Splittings of Relatively Hyperbolic Groups

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THESIS

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Chicago, 2013

Chicago, Illinois

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To my parents,

For opening every door.

To Bree,

For being you.

ACKNOWLEDGMENTS

I would like to express my deepest gratitude for the continuous support of my advisor Daniel Groves. His patience, enthusiasm and vast expertise made my course of study and this thesis possible. He was the ideal advisor and mentor, and this work would have been much inferior without his assistance. I count myself tremendously fortunate for his involvement in my life these past three years.

I would also like to thank Alexander Dranishnikov for advising me while I studied at the University of Florida. His guidance during my mathematical adolescence helped me develop the abilities that carried me through this process.

In various ways, each member of my thesis committee contributed to my growth and experience during my time at UIC. My thanks go to Marc Culler, Benson Farb, Peter Shalen and Kevin Whyte.

Additionally, Ned Rosen, Paul Thie and especially Robert Gross made my experience at Boston College exceptional and made graduate school possible.

Lastly, I would like to thank my colleagues with whom I have shared so many enlightening and wonderful conversations. I have very many fond memories from time spent around the world discussing math in classrooms, cafes, and bars. Mathematicians are among the world's best people and I won't forget what an amazing experience it has been to participate in that culture.

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LIST OF ABBREVIATIONS

$\mathbb{Z}$	The group of integers
$\mathbb{R}$	The real numbers

SUMMARY

The primary goal of Geometric Group Theory is to understand the interplay between the geometric and algebraic properties of discrete groups. Often, surprising connections can be found which profoundly illuminate the algebraic structure of a group using tools relying on the geometric properties or vice versa. This cross-pollination of ideas and tools has yielded tremendous insights into group theory as well as geometry and topology. Examples demonstrating the power of this approach include Mostow-Prasad rigidity (Mostow, 1973; Prasad, 1973), the covering space proof of the Nielsen-Schreier Theorem, the Seifert-van Kampen Theorem, Poincaré Duality, see (Hatcher, 2002), Stallings' proof of the Sphere Theorem (Stallings, 1971), Agol's proof of the virtually Haken Conjecture (Agol, 2012), and the numerous approaches to group splittings (Bowditch, 1998a; Guirardel and Levitt, 2010a; Guirardel and Levitt, 2010b; Kropholler, 1990; Papasoglu, 2005; Papasoglu and Swenson, 2009; Rips and Sela, 1997; Scott and Swarup, 2003).

It is this last example with which this thesis is concerned. In 1998, Bowditch demonstrated that a maximal splitting of a hyperbolic group can be recognized by inspecting the topology of the group's boundary (Bowditch, 1998a). In particular, he realized that local cut points in the boundary should correspond to the endpoints of separating tubes in the Cayley graph. In turn, these correspond to neighborhoods of cosets of two-ended subgroups over which the group splits. He suggested that a similar analysis might provide insight into splittings of relatively hyperbolic groups.

We show that this is the case. One of the main benefits of Bowditch's work is that it also establishes that the splitting is invariant under quasi-isometries. This same fact is seen to be true in this context by the following results, which are the focus of Chapter 2.

Theorem 2.2.1. *Let Γ_1 and Γ_2 be finitely generated groups. Suppose that Γ_1 is hyperbolic relative to a finite collection $\mathcal{A}_1$ such that that no $A \in \mathcal{A}_1$ is properly relatively hyperbolic. Let $q : \Gamma_1 \to \Gamma_2$ be a quasi-isometry of groups. Then there exists $\mathcal{A}_2$, a collection of subgroups of Γ_2, such that the cusped space of $(\Gamma_1, \mathcal{A}_1)$ is quasi-isometric to that of $(\Gamma_2, \mathcal{A}_2)$.*

Corollary 2.3.1. *With $(\Gamma_1, \mathcal{A}_1)$ and $(\Gamma_2, \mathcal{A}_2)$ as in Theorem 2.2.1, the cusped spaces $X(\Gamma_1, \mathcal{A}_1)$ and $X(\Gamma_2, \mathcal{A}_2)$ have homeomorphic boundaries.*

Corollary 2.3.2. *With $(\Gamma_1, \mathcal{A}_1)$ as in Theorem 2.2.1, the tree describing the maximal peripheral splitting (Bowditch, 1998b) and the cut-point/cut-pair $\mathbb{R}$-tree (Papasoglu and Swenson, 2006) for the boundary of the cusped space are quasi-isometry invariant.*

In Chapter 3, we analyze the properties of the $\mathbb{R}$-tree obtained by applying the construction of (Papasoglu and Swenson, 2006). Foremost, this means showing that the topological structure of the boundary often allows for the construction of a tree which is actually simplicial. Complicating this step is the existence of cut points, absent in the context of hyperbolic groups. Nonetheless, we obtain

Theorem 3.1.5. *Let Γ be a finitely presented, one-ended group which is hyperbolic relative to the finite collection $\mathcal{A}$ such that for every $A \in \mathcal{A}$, A is not properly relatively hyperbolic and A contains no infinite torsion subgroup. Let $\mathcal{T}$ be the combined tree obtained by the action of Γ on its Bowditch boundary. Then $\mathcal{T}$ is simplicial.*

We also show that this tree is a JSJ-tree (Guirardel and Levitt, 2010a), ie it reflects a splitting which can be described as maximal in a suitable way (see Section 1.6).

Theorem 3.2.1. *The cut-pair/cut-point tree $\mathcal{T}$ is a JSJ tree over elementary subgroups relative to peripheral subgroups.*

Following this, we identify all of the vertex stabilizers, except for those which are 'rigid,' exposing much of the algebraic structure of the group. In fact, we prove that the vertex groups are individually quasi-isometry invariant. The various types of vertex groups listed are described in the introduction.

Theorem 3.3.5. *Let Γ_1 and Γ_2 be finitely generated groups. Suppose additionally that Γ_1 is one-ended and hyperbolic relative to the finite collection $\mathcal{A}_1$ of subgroups such that no $A \in \mathcal{A}_1$ is properly relatively hyperbolic or contains an infinite torsion subgroup. Let $\mathcal{T}$ be the cut-point/cut-pair tree of $\partial(\Gamma_1, \mathcal{A}_1)$. If $f : \Gamma_1 \to \Gamma_2$ is a quasi-isometry then*

- *T is the cut-point/cut-pair tree for Γ_2 with respect to the peripheral structure induced by Theorem 2.2.1,*
- *if $Stab_{\Gamma_1}(v)$ is one of the following types then $Stab_{\Gamma_2}(v)$ is of the same type,*

    1. *hyperbolic 2-ended,*
    2. *peripheral,*
    3. *relatively QH with finite fiber.*

Given all of this information regarding quasi-isometries of these groups, an obvious question is how much information about $\mathcal{QI}(\Gamma)$ can be gained by this analysis. It turns out that all of

the structure of $\mathcal{QI}(\Gamma)$ is preserved in its action on $\mathcal{T}(\Gamma, \mathcal{A})$. This is the content of the final chapter.

Theorem 4.1.1. *With the conditions on $\Gamma, \mathcal{A}, \mathcal{T}$ as in Theorem 3.3.5, the action of $\mathcal{QI}(\Gamma)$ on $\mathcal{T}(\Gamma, \mathcal{A})$ is faithful, assuming that $\mathcal{T}$ is not a point.*

Lastly, we provide reasonable upper and lower bounds on the number of edges that this induced splitting of $\mathcal{QI}(\Gamma)$ may have in terms of the number of edges of this JSJ-splitting of Γ. Let $\Lambda = |E(\mathcal{T}/\Gamma)|$ and let $\mathrm{Aut}_{qi}(\mathcal{T}/\Gamma)$ be the group of graph automorphisms which respect the quasi-isometry type of each edge and vertex stabilizer.

Theorem 4.2.1. *The graph of groups decomposition of $\mathcal{QI}(\Gamma)$ induced by the JSJ-decomposition of Γ has at most Λ edges and at least $\Lambda/|\mathrm{Aut}_{qi}(\mathcal{T}/\Gamma)|$ edges.*

CHAPTER 1

INTRODUCTION

In this chapter we provide the historical context for the main results as well as a comprehensive review of the necessary background material. The first section will be solely descriptive and so all definitions and precise statements will be deferred to the following sections. See the following for more detailed background: (Bestvina and Feighn, 1995; Bridson and Haefliger, 1999; Groves and Manning, 2008; Kapovich and Benakli, 2002; Papasoglu and Swenson, 2006; Osin, 2006; Swenson, 2005).

1.1 Historical Notes

Geometric group theory can arguably be said to have started with the work of Max Dehn, as part of his attempts to answer questions in low dimensional topology and combinatorial group theory. In 1911, he summarized the main problems of this field concerning the algorithmic properties of discrete groups and their presentations. These can be stated as follows:

1. *the isomorphism problem*: is there an algorithm which determines in a finite number of steps whether two arbitrary group presentations represent isomorphic groups?

2. *the conjugacy problem*: is there an algorithm which, given a presentation, determines in a finite number of steps whether two arbitrary elements of a group are conjugate?

3. *the word problem*: is there an algorithm which determines in a finite number of steps whether a word in the free group $w \in F(S)$ is in the kernel $F(S) \twoheadrightarrow \langle S|R \rangle$, or equivalently if w is in the normal closure $\langle\langle R \rangle\rangle \subset F(S)$?

It is known that there can be no solution for any of these questions which works for all finitely presented groups. In fact, Adyan (Adyan, 1955) and Rabin (Rabin, 1958) proved that there is no algorithm which even decides whether an arbitrary presentation represents the trivial group. Still, Dehn did provide a solution for questions (2) and (3), now called Dehn's Algorithm, which he applied to the fundamental groups of closed hyperbolic surfaces (Dehn, 1911). His insight demonstrated the strong connections between geometry, topology, group theory and decidability. His result has been generalized significantly, most directly to groups satisfying the $C'(1/6)$ small cancellation condition (Lyndon, 1966) and to fundamental groups of closed negatively curved manifolds (Cannon, 1984).

In (Gromov, 1987), Gromov went further and showed that the class of groups for which Dehn's algorithm can be applied corresponds exactly to a class of groups which he defined, *δ-hyperbolic groups.* These groups are defined by their large scale geometry which is required to admit a type of coarse negative curvature. They occur in many natural contexts, including free groups, almost all surface groups and triangle groups, fundamental groups of closed, negatively curved manifolds and groups which act properly discontinuously and co-compactly by isometries on any CAT(κ)-space, with $\kappa < 0$, among others. In fact, with a suitable probability model, a group chosen at random will almost surely be hyperbolic (Gromov, 1987).

Despite their prevalence, these groups have also provided a sufficiently rich context for many results to be proven. For instance, the isomorphism problem, which is generally much harder than the other of Dehn's problems, has been solved for hyperbolic groups by Dahmani and Guirardel (Dahmani and Guirardel, 2011), and in the torsion free case by (Sela, unpublished), see (Dahmani and Groves, 2008). In contrast, this problem is unsolvable in many other large classes of groups, including rather well-structured groups such as free-by-free groups (Miller, 1971). Moreover, hyperbolic groups are of independent interest and there are many results pertaining to hyperbolic groups beyond the solutions to Dehn's problems. For instance, a tremendous amount of work has gone into understanding their subgroups, algebraic structures, metric properties and finiteness properties.

Despite the power and flexibility of this notion, there is a strictly larger class of groups which admits a similarly impressive collection of results called *relatively hyperbolic groups*. These groups, also originally defined in (Gromov, 1987) but first seriously studied by Farb (Farb, 1994) and then Bowditch (Bowditch, 1997) and many others (Druţu and Sapir, 2005; Groves and Manning, 2008; Osin, 2006; Sisto, 2012b; Yaman, 2004), have a preferred collection of subgroups whose cosets are, at least intuitively, arranged in a 'hyperbolic manner.'[1] For example, the Bass-Serre tree for the group $G = A *_C B$ is a hyperbolic space on which G acts and this action reveals much of the geometry of G. If we also assume that C is finite, then the cosets of A and B are close together only along a set of finite diameter (the copies of C in each A or B coset).

[1] This description is actually best applied to groups which contain hyperbolically embedded subgroups.

Consequently, any two paths from a point in A (viewed as a graph) to any points in B must pass through C. If the points in B are far apart, this should remind the reader of geodesics diverging exponentially - a fundamental and important feature of hyperbolic spaces.

A critical property of this example is that we are not concerned whatsoever with the structures of A or B, only how their cosets are arranged in G. In general, the definition of relatively hyperbolic groups does not allow for much control over the structure of these exceptional subgroups. In particular, the geometric structure within a single coset may not resemble anything hyperbolic and there may be thick triangles or other wild geometry. On the other hand, triangles which span multiple cosets will appear slim whenever the sides cross a junction such as C in the above example. As we will see, this is often enough to obtain significant control over the global geometric structure of the group.

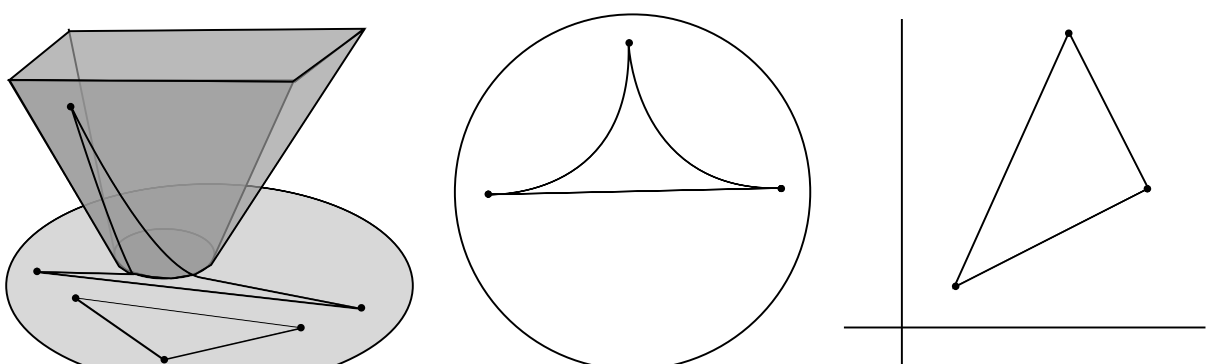

Figure 1. Triangles are 'thin between cosets' but may be 'thick within cosets.'

1.2 Coarse Geometry of Groups

The first hurdle to pass when attempting to study groups from a geometric perspective is that the geometry of a Cayley graph depends entirely on the chosen presentation. This is a serious issue even when restricted to finite presentations and easily understood groups. As a consequence, the class of maps which carry the geometric information of a group should not be isometries because two distinct presentations for the same group may not produce isometric Cayley graphs. Instead, a certain amount of flexibility is desired.

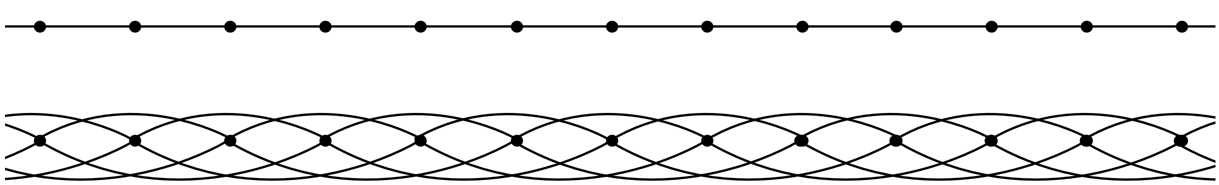

Figure 2. Cayley graphs for $\mathbb{Z}$ generated by $\{1\}$ and $\{2, 3\}$.

