as a 9-fold blowup at the base points of a pencil of cubics in $\mathbb{P}^2$. Considet a singular fiber F of Kodaira Type III, which consists of 2 tangent rational curves, its image $\tau_1(F)$ is a singular member of the pencil of cubics. Then $\tau_1(F)$ is either a cuspidal curve if τ_1 contracts one of the component of F, or $\tau_1(F)$ consists of a line and a conic, which are tangent to each other, if τ_1 dose not contract a component of F. Consider the intersections of plane curves, the images of two singular fibers of Kodaira Type III must be isomorphic. Then the pencil of pencil either contains two cuspidal curves or contains two singular members, which consist of a line and a conic tangent to each other. One can check that, in order to have another singular fiber of Kodaira type I_0^*, the pencil of cubics is either the one as in Prop 5.3 or the one as in Prop 5.6.

By Remark 5.5, two minimal elliptic surfaces induced from pencils of cubics in Prop 5.3 and Prop 5.6 are birational to each other and therefore isomorphic to each other by Corollay 2.11. Then the uniqueness of X is followed by the uniquenesses in Prop 5.3 or Prop 5.6. $\square$

Remark 5.8. U.Persson gave all the possible singular fiber configurations of minimal rational elliptic surfaces (Persson, 1990). Miranda and Persson classified the *extremal* rational elliptic surfaces. Also they found the corresponding pencil(s) of cubics for each extremal rational elliptic surface (Miranda and Persson, 1986).

CHAPTER 6

A SMOOTH BIRATIONAL MODEL OF E_1^3/μ_4

6.1 The Elliptic Threefold E_1^3/μ_4

In this section we consider the diagonal μ_4-action on the threefold E_1^3. We have the diagram

$$\begin{array}{ccc} E_1^3 & \xrightarrow{q_3} & E_1^3/\mu_4 \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle f} \\ E_1^2 & \xrightarrow{q_2} & E_1^2/\mu_4 \end{array}$$

where $\pi : E_1^3 \to E_1^2$ is the projection to the first two components, q_2 and q_3 are quotient maps with respect to the μ_4-actions. The map f is induced by π and its general fiber is isomorphic to E_1. We have seen in the previous chapter that E_1^2/μ_4 is a rational surface with cyclic quotient singularities. Our aim is to construct a smooth birational model of E_1^3/μ_4 that admits an elliptic fibration over $\mathbb{P}^2$.

6.2 A Smooth Elliptic Model with Two Fibrations

The construction is similar to the construction of the smooth model of E_0^3/μ_3. We make the following notations. Let's denote the π-preimages of (P_i, P_j), (P_i, Q_j), (Q_i, P_j) and (Q_i, Q_j) by

$$\begin{aligned} L_{P_iP_j} &= \pi^{-1}((P_i, P_j)) = \{(P_i, P_j, x) | x \in E_1\}, \\ L_{P_iQ_j} &= \pi^{-1}((P_i, Q_j)) = \{(P_i, Q_j, x) | x \in E_1\}, \\ L_{Q_iP_j} &= \pi^{-1}((Q_i, P_j)) = \{(Q_i, P_j, x) | x \in E_1\}, \\ L_{Q_iQ_j} &= \pi^{-1}((Q_i, Q_j)) = \{(Q_i, Q_j, x) | x \in E_1\}. \end{aligned}$$

We denote the blowup of E_1^3 along the 16 curves $L_{P_iP_j}$, $L_{P_iQ_j}$, $L_{Q_iP_j}$ and $L_{Q_iQ_j}$ for $i, j = 1, 2$ by

$$\epsilon_3 : \widetilde{E_1^3} \longrightarrow E_1^3.$$

Recall the blowup of E_1^2 at the 16 points (P_i, P_j), (P_i, Q_j), (Q_i, P_j) and (Q_i, Q_j) for $i, j = 1, 2$,

$$\epsilon_2 : \widetilde{E_1^2} \rightarrow E_1^2.$$

From a local construction as in Section 4.2, there is a map

$$\tilde{\pi} : \widetilde{E_1^3} \longrightarrow \widetilde{E_1^2}.$$

All the fibers of $\tilde{\pi}$ are isomorphic to E_1. We denote the exceptional divisors over $L_{P_iP_j}$, $L_{P_iQ_j}$, $L_{Q_iP_j}$ and $L_{Q_iQ_j}$ by $B_{P_iP_j}$, $B_{P_iQ_j}$, $B_{Q_iP_j}$ and $B_{Q_iQ_j}$ respectively. Since the normal bundle of $L_{P_iP_j}$ in E_1^3 is trivial, we have that $B_{P_iP_j} \cong L_{P_iP_j} \times \mathbb{P}^1$ is a product surface and the restriction of ϵ_3 on $B_{P_iP_i}$

$$\epsilon_3|_{B_{P_iP_j}} : B_{P_iP_i} \cong L_{P_iP_i} \times \mathbb{P}^1 \longrightarrow L_{P_iP_i}$$

is the projection to the first component. On the other hand the restriction of $\tilde{\pi}$ on $B_{P_iP_j}$

$$\tilde{\pi}|_{B_{P_iP_j}} : B_{P_iP_j} \cong L_{P_iP_j} \times \mathbb{P}^1 \longrightarrow E_{P_iP_j}$$

is the projection to the second component. Recall that $E_{P_iP_j}$ is the exceptional divisor in $\widetilde{E_1^2}$ over (P_i, P_j).

The curve $L_{P_iP_j}$ contains 4 points with non-trivial stabilizer with respect to the μ_4-action, which are (P_i, P_j, P_1), (P_i, P_j, P_2), (P_i, P_j, Q_1), and (P_i, P_j, Q_2). We denote their ϵ_3-preimages by

$$C_{P_iP_jP_k} = \epsilon_3^{-1}((P_i, P_j, P_k)) \cong \mathbb{P}^1,$$

$$C_{P_iP_jQ_k} = \epsilon_3^{-1}((P_i, P_j, Q_k)) \cong \mathbb{P}^1.$$

There are 64 such curves in $\widetilde{E_1^3}$, which are $C_{P_iP_jP_k}$, $C_{P_iP_jQ_k}$, $C_{P_iQ_jP_k}$, $C_{Q_iP_jP_k}$, $C_{P_iQ_jQ_k}$, $C_{Q_iQ_jP_k}$, $C_{Q_iP_jQ_k}$, and $C_{Q_iQ_jQ_k}$, for $i, j, k = 1, 2$. The quotient map $\widetilde{E_1^3} \to \widetilde{E_1^3}/\mu_3$ with respect to the μ_4-action ramifies along the 64 curves. In order to have a smooth quotient space, we further blow up $\widetilde{E_1^3}$ along the 64 curves as we did for $\widetilde{E_0^3}$. We denote the blowup by

$$\tau : Bl_C\widetilde{E_1^3} \longrightarrow \widetilde{E_1^3}.$$

A local discussion similar to Section 4.2 implies that the μ_4-action extends to $Bl_C\widetilde{E_1^3}$ and the quotient space $\left(Bl_C\widetilde{E_1^3}\right)/\mu_4$ is smooth. We denote the quotient map by

$$q_3 : Bl_C\widetilde{E_1^3} \longrightarrow \left(Bl_C\widetilde{E_1^3}\right)/\mu_4.$$

