Model Theory and Differential Algebraic Geometry

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THESIS

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To Kate and Sophie.

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The original motivation for chapter 8 was to prove a differential algebraic analogue of Bertini's theorem, an idea inspired by Dave Marker's exposition of a model theoretic proof of a portion of the classical theorem as presented by Poizat (68). At this point, I already knew that such a theorem could only be true for differential algebraic varieties of positive differential transcendence degree (from several geometric theorems proved in (70) and (20)). Later, after I

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had calculated the dimensions of intersections, Phyllis Cassidy directed me to (90), where parts of the ordinary case were handled (and some of the results of (70) or (20) were reproved in a different language). Thanks to Rahim Moosa and Omar Leön-Sanchez for discussions regarding the applications of results in chapter 8. Some of these were obtained during a visit to Waterloo, which was made possible thanks to an American Mathematical Society Mathematical Research Communities grant.

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SUMMARY

In this thesis we study problems in differential algebraic geometry and model theory. Of course, these fields enjoy a rich connection. The connection began with Robinson who introduced differentially closed fields and established their basic model-theoretic properties (74). Robinson's axiomatization was refined by Blum (7); in the ensuing years connections developed rapidly. Stability-theoretic tools saw useful applications in differential fields, and differential fields became a sort of testing ground for theorems about (ω-)stable theories (76). As both subjects developed, the connections became deeper. We will not attempt to tell this whole story (45; 11, for more details). We hope this work will contribute to those connections by applying techniques of stability theory in differential algebraic geometry and by generalizing results from differential algebraic geometry to purely model-theoretic settings. We will begin with brief introductory chapters on model theory and differential algebraic geometry.

In chapter 3 we develop a tool for studying differential algebraic groups which we call indecomposability. The notion is inspired by similar model theoretic results developed by Zilber in the finite Morley rank situation and Berline and Lascar in the general superstable setting. The general blueprint for indecomposability results is the following: if a subgroup is generated by a family (perhaps infinite) of definable subsets which satisfy certain connectivity requirements with respect to the group, then the subgroup is itself definable. We apply our technique to prove an indecomposability theorem in the setting of partial differential algebraic groups. We use the main theorem to solve a problem of Cassidy and Singer. In chapter 4, we continue to use indecomposability, adopting the general philosophy that, as groups, strongly connected differential algebraic groups of small typical differential dimension should behave like finite Morley rank groups of small Morley rank (though they are of infinite Morley rank!). Taking ideas of Cherlin and Zilber as inspiration, we prove classification results when the typical differential dimension is 3 or less. In particular, we prove that any nonsolvable strongly connected differential algebraic group of typical differential dimension three or less is isomorphic to the F-rational points of an algebraic group, where F is a definable subfield.

In chapter 5, we consider the Cassidy-Singer theorem for differential algebraic groups and prove a similar result in the superstable setting. Whether our result is a generalization of the Cassidy-Singer theorem or simply a result with similar flavor depends on open problems in differential algebraic geometry. The main contribution is to give the definition of isogeny for superstable groups; after proving some basic results about the notion, we prove a Jordan-Hölder-style decomposition theorem for superstable groups. Then we establish that this decomposition is unique up to isogeny, but not up to isomorphism.

In chapter 6, we consider the completeness problem for differential algebraic varieties. Completeness is a fundamental notion for algebraic varieties, and has been considered in the setting of ordinary differential algebraic geometry and the closely related category of algebraic D-varieties. We consider the notion for partial differential algebraic varieties, generalizing results of Pong concerning differential completeness in the ordinary case. In particular, we prove a valuative criterion for differential completeness and use this result to give several new examples of complete differential algebraic varieties.

In chapter 7, we consider an idea which we call relative Kolchin irreducibility. The Kolchin irreducibility theorem says that an irreducible algebraic variety is still irreducible as a differential algebraic variety. We prove a result of a similar spirit: if V is an irreducible differential algebraic variety in the Δ-topology, then V remains irreducible in the $\Delta \cup \{\delta\}$-topology. We give two proofs of the theorem. First, we apply Gillet's scheme-theoretic approach. Second, we use the method of characteristic sets of prime differential ideals, a classical technique of differential algebra. The classical technique (which may also be employed to prove Kolchin's theorem) also gives the Kolchin polynomial of V as a $\Delta \cup \{\delta\}$-variety.

In chapter 8, we develop a generic intersection theory for differential algebraic geometry and use it to prove Bertini-style theorems in the differential setting. We also give numerous applications of the intersection theory by proving results about the definability of rank and irreducibility in families of differential algebraic varieties. For instance, the intersection theory we develop allows us to generalize an observation of Hrushovski and Itai that the property of being generically irreducible is definable in families of uniformly defined differential algebraic varieties. There are forthcoming applications (joint work with William Simmons and Omar León-Sanchez) of the main ideas of chapter 8 to the completeness problem in differential algebraic geometry, which are not discussed in this thesis.

In chapter 9, we discuss model-theoretic and differential algebraic notions of genericity for points on differential algebraic varieties. The main thrust of the chapter is to give a specific differential algebraic variety and prove that it has a rather anomalous structure of subvarieties which enforce that the model-theoretically and differential algebraically generic points are actually disjoint. This example answers a question posed by Frank Benoist.

CHAPTER 1

A BRIEF INTRODUCTION TO DIFFERENTIAL ALGEBRAIC GEOMETRY

1.1 Differential rings

A differential field is a field K together with m commuting derivations, $\Delta = \{\delta_1, \ldots, \delta_m\}$, that is

$$\delta_i : K \to K$$

such that δ_i is an additive homomorphism and obeys a multiplicative Leibniz rule, that is for all $a, b \in K$,

$$\delta_i(ab) = \delta_i(a)b + a\delta_i(b)$$

A system of algebraic partial differential equations over K is a system of equations (say n_1 equations in n variables) of the form

$$\{f_k(\{\delta_1^{\alpha_1} \ldots \delta_m^{\alpha_m} x_j\}) = 0\}_{k \in n_1, 1 \leq j \leq n, \, \alpha_i \in \mathbb{N}},$$

where $f \in K[\{x_{j,\alpha_1,\ldots,\alpha_n}\}_{1 \leq j \leq n, \, \alpha_i \in \mathbb{N}}]$ with the correct correspondence between the unknowns in the polynomial equation and the derivatives of the variables. A solution to the system is a tuple $a \in L^n$ so that $f_k(a) = 0$ where L is a differential field extension of K. We denote by $k\{x\}$, the ring of differential polynomials over k. Throughout the paper, lowercase letters often denote tuples, unless specifically noted to be singletons. Sometimes when we want to emphasize that we may be talking about tuples we will write $\bar{x}$ or $\bar{a}$. We use the notation $k\langle x\rangle$ for the fraction field of $k\{x\}$.

1.2 Differential algebraic varieties

An *affine differential algebraic variety* (over K) is the zero set of a system of differential polynomial equations (in $K\{x\}$); so as in the algebraic case, a differential algebraic variety V is a functor from differential fields (over K) to sets. A set is constructible in the Kolchin topology if it is a boolean combination of differential algebraic varieties. The closed sets in the *Kolchin topology* (over K) on $\mathbb{A}^n$ are precisely the zero sets of systems of differential polynomial equations in n variables (over K). By the basis theorem of Ritt and Raudenbush, it is enough to consider only finitely many such equations:

Theorem 1.2.1. *Suppose that K is differentially closed.*

1. *If $X \subset \mathbb{A}^n(K)$, then*

$$V(I(X/K)) = \{f \in K\{x\} \mid f(a) = 0 \textit{ for all } a \in X\}$$

   is the Kolchin-closure of X.

2. *If $S \subset K\{x\}$, then $I(V(S))$ is the smallest radical ideal containing $[S]$, denoted $\{S\}$.*

3. *$\{S\}$ is finitely generated, as a differential ideal.*

More general differential algebraic varieties will occasionally be considered, for instance, in later chapters, we will consider projective differential algebraic varieties (closed sets in the

Kolchin topology on $\mathbb{P}^n$). Additionally, (35) considers an abstract formulation of "pre differential algebraic sets." There is also a (still emerging) differential scheme theory (see (38) (39), for instance). In this thesis, we will not consider Kolchin's abstract formulation nor differential schemes. On occasion, we will need to use classical scheme theory (for instance, in considering arc spaces).

A differential algebraic group is a differential algebraic variety V together with a differential rational map

$$\phi : V \times V \to V$$

which is a group operation. The theory of differential algebraic groups was developed in (13) in the affine case and (35) in an abstract formulation. See the discussion in 2.4 for more information.

Quantifier elimination gives a bijective correspondence between varieties, types, finitely generated differential field extensions, and radical differential ideals (see the next chapter if unfamiliar with quantifier elimination or types). Given a type $p \in S(K)$ and a realization of the type, $a \models p$ in some differential field extension L, we have the following correspondence:

$$p \in S(K) \leftrightarrow a \in L,\, a \models p \leftrightarrow I(a/K) = I_p = \{f \,|\, \text{“}f = 0\text{”} \in p\} \leftrightarrow V(I_p) \tag{1.1}$$

$$\text{types}/K \leftrightarrow \text{tuples}/K \leftrightarrow \text{prime } \Delta\text{-ideals} \leftrightarrow \Delta\text{-K-varieties}$$

We will use this correspondence implicitly throughout, including in the notation of *Kolchin polynomials*, which we will define next.

1.3 Dimension polynomials

Let Θ be the free commutative monoid generated by Δ. For $\theta \in \Theta$, if $\theta = \delta_1^{\alpha_1} \ldots \delta_m^{\alpha_m}$, then $ord(\theta) = \alpha_1 + \ldots + \ldots + \alpha_m$. The order gives a grading on the monoid Θ. We let

$$\Theta(s) = \{\theta \in \Theta : ord(\theta) \leq s\}$$

Theorem 1.3.1. *(Theorem 6, page 115, (34)) Let $\eta = (\eta_1, \ldots, \eta_n)$ be a finite family of elements in some extension of k. There is a numerical polynomial $\omega_{\eta/k}(s)$ with the following properties.*

1. *For sufficiently large $s \in \mathbb{N}$, the transcendence degree of $k((\theta\eta_j)_{\theta\in\Theta(s),\, 1\leq j\leq n})$ over k is equal to $\omega_{\eta/k}(s)$.*
2. *$deg(\omega_{\eta/k}(s)) \leq m$*
3. *One can write*

$$\omega_{\eta/k}(s) = \sum_{o\leq i\leq m} a_i \binom{s+i}{i}$$

In this case, a_m is the differential transcendence degree of $k\langle\eta\rangle$ over k.

4. *If $\mathfrak{p}$ is the defining differential ideal of the locus of η in $k\{y_1, \ldots, y_n\}$ and Λ is a characteristic set of $\mathfrak{p}$ with respect to an orderly ranking of $(y_1, \ldots, y_n)$, and if for each y_j we let*

E_j denote the set of all points $(e_1, \ldots, e_m) \in \mathbb{N}^m$ such that $\delta_1^{e_1} \ldots \delta_m^{e_m} y_j$ is a leader of an element of Λ, then

$$\omega_{\eta/k}(s) = \sum_{1 \le j \le n} \omega_{E_j} - b$$

where $b \in \mathbb{N}$.

Definition 1.3.2. The polynomial from the theorem is called the *Kolchin polynomial* or the *differential dimension polynomial.* With respect to item 3), the a_i are uniquely determined, but only the largest nonzero a_i is a differential birational invariant. We will always implicitely assume our Kolchin polynomials to be written in the form of item 3).

We remind the reader that for the rest of the thesis, we will be implicitly using the above correspondence Equation 1.1. The Kolchin polynomial of a differential variety is a birational invariant (since it depends only on transcendence degree calculations), but it is not a Δ-birational invariant. The leading coefficient and the degree of the Kolchin polynomial are Δ-birational invariants. We call the degree the *differential type* or Δ-type of V. We will use the notation $\tau(V)$ for the the differential type. When we wish to emphasize that the differential type is being computed over a certain field, k, then we write $\tau(V/k)$. Noting the above correspondence between tuples in field extensions (realizations of types) and varieties, we will occasionally write $\tau(p)$ or $\tau(a)$. Again, when we wish to emphasize the base set, we will write $\tau(a/k)$. Since types come over a specified set, this is never necessary for types.

The leading coefficient is called the *typical differential dimension* or the typical Δ-dimension. We will write $a_\tau(V)$ for the typical differential dimension of V. We will also write $a_\tau(a)$ and

$a_\tau(a)$ for a tuple of elements a in a field extension. Similarly, we write $a_\tau(p)$ and $a_\tau(p)$ for a complete type p. Similarly, we write $a_\tau(a/k)$ and $a_\tau(V/k)$ when we wish to emphasize that the calculation is being done over k. For further results on the significance of Kolchin polynomials, see (34) and (35). We write $\tau(a/k \cup \{b\}) = \tau(a/k\langle b\rangle)$ is the differential type of a over the differential field generated by b over k.

When working in a differential field with m derivations, when a tuple b/k has type m, then we will call the coefficient of t^m in $\omega_{b/k}(t)$ the Δ-*transcendence degree of* b *over* k or the Δ-*dimension of* b *over* k. In this case, we will occasionally write $dim(a/k)$. The analogous notation applies to types and varieties.

The following is elementary to prove, see (53).

Lemma 1.3.3. *For* a, b *in a field extension of* K.

$$\tau(a,b) = max\{\tau(a), \tau(tp(a/b))\}$$

If $\tau(a) = \tau(tp(b/a))$, *then*

$$a_\tau((a,b)) = a_\tau(a) + a_\tau(b/\{a\} \cup K)$$

If $\tau(a) > \tau(tp(b/a))$, *then*

$$a_\tau((a,b)) = a_\tau(a)$$

For additional results on the properties of differential type and typical differential dimension, see (16) and (34).

1.4 Characteristic sets

Let f be a Δ-polynomial in $K\{y_1, \ldots, y_n\}$. The order of f with respect to y_i, $ord(f, y_i)$, is defined to be the largest s such that for some $\theta \in \Theta(s)$, θy_i appears in f. The *order* of f is defined to be $max\{ord(f, y_i) \,|\, i = 1, \ldots, n\}$. A *ranking* is a total order (denoted $<_R$) on the set of differential operators applied to the variables (we occasionally call this the set of derivatives and denote it $\Theta(\bar{y}) = \Theta(y_1, \ldots, y_n) := \{\delta_1^{i_1}\delta_2^{i_2}\ldots\delta_n^{i_n} y_j \,|i_k \in \mathbb{N},\, j = 1, \ldots, n\})$ which satisfies two conditions (for all $u, v \in \Theta(y_1, \ldots, y_n)$ and all $\theta \in \Theta$):

1. $u \leq_R \theta u$
2. $u \leq_R v$ implies $\theta u \leq_R \theta v$.

A ranking is *integrated* if for each pair of derivatives, $\theta_1 y_i$ and $\theta_2 y_j$, there is $\theta \in \Theta$ such that $\theta\theta_1 y_i$ has higher ranking than $\theta_2 y_j$. A ranking is *sequential* if it is of order type ω. Every sequential ranking is integrated. An *orderly* ranking is one in which $ord(\theta_1 y_i) <_R ord(\theta_2 y_j)$ implies that $\theta_1 y_i <_R \theta_2 y_j$. Orderly rankings are sequential.

Elimination rankings are those such that $y_i >_R y_j$ implies that $\theta_1 y_i >_R \theta_2 y_j$. Of course, elimination rankings are not integrated. The *canonical orderly ranking* is simply ordering $\Theta(y_1, \ldots, y_n) = \{\delta_1^{i_1}\ldots\delta_m^{i_m} y_j \,|\, i_k \in \mathbb{N},\, j = 1, \ldots, n\}$ lexicographically with respect to $(\sum i_k, j, i_1, \ldots, i_m)$.

Now take an arbitrary Δ-polynomial $p \in K\{y_1, \ldots, y_n\}$ *and* fix a ranking of $\Theta(\bar{y})$. The largest ranking member of $\Theta(\bar{y})$ which appears in f is called the *leader* of f. We will use u_f to denote the leader of f. The differential polynomial has a certain degree, d, in u_f, and we can write

$$f = \sum_{i=0}^{d} I_i (u_f)^i$$

where we assume $I_d \neq 0$. We call the differential polynomial I_d the initial of f, and to avoid confusion when multiple differential polynomials appear, we denote it I_f. We call d the *rank* of f, denoted $rk(f)$. The formal derivative of f with respect to u_f is called the separant. It will be useful to extend our ranking to all of $K\{\bar{y}\}$ (but it will only be a pre-order). We do so by letting every element of K be lower than all elements of $K\{\bar{y}\}\backslash K$. If two differential polynomials f and g have the same leaders, then f is below g if g has higher degree in $u_g = u_f$. If f and g have different leaders, $f <_R g$ if and only if $u_f <_R u_g$.

We say that f is *partially reduced* with respect to g if no proper derivatives of u_g appear in f. We say f is *reduced* with respect to g if it is partially reduced with respect to g and $deg(f, u_g) < rk(g)$. A set, A, of differential polynomials is called *auto-reduced* if each element is reduced with respect to any other element of the set.

Let A be an auto-reduced set. Let H_A be the set of all initials and separants of A. H_A^∞ is the smallest multiplicative set containing A. The saturation ideal of A is:

$$sat(A) = [A] : H_A^\infty = \{f \in K\{\bar{y}\} \,|\, \exists h \in H_A^\infty \text{ so that } hf \in [A]\}.$$

The next theorem shows that autoreduced sets can be used to give what we call a reduction theory or a differential division algorithm.

Theorem 1.4.1. *Let Λ be an autoreduced set. Then for any $f \in R\{y\}$, there is $g \in R\{y\}$ such that g is a reduced with respect to Λ and $\Pi_{A\in\Lambda} I_A^{i_A} S_A^{j_A} f = g \, mod \, \Lambda$. Even more, $\Pi_{A\in\Lambda} I_A^{i_A} S_A^{j_A} f - g$ can be written as a linear combination of derivatives θA with $A \in \Lambda$ such that θu_A is lower than the leader of f.*

One can extend the pre-order on $R\{y\}$ to all finite subsets of $R\{y\}$. Given any finite subsetΛ, fix a nondecreasing ordering of the subset with respect to some ranking. Suppose $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ and $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ are nondecreasing with respect to $\leq_R$. Then $\Lambda <_R \Gamma$ if we have either:

- There is some $j \leq m$ such that for $i < j$, λ_i and γ_i have the same rank and $\lambda_j <_R \gamma_j$.
- $m > n$ and for $i < n$, λ_i has the same rank of γ_i.

If neither of the conditions holds, we say Λ and Γ have the same rank with respect to $<_R$. In *any collection* of autoreduced sets, there is a set of least rank. So, given a Δ-ideal, $\mathfrak{p}$, there is an autreduced subset of Λ of lowest rank such that $S_\Lambda = \Pi_{f\in\Lambda} S_f$ is not in $\mathfrak{p}$. Such a set is called a *characteristic set.*

Theorem 1.4.2. *Let K be a Δ-field and Λ a finite subset of $K\{\bar{y}\}$. Then the following are equivalent:*

1. *Λ is a characteristic set of a prime ideal $\mathfrak{p}$ of $K\{\bar{y}\}$*
2. *Λ is autoreduced and coherent, and $(\Lambda) : H_\Lambda^\infty$ is a prime ideal containing no nonzero element which is reduced with respect to Λ.*

Remark 1.4.3. As various authors must have remarked, the utility of the previous theorem lies (in large part) in that one needs only to consider the saturation of the *ideal* generated by Λ (in condition 2), rather than the *differential ideal*.

CHAPTER 2

A BRIEF INTRODUCTION TO MODEL THEORY

In this section we will attempt to give the reader not familiar with modern model theory some reminders as well as a roadmap of the technical developments which are related to the material of this thesis. This short chapter should not be regarded as a suitable introduction to the field as a whole, or even to the model theory of differential fields. The chapter is unlikely to be of interest to those familiar with model theory since many of the results we discuss are standard; however, we hope that, for non-model-theorists interested in the results of this thesis, this chapter will be of some use. We suggest the following references for the model theory used in this thesis (44, basic model theory) (45, model theory of ordinary differential fields) (48, model theory of partial differential fields) (61, stability theory).

2.1 Languages and formulas

One of the important aspects of model theory is working in a fixed formal language consisting of a set of function, relation, and constant symbols. The pertinent language for studying differential fields consists of the language of rings, $\{0, 1, +, \cdot\}$ along with symbols for the distinguished derivations, $\{\delta_1, \ldots, \delta_m\}$. With the exception of chapter 5, which takes place in a more general setting, we will work exclusively with either the language of rings or the language of differential rings.

In model theory, there is a distinction between the language (functions, relations, and constants, along with with logical symbols such as quantifiers and variables) and the interpretations of the language in structures (actual differential fields, rational points of differential algebraic varieties in those fields, etc.). This distinction is somewhat unfamiliar in the rest of mathematics (see for instance how differential indeterminants are handled in (90), floating back and forth between being variables and generators of field extensions), but this is an essential distinction in model theory.

Let $\mathcal{L}$ be a language. A *term* in $\mathcal{L}$ is a formal expression built out of finitely many formal variables, $\bar{x} = (x_1, \dots, x_n)$ and the symbols of the language. The set of $\mathcal{L}$-terms can be defined to be the smallest set which contains the following:

1. Each constant is a term.
2. Each variable is a term.
3. When $t_1, \dots, t_n$ are terms, and $f \in \mathcal{L}$ is an n-ary function symbol then $f(t_1, \dots, t_n)$ is a term.

Terms can be thought of as the basic expressions in a given language. In the language of differential rings, terms are simply names for differential polynomial functions in finitely many variables. For a term t, and $a \in \mathcal{M}$, and $\mathcal{L}$-structure, we denote by $t^{\mathcal{M}}(a)$ the interpretation of $t(a)$ in $\mathcal{M}$. When a term contains variables $v_1, \dots, v_n$, then it specifies a function $\mathcal{M}^n \to \mathcal{M}$ via $(a_1, \dots, a_n) = \bar{a} \mapsto t^{\mathcal{M}}(\bar{a})$. In the model theory of differential fields, this will necessarily be a differential polynomial function.

In general, *atomic* formulas in $\mathcal{L}$ are expressions of the form $R(t_1, \ldots, t_n)$ or $t_1 = t_2$ where t_i are terms and R is an n-ary relation in $\mathcal{L}$. In the model theory of differential fields, the atomic formulas are differential polynomial equations (recall there are no relation symbols in the language of differential rings).

A general formula is a combination of atomic formulas under logical operations. The set of formulas in a language $\mathcal{L}$ is the smallest set which contains

1. atomic formulas
2. boolean combinations of atomic formulas
3. $\exists x \phi$ and $\forall x \phi$ for any formula ϕ

So, formulas in the language of differential rings are boolean combinations of differential polynomial equations and inequations, along with quantified differential polynomial equations and inequations. The variables which appear in a formula, but are not quantified over are known as *free variables*. We will explain (in the coming sections) why one can often dispense with the quantifiers in the setting of differential fields.

2.2 Definable sets, groups, theories and types

Definition 2.2.1. Let $\phi(\bar{v})$ be an $\mathcal{L}$-formula and $\mathcal{M}$ an $\mathcal{L}$-structure with $\bar{a} \in \mathcal{M}$. We will define the satisfaction relation, $\mathcal{M} \models \phi(\bar{a})$ by induction on the complexity of the formula ϕ:

1. If ϕ is given by $t_1 = t_2$ for some terms t_1 and t_2, then $\mathcal{M} \models \phi(\bar{a})$ if $t_1^{\mathcal{M}}(a_1) = t_2^{\mathcal{M}}(a_2)$.
2. If ϕ is given by $R(t_1, \ldots, t_n)$, then $\mathcal{M} \models \phi(\bar{a})$ if $(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_n^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}}$.
3. If ϕ is given by $\neg\psi$, then $\mathcal{M} \models \phi(\bar{a})$ if and only if $\mathcal{M} \not\models \psi(\bar{a})$.

4. If ϕ is given by $\psi \wedge \theta$, then $\mathcal{M} \models \phi$ if and only if $\mathcal{M} \models \psi$ and $\mathcal{M} \models \theta$.

5. If ϕ is given by $\psi \vee \theta$, then $\mathcal{M} \models \phi$ if and only if $\mathcal{M} \models \psi$ or $\mathcal{M} \models \theta$.

6. If ϕ is given by $\exists w \psi(\bar{v}, w)$, then $\mathcal{M} \models \phi(\bar{a})$ if there is $b \in \mathcal{M}$ such that $\mathcal{M} \models \psi(\bar{a}, b)$

7. If ϕ is given by $\exists w \psi(\bar{v}, w)$, then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \models \psi(a\bar{a}, b)$ for all $b \in \mathcal{M}$.

A definable set is the solution set of a formula $\phi(x_1, \ldots, x_n)$ in an $\mathcal{L}$-structure $\mathcal{M}$. That is

$$\{\bar{a} \in \mathcal{M} \,|\, \mathcal{M} \models \phi(\bar{a})\}$$

Usually, we consider sets which are definable only once we use a set of additional parameters $A \subset \mathcal{M}$. One can set up the formalism in the following manner. Add to the language a constant for each $a \in A$ such that the interpretation of this constant in $\mathcal{M}$ is precisely the original element. This language is often denoted $\mathcal{L}_A$. We call $\mathcal{L}_A$-definable sets A-definable or definable over A or definable with parameters from A. One aspect of this construction which is pertinent to differential algebraic geometry is that for questions about a single differential algebraic variety, from the model-theoretic perspective it often does not matter if we augment the language and assume that the variety is definable over the empty set (see for instance, chapter 6).

An $\mathcal{L}$-*theory* T is a collection of $\mathcal{L}$-sentences. Theories are assumed to be *satisfiable*; that is, there exists an $\mathcal{L}$-structure $\mathcal{M} \models T$. A theory T is *complete* if for every sentence ϕ, either ϕ or its negation is satisfied by every $\mathcal{M} \models T$.

In particular, if $\mathcal{M}$ is an $\mathcal{L}$-structure, then the set of $\mathcal{L}$-sentences which are satisfied by $\mathcal{M}$ is a complete $\mathcal{L}$-theory, which we will denote by $Th(\mathcal{M})$. Generally, one does not consider all

$\mathcal{L}$-structures, but rather considers only those structures which satisfy a certain complete theory T.

There are several pertinent complete theories in this thesis:

- ACF_0. The theory of algebraically closed fields of characteristic zero in the language of rings, $\mathcal{L} = \{0, 1, +, \cdot\}$.
- $DCF_{m,0}$. The theory of differentially closed fields of characteristic zero with m commuting derivations in the language of differential rings, $\mathcal{L} = \{0, 1, +, \cdot, \delta_1, \ldots, \delta_m\}$.

Axiomatizing the latter in a first order manner is somewhat tricky in the case where $m > 1$ (48).

A group is *definable* in the theory T if there are formulas $g(\bar{x})$, $m(\bar{x}, \bar{y}, \bar{z})$ such that in any $\mathcal{M} \models T$, $\mu(\bar{x}, \bar{y}, \bar{z})$ is the graph of a group operation on $g(\mathcal{M})$. In particular, this means that the identity element and the inverse map is definable over the parameters used in $g(\bar{x})$ and $m(\bar{x}, \bar{y}, \bar{z})$.

A theory T *eliminates quantifiers* if for any $\mathcal{M} \models T$ and any definable set given by formula $\phi(x)$, there is $\psi(x)$, a boolean combination of atomic formulas, such that $\mathcal{M} \models \phi(x) \leftrightarrow \psi(x)$. Fortunately, the theories we consider in this thesis have quantifier elimination. So, in ACF_0 the definable sets are the constructible sets in the Zariski topology (this simply follows from Chevellay's theorem that projections of constructible sets are constructible (19)). Similarly, in $DCF_{m,0}$ the definable sets are the constructible sets in the Kolchin topology. See (45) for a proof in the ordinary case and (48) for the partial case.

Consider a theory T in a language $\mathcal{L}$. Then let $\psi(x, y)$ be a formula with $|x| = |y|$. Suppose that for any model $\mathcal{M} \models T$, $\psi(x, y)$ is an equivalence relation on $\mathcal{M}^{|x|}$. We call such a ψ a *definable equivalence relation*. T has *elimination of imaginaries* if for each definable equivalence relation $\psi(x, y)$, there is some m and a definable function $f : \mathcal{M}^{|x|} \to \mathcal{M}^m$ such that $\mathcal{M} \models \psi(x, y) \leftrightarrow f(x) = f(y)$. One particularly simple manner in which a first order theory might eliminate imaginaries is if each definable equivalence relation has a *definable* set of parameters. This means that for any definable equivalence relation, there is a definable set containing precisely one element from each equivalence class. An example of theory with this property is the theory of real closed fields. The situation is often more complicated. For instance, in an algebraically closed field or a differentially closed field, there is no possible way to distinguish between two square roots in a definable manner - for instance, there is a field automorphism (differential field automorphism) which sends $-\sqrt{2}$ to $\sqrt{2}$. Each of the theories ACF_0 and $DCF_{m,0}$ eliminate imaginaries in a different manner. The proof of elimination of imaginaries in (45, section 3 of Marker's article) proceeds via a more sophisticated model theoretic technique, showing that every formula has a canonical base by constructing minimal fields of definition. The arguments suffice to eliminate imaginaries in algebraically closed fields as well. A more constructive proof of elimination of imaginaries in algebraically closed fields is possible (28); we know of no work along these lines in differential fields. Finally, in light of the quantifier elimination results discussed in the previous paragraph, elimination of imaginaries implies that the constructible sets in the Zariski and Kolchin topologies are closed under taking quotients.

Let $\mathcal{M}$ be an $\mathcal{L}$-structure, and let p be a collection of $\mathcal{L}_A$-formulas with $A \subseteq \mathcal{M}$ in free variables $x = (x_1, \ldots, x_n)$. Then p is a *type* if $p \cup Th(\mathcal{M})$ is satisfiable. Sometimes we call p an A-type or a type over A. When p has the property that either $\phi \in p$ or $\neg\phi \in p$, we call p a *complete type.* Fix some complete theory T. Let $\bar{a}$ be a tuple and $K \subset \mathcal{M} \models T$ with K an $\mathcal{L}$-structure. We often consider the complete type

$$tp(\bar{a}/K) = \{\phi(\bar{x}) \in \mathcal{L}_K \,|\, \mathcal{M} \models \phi(\bar{a})\}$$

In differential fields, the (complete) type of $\bar{a}$ over K corresponds precisely (by quantifier elimination) to the isomorphism type of the differential field extension $K\langle\bar{a}\rangle/K$. We will often use the notation $a \models p$ where p is a type over K to mean that $tp(a/K) = p$.

There is a natural topology on the space of n-types over K, called the *Stone space,* $S_n(K)$. The basic open sets in the topology are given by $\{p \in S_n(K) \,|\, \phi(x) \in p\}$ where $\phi(x) \in \mathcal{L}_K$. The use of types in this thesis is ubiquitous; the reader unfamiliar with types might consult (44, chapter 4).

One of the reasons for the distinction between the formal language and the interpretations (mentioned in the previous section) is the first theorem of model theory:

Theorem 2.2.2. *(Compactness theorem) Assume that Σ is a set of sentences in the language $\mathcal{L}$ such that for any finite subset $\sigma \subset \Sigma$, there is an $\mathcal{L}$-structure $\mathcal{M}_\sigma \models \sigma$. Then there is an $\mathcal{L}$-structure $\mathcal{M} \models \Sigma$.*

We will use the compactness theorem somewhat subconsciously throughout much of this thesis.

2.3 (ω-)stability

Many of the most important tools in this thesis come from an area of model theory known as *stability theory*. Perhaps the earliest example of a result in this area is (54); the field owes many of its important tools to (79). Roughly, stability theory as a subject developed as a program to classify which first order theories have models which can be "classified" (generally, by cardinal invariants). For instance, algebraically closed fields can be classified by specifying their characteristic and the cardinality of their transcendence base. Perhaps more generally, one point of view is that stability theory and its variants and generalizations attempt to draw dividing lines among the first order theories.

We will not be considering such foundational questions in this thesis. As it turns out, much of the machinery developed for these foundational dividing line questions has found useful applications in mathematics. A good reference for much of the stability theory used in this thesis is (61) or chapters 11 through 19 of (67); some, but not all, of the necessary results could be found in chapter 6 of (44). For most of this thesis, a knowledge of general stable theories is *not necessary*; rather one could work in the more particular setting of *ω-stability* (described below). The exception is chapter 5, which takes place in the more general superstable setting.

Fix a complete theory T. T is ω-stable if whenever $\mathcal{M} \models T$, $A \subseteq \mathcal{M}$, and $|A| = \aleph_0$, then for each n, $|S_n(A)| = \aleph_0$. Differentially closed fields are ω-stable. The assumption of ω-stability allows us to work in a large fixed differentially closed field (a universal domain) instead of constantly considering arbitrary differential field extensions:

Theorem 2.3.1. *If T is a countable $\mathcal{L}$-theory which is ω-stable, then for any cardinal κ, there is $\mathcal{M}_\kappa \models T$ which is κ-saturated, κ-homogeneous, and $|\mathcal{M}_\kappa| = \kappa$.*

$\mathcal{M}$ is κ-saturated if for any subset $A \subset \mathcal{M}$ such that $|A| < \kappa$, if $p \in S_n(A)$, then there is $a \in \mathcal{M}$ such that $a \models p$. $\mathcal{M}$ is κ-homogeneous if $a_1, a_2 \models p$, then there is $\sigma \in Aut(\mathcal{M}/A)$ such that $\sigma(a_1) = a_2$. For the rest of this introduction, we work in a fixed κ-saturated model $\mathcal{M}$ of an ω-stable theory. All of the subsets A over which we consider types or definable sets are assumed to be of cardinality less than κ. Essentially no content will be lost if the reader assumes that this theory is $DCF_{m,0}$.

Throughout this thesis, we will use the ranks of stability theory extensively. *Morley rank* is an ordinal-valued dimension function on definable sets. Let $X = \phi(\mathcal{M})$ be a definable set.

Definition 2.3.2. The definition of Morley rank is inductive:

- $RM(X) \geq 0$ if $X \neq \emptyset$.
- $RM(X) \geq \beta$, where β is a limit if $RM(X) \geq \alpha$ for all $\alpha < \beta$.
- $RM(X) \geq \alpha + 1$, if there are definable sets $\{X_i\}_{i \in \mathbb{N}}$ such that $X_i \cap X_j = \emptyset$ for $i \neq j$ and $RM(X_i) \geq \alpha$ for each α.

$RM(X)$ is defined to be the minimum α such that $RM(X) \geq \alpha$ and $RM(X) \ngeq \alpha + 1$.

When $RM(X) = \alpha$, there is a maximum n such that there are n disjoint definable subsets $X_1, \ldots, X_n \subset X$ where $RM(X_i) = \alpha$. This number, n, is called the *Morley degree of X,* $DM(X)$. Morley rank can be extended to types:

$$RM(p(x)) = inf_{\phi(x)\in p(x)}\{RM(\phi(x)\}$$

From the perspective of differential algebraic geometry, Morley rank is a dimension function on differential algebraic varieties (or more generally, constructible sets). However, suppose that $RM(V) = \alpha+1$ and V_i is a collection of constructible sets witnessing that $RM(V) > \alpha$. There is no reason to believe that the collection V_i is a *uniform family,* that is, a family of constructible subsets cut out by some set of equations which vary in a parameter $\bar{a}$ in some moduli space having differential algebraic structure. In fact, anomalies occur (31) (64). In algebraically closed fields, Morley degree one definable sets are simply those whose Zariski closures are irreducible. This is not true in differentially closed fields. There are differential algebraic varieties which are irreducible and Morley degree two. For instance, the following subvariety of $\mathbb{A}^1$,

$$V = Z(xx'' - x') = \{x \in \mathcal{M} \,|\, xx'' - x' = 0\}.$$

The differential algebraic variety V has precisely one irreducible infinite subvariety, $Z(x')$. As definable sets, both $Z(x')$ and $V \backslash Z(x')$ are strongly minimal.