Definition 1.2.1. A map between metric spaces $f : X \to Y$ is called a *quasi-isometric embedding* if there exist constants A, B such that for all pairs $x_1, x_2 \in X$,

$$\frac{1}{A} d_X(x_1, x_2) - B \leq d_Y(f(x_1), f(x_2)) \leq A d_X(x_1, x_2) + B$$

One can easily see that quasi-isometric embeddings are generalizations of bi-Lipschitz maps (in which $B = 0$) and therefore of isometric embeddings (in which additionally $A = 1$). To have a reasonable generalization of isometry, we also need the map to be coarsely surjective.

Definition 1.2.2. If there also exists a constant C such that for every $y \in Y$ there exists $x \in X$ with $d_Y(y, f(x)) \leq C$, then f is called a *quasi-isometry.*

This resolves the conflict between geometric structures corresponding to different finite presentations of a single group $G = \langle S \mid R_S \rangle = \langle T \mid R_T \rangle$ - the identity map between any two will be a quasi-isometry, even though it will rarely be an isometry (it is an isometry if and only if they have the same symmetrized generating set, $S \cup S^{-1} = T \cup T^{-1}$). To see this, write out every generator of each presentation using the generating set of the other. Set A to be the maximum length of these words. One can easily verify that the identity map, $\mathrm{id} : (G, d_S) \to (G, d_T)$ is A-bi-Lipschitz.

An important fact is that quasi-isometries are always (quasi)-invertible. By this we mean that for every quasi-isometry $f : X \to Y$ there exists a quasi-isometry $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are bounded distance from id_Y and id_X, respectively, ie there exists D such that $\forall x \in X$, $d_X(x, g \circ f(x)) \leq D$. One candidate for such a map which always works is that which sends each $y \in Y$ to an x which minimizes $d_Y(y, f(x))$, choosing such an x at random in case of non-uniqueness.

Unfortunately, quasi-isometries do not have the structure of a group like $\mathrm{Isom}(X)$ or $\mathrm{Aut}(G)$, because they need not be bijections. However, because they are 'coarsely bijections,' we can find a group structure on the set of quasi-isometries modulo a certain equivalence relation. For any quasi-isometries $f, g : X \to X$ define $f \sim g$ if and only if $\sup_{x \in X}\{d(f(x), g(x))\}$ is finite. Define $\mathcal{QI}(X)$ to be the set of quasi-isometries modulo this relation.

Proposition 1.2.3. $\mathcal{QI}(X)$ is a group under the operation $[f] * [g] := [f \circ g]$.

We can similarly 'quasify' many other geometric notions.

Definition 1.2.4. A *quasi-geodesic line, arc or ray* is a quasi-isometric embedding of $\mathbb{R}$, $[0, 1]$ or $[0, \infty]$, respectively. We sometimes include the coefficients associated to the map and call such an embedding an (A, B)-quasi-geodesic line, arc or ray.

Definition 1.2.5. A subset $A \subset X$ of a geodesic metric space is called *quasi-convex* if there exists K such that for each $x, y \in A$, every geodesic connecting x to y is contained in the K-neighborhood of A, $N_K(A)$.

This last notion deserves a brief remark. While we will continue to concern ourselves with geodesics and quasi-geodesics both, the notion of convexity presents some more difficulties for studying discrete groups and needs some extra structure to remain useful. As an example, consider any G generated by a finite set S. Now consider the Cayley graph of G corresponding to the collection T of all elements of G with S-length either 1 or 4. The convex hull of the ball of radius 3 (with respect to T) is all of G (Bridson and Swarup, 1994, Proposition 3.1)!

Definition 1.2.6. Given x_1, x_2, x_3 in the geodesic metric space X and a triangle composed of geodesics connecting each pair, $\Delta = [x_1, x_2] \cup [x_2, x_3] \cup [x_3, x_1]$, we say that Δ is *δ-slim* if $[x_i, x_j] \subset N_\delta([x_i, x_k] \cup [x_j, x_k])$.

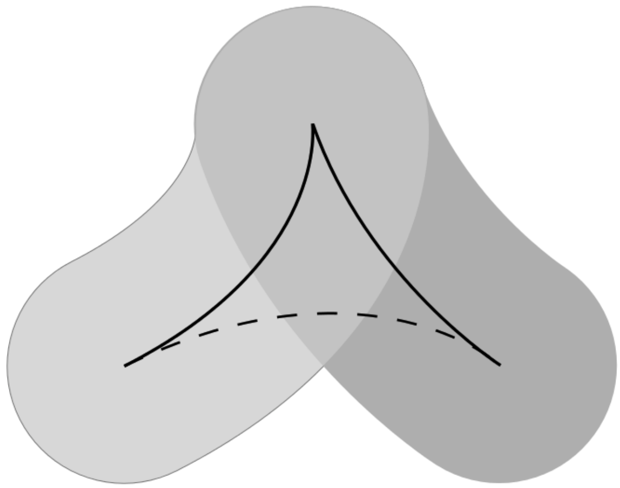

Figure 3. A slim triangle.

Definition 1.2.7. Let X be a geodesic metric space. If there exists a δ such that every triangle is δ-slim, then X is called a *δ-hyperbolic metric space.*

We note that there are numerous definitions for a space to be δ-hyperbolic and not all of them require that the space be geodesic or even connected. In the context of geodesic spaces, all of these definitions coincide (Bridson and Haefliger, 1999, Propositions III.H.1.17 and III.H.1.22), although the value of δ is not stable between definitions.

Definition 1.2.8. Let G be a group generated by the finite set S. If the Cayley graph of G with respect to S is δ-hyperbolic then G is called a *δ-hyperbolic group.*

Since Cayley graphs are combinatorial objects, there is little risk of confusing ourselves that it might admit a Riemannian metric of negative curvature. Therefore we often omit mention of δ and simply call such a group *hyperbolic*. Additionally, this property is a coarse metric condition and is preserved by quasi-isometries (Bridson and Haefliger, 1999, Theorem III.H.1.9). Consequently, if a group admits any δ-hyperbolic word metrics then all of its word metrics are δ'-hyperbolic, at least for finite generating sets. In these processes, the exact value of δ may change. However, we are generally concerned with qualitative properties of the geometry and so we will only be concerned with the binary matter of whether any such δ exists.

1.3 The Boundary of a Proper Hyperbolic Space

We begin with the definition of the boundary and then provide some simple but instructive examples. We first need to define the Gromov product, $(x|y)_z$, which roughly describes how long two geodesic arcs from z to x and to y stay close together in a metric space.

$$(x|y)_z := \frac{1}{2}\left[d(x,z) + d(y,z) - d(x,y)\right]$$

If the given metric space is a tree, then $(x|y)_z$ defines exactly the length that the arcs $[z,x]$ and $[z,y]$ coincide. In an arbitrary hyperbolic space, this idea still has plenty of power since triangles will be thin and can be approximated well by tripods.

The definition of the boundary of a δ-hyperbolic space relies on identifying sequences which 'converge to infinity' according to an equivalence relation based on the Gromov product. Let $\{x_i\}$ and $\{y_j\}$ be sequences in a metric space $(\mathbb{X}, d)$ and fix some $z \in \mathbb{X}$. If

$$\liminf_{i,j\to\infty}(x_i, y_j)_z = \infty$$

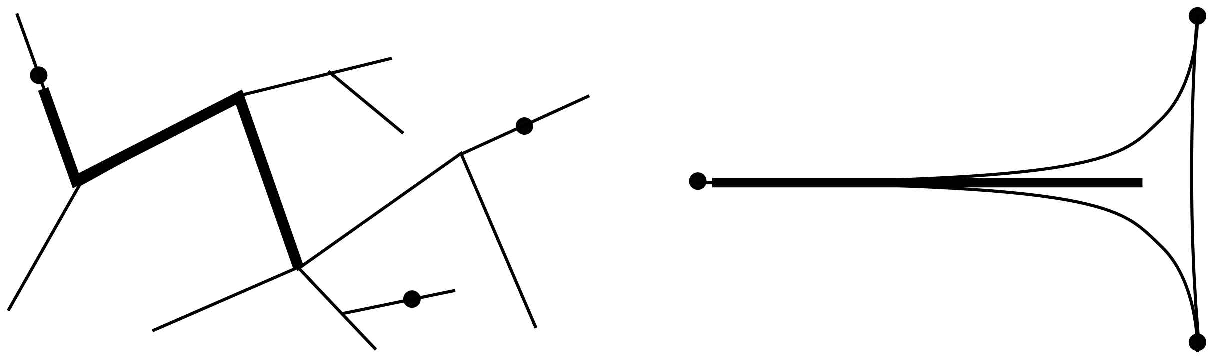

Figure 4. The length of the bold arc is exactly the Gromov product (based at the leftmost point) in a tree, and is very close in any δ-hyperbolic space.

then we say that $\{x_i\}$ and $\{y_j\}$ *converge at infinity* and $\{x_i\} \sim \{y_j\}$. It is a brief exercise to see that this is an equivalence relation and that convergence at infinity does not depend on the choice of z.

Definition 1.3.1. Let $\mathbb{X}$ be a proper δ-hyperbolic metric space. Given $x_0 \in \mathbb{X}$,

$$\partial\mathbb{X}(x_0) := \mathbb{X}^{\mathbb{N}} / \sim$$

The topology is defined via the following basis sets which describe collections of sequences such that geodesics from the product basepoint to the sequential points fellow travel for long distances:

$$U(p, r) := \{q \in \partial\mathbb{X} \mid \text{There are } \{x_i\} \to q \text{ and } \{y_j\} \to p \text{ with } \lim \inf_{i,j\to\infty} (x_i, y_j)_z \geq r\}$$

Here, changing the choice of z alters the basis sets but this change does not alter the topology. Different basepoints will dilate or constrict these sets and this may be important for applications involving the metric structure of the boundary but not here where we are only concerned with the topology.

The boundary has several very nice properties which we collect in the following proposition, noting that they may all be found separately in (Bridson and Haefliger, 1999, Section III.H.3).

Proposition 1.3.2. Given a proper δ-hyperbolic metric space $\mathbb{X}$, $\partial\mathbb{X}$ is

1. compact,
2. metrizable,
3. invariant (up to homeomorphism) under quasi-isometries,
4. connected when X is one-ended.

To expand on (3) above, let f be a quasi-isometry between proper δ-hyperbolic metric spaces $\mathbb{X}$ and $\mathbb{Y}$. This map will send sequences in $\mathbb{X}$ which fellow travel for a long distance to sequences in $\mathbb{Y}$ with the same property. In other words, f fully respects the topology induced by the basis elements $U(p, r)$. Since f has a quasi-inverse with the same property, we can see that f induces a homeomorphism $\partial\mathbb{X} \simeq \partial\mathbb{Y}$.

We conclude this section with some instructive examples. The first and simplest are the hyperbolic spaces, $\mathbb{H}^n$. Here, the boundary can be best described as the number of directions in which one can see from a given point, x_0. That is to say that the boundary turns out to be homeomorphic to the unit tangent sphere S^{n-1} of $x_0 \in \mathbb{H}^n$.

The next simplest example, for opposite reasons, are infinite regular proper trees. Here, the boundary is a Cantor set. This can be seen by associating each ray from a basepoint with a series of choices. Each choice can be interpreted as selecting a sub-interval in the next step in the construction of the Cantor set.

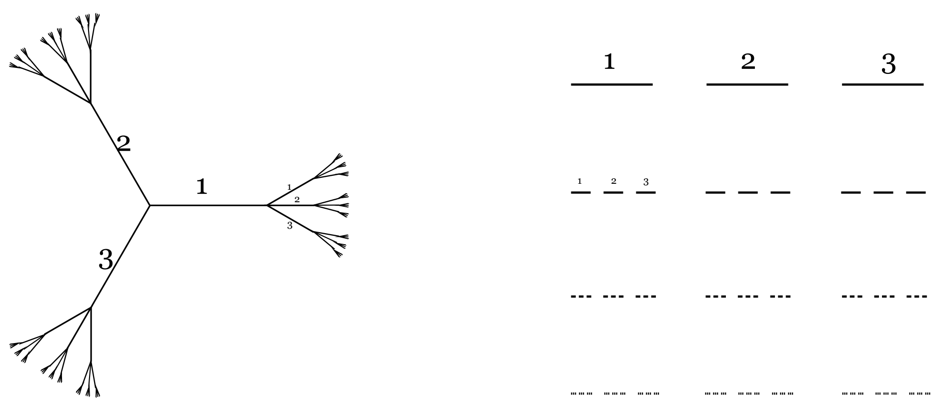

Figure 5. The boundary of a proper infinite regular tree is a Cantor set.

The last example is by far the most complex of the three under consideration. Consider two surfaces of genus 2, glued along their central curve.

We want to understand the boundary of the universal cover, $\widetilde{S_2 \coprod_\gamma S_2}$. We first consider a simpler but related space. Fix $\gamma_1 \in \widetilde{S_2}$ which is some particular lift of γ. Now we consider $\partial\left(\widetilde{S_2}\coprod_{\gamma_1}\widetilde{S_2}\right)$. The boundary is just two circles glued together along pairs of points,

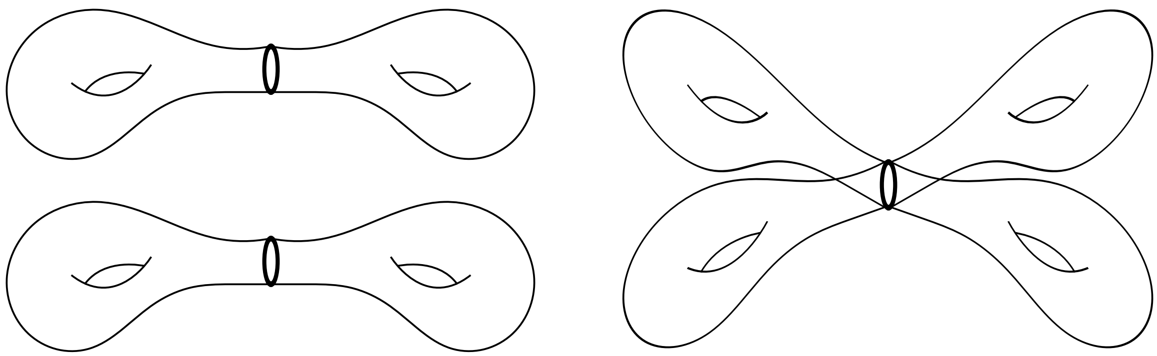

Figure 6. Surfaces glued along separating curves.

$S^1 \cup_{\{a,b\}\sim\{c,d\}} S^1$, where these pairs represent the endpoints of γ_1 in the closure of each copy of $\mathbb{H}^2$.