Since the composition map $\pi' = \tau \circ \tilde{\pi} : Bl_C\widetilde{E_1^3} \to \widetilde{E_1^2}$ is equivariant with respect to the μ_4-actions, it induces

$$f' : \left(Bl_C\widetilde{E_1^3}\right)/\mu_4 \longrightarrow \widetilde{E_1^2}/\mu_4,$$

which has a general fiber isomorphic to E_1. We recall that $\widetilde{E_1^2}/\mu_4$ is a rational elliptic surface and let S be its relative minimal model. Let $c : \widetilde{E_1^2}/\mu_4 \to S$ to be the contraction of (-1) fibral components. In Chapter 5, we have seen that S has two distinct representations as a 9-fold blowup of $\mathbb{P}^2$,

$$\tau_1 : S \longrightarrow \mathbb{P}^2,$$

$$\tau_2 : S \longrightarrow \mathbb{P}^2.$$

Composing f' with c and τ_1 or τ_2, we have two elliptic fibrations:

$$f_1 = \tau_1 \circ c \circ f' : \left(Bl_C\widetilde{E_1^3}\right) / \mu_4 \longrightarrow \mathbb{P}^2,$$

$$f_2 = \tau_2 \circ c \circ f' : \left(Bl_C\widetilde{E_1^3}\right) / \mu_4 \longrightarrow \mathbb{P}^2,$$

whose general fibers are isomorphic to E_1.

6.3 The Singular Fibers

In the previous section we constructed two distinct elliptic firations $f_1, f_2 : \left(Bl_C\widetilde{E_1^3}\right) / \mu_4 \to \mathbb{P}^2$. In this section we will study their discriminant locus $\Delta(f_1)$ and $\Delta(f_2)$, which are divisors in $\mathbb{P}^2$. Furthermore we will look into their singular fibers, especially over multiple points of discriminant loci. We will prove Theomre 1.6 by the end.

We start with the elliptic fiberation $f' : \left(Bl_C\widetilde{E_1^3}\right) / \mu_4 \to \widetilde{E_1^2}/\mu_4$. Before taking quotient with respect to the μ_4-actions, the morphism $\tilde{\pi} : Bl_C\widetilde{E_1^3} \to \widetilde{E_1^2}$ has disriminant locus $\Delta(\pi')$ the collection of the 16 exceptional curves of the blowup $\epsilon_2 : \widetilde{E_1^2} \to E_1^2$. Recall that in the previous chapter we denoted the exceptional curves over (P_i, P_j), (P_i, Q_j), (Q_i, P_j) and (Q_i, Q_j) by $E_{P_iP_j}$, $E_{P_iQ_j}$, $E_{Q_iP_j}$, and $E_{Q_iQ_j}$ for $i, j = 1, 2$. Then

$$\Delta(\pi') = \bigcup_{i,j=1,2} E_{P_iP_j} \bigcup_{i,j=1,2} E_{P_iQ_j} \bigcup_{i,j=1,2} E_{Q_iP_j} \bigcup_{i,j=1,2} E_{Q_iQ_j}.$$

One can check that the discriminant locus of f' is the the images of $\Delta(\pi')$ under the quotient map q_3. Recall our notations for $q_3(E_{P_iP_j})$, see Figure 9, we have that $\Delta(f')$ is the union of the 10 curves,

$$\Delta(f') = \bigcup_{i,j=1,2} D_{P_iP_j} \bigcup_{i=1,2} D_{P_iQ} \bigcup_{i=1,2} D_{QP_i} \bigcup D_{QQ} \bigcup D_{QQ'}.$$

We make the following notations. Recall the blowup of $\widetilde{E_1^3}$ along 64 curves, $\tau : Bl_C\widetilde{E_1^3} \to \widetilde{E_1^3}$, we denote the exceptional divisors over the 64 curves by

$$\begin{aligned}
B_{P_iP_jP_k} &= \tau^{-1}(C_{P_iP_jP_k}), & i,j,k = 1,2,\\
B_{P_iP_jQ_k} &= \tau^{-1}(C_{P_iP_jQ_k}), & i,j,k = 1,2,\\
B_{P_iQ_jP_k} &= \tau^{-1}(C_{P_iQ_jP_k}), & i,j,k = 1,2,\\
B_{Q_iP_jP_k} &= \tau^{-1}(C_{Q_iP_jP_k}), & i,j,k = 1,2,\\
B_{P_iQ_jQ_k} &= \tau^{-1}(C_{P_iQ_jQ_k}), & i,j,k = 1,2,\\
B_{Q_iP_jQ_k} &= \tau^{-1}(C_{Q_iP_jQ_k}), & i,j,k = 1,2,\\
B_{Q_iQ_jP_k} &= \tau^{-1}(C_{Q_iQ_jP_k}), & i,j,k = 1,2,\\
B_{Q_iQ_jQ_k} &= \tau^{-1}(C_{Q_iQ_jQ_k}), & i,j,k = 1,2.
\end{aligned}$$

We claim that

Lemma 6.1. All the 64 exceptional divisors $B_{P_iP_jP_k}, ..., B_{Q_iQ_jQ_k}$ are isomorphic to the Hirzebruch surface $\mathbb{F}_1$.

Proof. The argument is the same as the proof of Claim 4.2. $\square$

We also denote the strict transform of $B_{P_iP_j}$ under τ by $\widehat{B}_{P_iP_j}$, similar notations for $B_{P_iQ_j}$ and others. Then the morphism $\pi' : Bl_C\widetilde{E_1^3} \to \widetilde{E_1^2}$ restricts to

$$\begin{array}{rcl}
\pi' : \widehat{B}_{P_iP_j} \bigcup B_{P_iP_jP_1} \bigcup B_{P_iP_jP_2} \bigcup B_{P_iP_jQ_1} \bigcup B_{P_iP_jQ_2} & \longrightarrow & E_{P_iP_j}, \\
\pi' : \widehat{B}_{P_iQ_j} \bigcup B_{P_iQ_jP_1} \bigcup B_{P_iQ_jP_2} \bigcup B_{P_iQ_jQ_1} \bigcup B_{P_iQ_jQ_2} & \longrightarrow & E_{P_iQ_j}, \\
\pi' : \widehat{B}_{Q_iP_j} \bigcup B_{Q_iP_jP_1} \bigcup B_{Q_iP_jP_2} \bigcup B_{Q_iP_jQ_1} \bigcup B_{Q_iP_jQ_2} & \longrightarrow & E_{Q_iP_j}, \\
\pi' : \widehat{B}_{Q_iQ_j} \bigcup B_{Q_iQ_jP_1} \bigcup B_{Q_iQ_jP_2} \bigcup B_{Q_iQ_jQ_1} \bigcup B_{Q_iQ_jQ_2} & \longrightarrow & E_{Q_iQ_j}.
\end{array}$$