Again, recall that we are working in the ω-stable setting. The following definitions will not work in the superstable setting (so are not suitable for chapter 5 in full generality).

Definition 2.3.3. Let p and q be types such that $p \subset q$. In this case, we say that q *is an extension* of p. We say that q is a *nonforking* extension of p if $RM(q) = RM(p)$.

Now, consider $A \subset B$ and a tuple c. If $tp(c/B)$ is a nonforking extension of $tp(c/A)$, then c is said to be *independent* from B over A. In this case, we write $c \perp_A B$. This notion of independence obeys many nice properties in ω-stable theories. For instance,

Theorem 2.3.4. *With $A \subset B, C$ and a tuples b, c,*

1. *There is c' such that $tp(c/A) = tp(c'/A)$ and $c' \perp_A B$.*
2. *If $c \perp_A B$ and $B_1 \subseteq B$, then $c \perp_A B_1$.*
3. *$c \perp_A BC$ if and only if $c \perp_A B$ and $c \perp_{AB} C$*
4. *$c \perp_A B$ if and only if $c \perp_A B_0$ for all finite $B_0 \subset B$.*
5. *$c \perp_A b$ if and only if $b \perp_A c$*

We say that a type is *stationary* if it has unique nonforking extensions to any larger parameter set. Complete types over algebraically closed parameter sets are stationary. We denote the nonforking extension of a type $p \in S(A)$ to $S(C)$ via $p|_C$. In differential fields non-forking is characterized by the Kolchin polynomial (72):

$$\bar{a} \underset{A}{\perp} C \leftrightarrow \omega_{\bar{a}/A(t)} = \omega_{\bar{a}/A \cup C}(t)$$

Thinking on the level of differential algebraic varieties, this simply means that the locus of $\bar{a}$ over A is the same as the locus of $\bar{a}$ over $A \cup C$.

Let X be a definable set. A *canonical parameter* of X is a tuple c such that any $\sigma \in Aut(\mathcal{M})$ fixes X as a set if and only if $\sigma|_c = id$. If p is a type, then c is the *canonical base* of p if for any $\sigma \in Aut(\mathcal{M})$, σ fixes the set of realizations of p as a set if and only if $\sigma|_c = id$. Usually one must move to the multisorted setting in order to assure the existence of canonical bases and parameters. Generally, one needs to consider "imaginary sorts" which correspond to quotients of the original sorts by definable equivalence relations. However, for the majority of this thesis, we work in differentially closed fields or algebraically closed fields both of which have elimination of imaginaries - this gives canonical bases in the original sort and makes the aforementioned move unnecessary.

Canonical bases and parameters are only meaningful up to interdefinability, so we may as well take the definable closure of c. In differential fields, this corresponds to taking the differential field generated by c. Thus, in differential fields, one may think of canonical bases as minimal fields of definition of the locus of a tuple. Again, interdefinability of tuples in differential algebra means differential birationality. The following theorem holds in the ω-stable context:

Theorem 2.3.5. *Canonical bases and parameters exist, and are unique, up to interdefinability.*

Next we define *Lascar rank* to be the foundation rank on forking:

Definition 2.3.6. Let p be a type. Then,

- $RU(p) \geq 0$ if p is consistent.
- $RU(p) \geq \beta$, where β is a limit just in case $RU(p) \geq \alpha$ for all $\alpha < \beta$.
- $RU(p) \geq \alpha + 1$ just in case there is a forking extension q of p such that $RU(q) \geq \alpha$.

It is standard to write $RU(a/A)$ for $RU(tp(a/A))$.

It is worth noting what Lascar rank means in differential algebraic geometry. Let p be the type of the generic point (with respect to the Kolchin topology over K) on a differential algebraic variety V. Then to say that the Lascar rank of p is at least $\alpha+1$, means that there is a uniform family of subvarieties subvarieties such the generic point on each of the subvarieties has Lascar rank at least α. So, while Morley rank corresponds to information about configurations of subvarieties, Lascar rank corresponds to information about the uniform families of subvarieties.

Remark 2.3.7. We should note that the style of the above development of Morley rank, forking, and Lascar rank is not likely to be found in modern stability theory texts (61; 87; 2; 59). There are good reasons (alluded to above) for the difference, the main one being that these definitions do not work outside of the the ω-stable context. However, in the setting of differential fields, all notions of dimension which we consider are intertwined with forking (and witness forking). This exposition is designed to reflect that fact.

One of the most important properties of Lascar rank is that it obeys a version of the fiber dimension theorem:

Theorem 2.3.8. *(Lascar Inequality) For tuples a, b and a set A,*

$$RU(a/AB) + RU(b/A) \leq RU(ab/A) \leq RU(a/Ab) \oplus RU(b/A)$$

Any ordinal β may be expressed as a sum of powers of ω with coefficients in the natural numbers and ordinal exponents:

$$\beta = \omega^{\alpha_1} \cdot n_1 + \omega^{\alpha_2} \cdot n_2 + \ldots + \omega^{\alpha_k} \cdot n_k$$

such that for all i, $\alpha_i > \alpha_{i+1}$. This expression is known as the *Cantor normal form* of β. The cantor sum of two ordinals, $\oplus$, is obtained by simply adding the Cantor normal forms as if they were polynomials in ω over $\mathbb{N}$. Note that the differential context, the *exponents* are also always natural numbers.

We will use various other properties of Lascar rank throughout this thesis. A useful reference for these properties will be (67), chapter 19. We will also sometimes use technology from geometric stability theory, which we will not discuss in this introduction. A good reference is (61).

2.4 (More) Model theory of differential fields

The first interactions of model theory and differential algebra stem from (74), where the basic theory is laid out. The model theory of partial differential fields of characteristic zero with finitely many commuting derivations was developed in (48). In this setting, we have a model companion, which we denote $DCF_{0,m}$. As mentioned previously, the theory $DCF_{0,m}$ has quantifier elimination and elimination of imaginaries.

Historically, there were several notions of *differential algebraic groups* appearing in the literature. The principal notions were Kolchin's abstract formulation (34), a (more restrictive)

Weil-style notion in the differential category, differential algebraic subgroups of *algebraic groups*, and the class of definable groups over differentially closed fields (62). The groups in the Weil-style formulation turn out to be definable via elimination of imaginaries. Abstract "group chunk" arguments can be used to show that the Kolchin-style category, the Weil-style category, and the definable category are all the same(60). The version of the group chunk theorem used is due to Hrushovski, and applies in other more general model-theoretic settings, see (12).

Later proofs avoided the use of Hrushovski's generalization of Weil's theorem, only using the original Weil theorem. This was accomplished, in part, by proving that any differential algebraic group may be embedded as a subgroup of an algebraic group (see variations of the proof in (62) (41) (40)). The result is rather obvious for groups with generic element of constant Kolchin polynomial over the canonical base of the group. The general case requires more subtlety. Though Pillay's proof is in the ordinary differential setting, the arguments in any of the above referenced proofs work in the partial differential setting with little modification (and even in the difference-differential setting (49)). We will be using the above results of Pillay implicitly throughout much of this thesis.

Another useful result which we use often is the classification of definable subfields:

Theorem 2.4.1. *The fields interpretable in $DCF_{0,m}$ are precisely those (isomorphic) to kernels of definable derivations.*

The definable derivations are simply independent linear combinations of Δ over the constants, $\{x \in \mathcal{U} \,|\, \text{for all } \delta \in \Delta, \delta(x) = 0\}$ (for a proof, see (85) or (84)). We will use this characterization several times throughout the thesis.

CHAPTER 3

INDECOMPOSABILITY FOR DIFFERENTIAL ALGEBRAIC GROUPS

3.1 Introduction

The indecomposability theorem of Zilber generalized a known theorem for algebraic groups (75) to the setting of weakly categorical groups (92). The theorem is a powerful tool for definability results in groups of finite Morley rank. Zilber's theorem was generalized to the superstable (possibly infinite rank) setting by Berline and Lascar (6). Because $DCF_{0,m}$ is ω-stable, their results apply to differential algebraic groups. For differential algebraic groups over a partial differential field, the existing indecomposability theorems are not suitable for some applications because, at present, there is no known lower bound for Lascar rank in terms of several important differential birational invariants (85). Though the superstable version of the indecomposability theorem given by Berline and Lascar applies in this context, both the hypotheses and the conclusions the theorem are not clear from the perspective of differential algebra because of the lack of control of Lascar rank in partial differential fields. With that in mind, we give new indecomposability theorem in which both the hypotheses and conclusions are purely differential algebraic in nature. Additionally, we provide applications and examples of the ideas. These include the definability of commutators of differential algebraic groups with appropriate hypotheses. In particular, this allows us to generalize some results of Cassidy and Singer and answer a question they posed in (16).

The notion of indecomposability we consider is related to the notion of *strongly connected* studied by Cassidy and Singer (16). We also discuss open problems which could connect these results to those of Berline and Lascar (6). Generalizing this work to the difference-differential setting (or even more general settings) is of interest, but is not covered here. This generalization would be non-trivial in from the model theoretic perspective, since it would mean leaving the setting of stability for the more general setting of simplicity. Also, though there are well-developed theories of numerical polynomials in more general settings (36), work along the lines of (16) (in those settings) would seem to be a prerequisite for proving results like those below.

Section 2 covers stabilizers of types in differential algebraic groups. We use indiscernible sequences to calculate bounds on the Kolchin polynomials of stabilizers of certain types in differential algebraic groups. Section 3 gives the main theorem, a definability result for differential algebraic groups. Section 4 and section 5 give applications of the main theorem and show some specific examples. Section 6 discusses some open problems and possible generalizations of the work. Generalizations of the conditions of the theorem in the setting of differential fields are considered. We also discuss generalization of the setting itself, via adding more general operators to the fields.

3.2 Stabilizers

Next, we develop the notion of stabilizers of types in the differential algebraic setting. The basic setup is that of superstable groups (66), but the proofs of the results are easier or sometimes give more information in the setting of differential algebraic groups. In this section, G will be a differential algebraic group, that is, a definable group over some differential field,

k. For discussion of the category of differential algebraic groups, see (62). Again, recall that we assume K is a small differentially closed field containing k.

Definition 3.2.1. Let $p(x) \in S(K)$ be a complete type containing the formula $x \in G$. All of the complete types we deal with will contain this formula. Define

$$stab_G(p) = \{a \in G \mid \text{ if } b \models p,\, b \underset{K}{\perp} a, \text{ then } ab \models p\}$$

We emphasize to the reader that the type p is assumed to contain the formula $x \in G$. For the reader who prefers to avoid the use of types, one could see p as the isomorphism type of a tuple in a differential field extension of K. To say that we only consider types in the group G means that we only consider tuples which satisfy the equations defining G. If we work in a large saturated model, then *equality of types* over the base field K can simply be thought of as isomorphism of the differential field extensions over K generated by the elements in the field extensions realizing the types.

Example 3.2.2. Consider $G = Z(x'')$ in a model of the theory $DCF_{0,1}$. G is a subgroup of $\mathbb{G}_a$. In this case, we will write the group operation additively. Consider the generic type $p \in S_1(K)$ of G. That is, $p \models x'' = 0$, but not any lower order differential equations defined over K. Then by definition, the stabilizer of p in G is the definable subgroup of elements $g \in G$ for which if $b \models p$ and $b \perp_K g$ then $g + b \models p$. The independence of g and b ensures that $g + b$ satisfies no differential equations of order 1 (even over $K\langle g \rangle$). More generally, the stabilizer of the generic type of a connected ω-stable group must be the entire group (for details, see chapter 2 of (66)).

In this paper, we will be considering stabilizers of non-generic types of the differential algebraic group G. In certain cases, we try to control these subgroups to get definability results.

The next two results are standard for ω-stable groups (66) (only the first needs ω-stability; for the other superstability suffices).

Lemma 3.2.3. *$stab_G(p)$ is definable.*

Lemma 3.2.4. $RU(p) \geq RU(stab_G(p))$.

The first task is to put the last lemma into the differential algebraic context, with differential type and typical differential dimension playing the role that Lascar rank plays in the model theoretic context. The next two lemmas are preparation for this result.

Again, we are working in some fixed differential algebraic group G; *all types, elements and tuples in the following lemma are assumed to be in the differential algebraic group in which we are working.* Unless specially noted, multiplication of two elements or above as in the case of a type occurs with *respect to the group operation in G*. In differential fields, *canonical bases* correspond to fields of definition. In fact, we might as well assume that every canonical base is a differential field, because the choice of a canonical base matters only up to interdefinability, and definable closure of a given set in models of $DCF_{0,m}$ is obtained by taking differential field generated by that set.

Lemma 3.2.5. *Suppose that $c \perp_A b$, where we assume that A is a small set which contains the canonical base of the differential algebraic group G. Then $\tau(c/A) \leq \tau(cb/A)$ and in the case of equality, $a_\tau((c/A)) \leq a_\tau((cb/A))$.*

Proof. c is definable over $A \cup \{b, cb\}$ and cb is definable over $A \cup \{c, b\}$. Definable closure is the same as differential field closure in our setting. So,

$$\tau(c/A \cup \{b\}) = \tau(cb/A \cup \{b\}) \leq \tau(cb/A)$$

and in the case of equality in the previous line,

$$a_\tau((c/A \cup \{b\})) = a_\tau((cb/A \cup \{b\})) \leq a_\tau((cb/A)).$$

But, we know that

$$\omega_{c/A\cup\{b\}}(t) = \omega_{c/A}(t)$$

by the characterization of forking in partial differential fields (48), so the lemma is established.

□

Remark 3.2.6. As pointed out by both Cassidy and Sit, the difficulty with proving the previous lemma for the Kolchin polynomial itself (instead of the *coarser* differential type and typical differential dimension) is that the Kolchin polynomial is not a differential birational invariant. So, in the above lemma, when multiplying by group elements, we may be taking differential rational functions of those elements which might not preserve the Kolchin polynomials. While it is true that differential algebraic groups may be embedded in algebraic groups (62), relieving this potential problem, such an embedding (starting from our given differential algebraic group) need not preserve the Kolchin polynomial. One could state the above result with Kolchin polynomials, but only after assuming a specific embedding into an algebraic group.

Alternatively, a result of (82) shows that the Kolchin polynomials are well-ordered by eventual domination. For a given differential algebraic group, one could consider the set of Kolchin polynomials of groups differentially birationally isomorphic to G. Selecting the minimal Kolchin polynomial from this set would give every differential algebraic group a canonical polynomial. Analysis of this polynomial from a model theoretic perspective might be possible since it is an invariant of a tuple up to interdefinability. We avoid both this approach and the one mentioned in the previous paragraph, because our results are only require analysis of the differential type and typical differential dimension. Stronger bounds in terms of Kolchin polynomials are possible for many of the results in this paper, but they are not needed for our purposes.

The following lemma has been stated before (62) (4) (though, to the author's knowledge, not precisely in this form). For instance, the proof in (4) is for the case groups definable in ordinary differential fields. The proof in the partial differential version is not any harder, but we include its proof for convenience.

Lemma 3.2.7. *Suppose that G is an irreducible differential algebraic group. Then a type is generic in the sense of the Kolchin topology if and only if it is of maximal Lascar rank.*

Proof. We will refer to types which are generic in the Kolchin topology (in the sense that there is a realization of the type of Kolchin polynomial equal to the differential algebraic group) as a *topological generic*. We will refer to the types of maximal Lascar rank as *RU-generics*. We will refer to types for which any neighborhood covers the group G by finitely many left translates as *group generics*. In any superstable group, being group generic is equivalent to being RU-generic (66).

Suppose that $p(x)$ is RU-generic but not topological generic. Then finitely many left translates of any formula in $p(x)$ cover the group, but $p(x)$ is not topological generic, so the type is contained in a proper Kolchin closed subset of G. Take the formula witnessing this, $\phi(x)$. Now, finitely many left translates of $\phi(x)$ cover the group G, and each of these is clearly closed in the Kolchin topology (if a is a topological generic in $\phi(x)$ then $g\phi(x)$ is simply the zero set of the ideal of differential polynomials vanishing at ag). But, this is a problem. Now G is the finite union of proper closed subsets.

Now, assume that $p(x)$ is a type such that any realization a is topological generic. Then take any differential polynomial $P(x)$ vanishing at a. As a is topological generic, $P(x)$ vanishes everywhere in G. So, by quantifier elimination, then only possible non-group generic formula in $p(x)$ is the negation of a differential polynomial equality. Suppose that $P(x) \neq 0$ is not group generic. Then $P(x) = 0$ is group generic, so finitely many translates cover G, which is again a contradiction if $Z(P) \cap G$ is a proper closed subset of G. Thus $P(x) \neq 0$ is group generic. $\square$

From now on, we will simply refer to these types as generics. Note that this argument also shows that for a differential algebraic group, irreducibility in the Kolchin topology implies that Morley degree is one. In a general differential algebraic variety (with no group structure), there are examples in which the topological generics are disjoint from the U-generics and Morley degree (and even Morley rank) of a definable (constructible) set is not preserved by taking the closure of the set in the Kolchin topology (22). Also note that the previous lemma also holds for definable principal homogeneous spaces of a differential algebraic group (62).

Proposition 3.2.8. *For any complete type which includes the formula "$x \in G$", $\tau(stab_G(p(x))) \leq \tau(p(x))$ and in the case of equality, $a_\tau(stab_G(p(x))) \leq a_\tau(p(x))$.*

Proof. Suppose that $s(x)$ is a generic type of the stabilizer of $p(x)$. Take $b \models p(x)$ and $c \models s(x)$ such that $b \perp_K c$.

$$\tau(c/G(K)) \leq \tau(bc/G(K))$$

and if equality holds, then

$$a_\tau(c/K) \leq a_\tau(bc/K)$$

by Lemma 3.2.5. One needs only to argue that $tp(bc/G(K)) = p(x)$. This follows from the definition of $stab_G(p(x))$. □

In the next lemma, we will discuss a tuple called a *canonical base* of a type, see (45). Essentially, we aim to choose the generators of the field of definition of the corresponding differential variety of least Kolchin polynomial. We remind the reader that the Kolchin polynomials are ordered by eventual domination. In what follows, we will write the Kolchin polynomial of a type p in the following canonical form

$$\omega_p(t) = \sum_{0 \leq i \leq m} a_i \binom{t+i}{i}.$$

Lemma 3.2.9. *Suppose that $\tau(G) = n$. Suppose that $p(x) \in S(K)$ with "$x \in G$" $\in p(x)$ Then, suppose, for some finite $A \subseteq K$, that*

$$\omega_{p|_A}(t) < \omega_p(t) + \binom{t+n}{n}$$

Then there is a tuple $\bar{c} \in K$ such that $\omega_p(t) = \omega_{p|_{\bar{c}}}(t)$ and $\omega_{\bar{c}/A}(t) < \binom{t+n}{n}$.

Proof. Let $\langle b_k \rangle_{k \in \mathbb{N}}$ be a Morley sequence over K in the type of p. By the characterization of forking in $DCF_{0,m}$ this simply means that for all $k \in \mathbb{N}$,

$$\omega_p(t) = \omega_{b_k/K}(t) = \omega_{b_k/K \cup \{b_0, \ldots, b_{k-1}\}}(t)$$

We do not know, however, that the same holds over the (arbitrary) subset $A \subseteq K$. The sequence is still necessarily A-indiscernible, that is $tp(b_k/A)$ does not depend on k. It is not necessarily A-independent. In general, we simply know that $\omega_{b_k/A \cup \{b_0, \ldots, b_{k-1}\}}$ is a decreasing sequence of polynomials, again, ordered by eventual domination. By the well-orderedness of Kolchin polynomials we know that the sequence is eventually constant. Alternatively, this fact can be seen by noting the superstability of $DCF_{0,m}$ and the fact that decreases in Kolchin polynomial correspond to forking extensions. So, for the rest of the proof, we fix a k such that if $n \geq k$, the sequence is constant. That is, above k, we know that we have a Morley sequence over $A \cup \{b_0, \ldots, b_{k-1}\}$ in the type of p. Now, fix a model $K' \models DCF_{0,m}$ with K' containing K and $\{b_0, \ldots, b_{k-1}\}$. We let p' be the (unique) nonforking extension of p to K'.

We can get elements $\bar{c} \subseteq acl(A \cup \{b_0, \ldots, b_{k-1}\})$ such that p' does not fork over $\bar{c}$. In fact, by (79) (page 132) and the fact that $DCF_{0,m}$ eliminates imaginaries, we can assume that $\bar{c} \in K$. We know that

$$\omega_{p|_A}(t) = f(t) + h(t)$$

where

$$f(t) = \sum_{i=n+1}^{m} c_i \binom{t+i}{i}$$

and

$$h(t) = \sum_{i=0}^{n} c_i \binom{t+i}{i}.$$

By assumption, $\omega_{p|_A}(t) < \omega_p(t) + \binom{t+n}{n}$. Thus, $f(t) \leq \omega_p(t)$. By construction $\langle b_i \rangle$ was an indiscernible sequence, so if we define $\bar{b} := (b_0, \ldots, b_{k-1})$, then

$$k \cdot f(t) \leq \omega_{\bar{b}/K}(t)$$

Then we know that

$$k \cdot f(t) \leq \omega_{\bar{b}/A \cup \bar{c}}(t) \tag{3.1}$$

So, for all $i = 0, 1, \ldots, k-1$, we have that

$$\omega_{b_i/A \cup \{b_0, \ldots b_{i-1}\}}(t) \leq \omega_{p|_A}(t) = f(t) + h(t)$$

Further,

$$\omega_{\bar{b}/A}(t) \leq \omega_{b_0/A}(t) + \omega_{b_1/A\cup\{b_0\}}(t) + \ldots \omega_{b_{k-1}/A\cup\{b_0,\ldots b_{k-1}\}}(t)$$

But, this means that

$$\omega_{\bar{b}/A}(t) \leq kf(t) + kh(t) \tag{3.2}$$

By assumption, $\bar{c} \in acl(A \cup \bar{b})$ so $\omega_{\bar{b}/A}(t) = \omega_{\bar{b}\bar{c}/A}(t)$. Then

$$\omega_{\bar{b}/A\cup\bar{c}}(t) + \omega_{\bar{c}/A}(t) \leq \omega_{\bar{b}/A}(t).$$

Now, using Equation 3.1 and Equation 3.2, we know that

$$\omega_{\bar{c}/A}(t) < \binom{t+n}{n}.$$

□

The next lemma appears in (6):

Lemma 3.2.10. *Let $p(x) \in S(K)$ with "$x \in G$" $\in p(x)$. Suppose p does not fork over the empty set. Let b be an element of $G(K)$. Let $A \subset K$. If $\tilde{b} = b \,(mod\, stab_G(p))$ is not algebraic over A, then bp forks over A.*

Proposition 3.2.11. *Suppose that* $a \perp_K b$ *with* $a, b \in G$. *Let* $p = tp(a/K)$. *If*

$$\omega_{ba/K}(t) < \omega_{a/K}(t) + \binom{t+n}{n}$$

and $\tilde{b}$ *is the class of* b *mod* $stab_G(p)$ *then*

$$\omega_{\tilde{b}/K}(t) < \binom{t+n}{n}$$

Proof. We let K' be an elementary extension of K containing b such that $a \perp_K K'$. That is, $tp(a/K')$ is the unique nonforking extension of $tp(a/K)$. Then,

$$\omega_{a/K}(t) = \omega_{a/K'}(t).$$

Further, we know from Proposition 3.2.5 that $\tau(a/K) = \tau(ba/K)$ and $a_\tau(a/K) = a_\tau((ba/K))$. But, the same holds over K', since ba is interdefinable with a over K'. Then we know that

$$\omega_{ba/K}(t) < \omega_{ba/K'}(t) + \binom{t+n}{n}$$

By lemma 3.2.9 we can get $\bar{c} \in K'$ with $\omega_{\bar{c}/K}(t) < \binom{t+n}{n}$ such that $ba \perp_{K \cup \bar{c}} K'$. Then, applying lemma 3.2.10 we can see that $\tilde{b}$ is algebraic over $K \cup \bar{c}$. We know that $\omega_{\bar{c}/K}(t) < \binom{t+n}{n}$ and so $\omega_{\tilde{b}/K}(t) < \binom{t+n}{n}$. $\square$

Roughly, the next result says that if an element in a differential field extension of the base field lying in the differential algebraic group is sufficiently generic (in the differential algebraic sense), then the stabilizer of this element is large (again, in the differential algebraic sense). One might regard this as a sort of converse statement to Proposition 3.2.8.

Proposition 3.2.12. *Let $p(x) \in S(K)$, with "$x \in G$" a formula in $p(x)$. Let n be such that*

$$\omega_G(t) < \omega_p(t) + \binom{t+n}{n}.$$

Then

$$\omega_G(t) < \omega_{stab_G(p)}(t) + \binom{t+n}{n}$$

Proof. Choosing b to be a generic point on G over K and applying Proposition 3.2.11 gives that

$$\omega_G(t) \leq \omega_{stab_G(p)}(t) + \omega_{\tilde{b}/K}(t)$$

so

$$\omega_G(t) < \omega_{stab_G(p)}(t) + \binom{t+n}{n}$$

completing the proof. $\square$

3.3 Indecomposability

Definition 3.3.1. Let G be a differential algebraic group defined over K. Let X be a definable subset of G. For any $n \in \mathbb{N}$, X is n-indecomposable if $\tau(X/H) \geq n$ or $|X/H| = 1$ for any definable subgroup $H \leq G$. We use *indecomposable* to mean $\tau(G)$-indecomposable.

Elimination of imaginaries must be used to justify the notation $\tau(X/H)$. For any two elements x_1 and x_2 of X, say $x_1 \sim x_2$ if x_1 and x_2 are in the same left coset of H. This is a definable equivalence relation, so the set X/H has differential algebraic structure. So, one may apply 1.3.1 to assign $\tau(X/H)$. For more regarding elimination of imaginaries in differentially closed fields, see (45).

We should note the relationship of this notion to that of *strongly connected* considered by (16). A subgroup of G is strongly connected if and only if it is n-indecomposable where $n = \tau(G)$. We will occasionally use *n-connected* to mean n-indecomposable, but only in the case that the definable set being considered is actually a *subgroup*. In the next section, we will show some techniques for constructing indecomposable sets from indecomposable groups.

Definition 3.3.2. A differential algebraic group G is *almost simple* if for all normal differential algebraic subgroups H of G, we have that $\tau(H) < \tau(G)$.

We note that by 1.3.3, almost simple is a strengthening of strong connectedness. In the definition of strong connectedness, one could take the subgroup H to be normal without changing the definition. The reason for this is that, given a differential algebraic group G, then the set of subgroups such that $\tau(G/H) < l$ is closed under finite intersection. From this observation and the basis theorem for the Kolchin topology (or the Baldwin-Saxl condition), one can prove the existence of the strongly connected component of the identity for an arbitrary differential algebraic group. All of the preceding discussion in this section comes from (16).

In the definition of almost simple, taking the subgroup H to be normal is necessary. For example, every differential algebraic group G has an abelian differential algebraic subgroup of the same Δ-type (21).

Cassidy and Singer proved the following, showing the robustness of the notion of strong connectedness under quotients,

Proposition 3.3.3. *Every quotient X/H of a n-connected definable subgroup X by a definable subgroup is n-connected.*

For many other properties of strongly connected subgroups and numerous examples, see (16). The next proposition is used in the proof of the main theorem, but is stated separately because it applies more generally.

Proposition 3.3.4. *Let X_i, for $i \in I$, be a family of l-indecomposable sets for some $l \in \mathbb{N}$. Assume each X_i contains the identity. Let $H = \langle \cup_{i \in I} X_i \rangle$ and* suppose *that H is definable. Then H is l-indecomposable.*

Proof. Let $H_1 \leq G$ with $H \not\leq H_1$. Then there exists i such that $X_i \not\subset H_1$. For this particular i, we know, by l-indecomposability, that the coset space X_i/H_1 has differential type at least l. That is $\tau(X_i/H_1) \geq l$. But, then $\tau(H/H_1) \geq l$ since H contains X_i. $\square$

Note that there is no assumption in 3.3.4 that $\tau(G) = l$. This is assumed in the indecomposability theorem 3.3.5, but the proposition about the l-indecomposability of the generated subgroup holds more generally, assuming the group is definable. In general, we do not know about the definability of such a subgroup, unless additional assumptions are made.

Theorem 3.3.5. *Let G be a differential algebraic group. Let X_i for $i \in I$ be a family of indecomposable definable subsets of G. Assume that $1_G \in X_i$. Then the X_i's generate a strongly connected differential algebraic subgroup of G.*

Proof. Fix $n = \tau(G)$. Let Σ be the set of finite sequences of elements of I, possibly with repetition. Then for $\sigma \in \Sigma$ with $length(\sigma) = n_1$, we let $X_\sigma = X_{\sigma(1)} \cdot \ldots \cdot X_{\sigma(n_1)}$. Let $k_\sigma = a_\tau(X_\sigma)$. We note that $k_\sigma \leq a_\tau(G)$. For the remainder of the proof, we let $\sigma_1 \in \Sigma$ be such that

$$k_{\sigma_1} = Sup_{\sigma \in \Sigma}(k_\sigma)$$

is achieved. Now, let $p \in X_{\sigma_1}$ such that $\tau(p) = n$ and $a_p = k_{\sigma_1}$. We consider $stab_G(p)$. First, we note that

$$stab_G(p) \subseteq X_{\sigma_1} X_{\sigma_1}^{-1}$$

To see this, let $b \in stab_G(p)$ and $c \models p$. Then both c and bc satisfy X_{σ_1}. Then $b = bcc^{-1} \in X_{\sigma_1} X_{\sigma_1}^{-1}$. Next, we will show, for all i,

$$X_i \subset stab_G(p)$$

Let $b \in X_i$ and $c \models p$. We will also assume that c is K-independent from b. By this we mean simply that $b \perp_K c$. Then we have, $bc \models X_i \cdot X_{\sigma_1}$. And by assumption, $\tau(bc) = \tau(c) = n$. We claim that $\tau(\tilde{b}) < \tau(c) = n$ where $\tilde{b}$ is the class of b modulo $stab_G(p)$. This follows by applying Proposition 3.2.11 and noting that p has the properties needed for the hypothesis of that proposition, by the maximality of the differential type of $p \in X_{\sigma_1}$.

But, this holds for all $b \in X_i$ and X_i is indecomposable, which is a contradiction, unless $X_i \subseteq stab_G(p)$. This completes the proof, and strong connectedness follows by Proposition 3.3.4. $\square$

Of course, as in the more familiar case of groups of finite Morley rank we know more than that the group generated by the family is definable. We have constructed the definition of the group which gives it a very particular form. For further discussion see (44), chapter 7, section 3. Analogies between strongly connected differential algebraic groups and groups of finite Morley rank will be pursued in chapter 4 of this thesis and further in (21).

Remark 3.3.6. The notion of n-indecomposable presented here is similar (and inspired by) the notion of α-indecomposable considered by (6). It might be the case that this notion is a specialization of the notion considered there (note that for differential fields, the Lascar rank of a type is always less than ω^m, where m is the number of derivations). Proving that the notions coincide for differential fields would require finding a lower bound for Lascar rank in terms of Δ-type. There is currently no known lower bound for general differential varieties. For a discussion of this issue see (86). Even if such a lower bound is found, there are compelling reasons to develop indecomposability in this manner. In difference-differential fields, it is known that no such lower bound for Lascar rank holds, and so the notions of indecomposability coming from the analogue of the Kolchin polynomial and the notion coming from Lascar rank will be distinct.

3.4 Definability of Commutators of Strongly Connected Groups

In this section, we will first show some applications of the ideas and techniques for constructing indecomposable sets. Any group naturally acts on itself by conjugation, that is $x \mapsto gxg^{-1}$. Analysis of this action provides a way of transferring properties of the group doing the action to the set on which it acts. Now, fix a differential algebraic group G and a differential algebraic subgroup H. A subset $X \subseteq G$ is H-invariant if for all $h \in H$, conjugation by h is a bijection from X to itself.

First, we give the following example, due to Cassidy (35), of a differential algebraic group for which the commutator is not a differential algebraic group.

Example 3.4.1. Let $\Delta = \{\delta_1, \delta_2\}$. Then consider the following group G of matrices of the form:

$$\begin{pmatrix} 1 & u_1 & u \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\delta_i(u_i) = 0$. Of course,

$$\begin{pmatrix} 1 & u_1 & u \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & v_1 & v \\ 0 & 1 & v_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_1 & u \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & v_1 & v \\ 0 & 1 & v_2 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 0 & u_1v_2 - v_1u_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then one can see that the commutator is isomorphic $\mathbb{Q}[C_{\delta_1} \cup C_{\delta_2}]$, where C_{δ_i} is the field of δ_i-constants. This is not a differential algebraic group. This group is not strongly connected, however, since the subgroup of matrices of the form:

$$\begin{pmatrix} 1 & 0 & u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a subgroup of Δ-type and typical Δ-dimension equal to G. This means that the coset space has Δ-type strictly smaller than G. Of course, this means that G is not almost simple (or even strongly connected). Theorem 3.4.4 will show that this sort of example is impossible for strongly connected differential algebraic groups.

The next two lemmas have similar proofs in the context of groups of finite Morley rank (see Chapter 7 of (44)).

Lemma 3.4.2. *Let X be H-invariant. Suppose for all H-invariant differential algebraic subgroups $H_1 \leq G$, that $|X/H_1| = 1$ or $\tau(X/H_1) \geq n$. Then X is n-indecomposable.*

Proof. Suppose that there is a differential algebraic subgroup $H_2 \leq G$ with $\tau(X/H_2) < n$, but $|X/H_2| \neq 1$. Then, by the H-invariance of X, if $h \in H$ then $x^h \in X$. Thus h defines a map from $X/H_2 \to X/H_2^h$. In particular, $\tau(X/H_2^h) < n$, but $|X/H_2^h| \neq 1$. Then, set

$$H_1 = \bigcap_{h \in H} H_2^h$$

Then, by the Baldwin-Saxl condition, we know that H_1 is actually definable and is, in fact, the intersection of finitely many of the subgroups. But then, H_1 is clearly H-invariant and $\tau(X/H_1) < n$ and $|X/H_1| \neq 1$, contradicting the assumptions on X. □

Lemma 3.4.3. *If H is an indecomposable differential algebraic subgroup of G and $g \in G$, then g^H is indecomposable.*

Proof. The set g^H is H-invariant. Using the previous result, it is enough to prove the result for all $N \leq G$ which are H-invariant. So, to that end, suppose that N is such that $|g^H/N| \neq 1$ and $\tau(g^H/N) < n$. Now, we get, by the H-invariance of g^H and N, a transitive action of H on g^H/N,

$$h * g^{h_1}N \mapsto hg_1^hNh^{-1} = hg^hh^{-1}hNh^{-1} = g^{hh_1}N$$

Thus, this is a transitive action of H on a differential algebraic variety of differential type less than n. The kernel of the action must be a subgroup of H of differential type $\tau(H)$ and typical differential dimension equal to that of H. This is impossible, by the indecomposability of H, unless the kernel is simply H itself (see (16) or (21)). If that is the case, then by the transitivity of the action, $|g^H/N| = 1$. □

Cassidy and Singer make the following comment in (16), "We also do not have an example of a noncommutative almost simple linear differential algebraic group whose commutator subgroup is not closed in the Kolchin topology." The next result shows that such an example is not possible, even in the more general case of the group being strongly connected (with no assumption of linearity or almost simplicity).