To understand the more complex situation, we observe that the main difference is that this gluing process has to happen for every lift of γ. This causes a massive 'unfolding' effect to happen, as every plane conjoined with this space must itself have many planes glued to it along further lifts of γ, *ad infinitum.*

1.4 Relatively Hyperbolic Groups

As previously suggested, the class of relatively hyperbolic groups is characterized by the same coarse negative curvature as hyperbolic groups except for the cosets of preferred subgroups. To make this definition precise, we produce a construction which will dramatically alter the geometry of these cosets. By sufficiently distorting the geometry of the preferred cosets we force them to be hyperbolic, so that by performing this alteration on every such coset we can modify the entire Cayley graph to be hyperbolic. The approach we use is adapted from that

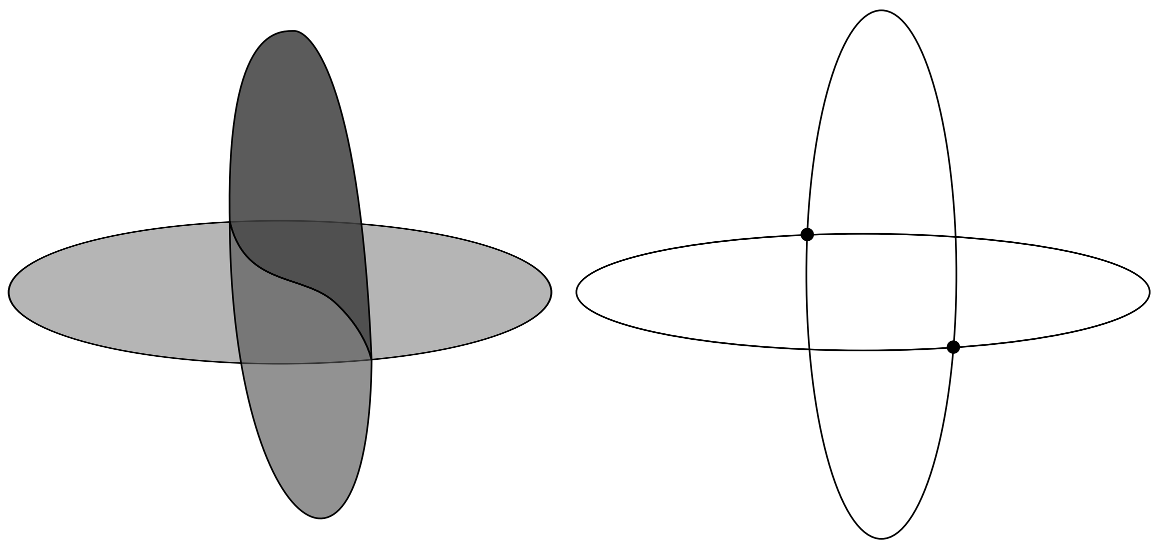

Figure 7. The boundary of conjoined hyperbolic planes.

of (Groves and Manning, 2008); we are not concerned with constructing a simply connected complex so we dispense with the 2-cells from their original construction.

Definition 1.4.1. (Groves and Manning, 2008, Definition 3.1) Given a connected graph Γ metrized with edges of length 1, we define the *combinatorial horoball* over Γ, $H(\Gamma)$, to be the graph with vertices $V(\Gamma) \times (\mathbb{N} \cup \{0\})$ and with the following collection of edges:

1. edges between (x, n) and (y, n) whenever $d_\Gamma(x, y) \leq 2^n$
2. edges between (x, n) and $(x, n + 1)$,

We refer to the edges of type (2) as *vertical*, those of types (1) as *horizontal* and the second coordinate of each pair as called its *depth*.

This construction has the effect on each horoball to which we have repeatedly alluded.

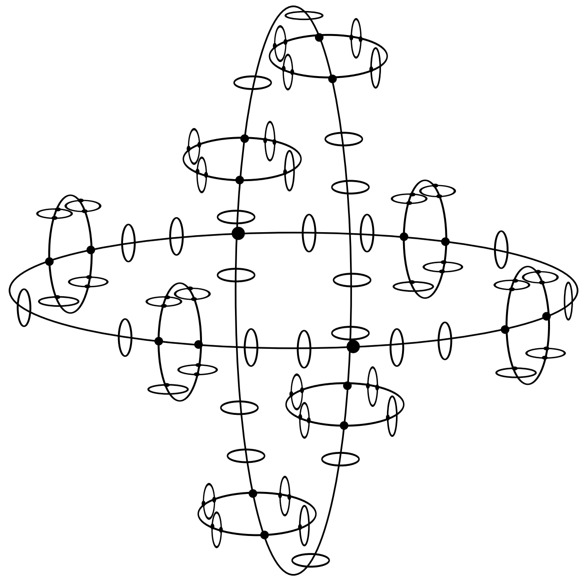

Figure 8. This boundary is much more complicated.

Theorem 1.4.2. *(Groves and Manning, 2008, Theorem 3.8) Let A be any connected graph with a length metric with edges length 1. The combinatorial horoball over A is δ-hyperbolic for some $\delta \leq 20$.*

This allows us to produce the main construction by which we will understand relatively hyperbolic groups. This construction requires that we have a generating set which also generates the cusps.

Definition 1.4.3. Let Γ be a group generated by a finite set S and let A be a finitely generated subgroup. We say that S is *compatible with* A if $\langle S \cap A \rangle = A$. Similarly, for a finite collection of finitely generated subgroups $\mathcal{A}$, we say that S *is compatible with* $\mathcal{A}$ if $\langle S \cap A_i \rangle = A_i$ for each $A_i \in \mathcal{A}$.

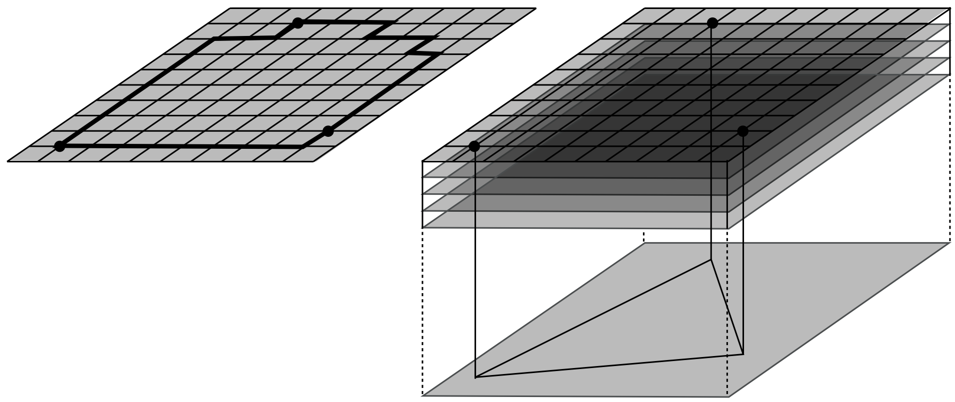

Figure 9. The Cayley graph of $\mathbb{Z} \times \mathbb{Z}$ (rescaled) and its combinatorial horoball.

The compatibility of a generating set is important because our definition of combinatorial horoball begins with a connected graph. From a mathematical perspective this is not strictly necessary and (Hruska, 2010) has extended the construction of horoballs to graphs which are not connected but inherit a proper metric from some other context. We choose to avoid the complications associated with this approach.

Definition 1.4.4. (Groves and Manning, 2008, Definition 3.12) Let $\mathcal{A}$ be a collection of subgroups of a group Γ generated by a finite set S compatible with $\mathcal{A}$. The *cusped space* $X(\Gamma, S, \mathcal{A})$ is the union of Γ with $\mathcal{H}(gA)$ for every coset of $A \in \mathcal{A}$, identifying gA with the depth 0 subset of $\mathcal{H}(gA)$. We suppress mention of $S, \mathcal{A}$ when they are clear from the context.

For points in $X(\Gamma)$, we do not distinguish between the depth functions of distinct horoballs because horoballs only overlap at depth 0 and so this convention is unambiguous. Thus, we can refer to the depth of any vertex in $X(\Gamma)$ without mention of the associated horoball or coset.

Definition 1.4.5. The elements of the collection of subgroups $\mathcal{A}$ are called *parabolic subgroups* and the subgroups which are conjugate to them are called *peripheral subgroups*.

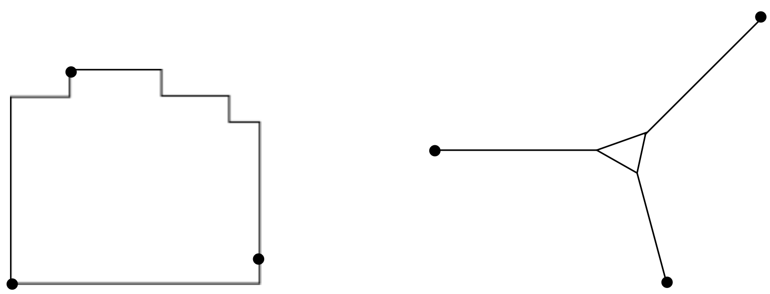

Figure 10. The effect of the combinatorial horoball on triangles. As in Figure 8, long edges tend to descend into the horoball where distances shrink exponentially.

Definition 1.4.6. (Groves and Manning, 2008, Theorem 3.25) A group Γ generated by the finite set S is *hyperbolic relative* to a collection of subgroups $\mathcal{A}$ if and only if $X(\Gamma, S, \mathcal{A})$ is δ-hyperbolic for some δ. We will often say that the pair $(\Gamma, \mathcal{A})$ is relatively hyperbolic or even that Γ is relatively hyperbolic when $\mathcal{A}$ is clear from context or unimportant.

This definition is not the original one. In fact, there have been several different but equivalent formulations of this notion, beginning with (Gromov, 1987) and (Farb, 1994) (in which the author uses the terminology 'relatively hyperbolic with bounded coset penetration') and followed by (Bowditch, 1997; Druţu and Sapir, 2005; Osin, 2006; Sisto, 2012b; Yaman, 2004). By (Szczepański, 1998) and (Groves and Manning, 2008, Theorem 3.25), these are equivalent.

It is worth considering how the cusped space can fail to be hyperbolic, as it is built of hyperbolic pieces (the horoballs). The simplest example of this is the fact that $\mathbb{Z} \times \mathbb{Z} = \langle a, b \rangle$ is not hyperbolic relative to either $\mathbb{Z}$ factor. In particular, given two adjacent cosets of $\langle a \rangle$, we can construct a thick triangle $\Delta((1,0),(a^n,0),(ba^{n/2},n))$ for some large n. Cosets which stay close together for large distances generally misbehave in this fashion and this is the basis of the description provided in the introduction - the cosets of the elements of $\mathcal{A}$ are 'hyperbolically arranged.'

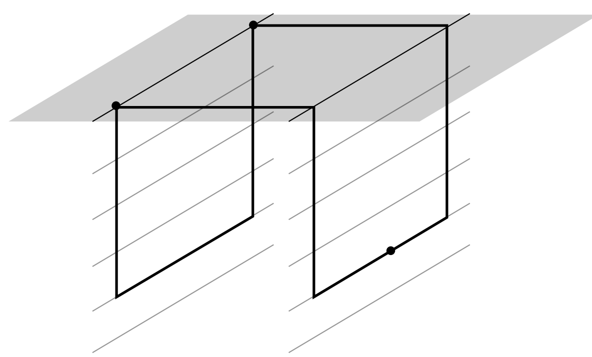

Figure 11. The cusped space can fail to be hyperbolic.

As is the case with hyperbolic groups and the value of δ, substituting S for some other finite generating set S' may change the topology of $X(\Gamma, S, \mathcal{A})$ and the value of δ, but does not affect the fact of the hyperbolicity of the cusped space.

Now that we have a definition for relatively hyperbolic groups, we should see why they are important. First of all, examples of relatively hyperbolic groups are abundant in geometric group theory.

- $\pi_1(M)$ for M a complete, finite volume manifold with pinched negative sectional curvature is hyperbolic relative to cusp subgroups (Farb, 1998; Bowditch, 1998b);
- the fundamental group of a graph of groups with finite edge groups is hyperbolic relative to vertex groups (Bowditch, 1998b);
- a limit group is hyperbolic relative to maximal non-cyclic abelian subgroups (Alibegović, 2005; Dahmani, 2003);
- a group acting geometrically on CAT(0) spaces with isolated flats is hyperbolic relative to the stabilizers of maximal flats (Druţu and Sapir, 2005; Hruska and Kleiner, 2005);
- a hyperbolic group is hyperbolic relative to $\{1\}$;
- a hyperbolic group is hyperbolic relative to a conjugacy-closed collection of infinite, quasi-convex subgroups, where there are finitely many conjugacy classes, each element is equal to its normalizer and each pair of conjugates of distinct elements has finite intersection (Bowditch, 1997, Proposition 7.11).

Moreover, relatively hyperbolic groups also have enough structure to allow us to understand them according to their structure at infinity, as (Bowditch, 1998a) did for hyperbolic groups.

Definition 1.4.7. (Bowditch, 1997, Definition 1) Given a group Γ hyperbolic relative to $\mathcal{A}$, the *Bowditch boundary*, $\partial(\Gamma, S, \mathcal{A})$ is the Gromov boundary of the associated cusped space, $\partial X(\Gamma, S, \mathcal{A})$. When there is no ambiguity we simply say the boundary.

In (Bowditch, 1997), the boundary is defined as the ideal boundary of a proper, hyperbolic space on which the group acts. This is largely motivated by the definition for relatively hyperbolic groups given in (Gromov, 1987). Part of the appeal of the characterization of relative hyperbolicity given in (Groves and Manning, 2008) is that it satisfies the conditions of the definition of (Gromov, 1987), and so it naturally fits with the notion of boundary developed in (Bowditch, 1997). In fact, Bowditch constructs a very similar space to the cusped space in which the edges at depth n are shrunk by a factor of 2^n instead of adding extra edges. The two spaces are quasi-isometric by the natural map on vertices, so the boundaries are the same. We note that peripheral subgroups present themselves in the following notable way in the boundary.

Lemma 1.4.8. (Groves and Manning, 2008, 3.11) If A is a combinatorial horoball, then the Gromov boundary of A consists of a single point, denoted e_A, which can be represented by a geodesic ray containing only vertical edges.

Because our proof of Theorem 2.2.1 can be simplified to prove an analogous result for the coned space, we include its definition.

Definition 1.4.9. (Farb, 1994) Let G be a group with a finite collection of subgroups $\mathcal{B}$. For every coset gB_i of an element of $\mathcal{B}$, we add to the Cayley graph a vertex v_{gB_i} and from every element of gB_i we add an edge to v_{gB_i}. We call this the *coned space* of $(G, \mathcal{B})$.

Definition 1.4.10. (Bowditch, 1997) A graph is *fine* if for every $n > 0$ and every edge e, the number of cycles of length at most n containing e is finite.

Definition 1.4.11 ((Bowditch, 1997) Alternate characterization of relative hyperbolicity)**.** Let C be the Cayley graph of a group G generated by a finite set S and let $\mathcal{A}$ be a collection of subgroups of G. If the coned space of $(G, \mathcal{A})$ is hyperbolic and fine, then $(G, \mathcal{A})$ is relatively hyperbolic (Bowditch, 1997, p. 27).

The following three definitions appear in Chapter 4 of (Osin, 2006). This author approaches relatively hyperbolic groups by understanding their relative presentations, ie those of the form $G = \langle S \cup \mathcal{A} \rangle$.

Definition 1.4.12. An element g of is called *hyperbolic* if g is not conjugate into any $A_i \in \mathcal{A}$.

Definition 1.4.13. Let $\Gamma = \langle S \mid R \rangle$ be a group hyperbolic relative to $\mathcal{A}$. A subgroup H is called *relatively quasi-convex* if there exists $\sigma > 0$ such that the following condition holds. Let f, g be two elements of H, and p an arbitrary geodesic path from f to g in $\mathrm{Cay}(\Gamma, S \cup \mathcal{A})$. Then for any vertex $v \in p$, there exists a vertex $w \in H$ such that $dist_S(u, w) \leq \sigma$.

Definition 1.4.14. If H is relatively quasi-convex and $H \cap A^g$ is finite for all $g \in \Gamma$ and $A \in \mathcal{A}$, then H is called *strongly relatively quasi-convex.*

Theorem 1.4.15 ((Osin, 2006), Theorem 4.19)**.** *The centralizer of a hyperbolic element in a relatively hyperbolic group is strongly relatively quasi-convex.*

Definition 1.4.16. A subgroup of a relatively hyperbolic group is called *elementary* if it is either virtually cyclic or a subgroup of a peripheral subgroup.