We further denote the images of $\widehat{B}_{P_iP_j}$ and $B_{P_iP_jP_k}$ under q_3 by

$$\begin{array}{rcl}
q_3(\widehat{B}_{P_iP_j}) & = & G_{P_iP_j}, \\
q_3(\widehat{B}_{P_iQ_1}) & = & q_3(\widehat{B}_{P_iQ_2}) = G_{P_iQ}, \\
q_3(\widehat{B}_{Q_1P_i}) & = & q_3(\widehat{B}_{Q_2P_i}) = G_{QP_i}, \\
q_3(\widehat{B}_{Q_1Q_1}) & = & q_3(\widehat{B}_{Q_2Q_2}) = G_{QQ}, \\
q_3(\widehat{B}_{Q_1Q_2}) & = & q_3(\widehat{B}_{Q_2Q_1}) = G_{QQ'},
\end{array}$$

and

$$
\begin{aligned}
q_3(B_{P_iP_jP_k}) &= G_{P_iP_jP_k}, \\
q_3(B_{P_iP_jQ_1}) &= q_3(B_{P_iP_jQ_2}) = G_{P_iP_jQ}, \\
q_3(B_{P_iQ_1P_k}) &= q_3(B_{P_iQ_2P_k}) = G_{P_iQP_k}, \\
q_3(B_{P_iQ_1Q_1}) &= q_3(B_{P_iQ_2Q_2}) = G_{P_iQQ}, \\
q_3(B_{P_iQ_1Q_2}) &= q_3(B_{P_iQ_2Q_1}) = G_{P_iQQ'}, \\
q_3(B_{Q_1P_iP_k}) &= q_3(B_{Q_2P_iP_k}) = G_{QP_iP_k}, \\
q_3(B_{Q_1P_iQ_1}) &= q_3(B_{Q_2P_iQ_2}) = G_{QP_iQ}, \\
q_3(B_{Q_1P_iQ_2}) &= q_3(B_{Q_2P_iQ_1}) = G_{QP_iQ'}, \\
q_3(B_{Q_1Q_1P_i}) &= q_3(B_{Q_2Q_2P_i}) = G_{QQP_i}, \\
q_3(B_{Q_1Q_1Q_1}) &= q_3(B_{Q_2Q_2Q_2}) = G_{QQQ}, \\
q_3(B_{Q_1Q_1Q_2}) &= q_3(B_{Q_2Q_2Q_1}) = G_{QQQ'}, \\
q_3(B_{Q_1Q_2P_i}) &= q_3(B_{Q_2Q_1P_i}) = G_{QQ'P_i}, \\
q_3(B_{Q_1Q_2Q_1}) &= q_3(B_{Q_2Q_1Q_2}) = G_{QQ'Q}, \\
q_3(B_{Q_1Q_2Q_2}) &= q_3(B_{Q_2Q_1Q_1}) = G_{QQ'Q'}.
\end{aligned}
$$

Then $f' : \left(Bl_C\widetilde{E_1^3}\right)/\mu_4 \longrightarrow \widetilde{E_1^2}/\mu_4$ restricts to the following union of surfaces

$$\begin{aligned}
f' : G_{P_iP_j} \bigcup G_{P_iP_jP_1} \bigcup G_{P_iP_jP_2} \bigcup G_{P_iP_jQ} &\longrightarrow D_{P_iP_j}, \\
f' : G_{P_iQ} \bigcup G_{P_iQP_1} \bigcup G_{P_iQP_2} \bigcup G_{P_iQQ} \bigcup G_{P_iQQ'} &\longrightarrow D_{P_iQ}, \\
f' : G_{QP_i} \bigcup G_{QP_iP_1} \bigcup G_{QP_iP_2} \bigcup G_{QP_iQ} \bigcup G_{QP_iQ'} &\longrightarrow D_{QP_i}, \\
f' : G_{QQ} \bigcup G_{QQP_1} \bigcup G_{QQP_2} \bigcup G_{QQQ} \bigcup G_{QQQ'} &\longrightarrow D_{QQ}, \\
f' : G_{QQ'} \bigcup G_{QQ'P_1} \bigcup G_{QQ'P_2} \bigcup G_{QQ'Q} \bigcup G_{QQ'Q'} &\longrightarrow D_{QQ'}.
\end{aligned}$$

The singular fibers of f' are as following.

- If $p \in D_{P_iP_j}$, its singular fiber $f'^{-1}(p)$ has 4 components, each of which is isomorphic to $\mathbb{P}^1$, as shown in Figure 14.
- If $p \in D_{P_iQ}, D_{QP_i}, D_{QQ}$ or $D_{QQ'}$, $f'^{-1}(p)$ has 5 components and it is of Kodaira Type I_0^*, see Table I.

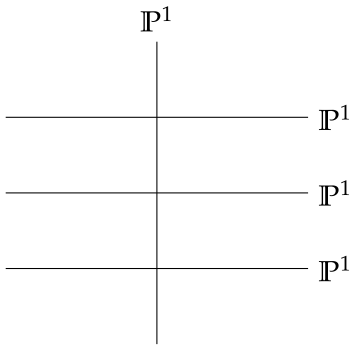


Figure 14: Singular Fiber of $f' : \left(Bl_C\widetilde{E_1^3}\right)/\mu_4 \to \widetilde{E_1^2}/\mu_4$ over A Point in $D_{P_iP_j}$.

Recall that $c : \widetilde{E_1^2}/\mu_4 \to S$ is the successive contraction of four (-1)-fibral components B_{P_1}, B_{P_2}, D_{P_1Q} and D_{P_2Q}, and S is the minimal elliptic surface with singular fibers $\{III, III, I_0^*\}$ as shown in Figure 10. Let

$$f_0 = c \circ f' : \left(Bl_C\widetilde{E_1^3}\right)/\mu_4 \to S.$$

From the discussion above, the discriminant locus of f_0 consists 8 curves

$$\Delta(f_0) = \widehat{D}_{P_1P_1} \bigcup \widehat{D}_{P_1P_2} \bigcup \widehat{D}_{P_2P_1} \bigcup \widehat{D}_{P_2P_2} \bigcup D_{QQ} \bigcup D_{QQ'} \bigcup D_{QP_1} \bigcup D_{QP_2}$$

For a general point of $\widehat{D}_{P_iP_j}, i, j = 1, 2$, its singular fiber of f_0 is as shown in Figure 14. For a general point of D_{QQ}, $D_{QQ'}$, D_{QP_1} and D_{QP_2}, its singular is of Kodair Type I_0^*, see Table I.