Theorem 3.4.4. *Commutator subgroups of strongly connected differential algebraic groups are differential algebraic subgroups and are strongly connected.*

Proof. Apply the previous lemma, noting that $g^{-1}g^G$ is indecomposable. As g varies, this family generates the commutator. Now apply Theorem 3.3.5. □

We should also note the result of Cassidy and Singer which says that if a strongly connected differential algebraic group is not commutative, then the differential type of the differential closure of commutator subgroup is equal to the differential type of the whole group (16). So, putting this together with the above theorem yields:

Corollary 3.4.5. *Let G be a strongly connected nonabelian differential algebraic group. Then the commutator of G is a strongly connected differential algebraic subgroup with $\tau([G,G]) = \tau(G)$.*

Because commutators are characteristic (thus normal), they are candidates to appear in the Cassidy-Singer decomposition of G (see (16). We also get the following generalization of a theorem of Cassidy and Singer (who proved it in the case of an almost simple linear differential algebraic groups of differential type at most one).

Theorem 3.4.6. *Let G be an almost simple differential algebraic group. Then G is either commutative or perfect.*

Proof. $\tau([G,G]) = \tau(G)$ implies that $[G,G] = G$, since G is almost simple. Otherwise $[G,G] = 1$. □

Explicit calculations of the Kolchin polynomial for linear differential algebraic groups are often easier than for general differential algebraic groups or varieties. We will briefly describe how to perform these calculations, and how they lead to many examples of indecomposable (strongly connected) differential algebraic groups. The techniques are completely covered by Kolchin in (34) and some appear in (16). The machinery is particularly easy to deal with in the case that G is the zero set of a single linear homogeneous differential polynomial in a single variable, that is, G is given as the zero set of $f(z) \in K\{z\}$. Note that G is a subgroup of the additive group, $\mathbb{G}_a$. Suppose that, for some orderly ranking of the free monoid Θ generated by $\Delta = \{\delta_1, \ldots, \delta_m\}$, the leader of $f(z)$ is $\delta_1^{i_1} \ldots \delta_m^{i_m} z$. Then the Kolchin polynomial, $\omega_G(t)$, is equal to the number of lattice points of $(n_1, \ldots, n_m) \in \mathbb{N}^m$ with

$$\sum_{i=1}^{m} n_i \leq x$$

and $(n_1, \ldots, n_m)$ not above $(i_1, \ldots, i_m)$ in the (partial) product order. Then we know that the Kolchin polynomial is given by,

$$\omega_G(t) = \binom{t+m}{m} - \binom{t - \sum_{j=1}^{n} i_j + m}{m} + a$$

where a is a constant. Letting N be the sum $\sum i_j$, we have

$$\omega_G(t) = \binom{t+m}{m} - \binom{t-N+m}{m} + a = Nt^{m-1} + g(t)$$

where $g(t)$ is lower degree in t. Then any subgroup of $H \leq G$ has, in its defining ideal, a differential polynomial g with the leader of g not above the *leader* of f in the lexicographic order (the highest differential monomial appearing in f with respect to the lexicographic order). This means that $\tau(H) < \tau(G)$ or the coefficient of x^{m-1} in the Kolchin polynomial of H is less than N. In either case, the coset space G/H must have Δ-type $m-1$. Thus, G is indecomposable. For the differential algebraic developments required to define rankings and leaders of differential polynomials, see (34).

Example 3.4.7. We will again work over a model of $DCF_{0,2}$. The following example was explored in (16) and was originally given in (10). Let G be the solution set of

$$(c_2\delta_1^3 - c_2\delta_1^2\delta_2 - 2c_2\delta_1\delta_2 + c_2^2\delta_2^2 + 2\delta_2)x = 0$$

where $\delta_1 c_2 = 1$ and $\delta_2 c_2 = 0$. By the above discussion of linear homogeneous differential equations, this is a strongly connected differential algebraic group. There are differential algebraic subgroups of Δ-type 1. In fact, since

$$\begin{aligned}
&c_2\delta_1^3 - c_2\delta_1^2\delta_2 - 2c_2\delta_1\delta_2 + c_2^2\delta_2^2 + 2\delta_2 \\
&= (c_2\delta_1 - c_2^2\delta_2 - 2)(\delta_1^2 - \delta_2) \\
&= (c_2\delta_1^2 - c_2\delta_2 - 2\delta_1)(\delta_1 - c_2\delta_2)
\end{aligned}$$

the solution sets to $\delta_1^2 x - \delta_2 x = 0$ and $\delta_1 x - c_2\delta_2 x = 0$ are differential algebraic subgroups. In (86) Suer showed the solution set of the first equation has Lascar rank ω by showing that every definable subset has finite transcendence degree. So, this subgroup is indecomposable. We will show that the subgroup given by the solutions to $\delta_1 x - c_2\delta_2 x = 0$ only has finite transcendence degree definable subsets. This subgroup is irreducible in the Kolchin topology, so the only definable proper subsets correspond to forking extensions of the generic type of subgroup. But, modulo, $\delta_1 x - c_2\delta_2 x = 0$, any differential polynomial can be expressed as a δ_2-polynomial or a δ_1-polynomial. So, this subgroup is also indecomposable.

3.5 Another Definability Result

In this section, we prove results inspired by work of Baudisch (3). As with many of the results of this paper, the relationship between the results here and the existing work on superstable and ω-stable groups would only become clear by getting control (or showing counterexamples) of Lascar rank in terms of differential type. The following lemma is easy to prove, see (16),

Lemma 3.5.1. *Suppose there is $H \leq G$ with $\tau(G/H) < n$. Then there is a normal subgroup L of G with $\tau(G/L) < n$.*

Theorem 3.5.2. *Suppose $\tau(G) = n$ and $H \lhd G_n$, the strongly connected component of G. Then if H is simple, H is definable.*

Proof. Let $h \neq 1$, $h \in H$. Then we will show $h^G \cup \{1\}$ is indecomposable. Note that by Lemma 3.5.1, we only need to show the indecomposability for quotients by normal subgroups. So, let N be a normal subgroup of G. First, suppose that $N \cap H \neq 1$. Then because H is simple, $H \lhd N$. In this case, the coset space $|h^G \cup \{1\}/N| = 1$. Thus, we may assume that $N \cap H = 1$. Now, to verify that h^G is indecomposable, we only need to show that $\tau(h^G) = n$.

There is a bijection between the elements of h^G and the G-cosets of $C_G(h)$. So, it would suffice to prove that $\tau(G/C_G(h)) = n$. We know that $H \not\leq C_G(h)$, because H is simple. But, then $|G_n/C_G(h)| \neq 1$. Because G_n is indecomposable, $\tau(G_n/C_G(h)) = n$. But, then $\tau(G/C_G(h)) = n$. Now, we know that the following family of definable sets $\langle h^G \rangle_h \in H$ is indecomposable. Now we apply Theorem 3.3.5 to see that H must be definable. □

Further definability consequences of indecomposability will be pursued in (21).

3.6 Generalizations of Strongly Connected

For differential algebraic groups, the notion of indecomposable matches the notion of strongly connected. But, in Definition 3.3.1 we defined *n-indecomposable.* In this section we will explore the notion in the case that $n \neq \tau(G)$. Consider the following family of proper differential algebraic subgroups,

$$\mathcal{G}_n := \{H < G \mid \tau(G/H) < n\}$$

We note that this family is closed under finite intersections (16). Since G is an ω-stable group,

$$\bigcap_{H \in \mathcal{G}_n} H$$

is a definable subgroup, which we will denote G_n. We note that H_n is a characteristic subgroup of G. We will refer to G_n as the n-connected component.

Example 3.6.1. It is entirely possibly that the subgroups H_n are different for every n. The following is a very simple example which readily generalizes. Consider the following group of matrices of the form

$$\begin{pmatrix} 1 & u_{12} & u_1 & u \\ 0 & 1 & u_{123} & u_2 \\ 0 & 0 & 1 & u_{23} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\delta_1\delta_2 u_{12} = 0$, $\delta_1 u_1 = 0$, $\delta_1\delta_2\delta_3 u_{123} = 0$, $\delta_2 u_2 = 0$, and $\delta_2\delta_3 u_{23} = 0$.

This is a group since

$$\begin{pmatrix} 1 & u_{12} & u_1 & u \\ 0 & 1 & u_{123} & u_2 \\ 0 & 0 & 1 & u_{23} \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & h_{12} & h_1 & h \\ 0 & 1 & h_{123} & h_2 \\ 0 & 0 & 1 & h_{23} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & h_{12}+u_{12} & u_1+u_{12}h_{123}+h_1 & h+u_{12}h_2+u_1h_{23}+u \\ 0 & 1 & u_{123}+h_{123} & u_2+u_{123}h_{23}+h_2 \\ 0 & 0 & 1 & u_{23}+h_{23} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the coordinates evidently satisfy the same differential equations as the original matrices. The group is 0-indecomposable. The reader should note that in the setting of differential algebraic groups, 0-indecomposable simply means connected, that is, there are no definable subgroups of finite index. The group is not 1-indecomposable. The 1-connected component is the subgroup of matrices of the form:

$$\begin{pmatrix} 1 & u_{12} & u_1 & u \\ 0 & 1 & 0 & u_2 \\ 0 & 0 & 1 & u_{23} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The 2-connected component is the subgroup of matrices of the form:

$$\begin{pmatrix} 1 & 0 & u_1 & u \\ 0 & 1 & 0 & u_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The 3-connected (in this case, strongly connected) component is the subgroup of matrices of the form:

$$\begin{pmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Though much of the analysis of this paper essentially works in the case of n-indecomposability with $n \neq \tau(G)$, by relativizing the appropriate statements (see Proposition 3.3.4, for instance), there are important pieces which are not immediate. For instance, when seeking definability results for a family of n-indecomposable subsets, the above techniques are only useful when the subsets can be contained in a strongly connected subgroup of differential type n.

The indecomposability theorem of Berline and Lascar (6) applies in the setting of superstable groups, so, specifically for groups definable in $DCF_{0,m}$. As we noted in the introduction, there is no known lower bound for Lascar rank in partial differential fields based on differential type and typical differential dimension. In fact, examples of (85) show that any such lower bound can not involve typical differential dimension (Suer constructs differential varieties of arbitrarily high typical differential dimension, differential type 1, and Lascar rank ω). It is not currently known if there is an infinite transcendence degree strongly minimal type. One should note that such examples are present in the difference-differential context (49), but have yet to be discovered in the partial differential context.

CHAPTER 4

GROUPS OF SMALL TYPICAL DIFFERENTIAL DIMENSION

4.1 Introduction

This chapter aims to apply techniques from ω-stable groups and groups of finite Morley rank to prove results about differential algebraic groups. Of course, this seems like a strange goal since differential algebraic groups *are* ω-stable and many are actually of finite Morley rank. But, our results are not stated model theoretically, nor do they have model theoretic hypotheses. We do not use Morley rank or U-rank (or any other model theoretic ranks) in the statements of the results, nor are there known lower bounds for ranks from stability theory with respect to the notions of dimension we use. For some discussion of this issue, see (85) and the previous chapter of this thesis.

Zilber's indecomposability theorem is a powerful tool for proving definability results in groups of finite Morley rank. We will show how to prove similar definability results in the differential algebraic setting using the indecomposability theorem. Indecomposability is one of the key tools for carrying out a detailed analysis of groups of small Morley rank. We aim for an analysis of differential algebraic groups of small typical differential dimension. Specifically, our analysis is similar in spirit to portions of chapter seven of (44) and (17) with Morley rank replaced by typical differential dimension. Of course, complications arise since the finiteness conditions in our setting are not nearly as strong as those when dealing with finite Morley rank objects. On the other hand, we have the benefit of working inside a fixed theory, $DCF_{m,0}$.

In section two, we consider some general interpretability and definability results in differential algebraic groups. Our analysis concentrates on the question of interpreting definable fields via given differential algebraic groups or differential algebraic group actions. Section three begins the analysis of differential algebraic groups with the view that Morley rank is to connected groups of finite Morley rank as typical differential dimension is to strongly connected differential algebraic groups. Groups of typical differential dimension one or two are considered in section three. In section four, we add some group theoretic assumptions and obtain stronger results. The final section consists of remarks on an open problem and points out how the results of chapter 2 can be used to give an answer in a special case.

The notation of this chapter comes from general model theoretic and differential algebraic conventions, but some of the notation was recently invented in (16) and (23) (see also chapter 2 of this thesis). In what follows, we will pay little attention to differential type. This approach is in contrast to the results of (16) in which work is done under the assumption of differential type one. Instead, we will restrict the typical differential dimension, but allow arbitrary differential type.

Strongly connected and almost simple differential algebraic groups abound in this setting; for instance, in order for G to have regular generic type, it is seems necessary that G be almost simple. However, the precise relationship between regularity and almost simplicity is not entirely clear (see the questions raised in (53), for instance). Every differential algebraic group

G has a characteristic subgroup which is the largest strongly connected differential algebraic group, called the *strongly connected component*. Any strongly connected group has a series of subnormal differential algebraic groups such that the successive quotients are almost simple.

Example 4.1.1. If H is a quasi-simple algebraic group and C' is a definable subfield, then $H(C')$ is an almost simple differential algebraic group. For a proof, see (16).

Example 4.1.2. The counterexamples of Suer (85) are almost simple. For instance, the zero locus of

$$\delta_1 x - \delta_2^2 x$$

In general, for a discussion of almost simple groups and linear differential operators, see (23).

Example 4.1.3. The following example is due to Cassidy and Singer. Consider the following matrix group, G_n:

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

where $a \neq 0$, and $a^{-1}\delta a = \delta^n(b)$. The example is especially interesting since the groups G_n are all nonisomorphic, but they are isogenous. The following is an isogeny from $G_n \to \mathbb{G}_a$:

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto a^{-1}\delta(a)$$

Example 4.1.4. The following matrix group is not almost simple (or strongly connected):

$$\begin{pmatrix} 1 & u_1 & u \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\delta_i(u_i) = 0$. The strongly connected component is the subgroup of matrices of the form:

$$\begin{pmatrix} 1 & 0 & u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Because of this, we can not apply many portions of the rich theory of superstable groups (for instance see (66)) without making assumptions about the model theoretic ranks. This chapter is about applying the techniques of stable groups with differential algebraic dimension functions rather than the ranks of stability theory.

4.2 Differential Algebraic Group Actions

In this section, we discuss differential algebraic groups in the sense of Kolchin (34). Pillay showed that these are the definable groups in the theory $DCF_{0,m}$, see (62). So, in this section, T will be $DCF_{0,m}$. Sometimes we will refer to the definable sets in this setting as *constructible sets of the Kolchin topology* or simply as constructible sets. Since there is no known lower bound for Lascar rank in terms of differential algebraic data (see (85)), there is not currently a way to apply the α-indecomposability theorem (6) to differential algebraic groups unless model theoretic hypotheses are assumed (or very special cases are considered). However, (23, or chapter 3 of this thesis) gave an indecomposability theorem in the differential algebraic setting, with purely differential algebraic hypotheses and conclusions. The theorems of this section are differential algebraic analogues of results originally proved by Zilber (91) and generalized by (6). We suggest Marker's exposition of these results, see chapter seven of (44). For many examples of the groups discussed in this section, see (16). The notation of this section follows (23) and (16). The reader should note that in this section the definitions of n-connected, n-indecomposable, etc. are differential algebraic in nature, and might not have anything to do with model theory. For further discussion of this issue, see (23). Since it is the degree of the Kolchin polynomial, n may be assumed to be a natural number, see (34) and (16).

Lemma 4.2.1. *Suppose that an strongly connected differential algebraic group Γ acts definably and transitively on a constructible set of Δ-type less than $\tau(\Gamma)$. Then the set is a singleton.*

Proof. The kernel of the action, H, must be of Δ-type and typical Δ-dimension equal to that of Γ. This is impossible since it forces $\tau(\Gamma/H) < \tau(\Gamma)$. □

Lemma 4.2.2. *Let G be a strongly connected differential algebraic group and $\sigma : G \to G$ a definable group homomorphism such that $(Ker(\sigma))$ has Δ-type less than $\tau(G)$. Then σ is surjective.*

Proof. Since $\tau(ker(\sigma)) < \tau(G)$, $\tau(\sigma(G)) = \tau(G)$ and $a_\tau(\sigma(G)) = a_\tau(G)$. This is impossible unless $\sigma(G) = G$. □

Theorem 4.2.3. *$(H, \cdot)$ and $(A, +)$ are infinite abelian differential algebraic groups such that H acts definably and faithfully on A where H acts as a group of automorphisms. Assume A and H are both Δ-type n. Assume that no subgroup $B \leq A$ of Δ-type n is H-invariant. Then $(H, \cdot)$ and $(A, +)$ interpret an algebraically closed field of the same Δ-type as H.*

Proof. The structure of the proof is much like that of Theorem 7.3.9 of (44). Several portions of the proof given there apply directly here, but are reproduced for convenience. Many of the dimension theoretic arguments must be converted to the differential algebraic setting. First, we know that A is strongly connected, since the strongly connected component is automorphism invariant. Now, without loss of generality, we may let $a \in A$ be generic (this would only require connectedness, not strong connectedness).

Claim 4.2.4. $\tau(Ha) \geq n$

Proof. Let $H^{(0)}$ be the strongly connected component. Suppose $H^{(0)}a$ is of type less than n. Then by 4.2.1 $H^{(0)}a = \{a\}$. But, since A is connected,

$$X = \{x \in A \,|\, H^{(0)}x = \{x\}\}$$

is generic. Any element in A is a product of generics, so $H^{(0)}$ must fix all $c \in A$. The fact that H acts faithfully means that $H^{(0)} = \{1\}$. But, this means that $\tau(H) < n$, contradicting the hypotheses of the theorem. □

Claim 4.2.5. $Ha \cup \{0\}$ is n-indecomposable.

Proof. 4.2.5 $Ha\cup\{0\}$ is H-invariant, so we need only test the indecomposability for H-invariant subgroups (see (23)). But, by the hypotheses of the theorem, each of these have type less than n, so the conclusion follows. □

By the indecomposability theorem (see (23)) the subgroup $\langle Ha \cup \{0\}\rangle$ is definable and H-invariant. But then $\langle Ha \cup \{0\}\rangle = A$. Further, the proof of the theorem tells us that any $a \in A$ is the sum of k elements from Ha, for some fixed k. Consider the ring of endomorphisms of A. Since $H \subseteq End(A)$, we may define R to be the subring generated by H. Since H is abelian, R is commutative. For all $b \in A$, we know,

$$b = \sum_{i=1}^{m} h_i a$$

for some $h_i \in H$ and $m \leq k$. Now,

$$r(b) = \sum_{i=1}^{m} r(h_i a) = \sum_{i=1}^{m} h_i(ra)$$

so if $r_1 \neq r_2 \in R$, then it must be the case that $r_1 a = r_2 a$. Further, since if $ra = b$, then as above there are $h_i \in H$, such that

$$ra = b = \sum_{i=1}^{m} h_i a,$$

and so we can see

$$r = \sum_{i=1}^{m} h_i.$$

That is, every element of R is the sum of less than or equal to k elements of $H \cup \{0\}$. Using this, we can show that R is interpretable. Consider, on $(H \cup \{0\})^n$ the equivalence relation defined by $(h_1, \ldots, h_k) \sim (g_1, \ldots, g_k)$ if and only if $\sum h_a = \sum g_i a$. Naturally, we define $\bar{h} \oplus \bar{g} = \bar{l}$ if and only if $\sum h_i a + \sum g_i a = \sum l_i a$. Also, we define $\bar{h} \otimes \bar{g} = \bar{l}$ if and only if $\sum \sum h_i g_i a = \sum l_i a$. Then $R \cong (H \cup \{0\})^n / \sim, \oplus, \otimes)$.

Claim 4.2.6. R is a field.

Proof. 4.2.6 Given any $r \in R$, with $r \neq 0$, take $b\ inA$ with $rb = 0$. Then $\forall h \in H$, $r(hb) = (rh)(b) = (hr)(b) = h(rb) = 0$. That is, $Ker(r)$ is H-invariant. But, we know that $\tau(Ker(r)) < n$. Now we can apply Lemma 4.2.2. So, there is some $c \in A$ with $rc = a$. Then for some $h_i \in H$,

$$c = \sum h_i a$$

so

$$r \sum h_i a = a.$$

But, as we have seen, an element of the ring R is uniquely determined by its action on a. But, then $r \sum h_i = 1$. □

By the superstable analogue of Macintyre's theorem, a superstable field is algebraically closed (see (18) or see (44) for the ω-stable version). In the differential setting, we know precisely that any interpretable field is actually the kernel of some set of C-linear combinations of the derivations of Δ, where C is the field of absolute constants (85). Thus, in this case, the group A is isomorphic to the additive group of such a definable field. $\square$

Remark 4.2.7. Moshe Kamensky notices that by the above construction, H is forced to be a subgroup of the multiplicative group of the field which we interpret. So, the interpretable fields are all kernels of some subset of linear combinations (over the absolute constants) of the distinguished derivations. The additive and multiplicative groups of these fields are almost simple. So, it is impossible for H to be a proper subgroup of the multiplicative group, thus, the only way to satisfy the hypotheses of the theorem is for H to be isomorphic to the multiplicative group of the field.

The proof of the following theorem almost identical to the proof given in (44) 7.3.12, with the appropriate changes ("finite" translates as "of lower Δ-type").

Theorem 4.2.8. *If G is a strongly connected solvable Δ-group with $Z(G)$ of Δ-type strictly less than G, then G interprets an algebraically closed field of the same Δ-type as G.*

Proof. We will do induction on the typical differential dimension of G. Since the type of $Z(G)$ is strictly less than that of G, we know that $G/Z(G)$ is a strongly connected (Cassidy and Singer prove that quotients of strongly connected groups are strongly connected (16)), centerless differential algebraic group of the same type and typical dimension as G. To see that $G/Z(G)$ is centerless: let $a \in G$ be such that $aZ(G)$ is in $Z(G/Z(G))$. Then for any $g \in G$, $a^{-1}g^{-1}ag \in Z(G)$, since a is central in the quotient. This means that $a^{-1}a^G$ is in the center of G. But, a^G is then of lower differential type, via the hypotheses of the theorem. But, we have a definable bijection $a^G \to G/C_G(a)$. Since G is strongly connected, $C/C_G(a)$ being of lower differential type means that it must be the identity. So, without loss of generality, we will assume that G itself is centerless (the quotient is clearly interpretable).

Now, we take A to be a minimal definable normal subgroup which is of the same differential type as G. By the minimality condition, A is almost simple. In (23), we showed that the commutator of a strongly connected group is a differential algebraic subgroup of the same type or the group is perfect. But, A is solvable and almost simple, so we must have that $[A, A] = 1$, that is A is commutative.

Next, we consider $C_G(A)$, which must not be all of G, since G is centerless and A is commutative. So, we let $G_1 = G/C_G(A)$. Then G_1 inherits the G action on A by conjugation, since under this action $C_G(A)$ is the kernel. Since we quotiented by $C_G(A)$, the action is faithful. By the minimality conditions on A, we know that there are no invariant subgroups of A with the same differential type. Of course, G_1 is also solvable and of typical differential dimension less than that of G, since $A \subseteq C_G(A)$ is of the same differential type as G (for behavior of typical differential dimension in quotients, see (53) or (16)).

Now, if G_1 has a center of strictly lower differential type, we may apply induction and interpret a field. So, assume $\tau(Z(G_1)) = \tau(G_1)$. Thus, we will let H be a minimal definable subgroup of $Z(G_1)$ of the same differential type. Then, again, H acts faithfully on A via conjugation. In the case that there are no proper H-invariant differential algebraic subgroups of A,, we may apply Theorem 4.2.3 to get the conclusion. If not, then we let B be a proper definable H-invariant subgroup of A. Then let H_0 be the subgroup of H which acts trivially on B. If $H = H_0$, then, because we know B is a minimal H-invariant subgroup, we know B is indecomposable (in the sense of (23)) and so are the groups B^g for all $g \in G_1$. But, since $H \subseteq Z(G_1)$, we know that for all $g \in G_1$, $h \in H$, and $b \in B$, $b^g = (b^g)^h$.

So, we can see H acts trivially on B^g. Then by the indecomposability theorem for differential algebraic groups, $\langle B^g \mid g \in G_1 \rangle$ is a definable G_1-invariant subgroup of A of the same differential type (since B is of the same type as A). This is impossible unless the group generated by the conjugates of B is actually A. But, H acts trivially on each conjugate of B and H does not permute the conjugates. So, H acts trivially on A.

So, now we know that H_0 is a proper subgroup of H. But, by the minimality assumptions on H, we know H_0 must be of lower type. So, taking the quotient H/H_0, we have a faithful H/H_0 action on B with no invariant subgroups of the same type as B. Note that since we are taking the quotient by a subgroup of lower type (H_0), if there were no invariant subgroups of differential type equal to B before taking the quotient, then there are none after taking the quotient. Now we apply Theorem 4.2.3 to interpret an algebraically closed field. $\square$

The next two lemmas are due to Berline (5) in the superstable context and previously appeared at least in the work of Zilber and Cherlin in the finite Morley rank context. The partial differential version of the first lemma appears in (16) and (23). The groups which appear in are assumed to be differential algebraic groups.

Lemma 4.2.9. *If H is a strongly connected subgroup of G is a strongly connected Δ-group, then for all $g \in G$, the Δ-type of g^H is either equal to the Δ-type of H or $H \leq C_G(g)$.*

Proof. There is a bijection between g^H and $H/C_G(g) \cap H$. H is strongly connected, so quotients of lower type are trivial. □

A natural consequence of this lemma is that every normal Δ-subgroup of smaller Δ-type in a strongly connected Δ-group is actually central, which was noted in (16).

Definition 4.2.10. A group is nilpotent if there is a chain of normal subgroups

$$G = G_0 \trianglerighteq \ldots \trianglerighteq G_n = 1$$

such that the successive quotients, $G_i/G_{i+1} \leq Z(G/G_{i+1})$. The lower central series of G is the chain of subgroups defined by $\Gamma_0(G) = G$ and $\Gamma_{n+1}(G) = [\Gamma_n(G), G]$. The upper central series is defined by letting $Z_0(G) = 1$ and $Z_n(G) = \{g \in G \mid g/Z_{n-1} \in Z(G/Z_{n-1})\}$.

It is a basic fact of group theory that a group is nilpotent if and only if the lower central series eventually reaches the identity if and only if the upper central series reaches G.

Lemma 4.2.11. *Let G be a strongly connected Δ-group. Then the ascending central series is eventually constant. For n such that the series has stabilized, $G/Z_n(G)$ is centerless.*

Proof. There is some m such that for all $n \geq m$, the Δ-type of $Z_{n+2}(G)/Z_n(G)$ is strictly less than the Δ-type of G. $G/Z_n(G)$ is strongly connected, and by the previous lemma, $Z_{n+2}(G)/Z_n(G)$ is normal and central. But, then $Z_{n+1}(G) = Z_{n+2}(G)$. □

One should note that (16) proves many basic facts about strongly connected Δ-groups, including that the class is stable under quotients. So, in the previous lemma $G/Z_n(G)$ is strongly connected. Assuming less connectivity is also viable for certain applications, see (23) for a discussion and examples. More generally, quotients of n-connected Δ-groups are n-connected.

Now we can prove the partial differential analogue of Theorem 7.3.15 of (44).

Theorem 4.2.12. *If G is an strongly connected, solvable, nonnilpotent differential algebraic group, then G (using only the group operation) interprets an algebraically closed field of the same Δ-type as G.*

Proof. Consider $G/Z_n(G)$ such that the upper central series has stabilized (previous lemma). Since G is not nilpotent, $G/Z_n(G)$ is a nontrivial strongly connected centerless Δ-group. Now we can apply Theorem 4.2.8 to get the result. □

Remark 4.2.13. Configuration theorems as given above in which one ends up interpreting an algebraically closed field might lead to impossibility theorems about the binding groups (definable groups of automorphisms) in the differential context.

4.3 Groups of typical dimension one or two

Cassidy and Singer take the approach of investigating almost simple differential algebraic groups of type one in additional detail (compared to the general case). Here, we will take what might be considered an orthogonal approach, allowing the type to be arbitrary, but assuming the typical dimension is small. The argument follows the outline of (17) or (6) with differential type and typical differential dimension replacing Morley rank (of (17)) or U-rank (of (6)). Even though some of the arguments are similar, we reproduce them here for convenience and because while the idea of using U-rank as an analogue of Morley rank is well-established, there have been relatively fewer attempts to apply model theoretic techniques to differential algebraic ranks. Specifically, the purely group theoretic arguments are almost entirely due to (17). The first result can be thought of as a more refined version of Reineke's theorem (in the differential setting), see section 7.2 of (44).

Proposition 4.3.1. *Suppose G is a Δ-group of Δ-type n. Then G has an abelian Δ-subgroup of type n.*

Proof. Consider the collection of Δ-subgroups of type n. Among this collection, choose one such connected subgroup H with minimal Kolchin polynomial. Now, assume that H is not commutative. Then take $a \in H$ which is not in the center of H. We know that $C_G(a) \cap H$ is of Δ-type less than n. This means that $H/C_G(a) \cap H$ is of Δ-type n and has the same typical Δ-dimension as H. Of course, this is also the Δ-type and typical Δ-dimension of a^H. Further, this is a generic subset of H (for a discussion of generic subsets in Δ-groups, see (23) or (22)). For any $b \in H$, a^H and b^H are either equal or disjoint. But, any two generic sets in a connected differential algebraic group intersect, the group H has only one conjugacy class which is not a singleton.

The rest of the argument would work in an arbitrary stable group, as (6) points out. Now we consider $H/Z(H)$. The action of this group on itself via conjugation is transitive, by the above arguments. Suppose that the elements are all of order 2; this forces H to be abelian, a contradiction. So, the square of any noncentral element of H is noncentral. Consider an element b such that $a = ba^{-1}b^{-1}$. Then $a = b^2ab^{-2}$. This means that $a \in C_H(b^2)$, but $a \notin C_H(b)$. Of course now we have a strictly decreasing sequence of differential algebraic groups given by $C_G(b^{2^n})$, a contradiction. □

Corollary 4.3.2. *Any almost simple differential algebraic group of typical dimension 1 is commutative.*

Lemma 4.3.3. *Suppose H is strongly connected and $a_\tau(H) = 2$. Then H is solvable (in two steps) or $H/Z(H)$ is simple.*

Proof. Consider the Cassidy-Singer series of H. This series is either length one or two. Suppose the series is length two. Then there is a strongly connected normal subgroup K (which is necessarily of typical dimension one). The quotient H/K is also of typical dimension one. By Proposition 4.3.1, we know that both K and H/K are abelian, so H is solvable.

In the case that such a K does not exist, every proper normal subgroup is of lower type. By Proposition 2.13 of (16), any normal subgroup of smaller type is central. Thus, $H/Z(H)$ is simple. □

Lemma 4.3.4. *There are no simple strongly connected differential algebraic groups with typical differential dimension 2.*

Proof. Suppose G is a counterexample. Then by Proposition 4.3.1 we know that there is a commutative differential algebraic subgroup of type 1. The typical dimension of this subgroup is one. Choose a minimal such subgroup, A, satisfying this criterium. Now, let $N = N_G(A)$ be the normalizer of A in G. By assumption, $N \neq G$. Now, take some $g \in N - A$. Consider $A \cap A^g$.

This differential algebraic subgroup must be of lower type than G by the hypotheses regarding the choice of A. So,

$$a_\tau(A.A^g) = 2$$

If $a \in A \cap A^g$, then $C_G(a)$ contains both A and A^g, and thus their product. This is a contradiction to the strong connectedness of G, since if $C_G(a)$ contains A and A^g it must be all of G, making $Z(G)$ nontrivial and contradicting the simplicity of G. So, the intersection must be the identity.

As we vary $g \in G - N$, the sets $A.g.A$ are either equal or distinct. But $A.g.A$ contains the generic type of G, so each of these sets must be equal. So,

$$G = N \cup A.g.A$$

Claim 4.3.5. $G - N$ contains an involution and $N = A$.

Proof. Take some $g \in G - N$. Then, we know $x^{-1} = a_1 x a_2$ for some $a_i \in A$, since x^{-1} is not in N. Now,

$$(xa_1)^2 = a_1 a_2^{-1}$$

So, $(xa_1)^2 \in A$. Then $(xa_1)^2 = ((xa_1)^2)^{xa_1} \in A \cap A^w = 1$. So, we have our involution. Consider $K = N \cap A^{xa_1}$. We claim $N = A \backslash K$. We know that K normalizes A, and that their intersection is the identity, so to determine that the group in question is the semidirect product, we need only to know that $N = AK$. Take $n \in N$. $nxa_1 \notin N$. But then

$$nxa_1 = a_3xa_1a_4$$

and $a_3^{-1}nxa_1 = a_4^{nxa_1} \in A^{nxa_1} \cap N = K$. So, $N = AK$. Next, we show that $K = 1$, which means that $N = A$. If $a \in K^{xa_1}$, then since xa_1 is an involution, $a \in A$ and $a^{xa_1} \in N$. Next we will show that for all $g \in G$, $a^g \in N$. So, take $g = a_3xa_1a_4$. Then $a^g = (a^{xa_1})^{a_2}$ which is in N.

Next, we will let $B = \langle a^g : g \in G \rangle$. $B \lhd G$ and $B \subseteq N$. B of the same type as G, since otherwise $G/C_G(a)$ is of type less than G, contradicting the strong connectedness of the group. By the above work A is of finite index in its normalizer, and so both $A \cap B$ and $A^{xa_1} \cap B$ are finite index in B. But then so is their intersection. Of course, this means that B must be finite, a contradiction. □

Claim 4.3.6. A is a maximal proper differential algebraic subgroup of G.

Proof. Let H be a differential algebraic subgroup containing A. Since G is strongly connected, $a_\tau(H) = 1$. But then for all $h \in H$,

$$a_\tau(A.h.A) \leq a_\tau(H) < 2.$$

But, this means that $h \in N_G(A) = A$. □

Claim 4.3.7. $G = \cup_g A^g$.

Proof. If we consider $b \in G - \cup A^g$, then $C_G(b)$ is of type one, since the conjugacy class of b is of the same type as G. Now, consider the strongly connected component of $C_G(b)$, and call this group B. Since b is not in the center of G, we know that $\tau(B) = 1$ and $a_\tau(B) = 1$. Now, B may be analyzed in the same manner as A. Now, we can see that both $\cup B^g$ and $\cup A^g$ are generic subsets of G. This means that they intersect generically, which implies that $A \cap B \neq 1$. Now, for an element in the intersection, b_1, we know that $C_G(b_1)$ is equal to both A and B, which, of course, contradicts the choice of b. □

Now, take $b_1 \in G - 1$. By the previous arguments, $C_G(b_1)$ is a strongly connected abelian differential algebraic subgroup. Take an involution $w \in G - C_G(b)$. By the last claim, we can see that w is conjugate via some element in g to an element $w_1 \in C_G(b)$. From here the argument is identical to (17):

$$w_1 w \neq w w_1$$

$$(w_1 w)^{w_1} = (w_1 w)^{-1}$$

Consider $B = C_G(w_1 w)$. Then we can see that $w_1 w \in B \cap B^{w_1}$, so we know that $B = B^{w_1}$. This means that $w \in B$. This is a contradiction, since then $w_1 w = w w_1$.