Some tremendously important and powerful research on relatively hyperbolic groups has been accomplished by considering their asymptotic cones, e.g. (Druţu and Sapir, 2005). While we will not define these, we will be using one of the results from this perspective - relatively hyperbolic groups are rigid.

Theorem 1.4.17. *(Druţu, 2009, Theorem 5.12) Let* Γ *be a group hyperbolic relative to a finite family of subgroups* $\mathcal{A}_1$*. If a group* Γ' *is* (L, C)*-quasi-isometric to* G *then* G' *is hyperbolic relative to* $\mathcal{A}_2$ *where each* $A_2^i \in \mathcal{A}_2$ *can be embedded quasi-isometrically in* $A_1^j \in \mathcal{A}_1$ *for some* j*.*

Lastly, a brief point on the subgroups of relatively hyperbolic groups.

Lemma 1.4.18. Let Γ be a relatively hyperbolic group. There are finitely many conjugacy classes of finite order subgroups F such that F is contained in a hyperbolic two-ended subgroup H with F fixing the ends of H.

Proof. The following argument is adapted from (Mosher, 2012). For every hyperbolic two-ended subgroup H, take $A_H \subset X(\Gamma, \mathcal{A})$ to be the set of all geodesics between the endpoints of H. There is a uniform width W for all A_H which is independent of the choice of H (Hruska, 2010, Corollary 8.16), ie for a point $p \in A_H \cap \mathrm{Cay}(\Gamma) \subset X(\Gamma, \mathcal{A})$, the set $H \setminus N_W(p)$ is not connected.

It follows that for some $k \in \mathbb{N}$, if $h \in H$ has the property that $[h.N_{kW}(p)] \cap N_{kW}(p) = \emptyset$ and h fixes the endpoints of H, then h has infinite order.

For F as above, it must be that the F-translates of $N_{kW}(p)$ are not disjoint from $N_{kW}(p)$. Conjugating by p sends every such F to a subgroup $F' \subset N_{kW}(1)$ of the same conjugacy class, and there are only finitely many possible F'. $\square$

1.5 Convergence Groups

Another definition for relatively hyperbolic groups is dynamical and characterizes these groups by means of *convergence actions* on a particular space (Bowditch, 1998b; Yaman, 2004). Roughly speaking, a convergence action is one which exhibits source-sink or North-South dynamics. We describe this property by means of subsequences.

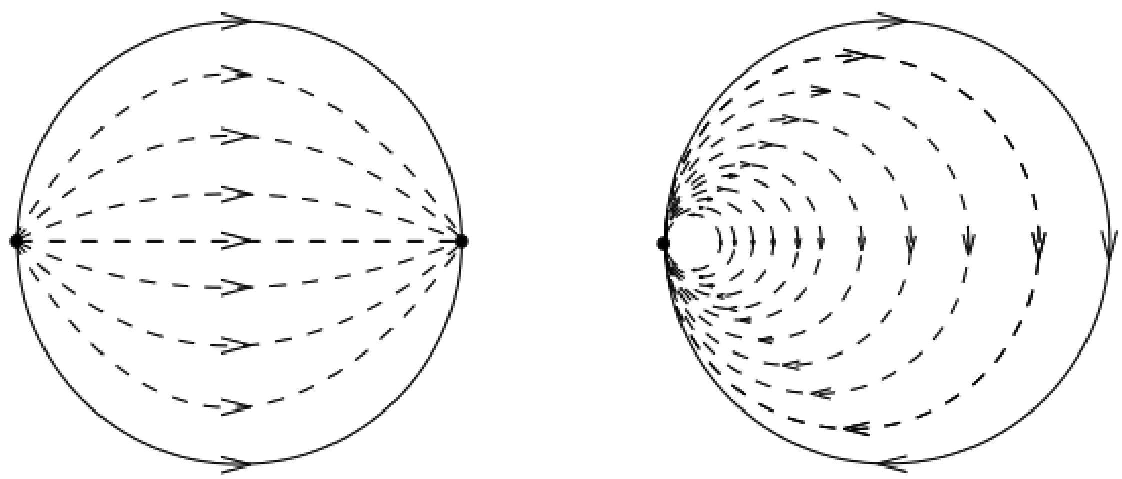

Figure 12. Source-sink dynamics - the source and sink can be the same point (right).

Definition 1.5.1. Suppose that G acts on the space X by homeomorphisms. The sequence of group elements $\{g_i\}$ is a *convergence sequence* if there exist points $x_1, x_2 \in X$ such that for any compact K with $x_1 \notin K$, $\{g_i(k)\} \to x_2$ for all $k \in K$.

Definition 1.5.2. If every sequence of elements of G contains a convergence subsequence then we say that G acts as a *convergence group* on X.

The importance of this definition is not immediately obvious yet it characterizes many important classes of groups. For instance, both hyperbolic and relatively hyperbolic groups are characterized by types of convergence actions which they exhibit on their boundaries. Suppose that M is a non-empty perfect metrisable compactum.

Theorem 1.5.3. *(Bowditch, 1998b, Theorem 0.1) Suppose that a group Γ acts by homeomorphism on M such that the induced action on the space of distinct triples, $\Sigma_3(M)$, is properly discontinuous and co-compact. Then, Γ is hyperbolic. Moreover, there is a Γ-equivariant homeomorphism $M \simeq \partial\Gamma$.*

Theorem 1.5.4. *(Yaman, 2004, Theorem 0.1) Let Γ be a group acting on M with the convergence property such that M consists only of conical limit points and bounded parabolic points*[1]*. Suppose also that the quotient of the set of bounded parabolic points by Γ is finite and that the stabilizer of each bounded parabolic point is finitely generated. Then Γ is hyperbolic relative to the set of its maximal parabolic subgroups and M is equivariantly homeomorphic to $\partial(\Gamma, \mathcal{A})$.*

This allows for the identification of stabilizers of certain topological features in these contexts and, while we will not apply the full power of this fact, we will use this to identify many vertex stabilizers in Chapter 3. We also require the classic Convergence Group Theorem:

Theorem 1.5.5. *(Casson and Jungreis, 1994; Gabai, 1992) A subgroup G of $\mathrm{Homeo}(S^1)$ acts as a convergence group on S^1 if and only if G is a discrete subgroup of Möbius transformations of D^2.*

---

[1] *These terms are somewhat technical and will not be used so we omit definitions*

In other words, G is a *Fuchsian* group or $\pi_1(\mathcal{O})$ for some 2-orbifold $\mathcal{O}$. As it turns out, this theorem is a specific instance of the power of the convergence property. The recipient space for a convergence action can, in some cases, identify the group acting with impressive precision. We summarize these relationships below. We denote by C a perfect, compact, connected metric space.

TABLE I

Groups which act on specific spaces with particular types of convergence actions.

Space	Additional Conditions?	Group
$\{x_1\}$	No	Any Group
$\{x_1, x_2\}$	No	Two-Ended
S^1	Yes	Fuchsian Group
C	Yes	Hyperbolic (Bowditch, 1998b)
C	Yes	Rel. Hyp. (Yaman, 2004)

Perhaps the best interpretation of C is as a stand-in for the boundary of a hyperbolic space. Since (Bowditch, 1998b) and (Yaman, 2004) are trying to deduce hyperbolicity and relative hyperbolicity, respectively, they cannot assume that they have a boundary *a priori*. The motivation for using Σ_3 is that ideal hyperbolic triangles in $\mathbb{H}^n$ uniquely represent their circumcenters up to rotation. For instance, $\Sigma_3(\partial\mathbb{H}^n)$ is the unit tangent bundle of $\mathbb{H}^n$, so we recover the original space up to the compact fiber. In the more general context, $\Sigma_3(C)$ actually still replicates the Cayley graph of the group quite closely.

1.6 Splittings of Groups

1.6.1 JSJ-Decompositions

The JSJ terminology is borrowed from 3-manifold topology. There, the concept of a JSJ decomposition comes from the work of Jaco-Shalen (Jaco and Shalen, 1979) and, independently, Johannson (Johannson, 1979). Their work established much of the structure of 3-manifolds by decomposing them into simpler pieces separated by a canonical collection of tori (and perhaps annuli). This is similar to studying general manifolds by understanding their prime pieces - those who do not admit any non-trivial connected sum decompositions. It is the next level of sophistication.

There is a natural duality between these canonical decompositions for manifolds and splittings of groups. The prime decomposition of a 3-manifold corresponds to the Grushko decomposition of its fundamental group and the JSJ decomposition corresponds to splittings over either $\mathbb{Z} \times \mathbb{Z}$ or over $\mathbb{Z}$. The latter situation arises in manifolds with boundary with an embedded annulus between boundary components.

This idea was popularized by (Rips and Sela, 1997) after Kropholler applied it to the class of groups closest to 3-manifold groups: $PD(3)$ groups (Kropholler, 1990). Rips and Sela borrowed the JSJ terminology and restricted it to splittings over $\mathbb{Z}$; later work extends this to arbitrary two-ended subgroups (Bowditch, 1998a), and other types of groups (Dunwoody and Sageev, 1999; Fujiwara and Papasoglu, 2006; Guirardel and Levitt, 2010a; Guirardel and Levitt, 2010b). The main properties which characterize JSJ decompositions of groups are that they are maximal and the splittings are all mutually compatible - see Definition 1.6.1 below. In other words, this type of decomposition is the optimal way of understanding all of the potential splittings of a group simultaneously. A particularly nice perk is that these splittings are often detectable geometrically (Bowditch, 1998a; Papasoglu, 2005).

Attempts to understand groups from this JSJ perspective have been quite successful and have provided a rich understanding of the interplay between the geometric and algebraic properties of discrete groups. Numerous authors have worked in this direction, including (Papasoglu, 2005; Papasoglu and Swenson, 2009; Sela, 1997; Bowditch, 1998a; Kropholler, 1990; Scott and Swarup, 2003). We note that the work of Scott and Swarup (Scott and Swarup, 2003) is similar in spirit to these but of a fairly distinct character and is, in fact, more closely related to the 3-manifold JSJ-decomposition than the others, perhaps excepting (Kropholler, 1990).

Much of the language developed by these various authors differs significantly and only recently has a unifying description been presented by (Guirardel and Levitt, 2010a; Guirardel and Levitt, 2010b). It is the language presented here which we adopt. Since we are interested only in the particular type of JSJ-decomposition most naturally associated to relatively hyperbolic groups we do not include the most general definitions.

Definition 1.6.1. (Guirardel and Levitt, 2010a, Definition 2 and Section 5) Let Γ be hyperbolic relative to $\mathcal{A}$. An *elementary JSJ splitting relative to* $\mathcal{A}$ is a tree, T, with a Γ-action such that the following hold.

1. all edge stabilizers are elementary subgroups;
2. (universally elliptic) any edge stabilizer of T fixes a point in any other tree with property (1);

3. (maximal for domination) for any tree T' satisfying (2), every vertex stabilizer of T stabilizes a vertex of T'; and

4. (relative to $\mathcal{A}$) all subgroups of elements of $\mathcal{A}$ fix a point in T.

Trees with these properties have been shown to exist in a wide range of scenarios (Guirardel and Levitt, 2010a; Guirardel and Levitt, 2010b). However, it should be mentioned that the trees are not unique, suggesting that a different principle object of study may be worthwhile. In fact, there are a number of permissible transformations of these trees which produce new trees that describe the same group. Instead, the focus is on collections of these trees.

Definition 1.6.2 ((Guirardel and Levitt, 2010a))**.** Having the same elliptic subgroups is an equivalence relation on the set of trees with elementary edge stabilizers. An equivalence class is called a *deformation space*.

In the study of JSJ decompositions, one important focus is on understanding those subgroups in which there are many mutually incompatible splittings. Such pairs of splittings are called *hyperbolic-hyperbolic* in (Rips and Sela, 1997) and are best understood as analogous to splittings of a surface group $G = \pi_1(S_g)$ over two simple closed curves, γ_1 and γ_2, with an essential intersection. Each curve gives a different splitting of G over $\mathbb{Z} = \pi_1(\gamma_i)$. The graph of groups with respect to the splitting over $\langle\gamma_1\rangle$ can be seen by first inspecting the universal cover of S_g. Here, the lifts of γ_1 provide a tesselation of $\mathbb{H}^2$ with the property that each component is stabilized by a conjugate of $\pi_1(C_i)$, with C_i a component of $S_g \setminus \gamma_1$.

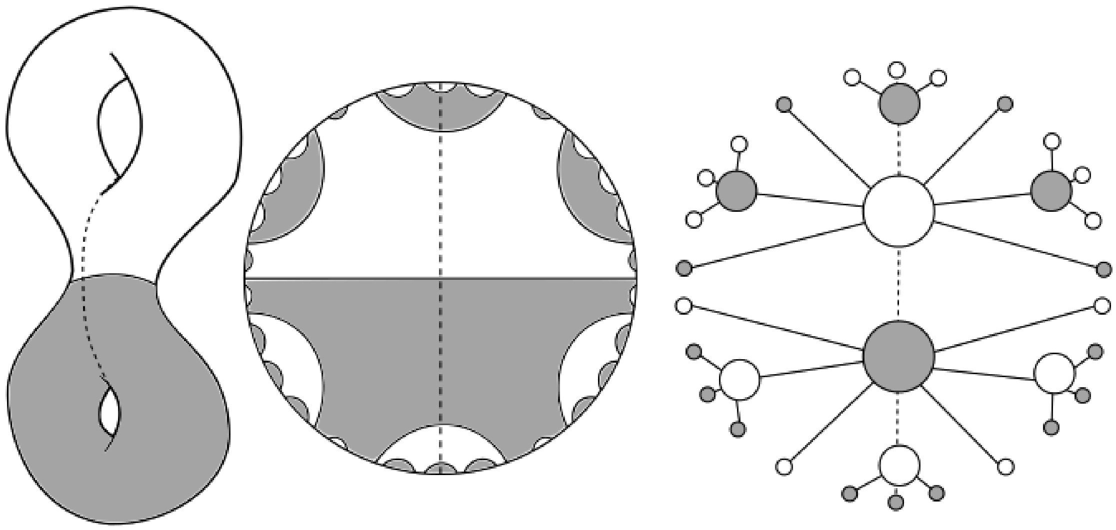

Figure 13. Hyperbolic-hyperbolic splittings.

For a freely indecomposable group, it is impossible to realize both splittings simultaneously with a common refinement of the graphs of groups. These are the hyperbolic-hyperbolic splittings identified above and they derive their name from the fact that each edge group acts hyperbolically on the tree associated to the other splitting, as illustrated by Figure 13. Give two curves which do not intersect, this issue does not arise and a common splitting is easy to produce. To combat the difficulties associated with hyperbolic-hyperbolic splittings, we simply accept that it is impossible to see these splittings in one graph of groups and we instead consider splittings over the subgroups which naturally contain many of these crossings.

The subgroups with this property have various names in different contexts, including *quadratically hanging* (Rips and Sela, 1997), *maximal hanging Fuchsian* (Bowditch, 1998a), *orbifold* *hanging vertex* (Papasoglu, 2005). These names all seek to describe the same central idea: many pairs of simple closed curves on surfaces intersect. The essential power of these ideas is that whenever these hyperbolic-hyperbolic splittings occur in finitely presented groups, the situation is always very close to the surface case. There is a more general definition which happens to encompass all of the above.