As shown in Figure 10, $\widehat{D}_{P_1P_1}$ and $\widehat{D}_{P_1P_2}$ are tangent to each other. We denote the two tangency points by

$$B_1 = \widehat{D}_{P_1P_1} \cap \widehat{D}_{P_1P_2},$$
$$B_2 = \widehat{D}_{P_2P_1} \cap \widehat{D}_{P_2P_2}.$$

We need to look into the singular fibers of f_0 over the two tangency points B_1 and B_2. Since c contracts B_{P_i} and D_{P_iQ} to B_i for $i = 1, 2$, we have

$$\begin{aligned} f_0^{-1}(B_i) &= f'^{-1}(B_{P_i}) \bigcup f'^{-1}(D_{P_iQ}) \\ &= f'^{-1}(B_{P_i}) \bigcup G_{P_iQ} \bigcup G_{P_iQP_1} \bigcup G_{P_iQP_2} \bigcup G_{P_iQQ} \bigcup G_{P_iQQ'}. \end{aligned}$$

Let's denote $G_i = f'^{-1}(B_{P_i})$. We claim that

Claim 6.2. *G_i is isomorphic to $\widetilde{E_1^2}/\mu_4$. Moreover, $f'|_{G_i} : G_i \to B_{P_i}$ is the elliptic surface $\widetilde{E_1^2}/\mu_4 \to \mathbb{P}^1$.*

Proof. Recall that $B_{P_i} = \tilde{q}_2(\hat{C}_{P_1})$ and $\hat{C}_{P_i} \cong \{(P_i, x) | x \in E_1\}$. Then $\pi^{-1}(\hat{C}_{P_i}) \cong \{(P_i, x, y) | x, y \in E_1\}$ and $q_3(\pi^{-1}(\hat{C}_{P_i})) \cong \{(P_i, x, y) | x, y \in E_1\}/\mu_4 \cong E_1^2/\mu_4$. One can check that G_i is the minimal resolution of $q_3(\pi^{-1}(\hat{C}_{P_i}))$, which is isomorphic to $\widetilde{E_1^2}/\mu_4$.

For the second statement, the restriction

$$\pi|_{\pi^{-1}(\hat{C}_{P_i})} : \pi^{-1}(\hat{C}_{P_i}) \longrightarrow \hat{C}_{P_i}$$

induces $f'|_{G_i} : G_i \to B_{P_i}$, which is the elliptic fibration $\widetilde{E_1^2}/\mu_4 \to \mathbb{P}^1$. $\square$

Recall that $G_i \to B_{P_i}$ has one singular fiber of Kodaira Type I_0^*, see Figure 10. Furthermore, G_i intersect the other 5 components of the singular fiber $f_0^{-1}(B_i)$ along its singular of Kodaira Type I_0^* as shown in Figure 15.

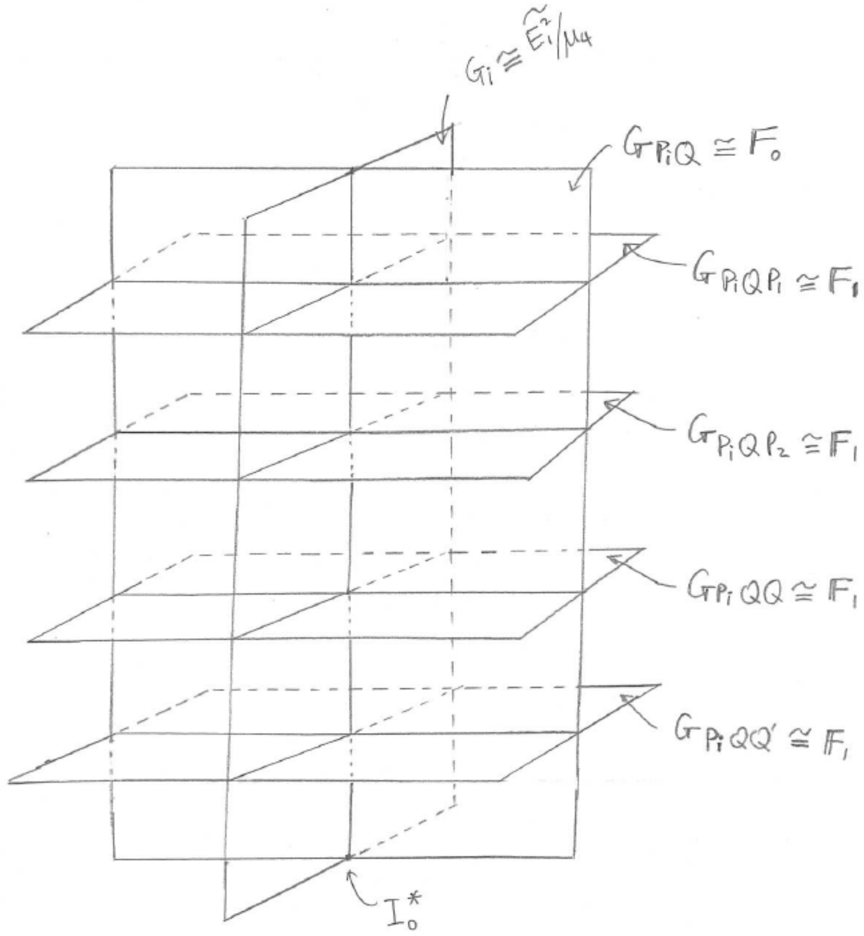


Figure 15: The Singular Fiber of f_0 over B_i

For the next step, we recall the two representations of S as a 9-fold blowup of $\mathbb{P}^2$,

$$\tau_1, \tau_2 : S \longrightarrow \mathbb{P}^2,$$

where τ_1 is a successive contraction of 9 curves of S in the following order

$$s_{[1]},\ s_{[2]},\ b_{[1]},\ b_{[2]},\ c_{[2]},\ d_{[2]},\ D_{QP_1},\ D_{QP_2},\ D_{QQ},$$

and τ_2 contracts successively 9 curves of S in the following order

$$b_{[2]},\ D_{QQ},\ B_Q,\ c_{[2]},\ d_{[2]},\ s_{[2]},\ \hat{D}_{P_2P_2},\ s_{[1]},\ \hat{D}_{P_1P_1}.$$

All the curves in S are shown in Figure 11. Composing with f_0, we have the two elliptic fibrations f_1 and f_2 over $\mathbb{P}^2$.