□

Now we have also proved:

Theorem 4.3.8. *There are no nonsolvable strongly connected differential algebraic groups of typical differential dimension 2.*

4.4 Typical differential dimension two and nilpotence

Cherlin's analysis (17) continues via analyzing the nonnilpotent groups of Morley rank 2. In some essential ways, that analysis uses the finiteness conditions imposed by the hypotheses of finite Morley rank. Those finiteness conditions are not available in our setting, but the hypothesis of strong connectedness is stronger than the condition of connectedness.

Theorem 4.4.1. *Let G be a strongly connected differential algebraic group of typical differential dimension 2 which is centerless. Then for some definable field F, G is the semidirect product of F_+ and $F^{\cdot}$ with $F^{\cdot}$ acting on F_+ via multiplication.*

Proof. By Proposition 4.3.1, we can find an abelian differential algebraic subgroup H of G with $\tau(H) = \tau(G)$. Further, we may assume that H is strongly connected, normal (by the results of section three), and $a_\tau(H) = 1$. In particular, this means that H is almost simple. Now consider $b \in G - C_G(H)$. By the arguments of Lemma 4.3.4, we know that $\tau(C_G(b)) = \tau(G)$. So, in particular, we may take T to be the (nontrivial) $\tau(G)$-connected component of $C_G(b)$. Since they are both of the same type, T is the strongly connected component of $C_G(b)$. Both H and T are strongly connected of typical differential dimension 1. So,

$$\tau(G \cap H) < \tau G.$$

Then by Lemma 3.1 of (53), we know

$$a_\tau(HT) = a_\tau(G).$$

But then HT is a closed set of differential type and typical differential dimension equal to that of G, so by the irreducibility of G, $HT = G$.

Since $Z(G) = 1$, $H \cap T = 1$. Now consider $C_G(h)$ for $h \in H - 1$. Because G is strongly connected and centerless, we know that conjugacy classes of elements of G under conjugation are of differential type $\tau(G)$. So, by the almost simplicity of H, every element $h_1 \in H$ is of the for h^t for some $t \in T$. Now suppose that $t \in C_G(h)$ for some $t \in T$. T is abelian by 4.3.1, so $t = t^{t_1}$ for any $t_1 \in T$. Then t centralizes h_1^t for all $t_1 \in T$, which contradicts the fact that T acts transitively on $H - 1$ via conjugation. Thus,

$$t \mapsto h^t$$

is a bijection from T to $H - 1$.

We will define addition on T via

$$x + y = z$$

if and only if

$$h^x h^y = h^z.$$

We can, of course, add a symbol to T for 0, and assume that $h^0 = 1$. This is a commutative and associative operation. We would like to prove that T is a ring under this operation and the multiplication given by the group operation. Define $-x$ to be z such that $-h^x = h^z$. We must prove that the operations are distributive, that is,

$$z(x+y) = zx + zy$$

Suppose that $x + y = z_1$. Then

$$\begin{aligned} h^{zz_1} &= zz_1hz_1^{-1}z^{-1} \\ &= z(u^x u^y z^{-1} \\ &= zu^x z^{-1} zu^y z^{-1} \\ &= u^{zx} u^{zy} \end{aligned}$$

So, $z(x+y) = zx+zy$. Now superstability theory does the rest. By (18) this ring is actually an algebraically closed field. Further, in differentially closed fields, the definable fields are actually the kernels of subsets of definable derivations (see (85)). We know, of course that $T \cup \{0\}$ and H are isomorphic as differential fields (with the definable isomorphism given above by the group operation). $\square$

Next, as in the analysis of the previous section, we drop the condition of centerless.

Theorem 4.4.2. *Let G be a nonnilpotent strongly connected differential algebraic group with $a_\tau(G) = 2$. Then $G = H \setminus T$ with $H = F_+$ and $T/Z(G) = F^{\cdot}$ where F is an algebraically closed (definable) field. Conjugation of H by T is given via multiplication in F.*

Proof. We know that $G/Z(G)$ satisfies the hypotheses of the previous theorem, so

$$G/Z(G) \cong F_+ \setminus F^{\cdot}$$

as in that theorem. So, let H and T be the strongly connected components of the inverse images of F_+ and $F^{\cdot}$, respectively, under the natural quotient map. Then HT is of typical differential dimension two (see the proof in the previous theorem). So, $HT = G$. We claim that $Z(G) \cap H = 1$. Suppose not and take $h \in Z(G) \cap H$. . Then for some t_1 and t_2 in T,

$$h^{t_1} = z + h^{t_2}$$

so $t_1 t_2^{-1} \in Z(G) \cap T$, since modulo $Z(G)$ they are the same element in $F^{\cdot}$. But then $u^{t_1} = u^{t_2}$ so $z = 1$. □

4.5 Typical Δ-dimension 3

Throughout this entire section, we assume that G is a strongly connected Δ-group of typical Δ-dimension 3. The techniques are adapted from (5) and (17). We should mention that since the main theorem in this section reduces to the case of analysis of an *almost simple* differential algebraic group, one can shorten the presentation considerably by assuming Cassidy's theorem that every simple group definable in $DCF_{0,m}$ is isomorphic to the C'-rational points of an algebraic group where C' is a definable subfield (45, see Pillay's article). We have chosen not to do this for several reasons which are discussed below.

Proposition 4.5.1. *G is either solvable or G is almost simple.*

Proof. This is clear since a normal subgroup which witnesses non-almost simplicity means that we have the group separated into an Abelian piece and a solvable piece by a simple dimension count and the results of the previous sections. □

Lemma 4.5.2. *If G has a nilpotent subgroup H with $\tau(H) = \tau(G)$ and $a_\tau(H) = 2$, then G is solvable.*

Proof. Assume H is strongly connected (if not, replace H by its strongly connected component). If H is normal in G, then $a_tau(G/H) = 1$ and G/H is strongly connected and thus abelian. So, G is solvable.

If H is not normal then let $a \in G\backslash N_(H)$. By **??** $a_\tau(H \cdot H^a) = 3$. The only elements of G which centralize H and H^a are those in $Z(G)$. $\tau(H \cap H^a) = \tau(G)$. Let H_1 be the strongly connected component of $H \cap H^a$. Either H_1 is not in $C_G(H)$ or $C_G(H^a)$. Then $\tau(H_1 \cap Z(H)) < \tau(G)$. H is nilpotent, so $\tau(Z(H)) = \tau(H)$ and $a_\tau(H_1 \cdot Z(H)) = 2$. So, $H = H_1 \cdot Z(H)$. But, this forces H to be abelian and $H_1 \subseteq H \cap H^a$. □

In order to proceed in the manner of (5; 17), one must either prove or assume that G has a subgroup H with $\tau(H) = 2$ and $a_\tau(H) = 2$. This condition is the analogue of what is called *good* by both (5; 17). Let us briefly explain why any almost simple group G of typical dimension 3 has this property. This is the one place in which we use Cassidy's theorem. Proving the result independently ought to be possible, but we will not pursue it here. Let G be almost simple with $a_\tau(G) = 3$. Then we have the exact sequence (see chapter 3 of this thesis):

$$1 \to Z(G) \to G \to G_1 \to 1$$

where G_1 consists of the $\mathcal{F}$ rational points of a simple algebraic group of dimension 3 for some definable subfield $\mathcal{F} \subseteq \mathcal{U}$. Such a G_1 is good in the notation of (5; 17). Let H be the strongly connected component of the inverse image of a dimension 2 algebraic subgroup. Then, $a_\tau(H) = 2$ and $\tau(H) = \tau(G)$. By results of the previous section, H is solvable, and by the previous lemma, we may assume that H is not nilpotent.

But then we can get H and T as in Theorem 4.4.2. Then letting T_1 be the strongly connected component of the identity of T, we can see that $G_2 = HT_1$ is a definable subgroup of G_1 (by the Δ-indecomposability theorem of (23)). In fact, G_2 is the strongly connected component of the identity in G_1. So, in what follows, simply assume G_1 was chosen to be strongly connected. Then the commutator $[G_1, G_1] = H$. and $Z(G_1) = Z(G_1) \cap H$. The proof of the following theorem roughly follows the proof of an analogous theorem in (17). It is also similar to a proof of an analogous result in (5), which is also based on the argument in (17).

Theorem 4.5.3. *Let G be a nonsolvable strongly connected group of typical Δ-dimension 3. Then G is isomorphic to $SL_2(\mathcal{F})$ or $PSL_2(\mathcal{F})$, where $\mathcal{F}$ is a definable subfield of $\mathcal{U}$.*

Remark 4.5.4. The definable subfields of $\mathcal{U}$ are completely classified. See chapter 2 section 4 of this thesis.

Proof. By the previous two lemmas, G can be assumed to have a solvable nonnilpotent subgroup G_1 with $a_\tau(G_1) = 2$. Assuming G_1 is strongly connected, along with the results of the previous section means we know $G_1 \cong H \setminus T$ with $H \cong F_+$ for some definable field F, $Z(G_1) \subset T$, $T/Z(G_1) = F.$, and the action of $T/Z(G_1)$ on H is given by field multiplication.

Claim 4.5.5. The strongly connected Δ-subgroups of G_1 with Δ-type equal to $\tau(G_1)$ either contain H or are equal to T^g for some $g \in H$.

Proof. (Claim 4.5.5) It suffices to show that two statements:

1. $T/Z(G_1) \cong F.$. T is a maximal proper definable subgroup of G_1.
2. Any two definable strongly connected groups of type $\tau(G_1$ which are not equal to H are conjugate.

To prove the first statement, let $C \supseteq T$ be a Δ-subgroup of G_1. Then since $G_1 = HT$, so if $C \neq T$, there is $v \in C \cap V$ which is not the identity. By results of the previous section, any two nontrivial elements of H are conjugate by an element of T. Thus $H = v^H \cup \{1\}$. So, $H \subseteq C$ and $C = B$.

For the second statement, let $D \neq H$ be strongly connected. Then $HD = B$ by the indecomposability theorem. $Z(B) \subseteq D$, since $Z(B) \cap H = \{1\}$. We claim that $H \cap D = H \cap N_B(D) = \{1\}$. Every element of $H \cap D$ centralizes both V and D (these are abelian groups, by our analysis of groups of typical dimension 1). So, if $v \in V \cap N_B(D)$, then $[v, D] \subseteq H \cap D$, so v centralizes D. Thus v is trivial. Now note that

$$\tau(\bigcup_{v \in H} D^v - Z(B)) = \tau(G_1),$$

$$a_t au(\bigcup_{v \in V} -Z(B)) = 2$$

The above arguments also apply to T, so

$$(\bigcup_{v \in H} D^v - Z(B)) \cap (\bigcup_{v \in H} T^v - Z(B)) \neq \emptyset$$

Thus there are $v_1, v_2 \in H$ so that $T^{v_1} \cap D^{v_2} \neq Z(G_1)$. Then there is $d \in D \cap T^v) - Z(G_1)$ where $v = v_2 v_1^{-1}$. Note that $d \in C_{G_1}(T^v)$ and $d \in C_{G_1}(D)$, but $d \notin Z(G_1)$, so it must be that $T^v = D$. So, we have established the claim. □

Claim 4.5.6. $\forall x \notin N_G(G_1)$, $a_\tau(BxB) = 3$ and $\tau(B \cap B^x) = \tau(G)$ and $a_\tau(B \cap B^x) = 1$.

Proof. (Claim 4.5.6) The second statement follows from the assumptions, and the first statement follows from noting that BxB is in definable bijection with $B \cdot B^x$. □

Claim 4.5.7. $\forall x \notin N_G(G_1)$, $G = N_G(G_1) \cup G_1 x G_1$.

Proof. (Claim 4.5.7) If $x \in N_G(G_1)$ then BxB contains the generic of G. Then since two double cosets are necessarily disjoint or equal, and generic subsets can not be disjoint, $\forall x, y \in N_G(G_1)$,

$$G_1 x G_1 = G_1 y G_1$$

Now the claim follows since the union of the double cosets always covers $G - N_G(G_1)$. $\square$

Claim 4.5.8. $N_G(T) \not\subset N_G(G_1)$.

Proof. (Claim 4.5.8) The smallest normal subgroup of G containing H is G. $\exists g \in G$ such that $H^g \not\subset G_1$. By work of the previous section, $H = G_1'$ is characteristic in G_1, so such a g is not in $N_G(G_1)$. Applying 4.5.7 gives $b, c \in G_1$ such that $g^{-1} = bgc$. Then we get that $H^{g^{-1}} \not\subset G_1$. So, $G_1^g \cap G_1 \neq G_1$. By 4.5.6,

$$\tau(G_1^g \cap G_1) = \tau(G)$$

so by 4.5.5 there is $v \in H$ such that $G_1^g \cap G_1 = T^v$. Thus $T^v \subset G_1^g$ so $T^{g^{-1}v} \subset B$. Also, $T^{g^{-1}v}$ cannot contain H, since otherwise T^v would contain H^g, which (as we showed above) cannot happen.

Apply 4.5.5 to get $u \in H$ such that

$$T^{g^{-1}v} = T^u$$

and thus

$$w = v^{-1} g u \in N_G(T)$$

then $u, v^{-1} \in G_1$ and $g \notin N_G(G_1)$, so $w \in N_G(T) - N_G(G_1)$, proving the lemma. $\square$

Claim 4.5.9. $\forall w \in N_G(T) - N_G(G_1)$, $G_1 \cap G_1^w = T$ and $G = G_1 \cup G_1 wH$. The decomposition of any element of G, but not in G_1 in the form $gwu \in G_1 wH$ is unique.

Proof. (Claim 4.5.9) Fix $w \in N_G(T) - N_G(G_1)$. Then $T \subset G_1 \cap G_1^w$, and by 4.5.5, $B \cap G_1^w = T$. Then $G_1 = HT = TH$, so $G = N_G(G_1) \cup G_1 w G_1$ by 4.5.8. So,

$$G = N_G(G_1) \cup G_1 wH$$

Further, assume that $gwv = g'w'v' \in G_1 wH$. Then $vv'^{-1} = w^{-1} g^{-1} b' w' \in H \cap G_1^w = H \cap T = \{1\}$. Thus the decomposition is unique. $\square$

Claim 4.5.10. $N_G(G_1) = G_1$ and $N_G(T) \cap G_1 = T$.

Proof. (Claim 4.5.10) If $b \in N_G(T)$, then $[b, T] \subset T$. Moreover $b \in G_1$ implies $[b, T \subset H = G_1'$. So, $b \in N_G(T) \cap G_1$ implies that $[b, T] = \{1\}$ and $b \in G_1 \cap C_G(T) = T$. Thus, $N_G(T) \cap G_1 = T$.

For any $c \in N_G(T) \cap N_G(G_1)$, $cw \in N_G(T) - N_G(G_1)$. Thus by 4.5.7,

$$cw = b^{-1} x v$$

for some $b \in G_1$ and $v \in H$. Then $T^b = T^{wv}$ is in $G_1 \cap G_1^w$ and is thus equal to T, by arguments in the proof of 4.5.7 and claim 4.5.5.

So, b and wv are in $N_G(T)$. Then $v \in N_G(T)$. But, then, by the above arguments, $b, v \in T$. But, then we know $v = 1$ and $c = b$. Thus, we see

$$N_G(T) \cap N_G(G_1) = T.$$

□

Claim 4.5.11. $Z(G) \subset T$

Proof. (Claim 4.5.11) $Z(G)$ normalizes G_1 and hence $Z(G) \subset G_1$. So, $Z(G) \subset Z(G_1) \subset T$. □

Claim 4.5.12. Let $w \in G$ be such that $w^2 \in T$. Then $\forall t \in T$, $t^w = t$ or for all $t \in T$, $t^w = t^{-1}$.

Proof. (Claim 4.5.12) Define $\sigma : T \to T$ by $\sigma(t) = t^w t$. Then $\sigma(t)^w = \sigma(t)$. Now $\tau(Ker\sigma) < \tau(G)$ implies that $im\sigma = T$ and $\sigma(t) = t$. On the other hand, if the kernel of $sigma$ is fo the same type as T, then $T = ker\sigma$ and $\sigma(t) = t^{-2}$. □

The next claim is proved in (17). It involves nothing about type or typical differential dimension. The claim depends only on the group decompositions we have set up. We reproduce the proof here for convenience.

Claim 4.5.13. There is $w \in N_G(T) - T$ such that $\forall t \in T$, $t^w = t^{-1}$, $w^2 \in T$, and $w^2/Z(B)$ is either -1 or 1 as an element of the interpretable field F..

Proof. (Claim 4.5.13) Let $x \in N_G(T) - T$ and let $b_1, b_2 \in G_1$ be such that $x^{-1} = b_1 x b_2$. $T^{b_2^{-1}} = T^{x^{-1} b_2^{-1}} = T^{b_1 x} \subseteq G_1 \cap G_1^x$. Thus, $T = T^{b_2^{-1}} = T^{b_1 x}$, so $b_1, b_2 \in T$. Then let $w = x b_1$. Then $w^2 = b_2^{-1} b_1 \in T$. Now $w \in C(T)$ or $t^w = t^{-1}$.

If $t = w^2$, then recall that $T/Z(G_1)$ is isomorphic to the multiplicative group of an algebraically closed field. So, let b be such that $b^2 + Z(G_1) = 1/(t + Z(G_1))$. Then $(wb)^2 = 1$.

Now, if $t^w = t^{-1}$ then if $t = w^2$, then $t = t^w = t^{-1}$, and so $t^2 = 1$. Then $t + Z(G_1) = \pm 1$.

Now we prove that we can take $w \notin C_G(T)$. Take $a \in H$ with $waw \notin G_1$. Then $waw = u_1 t_1 w u_2$ with

□

Claim 4.5.14. $G/Z(G) \cong PSL_2(F)$.

Proof. (Claim 4.5.14) The proof of theorem 1 of section 5 (page 23) of (17) works in this setting without modification. We have essentially constructed the *Bruhat* decomposition of the group. □

The next claim is proven by (5) and is implicitly used by (17).

Claim 4.5.15. $G \cong SL_2(F)$ or $PSL_2(F)$.

Proof. (Claim 4.5.15) We now know that G is a perfect central extension of $PSL_2(F)$. By 4.5.13 and 4.5.11 we know that $Z(G)$ is a group of exponent two. By results of (46), any perfect group G which is a central extension of $PSL_n(F)$ and is of bounded exponent is a homomorphic image of $SL_n(F)$. From this, the claim follows and G is either $SL_n(F)$ or $PSL_2(F)$. □

This also completes the proof of Theorem 4.5.3.

□

Remark 4.5.16. One can replace all but the last claim of the theorem (which needs algebraic K-theory) by Cassidy's theorem (as discussed above) along with the classification of quasi-simple algebraic groups. Of course, the final claim uses several preliminary structure results. In fact, the entire argument can be replaced by Cassidy's theorem, and the results of (5) after noting that Cassidy's theorem implies that for this group almost simplicity is identical to indecomposability in the sense of (6). In fact, if one is only interested in noncommutative almost simple differential algebraic groups many techniques in the present chapter can be replaced by Cassidy's theorem (14) and the machinery of superstability theory.

We chose the above method of proof for several reasons. This method is fairly self-contained with respect to this thesis; the proof of Cassidy's theorem is quite involved. The argument also shows how directly one can translate theorems from the superstable or finite Morley rank case to the case of differential algebraic groups when we assume strong connectedness. The analogy goes through considering the leading coefficient of the Kolchin polynomial in the same manner we consider the leading coefficient of the highest power of the ω contained in the Cantor normal form of the Lascar rank of the group. The analogy has potential for extension beyond the realm of superstability or even supersimplicity. Even in the (supersimple) case of difference-differential fields, we know that there are no bounds implied on the degree and leading coefficient of the analogue of the Kolchin polynomial in terms of the Lascar rank (for general definable sets, at least). In settings outside of differential fields, when there are multiple operators, we know of no examples of analogues of Cassidy's theorem (14), which is what would make the application of superstability theory possible.

CHAPTER 5

ISOGENY FOR SUPERSTABLE GROUPS

5.1 Introduction

The main goal of this chapter is to develop notions of *strong connectedness*, *almost simplicity* and *isogeny* for the class of superstable groups, in analogy to the related notions for algebraic groups. In this chapter, notions like simple, quasi-simple, and almost simple are group theoretic notions and have nothing to do with the similarly named model theoretic property of first order theories. We will then use the notion of isogeny to prove results of the form "the construction is unique up to isogeny". For an example from algebraic groups, see the Jordan-Hölder theorem. The other guiding example will be the Cassidy-Singer analysis of differential algebraic groups. There are two steps in (16) which provide a compelling reason to view *almost simple* differential algebraic groups as the basic building blocks of all differential algebraic groups. First, every differential algebraic group has a subnormal series in which the quotients are almost simple. Second, this decomposition is unique up to isogeny (not isomorphism) and permutation of the quotients.

As stated in the introduction, the model-theoretic prerequisites are slightly greater for this chapter, where we work in the superstable setting, which was not discussed in the introduction. The results here specialize to the known results in algebraic groups. The algebraic groups case also inspired the work of Cassidy and Singer in the differential setting. Many of the proofs in this chapter are translations of proofs from these cases, generalized and modified appropriately. One interesting note is that while U-rank specializes to (Krull) dimension in algebraic groups, the notion of dimension that Cassidy and Singer use in their analysis is not U-rank in differentially closed fields. Cassidy and Singer use the gauge of the differential algebraic group, that is the pair (a_τ, τ), where a_τ is the typical differential dimension and τ is the differential type. From these differential birational invariants, one can formulate an upper bound for Lascar rank in differential fields. There is no known lower bound for Lascar rank in terms of these invariants (85). We will define a similar notion of gauge in the superstable setting.

Strong results on the structure of infinite rank superstable groups were first established in (6). Further model theoretic analysis continued over the next several years and is recalled in (66). Our purpose here is somewhat different from the existing model theoretic analysis. The basic notion we consider is *isogeny*. The notion is interesting in its own right, and we prove several results about the properties of isogeny. We hope to illustrate how to import techniques from differential algebraic groups into superstable groups, even when (as in this case) the results are not necessarily generalizations. This translation goes via thinking about Lascar rank in the way that differential algebraists think about the gauge of a differential algebraic group. Further, we hope this will lead to future work in model theory of fields with more general operators in which Lascar rank is either difficult to understand and calculate or is simply not available. The decomposition theorem proved here is close to the one proved by Baudisch (3). The quotients in our decomposition are almost simple and might have infinite centers; our decomposition is coarser than Baudisch's decomposition. Baudisch's paper does not mention the issue of uniqueness of the decomposition. The style and techniques for proving the decomposition theorem in this chapter follow developments from algebraic groups and differential algebraic groups much more closely than then development contained in (3).

5.2 Notation and Preliminaries

Throughout this note, G is a group definable or even type-definable in a monster model of a superstable theory T. We will heavily use the notion of *Lascar rank* on types, denoted $RU(p)$. Though this is a rank on types, one can abuse notation and denote, by $RU(G) = RU(p_G)$, where p_G is a generic type of G. For certain technical reasons, this might be somewhat problematic when dealing with arbitrary definable sets, but not when dealing with (type-)definable groups. For this paper, we will assume that α and β are ordinals such that $RU(G) = \omega^\alpha \cdot n + \beta$ where $\beta < \omega^\alpha$ (note that this is no restriction at all on the group G). Lascar rank (U-rank, RU) is the main tool used in this paper, and properly it is a rank on types. We abuse notation in a standard way and write $RU(X)$, where X is a (type-)definable set (usually a group, in fact). In this case, the Lascar rank of the set is the supremum of the Lascar ranks of the complete types which include the formula "$x \in X$."

A group is called type-definable if it is an intersection of definable subgroups. We will be assuming standard notation from superstable group theory except where we define new notation. Poizat's Stable Groups (66) is suggested as a reference for the notation which is not explicitly defined. The reader is advised that we will make frequent use of the Lascar inequality in particular. We emphasize that we are working in some fixed superstable theory T, and are calculating Lascar rank within that theory.

Definition 5.2.1. Define $\tau_U : \{Def(G)\} \to On$ to be the highest power α, such that ω^α appears in the Cantor normal form of the Lascar rank of definable set in question. G is *α-connected* if for every proper type-definable normal subgroup H of G, $\tau_U(G/H) = \tau_U(G)$.

We will also call α-connected groups *strongly connected.* G is *almost simple* if there is no type-definable normal subgroup H of G, with $\tau_U(G/H) < \tau_U(G)$.

Every group G has (type-definable) subgroups which are α-connected (though this condition may imply such a subgroup is the identity). When G is a saturated model of a superstable group, there is a unique maximal such subgroup, called the α-connected component of the identity (which is definably characteristic). Let

$$\mathcal{S} = \{H \subset G \,|\, RU(G/H) < \omega^\alpha\} = \{H \subset G \,|\, \tau_U(G/H) < \tau_U(G)\}.$$

Then the α-connected component is the minimal such subgroup (it is easy to show that S is closed under intersection). For more details and proofs, see (6, section IV).

Remark 5.2.2. Much of the above notation is not standard, but it is convenient for the purposes here. It is inspired by the notation of (16). The definition of α-connected agrees with that of (6). The following open question depends on the relationship between Lascar rank and gauge in differential algebraic groups:

Question 5.2.3. *Is a strongly connected differential algebraic group (strongly connected in the sense of differential gauge) actually strongly connected in the sense of Lascar rank?*

It is known, by results of Berline and Lascar (6, section IV), that G is α-connected if and only if $RU(G) = \omega^{\alpha_1} \cdot n_1 + \ldots + \omega^{\alpha_k} \cdot n_k + \omega^{\alpha} \cdot n$ and G is connected (in the traditional sense that there is no type-definable subgroup of finite index). So, when we consider strongly connected groups, we are limited to groups of monomial valued U-rank. In that case, being strongly connected is equivalent to being connected (i.e., no finite index definable subgroups).

Proposition 5.2.4. *Suppose that G is α-connected. Every type-definable normal subgroup, N, with $\tau_U(N) < \tau_U(G)$ is central.*

Proof. Consider the map $\alpha : G \times N \to N$ given by $(g, a) \mapsto gag^{-1}$. For any fixed $a \in N$, $\alpha_a(g) := gag^{-1}$ is a definable map from G to N, such that α_a is constant on left cosets of the centralizer of a, $Z_G(a)$. So, there is a definable map β, such that the diagram commutes,

$$\begin{tikzcd} G \arrow[r, "\alpha_a"] \arrow[d, "\pi"'] & N \\ G/Z_G(a) \arrow[ur, dotted, "\beta"'] & \end{tikzcd}$$

We note that $\alpha_a(g) = \alpha_a(h)$ implies that $h^{-1}g \in Z_G(a)$. Thus, β is injective. But, then $\tau_U(G/Z_G(a)) \leq \tau_U(N) < \tau(G)$, so $Z_G(a)$ must be all of G, since otherwise we have found a subgroup such that the U-rank of the coset space has leading monomial in its Cantor normal form less than $\tau_U(G)$. This means that the rank of $Z_G(a)$ is at least equal to the leading monomial. On the face of things, this should not force $Z_G(a)$ to be all of G, since we do not know that $Z_G(a)$ is a normal subgroup of G. But, in general, one now knows that the set S of subgroups H of G such that the coset space has rank less than ω^{α} is nonempty. The set

S is closed under intersections and the minimal element will be a type-definable characteristic (so normal) subgroup of G which shows that G is not α-connected. So, it must be that $G = Z_G(a)$. □

Proposition 5.2.5. *The image of a strongly connected group under a definable homomorphism is strongly connected or trivial.*

Proof. Suppose that the image is nontrivial and not strongly connected. Then taking the inverse image of the definable subgroup of the image which shows non-strong connectedness would show the non-strong connectedness of G itself. □

5.3 Isogeny

We remind the reader that we are working within a monster model of a superstable theory. As usual, all of the groups and maps between them are (type-)definable (perhaps with parameters) in the monster model. The notion of strongly connected (or α-connected - recall we assume that the Lascar rank of G has ω^α appearing as the leading term in its Cantor normal form) plays the role that connected plays in algebraic groups. Almost simple plays the role of quasi simple. Now we define isogeny in this setting.

Definition 5.3.1. Suppose that G and H are α-connected. Then a definable group homomorphism $\phi : G \to H$ is an isogeny if ϕ is surjective and $\tau_U(Ker\phi) < \tau_U(G)$. We say that H_1 and H_2 are isogenous if there are $\phi_i : G \to H_i$ which are isogenies.

We are not generally dealing with definability problems in this paper, so even if we do not explicitly say so, groups and homomorphisms are assumed to be type-definable.

Proposition 5.3.2. *Let G_1 and G_2 be α-connected subgroups. The following are equivalent:*

1. *There is an α-connected group H and isogenies $\phi_i : H \to G_i$:*

$$\begin{tikzcd} & H \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_2"] & \\ G_1 & & G_2 \end{tikzcd}$$

2. *There is an α-connected group K and isogenies $\psi_i : G_i \to K$:*

$$\begin{tikzcd} G_1 \arrow[dr, "\psi_1"] & & G_2 \arrow[dl, "\psi_2"'] \\ & K & \end{tikzcd}$$

Proof. Let H and ϕ_i be as in condition 1). Let $H_1 = \phi_1(ker\phi_2)$ and $H_2 = \phi_2(ker\phi_1)$. Then $H_1 = \phi_1(ker\phi_1 ker\phi_2)$, and $H_2 = \phi_2(ker\phi_1 ker\phi_2)$. Then

$$G_1/H_1 = \phi_1(H)/\phi_1(ker\phi_2) = \phi_1(H)/\phi_1(ker\phi_1 ker\phi_2) = H/(ker\phi_1 ker\phi_2).$$

$$G_2/H_2 = \phi_2(H)/\phi_2(ker(\phi_2)) = \phi_2(H)/\phi_2(ker\phi_1 ker\phi_2) = H/(ker\phi_1 ker\phi_2).$$

So, let $K = H/(ker\phi_1 ker\phi_2)$. K, being the image of an α-connected group H is α-connected. Further, $\tau_U(ker\phi_i) < \alpha$, so $\tau_U(ker\phi_1 ker\phi_2) < \alpha$. But, then letting ψ_i be the projection map $G_i \to G_i/H_i = K$, we have shown that ψ_i is an isogeny.

Now, assume condition 2). We let $G = \{(g_1, g_2) \in G_1 \times G_2 \,|\, \psi_1(g_1) = \psi_2(g_2)\}$. Then there are natural surjective projections $\phi_i : G \to G_i$. But, then we see that $\tau_U(G) \geq \tau_U(G_i)$. As the kernel of the projection maps, ϕ_i, are contained in $ker\psi_1 \times ker\psi_2$, the Lascar rank of the kernels of the maps is less than ω^α, since both of the groups in the product are (by virtue of ψ_i being an isogeny). So, ϕ_i is an isogeny. $\square$

Proposition 5.3.3. *Isogeny is an equivalence relation on the strongly connected type-definable groups. Let G and K be type-definable, strongly connected, isogenous groups. Then,*

1. *There is a bijection, r, between the type-definable strongly connected subgroups $G_1 \leq G$ with $\tau_U(G_1) = \tau_U(G)$ and those $K_1 \leq K$ with $\tau_U(K_1) = \tau_U(K)$.*

2. *Let $G_1, G_2 \leq G$ and $K_1, K_2 \leq K$ be strongly connected. Suppose that $r(G_1) = K_1$ and $r(G_2) = K_2$.*

   $G_1 \leq G_2$ if and only if $K_1 \leq K_2$.

   $G_1 \lhd G_2$ if and only if $K_1 \lhd K_2$.

3. *Let G_1, G_2, K_1, K_2 be as in 2). If $G_1 \lhd G_2$, then $\tau_U(G_2/G_1) = \alpha$, G_2/G_1 is strongly connected, and G_2/G_1 is isogenous to K_2/K_1.*

4. *Products of isogenous groups are isogenous.*

Proof. Reflexivity and symmetry of the isogeny relation are clear. Now, Suppose that H_1 is isogenous to H_2 and H_2 is isogenous to H_3. Then, we have a diagram of isogenies with α-connected K_1 and K_2:

$$\begin{tikzcd} & K_1 \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_2"] & & K_2 \arrow[dl, "\psi_1"'] \arrow[dr, "\psi_2"] & \\ H_1 & & H_2 & & H_3 \end{tikzcd}$$

But, by 5.3.2, we get the following diagram, with isogenies and α-connected L :

$$\begin{tikzcd} & & L \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\ & K_1 \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_2"] & & K_2 \arrow[dl, "\psi_1"'] \arrow[dr, "\psi_2"] & \\ H_1 & & H_2 & & H_3 \end{tikzcd}$$

For H_1 to be isogenous to H_3, we would require that $\phi_1 \circ \pi_1$ and $\psi_2 \circ \pi_2$ are isogenies. Surjectivity is obvious. We check that the kernel of either of the compositions is of U-rank less than ω^α. The fiber of π_1 over any point of K_1 is a coset of the kernel of π_1. Therefore, by the Lascar inequality, the kernel of the map $\phi_1 \circ \pi_1$ is bounded above by $RU(a) \oplus RU(ker(\pi_1))$, where a is an element of the kernel of ϕ_1. Of course, this implies that $RU(a) < \omega^\alpha$. So, $\tau_U(ker(\phi_1 \circ \pi_1)) < \alpha$. Then, by a symmetric argument on $\psi_2 \circ \pi_2$, both maps are isogenies.

Now we prove item one of the proposition. Suppose that we have the following diagram:

$$\begin{tikzcd} & H \arrow[dl, "\phi_G"'] \arrow[dr, "\phi_K"] & \\ G & & K \end{tikzcd}$$

Then we claim there is a bijection between the sets of α-connected subgroups $G_1 \leq G$ and $K_1 \leq K$ with $\tau_U(G_1) = \tau_U(G) = \tau_U(K_1) = \tau_U(K)$. We will now set up a correspondence between these two types of subgroups. Let $r(G_1) = K_1$ if there is a type-definable α-connected subgroup $H_1 \leq H$ with $\phi_G(H_1) = G_1$. To show that the map r is well-defined and bijective, by the symmetry of the situation for G and K, it suffices to show that there is a unique choice of α-connected subgroup $H_1 \leq H$ with $\phi_G(H_1) = G_1$ and that $\phi_G(H_1)$ is a strongly connected subgroup of G. The second part follows from Proposition 5.2.5. For the first part, there is one natural candidate, namely, the α-connected component of the inverse image of G_1, which we will denote $\phi_G^{-1}(G_1)^{(\alpha)}$. Certainly, by the Lascar inequality and the fact that $\tau_U(ker\phi_G) < \alpha$, we know that $\tau(\phi_G^{-1}(G_1)) = \alpha$. So, at least $\phi_G^{-1}(G_1)^{(\alpha)}$ is a type-definable group which is α-connected and of suitable rank. We claim that $\phi_G(\phi_G^{-1}(G_1)^{(\alpha)}) = G_1$. Of course, the image is contained in G_1. Suppose that $RU(G_1) = \omega^\alpha \cdot n$. Then $RU(\phi^{-1}(G_1)) = \omega^\alpha \cdot n$. That the Lascar rank of the inverse image is at least this big is trivial. That it is at most this big follows from the Lascar inequality and the fact that $RU(ker\phi_G \cap \phi_G^{-1}(G_1)) \leq RU(ker\phi_G) < \omega^\alpha$. So, the image of $\phi_G^{-1}(G_1)^{(\alpha)}$ is a strongly connected subgroup of G_1 of the same leading monomial U-rank. This implies that the image is G_1. Now, we claim that there is no other choice of H_1. If there was, it would have to be a proper type-definable subgroup of $\phi_G^{-1}(G_1)^{(\alpha)}$. But, we know that all such subgroups have leading monomial U-rank less than $\phi_G^{-1}(G_1)^{(\alpha)}$ by virtue of α-connectedness. Of course, then the image of such a group can not be all of G_1, simply by virtue of rank. The correspondence is bijective, since the image of a α-connected subgroup H_1 under ϕ_G is an α-connected 5.2.5 subgroup G_1 of G with the same U-rank.