Definition 1.6.3. (Guirardel and Levitt, 2010a, Definition 4.2) Let Γ_v be a vertex group for a JSJ tree. If Γ_v is not universally elliptic, then Γ_v is called *flexible*.

For our purposes, this definition falls somewhat short. We are interested in identifying subgroups up to quasi-isometry and flexibility is not preserved under such maps. For example, compare closed hyperbolic surface groups with hyperbolic triangle groups. Here, a hyperbolic triangle group is the fundamental group of a hyperbolic 2-orbifold whose fundamental group is generated by three cone points. Both groups are the result of a geometric group action on $\mathbb{H}^2$, so the groups are quasi-isometric. However, surface groups have an infinite number of splittings whereas triangle groups have none!

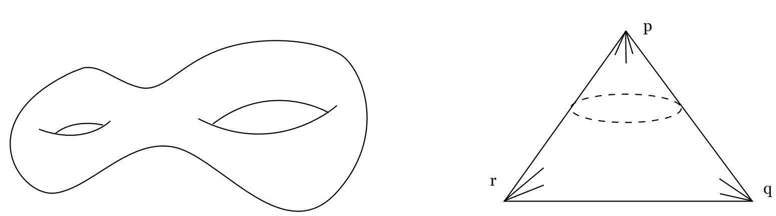

Figure 14. Hyperbolic Surface and triangle groups are quasi-isometric.

In the context of relatively hyperbolic groups, there is a larger class of subgroups which we can use instead of flexible subgroups and which will reflect the coarse geometry directly: relatively QH subgroups with finite fiber.

Definition 1.6.4. (Guirardel and Levitt, 2010a, Definition 7.3) Given a group with a JSJ tree relative to $\mathcal{A}$, a vertex stabilizer Q is a *relatively QH-subgroup* if it satisfies the following:

1. there is an exact sequence, with $\mathcal{O}$ a hyperbolic 2-orbifold and a subgroup F called the *fiber*:
$$1 \to F \to Q \to \pi_1(\mathcal{O}) \to 1$$
2. the images in $\pi_1(\mathcal{O})$ of edge groups incident to Q are either finite or contained in a boundary subgroup of $\pi_1(\mathcal{O})$.
3. every conjugate of an element of $\mathcal{A}$ intersects Q with image in $\pi_1(\mathcal{O})$ either finite or contained in a boundary component of $\pi_1(\mathcal{O})$.

The main reason that we prefer these to flexible subgroups is that these subgroups are detectable in the boundary of the relatively hyperbolic group. In Theorem 3.3.5, they appear as necklace stabilizers in the cut-point/cut-pair tree.

1.6.2 Peripheral Splittings of Relatively Hyperbolic Groups

In (Bowditch, 2001), the topological structure of the Bowditch boundary was analyzed to understand splittings of relatively hyperbolic groups over their peripheral subgroups. As might be guessed from Lemma 1.4.8, these splittings are detectable in the boundary as singletons which separate the boundary.

Theorem 1.6.5. *(Bowditch, 2001, Theorem 1.2) Suppose that $\partial\Gamma$ is connected. If Γ admits a non-trivial peripheral splitting, then $\partial\Gamma$ contains a global cut point.*

We ensure a connected boundary by requiring one-endedness of the group. Moreover, Bowditch remarks that every such cut point is a parabolic fixed point under some mild topological constraints. The full technical statement is:

Theorem 1.6.6. *(Bowditch, 1999, Theorem 0.2) Suppose that Γ is a relatively hyperbolic group whose boundary, $\partial\Gamma$, is connected. Suppose that each peripheral subgroup is finitely presented, either one-ended or two-ended, and contains no infinite torsion subgroup. Then every global cut point of $\partial\Gamma$ is a parabolic fixed point.*

Lastly, we can find a complete description of these cut points and express this as a splitting of Γ.

Theorem 1.6.7. *(Bowditch, 2001, Theorem 1.4) : Suppose that Γ is relatively hyperbolic with connected boundary. Then Γ admits a (possibly trivial) peripheral splitting which is maximal in the sense that it is a refinement of any other peripheral splitting.*

We further note that in a maximal splitting every cut point of the boundary corresponds to an edge stabilizer in the tree. This is because a maximal splitting has exclusively 2-connected (ie contains no cut points) components by definition, (Bowditch, 2001, p. 10).

1.7 $\mathbb{R}$-Trees and the Rips Machine

Part of the difficulty of (Bowditch, 2001; Bowditch, 1998a) is that the main constructions do not produce simplicial trees, but rather $\mathbb{R}$-trees.

Definition 1.7.1. Let $\mathbb{X}$ be a geodesic metric space. If $\mathbb{X}$ is 0-hyperbolic then $\mathbb{X}$ is called an $\mathbb{R}$-*tree.*

The Rips Machine is a collection of theorems which describe actions of groups on $\mathbb{R}$-trees. It is often used to prove that an $\mathbb{R}$-tree admitting a particular group action is simplicial. Most of the details can be found in (Bestvina and Feighn, 1995) and we record the most relevant definitions and theorems here.

Definition 1.7.2. Let Γ act on the $\mathbb{R}$-tree T by homeomorphisms. We say the action is *nesting* if there exists a $g \in \Gamma$ and an interval $I \subset T$ such that $g(I)$ is properly contained in I. Otherwise, we say the action is *non-nesting.*

A non-degenerate arc I is called *stable* if there is a non-degenerate sub-arc such that for any non-degenerate arc $K \subset J$, $\mathrm{Stab}(K) = \mathrm{Stab}(J)$. An action is called *stable* if any closed arc I of T is stable.

An action of a group on an $\mathbb{R}$-tree is called *minimal* if there are no proper invariant subtrees.

Theorem 1.7.3 ((Levitt, 1998) Theorem 1)**.** *If a finitely presented group G admits a non-trivial non-nesting action by homeomorphisms on an $\mathbb{R}$-tree T, then it admits a non-trivial isometric action on some $\mathbb{R}$-tree T_0. A subgroup fixing an arc in T_0 fixes an arc in T.*

Theorem 1.7.4 ((Bestvina and Feighn, 1995) Theorem 9.4)**.** *Let T be a minimal pure stable G-tree with G finitely presented. Then one of the following holds.*

1. *(surface) G maps onto a cone-type 2-orbifold group that splits over $\mathbb{Z}$ such that the kernel of this map is in the kernel of the action of G on T_G.*

2. *(toral) T is a line and G maps onto a finitely generated subgroup of* $\mathrm{Isom}(\mathbb{R})$ *such that the kernel of this map is in the kernel of the action of G on T.*

3. *(thin) G splits over a subgroup that fixes an arc of T.*

Theorem 1.7.5 ((Bestvina and Feighn, 1995) Theorem 9.5)**.** *Let G be a finitely presented group with a stable and minimal action on a tree T. Then either*

1. *G splits over an extension E-by-cyclic where E fixes an arc of T or*

2. *T is a line. In this case, G splits over an extension of the kernel of the action by a free abelian group.*

In any case, G has a nontrivial action on a simplicial tree.

The essential argument which is associated to our application of the Rips Machine is adapted from that outlined in (Swenson, 1999) and is similar to that of (Papasoglu and Swenson, 2009). We include citations for the tools which we use to accomplish each step. In summary:

1. Construct an $\mathbb{R}$-tree T with a G action (Papasoglu and Swenson, 2006)

2. Show that this action is non-trivial, stable and non-nesting (Lemmas **??**, 3.1.3, 3.1.4)

3. Construct from T an $\mathbb{R}$-tree S with an isometric G-action (Levitt, 1998, Theorem 1)

4. If S is not simplicial, obtain a contradiction with the Rips machine (Bestvina and Feighn, 1995)

1.8 Continua and $\mathbb{R}$-Trees

A major step in (Bowditch, 1998a) (and here as well) is converting the action of a group on its boundary into an action on a tree. The work of (Papasoglu and Swenson, 2006) systematizes this process for any group action on a continuum and it is their approach which we adopt.

Definition 1.8.1. A *continuum* is a compact connected metric space.

For us, the main examples of continua are group boundaries and certain subsets thereof. The $\mathbb{R}$-trees constructed using (Papasoglu and Swenson, 2006) have points representing the topological features of the continuum (cut points and cut pairs). The tree inherits the action of the group on the continuum. We condense their exposition and will later demonstrate that this tree is of JSJ type and often simplicial in the context of relative hyperbolicity. For the remainder of this section we assume that Γ is a relatively hyperbolic group.

Definition 1.8.2. Given a continuum X, a point $x \in X$ is a *cut point* if $X \setminus \{x\}$ is not connected. If $\{a, b\} \subset X$ contains no cut points and $X \setminus \{a, b\}$ is not connected, then $\{a, b\}$ is a *cut pair*. A subset Y is called *inseparable* if no two points of Y lie in different components of the complement of any cut pair of X.

These separating features occur as the fixed points of peripheral or hyperbolic two-ended subgroups over which Γ splits. Given that we want to also understand when there are many mutually incompatible splittings (as in Definitions 1.6.3 and 1.6.4), we have terminology reflecting interlocking cut pairs. These pairs arise in our context as the endpoints of pairs of hyperbolic two-ended subgroups over which the group splits but which admit no common refinement.

Definition 1.8.3. Let X be a continuum without cut points. A finite set S is called a *cyclic subset* if there is an ordering $S = \{s_1, s_2, \ldots, s_n\}$ and continua $M_1, \ldots M_n$ such that

1. $M_i \cap M_{i+1} = \{s_i\}$, subscripts mod n
2. $M_i \cap M_j = \emptyset$ whenever $|i - j| > 1$
3. $\cup M_i = X$

An infinite subset in which all finite subsets of cardinality at least 2 are cyclic is also called cyclic.

Definition 1.8.4. (Papasoglu and Swenson, 2006, p. 1769) A maximal cyclic subset with at least 3 elements is called a *necklace*.

Cyclic subsets arise as collections of mutually separable cut pairs. We also note that an inseparable cut pair can be in the closure of more than one necklace, but if the cut pair is not inseparable then that necklace is unique.

Definition 1.8.5. (Papasoglu and Swenson, 2006, p. 1762) Given a continuum X, we define an equivalence relation $\sim$ such that any cut point is equivalent only to itself and for x, y which are not cut pairs, $x \sim y$ if and only if there is no cut point z such that x and y are in different components of $X \setminus z$.

We would like to define a similar notion for cut pairs but the extra structure (particularly the interaction between maximal inseparable sets and necklaces) makes this difficult. Instead, we directly construct subsets of the powerset of X which reflect the topology. Let $\mathcal{R}$ be the

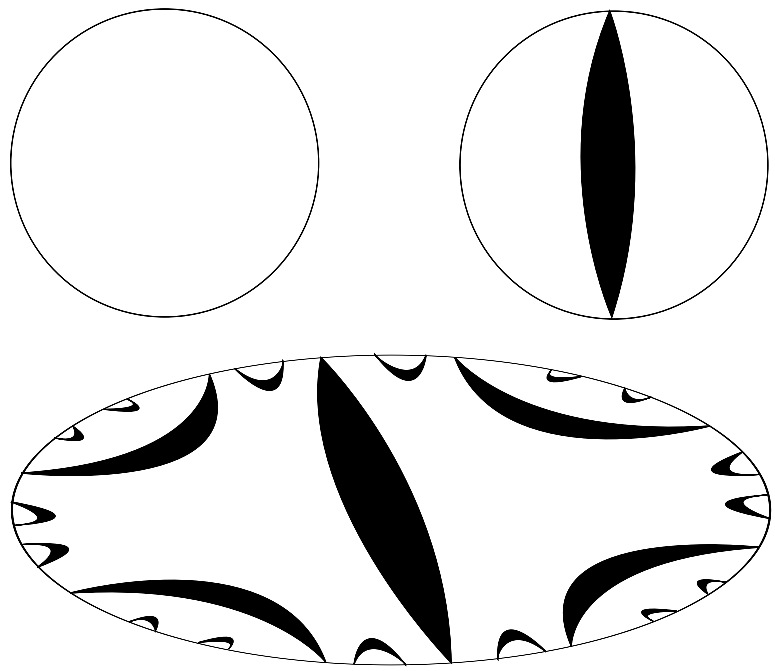

Figure 15. Necklaces can be much more complicated than circles or arcs. Attaching thickened segments to S^1 as above results in a continuum with several necklaces, all of which are Cantor sets.

collection containing all inseparable cut pairs, necklaces and maximal inseparable sets of X. We claim that this structure is compatible with $\sim$, ie that $\mathcal{R}$ is the union of sets defined similarly on each class of $\sim$. This follows from the following lemma, which immediately implies that cut points do not separate cut pairs.

Lemma 1.8.6. Suppose that T is a connected topological space with a continuum $A \subset T$ with the property that $T \setminus A$ is not connected. If $\{y, z\}$ is a cut pair then y and z are in the same component of $T \setminus A$.

Proof. First, replace T by the quotient space with all elements of A identified and let x be the image of A in the quotient. Let C_1 be the component of $T \setminus \{y, z\}$ containing x. Let w be a point in another component, C_2. Clearly x separates C_1 but not C_2. Thus, w and y are in the same component of $T \setminus \{x, z\}$ and w and z are in the same component of $T \setminus \{x, y\}$. Thus, y and z are in the same component of $T \setminus x$. $\square$

In (Papasoglu and Swenson, 2006, Theorems 12, 13, 14), $\sim$ is shown to satisfy a 'betweenness' property so that a process of 'connecting the dots' can fill it in to an $\mathbb{R}$-tree. By this we mean that we can associate an arc to every pair of points and glue the arcs together according to the betweenness relation. This produces the $\mathbb{R}$-tree from the betweenness relation. For more information see (Bestvina, 2002).

(Papasoglu and Swenson, 2006, Corollary 31) serves the same purpose for $\mathcal{R}$. The cut point tree which they construct is of course the peripheral splitting produced by (Bowditch, 2001). We remark that this tree (ie the tree produced by connecting the dots for $\sim$) is simplicial whenever the boundary is connected and locally connected (Bowditch, 2001, Theorem 9.2). This can be achieved by the following mild constraints on $\mathcal{A}$.

Theorem 1.8.7 ((Bowditch, 2001), Theorem 1.5)**.** *Suppose that Γ is relatively hyperbolic and that each peripheral subgroup is one- or two-ended and contains no infinite torsion subgroup. If $\partial\Gamma$ is connected then it is locally connected.*

Moreover, by the work of (Papasoglu and Swenson, 2011, Theorem 6.6), the cut pair tree is simplicial for any continuum without cut points.

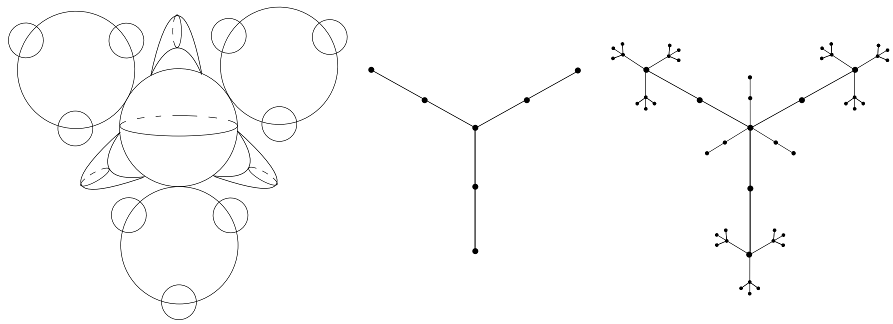

Figure 16. A continuum with the associated cut-point tree and combined tree.