6.3.1 The Singular Fibers of f_1

Let's first look into singular fibers of f_1. Since τ_1 contracts D_{QP_1}, D_{QP_2} and D_{QQ}, the discriminant locus of f_1 consists of the images of $\widehat{D}_{P_1P_1}$, $\widehat{D}_{P_1P_2}$, $\widehat{D}_{P_2P_1}$, $\widehat{D}_{P_2P_2}$, and $D_{QQ'}$ under τ_1. Recall our notations in Section 5.5,

$$\begin{aligned}
\tau_1(\widehat{D}_{P_1P_1}) &= l_1, \\
\tau_1(\widehat{D}_{P_1P_2}) &= Q_1, \\
\tau_1(\widehat{D}_{P_2P_1}) &= Q_2, \\
\tau_1(\widehat{D}_{P_2P_2}) &= l_2, \\
\tau_1(B_Q) &= L, \\
\tau_1(D_{QQ'}) &= l.
\end{aligned}$$

where l_1, l_2, l, L are lines in $\mathbb{P}^2$ and Q_1 and Q_2 are smooth plane conics satisfying the configuration $(\dagger)$ in Claim 5.2. Then we have the discriminant locus of f_1,

$$\Delta(f_1) = l_1 \cup l_2 \cup l \cup Q_1 \cup Q_2.$$

The singular fibers over smooth points of $\Delta(f_1)$ are the same as singular fibers of f'. Now we need look into the singular fibers over multiple points of $\Delta(f_1)$.

We further make the following notations. Let B_1 be the tangency point of l_1 and Q_1 and B_2 be the tangency point of l_2 and Q_2. Let B_3 be the tangency point of l_1 and Q_2, and B_4 be the tangency point of

l_2 and Q_1. Let A_3 be the common point of l_1, l_2 and l, and M_0 be the tacnode of Q_1 and Q_2. Let A_1 and A_2 be the two transverse points of $Q_1 \cap Q_2$. The curves and points are shown in Figure 12.

The singular fibers over the two tangency points B_1 and B_2 are the same as $f_0^{-1}(B_1)$ and $f_0^{-1}(B_2)$ as shown in Figure 15. For the other multiple points, we notice the following

- τ_1 contracts $s_{[1]}$ and D_{QP_1} to B_3,
- τ_1 contracts $s_{[2]}$ and D_{QP_2} to B_4,
- τ_1 contracts $b_{[1]}$ to A_3,
- τ_1 contracts $b_{[2]}$ and D_{QQ} to M_0,
- τ_1 contracts $c_{[2]}$ to A_1,
- τ_1 contracts $d_{[2]}$ to A_2.

Therefore we have that

$$\begin{aligned}
f_1^{-1}(B_3) &= f'^{-1}(s_{[1]}) \cup f'^{-1}(D_{QP_1}), \\
f_1^{-1}(B_4) &= f'^{-1}(s_{[2]}) \cup f'^{-1}(D_{QP_2}), \\
f_1^{-1}(A_3) &= f'^{-1}(b_{[1]}), \\
f_1^{-1}(M_0) &= f'^{-1}(b_{[2]}) \cup f'^{-1}(D_{QQ}), \\
f_1^{-1}(A_1) &= f'^{-1}(c_{[2]}), \\
f_1^{-1}(A_2) &= f'^{-1}(d_{[2]}).
\end{aligned}$$

Similar to Claim 6.2, we have that

$$f'^{-1}(S_{[1]}) \cong f'^{-1}(S_{[2]}) \cong f'^{-1}(b_{[1]}) \cong f'^{-1}(b_{[2]}) \cong f'^{-1}(c_{[2]}) \cong f'^{-1}(d_{[2]}) \cong \widetilde{E_1^2}/\mu_4.$$

and

$$\begin{aligned}
f'^{-1}(D_{QP_1}) &= G_{QP_1} \bigcup G_{QP_1P_1} \bigcup G_{QP_1P_2} \bigcup G_{QP_1Q} \bigcup G_{QP_1Q'}, \\
f'^{-1}(D_{QP_2}) &= G_{QP_2} \bigcup G_{QP_2P_1} \bigcup G_{QP_2P_2} \bigcup G_{QP_2Q} \bigcup G_{QP_2Q'}, \\
f'^{-1}(D_{QQ}) &= G_{QQ} \bigcup G_{QQP_1} \bigcup G_{QQP_2} \bigcup G_{QQQ} \bigcup G_{QQQ'}.
\end{aligned}$$

We have that $f_1^{-1}(B_3), f_1^{-1}(B_4), f_1^{-1}(M_0)$ are all isomorphic to $f_0^{-1}(B_1)$ containg 6 components as we described above. We have seen that G_{QP_i} and G_{QQ} are ismorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, which is denoted by $\mathbb{F}_0$. Due to Lemma 6.1, the sufaces $G_{QP_iP_j}, ..., G_{QQQ'}$ and G_{QQQ} are isomorphic to $\mathbb{F}_1$.

We summerize the singular fibers of f_1 over multiple points of $\Delta(f_1)$ as following:

- Over the concurrent points A_1, A_2 and A_3, the singular fiber is isomorphic to the elliptic surface $\widetilde{E_1^2}/\mu_4$.
- Over the tangency points B_1, B_2, B_3, B_4 and the tacnode M_0, the singular fiber has 6 components. One is isomorphic to the elliptic surface $\widetilde{E_1^2}/\mu_4$, one is isomorphic to the Hirzebruch surface $\mathbb{F}_0$ and the other four are isomorphic to the Hirzebruch surface $\mathbb{F}_1$. Furthermore, the elliptic surface intersects the 5 Hirzebruch surfaces along its singular fiber of Kodaria Type I_0^*. See Figure 16.

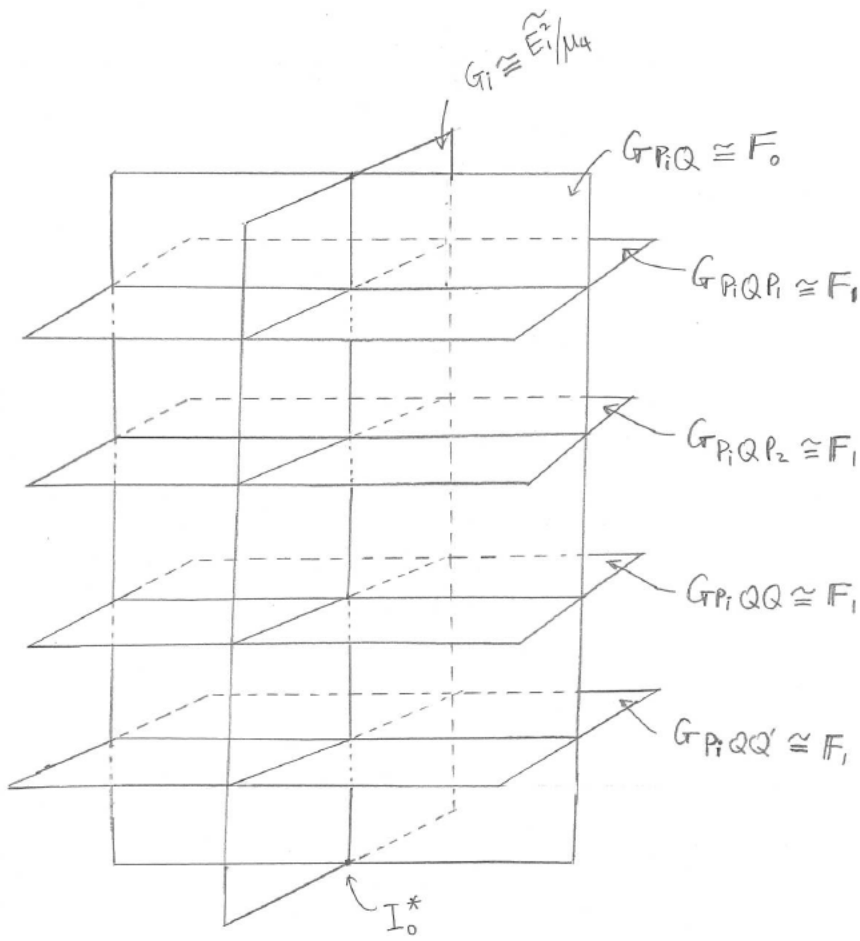


Figure 16: The Singular Fiber of f_1 over B_i for $i = 1, 2$,

Which Is Isomorphic To The Singular Fibers over B_3, B_4 And M_0.