Now we move on to item two. All of the subgroups in the following paragraph are α-connected. Suppose that $r(G_1) = K_1$ and $r(G_2) = K_2$. Now suppose that $G_1 \leq G_2$. Then $\phi_G^{-1}(G_1)^{(\alpha)} \leq \phi_G^{-1}(G_2)^{(\alpha)}$, because α-connected subgroups of $\phi_G^{-1}(G_2)$ must be contained in the α-connected component.

Of course, this implies that $K_1 = \phi_K(\phi_G^{-1}(G_1)^{(\alpha)}) \leq \phi_K(\phi_G^{-1}(G_2)^{(\alpha)}) = K_2$. Now we assume that $G_1 \lhd G_2$. Then $\phi_G^{-1}(G_1) \lhd \phi_G^{-1}(G_2)$. Since the α-connected component of a group is definably characteristic, $\phi_G^{-1}(G_1)^{(\alpha)} \lhd \phi_G^{-1}(G_2)$. So, $\phi_G^{-1}(G_1)^{(\alpha)} \lhd \phi_G^{-1}(G_2)^{(\alpha)}$. But, then $K_1 = \phi_K(\phi_G^{-1}(G_1)^{(\alpha)}) \lhd \phi_K(\phi_G^{-1}(G_2)^{(\alpha)}) = K_2$.

Now we prove the third item. The maps induced by ϕ_G and ϕ_K on the quotient H_2/H_1 are isogenies, since they are surjective onto their image, their kernels of the maps are quotients of the kernels of an isogenies, and $\tau_U(H_2/H_1) = \tau_U(H)$ and $\tau_U(G_2/G_1) = \tau_U(G)$.

For the final item, first note that products of α-connected groups are α-connected. Products of isogenous groups are isogenous, because taking a product of the isogeny maps gives an isogeny map (surjectivity is clear and the U-rank of the kernel is bounded by the Cantor sum of the U-rank of the kernels in the product). $\square$

Remark 5.3.4. For more details on the following brief remarks, see (66). In superstable theories, all types are coordinatized by *regular* types. One often considers the equivalence relation of *nonorthogonality* of the regular types. The strongly connected groups considered here have generics which are a product of regular types, each nonorthogonal to a type of rank ω^α. The equivalence relation of nonorthogonality is much coarser than isogeny. The isogeny relation on almost simple groups is finer, and takes into account the group theoretic properties of the definable group in ways which nonorthogonality does not.

Let G be a (non-commutative) quasi-simple algebraic group. In algebraically closed fields, the nonorthogonality relation is rather trivial, since any two positive rank types are nonorthogonal. The isogeny relation is nontrivial, and it matches the classical definition. Even in settings in which the nonorthogonality relation is highly nontrivial (for instance differentially closed fields), the isogeny relation is finer. Of course, almost simplicity is not a sufficient condition for a connected group to have regular generic type. In the setting of differential algebraic groups, is it necessary?

Lemma 5.3.5. *Let G be a strongly connected and non-commutative group. Then $\tau_U([G,G]) = \tau_U(G)$.*

Proof. Let $H := [G,G]$. Implicit in the lemma is the fact that H is type-definable. This follows from α-indecomposability theorem of (6). In fact, if G is actually definable, then so is the commutator subgroup. Suppose $\tau_U(H) < \tau_U(G)$, then $H \leq Z(G)$ by 5.2.4. Choose $a \in G \backslash Z(G)$, and define

$$c_a : G \to G$$

$$x \mapsto axa^{-1}x^{-1}.$$

Since $H \leq Z(G)$, c_a is a definable homomorphism from G to H. So, the kernel of the map is a subgroup of G with the property that $RU(Ker(c_a)) \oplus RU(H) \geq RU(G)$ by the Lascar inequality. As $\tau_U(H) < \tau_U(G)$, this implies that $RU(Ker(c_a)) \geq \omega^\alpha n$. This is impossible since G is α-connected where $RU(G) = \omega^\alpha \cdot n$. □

Remark 5.3.6. Even in the case that $[G,G]$ (or another normal abstract subgroup) is not definable, one can consider the smallest type-definable subgroup, H, containing the $[G,G]$. One can still show H is normal. It appears that Cassidy and Singer (16) need this fact for their lemma 2.24, since they did not know until (23) that the commutator subgroup is definable). I will offer a proof. Take $A \lhd G$ where there are no definability conditions on A. Then, let H be the smallest definable subgroup containing A (differentially closed fields are ω-stable, so we have the descending chain condition on definable groups). Now, consider a G-conjugate H^g of H. Since A is normal, H^g is still a definable subgroup containing A. So, $H \cap H^g$ is a definable subgroup containing A. By the minimality of H, $H = H^g$. Thus, $H \lhd G$.

Proposition 5.3.7. *Let G and H be isogenous α-connected groups. Both are almost simple or neither is. Both are commutative or neither is.*

Proof. We have the following diagram, since G and H are isogenous,

$$\begin{tikzcd} G \arrow[dr, "\phi_G"] & & H \arrow[dl, "\phi_H"'] \\ & K & \end{tikzcd}$$

G commutative implies K is commutative. Let $H_1 = [H, H]$. We know that $\tau_U(H_1) = \tau_U(H)$ by 5.3.5. But, $\tau_U(ker\phi_H) < \tau_U(H_1)$, so the image is nontrivial. On the other hand, $\phi_H(H_1) \subseteq [K, K] = \{e\}$, a contradiction. $\square$

The main reason for the notion of isogenous in this paper is to utilize it to prove uniqueness results of the form "up to isogeny" similar to the case of algebraic groups or differential algebraic groups. In particular, we will start, in the next section with a theorem similar to Baudisch's Jordan-Hölder style decomposition based on Berline-Lascar analysis of superstable groups.

5.4 Jordan-Hölder Theorem

The proof of the following theorem follows the proof of the Jordan-Hölder theorem in the case of partial differential fields due to Cassidy and Singer. We should mention that though Lascar rank is *not* the same as the notions of dimension that Cassidy and Singer use, it shares enough of the same properties to make the proofs work similarly after the correct translation of the statements is known.

Theorem 5.4.1. *Let G be an α-connected superstable group. Then there exists a normal sequence*

$$1 = G_r \lhd G_{r-1} \lhd \ldots \lhd G_1 \lhd G_0 = G.$$

For each $i \in \{0, \ldots, r-1\}$:

1. *G_i is strongly connected and $\tau_U(G_i) = \tau_U(G)$.*
2. *$RU(G_i) > RU(G_{i+1})$.*
3. *G_i/G_{i+1} is almost simple and $\tau_U(G_i/G_{i+1}) = \tau_U(G)$. $RU(G_i/G_{i+1}) = \omega^\alpha \cdot (n-m)$, where $RU(G_i) = \omega^\alpha \cdot n$ and $RU(G_{i+1}) = \omega^\alpha \cdot m$.*

If

$$1 = H_s \lhd H_{s-1} \lhd \ldots \lhd H_1 \lhd H_0 = G$$

is another sequence which satisfies the above properties, then the sequences must be of the same length ($r = s$). There is a permutation (call it σ) of the indices so that the quotients are isogenous. That is, $G_{\sigma(i)}/G_{\sigma(i)+1}$ is isogenous to H_i/H_{i+1}.

Proof. If G is already almost simple, then there is nothing to do. If this is not the case, then there is a nonempty collection of proper, type-definable, normal subgroups H of G with $\tau_U(G) = \tau_U(H)$. Pick any such H so that if $RU(H) = \omega^\alpha \cdot n_0 + \beta$, then there is no other H_1 in the collection so that $RU(H) = \omega^\alpha \cdot n_1 + \beta_1$, where $n_1 > n_0$. We let G_1 be the α connected component of H, $G_1 = H^{(\alpha)}$. G_1 is a definably characteristic subgroup of $H \lhd G$, so $G_1 \lhd G$. Since the Cantor sum on ordinals is equal to the sum when the ordinals in question are monomials of the same "degree", $RU(G/G_1) = \omega^\alpha \cdot (n - m)$, where $RU(G) = \omega^\alpha \cdot n$ and $RU(G_1) = \omega^\alpha \cdot m$. Suppose that the quotient G/G_1 is not almost simple. Then, there is a proper, type-definable, normal subgroup $H_1 \lhd G/G_1$ with $\tau_U(G) = \tau_U(G/G_1) = \tau_U(H_1)$. But, then the preimage of H_1 under the quotient map is a subgroup of G which violates the maximality condition with which H was chosen, namely, the leading monomial of the Lascar rank of the preimage of H_1 is larger than that of H. So, the quotient is almost simple. Continuing in this way, we can find $G_2, G_3, \ldots$ having the properties prescribed in the statement of the theorem.

The proof of uniqueness proceeds in a similar manner to the proof of Cassidy-Singer decomposition in the differential field context. In turn, that proof follows the one in (43, chapter 1, section 3). So, suppose we have two sequences as above $\langle G_i \rangle_{i \leq r}$ and $\langle H_j \rangle_{j \leq s}$. For each pair (i, j) with $i < r$ and $j < s$, we define:

$$G_{i,j} := G_{i+1}(H_j \cap G_i).$$

Notation:

$$G_{i,s} := G_{i+1,0}$$

Then,

$$1 \lhd G_{r-1,s-1} \lhd G_{r-1,s-2} \lhd \ldots \lhd G_{r-1} \lhd G_{r-2,s-1} \lhd \ldots G_1 \lhd G_{0,s-1} \lhd \ldots G_{0,0} = G.$$

Of course, one can apply the definition in the opposite way as well, so get a refinement of $\langle H_j \rangle$,

$$H_{j,i} = H_{j+1}(G_i \cap H_j).$$

By 3.3 from Lang's algebra, $G_{i,j}/G_{i,j+1}$ is isomorphic to $H_{j,i}/H_{j,i+1}$. Further, the isomorphism is definable.

Claim 5.4.2. For and $i = 0 \ldots r-1$, there is precisely one j so that $\tau_U(G_{i,j}/G_{i,j+1}) = \tau_U(G)$. Further, for this specific value of j, we have that $G_{i,j}/G_{i,j+1}$ is isogenous to G_i/G_{i+1}.

Assume that we have established the claim. Then, since the symmetric statement holds for the $H_{j,i}$ we know that $r = s$ and the theorem follows.

Now we prove the claim. By the Lascar inequality,

$$RU(G_{i+1}) + \sum_{j=s-1}^{0} RU(G_{i,j}/G_{i,j+1}) \leq RU(G_i) \leq RU(G_{i+1}) \oplus \bigoplus_{j=s-1}^{0} RU(G_{i,j}/G_{i,j+1})$$

So, for some j,

$$\tau_U(G_{i,j}/G_{i,j+1}) = \alpha.$$

Now, let j be minimal so that the condition holds.

$$\tau_U(G) = \tau_U(G_{i,j}/G_{i,j+1}) \leq \tau_U(G_{i,j}/G_{i+1}) \leq \tau_U(G_i/G_{i+1}) = \tau_U(G).$$

Then note that for each $k < j$,

$$\tau_U(G_{i,j}/G_{i+1}) \leq \tau_U(G_{i,k}/G_{i+1}) \leq \tau_U(G_i/G_{i+1}).$$

Thus, for all $k < j$, $\tau_U(G_{i,k}/G_{i+1}) = \tau_U(G)$. We also know that $\tau_U(G_{i,k}/G_{i,k+1}) < \alpha$. Since G_i/G_{i+1} is almost simple, all of this forces $G_{i,0} = G_i = G_{i,1}$. Continuing in the same way, we can see

$$G_{i,0} = \ldots = G_{i,j}.$$

We have the canonical projection map

$$G_i/G_{i+1} \to G_{i,j}/G_{i,j+1}.$$

The kernel is a proper normal subgroup of an almost simple group, so the map is an isogeny. Now, suppose that for some $t > j$, we have that $\tau_U(G_{i,t}/G_{i,t+1}) = \tau_U(G)$. Then $\tau_U(G_{i,t}/G_{i+1}) =$

$\tau_U(G)$. This contradicts the almost simplicity of G_i/G_{i+1}. So, we have the desired uniqueness result. $\square$

5.5 Final remarks

In this section, we take $\mathcal{U} \models DCF_{0,m}$ to be a differentially closed field in m commuting derivations, $\Delta = \{\delta_1, \ldots, \delta_m\}$, which we assume to be sufficiently saturated. In differential fields, there are examples, due to Cartan, Cassidy and Singer, which show that some sort of weaker notion of correspondence than isomorphism is necessary for the sort of theorem of the previous section to be true. We discuss the model theoretic aspects of an example of Cassidy and Singer here. For the next portion of the discussion, we assume $m = 2$, that is $\Delta = \{\delta_1, \delta_2\}$. The purpose of this assumption is mainly to simplify the discussion. Let $a \in \mathcal{U}$ be such that $\delta_1(a) = 1$ and $\delta_2(a) = 0$. Let G_1 be the zero set of $(\delta_1^2 z - \delta_2 z)$. Let G_2 be the zero set of $(\delta_1 z - a\delta_2 z)$. We consider these groups as subgroups of the additive group, $\mathbb{G}_a(\mathcal{U})$. Both of these differential algebraic groups have Lascar rank ω.

One can quickly see that ω is a lower bound for the Lascar rank as follows. The solution sets to the equations are infinite dimensional vector spaces over the field $\{c \in \mathcal{U} \mid \delta_1(c) = \delta_2(c) = 0\}$. To see this, consider, for $j \in \mathbb{N}$, the definable subspaces of G_1 given by $\delta_1^j z = 0$. Then, for all j,

$$\{z \in G_1 \mid \delta_1^j z = 0\} \quad \{z \in G_1 \mid \delta_1^{j+1} z = 0\} \quad G_1$$

The strictness of the containments follows easily from the axioms for differential fields (48). Specifying containment in a particular coset of one of the subspaces $\{z \in G_1 \,|\, \delta_1^j z = 0\}$ naturally gives arbitrarily long finite chains of forking extensions of the generic type of G_1.

Seeing that ω is an upper bound takes slightly more work. We will work with a slightly more general class of examples. Lets show $Z(\delta_1 y - f(y))$ where $f \in K[\delta_2]$ is almost simple. A proper subgroup of the additive group, H, is a kernel of a linear operator $g \in K[\Delta]$ such that $g(y) \notin \{\delta_1 y - f(y)\}$ in $K\{y\}$ (85, Proposition 3.45). We may assume that $g \in K[\delta_2]$ (and is linear), since we are only interested in working modulo the relation $\delta_1 y = f(y)$, where f is linear as a Δ-polynomial. If g has order d, then for any $y \in H$, we have $k(y, \delta_2 y, \delta_2 y, ...) = k(y, \delta_2 y, ..., \delta_2^{d-1} y)$. So, H has Δ-type 0, which implies finite Lascar rank. In partial differential fields, the Δ-type of differential algebraic group is the degree of the Kolchin polynomial of a generic point on the group; Δ-type zero means that the Kolchin polynomial is a constant, which is equivalent to saying that if a is a generic point on the group over some differential field K, then $\{\delta_1^{j_1} \delta_2^{j_2}(a) \,|\, j_i \in \mathbb{N}\}$ has finite transcendence degree over K. In differentially closed fields, forking extensions correspond to proper differential algebraic subvarieties. So, we have proved that every forking extension has finite Δ-type, which implies finite Lascar rank (48).

$G = G_1 + G_2$ is strongly connected and the series decomposition as above may be given $1 \lhd G_1 \lhd G$ or $1 \lhd G_2 \lhd G$. Cassidy and Singer (16) show that G_1 is not isomorphic to either G_2 or G/G_2. However, G_1 is isogenous to G/G_2.

All currently known non-commutative almost simple differential algebraic groups actually have finite center. Such groups are, by the results of (23; 13), perfect central extensions of the C' points of an algebraic group, where C' is some definable subfield. Let H denote the C' points of the algebraic group. So, any almost simple G has the following exact sequence:

$$1 \to Z(G) \to G \to H \to 1$$

Now, in an arbitrary superstable theory T work with an arbitrary definable perfect central extension of an algebraic group H, perhaps restricted to a definable subfield, which is almost simple. Is G a finite extension of H? The assumptions are weak enough so that one should guess that the answer is no. However, examples which show the negative conclusion would be of great interest if they could be translated to the setting of differential algebraic groups.

There are suitable theories of numerical polynomials in other algebraic settings from which a theory similar to that of Cassidy and Singer might be developed. An example of model theoretic interest is the setting of difference-differential fields (49). In that setting, there is no nontrivial lower bound Lascar rank in terms of the appropriate generalization of differential gauge (there are definable sets of Morley rank one with infinite difference-differential transcendence degree). The results in this paper would have to be generalized to the supersimple setting in order to compare potential model theoretic and algebraic notions of strong connectedness.

In the setting of differential fields, the decomposition series of Cassidy-Singer theorem are connected to the factorization of differential operators and with differential galois theory. Subsequent work (51; 52) related to that of Cassidy and Singer has used this connection. Of course, there is no analogue of this connection in the level of generality which we currently work; however such connections are plausible in various other model theory of fields settings. We hope that this paper and future work on generalizing the setting might set the stage for the transfer of the ideas of the Cassidy-Singer theorem to the groups definable in other settings.

CHAPTER 6

COMPLETENESS

6.1 Introduction

Completeness is a fundamental notion in algebraic geometry. In this chapter, we examine the analogue in differential algebraic geometry. The chapter builds on Pong's (70) and Kolchin's (33) work on differential completeness in the case of differential varieties over ordinary differential fields and generalizes to the case of differential varieties over partial differential fields with finitely many commuting derivations. Many of the proofs generalize in the straightforward manner, given that one sets up the correct definitions and attempts to prove the correct analogues of Pong's or Kolchin's results. Of course, some of the results are harder to prove because our varieties may be infinite transcendence degree. Nevertheless, in some instances, we prove stronger results. For instance, Proposition 6.4.3 generalizes a theorem of (72) even when we restrict to the ordinary case. The proposition also generalizes a known result projective varieties. We also give many examples, which we hope shed some light on the nature of completeness in the differential setting.

The model theory of partial differential fields was developed in (48). For a recent alternate (geometric) axiomatization of partial differentially closed fields, see (77). For a reference in differential algebra, we suggest (34) and (45). The differential varieties we consider will be embedded in projective space. Generalizations to differential schemes are of interest, but are not treated here (38; 39, for instance). Pillay also considers differential completeness for a slightly different category in (63). Though Pillay's conditions for differential completeness are implied by the conditions here, their precise relationship is not clear. Generalizing this work to the difference-differential topology is of interest, but there are important model theoretic obstacles. Specifically, when working in the setting of (49), we would not have quantifier elimination. We would also leave the model theoretic setting of superstability for the more general setting of supersimplicity. Nevertheless, the difference-differential category is of particular interest because we know that the natural analogues of questions 6.4.10 and 6.4.11 are actually distinct.

The definition of Δ-completeness 6.2.4 is a straightforward generalization of the definition of completeness in the category of abstract algebraic varieties (integral separated schemes of finite type over an algebraically closed field). Section 2 is devoted to giving the basic definitions and particular notation of this chapter. After the basic examples, we turn to an example of Kolchin in section 3, which highlights one of the essential differences between Δ-completeness and completeness for abstract algebraic varieties: $\mathbb{P}^n$ is not Δ-complete. We should note that this chapter only considers quasiprojective differential algebraic varieties over a differentially closed field. The completeness question for more abstract differential varieties (37) is of interest. In the category of abstract algebraic varieties, there are complete varieties which are not projective (56) (78, chapter 6, section 2.3). We do not know about the existence of such complete differential schemes (which are provably non-projective in the Δ-topology). We also do not know of any projectivity criteria for complete differential schemes. For instance, in the setting of abstract algebraic varieties, see the Chevalley-Kleiman Criterion (78, chapter 6, section 2.4).

Even in the quasiprojective case, the notion of Δ-completeness sits in stark contrast to its algebraic counterpart. Pong (70) proved that all Δ-complete *ordinary* differential algebraic varieties are of finite transcendence degree (δ-transcendence degree zero). Using the model-theoretic tool of Lascar rank, one can prove that any finite transcendence degree ordinary differential algebraic variety is affine and may in fact be embedded in $\mathbb{A}^1$. This fact blurs the lines between projective and affine differential algebraic varieties; projective differential algebraic varieties like the one given by

$$Z(zy^2 - x^3 - azx^2 - bz^3, z\delta(x) - x\delta(z), z\delta(y) - y\delta(z)) \subset \mathbb{P}^2$$

are isomorphic to *affine* differential algebraic varieties, even subvarieties of $\mathbb{A}^1$.

In this chapter, we generalize Pong's results in several ways. The main generalization is the setting: we work with partial differential algebraic varieties. In this case, we prove the analogues of the results mentioned above; the Δ-complete varieties are of Δ-transcendence degree zero and may be embedded in $\mathbb{A}^1$. We also prove several embedding theorems for positive Δ-transcendence degree differential algebraic varieties, which generalize Pong's results in the ordinary case (he only considers varieties of δ-transcendence degree zero). A very simple instance of the embedding theorems discussed above will be slowly developed throughout the text via a series of examples 6.2.3 6.2.6 6.4.12 where we consider the constant points of an elliptic curve as above. The examples reveal some of the exotic nature of Δ-complete differential algebraic varieties as the interesting potential examples coming from the embedding theorems of section 4 indicate.

In section 5, we turn to a criterion for proving differential algebraic varieties are complete. The first ingredient is a result of van den Dries (88), which relates sets definable by positive quantifier free formulas to homomorphisms of substructures:

Proposition. *T a complete $\mathcal{L}$-theory and $\phi(v_1 \dots v_n)$ an $\mathcal{L}$-formula without parameters. Then the following are equivalent:*

1. *There is a positive quantifier free formula ψ such that T proves $\forall v \phi(v) \leftrightarrow \psi(v)$.*
2. *For any $K, L \models T$ and $f : A \to L$ an embedding of a substructure A of K into L, if $a \in \mathbb{A}^m$ and $K \models \phi(a)$ then $L \models \phi(f(a))$.*

In the differential context, such definable sets as in condition 1) above are Δ-closed subsets of affine space. It is worth mentioning for the non-model theorist that assumption in the above proposition that ϕ is definable without parameters (the corresponding variety would then be over $\mathbb{Q}$ in the natural language of differential rings) is a red herring. A standard trick fixes the problem: one may add formal constants (*not* to be confused with constants of Δ - here constant is a logical term) to the language, whose interpretations in any model are precisely the parameters used in ϕ. The proposition applies equally well to the new theory T^* of a model of T in the language $\mathcal{L}$ extended by the new constant symbols.

In the differential context, such differential homomorphisms as in condition 2) above were studied by Blum (7) and Morrison (55). Thanks to an anonymous referee for pointing out (8), where the study of completeness in the differential setting was initiated. Morrison and Blum considered the problem of extending differential homomorphisms; in particular, they exhibit examples of differential homomorphisms which cannot be extended. This is in contrast to the case of (non-differential) fields where any valuation on a field F has an extension to a valuation on any extension field K/F. In the differential context, it is the domains of nonextendible differential homomorphisms which form the basis for our valuative criterion. Our valuative criterion for a variety $V \subset \mathbb{A}^n$ involves considering the R-rational points of the differential algebraic variety, where R is a *maximal* Δ*-subring* of a differentially closed field. We do not establish any result regarding the nature of maximal Δ-subrings, but rather use them as a tool via the results of Blum and Morrison (providing examples showing their utility). Further results regarding maximal Δ-rings would be of interest, in part because they might be brought to bear on the completeness problem. On the other hand, further results on complete Δ-varieties would be of purely algebraic interest even if obtained via completely different methods, because they may shed light on the problem of extending differential homomorphisms. We should note that the approach we describe here has been taken up in much greater detail in the ordinary case in some recent work (80).

In section 6, we use the criterion to construct some new examples of Δ-complete varieties. When moving from the ordinary setting to the partial setting, there are two fairly natural generalizations of the finite transcendence degree ordinary differential algebraic varieties: the finite transcendence degree partial differential algebraic varieties and the partial differential algebraic varieties with Δ-type less than m, the number of derivations. For proving the results of this chapter, the second generalization turns out to be the appropriate one. As we have mentioned above, many of the results have nearly the same proof as some of the analogous results of (70) once one sets up the correct definitions. We reproduce even those proofs here for two primary reasons. First, this keeps the chapter more self-contained and shows the easy transition to the partial case once the assumptions and definitions are in place. We also take the opportunity to explain some of the model-theoretic notation in more algebraic terms and give some specific examples and discussion about completeness. We hope this chapter strikes a balance and is accessible to both model theorists and differential algebraic geometers.

Though we often consider partial differential algebraic varieties with Δ-type less than m as the natural generalization of finite transcendence degree ordinary differential algebraic varieties, we also raise some questions regarding finite transcendence degree partial differential algebraic varieties. Our examples focus on this case, because there are no known complete infinite transcendence degree partial differential algebraic varieties. On the other hand, the following question is open:

Question 6.1.1. *Let $V \subseteq \mathbb{P}^n$ be an irreducible Δ-closed variety of finite transcendence degree. Is V complete in the Kolchin topology?*

6.2 Definitions and basic results

A subset of $\mathbb{A}^n$ is Δ-*closed* if it is the zero set of a collection of Δ-polynomials in n variables. We use $F\{y_1, \ldots, y_n\}$ to denote the ring of Δ-*polynomials* over F in $y_, \ldots, y_n$. When we want to emphasize that the collection of Δ-polynomials is over F, we say Δ-F-closed. For a thorough development, see (34) or (45).

Definition 6.2.1. A (non-constant) Δ-polynomial in $F\{y_0, \ldots, y_n\}$ is Δ-*homogeneous of degree* d if

$$f(ty_0, \ldots ty_n\} = t^d f(y_0, \ldots, y_n),$$

where t is a Δ-F-indeterminate.

Remark 6.2.2. The reader should note that Δ-homogeneity is a stronger notion that homogeneity of a differential polynomial as a polynomial in Θx. For instance,

$$\delta x - x$$

is a homogeneous Δ-polynomial, but not a Δ-homogeneous Δ-polynomial. The reader may verify that the following is a Δ-homogeneous Δ-polynomial:

$$y\delta x - x\delta y - xy.$$

The vanishing of Δ-homogeneous Δ-polynomials in $n+1$ variables is well-defined on $\mathbb{P}^n$. Generally, we can easily homogenize an arbitrary Δ-polynomial in x with respect to a new variable y. Homogenization in the partial differential case works identically to the ordinary case. For more details and examples see (70). However, next we will develop one specific example which will be used throughout this paper.

Example 6.2.3. For this example, we assume that the differential field is ordinary with $\Delta = \{\delta\}$. We wish to consider, as a differential variety, the constant points of an elliptic curve. That is, we are considering the projective closure of:

$$V = Z(y^2 - x^3 - ax^2 + b) \subseteq \mathbb{A}^2(\mathcal{C}),$$

where $4a^3 + 27b^2 \neq 0$. As a differential variety, V is the zero set of the above equation and $\delta(x) = \delta(y) = 0$. The projective closure of V in $\mathbb{P}^2$ is given by the zero set of the differential homogenizations of the three above equations. Of course, the algebraic equation is homogenized in the standard way:

$$zy^2 = x^3 + azx^2 + bz^3.$$

When $f(x, y) \in F\{x, y\}$ is a differential polynomial in x and y, then for some sufficiently large $d \in \mathbb{N}$,

$$z^d f\left(\frac{x}{z}, \frac{y}{z}\right)$$

is differentially homogeneous of degree d. When f is irreducible over F, the minimal such d produces a homogeneous irreducible differential polynomial. In the case $f(x, y) = \delta(x)$,

$$z^2 \delta\left(\frac{x}{y}\right) = z\delta(x) - x\delta(z)$$

is differentially homogeneous of degree 2. Similarly, the differential homogenization of $\delta(y)$ produces the degree 2 differential polynomial $z\delta(y) - y\delta(x)$. The projective closure of V in $\mathbb{P}^2$ is the locus of the three equations. We will let

$$E := Z(zy^2 - x^3 - azx^2 - bz^3, z\delta(x) - x\delta(z), z\delta(y) - y\delta(z)) \subset \mathbb{P}^2.$$

This differential algebraic variety will be used as an example several times in the coming sections. We should note that calculating the projective closure of an affine differential algebraic variety can be slightly tricky. For instance, in the above variety, the hyperplane at infinity contains only the point $[0 : 1 : 0]$, so it is not necessary to include the differential equation $x\delta(y) - y\delta(x) = 0$. On the other hand, consider the projective closure of

$$\mathbb{A}^2(\mathcal{C}) = \{(x, y) \,|\, \delta(x) = \delta(y) = 0\}.$$

In this case, the projective closure is given by

$$\mathbb{P}^2(\mathcal{C}) = Z(z\delta(x) - x\delta(z), z\delta(y) - y\delta(z), x\delta(y) - y\delta(x)).$$

Failing to include the last differential polynomial would result in including the entire line at infinity in $\mathbb{P}^2$ instead of only the constant points, or, in terms of the rational points of our universal domain, $\mathbb{P}^2(\mathcal{U})$ rather than $\mathbb{P}^2(\mathcal{C})$.

In general, the Δ-closed subsets of $\mathbb{P}^n$ are the zero sets of collections of homogeneous differential polynomials in $F\{y_0, \dots, y_n\}$. Δ-closed subsets of $\mathbb{P}^n \times \mathbb{A}^m$ are given by the zero sets of collections of differential polynomials in $F\{y_0, \dots, y_n, z_1, \dots, z_m\}$ which are homogeneous in $\bar{y}$. Usually we will consider *irreducible* Δ-closed sets, that is, those which are not the union of finitely many proper Δ-closed subsets. Because differentially closed fields have quantifier elimination, the *definable sets* over F are simply boolean combinations of closed sets, that is, the constructible sets of the Δ-topology. A constructible set is irreducible if its closure is irreducible.

Definition 6.2.4. A Δ-closed $X \subseteq \mathbb{P}^n$ is Δ-complete if the second projection $\pi_2 : X \times Y \to Y$ is a Δ-closed map for every quasi projective Δ-variety, Y.

We will simply say *complete* rather than Δ-complete. This will cause no confusion with the analogous term in the algebraic category, because in this paper we work exclusively in the category of differential algebraic varieties (except for remarks). It is not a restriction to consider only irreducible Δ-closed sets. To see this, note that in proving that $X \times Y \to Y$ is Δ-closed, it is enough to prove that the map is Δ-closed on each irreducible component of X.

Proposition 6.2.5. *If X is Δ-complete and Y is a quasi projective Δ-variety,*

1. *Suppose $f : X \to Y$ continuous. Then $f(X)$ is Δ-closed in Y and Δ-complete.*
2. *Any Δ-closed subset of X is Δ-complete.*
3. *Suppose that Y is Δ-complete. Then $X \times Y$ is Δ-complete.*

Proof. Let $f : X \to Y \subseteq \mathbb{P}^m$. Then $f \times id : X \times \mathbb{P}^m \to \mathbb{P}^m \times \mathbb{P}^m$ is a continuous map. The diagonal of $\mathbb{P}^m \times \mathbb{P}^m$ is Δ-closed. By virtue of the completeness of X, $\pi_2(gr(f))$ is Δ-closed. This is $f(X)$. Now, we get the following commuting diagram, giving the completeness of the image, $f(X)$.

$$\begin{tikzcd} X \times Y \arrow[r, "f \times id"] \arrow[d, "\pi_2"'] & f(X) \times Y \arrow[dl, "\pi_2"] \\ Y & \end{tikzcd}$$

For 2), simply note that if Z is and Δ-closed subset of X, then we have the natural injective map $Z \times Y \to X \times Y$. Further, the π_2 projection map clearly factors through this map. So, Z must be Δ-complete. Similarly, for 3), we can simply note that if we have $X \times Y \times Z$, then the projection $X \times Y \times Z \to Y \times Z$ is closed by the completeness of X. $Y \times Z \to Z$ is closed by the completeness of Y. The composition of closed maps is closed. $\square$

When we wish to verify that a differential algebraic variety, X, is complete, we need only show that

$$\pi : X \times Y \to Y$$

is Δ-closed for affine Y. This fact is true for the same reason as in the case of algebraic varieties. Δ-closedness is a local condition, so one should cover Y by finitely many copies of $\mathbb{A}^m$ for some m and verify the condition on each of the affine pieces.

Example 6.2.6. We continue our elliptic curve example 6.2.3. Kolchin (33) proved that

$$\mathbb{P}^n(\mathcal{C}) = Z(\{x_j\delta(x_i) - x_i\delta(x_j)\}_{i,j=1,\ldots,n}) \subset \mathbb{P}^n$$

is differentially complete. Pong (70) gave an alternate proof of this fact. It follows from Proposition 6.2.5 part 2 that E is differentially complete, being a Δ-closed subset of $\mathbb{P}^2(\mathcal{C})$.

6.3 $\mathbb{P}^n$ is not Δ-complete

There are more closed sets in Δ-topology than in the Zariski topology, so in the differential setting, there are more closed sets in both the image of the projection map (which makes completeness "easier" to achieve) and more closed sets in the domain (making completeness "harder" to achieve). Though the Δ-topology is richer than the Δ-topology for ordinary differential fields, Pong's exposition (70) of Kolchin's theorem that $\mathbb{P}^n$ is not complete holds in this setting. The techniques are model-theoretic, as in (70), but use the model theory of Δ-fields.

Consider, for some $\delta \in \Delta$, the closed set in $\mathbb{P}^1 \times \mathbb{A}^1$ given by solutions to the equations

$$z(\delta y)^2 + y^4 - 1 = 0 \tag{6.1}$$

$$2z\delta^2 y + \delta z \delta y + 4y^3 = 0 \tag{6.2}$$

Note that the projective closure with respect to the y variable, does not contain the point at infinity. So, we can work with the above local equations. Let $b \in \mathcal{U}$ be a Δ-transcendental over F. Then, we have a solution, in $\mathcal{U}$, to the equation

$$b(\delta y)^2 + y^4 - 1 = 0$$

Further, we can demand that $4y^3 - 1 \neq 0$. Call this solution a. This means that $\delta a \neq 0$. But, since we chose b to be a Δ-transcendental over F, we know, by quantifier elimination, that if $\pi_2 Z$ is Δ-closed, then it is all of $\mathbb{A}^1$. But $\pi_2 Z$ can not be all of $\mathbb{A}^1$, since $\pi_2 Z$ can not contain 0. Thus, $\pi_2 Z$ is not Δ-closed and $\mathbb{P}^1$ is not Δ-complete. Since $\mathbb{P}^1$ is a Δ-closed subset of $\mathbb{P}^n$ for any n, we have the following result of Kolchin,

Proposition 6.3.1. *$\mathbb{P}^n$ is not Δ-complete.*

Remark 6.3.2. Restrict to the case of ordinary differential algebra for this remark. When dealing with differential algebraic varieties, there are two projective n-spaces. Regarded via their $\mathcal{U}$-rational points, one often considers $\mathbb{P}^n(\mathcal{U})$ and $\mathbb{P}^n(\mathcal{C})$. Because differential algebraic varieties are definable sets, we write $\mathbb{P}^n$ for the differential algebraic variety which has rational points $\mathbb{P}^n(\mathcal{U})$. Occasionally we write $\mathbb{P}^n(\mathcal{C})$ for the differential algebraic variety having those rational points. This is an abuse of notation, of course.