It is the combination of the two (which Lemma 1.8.6 justifies) which we show is simplicial and which is originally discussed in Section 5 of (Papasoglu and Swenson, 2006), see Figure 16. As the two types of tree are independently simplicial, they can be combined by 'blowing up' the vertices of the cut point tree which correspond to maximal cut-point free subsets, along with their incident edges according to their individual cut pair structures. We call this the combined or cut point/cut pair tree and denote it by $\mathcal{T}$.

Theorem 1.8.8. *(Papasoglu and Swenson, 2006) For a locally connected continuum the combined* $\mathbb{R}$*-tree exists.*

As we stated, the most important question about an action of a group on an $\mathbb{R}$-tree is whether it can be promoted to an action on a simplicial tree, or whether the $\mathbb{R}$-tree itself is simplicial. This is often accomplished via the Rips machine and we will see this in Theorem 3.1.5.

There are topological examples in which this process produces $\mathbb{R}$-trees which cannot be made simplicial. This occurs when a cut point vertex appears as an end of the cut pair tree of one of its adjacent vertices. This pathology is clearly visible in the following example provided by Eric Swenson.

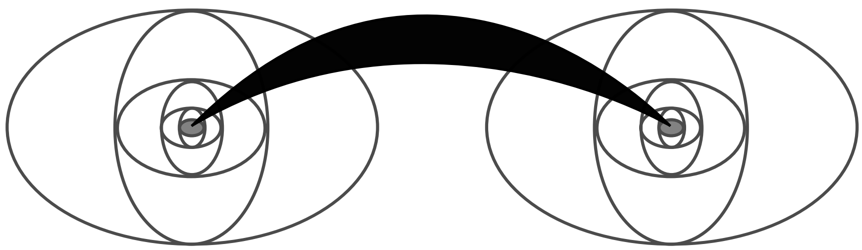

Figure 17. Swenson's Spectacles.

Let M be the closure of a union of infinitely many increasingly small ellipses centered at the origin which pairwise intersect in exactly two points. For instance, take the minor axis of one to be the same length as the major axis of its successor and arrange them at an angle of $\pi/2$. Let X be the union of two copies of M, connected at their center points by a thickened arc. The combined tree can never be simplicial because between any cut pair and a cut point there are infinitely many cut pairs.

We will show in Theorem 3.1.5 that this pathology can never occur in the boundary of a relatively hyperbolic group. This example shows that even if the cut point tree is simplicial and

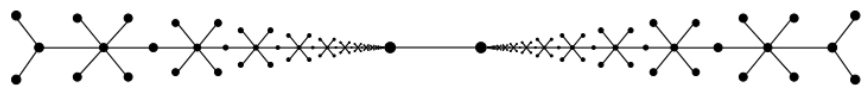

Figure 18. The $\mathbb{R}$-tree constructed from Swenson's Spectacles.

all cut pair trees are simplicial, we still require the group action to prove that the combined tree is simplicial.

CHAPTER 2

RIGIDITY OF THE CUSPED SPACE

In this chapter we prove our first main result:

Theorem 2.2.1. *Let Γ_1 and Γ_2 be finitely generated groups. Suppose that Γ_1 is hyperbolic relative to a finite collection $\mathcal{A}_1$ such that that no $A \in \mathcal{A}_1$ is properly relatively hyperbolic. Let $q : \Gamma_1 \to \Gamma_2$ be a quasi-isometry of groups. Then there exists $\mathcal{A}_2$, a collection of subgroups of Γ_2, such that the cusped space of $(\Gamma_1, \mathcal{A}_1)$ is quasi-isometric to that of $(\Gamma_2, \mathcal{A}_2)$.*

2.1 Quasi-Isometries between Horoballs

We first show that horoballs over quasi-isometric spaces are themselves quasi-isometric. To that end, we distinguish among types of geodesics which exist in horoballs. We assume that $n_2 > n_1$.

Definition 2.1.1. Let $H(T)$ be a horoball over the graph T with (t_1, n_1) and (t_2, n_2) vertices of $H(T)$. We say that the geodesic segment $[(t_1, n_1), (t_2, n_2)]$ is *vertical* or a *vertical geodesic segment* if n_2 is the maximal depth among vertices of $[(t_1, n_1), (t_2, n_2)]$. See Figure 19.

Lemma 2.1.2. Let $q : T \to S$ a (k, c)-quasi-isometry between graphs. There is a $(1, C)$-quasi-isometry $\hat{q} : H(T) \to H(S)$ between combinatorial horoballs such that $\hat{q}$ extends q. Furthermore, C depends only on k and c.

Proof. Extend q to $\hat{q}$ by defining $\hat{q}(v, n) = (q(v), n)$ and let $s_i = q(t_i)$. The proof proceeds by comparing lengths of geodesics; we show that the length of a segment in $H(S)$ is less than

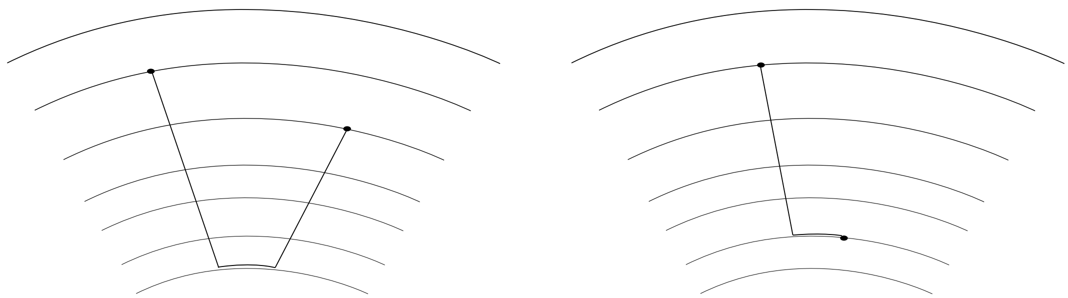

Figure 19. A vertical geodesic (right) and a non-vertical geodesic.

a linear function of the length of the corresponding segment in $H(T)$. Since we start with a quasi-isometry, this argument is also valid in the reverse direction. Moreover, the quasi-inverse has coefficients which obey the same dependencies which establishes the proof. We partition the proof into cases by which geodesics are vertical.

$[(s_1, n_1), (s_2, n_2)]$ *vertical:* $d_{H(S)} \in \{n_2 - n_1, n_2 - n_1 + 1, n_2 - n_1 + 2, n_2 - n_1 + 3\}$ and $d_{H(T)} + 3$ is clearly at least as large.

For the remaining two cases we assume that $[(s_1, n_1), (s_2, n_2)]$ is not vertical.

$[(t_1, n_1), (t_2, n_2)]$ *not vertical:*

$$\begin{aligned}
d_{H(S)}((s_1, n_1), (s_2, n_2)) &\leq 2\log_2 [d_S(s_1, s_2)] + 3 - n_2 - n_1 \\
&\leq 2\log_2[kd_T(t_1, t_2) + c] + 3 - n_2 - n_1 \\
&\leq 2\log_2[(k + c)d_T(t_1, t_2)] + 3 - n_2 - n_1 \\
&\leq 2\log_2(k + c) + 2\log_2[d_T(t_1, t_2)] + 3 - n_2 - n_1 \\
&\leq \hat{d}_T((t_1, n_1), (t_2, n_2)) + 2\log_2(k + c) + 3
\end{aligned}$$

$[(t_1, n_1), (t_2, n_2)]$ *vertical:*

$$\begin{aligned}
d_{H(S)}((s_1, n_1), (s_2, n_2)) &\leq 2\log_2(d_S(s_1, s_2)) - n_2 - n_1 + 3 \\
&\leq 2\log_2(k + c) + 2\log_2(d_T(t_1, t_2)) - n_2 - n_1 + 3 \\
&\leq 2\log_2(k + c) + n_2 - n_1 + 3 \qquad (2.1) \\
&\leq 2\log_2(k + c) + \hat{d}_T(t_1, t_2) + 3
\end{aligned}$$

Since $[t_1, t_2]$ is not descending, $\log_2(d_T(t_1, t_2)) \leq n_2$, justifying (2.1).

The coarse density of the image is clear. Since q is a quasi-isometry, we can also get identical results in the reverse direction with a symmetric argument so that geodesics in $H(T)$ are bounded by a linear function of the lengths in $H(S)$. Thus, $\hat{q}$ is a $(1, 2\log_2(k + c) + 3)$-quasi-isometry. $\square$

2.2 Reconstructing Geodesics

Theorem 2.2.1. *Let Γ_1 and Γ_2 be finitely generated groups. Suppose that Γ_1 is hyperbolic relative to a finite collection $\mathcal{A}_1$ such that that no $A \in \mathcal{A}$ is properly relatively hyperbolic. Let $q : \Gamma_1 \to \Gamma_2$ be a quasi-isometry. Then there exists $\mathcal{A}_2$, a collection of subgroups of Γ_2, such that the cusped space of $(\Gamma_1, \mathcal{A}_1)$ is quasi-isometric to that of $(\Gamma_2, \mathcal{A}_2)$.*

Proof. We extend q to a map $Q : X(\Gamma_1) \to X(\Gamma_2)$. First, let $Q = q$ on $\mathrm{Cay}(\Gamma_1)$. By the proof of Theorem 5.12 of (Druţu, 2009), q induces a quasi-isometric embedding of cosets of elements of $\mathcal{A}_1$ into those of $\mathcal{A}_2$ and we can take these to have uniform constants. We observe that this can be made coarsely surjective because no peripheral subgroup is properly relatively hyperbolic. The reason for this is that if we had a quasi-isometry which was not coarsely surjective then this would give an element of $\mathcal{A}_2$ that would be hyperbolic relative to the image of q of some coset of an element of $\mathcal{A}_1$, again by (Druţu, 2009), contrary to our hypothesis. When the map is coarsely surjective, we get that this element is hyperbolic relative to itself, which is the trivial case that we allow.

Now, q might only take a coset to within a bounded distance of the corresponding coset in Γ_2, rather than directly to it as in Lemma 2.1.2. We can still use the induced quasi-isometry on the subsets of the horoballs which have positive depth but at depth 0 we have to make an adjustment.

However, the proof of (Druţu, 2009, Theorem 5.12) shows us that there exists a bound T such that the image of A_i is at most T from the coset to which it is quasi-isometric. Thus, we have to account only for an extra additive T in the constants.

Thus we know that q induces quasi-isometries between cosets of peripheral subgroups and that the constants of these quasi-isometries do not depend on the particular cosets but only on the constants of q and on $(\Gamma_1, \mathcal{A})$. Therefore, Q restricts to a quasi-isometry on each individual horoball and on the Cayley graph. We only need to show that these can be combined across all of X.

Now let $[x, y]$ be a geodesic arc between points of $X(\Gamma_1)$. We divide this arc into several subarcs by taking the collection of maximal subarcs I_1 which have every vertex of depth 0 and also take the complimentary collection of segments I_2. We note that some of these arcs may be degenerate (length 0, just a singleton) and that our methods account for this. In other words, we have

$$[x, y] = [x_0, x_1] \cup [x_1, x_2] \cup \ldots \cup [x_{n-1}, x_n] = [x_0, x_n]$$

with $[x_{2i}, x_{2i+1}] \in I_1$ and $[x_{2i+1}, x_{2i+2}] \in I_2$. Essentially, we have divided $[x_0, x_n]$ into segments which go between two different horoballs (again, possibly with length 0) and segments which traverse individual horoballs. We should mention that the subdivision used here has a parity which suggests that the terminal points x_0 and x_n must have depth 0, but that this is easily surmounted. Simply attach a vertical segment to a depth 0 vertex whenever necessary.

We have the following expression:

$$\hat{d}_1(x_0, x_n) = \sum_{i=0}^{n-1} \hat{d}(x_i, x_{i+1}) = \sum_{I \in I_1} \text{length}(I) + \sum_{I \in I_2} \text{length}(I)$$

We construct a path in Γ_2 which tracks the image of $[x_0, x_n]$. For each x_i, x_{i+1}, we take any geodesic in $X(\Gamma_2)$, $[Q(x_i), Q(x_{i+1})](= [q(x_i), q(x_{i+1})])$, see Figure Figure 20. Because q is a quasi-isometry between Cayley graphs, $d_1 = \hat{d}_1$ for any segment in I_1, and $d_2 \geq \hat{d}_2$, we get the following estimate on the lengths of the images of endpoints of segments of I_1:

$$\sum_{i=0}^{(n-1)/2} \hat{d}_2(q(x_{2i}), q(x_{2i+1})) \leq \sum_{i=0}^{(n-1)/2} d_2(q(x_{2i}), q(x_{2i+1}))$$
$$\leq \sum_{i=0}^{(n-1)/2} [kd_1(x_{2i}, x_{2i+1}) + c] = \sum_{i=0}^{(n-1)/2} [k\hat{d}_1(x_{2i}, x_{2i+1}) + c]$$

By Lemma 2.1.2, we know that the horoballs paired by q are quasi-isometric and, by (Druţu, 2009), that the constants can be chosen uniformly. If we let $\Lambda - 2T$ be the maximum among those constants and k, c, we get the following length estimate:

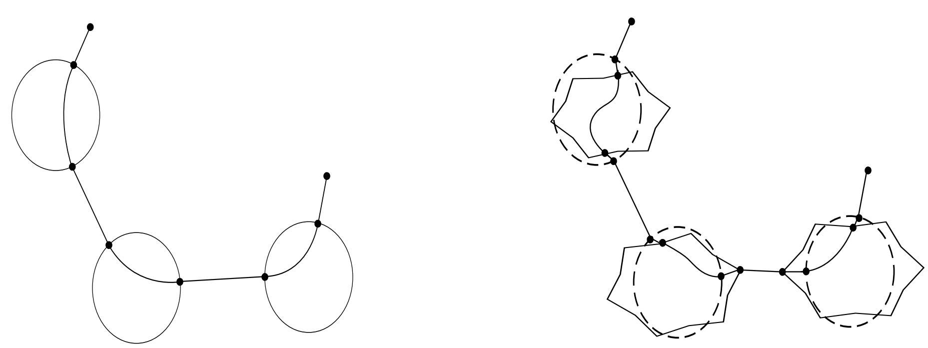

Figure 20. A typical geodesic in $X(\Gamma_1)$ and a reconstructed piecewise geodesic in $X(\Gamma_2)$.

$$\begin{aligned}
\hat{d}_2(q(x_0), q(x_n)) &\leq \sum_{i=0}^{(n-1)/2} \hat{d}_2(q(x_{2i}), q(x_{2i+1})) &&+ \sum_{i=1}^{(n-1)/2} \hat{d}_2(q(x_{2i-1}), q(x_{2i})) \\
&\leq \sum_{i=0}^{(n-1)/2} [\Lambda \hat{d}_1(x_{2i}, x_{2i+1}) + \Lambda] &&+ \sum_{i=1}^{(n-1)/2} [\Lambda \hat{d}_1(x_{2i-1}, x_{2i}) + \Lambda]
\end{aligned} \tag{2.2}$$

Now, since we know that the horoball-transversals have length ≥ 1, we can move the additive constant Λ from the first sum to the second sum, except for a single summand. We do this to account for scenarios in which a geodesic is constructed from several paths contained entirely in horoballs, making the first sum $\sum \Lambda$.

$$\begin{aligned}
\hat{d}_2(q(x_0), q(x_n)) &\leq \Lambda + \sum_{i=0}^{(n-1)/2} [\Lambda \hat{d}_1(x_{2i}, x_{2i+1})] &&+ \sum_{i=1}^{(n-1)/2} [\Lambda \hat{d}_1(x_{2i-1}, x_{2i}) + 2\Lambda] \\
&\leq \Lambda + \sum_{i=0}^{(n-1)/2} [\Lambda \hat{d}_1(x_{2i}, x_{2i+1})] &&+ \sum_{i=1}^{(n-1)/2} [3\Lambda \hat{d}_1(x_{2i-1}, x_{2i})] \\
&\leq \Lambda + 3\Lambda \left[\sum_{i=0}^{(n-1)/2} [\hat{d}_1(x_{2i}, x_{2i+1})] \right. &&\left. + \sum_{i=1}^{(n-1)/2} [\hat{d}_1(x_{2i-1}, x_{2i})] \right] \\
&= 3\Lambda \hat{d}_1(x_0, x_n) + \Lambda
\end{aligned} \tag{2.3}$$

We still need to establish coarse density. However, this is clear for depth 0 vertices since the Cayley graphs are quasi-isometric and it is clear for the positive depth vertices since the horoballs are in bijection and this pairing is by quasi-isometries.