6.3.2 The Singular Fibers of f_2

Next we are going to study the singular fibers of f_2. Recall our notations in the previous chapter

$$\tau_2(\widehat{D}_{P_1P_2}) = C_2,$$

$$\tau_2(\widehat{D}_{P_2P_1}) = C_1,$$

$$\tau_2(\widehat{D}_{QP_1}) = L_1,$$

$$\tau_2(\widehat{D}_{QP_2}) = L_2,$$

$$\tau_2(\widehat{D}_{QQ'}) = L_3.$$

where C_1, C_2, L_1, L_2, L_3 satisfy the configuration $(\dagger\dagger)$ as stated in Claim 5.4. The discriminant locus of f_2 is

$$\Delta(f_2) = C_1 \cup C_2 \cup L_1 \cup L_2 \cup L_3.$$

The singular fibers of f_2 over smooth points of $\Delta(f_2)$ is the same as singular fibers of f'. Now we need to look into the singular fibers over multiple points of $\Delta(f_2)$. There are 5 multiple points. Let R_1 be the cusp of C_1 and R_2 be the cusp of C_2. Let S_1 and S_2 be the two transverse points of $C_1 \cap C_2$, and let T_0 be the common point of L_1, L_2 and L_3. The curves and points are shown in Figure 13.

τ_2 contracts curves to multiple points of $\Delta(f_2)$ as following: (see Figure 11)

- τ_2 contracts $b_{[2]}$, D_{QQ} and B_Q to T_0,
- τ_2 contracts $c_{[2]}$ to S_2,

- τ_2 contracts $d_{[2]}$ to S_1,
- τ_2 contracts $s_{[2]}$ and $\widehat{D}_{P_2P_2}$ to R_2
- τ_2 contracts $s_{[1]}$ and $\widehat{D}_{P_1P_1}$ to R_1.

Then we have the following singular fibers

$$\begin{aligned}
f_2^{-1}(T_0) &= f'^{-1}(b_{[2]}) \cup f'^{-1}(D_{QQ}) \cup f'^{-1}(B_Q), \\
f_2^{-1}(R_1) &= f'^{-1}(s_{[1]}) \cup f'^{-1}(\widehat{D}_{P_1P_1}), \\
f_2^{-1}(R_2) &= f'^{-1}(s_{[2]}) \cup f'^{-1}(\widehat{D}_{P_2P_2}), \\
f_2^{-1}(S_1) &= f'^{-1}(d_{[2]}), \\
f_2^{-1}(S_2) &= f'^{-1}(c_{[2]}).
\end{aligned}$$

As we have seen above, the preimage of a section is isomorphic to the elliptic surface $\widetilde{E_1^2}/\mu_4$,

$$f'^{-1}(b_{[2]}) \cong f'^{-1}(s_{[1]}) \cong f'^{-1}(s_{[2]}) \cong f'^{-1}(d_{[2]}) \cong f'^{-1}(c_{[2]}) \cong \widetilde{E_1^2}/\mu_4.$$

Also, we have

$$\begin{aligned}
f'^{-1}(D_{QQ}) &= G_{QQ} \bigcup G_{QQP_1} \bigcup G_{QQP_2} \bigcup G_{QQQ} \bigcup G_{QQQ'}, \\
f'^{-1}(\widehat{D}_{P_1P_1}) &= G_{P_1P_1} \bigcup G_{P_1P_1P_1} \bigcup G_{P_1P_1P_2} \bigcup G_{P_1P_1Q}, \\
f'^{-1}(\widehat{D}_{P_2P_2}) &= G_{P_2P_2} \bigcup G_{P_2P_2P_1} \bigcup G_{P_2P_2P_2} \bigcup G_{P_2P_2Q}.
\end{aligned}$$

Now we denote $G_Q = f'^{-1}(B_Q)$. Then f' restrcit to $f'|_{G_Q} : G_Q \to B_Q$ is an elliptic fibration. By looking into the μ_4-actions on $\widetilde{E_1^2}$, we have that

$$B_Q \cong E_1/\mu_2 \cong \mathbb{P}^1,$$

where $\mu_2 =< e, g^2 >$ and g is a generator of μ_4. In particular, $g^2 = -Id_{E_1}$ is the involution on E_1. On the other hand, by looking into the μ_4-actions on E_1^3 and its blowups $\widetilde{E_1^3}$ and $Bl_C\widetilde{E_1^3}$, we have that the preimage of B_Q is

$$G_Q \cong \widetilde{E_1^2}/\mu_2.$$

Recall that $\widetilde{E_1^2} \to E_1^2$ is the blowup at the 16 points fixed by g^2, i.e. the 16 2-torsion points of E_1^2. Since $\widetilde{E_1^2}/\mu_2$ is a minimal resolution of E_1^2/μ_2, we have that G_Q is a *Kummer surface* , which is typically denoted by $Km(E_1 \times E_1)$ (Barth et al., 2015). In particular, G_Q is a K3 surface. Furthermore,

$$f'|_{G_Q} : G_Q \longrightarrow B_Q$$

is an elliptic K3 surface (see the classification of elliptic surfaces Lemma 2.17), and it has four singular fibers of Kodaira Type I_0^* (see Table I).

We summarize the singular fibers over multiple points of $\Delta(f_2)$ as following:

- For S_1 and S_2, $f_2^{-1}(S_i)$ is isomorphic to the elliptic surface $\widetilde{E_1^2}/\mu_4$.
- For R_1 and R_2, $f_2^{-1}(R_i)$ has 5 components. One is isomorphic to the elliptic surface $\widetilde{E_1^2}/\mu_4$, one is isomorphic to Hirzebruch surface $\mathbb{F}_0$, the other three are isomorphic to Hirzebruch surface $\mathbb{F}_1$.

Furthermore, the four Hirzebruch surfaces intersect the elliptic surface along one of its singular fibers with 4 components, see Figure 17.