There are infinite rank definable subfields in partial differential fields (they are all given as the constant field of a set of independent definable derivations, see (85)). Suppose the set of definable derivations of cutting out the constant field, K_0, is of size $m - m_1$. $\mathbb{P}^n(K_0)$ is not Δ-complete. To see this, one can simply repeat the above techniques in a model of DCF_{0,m_1} for $m_1 < m$.

6.4 Embedding Theorems

In this section, we think of the differential algebraic varieties which appear as synonymous with their $\mathcal{U}$ rational points where $\mathcal{U}$ is a very saturated differentially closed field. We will use the model-theoretic tool of Lascar rank to show that every Δ-complete set can be embedded in $\mathbb{A}^1$. We may identify hypersurfaces of degree d in $\mathbb{P}^n$ with points in $\mathbb{P}^{\binom{d+n}{d}-1}$ via the following correspondence,

$$\bar{a} = [a_1 \dots a_{\binom{d+n}{d}}] \in \mathbb{P}^{\binom{d+n}{d}-1} \leftrightarrow H_{\bar{a}} = Z(\sum_{i=1}^{\binom{d+n}{d}} a_i M_i)$$

where M_i is the ith monomial of degree d in $x_0 \dots x_n$ ordered lexicographically. Notice the unusual numbering on the parameter space of $a_i's$. This follows Hartshorne (26); let $N = \binom{d+n}{d} - 1$ for the remainder of the section. We say that a hypersurface is generic if the associated $\bar{a}$ is a tuple of differential transcendentals over F. This is equivalent to saying that $RU(a/F) = \omega^m \cdot N$.

Proposition 6.4.1. *Suppose $X \subset \mathbb{P}^n$ is a definable set of Lascar rank less than ω^m. Then any generic hypersurface does not intersect X.*

Proof. Let

$$H = Z(\sum_{i=1}^{\binom{d+n}{d}} a_i M_i)$$

where $\bar{a}$ is a generic point in $\mathbb{P}^N$. Now, suppose that $x \in X \cap H$. The x specifies a proper subvariety of $\mathbb{P}^N$, namely the hyperplane given by $\sum a_i M_i(x) = 0$ This hyperplane is, of course, isomorphic to $\mathbb{P}^{N-1}$. But, we know that the rank of a generic point in $\mathbb{P}^{N-1}$ is not $\omega^m \cdot N$, namely we know $RU(a/x) \leq \omega^m \cdot (N-1)$. Now, using the Lascar inequality,

$$RU(a) \leq RU(a, x) \leq RU(a/x) \oplus RU(x) < \omega^m \cdot N.$$

So, $\bar{a}$ must not have been generic in $\mathbb{P}^N$. $\square$

Proposition 6.4.2. *Let $X \subseteq \mathbb{P}^n$ be a proper Δ-closed subset. Let $p \in \mathbb{P}^n - X$ be generic. Suppose $RU(X) \geq \omega^m$. Then $RU(\pi_p(X)) \geq \omega^m$, where π_p is projection from the point to any hyperplane not containing the point.*

Proof. Take $b \in X$ generic (of full RU-rank). Then let $c = \pi_p(b)$. Since we assumed that $p \in \mathbb{P}^n - X$, we know that the intersection of the line $\bar{pb}$ and X is a proper Δ-closed subset of the affine line. Thus, $RU(b/cp) < \omega^m$. But then

$$\omega^m \leq RU(b) = RU(b/p) \leq RU(b, c/p) \leq RU(b/p) \oplus RU(c/bp).$$

Of course, this implies that $RU(c) \geq \omega^m$. □

Proposition 6.4.3. *Let X be a Δ-variety with Lascar rank less than $\omega^m(k+1)$. Then X is isomorphic to a definable subset of $\mathbb{P}^{2k+1}$.*

Proof. Suppose that $X \subseteq \mathbb{P}^n$. Let p be a generic point of $\mathbb{P}^n$. Let H be any hyperplane not containing p. We claim that projection from p to H, restricted to X, is an injective map. Suppose not. Then there are two points X_1, x_2 on X, so that p is on the line joining x_1 and x_2. Then

$$RU(p) \leq RU(p, x_1, x_2) \leq RU(p/x_1x_2) \oplus RU(x_1x_1) < \omega^m \cdot (2k+2)$$

This is a contradiction unless $n < 2k+1$. Iteratively projecting from a generic point gives the result. □

Remark 6.4.4. In the special case that X is an algebraic variety, then this simply says that we can construct a definable isomorphism to some constructible set in $\mathbb{P}^{2dim(X)+1}$.

From the previous two propositions, we get,

Corollary 6.4.5. *Suppose that X is of Lascar rank less than ω^m. Then X is definably isomorphic to a definable subset of $\mathbb{A}^1$.*

Proof. Using Proposition 6.4.3, we get an embedding of X into $\mathbb{P}^1$. We know that X avoids any generic point of $\mathbb{P}^1$ by Proposition 6.4.1. $\square$

Remark 6.4.6. The use of Proposition 6.4.1 in the above proof is gratuitous, since it is clear that the projection of X is a proper subset of $\mathbb{P}^1$, by simple rank computations. The proposition is less obvious when the variety is embedded in higher projective spaces.

Theorem 6.4.7. *Any Δ-complete set is of RU-rank strictly less than ω^m.*

Proof. Suppose X is complete and of rank larger than ω^m. Projection from any generic point gives a Δ-complete, Δ-closed set (by Proposition 6.2.5) of rank at least ω^m (by Proposition 6.4.2). Iterating the process yields such a set in $\mathbb{P}^1$. The only Δ-closed subset of rank ω^m in $\mathbb{P}^1$ is all of $\mathbb{P}^1$. This is a contradiction since $\mathbb{P}^1$ is not Δ-complete. $\square$

Corollary 6.4.8. *Every Δ-complete subset of $\mathbb{P}^n$ is isomorphic to a Δ-closed subset of $\mathbb{A}^1$.*

Remark 6.4.9. More results along the lines of those in (69) computing bounds on ranks of various algebraic geometric constructions on differential varieties are certainly possible, but the above results suffice our purposes here.

The following two questions may or may not be distinct. For discussion of this see (23) and (85).

Question 6.4.10. *Are there infinite Lascar rank Δ-complete sets?*

Question 6.4.11. *Are there infinite transcendence degree Δ-complete sets?*

Example 6.4.12. We continue with the elliptic curve example 6.2.3. We wish to give concrete examples of Proposition 6.4.1 and Proposition 6.4.2. These propositions are used to give differential algebraic embeddings of E (or any differential algebraic group of *differential transcendence degree* zero) into affine space (6.4.1) and $\mathbb{P}^1$ (6.4.2).

Proposition 6.4.1 will give an embedding of E into $\mathbb{A}^2$. Generally, the content of the proposition in the ordinary case is that any closed set of $\mathbb{P}^n$ with finite transcendence degree defined over F does not intersect a line with coefficients which are generic over F. To that end, consider the line L determined by

$$\alpha x + \beta y + z = 0$$

where α and β are δ-F-transcendentals. Then $L \cap E = \emptyset$, so the map $\phi E \to \mathbb{A}^2$ given by

$$[x, y, z] \mapsto \left(\frac{x}{\alpha x + \beta y + z}, \frac{y}{\alpha x + \beta y + z} \right)$$

is a well-defined injective map. The inverse map $\phi^{-1} : \mathbb{A}^2 \to E$ is given by

$$(u, v) \mapsto [u : v : 1 - \alpha u - \beta v].$$

The image $\phi(E) \subset \mathbb{A}^2$ is given by the zero set of the following differential polynomials over $F\langle \alpha, \beta \rangle$:

$$(1 - \alpha u - \beta v)v^2 - u^3 a(1 - \alpha u - \beta v)u^2 + b(1 - \alpha u - \beta v)^3$$

$$\delta u - \beta v \delta u + \delta \alpha u^2 + \delta \beta uv + \beta u \delta v$$

$$\delta u - \alpha u \delta v + \delta \alpha uv + \alpha v \delta u + \delta \beta v^2.$$

In this specific example, E is actually a *differential algebraic group*. More specifically, it is a differential algebraic subgroup of an algebraic group. Let $+_E$ be the group operation on E. Then we obtain a group operation on the image of E in $\mathbb{A}^2$ via

$$\begin{array}{ccc} E & \xrightarrow{+_E} & E \\ \phi \big\downarrow & & \big\downarrow \phi \\ \mathbb{A}^2 & \xrightarrow{\phi \circ +_E \circ \phi^{-1}} & \mathbb{A}^2. \end{array}$$

Recall that we assume that F is differentially closed. Since the set of lines which intersect E is a definable subset of the Grassmannian, $\mathbf{GR}(1,2)$, by the model completeness of $DCF_{0,m}$ (see (45) for the ordinary case or (48) for the partial case), there is a line defined over F which does not intersect E. With more care, one could construct a morphism with all of the pertinent properties of ϕ which is defined over F. For instance, in the particular case given above, any line of the form $x + \gamma y = 0$ where γ is not a constant does not intersect E.

More generally, using Proposition 6.4.1 and a suitable generalization of the diagram and the model completeness mentioned in the previous paragraph:

Corollary 6.4.13. *Every differential algebraic variety of Δ-transcendence degree zero defined over F is isomorphic to an affine differential algebraic variety over F. Every differential algebraic group of Δ-transcendence degree zero defined over F is isomorphic to an affine differential algebraic group over F.*

Next, we will use proposition 6.4.2 to give an embedding of E into $\mathbb{P}^1$. Take $[\alpha : \beta : 1]$ generic in $\mathbb{P}^2$ over F. By proposition 6.4.2 the morphism, π, obtained by projection (of E) from this point to any hyperplane in $\mathbb{P}^2$ is injective. For instance, we might choose the hyperplane given by $z = 0$. In this case, the map $\pi : E \to \mathbb{P}^1$ is given by

$$[x : y : z] \mapsto [x - \alpha z : \beta z - y]$$

The map π^{-1} is given by

$$[u : v] \mapsto [uv + \frac{\alpha}{\delta(\alpha)}(v\delta(u) - u\delta(v)) : \frac{u\delta(v) - v\delta(u) - \frac{\delta(\beta)}{\beta}uv}{\frac{\delta(\beta)}{\beta}} : \frac{(v\delta(u) - u\delta(v))}{\delta(\alpha)}].$$

Any finite transcendence degree differential algebraic variety in $\mathbb{P}^1$ must be a proper subvariety. Naturally, the image of E under the above map has this property. Indeed, the second coordinate of the image never vanishes for $[x : y : z] \in E$. To see this, note that y and z do not simultaneously vanish on E; the unique point of E on the line at infinity is $[0 : 1 : 0]$. On the other hand, in E, $\frac{y}{z}$ is never equal to β, since β is a Δ-F-transcendental (remember, E has finite transcendence degree). So, the map $\psi : E \to \mathbb{A}^1$ is given by

$$[x : y : z] \mapsto \frac{x - \alpha z}{\beta z - y}.$$

Thus, as above, we obtain a commutative diagram.

$$\begin{array}{ccc} E & \xrightarrow{+_E} & E \\ \psi \downarrow & & \downarrow \psi \\ \mathbb{A}^1 & \xrightarrow{\psi \circ +_E \circ \psi^{-1}} & \mathbb{A}^1 \end{array}$$

Naturally, this gives another result along the lines of 6.4.13:

Corollary 6.4.14. *Every differential algebraic variety of Δ-transcendence degree zero defined over F is isomorphic to a constructible set in $\mathbb{A}^1$. Every differential algebraic group of Δ-transcendence degree zero defined over F is isomorphic to a differential algebraic group embedded in $\mathbb{A}^1$.*

Question 6.4.15. *Restrict to the ordinary case. Are all projective finite Morley rank differential algebraic varieties complete? In the partial case, this is a special case of the following two questions. Are all projective differential transcendence degree zero differential algebraic varieties complete? Are all projective finite transcendence degree differential algebraic varieties complete?*

Remark 6.4.16. Given the stark contrast of differential completeness to the classical notion in algebraic geometry, one should not expect theorems from the algebraic category which use the essentially projective nature of complete algebraic varieties to hold for complete differential algebraic varieties. For instance, complete algebraic varieties have so few regular functions that:

Proposition 6.4.17. *(Rigidity Theorem) Let $\alpha : V \times W \to U$ be a regular map, let V be complete and $V \times W$ geometrically irreducible. Then if there are $u_0 \in U(k)$, $v_0 \ inV(k)$, and $w_0 \in W(k)$ so that*

$$\alpha(V \times \{w_0\}) = \{u_0\} = \alpha(\{v_0\} \times W)$$

then $\alpha(V \times W) = \{u_0\}$.

For a proof, see (50). Of course, this theorem does not hold for differential algebraic varieties. For a simple counterexample, consider the projective closure of the affine differential algebraic variety $Z(x' - x^3) \subset \mathbb{A}^1$. The closure, in $\mathbb{P}^1$ is given by the zero set of differential homogenization of the differential polynomial, that is $V = Z(y(x'y - xy') - x^3) \subseteq \mathbb{P}^1$. V is complete (80) Notice that the point at infinity, $[1:0]$ is not a point on this variety. So, $\frac{x}{y}$ is a regular function on V. Now, consider

$$\alpha : V \times \mathbb{A}^1 \to \mathbb{A}^1$$

given by $\alpha([x, y], z) = \frac{x}{y} z$. The coordinate axes are mapped to zero, but the image of the map is all of $\mathbb{A}^1$.

6.5 A Valuative Criterion for Δ-completeness

The following is a proposition of van den Dries (88) mentioned in the introduction, which Pong used in the case of ordinary differential fields to establish a valuative criterion for completeness. We take a similar approach here.

Proposition 6.5.1. *T a complete $\mathcal{L}$-theory and $\phi(v_1 \ldots v_n)$ an $\mathcal{L}$-formula without parameters. Then the following are equivalent:*

1. *There is a positive quantifier free formula ψ such that T proves $\forall v \phi(v) \leftrightarrow \psi(v)$.*
2. *For any $K, L \models T$ and $f : A \to L$ an embedding of a substructure A of K into L, if $a \in \mathbb{A}^m$ and $K \models \phi(a)$ then $L \models \phi(f(a))$.*

Proposition 6.5.2. *Let $R \supseteq \mathbb{Q}$ be a Δ-ring. Every proper Δ-ideal I is contained in some prime Δ-ideal.*

Proof. (32, see II.5). $\square$

The next definition is essential for the criterion we give for completeness.

Definition 6.5.3. Let K be a Δ-field.

$$H_K := \{(A, f, L) : A \text{ is a } \Delta\text{-subring of } K, L \text{ is a } \Delta\text{-field}, f : A \to L \text{ a } \Delta\text{-homomorphism}\}.$$

Given $(A_1, f_1, L_1), (A_2, f_2, L_2) \in H_K$. Then f_2 extends f_1 if $A_1 \subset A_2$, $L_1 \subseteq L_2$, and $f_2|_{A_1} = f_1$. In this case, one could write $(A_1, f_1, L_1) \leq (A_2, f_2, L_2)$. With respect to this ordering, H_K has maximal elements. These will be called *maximal Δ-homomorphisms of K*.

Remark 6.5.4. There are no assumptions (yet) about these homomorphisms being over F. Eventually, we will enforce this condition model-theoretically by changing the formal language.

Definition 6.5.5. A Δ-subring is maximal if it is the domain of a maximal Δ-homomorphism. A Δ-ring is called a local Δ-ring if it is local and the maximal ideal is a Δ-ideal.

Proposition 6.5.6. *Let (R, f, L) be maximal in H_K. Then,*

1. *R is a local Δ-ring and $ker(f)$ is the maximal ideal.*

2. *$x \in K - R$ if and only if $\mathfrak{m}\{x\} = R\{x\}$.*

Proof. Kernels of Δ-homomorphisms are Δ-ideals. Let $x \notin ker(f)$. Then we extend f to $R_{(x)}$ by letting $x^{-1} \mapsto f(x)^{-1}$. By maximality, $x^{-1} \in R$. Thus, x is a unit. This establishes the first statement.

If $\mathfrak{m}\{x\} = R\{x\}$ then $1 = \sum_{i=1}^{k} m_i p_i(x)$. Then if $x \in R$, $f(1) = f(\sum m_i p_i(x)) = 0$, a contradiction.

For the converse: if $\mathfrak{m}\{x\} \neq R\{x\}$, then there is a prime Δ-ideal $\mathfrak{m}'$ which contains $\mathfrak{m}\{x\}$. So, we let K' be the fraction field of $R\{x\}/\mathfrak{m}'$.

$$\begin{tikzcd} & L' \arrow[dl] \arrow[dr] & \\ K' \arrow[dr] & & L \arrow[dl] \\ & K & \end{tikzcd}$$

Further, we have:

$$\begin{tikzcd} R\{x\} \arrow[r] & K' \arrow[r] & L' \\ R \arrow[u] \arrow[r] & K \arrow[u] \arrow[r] & L \arrow[u] \end{tikzcd}$$

But, one can see that the maximality condition on R means that $x \in R$. $\square$

Remark 6.5.7. We are about to give the valuative criterion for completeness. The varieties in question are given as closed subsets of affine space. This is slightly deceptive, especially if one wishes to think in analogy to classical results of algebraic geometry. In light of the results of chapter 3, any complete differential algebraic variety might be equipped with an affine embedding.

For the previous results of this section, we considered substructures in the language of differential rings, that is Δ-subrings. So, there was no assumption about the differential homomorphisms fixing F. For the next result, we take T to be $Th(F)$ in the language of differential rings, $\{\Delta, 0, 1, +, \cdot\}$ augmented by *constant symbols* $\{d\}_{d \in F}$ where F is a differentially closed field (this approach was mentioned in the introduction). The substructures in this augmented language are Δ-F-algebras. Homomorphisms of substructures in this language are differential ring homomorphisms which fix each element of F. The models of T are differentially closed fields which contain F.

Theorem 6.5.8. *Let X be a Δ-closed subset of $\mathbb{A}^n$ Then X is Δ-complete if and only if for any $K \models T$ and any R, a maximal Δ-subring of K containing F, we have that every K-rational point of X is and R-rational point of X.*

Proof. Suppose the valuative criterion holds for X. Then, to show that X is Δ-complete, it suffices to show that for any Δ-closed set $Z \subseteq X \times \mathbb{A}^n$, $\pi_2(Z)$ is Δ-closed. Given K and f with $K, L \models DCF_{0,m}$, let $f : A \to L$ be a Δ-homomorphism where A is a substructure of K. We let $\phi(y)$ be the formula saying $y \in \pi_2 Z$. We will show that if $a \in \mathbb{A}^n$ and $K \models \phi(a)$ then $L \models \phi(f(a))$. So, we assume there is a $x \in X(K)$ with $(x, a) \in Z$. Now, extend f to $\tilde{f} : R \to L'$

a maximal Δ-homomorphism. One can always assume that $L' \models DCF_{0,m}$, since if this does not hold, we simply take the Δ-closure. At this point, we have $F \subseteq A \subseteq R$. By the assumption, we know already that $x \in R$.

$$L' \models (\tilde{f}(x), \tilde{f}(a)) \in Z \wedge \tilde{f}(x) \in X.$$

So, $L' \models \tilde{f}(a) \in \pi_2 Z$. But, then $L \models f(a) \in \pi_2 Z$, since $\tilde{f}(a) = f(a)$. Now, using van Den Dries' condition, 6.5.1, we can see that $\pi_2 Z$ is Δ-closed.

Now, we suppose that the valuative criterion does not hold of X. So, we have some $f : R \to L$ a maximal Δ-homomorphism of K, with $R \supseteq F$. There is some point $x \in X(K)$, so that $x \notin R$. Then for one of the elements x_i in the tuple x, we know, by 6.5.6 that $1 \in \mathfrak{m}\{x_i\}$. Then, we know that there are $m_j \in \mathfrak{m}$ and $t_j \in R\{y\}$ so that

$$\sum_j m_j t_j(x) = 1.$$

We let $m := (m_1 \ldots m_k)$. So, we let Z be the differential algebraic variety given by the conditions

$$\sum_j z_j t_j(y_i) - 1 = 0 \text{ and } y \in X.$$

Then we take L to be the Δ closure of the Δ-field $R/\mathfrak{m}$. If we let g be the quotient map then $g|_F$ is an embedding. Then we have that $K \models \exists y \phi(y, m)$ since x is a witness. But, we see that

L can not have a witness to satisfy this formula, $m \in ker(f)$. Then again, by 6.5.1, we see that $\pi_2 Z$ is not Δ-closed. □

One can rephrase the criterion for an affine Δ-closed subset X, a fact noticed by Pong, in the ordinary case. For any $K \models DCF_{0,m}$, let R be a maximal Δ-subring of K which contains F. Let $A = K\{y_1, \ldots, y_n\}/I(X)$. Suppose we are given a commutative diagram,

$$\begin{tikzcd} SpecK \arrow[r] \arrow[d] & SpecA \arrow[d] \\ SpecR \arrow[r] & SpecF \end{tikzcd}$$

then we have the diagonal morphism,

$$\begin{tikzcd} SpecK \arrow[r] \arrow[d] & SpecA \arrow[d] \\ SpecR \arrow[r] \arrow[ur, dashed] & SpecF \end{tikzcd}$$

6.6 Δ-complete varieties

In this section, we will use some commutative algebra along with the valuative criterion given above to give examples of Δ-complete sets. Let $\bar{y} = (y_1, \ldots, y_n)$ and consider a system of linear differential equations of the form $\{\delta_i \bar{y} = A_i \bar{y}\}_{i=1}^m$ where $A_i \in gl_n(K)$. The system is *integrable* if

$$\delta_i A_j - \delta_j A_i = A_i A_j - A_j A_i$$

for $1 \leq i, j \leq m$. In the case $n = 1$, one can quite clearly see that the conditions are a special case of the notion of coherence present in the axioms for differentially closed fields:

Example 6.6.1. Let $\Delta = \{\delta_1, \delta_2\}$. Let n=1. Then we are considering the system of equations given by the zero set of two differential polynomials, f_1, f_2:

$$f_1(y) = \delta_1 y - a_1 y = 0$$

$$f_2(y) = \delta_2 y - a_2 y = 0.$$

In this case, the system is coherent if $\delta_2(f_1(y)) - \delta_1(f_2(y)) \in I(f_1, f_2)$. But

$$\begin{aligned} \delta_2(f_1(y)) - \delta_1(f_2(y)) &= \delta_2(\delta_1 y - a_1 y) - \delta_1(\delta_2 y - a_2 y) \\ &= \delta_2\delta_1(y) - \delta_2(a_1)y - a_1\delta_2(y) - \delta_1\delta_2(y) + \delta_1(a_2)y - a_2\delta_1(y) \end{aligned}$$

Taking into account the relations of $I(f_1, f_2)$, we see:

$$\delta_2(f_1(y)) - \delta_1(f_2(y)) \quad =_{mod(I(f_1,f_2))} \quad \delta_2(a_1)y - \delta_1(a_2)y$$

Clearly, the expression on the right hand side is zero only if $\delta_2(a_1) - \delta_1(a_2) = 0$. The integrability conditions given here are standard hypotheses in differential galois theory (15, section two). If the conditions are not satisfies, then the only point in the solution set of the system is zero. Since we are looking for examples of complete Δ-varieties, we want to rule out the scenario that the variety is simply a point. Generally speaking, this example is indicative with the complications with specifying classes of partial differential varieties:

The class of examples we wish to consider in this section will be both more and less general than the preceding example. We consider systems of equations in one variable which are nonlinear.

Theorem 6.6.2. *Let V be the Δ-closure of $Z(\{\delta_i y - P_i(y)\}_{i=1}^m)$ in $\mathbb{P}^1$, where $P_i(y) \in F[y]$, the collection $\{\delta_i y - P_i(y)\}_{i=1}^m$ is coherent as a collection of differential polynomials, and at least one of the P_i is of degree greater than one. Then V is Δ-complete.*

Remark 6.6.3. Thanks to an anonymous referee for pointing out that "integrability" conditions are necessary in the statement of the theorem.

Proof. One should homogenize $\delta_i y - P_i y$ in $\mathbb{P}^1$ to calculate the Δ-closure. If $deg P_i = d_i > 1$ then $H_i(y, y_1) = y_1^{d_i}(\delta_i(\frac{y}{y_1}) - P_i(\frac{y}{y_1}))$ is homogeneous. One can easily see (by examining the leading monomial of $y_1^{d_i} P_i(\frac{y}{y_1})$) that $[1, 0] \notin Z(H_i(y, y_1))$. Thus, the set V in affine space is equal to its projective closure. Now, we can use the valuative criteria to establish the completeness of the variety.

Let $x \in V(K)$ and R a maximal Δ-subring containing F. It is enough to show that $\mathfrak{m}_K\{x\} \neq R\{x\}$. Since $\delta_i x = P_i(x)$,

$$\mathfrak{m}\{x\} = \mathfrak{m}[x]$$

$$R\{x\} = R[x].$$

So, since by classical commutative algebra, either $\mathfrak{m}[x]$ or $\mathfrak{m}[x^{-1}]$ is the unit ideal, see (1), it is enough to show that $\mathfrak{m}[x^{-1}]$ is the unit ideal. Now, we take the approach of Pong (70), using a result of Blum (9) which is also contained in (55).

Definition 6.6.4. Let R be a δ ring. An element of $R\{y\}$ is monic if it is of the form $y^n + f(y)$ where the total degree of f is less than n. An element in a δ-field extension of R is monic over R if it is the zero of a monic δ-polynomial.

Proposition 6.6.5. *If $(R, \mathfrak{m})$ is a maximal δ-subring of K, then $x \in K$ is monic over R if and only if $x^{-1} \in \mathfrak{m}$.*

Remark 6.6.6. Though the results of Blum and Morrison are in the context of ordinary differential algebra, Proposition 6.5.6 lets us apply their results, since a local Δ-ring is a local δ-ring for any $\delta \in \Delta$.

Since $degP_i \geq 2$,

$$a^{-1}(\delta_i y - P_i(y))$$

is monic, where a is the coefficient of the leading monomial of P_i. x monic implies $x^{-1} \notin \mathfrak{m}$, by Blum's theorem (9).

So, if $x^{-1} \in R$, then $x \in R$. Thus, we assume $x^{-1} \notin R$. Then $\mathfrak{m}\{x^{-1}\}$ is the unit ideal.

$$\delta_i(x^{-1}) = -x^{-2}P_i(x).$$

This means that we can get some expression,

$$1 = \sum_{j=r}^{s} m_j x^j \tag{6.3}$$

with $m_j \in \mathfrak{m}$ and r and s integers. Then, applying δ_i to both sides of the equation yields:

$$1 = \sum_{j=r}^{s} \delta_i(w_j)x^j + \frac{\partial x^j}{\partial x} P_i(x)$$

The leading term of the sum is $am_s s x^{s-1} x^d$. But, a is a unit and so we can divide both sides of the equation by sax^{d-1} to obtain an expression for $m_s x^s$ as a sum of lower degree terms. Substituting this expression into Equation 6.3, we get an expression

$$1 = \sum_{j=r_1}^{s_1} n_j x^j \tag{6.4}$$

Continuing in this manner, one can assume that we have an expression of the form

$$1 = \sum_{j=r}^{0} w_j x^j.$$

Then we see that $1 \in \mathfrak{m}[x^{-1}]$. □

Remark 6.6.7. Beyond order 1 the techniques as shown above are not as easy to apply. For techniques in that situation, at least in the case of linear ordinary differential varieties, see (80).

One can combine the above techniques of that paper with the above techniques to give a wider class of complete partial differential varieties.

CHAPTER 7

RELATIVE KOLCHIN IRREDUCIBILITY

7.1 Introduction

Kolchin's irreducibility theorem says that if V is an algebraic variety over an algebraically closed field of characteristic zero, then V is also irreducible in the (finer) Kolchin topology. Here is the central question we concern ourselves with here:

Question 7.1.1. *Suppose K is a Δ-closed field, where $\Delta = \{\delta_1, \dots, \delta_m, \delta_{m+1}\}$. Now, suppose that V is an irreducible $\Delta \backslash \{\delta_{m+1}\}$ variety over K in the $\Delta \backslash \{\delta_{m+1}\}$-Kolchin topology. Is V irreducible in the Δ-Kolchin topology over K? If so, what is the Kolchin polynomial of V in the Δ-topology?*

We will call this a relative version of the Kolchin irreducibility question. This question has implications regarding how one might axiomatize differentially closed fields (in the partial case, see (77)), but we do not touch on that here. Also, the connections between Kolchin's irreducibility theorem and the irreducibility of arc schemes are well documented (see for instance (24) (58) (57)). Let X be a variety over k. The structure structure of $\varprojlim \mathcal{A}_m(X/k)$ as a k-scheme corresponds for the structure of X as a δ-variety. A reasonable future direction of this work would be to compare the structure of the arc space of a Δ-variety V, $\varprojlim \mathcal{A}^{\Delta}_m(V/k)$ as a Δ-k-scheme (no longer of finite type) to the structure of V itself in the richer $\Delta \cup \{\delta_{m+1}\}$-topology. We leave these issues as avenues for future research.

Gillet (24) proves Kolchin's theorem using commutative algebra. An alternate approach can be found in (45) or (34). In the next section, we will follow the approach taken by Gillet. The introduction does not cover the prerequisites for understanding Gillet's proof of Kolchin's theorem (and thus the results of this chapter). See (24) for complete references. We will also use facts from category theory rather freely; everything we use is covered in (42). The proof follows almost trivially from Gillet's methods if one assumes Kolchin's theorem. In the third section, we give an alternate approach using characteristic sets and differential algebraic techniques.

We will use both of the approaches mentioned in the previous paragraph to answer our question in the coming sections. The second approach was completed only a few weeks before Phyllis Cassidy informed the author that this relative version of the Kolchin irreducibility theorem was actually proved by Kolchin (35, Chapter 1, section 6). Our proof in section three of this chapter turned out to be nearly identical to Kolchin's original proof.

7.2 Relative Kolchin irreducibility

We adopt the following notation: $K\{\bar{x}\}_\Delta$ is the Δ-polynomial ring over K. $[S]_\Delta$ is the Δ-ideal generated by S.

Following Gillet, we define the functor

$$(-)^\infty : K\text{-algebras} \to \delta\text{-}K\text{-algebras}$$

to be the left adjoint of the forgetful functor from δ-K-algebras to K-algebras.

For a proof of the existence and representability, along with many properties of this functor, see (24). For the remainder of this chapter, we will refer to this functor as the prolongation functor; please note that prolongation functors are used elsewhere in this thesis following the notation of (53) where the definition is different.

Theorem 7.2.1. *(Kolchin's irreducibility theorem (34)) Let R be an integral domain of finite type over a differential field k. Then the associated differential algebraic variety is irreducible; that is R^∞ is an integral domain.*

Remark 7.2.2. Kolchin's proof uses characteristic sets and reduction theory; for a proof of relative Kolchin irreducibility in this style, see the next section. From the perspective of differential algebra, this has the advantage of computing the Kolchin polynomial of the differential variety. Gillet's proof uses various developments from commutative algebra and category theory; it applies to more general rings than Kolchin's original proof.

For simplicity, for the remainder of this section, we fix the notation $\delta := \delta_{m+1}$.

Theorem 7.2.3. *Let V be an affine irreducible $\Delta\backslash\{\delta\}$-$K$-variety. Then V is irreducible in the Δ-K-topology.*

Proof. V is irreducible over K if and only if the ideal of $\Delta\backslash\{\delta\}$-K polynomial functions which vanish on V is prime. Let $\mathfrak{p} := I(V)$. This means that $K\{\bar{x}\}_{\Delta\backslash\{\delta\}}/\mathfrak{p}$ is an integral domain.

Now, regarding this integral domain as a K-algebra (via the forgetful functor, if you like), $(K\{\bar{x}\}_{\Delta\backslash\{\delta\}}/\mathfrak{p})^\infty$ is isomorphic to $K\{\bar{x}\}_\Delta/[p]_\Delta$, the coordinate ring of V in the Δ-topology (we mean isomorphic as K-algebras, of course). Left adjoints always commute with direct limits, so $(K\{\bar{x}\}_{\Delta\setminus\{\delta\}}/\mathfrak{p})^\infty$ is the direct limit

$$\varinjlim A^\infty$$

where A ranges over the finitely generated subalgebras of the K-algebra $K\{\bar{x}\}_{\Delta\setminus\{\delta\}}/\mathfrak{p}$. To each such A, we may apply Kolchin's theorem 7.2.1, so A^∞ is an integral domain. A direct limit of integral domains is an integral domain. Then $[\mathfrak{p}]_\Delta$ is a prime ideal in $K\{\bar{x}\}_\Delta$ this implies $V([\mathfrak{p}]_\Delta)$ is irreducible. $\square$

7.3 An alternate approach to relative irreducibility

The goal of this section is to (re-)prove the relative Kolchin irreducibility theorem in a more detailed form.

Theorem 7.3.1. *Let V be an affine irreducible $\Delta\setminus\{\delta\}$-$K$-variety. Then V is irreducible in the Δ-K-topology. Further, if*

$$\omega_{V_{\Delta\setminus\{\delta\}}}(t) = \sum_{i=0}^{m} a_i \binom{x+i}{i}$$

then

$$\omega_{V_\Delta}(t) = \sum_{i=0}^{m+1} a_{i-1} \binom{x+i}{i}$$

Proof. Adopt the notation of the previous section. Specifically, let $\mathfrak{p} \in K\{y\}_{\Delta\setminus\{\delta\}}$ be the defining ideal of V, an irreducible $\Delta\setminus\{\delta\}$-variety. Suppose that $\bar{a} = a_1, \ldots, a_n$ is a generic zero of V over some $\Delta\setminus\{\delta\}$-closed field K. Fix the canonical orderly ranking on $\Theta_\Delta y = \Theta_\Delta(y_1, \ldots, y_n) = \{\delta_1^{e_1} \ldots \delta_{m+1}^{e_{m+1}} y_j \mid e_i, j \in \mathbb{N}\}$ given by the lexicographic order on $(\sum e_i, j, e_1, \ldots, e_{m+1})$. Consider the induced subranking on $\Theta_{\Delta\setminus\{\delta\}}y$. Then let Λ be a characteristic set of $\mathfrak{p}$ with respect to this subranking. Then Λ is autoreduced with respect to the full ranking on $\Theta_\Delta y$.

Suppose that the leaders of $f_1, f_2 \in \Lambda$ have a common derivative. Then $\theta_1 f_1 = \theta_2 f_2$. First, note that if $\theta_1 = \delta_1^{e_1} \dots \delta_m^{e_m} \delta^{e_{m+1}}$ and $\theta_2 = \delta_1^{d_1} \dots \delta_m^{d_m} \delta^{d_{m+1}}$, then $d_{m+1} = e_{m+1}$. So, suppose first that $e_{m+1} = 0$. Then the fact that Λ is a characteristic set implies that $S_{f_2}\theta_1 f_1 - S_{f_1}\theta_2 f_2 \in (\Lambda_{\theta_1 f_1}) : H_\Lambda^\infty$. We now observe that this special case is sufficient to prove that Λ is coherent in $K\{y\}_\Delta$.