We also need to produce the other bound, but by a symmetric argument using the quasi-inverse r,

$$\hat{d}_1(r(x_0), r(x_n)) \leq 3\Lambda' \hat{d}_2(x_0, x_n) + \Lambda'$$

Therefore,

$$\hat{d}_1(r(q(x_0)), r(q(x_n))) \leq 3\Lambda' \hat{d}_2(q(x_0), q(x_n)) + \Lambda'$$

$$\leq 3\Lambda'[3\Lambda \hat{d}_1(x_0, x_n) + \Lambda] + \Lambda' = 9\Lambda\Lambda' \hat{d}_1(x_0, x_n) + 3\Lambda\Lambda' + \Lambda$$

Because q and r are quasi-inverse, there exists an $a > 0$ such that

$$\hat{d}_1(x_0, x_n) \leq \hat{d}_1(r(q(x_0)), r(q(x_n))) + a$$

Combining these, we get

$$\frac{1}{3\Lambda'} \hat{d}_1(x_0, x_n) - \frac{a + \Lambda'}{3\Lambda'} \leq \hat{d}_2(r(x_0), r(x_n)) \leq 3\Lambda' \hat{d}_1(x_0, x_n) + \Lambda'$$

We conclude by maximizing among constants. □

As indicated in the introduction, our proof of Theorem 2.2.1 simplifies to prove an analogous result for the coned space.

Theorem 2.2.2. *Let $\Gamma_1, \Gamma_2, \mathcal{A}_1, q$ be as above. Then there exists a collection of subgroups $\mathcal{A}_2$ of Γ_2 such that the coned spaces of $(\Gamma_1, \mathcal{A}_1)$ and $(\Gamma_2, \mathcal{A}_2)$ are quasi-isometric.*

Proof. Adjust the proof of Theorem 2.2.1 by replacing the intra-horoball arcs with arcs through the cone-points. These all have length 2 so simply change (2.2) to

$$\hat{d}_2(q(x_0), q(x_n)) \leq \sum_{i=0}^{(n-1)/2} \hat{d}_2(q(x_{2i}), q(x_{2i+1})) + \sum_{i=1}^{(n-1)/2} 2$$

□

2.3 Some Important Corollaries

Corollary 2.3.1. With $(\Gamma_1, \mathcal{A}_1)$ and $(\Gamma_2, \mathcal{A}_2)$ as in Theorem 2.2.1, the cusped spaces $X(\Gamma_1, \mathcal{A}_1)$ and $X(\Gamma_2, \mathcal{A}_2)$ have homeomorphic boundaries.

This is the most important corollary for our purposes. Because the $\mathbb{R}$-trees of (Bowditch, 1998a; Papasoglu and Swenson, 2006) are constructed from the topological structure of the boundary and do not incorporate any metric properties, this allows us to use the structure of the boundary to acquire splittings which are invariant under quasi-isometries.

Corollary 2.3.2. With $(\Gamma_1, \mathcal{A}_1)$ as in Theorem 2.2.1, the tree describing the maximal peripheral splitting (Bowditch, 1998b) and the cut-point/cut-pair $\mathbb{R}$-trees (Papasoglu and Swenson, 2006) for the boundary of the cusped space are quasi-isometry invariant up to homeomorphism.

Corollary 2.3.3. Let Γ be a group hyperbolic relative to $\mathcal{A}$ with finite compatible generating sets S and T. Then $X(\Gamma, S, \mathcal{A})$ and $X(\Gamma, T, \mathcal{A})$ are quasi-isometric. The analogous result for the coned space also holds.

Proof. We only need to show that we can drop the condition on peripheral subgroups. Inspecting the proof of Theorem 2.2.1, the requirement that no peripheral subgroup is properly relatively hyperbolic is used to establish a bijection between cosets of peripheral subgroups from (Druţu, 2009). Because peripheral cosets will be mapped to themselves under the identity map, this bijection is automatic and the hypothesis is unnecessary. □

We note that this corollary is also new for hyperbolic groups with a non-trivial relatively hyperbolic structure. See Section 1.4 for conditions characterizing when this can happen.

CHAPTER 3

THE CANONICAL JSJ-TREE

3.1 $\mathcal{T}(\Gamma, \mathcal{A})$ is Simplicial for Finitely Presented Γ

As mentioned in Section 1.7, we seek to apply the Rips machine to the Γ action on the combined tree $\mathcal{T}$ so we first demonstrate that the action is stable, minimal and non-nesting. Because the cut point tree is simplicial, the only intervals which can be unstable are those which contain multiple inseparable cut pairs. Thus, we restrict our attention to those.

Lemma 3.1.1. There is a uniform bound on the order of the stabilizer of an interval containing at least two vertices corresponding to cut pairs.

Proof. Let I be any interval containing at least two inseparable cut pairs, A, B. We show that $\mathrm{Stab}(I)$ is finite. If $\{g_n\} \subset \mathrm{Stab}(I)$ is an infinite sequence of elements then we may assume that $g_n(x) \to p$ for all $x \in \partial\Gamma$, perhaps after passing to a subsequence. However, each g_n must fix all of the points of both cut pairs. This is a contradiction.

Additionally, this finite subgroup satisfies the hypotheses of Lemma 1.4.18 because it is a subgroup of $\mathrm{Stab}(A)$. There are only finitely many conjugacy classes of such subgroups so there is a uniform bound on the order of $\mathrm{Stab}(I)$. It follows immediately that the action is stable. $\square$

Corollary 3.1.2. The action of Γ on $\mathcal{T}$ is stable.

Lemma 3.1.3. The action of Γ on $\mathcal{T}$ is non-nesting.

Proof. Assume not. Then there exists an interval $I \subset T$ and $g \in \Gamma$ (replaced by g^2 if necessary) such that $g(I)$ is a proper subset of I. By the Brouwer fixed point theorem, there is a fixed point of g in I, call this A. We may assume $I = [A, B]$, $g(B) \in [A, B)$. Note that g has infinite order.

By convergence, there exists $p, q \in \partial\Gamma$ such that $g^n(x) \to p$ for every $x \neq q$. Clearly, $p \in A$. Because $g^{-n}(x) \to q$, also $q \in A$ otherwise $q \in C \in (A, B)$ but $g^{-n}(C) \neq C$. However, this implies that for all $x \neq p$, $g^{-n}(x) \to q \in A$. Yet $g^{-n}(B) \notin [A, B]$ for any n, a contradiction. $\square$

Lemma 3.1.4. This action is also minimal.

Proof. Every cut point is stabilized by a peripheral subgroup and every cut pair is stabilized by a finite or two-ended subgroup. Since the group is one-ended, cut pair stabilizers must be two-ended, showing that a 2-dense collection of vertices of the cut point tree and of every cut pair tree have non-trivial stabilizers. Thus, no proper invariant subtree exists. $\square$

Theorem 3.1.5. *Let Γ be a finitely presented, one-ended group, hyperbolic relative to $\mathcal{A}$ such that for every $A \in \mathcal{A}$, A is not properly relatively hyperbolic and A contains no infinite torsion subgroup. Let $\mathcal{T}$ be the combined tree obtained by the action of Γ on its Bowditch boundary. Then $\mathcal{T}$ is simplicial.*

Proof. Suppose not. Then because the action is non-nesting, by (Levitt, 1998), there is an $\mathbb{R}$-tree $\mathcal{T}'$ equipped with an isometric Γ-action and an equivariant quotient map $\mathcal{T} \to \mathcal{T}'$. Furthermore, stabilizers of non-simplicial segments in $\mathcal{T}'$ stabilize segments in $\mathcal{T}$, and so are finite of uniformly bounded order by Lemma 3.1.1. Therefore, as in Corollary 3.1.2, the Γ-action is stable.

In all cases of (Bestvina and Feighn, 1995, Theorem 9.5) other than the pure surface case, one obtains a splitting over a finite group. However, Γ is one-ended, so we reduce to this case. By (Bestvina and Feighn, 1995, Theorems 9.4(1) & 9.5) Γ admits a splitting over a two-ended group V, and this two-ended group corresponds to an essential, non-boundary parallel simple closed curve in the associated orbifold. If $g \in V$ corresponds to this curve, then since the associated lamination on the orbifold has no closed leaves g must act hyperbolically on $\mathcal{T}'$. This implies that g also acts hyperbolically on $\mathcal{T}$. However, a splitting of Γ over a two-ended group must induce a cut pair corresponding to the endpoints of the axis of $\langle g \rangle$. This cut pair must be stabilized by g, so g cannot act hyperbolically. This is a contradiction. $\square$

In summary, the combined tree $\mathcal{T}$ is simplicial and has one vertex for each of the following topological structures in the continuum:

1. cut points
2. inseparable cut pairs
3. necklaces
4. equivalence classes of points not separated by cut points or cut pairs

Additionally, there is an edge between two vertices if the corresponding sets in the continua have intersecting closures. We note that, by the construction of $\mathcal{R}$, points of the continuum can be contained in multiple elements of $\mathcal{R}$.

3.2 $\mathcal{T}(\Gamma, \mathcal{A})$ is a JSJ-Tree

Now that we know that $\mathcal{T}$ is simplicial in common situations, we are left with the task of classifying it as a JSJ tree. This effectively labels the splitting as the ideal splitting for understanding the algebraic content of the group.

Theorem 3.2.1. *With* $\Gamma, \mathcal{A}$ *as in Theorem 3.1.5,* $\mathcal{T}$ *is a JSJ tree over elementary subgroups relative to peripheral subgroups.*

Proof. We show that $\mathcal{T}$ satisfies the conditions of Definition 1.6.1. By the construction of the tree every edge group must be the stabilizer of either a cut point or a cut pair. Because relatively hyperbolic groups act on their boundaries with a convergence action, these stabilizers must be elementary subgroups (condition (1)). Every peripheral subgroup fixes a point in the tree because it fixes a point in the boundary (this point is e_A of Lemma 1.4.8), which implies condition (4). Furthermore, this tree satisfies (3) because every elementary splitting always has a topological expression in the boundary. In particular, (Bowditch, 2001) implies the existence of a cut point and (Papasoglu, 2005) implies the existence of a cut pair whenever there is an peripheral or hyperbolic two-ended splitting, respectively. Thus, every vertex in every such tree comes from one of these structures and hence is already a vertex stabilizer in T. Finally, every splitting of this group must reflect the topology of the boundary. In particular, every edge represents the intersection of topological features. Fixing these features means fixing a vertex in a tree for another splitting. In essence, the argument that this tree satisfies (3) goes in both directions. This is (2). $\square$

3.3 Vertex Stabilizers

The first step towards identifying the vertex stabilizers is establishing the quasi-convexity of vertex groups.

Lemma 3.3.1. Let Γ be finitely presented and one-ended. Additionally suppose that $(\Gamma, \mathcal{A})$ is relatively hyperbolic with $\mathcal{A}$ finite such that no $A \in \mathcal{A}$ is properly relatively hyperbolic and no A contains an infinite torsion subgroup. Let $\mathcal{T}$ be the combined tree from the boundary. If Γ_v is a vertex group of $\mathcal{T}$ then Γ_v is relatively quasi-convex.

Proof. This is clearly true for vertex groups which are peripheral. It is also true for hyperbolic two-ended vertex groups by Theorem 1.4.15. Assume Γ_v is not of these types.

$\mathcal{T}$ is a bipartite graph in which all vertices of one color have corresponding groups which are either peripheral or hyperbolic two-ended. To see this, note that maximal inseparable sets and necklaces must only be adjacent to cut pairs or points. Furthermore, cut pairs and points must come between these larger sets. Now, let $\{Z_1, \ldots, Z_n\}$ and $\{P_1, \ldots, P_m\}$ be the collection of all hyperbolic two-ended subgroups and the collection of all peripheral subgroups incident to Γ_v, respectively.

Let γ be any geodesic between points in Γ_v. Decompose γ into maximal segments of length ≥ 1 which have endpoints in cosets of some Z_i, P_j or are contained completely within Γ_v. The segments contained in each Z_i must stay within a bounded distance of Γ_v because hyperbolic two-ended subgroups are strongly relatively quasi-convex (Theorem 1.4.15). Additionally the P_j segments stay within bounded distance of P_j (Druţu, 2009, Theorem 4.21). Since they are peripheral and we are investigating relative quasi-convexity, we are not concerned with how far they stray from $P_j \cap \Gamma_v$ inside P_j. $\square$

Proposition 3.3.2. With $\Gamma, \mathcal{A}, \mathcal{T}$ as in Lemma 3.3.1, a vertex group Γ_v of $\mathcal{T}$, is relatively QH with finite fiber if and only if Γ_v is the stabilizer of a necklace in T.

Proof. If Γ_v is relatively QH with finite fiber, then by Definition 1.6.4 there is a short exact sequence

$$1 \to F \to \Gamma_v \to \pi_1(\mathcal{O}) \to 1$$

with F finite and $\mathcal{O}$ a hyperbolic orbifold. We first determine that $\Gamma_v \notin \mathcal{A}$. Because F is finite, $\Gamma_v \simeq_{qi} \pi_1(\mathcal{O})$ and because $\mathcal{O}$ is a hyperbolic 2-orbifold, Γ_v is itself hyperbolic relative to $\{1\}$. By our condition on peripheral subgroups, $\Gamma_v \notin \mathcal{A}$.

Now, by definition Γ_v is virtually Fuchsian. Let $\mathcal{C}$ be the set of bi-infinite curves in the universal cover $\widetilde{\mathcal{O}}$ which are not homotopic to a boundary component of $\widetilde{\mathcal{O}}$. Since F is finite, Γ_v is quasi-isometric to $\pi_1(\mathcal{O})$ and so $\partial\Gamma_v \simeq \partial\pi_1(\mathcal{O})$ with respect to the relatively hyperbolic structure induced by $\mathcal{A}$. Call this set N. Let $\mathcal{N}$ be the image of N in $\partial\Gamma$ induced by the inclusion of Γ_v in Γ. This map is well-defined because of relative quasi-convexity of Γ_v. Specifically, by (Manning and Martínez-Pedroza, 2010), relative quasi-convexity is equivalent to quasi-convexity in the cusped space and by general results on δ-hyperbolic spaces the boundary will embed.