- For T_0, $f_2^{-1}(T_0)$ has 7 components. One is isomorphic to the elliptic surface $\widetilde{E_1^2}/\mu_4$, one is G_Q isomorphic to the K3 surface $\widetilde{E_1^2}/\mu_2$, one is G_{QQ} isomorphic to Hirzebruch surface $\mathbb{F}_0$ and the other four are isomorphic to Hirzebruch surface $\mathbb{F}_1$. Furthermore, the 5 Hirzebruch surfaces intersect the elliptic surfaces $\widetilde{E_1^2}/\mu_4$ and $\widetilde{E_1^2}/\mu_2$ along their singular fiber of Kodaira Type I_0^*. See Figure 18.

The results of this chapter is summarized to Theorem 1.6.

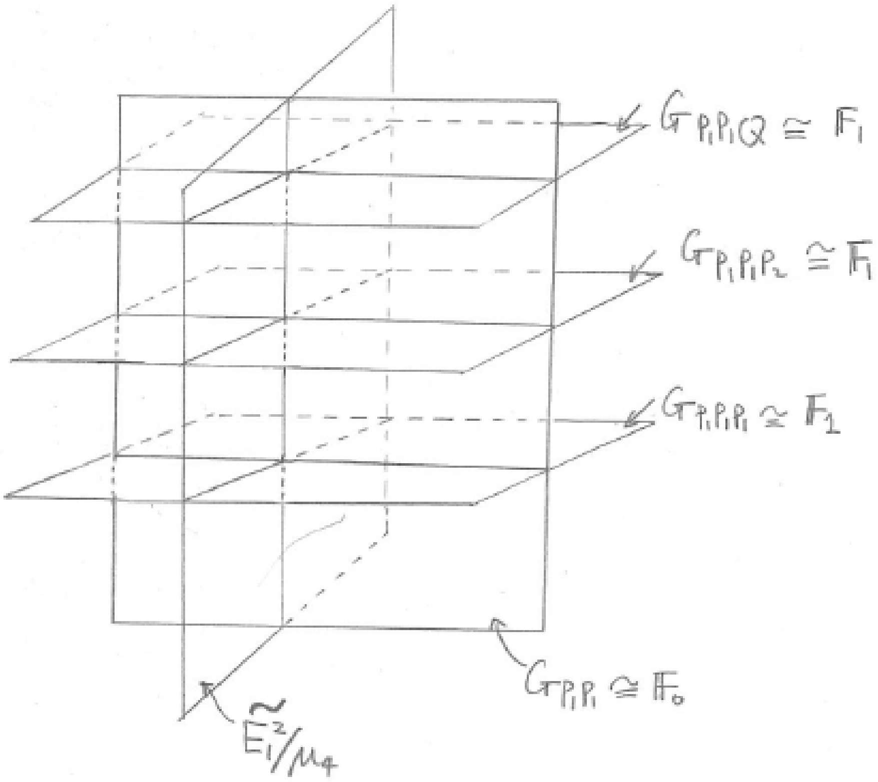


Figure 17: The Singular Fiber of f_2 over R_1.

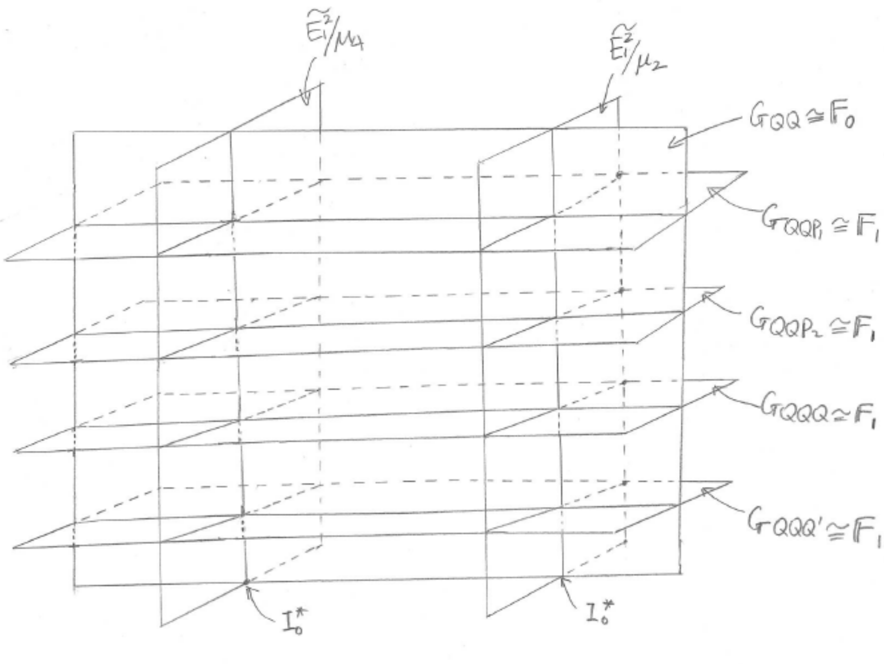


Figure 18: The Singular Fiber of f_2 over T_0.

APPENDIX

A. AN INTRODUCTION TO INTERSECTION THEORY

Let X be a scheme. The group of cycles $Z(X)$ on X is the free abelian group generated by the set of subvarieties(reduced and irreducible subschemes) of X. The group is graded by dimension:

$$Z(X) = \bigoplus_k Z_k(X)$$

where $Z_K(X)$ is generated by subvarieties of dimension k.

Definition A.1. The *Chow group* of X is the quotient

$$A_*(X) = Z(X)/Rat(X)$$

Here $Rat(X) \subset Z(X)$ is generated by $A - B$, where A and B are rational equivalent subvarieties.

The Chow group is graded by dimension:

$$A_*(X) = \bigoplus_{k=0}^{dimX} A_k(X)$$

where $A_k(X)$ is the group of rational equivalence classes of k-cycles. If $Y \subset X$ is a subvariety, we denote $[Y]$ its rational equivalence class.

We say that subvarieties A, $B \subset X$ are *generically transverse* if they meet transversely at a general point of each component of $A \cap B$. The Chow group $A(X)$ has a ring structure with respect to intersections:

Theorem A.2. If X is a smooth quasi-projective variety, then there is a unique product structure on $A(X)$ satisfying:

$$[A][B] = [A \cap B].$$

This structure makes

$$A^*(X) = \bigoplus_{c=0}^{dimX} A^c(X)$$

into an associative, commutative ring, graded by codimension. Where $A^c(X)$ is generated by the rational equivalence classes of codimensional c. $A^*(X)$ is called the **Chow ring** of X

Definition A.3. Let $f : X \to Y$ be a proper morphism of schemes and let $A \subset Y$ be a subvariety. Then the *push-forward for cycles* induced by f is a linear map $f_* : Z(X) \longrightarrow Z(Y)$ defined as following,

(a) If $dimf(A) < dimA$, then we set $f_*A = 0$.

(b) If $dimf(A) = dimA$ and $f|_A$ has degree n, then we set $f_*A = n(f(A))$.

(c) We extend f_* to all cycles on X by linearity.