Indeed, to verify that a set is coherent, one only needs to verify the hypothesis for the lowest common derivatives of any of the leaders of pairs of members of the set. This point is made on page 167 of (34), but we will sketch it here. It is clear that if

$$S_{f_2}\theta_1 f_1 - S_{f_1}\theta_2 f_2 \in (\Lambda_{\theta_1 f_1}) : H_\Lambda^\infty$$

, then

$$\delta(S_{f_2}\theta_1 f_1 - S_{f_1}\theta_2 f_2) \in (\Lambda_{\delta\theta_1 f_1}) : H_\Lambda^\infty$$

But, then

$$S_{f_2}\delta\theta_1 f_1 - S_{f_1}\delta\theta_2 f_2 = \delta(S_{f_2}\theta_1 f_1 - S_{f_1}\theta_2 f_2) - \delta(S_{f_2})\theta_1 f_1 + \delta(S_{f_1})\theta_2 f_2 \in (\Lambda_{\delta\theta_1 f_1}) : H_\Lambda^\infty$$

and the result follows by induction on the order of the common derivative of the leaders. Then we know that Λ is the characteristic set of a prime differential ideal in $K\{y\}$. So, $\mathfrak{q} = [\Lambda]_\Delta : H_\Lambda^\infty$

is a prime differential ideal. Further, since we have done the computations with respect to an orderly ranking, we know, using Theorem 6, part d) of (34), page 115 along with simply computing the quantity $V(r)$ which appears in that proof, that

$$\omega_{\mathfrak{q}}(t) = \sum_{i=0}^{m+1} a_{i-1} \binom{x+i}{i}$$

where a_i are such that

$$\omega_{\mathfrak{p}}(t) = \sum_{i=0}^{m} a_i \binom{x+i}{i}$$

Now, it is clear that $\{\mathfrak{p}\}_\Delta \subseteq \mathfrak{q}$. But, a generic solution $\bar{a}$ to $\mathfrak{p}$ has the property that H_Λ does not vanish at $\bar{a}$. But, then $\bar{a} \in V(\mathfrak{q})$, so $\mathfrak{p} = \mathfrak{q}$. $\square$

CHAPTER 8

INTERSECTION THEORY

8.1 Introduction

Proposition 8.1.1. *(7.1 pg 48, (26)) Let Y, Z be irreducible algebraic varieties of dimensions r, s in $\mathbb{A}^n$ Then every irreducible component W of $Y \cap Z$ has dimension $r + s - n$.*

This theorem fails for differential algebraic varieties embedded in projective space, as the next example shows. We should note that there are some positive partial results in the case that one considers ordinary differential algebraic varieties embedded in a finite Morley rank differential algebraic variety (29; 30) or in the case of linear equations (81).

Example 8.1.2. (Ritt's example). We work in $\mathbb{A}^3$ over an ordinary differential field, k. Let $V = Z(f)$, where

$$f(x, y, z) = x^5 - y^5 + z(x\delta y - y\delta x)^2$$

In fact, though V is the zero set of an absolutely irreducible differential polynomial, it is not irreducible in the Kolchin topology. V has six components. For each of the fifth roots of unity, $x - \zeta y$ cuts out a subvariety of V. To show that each of these is actually a component of V, one can use the Low Power theorem (see chapter 7 of (73)). The general component is given by the saturation by the separant (with respect to some ranking) of $[f]$.

The general component of V intersects the hyperplane $z = 0$ in precisely one point, $(0, 0, 0)$. Seeing that the origin is in the intersection is relatively straightforward, however, proving it is the only point in the intersection seems to require the use of Levi's lemma (34) page 177. For a complete exposition of this example, see (83).

The motivating question of the chapter is:

Question 8.1.3. *Is Ritt's anomalous example exceptional or generic for intersections of differential algebraic varieties? More technically: in moving families of differential algebraic varieties, what is the locus on which the intersection theorem fails with a given arbitrary differential algebraic variety? Also what is the locus on which the intersection is reducible? Do these loci even have differential algebraic structure?*

The main thrust of the chapter is to provide an answer (of sorts) to the question by proving a differential analogue of Bertini's theorem. After this, we point out several applications of the main theorem, as well as future directions. In practice, anomalous intersections cause two problems with respect to Bertini-style theorems. First, hyperplane sections might not be codimension one. Second, the potential for anomalous intersections makes proving irreducibility results more difficult in differential algebraic geometry. The possibility of small dimensional components in intersections are a worry which can be more easily dismissed in the algebraic case, by applying the intersection theorem.

Dispensing with the first problem is reasonably straightforward. The second problem is slightly more involved. Overcoming it essentially involves several steps: applying differential algebraic reduction theory to establish the primality of a differential ideal over a specific field, using a differential lying over theorem for prime differential ideals, followed by geometric arguments.

We saw the idea for the first step in (90), in the ordinary case. In that paper, the authors also give an algebraic solution to calculating the dimensions of intersections. To argue about the dimensions of intersections here, we use model theoretic tools as well as the ideas in (90) (generalized to the partial case). In some cases, model theoretic tools allow proofs to be shortened significantly. Further, in the case of hyperplane sections, we pursue stronger irreducibility results; we are not ultimately interested in irreducibility over a specific differential field. Rather, our goal is to prove *geometric irreducibility* results:

Definition 8.1.4. An affine differential algebraic variety, V over k, is geometrically irreducible if $I(V/k')$ is a prime differential ideal for *any* k', a differential field extension of K.

In model theoretic terms, this corresponds to the generic type of V over k being stationary. The versions of Bertini's theorem proved in (90) are purely algebraic (27, see page 54), that is, they apply to hyperplanes sections cut out by hyperplanes with coefficients which are new generic indeterminants - the results are not valid when the coefficients are not independent differential transcendentals. However, for applications, one often wishes to take to the coefficients of the hyperplanes in some field extension and then consider irreducibility over that field extension. Unfortunately, the irreducibility results of (90) would not be true in that setting even for algebraic varieties (without the hypothesis that the dimension of V is greater than one). In fact, we will show that algebraic curves are the only potential counterexamples to geometric irreducibility.

Most of the intersection theory results in (90) have generalizations to the partial differential case, but some of the main tools (e.g. *characteristic sets*) are trickier in the partial differential setting. We should point out that some of the results of (90) are directly implied by earlier results of (70), but since that paper is written from a model theoretic point of view, the connections may not be obvious to the reader not familiar with the languages of both model theory and differential algebra. The main thrust of (90) is to develop differential Chow varieties in the ordinary setting. Though this direction is fascinating, we pursue applications of the theory in a different direction, and leave the exploration of partial differential Chow varieties for the future.

From the model theoretic point of view, most of the interesting questions about *ordinary* differential fields occur at the level of the finite rank types (equivalently field extensions or varieties). This is because there is precisely one infinite rank type in $S_1(K)$ - namely that of a δ-transcendental. Of course, there are interesting questions about the infinite rank definable sets, but these fall outside of much of the detailed modern model theoretic analysis (i.e. geometric stability theory), because nonorthogonality is not sufficiently sensitive to detect variations on this level.

The situation is similar in partial differential fields, but the line is drawn *not* at the finite-infinite level, but rather at the level of the degree of the Kolchin polynomial being at or below m (the number of derivations). The model theoretic-differential algebraic correspondence is just as strong at this level in the partial case as it is at the finite-infinite level in the ordinary case. Most of the strength of this correspondence is present at other levels of Lascar rank as well (for instance, for $m_1 < m$, considering the types of Lascar rank less than ω^{m_1}). For these intermediate levels of rank, it is not the model theoretic correspondence which breaks down (and prevents results like those of this chapter); rather, these differential algebraic varieties, though complicated and in a certain sense even often infinite dimensional, behave like zero-dimensional algebraic varieties with respect to generic intersections.

After setting up the intersection theory in the partial differential case, we will show several applications of model theoretic or geometric interest. The first is an infinite-dimensional analogue of a model theoretic idea in strongly minimal theories. In a strongly minimal theory Morley rank is definable; take a formula $\phi(\bar{x}, \bar{y})$ over the empty set with parameters $\bar{y}$ from a model $\mathcal{M} \models T$. Then $\{\bar{y} \,|\, RM(\phi(x, y)) = n\}$ is a definable set. We prove the same fact for differential fields with Δ-transcendence degree playing the role of Morley rank. This allows an analysis of the exceptional sublocus of the Grassmannian in our version of Bertini's theorem, at least with respect to the dimension conclusion. Namely, we prove that intersections have appropriate dimension on a Kolchin open subset of the Grassmannian. We do not provide any characterization of the exceptional sublocus with respect to irreducibility. In fact, proving that the exceptional sublocus is a differential algebraic subvariety of the Grassmannian would yield a nontrivial reduction to the Ritt problem, described in the final section.

In the final section, we generally consider the problem of irreducibility in families. Let

$$\phi : V \to S$$

be a morphism of differential algebraic varieties. Is the set

$$\{s \,|\, \phi^{-1}(s) \text{ is irreducible}\}$$

a constructible set in the Kolchin topology? This problem is equivalent to several well-known and long-standing open problems in differential algebraic geometry, and we do not solve it here. We solve a related problem in the ordinary case. Namely, the answer to the above question is yes when irreducibility is replaced by *generic irreducibility*.

8.2 Differential specializations

Definition 8.2.1. Let $\Delta = \{\delta_1, \dots, \delta_m\}$. Let $\Delta' = \{\delta_1', \dots, \delta_m'\}$. A homomorphism ϕ from Δ-ring (R, Δ) to Δ'-ring (S, Δ') is called a differential homomorphism if for each i, $\phi \circ \delta_i = \delta_i' \circ \phi$. When R is an integral domain and S is a field, then such a map is called a Δ-*specialization.*

Remark 8.2.2. Differential algebraists sometimes use $\bar{y}$ to denote a specialization of y. Since model theorists often denote tuples of variables in this way, and we will avoid this notation. For instance, if the specialization is given by a homomorphism ϕ, then when necessary we will write $\phi(y)$ for the specialization of y. The following proposition is proved in a constructive manner in (90, Theorem 2.16), but seems to be a natural consequence of the model-theoretic setup.

Proposition 8.2.3. *Let $\bar{u} = (u_1, \dots, u_r) \subset \mathfrak{U}$ be a set of Δ-independent differential transcendental elements over K. Let $\bar{y} = (y_1, \dots, y_n)$ be a set of differential indeterminants. Let $P_i(\bar{u}, \bar{y}) \in K\{\bar{u}, \bar{y}\}$ for $i = 1, \dots, n_1$. Suppose $\phi : K\{\bar{y}\} \to \mathfrak{U}$ be a differential specialization into $\mathfrak{U}$ such that $\phi(y_i) \mathop{\smile\!\!\!\!\!\!\mid}_K K\langle \bar{u} \rangle$. Suppose that $P_i(\bar{u}, \phi(\bar{y}))$ are (as a collection), not independent over $K\langle \bar{u} \rangle$. Then let ψ be a differential specialization from $K\langle \bar{u} \rangle \to K$. The collection $\{P_i(\psi(\bar{u}), \phi(\bar{y}))\}_{i=1,\dots,n_1}$ are not independent over K.*

Proof. The positive atomic formulas which witness non-independence are pushed forward by the differential homomorphism, witnessing non-independence in the image. $\square$

8.3 Intersection theory

In this section we develop an intersection theory for differential algebraic varieties with generic Δ-polynomials. The influence of (90) for proving statements about irreducibility over specific differential fields is obvious; we have adapted their techniques to the partial differential setting. Our arguments about dimensions of intersections were done earlier from a more model-theoretic point of view. The following definition matches that of (90, if restricted to the ordinary case).

Definition 8.3.1. Let X be a Δ-K-variety. Denote, by $dim(X/K)$ the Δ-transcendence degree of a generic point on X over K. As usual, via the correspondence between types, tuples, and differential varieties we will abuse notation and write $dim(p)$ for $p \in S(K)$ or $dim(\bar{a}/K)$ for some tuple in a Δ-field extension (see chapter 2, section 2 of this thesis).

We will be using Lascar rank at various points, and remind the reader of the following result, which we use implicitly throughout the section:

Theorem 8.3.2. *((48)) Let b be a tuple in a differential field extension of k. Then*

$$dim(b/k) = n \text{ if and only if } \omega^m \cdot n \leq RU(tp(b/k)) < \omega^m \cdot (n+1)$$

8.3.1 Intersections with generic hypersurfaces

Definition 8.3.3. In $\mathbb{A}^n$, the differential hypersurfaces are the zeros of a Δ-polynomial of the form

$$a_0 + \sum a_i m_i$$

where m_i are differential monomials in $\mathcal{F}\{y_1, \ldots, y_n\}$. For convenience, in the following discussion, we do not consider 1 to be a monomial. A *generic Δ-polynomial* of order s and degree r over K is a Δ-polynomial

$$f = a_0 + \sum a_i m_i$$

where m_i ranges over all differential monomials of order less than or equal to s and degree less than or equal to r and $\bar{a} = (a_0, a_1, \ldots, a_n)$ is Δ-transcendental over K. A *generic Δ-hypersurface* of order s and degree r is the zero set of a generic Δ-polynomial of order s and degree r. When f is given as above, we let $\bar{a}_f$ be the tuple of coefficients of f. Throughout, we adopt the notation $\bar{a}_f = a_f \backslash \{a_0\}$.

The next lemma is proved in the ordinary case in (90) (Lemma 3.5). The proof in this case works similarly, assuming that one sets the stage with the proper reduction theory in the partial case. One might notice that Lemma 3.5 of (90) has a second portion. For now, we will concentrate only on the irreducibility of the intersection. Necessary and sufficient conditions for the intersection to be nonempty will be given later.

Lemma 8.3.4. *Let $\mathfrak{I}$ be a prime Δ-ideal in $K\{\bar{y}\}$ with differential transcendence degree d and let $f = y_0 + \sum_{i=1}^n a_i m_i$ with $(a_1, \ldots, a_n)$ differentially independent over K. Then $\mathfrak{I}_0 = \{\mathfrak{I}, f\}$ is a prime Δ-ideal of $K\langle \bar{a}_f \rangle\{\bar{y}, y_0\}$.*

Proof. Let $\bar{b} = (b_1, \ldots, b_n)$ be a generic point of $V(I)$ over K such that $\bar{b}$ is independent from $\bar{a}$ over K. In model theoretic terms, $\bar{b} \mathop{\smile\hspace{-0.9em}\mid\hspace{0.3em}}_K \bar{a}$. Let $f = y_0 + \sum_{i=1}^n a_i m_i$. Consider the tuple $(b_1, \ldots, b_n, -\sum_{i=1}^n a_i m_i(\bar{b}))$. Let $\mathfrak{I}_0 = [\mathfrak{I}, f]$. We show irreducibility of the variety $V(\mathfrak{I}_0)$ in

$\mathbb{A}^{n+1}$ via showing that it is the Kolchin closure of $(b_1, \ldots, b_n, -\sum_{i=1}^{n} a_i m_i(\bar{b}))$ over K. Since only irreducible sets over K have K-generic points in the Kolchin topology, this will complete the proof (said another way, being the locus over K of a tuple in a differential field extension is precisely equivalent to being an irreducible Δ-K-closed set).

Suppose g is a Δ-polynomial over $K\langle \bar{a}_f \rangle$ which vanishes at $(b_1, \ldots, b_n, a_0)$. Consider the f as a Δ-polynomial of $K\langle \bar{a}_H \rangle\{\bar{y}, y_0\}$. Fix a ranking so that y_0 is the leader of f. Then reducing g with respect to f gives some g_0 (which is equivalent to g modulo f). This g_0 must be in $K\langle \bar{a}_f \rangle\{y_1, \ldots, y_n\}$. Of course, since $\bar{b}$ is generic for $\mathfrak{I}$, we must have that $g_0 \in K\langle \bar{a}_f \rangle \cdot \mathfrak{I}$. But then $g \in \mathfrak{I}_0$ and the claim follows. $\square$

Lemma 8.3.5. *Following the notation of the previous lemma assume $d > 0$; then $RU(V(\mathfrak{I}_0)) = RU(V(\mathfrak{I}))$.*

Proof. Take a nonforking extension of a generic type of $V(I)$ to $K\langle \bar{a}_f \rangle$. Let $\bar{b} = (b_1, \ldots, b_n)$ be a realization of this type. Then $(\bar{b}, -\sum_{i=1}^{n} a_i m_i(\bar{b}))$ is a point on $V(\mathfrak{I}_0))$. So, $RU(V(\mathfrak{I}_0)) \geq RU(V(\mathfrak{I}))$. On the other hand, for any point $\bar{c} \in V(\mathfrak{I}_0)$, c_{n+1} is in the field $K\langle \bar{a}_f \rangle(c_1, \ldots, c_n)$. In model theoretic terms, the $n+1$st coordinate is necessarily in the definable closure of the first n. $RU(\bar{c}/K\langle \bar{a}_f \rangle) = RU((c_1, \ldots, c_n)/K\langle \bar{a}_f \rangle)$. This establishes $RU(V(\mathfrak{I}_0)) = RU(V(\mathfrak{I}))$. $\square$

Corollary 8.3.6. *Suppose that the Δ-transcendence degree of $\mathfrak{I}$ is equal to d. Then the Δ-transcendence degree of $\mathfrak{I}_0$ is equal to d.*

Now we turn towards establishing necessary and sufficient conditions for the intersection to be nonempty when we relax the sorts of intersections under consideration. In the case that the intersection is nonempty, we calculate the differential transcendence degree.

Lemma 8.3.7. *Suppose that V is a differential algebraic variety such that $RU(V/K) < \omega^m$. Then $V \cap V(f(\bar{x})) = \emptyset$ for any generic differential polynomial $f(\bar{x})$.*

Proof. This was originally proved in (71) in the ordinary case, and was reproved in (90) in the ordinary case. The proof for hypersurfaces in the partial case can be found in chapter six of this thesis, Proposition 6.4.1. □

Lemma 8.3.8. *Suppose that V is a differential algebraic variety embedded in $\mathbb{A}^n$ with $\omega^m \cdot n_1 \leq RU(V/K) < \omega^m \cdot (n_1+1)$ where $n_1 \geq 1$. If $f(\bar{x})$ is a generic differential polynomial, then*

$$\omega^m \cdot (n_1 - 1) \leq RU(V \cap V(f)/K_1) < \omega^m \cdot n_1,$$

where K_1 is the differential field extension of K generated by the coefficients appearing in f.

Proof. Let us prove that the Lascar rank is at least $\omega^m \cdot (n_1 - 1)$. Suppose that for a realization of the generic type, $b_1, \ldots, b_{n_1-1}$ are Δ-dependent mod $\mathfrak{I}_1$, where perhaps we rearrange the coordinates so that $y_1, \ldots, y_{n_1}$ is such that for a generic realization on $\bar{b} \in V(\mathfrak{I})$, $b_1, \ldots, b_{n_1}$ are a Δ-transcendence base for the differential function field generated by $\bar{b}$. Then we get some $f \in K\{\bar{a}_H, b_1, \ldots, b_{n_1-1}, -\sum a_i b_i\}$ and we see that $b_1, \ldots, b_{n_1-1}, -\sum a_i b_i$ are dependent over $K\langle \bar{a}_H \rangle$. Now specialize a_d to -1 and specialize all other $a_i \in \bar{a}_H$ to 0. But then $b_1, \ldots, b_d$ are dependent over K by 8.2.3, a contradiction.

The upper bound follows easily from considering the projection of the last coordinate of the variety in 8.3.5 and applying the Lascar inequality. $\square$

Lemma 8.3.9. *Let $\mathfrak{I}$ be a prime Δ-ideal in $K\{y_1, \dots, y_n\}$. Let $f = y_0 + \sum_{i=1}^{n} a_i m_i$ give a generic hypersurface. Then $\mathfrak{I}_1 = \{\mathfrak{I}, f\}$ is a prime Δ-ideal in $K\langle a_0, a_1, \dots, a_n\rangle\{y_1, \dots, y_n\}$.*

Proof. In the case that $\tau(V(\mathfrak{I})) < m$, that is, the differential transcendence degree is zero, $V(I) \cap V(f) = \emptyset$ by Lemma 8.3.7. Thus $\{\mathfrak{I}, f\} = K\langle y_0, a_1, \dots, a_n\rangle\{y_1, \dots, y_n\}$.

Now suppose that $\tau(V(I)) = m$, that is, the differential transcendence degree is at least one. Then by Lemma 8.3.8, $V(I) \cap V(f) \neq \emptyset$. Recall the notation of $\mathfrak{I}_0$ from Lemma 8.3.4. We will show that $\mathfrak{I}_1 \cap K\langle a_1, \dots, a_n\rangle\{y_1, \dots, y_n, y_0\} = \mathfrak{I}_0$. Suppose that we have $g, h \in K\langle a_1, \dots, a_n, y_0\rangle\{y_1, \dots, y_n\}$ such that $g \cdot h \in \mathfrak{I}_1$. Since we are taking a field extension over K, the coefficients of the differential polynomials might involve differential rational functions in $a_1, \dots, a_n, y_0$ over K. This is easily dispensed with since if we multiply by suitable differential polynomials in $a_0, \dots, a_n, y_0$ over K, we will get $g, h \in K\{a_1, \dots, a_n, y_0, y_1, \dots, y_n\}$ such that $g \cdot h \in \mathfrak{I}_0$. But, $\mathfrak{I}_0$ is prime by Lemma 8.3.4. So, we have a contradiction and $\mathfrak{I}_1$ is prime. Further, we can see (again, simply by clearing denominators) that $\mathfrak{I}_1$ lies over $\mathfrak{I}_0$, when we regard $\mathfrak{I}_0$ as an ideal of $R\{y_1, \dots, y_n\}$ where $R = K\langle a_1, \dots, a_n\rangle\{y_0\}$. $\square$

Proposition 8.3.10. *Using the notation of the previous lemma and assuming that $dim(V(\mathfrak{I})) = d$ implies that $dim(V(\mathfrak{I}_1)) = d - 1$.*

Proof. Now suppose without loss of generality that $b_1, \dots, b_d$ are independent Δ-transcendentals where $\bar{b}$ is a generic point on $V(\mathfrak{I})$ and $\bar{b} \in V(f)$. We know that $y_0, y_1, \dots, y_d$ are not independent mod $\mathfrak{I}_0$, so we know $b_1, \ldots, b_d$ are not independent over $K\langle \bar{a} \rangle$. We claim that $\bar{b}_1, \ldots, b_{d-1}$ are independent over $K\langle \bar{a} \rangle$. Suppose not. Then there is some Δ-polynomial $f(x_1, \ldots, x_{d-1}) \in \mathfrak{I}_1$. But then there is some Δ-polynomial $\tilde{f}(x_1, \ldots, x_{d-1}, y_0) \in \mathfrak{I}_0$, which contradicts lemma 8.3.4. □

When we assume that the hypersurface is actually a hyperplane, we can make a more detailed calculation:

Lemma 8.3.11. *Following the notation of the previous lemma, let* $d > 0$*. Let* f *be order* 0 *and degree* 1 *(that is* $V(f)$ *is a generic hyperplane). Then,*

$$\omega_{V([\mathfrak{I},f])/K\langle y_0,a_1,\ldots,a_n\rangle}(t) = \omega_{V(\mathfrak{I})/K}(t) - \binom{t+m}{m}$$

Proof. In the notation of 8.3.9, b_d rational over $b_1, \ldots, b_{d-1}$ in $K\langle a_f \rangle$. So, $\omega_{\bar{b}/K\langle a_f \rangle}(t) = \omega_{b_1,\ldots,b_{d-1},d_{d+1},\ldots,b_n/K\langle a_f \rangle}(t)$. Further, by inspecting the differential ideals, we can see

$$\omega_{b_1,\ldots,b_{d-1},d_{d+1},\ldots,b_n/K\langle a_f \rangle}(t) = \omega_{b_1,\ldots,b_{d-1},d_{d+1},\ldots,b_n/K}(t) = \omega_{\bar{b}/K}(t) - \binom{t+m}{m}$$

□

Putting together results 8.3.4, 8.3.5, 8.3.7, 8.3.8, 8.3.9, and 8.3.11 we have proved the following theorem,

Theorem 8.3.12. *Let* V *be a Kolchin-closed (over* K*) subset of* $\mathbb{A}^n$ *with differential transcendence degree* d*. Let* H *be a generic (with respect to* K*) hypersurface corresponding to the tuple* $\bar{a}$

(as above). Then $V \cap H$ is a Kolchin-closed subset of $\mathbb{A}^n$ with differential transcendence degree $d-1$. $V \cap H$ is irreducible over $K\langle \bar{a} \rangle$. In the case that $d = 0$, $V \cap H = \emptyset$. If $d > 0$ and H is a generic hyperplane, then the Kolchin polynomial of $V \cap H$ is given by

$$\omega_{V \cap H / k\langle a_H \rangle}(t) = \omega_{V/k}(t) - \binom{t+m}{m}$$

Remark 8.3.13. Note that we are considering the Kolchin topology over a *specific field*, and not its differential (or even algebraic) closure. Irreducibility over the algebraic closure is akin to geometric irreducibility in algebraic geometry. We have not proved this yet, nor do the authors of (90). In fact, at least one additional hypothesis is necessary for that result: if the hypothesis is *purely in terms of dimension*, we would have to restrict to the situation $d \geq 2$. After all, take any degree $d_1 > 1$ plane curve. This curve meets the generic hyperplane in precisely d_1 points, so the intersection is not irreducible over any algebraically closed field. In fact, in the next section, we show that this is the only potential problem by proving a more detailed result which applies to any differential algebraic variety for which the intersection with a generic hyperplane is infinite.

Also note that we have not computed the Kolchin polynomial of the intersection except in the special case of a hyperplane. The computation is carried out in (90) for the ordinary case and generic hypersurfaces. A similar result ought to be possible in this setting, but we will not pursue it here.

8.4 Geometric irreducibility

Before discussing geometric irreducibility, we will require some results about the Kolchin polynomials of prime differential ideals lying over a fixed prime differential ideal in extensions.

Proposition 8.4.1. *((34) pg131, proposition 3, part b) Let $\mathfrak{p}$ be a prime differential ideal in $\mathcal{F}\{y_1, \ldots, y_n\}$ and let $\mathcal{K}$ be a differential field extension of $\mathcal{F}$. Then $\mathcal{K}\mathfrak{p}$ has finitely many prime components in $\mathcal{K}\{y_1, \ldots, y_n\}$. If $\mathfrak{q}$ is any of the components, then a generic type of the variety $V(\mathfrak{q})$ has the same Kolchin polynomial as the generic type of $V(\mathfrak{p})$.*

Remark 8.4.2. In model theoretic terms, the generic types of the components $V(\mathfrak{p}_1), \ldots, V(\mathfrak{p}_n)$ of $V(\mathfrak{p})$ are each nonforking extensions of the generic type of $V(\mathfrak{p})$. Assuming that the base field $\mathcal{F}$ is algebraically closed would ensure that the generic type of $V(\mathfrak{p})$ is stationary; consequently $\mathcal{K}\mathfrak{p}$ is a prime differential ideal for any field extension $\mathcal{K}$ of $\mathcal{F}$.

Definition 8.4.3. Let V be a differential algebraic variety. Then V is *geometrically irreducible* if V is irreducible over any differential field K containing the field of definition of V.

Remark 8.4.4. The previous proposition and remark show that it is enough to consider irreducibility over K^{alg}, the algebraic closure of K. To put geometric irreducibility in the language of differential algebra, if $V = \Delta Spec(K\{x_1, \ldots, x_n\}/\mathfrak{p})$ where $\mathfrak{p}$ is a prime differential ideal, then V is geometrically irreducible if $V = \Delta Spec(K\{x_1, \ldots, x_n\}/\mathfrak{p}) \times_{\Delta Spec(K)} \Delta Spec(K^{alg})$. In terms of rings, $K\{x_1, \ldots, x_n\}/\mathfrak{p} \otimes_K K^{alg}$ should be an integral domain.

Theorem 8.4.5. *Let V be a (geometrically) irreducible Kolchin-closed (over K) subset of $\mathbb{A}^n$ with Kolchin polynomial $\omega_V(t) > \binom{t+m}{m}$. Let H be a generic hyperplane. Then $V \cap H$ is geometrically irreducible and $\omega_{H\cap V}(t) = \omega_V(t) - \binom{t+m}{m}$.*

Remark 8.4.6. The strict inequality $\omega_V(t) > \binom{t+m}{m}$ in the hypothesis is necessary. This hypothesis prevents V from being an algebraic curve (in fact, it is equivalent to this). The fact we get geometric irreducibility of the intersection even in the case that V is dimension one (as long as V is not an algebraic curve) is in contrast to the case of algebraic geometry.

Proof. Consider the the differential algebraic variety $W = \{(v_1, v_2, \beta) \,|\, v_i \in V, v_i \in H_\beta\} \subseteq V \times V \times \mathbb{A}^n$ where H_β is the hyperplane given by $\sum \beta_i x_i = 1$. It might be the case that W is reducible in the Kolchin topology, but we will not be concerned with this specifically.

Consider $V \cap H_\beta$. When β is generic over K, we know that $V \cap H$ is irreducible over $K\langle\beta\rangle$, so by the Proposition 8.4.1, all of the components of V over the algebraic closure of K have Kolchin polynomial equal to $\omega_{V\cap H_\beta/K\langle\beta\rangle}(t)$. If $V \cap H$ has more than one component, then W has more than one component with Kolchin polynomial $2\omega_{V/k} + (n-2)\binom{t+m}{m}$. To see this, consider the complete types given by independent generic points on $V \cap H$, and β generic subject to the requirement that $v_i \in H_\beta$. Then there is more than one option for the type (v_1, v_2, β), depending on if v_1 and v_2 are in the same component of $V \cap H_\beta$ over $K\langle\beta\rangle^{alg}$. Now, we only consider components of W with Kolchin polynomial $2\omega_{V/k} + (n-2)\binom{t+m}{m}$ and show that $V \cap H_\beta$ is irreducible over $K\langle\beta\rangle^{alg}$.

Suppose v_1 and v_2 are independent generic points on V, and β is generic subject to the condition that H_β contains v_1, v_2. Then the triple v_1, v_2, β is generic on W over K. We will show that this is the only way to construct a generic type on W. If $v_1 \neq v_2$ and β is generic over v_1, v_2, subject to $v_i \in H_\beta$, then $\omega_{\beta/v_1v_2}(t) = \binom{t+m}{m} \cdot (n-2)$. If $v_1 = v_2$, then $\omega_{\beta/v_1}(t) = \binom{t+m}{m} \cdot (n-1)$. But, if $v_1 = v_2$, $\omega_{v_1v_2\beta}(t) < 2\omega_V + \binom{t+m}{m} \cdot (n-2)$, because $\omega_{v_1/K} > \binom{t+m}{m}$. So, there is a unique type in W with Kolchin polynomial $2\omega_{V/k} + (n-2)\binom{t+m}{m}$, so there is only one component with Kolchin polynomial $2\omega_{V/k} + (n-2)\binom{t+m}{m}$.

By our earlier arguments, there is a unique component of $V \cap H$ with Kolchin polynomial $\omega_V - \binom{t+m}{m}$. But, by Proposition 8.4.1, we know any of component of $V \cap H$ must have Kolchin polynomial $\omega_V - \binom{t+m}{m}$. So, $V \cap H$ is geometrically irreducible. $\square$

Notice that the main intersection theory results of the last section applied to subvarieties cut out by generic differential polynomials, not just generic hyperplane sections. The exception is the calculation of the Kolchin polynomial of $V \cap H$ in Theorem 8.3.12. In order to replicate the methods used in this section for arbitrary generic differential hypersurfaces, one would have to provide calculations of the Kolchin polynomial of the intersection of V with arbitrary generic hypersurfaces; in principle this should be possible.

As it stands, the intersection theory developed in the previous section would be sufficient to carry out the development of Chow forms for partial differential varieties along the lines of (90), but would not be sufficient to carry out the development of differential Chow forms for partial differential varieties. We do not carry out this development here, though this sort of development has been of interest to model theorists (see the second paragraph of page four of (65), for instance). We have several different applications in mind, and pursue those in the remaining sections after discussing smoothness of the intersections.

8.5 Smoothness

8.5.1 Arc spaces

Before discussing the smoothness of generic intersections, we will review the construction of differential arc spaces, following (53) for the first part, and (35) thereafter. *None of the results in this subsection are new*; see (53) and (35) for complete references. Throughout this subsection: S is a scheme and $T \to S$ is a scheme over S. Given a scheme Y over T, we let

$$\mathcal{R}_{T/S}Y : \{\text{Schemes over } S\} \to \{\text{Sets}\}$$

be the functor given by

$$\mathcal{R}_{T/S}(U) = Hom_T(U \times_S T, Y)$$

In some situations, $\mathcal{R}_{T/S}$ is a representable functor. When this is the case, we will let $\mathcal{R}_{T/S}(Y)$ be the representing object, that is, the scheme over S. When T is finite over S, the functor is representable; this is the case in which we will work exclusively (again, for more details, see (53)).

Example 8.5.1. Here is a concrete and pertinent example of Weil restriction. Let k' be a finitely generated field extension of k and let X' be an affine scheme of finite type over k'. Then the *Weil restriction* $\mathcal{R}_{k'/k}X'(A) = X'(A \otimes_k k')$ for any finitely generated commutative k-algebra A. So,

$$\mathcal{R}_{k'/k}(-) : \{\text{affine k'-schemes}\} \to \{\text{affine k-schemes}\}$$

is right adjoint to base change from k to k'. Fix X' an affine k'-scheme with coordinate ring $k'[X']$ given by $k'[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Then the coordinate ring of the affine k-scheme given by the Weil restriction, $\mathcal{R}_{k'/k}X'$ is given as follows. Let $y_0, y_1, \ldots, y_d$ be a basis for k' over k as a vector space (let us assume $y_0 = 1$). Then

$$k[\mathcal{R}_{k'/k}X'] = k[x_{1,1}, \ldots, x_{1,d}, \ldots x_{n,1}, \ldots, x_{n,d}]/(f_{1,1}, \ldots, f_{m,d})$$

where $f_{j,k} \in k[x_{1,1}, \ldots, x_{1,d}, \ldots, x_{n,1}, \ldots, x_{n,d}]$ are such that:

$$f_j \left(\left(\sum_{j=0}^{m} x_{i,k} y_k \right) \right) = \sum_{k=0}^{m} f_{j,k} y_k$$

Note that this uniquely defines $f_{j,k}$.

Next, consider the case in which again, $S = Spec(k)$. Now let $T = Spec(k^{(m)})$ where

$$k^{(m)} := k[\epsilon]/(\epsilon^{m+1})$$

Let $Y = X \times_S T$, where X is an affine scheme over k. $k^{(m)}$ is a k-algebra with the natural map,

$$a \to a + +0\epsilon + \ldots + 0\epsilon^m.$$

The m^{th} *arc bundle of* X over k is $R_{k^{(m)}/k}(X \times_k k^{(m)})$, the scheme representing the Weil restriction of $X \times_S T$ from T to S. Throughout, we will denote this particular Weil restriction as $\mathcal{A}_m(X/k)$ or $\mathcal{A}_m(X)$ when k is implicit. Of course, for any k-algebra R, $\mathcal{A}_m X(R)$ can be naturally identified with $X(R[\epsilon]/(\epsilon^{m+1}))$.

Now, recalling the remarks in example 8.5.1, take $X \subseteq \mathbb{A}^l$. Suppose

$$X := Spec(k[x_1, \ldots, x_l]/(\{f_j\}_{j \in J}\})$$

Then

$$\mathcal{A}_m(X) = Spec(k[\{x_{i,s}\}_{1, \leq i \leq l, 0 \leq s \leq m}]/(\{f_{j,t}\}_{0 \leq t \leq m})$$

where $f_{j,t} \in k[\{x_{i,s}\}_{1, \leq i \leq l, 0 \leq s \leq m}]$ is defined by,

$$f_j((\sum_{i=0}^{m} x_{i,t}\epsilon^t)_{1 \leq i \leq l}) = \sum_{t=0}^{m} f_{j,t}\epsilon^t,$$

calculated in the ring $k[\{x_{i,s}\}, \epsilon]/(\epsilon^{m+1})$.