We claim that $\mathcal{N}$ is a necklace in $\partial\Gamma$. By definition, every edge group must be either finite or contained in a boundary component. Because Γ is one-ended, the finite case is excluded. Consequently, for any $\gamma \in \mathcal{C}$ each coset of an edge group is contained in a single component of

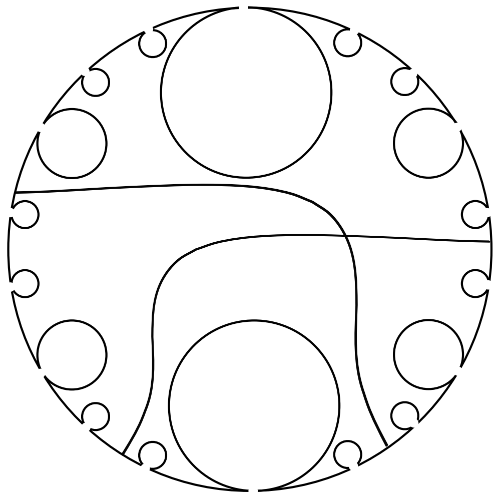

Figure 21. Interlocking geodesics separate boundary components.

$\widetilde{\mathcal{O}} \setminus \gamma$. Let $\eta \in \mathcal{C}$ be any curve which has an essential crossing with γ. Such η exists because if not then γ must be boundary parallel, but $\mathcal{C}$ contains no boundary parallel curves.

Since η^+ and η^- are in different components of $N \setminus \{\gamma^+, \gamma^-\}$ and each edge group is attached to only a single boundary component, it must be that every edge group has image contained in either the same component of $N \setminus \gamma$ as η^+ or η^-, but never both. Thus, the image of $\{\gamma^+, \gamma^-\}$ also separates the image of $\{\eta^+, \eta^-\}$ in $\mathcal{N}$ and the endpoints of γ form a cut pair in $\partial\Gamma$.

In the reverse direction, if Γ_v stabilizes a necklace $\mathcal{N}$ then, by the last paragraph of the proof of (Papasoglu and Swenson, 2006, Theorem 22), it has an action on S^1 which preserves the cyclic order. Since $\Gamma_v = \mathrm{Stab}(\mathcal{N})$, Γ_v inherits the convergence property of Γ and this property must be realized on $\mathcal{N}$. Any sequence of group elements contained in the kernel of this action is not a convergence sequence, so the kernel must be finite. The fiber, F, is the kernel of this action. By Theorem 1.5.5, the quotient must be a Fuchsian group. Let $\mathcal{O}$ be the quotient of $\mathbb{H}^2$ by the action of Γ_v/F, truncating cusps so that $\mathcal{O}$ is compact. We are left with showing that edge groups are boundary parallel.

First, we show that no element of $\mathcal{C}$ (again defined as the set of non-boundary bi-infinite curves on $\mathcal{O}$ - those which cross other bi-infinite curves and interlock in ∂N) can be contained in the image of a peripheral edge group. Because peripheral subgroups have a unique boundary point (Lemma 1.4.8), such a curve would induce a cut point in $\mathcal{N}$. However, by Lemma 1.8.6 no cut pair can be separated by a cut point. In this context, every cut pair separates another cut pair with the only exception arising from those cut pairs which are end points of boundary curves of $\widetilde{\mathcal{O}}$. In other words, for every cut pair C of $\mathcal{N}$ there is an interlocked cut pair unless C forms the endpoints in $\widetilde{\mathcal{O}}$ of a curve homotopic to a boundary component of $\mathcal{O}$.

Now we are left with only the possibility that some $\gamma \in \mathcal{C}$ is identified with a hyperbolic two-ended edge group. Let $\{x_1, x_2\}$ be any cut pair interlocked with $\{\gamma^+, \gamma^-\}$. We claim that $\{x_1, x_2\}$ is not actually a cut pair in $\partial\Gamma$. $\partial\Gamma \setminus \{\gamma^+, \gamma^-\}$ must have at least 3 components in this situation. Let Y be a component which does not contain any points of $\mathcal{N}$. In particular, $Y \cap \mathcal{N} = \{\gamma^+, \gamma^-\}$. Thus, no cut pair of $\mathcal{N}$ separates γ^+ from γ^- as both are contained in the component Y. However, then $\overline{Y}$ must separate $\{x_1, x_2\}$, contrary to Lemme 1.8.6. Thus, we must have that no such γ exists in $\mathcal{C}$.

$\square$

TABLE II

Subgroups which stabilize particular topological features in the boundary.

Stabilizer		Topological Feature
hyperbolic 2-ended	$\longleftrightarrow$	cut-pair
peripheral	$\longleftrightarrow$	cut-point
relatively QH with finite fiber	$\longleftrightarrow$	necklace

Lemma 3.3.3. If $\{x, y\}$ is an inseparable cut pair in $\partial\Gamma$ then $\mathrm{Stab}(\{x, y\})$ is a hyperbolic two-ended subgroup of Γ.

Proof. Let $Z = \mathrm{Stab}(\{x, y\})$. Z must be infinite else Γ would not be one-ended. As a subgroup of Γ, Z acts on $\partial\Gamma$ with the convergence property. Since Z fixes $\{x, y\}$, this implies that Z is two-ended. □

Corollary 3.3.4. With $\Gamma, \mathcal{A}, \mathcal{T}$ as in Lemma 3.3.1, there is a correspondence between vertex groups of $\mathcal{T}$ tree and the topological features of the boundary given by Table 2.

Proof. Since hyperbolic 2-ended subgroups are strongly relatively quasi-convex (Osin, 2006), their boundaries embed (Hruska, 2010). Similarly, (Bowditch, 2001) demonstrates that cut points correspond to peripheral splittings. As vertex groups, these features must be separating. The last point is Proposition 3.3.2. □

With this in place, we are ready to show:

Theorem 3.3.5. *Let Γ_1 and Γ_2 be finitely generated groups. Suppose additionally that Γ_1 is one-ended and hyperbolic relative to the finite collection $\mathcal{A}_1$ of subgroups such that no $A \in \mathcal{A}$*

is properly relatively hyperbolic or contains an infinite torsion subgroup. Let $\mathcal{T}$ be the cut-point/cut-pair tree of $\partial(\Gamma_1, \mathcal{A}_1)$. If $f : \Gamma_1 \to \Gamma_2$ is a quasi-isometry then

- *T is the cut-point/cut-pair tree for Γ_2 with respect to the peripheral structure induced by Theorem 2.2.1,*
- *if $Stab_{\Gamma_1}(v)$ is one of the following types then $Stab_{\Gamma_2}(v)$ is of the same type,*
    1. *hyperbolic 2-ended,*
    2. *peripheral,*
    3. *relatively QH with finite fiber.*

Proof. By Corollary 2.3.1, there exists a relatively hyperbolic structure for Γ_2 such that the boundaries of the cusped spaces are homeomorphic. Since T depends only on the topology of this continuum, $\mathcal{T}$ is the cut-point/cut-pair tree for Γ_2.

By the correspondence given in Corollary 3.3.4, these vertex types depend only on the topology of the boundary. Since these topological features are preserved, the vertex group types are preserved as well.

□

We conclude with a consequence of the fact that $\mathrm{Out}(\Gamma)$ acts on $\partial\Gamma$ by homeomorphisms.

Corollary 3.3.6. Let $(\Gamma, \mathcal{A})$ be relatively hyperbolic with no $A \in \mathcal{A}$ properly relatively hyperbolic. Then, the $\mathrm{Out}(\Gamma)$ action on the JSJ-deformation space over elementary subgroups relative to peripheral subgroups fixes T.

Proof. The action passes to an action on the boundary by homeomorphisms which induces an action on the combined tree so that vertex groups map to vertex groups and adjacencies are preserved. Furthermore, because maximal relatively hyperbolic structures are unique (Matsuda et al., 2012), there is no change in the choice of peripheral structure. □

CHAPTER 4

THE ACTION OF $\mathcal{QI}(\Gamma)$ ON $\mathcal{T}(\Gamma)$

As demonstrated in Theorem 2.2.1, the action of $\mathcal{QI}(\Gamma)$ on the Cayley graph of Γ induces an action on the cusped space $X(\Gamma, \mathcal{A})$ and by Proposition 1.3.2 on the boundary $\partial(\Gamma)$. As stated in Corollary 2.3.2, this naturally yields an action on the tree $\mathcal{T}$. In this chapter we describe this action.

4.1 Faithfulness

Theorem 4.1.1. *With $\Gamma, \mathcal{A}, \mathcal{T}$ as in Lemma 3.3.1, the action of $\mathcal{QI}(\Gamma)$ on $\mathcal{T}(\Gamma, \mathcal{A})$ is faithful, assuming that $\mathcal{T}$ is not a point.*

Proof. Given a (k, c)-quasi-isometry $\phi \in \mathcal{QI}(\Gamma)$ which has a trivial induced action on $\mathcal{T}(\Gamma)$, it must be that ϕ coarsely fixes all peripheral subgroups over which G splits. For now, assume one such subgroup exists and call this A. By the same citation of (*Druţu*, 2009) applied to Theorem 2.2.1, ϕ must map every coset gA within $N_T(gA)$. However, given two adjacent cosets, hA and shA, both must satisfy this condition. Consider adjacent points $a \in hA$, $b \in shA$. It must be that $\phi(a) \in N_T(hA)$ and $\phi(b) \in N_T(shA)$. Further, as $d(a, b) = 1$, $d(\phi(a), \phi(b)) \leq k+c$. Consequently, $\phi(a) \in N_{k+c+T}(shA)$ and $\phi(b) \in N_{k+c+T}(hA)$ and $\phi(a), \phi(b) \in N_{k+c+T}(hA) \cap N_{k+c+T}(shA)$. This intersection must have a finite diameter which is uniformly bounded over any chosen pair of cosets. Therefore, $d(a, \phi(a))$ is bounded by this diameter. Since every point can be found within such a neighborhood by an appropriate choice of such cosets, ϕ must be a finite distance from the identity map.

The two-ended case is similar. These subgroups have been shown to satisfy a similar divergence property called *hyperbolically embedded* (Dahmani et al., 2012; Sisto, 2012a) and they are strongly relatively quasi-convex, so the argument is nearly identical.

$\square$

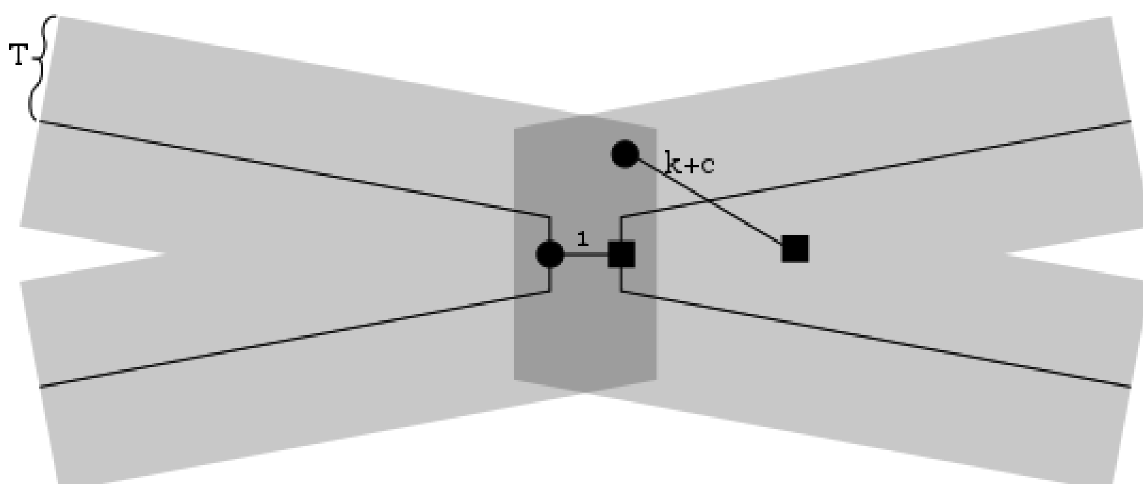

Figure 22. $\mathcal{QI}(\Gamma)$ acts faithfully on the JSJ tree.

4.2 Bounding the Number of Edges of $\mathcal{T}/\mathcal{QI}(\Gamma)$

We have that Γ and $\mathcal{QI}(\Gamma)$ act on the same tree so that $\mathcal{QI}$ splits whenever Γ has a nontrivial JSJ as described here. We can also control the number of edges which $\mathcal{QI}$ admits in this induced splitting. Let Λ be $|E(\mathcal{T}/\Gamma)|$ and let $\mathrm{Aut}_{qi}(\mathcal{T}/\Gamma)$ be the group of graph automorphisms which respect the quasi-isometry type of each edge and vertex stabilizer.

Theorem 4.2.1. *The graph of groups decomposition of $\mathcal{QI}(\Gamma)$ induced by the JSJ-decomposition of Γ has at most Λ edges and at least $\Lambda/|\mathrm{Aut}_{qi}(\mathcal{T}/\Gamma)|$ edges.*

Proof. First, note that Γ acts on itself by quasi-isometries and if $\mathcal{T}$ is not a point then this action will have no global fixed points so obviously the same is true for the $\mathcal{QI}(\Gamma)$ action.

Now, each $g \in \Gamma$ acts by (quasi)-isometries on $X(\Gamma, \mathcal{A})$ which gives the upper bound, and $\mathcal{QI}(\Gamma)$ acts on $\mathcal{T}/\Gamma$ by graph automorphisms which must preserve the quasi-isometry class of each stabilizer because the action is filtered through the action on $X(\Gamma, \mathcal{A})$, thus giving the lower bound. $\square$

CHAPTER 5

CONCLUSION

The theorems proved in this dissertation provide several significant methods for understanding relatively hyperbolic groups and their quasi-isometries. We have seen that the canonical splitting observed in the boundary of a relatively hyperbolic group is a JSJ-splitting which exposes much of the algebraic structure of these groups. Furthermore, this allows us to understand the quasi-isometries of these groups and splittings of the group of quasi-isometries since they act faithfully on the same tree.

Nonetheless, this leaves open some compelling questions. Exactly how much more structure can we deduce about $\mathcal{QI}(\Gamma)$ by understanding the action of Γ on $\partial\Gamma$? We have only used topological features of $\partial\Gamma$ here, but there is a rich theory of the metric properties of boundaries of hyperbolic spaces. Perhaps a great deal more can be said about the structure of $\mathcal{QI}(\Gamma)$. In fact, if we replace $\mathrm{Aut}_{qi}(\mathcal{T}/\Gamma)$ in Theorem 4.2.1 by the subgroup which also respects the edge group inclusion maps, it may be possible to obtain a complete description of the splitting of $\mathcal{QI}(\Gamma)$.

Moreover, there is a definition of the boundary which does not require that the hyperbolic space be proper. It may be reasonable to expect that the same analysis carried out here could be redone with some modifications to understand group splittings in this scenario. In fact, there is a broad class of groups for which this would be the very natural question to ask: groups containing hyperbolically embedded subgroups (Dahmani et al., 2012; Sisto, 2012b).

Even further, Sisto (Sisto, 2012a) has provided a definition for these groups which effectively mirrors the Groves-Manning definition of relatively hyperbolic groups via the cusped space. Do groups in this class admit a similar canonical splitting?

Perhaps more directly, we suspect that the theorems contained herein can be applied to subcategories of groups which fall within the umbra of relative hyperbolicity. A specific example could be limit groups, which are studied very closely by means of their JSJ-decompositions. Perhaps the study of these groups can be expanded to the entirety of the quasi-isometry class of limit groups, since these may share many properties with limit groups via our theorems. Perhaps the class of limit groups is even ‘coarsely closed’ under quasi-isometries. This is of course supposing the existence of an author willing to properly define such a notion.

Answers to any or all of these questions, as unlikely as some of them may be, would certainly help advance our understanding of groups and their geometric properties.

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  Bass-Serre Theory (2011)

  Bieri-Stallings Groups (2011)

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