Theorem A.4. If $f : X \to Y$ is a proper map of schemes, then the pushforward map $f_* : Z(X) \to Z(Y)$ induces a map of Chow groups $f_* : A_k(X) \to A_k(Y)$ for each $k \in \mathbb{N}$.

APPENDIX (Continued)

Definition A.5. Let $f : X \to Y$ be a morphism of smooth varieties. We say a subvariety $A \subset Y$ is *generically transverse* to f if the preimage $f^{-1}(A)$ is generically reduced and $codim_X(f^{-1}(A)) = codim_Y(A)$.

Theorem A.6. Let $f : X \to Y$ be a map of smooth quasi-projective varieties. There is a unique map of groups

$$f^* : A^c(Y) \to A^c(X)$$

such that whenever $A \subset Y$ is a subvariety generically transverse to f we have

$$f^*([A]) = [f^{-1}(A)].$$

Moreover, f^* is a ring homomorphism.

Theorem A.7. Let $f : X \longrightarrow Y$ be a map of smooth quasi-projective varieties. If $\alpha \in A^k(Y)$ and $\beta \in A_l(X)$, then

$$f_*(f^*\alpha \cdot \beta) = \alpha \cdot f_*\beta \in A_{l-k}(Y).$$

CITED LITERATURE

[Artebani and Dolgachev, 2006]Artebani, M. and Dolgachev, I.: The Hesse pencil of plane cubic curves. arXiv preprint math/0611590, 2006.

[Barth et al. , 2015]Barth, W., Hulek, K., Peters, C., and Van de Ven, A.: Compact complex surfaces, volume 4. Springer, 2015.

[Beauville, 1996]Beauville, A.: Complex algebraic surfaces. Number 34. Cambridge University Press, 1996.

[Candelas et al. , 1993]Candelas, P., Derrick, E., and Parkes, L.: Generalized Calabi-Yau manifolds and the mirror of a rigid manifold. Nuclear Physics B, 407(1):115–154, 1993.

[Cattaneo et al. , 2016]Cattaneo, A., Garbagnati, A., et al.: Calabi-Yau 3-folds of Borcea-Voisin type and elliptic fibrations. Tohoku Mathematical Journal, 68(4):515–558, 2016.

[Cogolludo-Agustín and Libgober, 2014]Cogolludo-Agustín, J.-I. and Libgober, A.: Mordell-Weil groups of elliptic threefolds and the alexander module of plane curves. Journal für die reine und angewandte Mathematik (Crelles Journal), 2014(697):15–55, 2014.

[Dold, 2012]Dold, A.: Lectures on algebraic topology, volume 200. Springer Science & Business Media, 2012.

[Eisenbud and Harris, 2016]Eisenbud, D. and Harris, J.: 3264 and all that: A second course in algebraic geometry. Cambridge University Press, 2016.

[Filippini et al. , 2011]Filippini, S. A., Garbagnati, A., et al.: A rigid Calabi-Yau three-fold. Advances in Theoretical and Mathematical Physics, 15(6):1745–1787, 2011.

[Fujita, 1986]Fujita, T.: Zariski decomposition and canonical rings of elliptic threefolds. Journal of the Mathematical Society of Japan, 38(1):19–37, 1986.

[Fulton, 2013]Fulton, W.: Intersection theory, volume 2. Springer Science & Business Media, 2013.

[Grassi, 1991]Grassi, A.: On minimal models of elliptic threefolds. Mathematische Annalen, 290(1):287–301, 1991.

[Grauert, 1962]Grauert, H.: Über modifikationen und exzeptionelle analytische mengen. Mathematische Annalen, 146(4):331–368, 1962.

[Hartshorne, 1977]Hartshorne, R.: Algebraic geometry, volume 52 of graduate texts in mathematics, 1977.

[Kawamata, 1983]Kawamata, Y.: Kodaira dimension of certain algebraic fiber spaces. Journal of the Faculty of Science, the University of Tokyo. Sect. 1 A, Mathematics, 30(1):1–24, 1983.

[Kirwan, 1992]Kirwan, F. C.: Complex algebraic curves, volume 23. Cambridge University Press, 1992.

[Kodaira, 1963]Kodaira, K.: On compact analytic surfaces: II. Annals of Mathematics, pages 563–626, 1963.

[Lamotke, 2013]Lamotke, K.: Regular solids and isolated singularities. Springer-Verlag, 2013.

[Libgober, 1982]Libgober, A.: Alexander polynomial of plane algebraic curves and cyclic multiple planes. Duke Mathematical Journal, 49(4):833–851, 1982.

[Libgober, 2012]Libgober, A.: On combinatorial invariance of the cohomology of Milnor fiber of arrangements and catalan equation over function field. Arrangements of hyperplanes-Sapporo 2009, 175-187, Adv. Stud. Pure Math., 62, Math. Soc. Japan, Tokyo, 2012.

[Miranda, 1981]Miranda, R.: The moduli of Weierstrass fibrations over $\mathbb{P}^1$. Mathematische Annalen, 255(3):379–394, 1981.

[Miranda, 1983]Miranda, R.: Smooth models for elliptic threefolds. The birational geometry of degenerations (Cambridge, Mass., 1981), 29:85–133, 1983.

[Miranda, 1989]Miranda, R.: The basic theory of elliptic surfaces. ETS, 1989.

[Miranda and Persson, 1986]Miranda, R. and Persson, U.: On extremal rational elliptic surfaces. Mathematische Zeitschrift, 193(4):537–558, 1986.

[Motzkin, 1951]Motzkin, T.: The lines and planes connecting the points of a finite set. Transactions of the American Mathematical Society, pages 451–464, 1951.

[Oguiso and Truong, 2013]Oguiso, K. and Truong, T. T.: Explicit examples of rational and Calabi-Yau threefolds with primitive automorphisms of positive entropy. arXiv preprint arXiv:1306.1590, 2013.

[Persson, 1990]Persson, U.: Configurations of Kodaira fibers on rational elliptic surfaces. Mathematische Zeitschrift, 205(1):1–47, 1990.

[Prill and others, 1967]Prill, D. et al.: Local classification of quotients of complex manifolds by discontinuous groups. Duke Mathematical Journal, 34(2):375–386, 1967.

[Serre, 2012]Serre, J.-P.: A course in arithmetic, volume 7. Springer Science & Business Media, 2012.

[Shioda and Schütt, 2010]Shioda, T. and Schütt, M.: Elliptic surfaces. arXiv preprint arXiv:0907.0298, 2010.

[Silverman, 2009]Silverman, J. H.: The arithmetic of elliptic curves, volume 106. Springer Science & Business Media, 2009.

[Smith, 1983]Smith, L.: Transfer and ramified coverings. In Mathematical Proceedings of the Cambridge Philosophical Society, volume 93, pages 485–493. Cambridge University Press, 1983.

[Voisin, 2002]Voisin, C.: Hodge theory and complex algebraic geometry. i, volume 76 of cambridge studies in advanced mathematics, 2002.

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