Example 8.5.2. Let us consider the very simplest nontrivial case of the construction. Let C be a smooth affine plane curve, given by $f(x, y) = 0$. We calculate $\mathcal{A}_1(C)$. Pick any point (a, b) on the curve and consider the taylor expansion of f around point (a, b), truncated to the first order terms. Then

$$f(x - a, y - b) = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b).$$

Then

$$f(a + \alpha\epsilon, b + \beta\epsilon) = 0$$

if and only if $\frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$ vanishes at $(a+\alpha\epsilon, b+\beta\epsilon)$. In other words, the tangent space at (a,b) is

$$\{(w,z) \,|\, \frac{\partial f}{\partial x}(a,b)(w-a) + \frac{\partial f}{\partial y}(a,b)(z-b) = 0\}.$$

More generally, $\mathcal{A}_1(V)$ is simply the tangent bundle.

Now, suppose $f : X \to Y$ is a map of k-varieties. It is a fact that f induces a map on the arc spaces denoted $\mathcal{A}_m(f) : \mathcal{A}_m(X) \to \mathcal{A}_m(Y)$. Suppose $X \subseteq \mathbb{A}^l, Y \subseteq \mathbb{A}^r$. Let us consider this map in more detail. Say $f = (f_1, \ldots, f_r)$. Then we can compute the map $\mathcal{A}_m(f)$ by considering $\mathcal{A}_m(X)$ as points in $\mathbb{A}^l(k[\epsilon]/(\epsilon^{m+1}))$. Then to compute the image of some point $b \in \mathbb{A}^l(k[\epsilon]/(\epsilon^{m+1}))$, one simply computes $f_i(b) \in k[\epsilon]/(\epsilon^{m+1})$. Further, we remark that there is a natural map of arc spaces $\rho_{l,m} : \mathcal{A}_l \to \mathcal{A}_m$ for $l > m$ which is induced by the quotient map on the ring $k[\epsilon]/(\epsilon^l)$. When we refer to m^{th} arc space at a point $a \in X$, we mean the fiber of the map $\rho_{m,0}$.

Next, we summarize some of the pertinent results for arc spaces; complete proofs of these standard results can be found in (53).

Proposition 8.5.3. *Suppose X is an algebraic variety over a field k and $a \in X(k)$ a smooth point. Then for any pair of natural numbers $l > m \geq 0$ the restriction of the map $\rho_{l,m} : \mathcal{A}_l X \to \mathcal{A}_m X$ to $\mathcal{A}_l W_a(k)$ is surjective onto $\mathcal{A}_m X_a(k)$.*

Proposition 8.5.4. *Let $f : X \to Y$ be a map of algebraic varieties over k. Let $a \in \mathcal{A}_m(k)$. Let $\tilde{X}$ be the fiber of $\rho_{m+1,m} : \mathcal{A}_{m+1}X \to \mathcal{A}_m X$ over a. Let $\bar{a} := \rho_m(a)$. Then there are biregular $\psi_X : \tilde{X} \to T_{\bar{a}}X$ and $\psi_Y : \tilde{Y} \to T_{f(\bar{a})}T$ such that,*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\mathcal{A}_{m+1}f} & \tilde{Y} \\ \psi_X \big\downarrow & & \big\downarrow \ \psi_Y \\ T_{\bar{a}}X & \xrightarrow{df_{\bar{a}}} & T_{f(\bar{a})}Y \end{array}$$

Proposition 8.5.5. *Let $f : X \to Y$ be a dominant map of algebraic varieties (all over k). Suppose $a \in X(k)$ is a smooth point and $f(a)$ is smooth on Y. Assume df_a has rank equal to the dimension of Y. Then for all m $\mathcal{A}_m(f) : \mathcal{A}_m X_a(k) \to \mathcal{A}_n Y_{f(a)}(k)$ is onto.*

Proposition 8.5.6. *Let k be algebraically closed and $X, Y \subseteq Z$ are irreducible varieties over k. If $a \in X(k) \cap Y(k)$ then $X = Y$ iff $\mathcal{A}_m X_a(k) = \mathcal{A}_m Y_a(k)$ for all $m > 0$.*

We will now review the construction of Δ-arc spaces for affine Δ-varieties; again, we follow (53), where complete details are given. Let

$$k_m := k[\eta_1, \ldots, \eta_n]/(\eta_1, \ldots, \eta_n)^{m+1}.$$

The ring is a k-algebra via the map

$$a \mapsto \sum_{0 \leq \alpha_1 +, \ldots, +\alpha_n \leq m} \frac{1}{\alpha_1! \cdot \ldots \cdot \alpha_n!} \delta_1^{\alpha_1} \ldots \delta_n^{\alpha_n}(a) \eta_1^{\alpha_1} \ldots \eta_n^{\alpha_n}.$$

Now let $S = spec(k)$, $T = spec(k_m)$. For an algebraic variety, the *m^{th} prolongation $\tau_m X$ of X* is the Weil restriction of $X \times_S T$ from $Spec(k_m)$ to $Spec(k)$.

Note that if $m = 1$ and δ_1 is the trivial derivation, τ_m is the same as $\mathcal{A}_m$. As in the case of $k^{(m)}$, we have reduction maps (given by quotients) from $k_l \to k_m$, and $\pi_{l,m} : \tau_l \to \tau_m$. We denote $\pi_{l,0}$ by π_l. Let $\nabla_m : X \to \tau_m X$ be given by

$$x \mapsto \sum_{0 \leq \alpha_1 + \ldots + \alpha_n \leq m} \frac{1}{\alpha_1! \cdot \ldots \cdot \alpha_n!} \delta_1^{\alpha_1} \ldots \delta_n^{\alpha_n}(x) \eta_1^{\alpha_1} \ldots \eta_n^{\alpha_n}.$$

Then ∇_m is a Δ-regular section of π_m.

Recall, we only consider affine Δ-varieties, so, throughout, we let X be a Δ-variety over k and $\bar{X}$ be the Zariski closure of X over k. Now we define $\tau_m X$ as the *Zariski closure* of $\nabla_m X(k)$ as a subvariety of $\tau_m \bar{X}(k)$. Note that X is determined completely by the sequence

$$\langle \pi_{l,m} : \tau_l X \to \tau_m X \mid l > m \rangle.$$

Note that

$$X(k) = \{a \in \tau_0 X(k) \, : \, \nabla_l(a) \in \tau_l X(k), \forall l > 0\}$$

On the other hand, define

$$\langle X_l \subseteq \tau_l \bar{X} \mid l \geq 0 \rangle,$$

a sequence of irreducible algebraic subvarieties to be a *prolongation sequence* if

- $\pi_{l+1,l}$ is a dominant map from X_{l+1} to X_l

- Considering $\tau_{l+1}\bar{X}$ as a subvariety of $\tau^{l+1}\bar{X}$, X_{l+1} is a closed subvariety of τX_l

In this case, there is a unique Δ-subvariety X of $\bar{X}$ so that $\tau_l X = X_l$. So, there is a natural equivalence of categories between affine Δ-varieties (over k) and prolongation sequences (over k).

Proposition 8.5.7. *Let $k \models DCF_{0,m}$ and let $X \subseteq \bar{X}$ be irreducible Δ-subvariety of $\bar{X}$, an algebraic subvariety both defined over k.*

$$\langle \mathcal{A}_m(\pi_{s,t}) : \mathcal{A}_m\tau_s X \to \mathcal{A}_m\tau_t X \mid s \geq t\rangle$$

is the prolongation sequence for a Δ-subvariety of $\mathcal{A}_m\bar{X}$.

Definition 8.5.8. The m^{th} arc bundle of X is $\mathcal{A}_m X$ is the Δ-subvariety specified by the prolongation sequence from the previous proposition.

$\mathcal{A}_1 X$ is naturally isomorphic to the Δ-tangent bundle, which is described below, and for which we use Kolchin's notation $T^\Delta X$.

In (53), the authors give the following definition:

Definition 8.5.9. $a \in X(k)$ is a *smooth point* if $\nabla_s(a)$ is a smooth point on $\tau_s X(k)$ for each s and $d(\pi_{s,t})_{\nabla_s(a)}$ has full rank for every $s \geq t$. When $a \in X(k)$ is smooth, $\tau_s(\mathcal{A}_m X_a) = \mathcal{A}_m(\tau_s X)_{\nabla_s(a)}$.

The remainder of the section follows Kolchin's development of the differential tangent space (35). For this section, let $X = V(\mathfrak{p})$, a prime differential ideal in the ring $K\{y_1, \ldots, y_n\}$. Let

$R = K\{y\}/\mathfrak{p}$. Then for $x \in X(K)$, let $\mathcal{O}_{x,X}$ be the localization of R at the maximal ideal of differential polynomials which vanish at x.

Definition 8.5.10. A local Δ-derivation on $\mathcal{O}_{X,x}$ is a derivation from $\mathcal{O}_{X,x}$ to K which commutes with the elements of Δ. The set of all Δ-derivations on $\mathcal{O}_{X,x}$ is a vector space over the constants. This space is denoted by $T_x^{\Delta}(X)$.

We have the following natural map:

$$d_x : T_x^{\Delta}(X) \to \mathbb{G}_a^n$$

where $T \mapsto (T\bar{y}_1, \ldots, T\bar{y}_n)$ where $\bar{y}_i$ is the image of y_i in the quotient map $\mathcal{O}_{\mathbb{A}^n,x}/\mathfrak{p}\mathcal{O}_{\mathbb{A}^n,x}$. For $f \in K\{y\}$, define

$$D(f) := \sum_{\theta \in \Theta, i=1,\ldots,n} \frac{\partial f}{\partial \theta y_i} \theta y_i$$

and

$$D(f)_x := \sum_{\theta \in \Theta, i=1,\ldots,n} \frac{\partial f}{\partial \theta y_i}(x) \theta y_i.$$

Then let $\mathfrak{p}_x := [\{D(f)_x \mid f \in \mathfrak{p}\}]$. $T_x^{\Delta}(X)$ is isomorphic to the subgroup of $\mathbb{G}_a^n$ defined by $\mathfrak{p}_x$. So, given ideal of a differential algebraic variety, it is easy to construct the ideal of the differential tangent space at a point.

This philosophy *almost* works for characteristic sets. In order to ensure that the characteristic set of the ideal of the differential tangent space at a point $\bar{a}$ has the same leaders as the characteristic set of the ideal of the original variety, it is necessary to assume that the initials and separants of a characteristic set of the variety do not vanish at $\bar{a}$. If one only considers orderly rankings, this leads to an easy proof that at points where the initials and separants do not vanish, the differential tangent space and the original variety have the same Kolchin polynomial. For complete details, see (85).

But while the conditions described in the above paragraph are sufficient for the differential tangent space to have the same Kolchin polynomial; they are certainly not necessary. One might take the definition of smooth to depend on the nonvanishing of initials and separants under a given orderly ranking, but this approach has the unwanted property of depending on the ranking chosen. Perhaps it would be interesting to quantify over all orderly rankings for the definition of smoothness, but we do not pursue this line here.

Now, we give a second very natural definition of smoothness for differential algebraic varieties:

Definition 8.5.11. A point $a \in V$ is smooth if $T_a^{\Delta} V$ has the same Kolchin polynomial as V, that is:

$$\omega_{V/k}(t) = \omega_{T_a^{\Delta} V/k\langle a \rangle}(t)$$

Remark 8.5.12. This definition of smoothness is not equivalent to the earlier one given 8.5.9. In the remaining portion of the chapter, we pursue results for this 8.5.11 notion of smothness, leaving analagous results for the earlier notion to future work.

8.5.2 Inheriting orthogonality from Δ-tangent spaces

In this subsection we take a short detour and consider the model theoretic relation of orthogonality and its relationship with arc spaces. Of course, work along these lines was undertaken in several previous works (53) (65), where arc spaces were used to prove dichotomy-style theorems dividing differential algebraic varieties into two nonorthogonality classes: those whose geometries are complicated, but whose nonorthogonality classes have varieties with defining ideals that are linear and those whose geometries are simple. We will not pursue such goals in this subsection, but we wish to show a few applications of arc spaces, especially in light of the work (85) regarding the heat equation.

Recall that for two types p, q, we say p is *orthogonal* to q and write $p \perp q$ if for all A containing the domains of p and q, and realizations $a \models p|_C, b \models q|_C$ of the nonforking extensions, we have $a \perp\!\!\!\perp_C b$. A type is *regular* if it is orthogonal to any forking extension. In this subsection we will pursue the following idea: nonorthogonality of two Kolchin-closed sets should induce nonorthogonality of their differential tangent spaces.

Suppose that $X_1 \not\perp X_2$. Then, over some sufficiently large parameter set, A, we can find $a_1 \in X_1$ and $a_2 \in X_2$ such that $Z = V(I(a_1, a_2))$ is a proper Kolchin-closed subset of $X_1 \times X_2$, and the projections $\pi_1 : Z \to X_1$, $\pi_2 Z \to X_2$ are Kolchin dense.

So, there is a natural inclusion $T^\Delta Z$ in $T^\Delta(X_1 \times X_2)$. Now, suppose we could find $c \in Z$ so that $\pi_1(c)$ and $\pi_2(c)$ are generic in $X_1 \times X_2$. Further, demand that $\nabla(\pi_i(c)) \in T^\Delta X_i$ is of full rank (this is an open condition on X_i, certainly, this is implied by taking $\pi_i(c)$ to be smooth in the sense of definition 8.5.9).

Then $\nabla(\pi_1(c)), \nabla(\pi_2(c))$ are generic in $T^{\Delta}_{\pi_i(c)}(X_i)$, respectively. But, the Kolchin polynomial of the pair $(\nabla(\pi_1(c)), \nabla(\pi_2(c))) \in T^{\Delta}_{(\pi_1(c),\pi_2(c))}(X_1 \times X_2)$ is bounded by the Kolchin polynomial of a generic point in $T^{\Delta}_c(Z)$. Assuming c is a smooth point (see 8.5.9 - again, this is an open condition) on Z, the Kolchin polynomial for our pair is bounded by the Kolchin polynomial for a generic point in Z. Of course, this is strictly less than the Kolchin polynomial for a generic point on $X_1 \times X_2$, which, at least at smooth points, has the same Kolchin polynomial as $T^{\Delta}_{(a_1,a_2)}(X_1 \times X_2)$. Thus, it must be the case that $V(I((\nabla(\pi_1(c)), \nabla(\pi_2(c)))))$ is a proper subvariety of $T^{\Delta}_{(\pi_1(c),\pi_2(c))}(X_1 \times X_2)$. So, $\nabla(\pi_1(c)) \not\perp_A \nabla(\pi_2(c))$. This means that as definable sets, $T^{\Delta}_{\pi_1(c)}X_1 \not\perp T^{\Delta}_{\pi_2(c)}X_2$.

So, we have found a sufficient condition for orthogonality of two types based on orthogonality of their Δ-tangent spaces above sufficiently general points:

Proposition 8.5.13. *Suppose that the generic types of* $T^{\Delta}_{x_1}X_1$ *and* $T^{\Delta}_{x_2}X_2$ *are orthogonal where* $x_i \in X_i$ *generic. Then the generic types of* X_1 *and* X_2 *are orthogonal.*

Remark 8.5.14. This condition does not manifest in a meaningful way for ordinary differential fields, because in that setting $T^{\Delta}_{\pi_1(c)}X_1 \not\perp T^{\Delta}_{\pi_2(c)}X_2$ holds for all X_1 and X_2. In that setting, for finite rank differential algebraic varieties, differential tangent spaces (like all linear differential algebraic varieties) are finite dimensional vector spaces over the constants. However, in partial differential algebraic geometry, there is a greater diversity of types modulo nonorthogonality, assuming we allow types with nonconstant Kolchin polynomials.

We will give several examples later in this section. First, we prove another corollary which is nontrivial for partial differential algebraic geometry and completely trivial for ordinary differential algebraic geometry.

Corollary 8.5.15. *Let V be a differential algebraic variety over k. Let $a \in V$ be generic over K. Let $c \in T_a^\Delta V$ be generic over $k\langle a\rangle$. If $tp(c/k\langle a\rangle)$ is regular then $tp(a/k)$ is regular. Also,*

$$RU(a/k) \leq RU(c/k\langle a\rangle)$$

Proof. If $tp(a/K)$ is nonorthogonal to some forking extension, then this induces nonorthogonality of $tp(c/K)$ to some forking extension - namely, the generic type of differential tangent space (at a generic point) of the locus of a realization of the forking extension of $tp(a/K)$. □

Example 8.5.16. Let $K_0 \models DCF_{0,\Delta}$ with $\Delta = \{\delta_1, \delta_2\}$. We will study the differential variety, X, defined by the equation:

$$(\delta_1^2 x)^2 = (\delta_2^3 x)^3.$$

Of course, the Δ-tangent space of this variety at a smooth point is isomorphic to the Heat equation, which, by (86), has Lascar rank equal to ω. Throughout this discussion E will be the curve $y^2 = x^3$.

Let $c \in \mathcal{U}$ be a δ_2 transcendental over K_0. Further, let $d^2 = c^3$. Let $b \in \mathcal{U}$ be an element such that $b \models \delta_2^3(x) = \delta_1(c) \wedge \delta_1^2(x) = d$. In fact, we assume that the positive type of b in $S_1(K_0\{c, d\})$ is isolated by the given formulas (equivalently, b is a generic point on the given Δ-variety, in the Kolchin topology).

Now, take $e \in \mathcal{U}$ to be a generic solution to the equation $\delta_2^3(x) = c$. We let $F = K_0\{b, c, d\}$ and we consider

$$I_F^e(\delta_1) := \{T : F\{e\} \to F\{e\} \,|\, T \text{ is a } \Delta\text{-derivation and } T|_F = \delta_1\}.$$

From work of Kolchin ((34) chapter 2), we know $I_F^e(\delta_1)$ actually has the form of a $\Delta-$variety, $\mathcal{G}(\delta_1)$. In this case, since the positive type of e over F is implied by $\delta_2^3 x = c$, we have that $\mathcal{G}(\delta_1)$ is the zero set of the differential ideal generated by:

$$\mathfrak{p}_{F,\delta_1}^e := \{\delta_2^3 x - \delta_1(c)\}.$$

So, there is $T \in I_F^e(\delta_1)$ with $T(e) = b$ if and only if $b \in \mathcal{G}(\delta_1)$. But, indeed, by the choice of b, we know that $b \in \mathcal{G}(\delta_1)$. So, there is some $T \in I_F^e(\delta_1)$ with $T(e) = b$. Then, $K_0 \models DCF_{0,\Delta'}$ where $\Delta' = \{T, \delta_1\}$. But, this means that c is a δ_2-transcendental over K_0 and $c \models (T^2 x)^2 = (\delta_2^3 x)^3$. But this means that the original differential variety has $RU(X) \geq \omega$, since $RU(tp(c/K_0)) \geq \omega$.

By work of Suer (85), the Lascar rank of the Heat equation is ω, and by corollary 8.5.15, we know that the Lascar rank of the variety is bounded by the Lascar rank of its differential tangent space. So, $RU(X) = \omega$.

Similar analysis of other varieties whose generic differential tangent spaces are subgroups of the additive group which are rank ω easily lead to other examples of generic types in differential fields. For other subgroups of the additive group of rank ω, see (85, Proposition 3.45, for instance).

8.5.3 Smoothness and hyperplane sections

We now return to our development of Bertini-style theorems in differential algebraic geometry.

Question 8.5.17. *Suppose that a differential algebraic variety V is smooth. Then, for sufficiently general hyperplane H, is $V \cap H$ smooth?*

We will answer this question affirmatively for definition 8.5.11, and leave analogous question for definition 8.5.9 for future work.

Theorem 8.5.18. *Let V be an irreducible smooth Kolchin-closed (over K) subset of $\mathbb{A}^n$ 8.5.11. Let H be a generic (with respect to K) hypersurface. Then $V \cap H$ is smooth.*

Proof. In the following proof, we are using a technique similar to that of lemmas 8.3.4 through 8.3.11. We will be using the calculations of Kolchin polynomials of those results, as well as the irreducibility results in those lemmas.

Consider, as a Kolchin-closed subset of $\mathbb{A}^{n+1}$, the locus of $f = y_0 + \sum_{i=1}^{n} a_i y_1$ and $I(V/K\langle a_1, \ldots, a_n\rangle)$. We will call this differential algebraic variety W. We note that by Lemma 8.3.4, W is irreducible. Let $\bar{b} = (b_0, \ldots, b_n) \in W$. Then consider $T^{\Delta}_{\bar{b}} W$.

Let $\hat{b} = (b_1, \ldots, b_n)$. Then fix some orderly ranking $<_1$ on $y_1, \ldots, y_n$ and let $\ngeq_{\hat{b}}$ be a characteristic set of the differential tangent space $T^{\Delta}_{\hat{b}} V$. We note that by (34) Theorem 6, section 2.6, the Kolchin polynomial of a variety is determined by the leaders of its characteristic set with respect to any orderly ranking. But, now consider orderly ranking $<_2$ for which $y_i <_2 y_0$ for all $i > 0$ and for all differential monomials $m_1, m_2 \in \Theta(y_1, \ldots, y_n)$, $m_1 <_1 m_2$ if and only if

$m_1 <_2 m_2$. Then $\Lambda_{\hat{b}}$ together with f is a characteristic set for $T_{\bar{b}}^{\Delta}W$. To see this, note that the set is autoreduced and coherent with respect to $<_2$. $\Lambda_{\hat{b}}$ is a characteristic set under $<_1$, and no derivative of the leader of f appears in $\Lambda_{\hat{b}}$.

We note that by (34) Theorem 6, section 2.6, the Kolchin polynomial of a variety is determined by the leaders of its characteristic set with respect to any orderly ranking. Then

$$\omega_{T_{\bar{b}}^{\Delta}W/k\langle\bar{b}\rangle}(t) = \omega_{T_{\hat{b}}^{\Delta}V/K\langle\hat{b}\rangle}(t)$$

Now, let H be the hyperplane in $\mathbb{A}^n$ given by f over $K\langle\bar{a}, y_0\rangle$. It is obvious that

$$\omega_{T_{hatb}^{\Delta}(V\cap H)/K\langle\bar{a},\hat{b}\rangle}(t) = \omega_{T_{\bar{b}}^{\Delta}W/k\langle\bar{b}\rangle}(t) - \binom{t+m}{m}$$

We can also see that

$$\omega_{V\cap H/K\langle\bar{a},y_0\rangle}(t) = \omega_{W/K\langle\bar{a}\rangle}(t) - \binom{t+m}{m} = \omega_{V/K}(t) - \binom{t+m}{m}$$

So,

$$\omega_{V\cap H/K\langle\bar{a},y_0\rangle}(t) = \omega_{T_{\hat{b}}^{\Delta}(V\cap H)/K\langle\bar{a},\hat{b}\rangle}(t)$$

follows from the smoothness assumption on V and the above equations. $\square$

8.6 Definability of rank

We next turn to a couple of applications of geometric or model theoretic nature. First, we will prove differential transcendence degree is a constructible condition in the Kolchin topology:

Lemma 8.6.1. *Given a family of differential algebraic quasi-varieties, $\phi : X \to S$, with $a_m(S) = 0$, the set $\{s \in S \,|\, a_m(X_s) = d\}$ is a constructible subset of S.*

Proof. Fix d $n+1$-tuples $(c_{i,j})_{1 \leq i \leq d, 1 \leq j \leq n+1}$ such that the elements in the tuple are independent Δ-transcendentals over the canonical bases of X, S, and ϕ. Then by Theorem 8.3.12,

$$\{s \,|\, a_m(\phi^{-1}(s)) \geq d\} = \{s \,|\, \phi^{-1}(s) \cap Z(\sum_{j=1}^{n} c_{1,j}y_j - c_{1,n+1}, \dots, \sum_{j=1}^{n} c_{d,j}y_j - c_{d,n+1}) \text{ is nonempty}\}$$

One should note that Theorem 8.3.12 applies in this case only because over the canonical base of S, we know that any point on S is of differential transcendence degree 0. So, choosing a generic hyperplane over the base of all of the definable sets at the beginning ensures that the hyperplane remains generic over any given fiber of the definable map ϕ. The set on the right is obviously first order definable (which implies Kolchin constructible by quantifier elimination when K is differentially closed). $\square$

Remark 8.6.2. When S is positive differential transcendence degree, more care is clearly needed, for instance, consider the following example. In the previous theorem, suppose that $S = \mathbb{A}^{n+1}$ and $X \subseteq \mathbb{A}^n$. Let the fiber above a point in S be the hyperplane cut out in $\mathbb{A}^n$ by the coordinates of the point. For instance, fix coordinates $y_0, \dots, y_n$ for S, and now fix the system of generic hyperplanes which we propose to use as in the above theorem. Then let $\bar{c}$ be such that $H_{\bar{c}}$ is in the collection. Of course, $\bar{c}$ is a point on S. In the fiber above this point, this hyperplane is not generic, and in fact is precisely the set $\phi^{-1}(\bar{c})$, so intersections with this hyperplane are useless in this fiber. Our solution to this potential problem is completely combinatorial in nature.

Theorem 8.6.3. *Given a family of differential algebraic quasi-varieties, $\phi : X \to S$, the set $\{s \in S \,|\, a_m(X_s) = d\}$ is a constructible subset of S.*

Proof. Adopt the notation of Lemma 8.6.1. Suppose that $a_m(S) = n_1$. Then pick $2n_1 + 1$ systems of $d+1$ $(n+1)$-tuples of mutually independent Δ-transcendentals (equivalently, fix an indiscernible set in the generic type, over K with the canonical bases of X, S, and ϕ; then pick any $(2n_1+1)(d)(n+1)$ elements). Denote the chosen elements

$$\{c_{k,i,j} \,|\, 1 \leq k \leq 2n_1 + 1,\, 1 \leq i \leq d,\, 1 \leq j \leq n+1\}$$

Of course, over any given fiber of S, some of the $2n_1 + 1$ systems do not determine generic hyperplanes. But, because $a_m(S) = n_1$ and the systems are mutually independent, at least $n+1$ of the systems are generic over any given fiber $\phi^{-1}(s)$.

Now, the requirement that $a_m(\phi^{-1}(s)) \geq d$ is equivalent to the condition that for at least $n_1 + 1$ values of k,

$$\phi^{-1}(s) \cap Z(\sum_{j=1}^{n} c_{k,1,j} y_j - c_{k,1,n+1}, \ldots, \sum_{j=1}^{n} c_{k,d,j} y_j - c_{k,d,n+1}) \neq \emptyset$$

$\square$

Remark 8.6.4. There are numerous other routes to the previous proposition, but we prove it as above to demonstrate an application of the intersection theory developed earlier. Also, the proposition tells us about the nature of the intersection of our variety V with arbitrary linear subspaces. The proposition says that the subset of the Grassmannian which intersects V with fixed dimension n is actually a constructible set in the Kolchin topology.

Note that this would not simply follow from the main generic intersection theorem. After all, there is, a priori, no reason that the exceptional locus could not be an infinite union of differential algebraic subvarieties. In fact, when working with other ranks on differential algebraic varieties, exceptional loci need not have differential algebraic structure. For instance, in (64), a family of differential algebraic varieties in a complex parameter α is shown to be strongly minimal if and only if α is in a certain discrete subset of the real axis (this exceptional locus is far from being a constructible set in the Kolchin topology). The above proposition means that this can not happen when considering the differential dimension. We note that the above result is obviously related to remark 4.44 of (90). Adopting their notation, we have shown that X is a constructible set.

The authors of (90) also prove an interesting geometric result which also generalizes to the partial differential setting. Our approach here is rather different, and we will not use differential specializations to achieve the result, though a proof by suitably generalized methods of this kind is possible. Our proof is shorter, but as usual, we are using the machinery of stability theory.

Theorem 8.6.5. *Let V be a differential algebraic variety of dimension d. If the set of $d+1$ independent generic hyperplanes through $\bar{a}$ intersects V, then $\bar{a} \in V$.*

Proof. We note that the hypotheses imply that $\omega^m \cdot d \leq RU(V/K) < \omega^m \cdot (d+1)$. Let $\bar{a} \notin V$. First, we will argue the result in the case that $\bar{a} = (0, \ldots, 0)$. Any hyperplane through the origin is of the the form $\sum c_i y_i = 0$. We assume that the c_i are independent differential transcendentals over K. We denote this hyperplane by $H_{\bar{c}}$. Suppose that $\bar{b}$ is a generic point on one of the irreducible component of $V \cap H_{\bar{c}}$ over $K\langle \bar{c} \rangle$. If $\bar{d}$ is a generic point on V over K, and for $I \subseteq \{1, 2, \ldots, n\}$ we have that $D = \{d_i \,|\, i \in I\}$ is a differential transcendence base for the field extension $K\langle \bar{d} \rangle / K$, then the same property holds for $\bar{b}$, that is $B = \{b_i \,|\, i \in I\}$ is a differential transcendence base for the field extension $K\langle \bar{b} \rangle / K$. Thus, $RU(\bar{b}/K\langle B \rangle) < \omega^m$.

Now since $\bar{b} \in H_{\bar{c}} \cap V$, we know that $\sum c_i b_i = 0$. We will bound $RU(\bar{b}/K\langle \bar{c} \rangle)$. Since over K, $\bar{c}$ is an independent differential transcendental,

$$RU(\bar{c}/K\langle \bar{b} \rangle) + \omega^m = RU(\bar{c}/K)$$

But, then by Lascar's symmetry lemma (see (67), chapter 19)

$$RU(\bar{b}/K\langle \bar{c} \rangle) + \omega^m \leq RU(\bar{b}/K)$$

Thus, the differential transcendence degree of $\bar{b}/K\langle \bar{c} \rangle$ is at least one less than that of $\bar{d}/K$.

In the case that $RU(V/K) < \omega^m$, the above argument using Lascar's symmetry lemma shows that $V \cap H_{\bar{c}} = \emptyset$.

Now, suppose that $\bar{a}$ is some point besides $(0, \ldots, 0)$. If so, adjoin $\bar{a}$ to the field K and consider $K\langle\bar{a}\rangle$. A priori, perhaps V is no longer irreducible; if not, arguing about each irreducible component would suffice.

Remark 8.6.6. This is not necessary though: take the differential closure of K. The generic tuple used to define the hyperplane is not in the differential closure of K. But, any parameters witnessing reducibility must come from the differential closure (in fact, from the algebraic closure).

Now, by translating the variety V and the point $\bar{a}$, one can assume $\bar{a} = (0, \ldots, 0)$.

□

8.7 Irreducibility in families

In this section, we will produce new results in ordinary differential fields, but we discuss the partial case.

Question 8.7.1. *Let $\phi : V \to S$ be a morphism of differential algebraic varieties. Is the set $\{s \mid \phi^{-1}(s)$ is irreducible$\}$ a constructible set in the Kolchin topology?*

This question and several other equivalent statements were addressed in (47), but a complete answer to the irreducibility problem was not obtained. We will not address this question of irreducibility in particular, but rather a related one, which is considered in appendix 3.1 of (30).

Definition 8.7.2. Let V be a differential algebraic variety. We say that V is *generically irreducible* if V has one component of maximal Kolchin polynomial.

In (30), this notion was considered for finite transcendence degree ordinary differential algebraic varieties where the following result was proved (in slightly different language):

Proposition 8.7.3. *Let $\phi : V \to S$ be a morphism of differential algebraic varieties such that the fibers of ϕ are dimension zero. Then the set $\{s \mid \phi^{-1}(s)$ is generically irreducible$\}$ is Kolchin-constructible.*

The proof involves two ideas. First, any fiber V_a, being finite rank, is naturally associated with an algebraic D-variety, that is, an algebraic variety W_a along with s_a, a section of the twisted tangent bundle. V_a is *generically* irreducible if and only if W_a is irreducible as an algebraic variety. The construction of W_a from V_a is uniform in families, and irreducibility of W_a is constructible in families (89). Note that the approach to the problem of irreducibility in families via ultraproducts in (89) (for the algebraic case) was the basic blueprint for the analogous approach in (47) in the differential case where several interesting equivalencies were proved (but the problem was not completely settled). An alternate proof of irreducibility in families can be found in (25, 15.5.3).

Proposition 8.7.4. *Let $\phi : V \to S$ be a morphism of differential algebraic varieties. Then the set $\{s \mid \phi^{-1}(s)$ is generically irreducible$\}$ is Kolchin-constructible.*

Proof. The set $\{s \mid dim(\phi^{-1}(s)) = k\}$ is a constructible set by 8.6.3. So, we need only establish the result for each value of k separately.

For the zero dimensional fibers, we apply 8.7.3. Suppose that $k > 0$. Recalling the technique and notation in the proof of 8.6.3, we fix a system of $\{H_{i,j}\}_{i=1,\dots,k,\, j=1,\dots,2dim(S)+1}$ of $2dim(S)+1$

generic independent systems of k hyperplanes. Now, because the intersection of $\phi^{-1}(s)$ with any system of k independent generic hyperplanes over s is dimension zero, if we could fix a single system of k generic independent hyperplanes which were independent from each $s \in S$, we could reduce to the problem to 8.7.3. Of course, this is impossible when S is not dimension zero. Thus, we must apply the trick from 8.6.3. A fiber $\phi^{-1}(s)$ of dimension k is generically irreducible if and only if either

1. $\phi^{-1}(s) \cap H_{1,j} \cap \ldots \cap H_{k,j}$ is zero dimensional and generically irreducible for at least $dim(S) + 1$ values of $j \in \{1, \ldots, 2dim(S) + 1\}$
2. For at least $dim(S) + 1$ values of $j \in \{1, \ldots, 2dim(S) + 1\}$, the variety $\phi^{-1}(s) \cap H_{1,j} \cap \ldots \cap H_{k,j}$ consists of finitely many points (this implies that the Kolchin polynomial of any component of $\phi^{-1}(s)$ is $d \cdot (t + 1)$, which implies that $\phi^{-1}(s)$ is an algebraic variety) and $\phi^{-1}(s)$ is an irreducible algebraic variety

Consider the first case: $\phi^{-1}(s) \cap H_{1,j} \cap \ldots \cap H_{k,j}$ being dimension zero is a definable condition; then applying 8.7.3 for the values of j such that $\phi^{-1}(s) \cap H_{1,j} \cap \ldots \cap H_{k,j}$ is zero dimensional gives this is a constructible condition.

Consider the second case: here $\phi^{-1}(s)$ is an algebraic variety, because the differential field generated by a generic point is algebraic over $dim(\phi^{-1}(s)/k\langle s \rangle)$ differential transcendentals. The number of points in the intersection of the variety with $dim(\phi^{-1}(s))$ generic hyperplanes may be bounded in terms of the degree of the equations. In model-theoretic terms, this is known as uniform boundingapplies in differentially closed fields (45, Corollary 2.15). So, as s

varies, there is a uniform upper bound, n_1. So, checking that there are at least $dim(S)$ such families of hyperplanes which intersect $\phi^{-1}(s)$ in at most n_1 points is a constructible condition. Irreducibility within this constructible subfamily is given by the result of (89) discussed above.

□

The problem of irreducibility in families is related to several important problems in differential algebra. In (47), the following theorem is given.

Theorem 8.7.5. *The following are equivalent:*

1. *Irreducibility is definable in families.*
2. *For every d there exists $r(d, n, m)$ such that for every Δ-field k if P is a prime Δ-ideal of $k\{x\}$ with characteristic set whose elements are of degree and order less that or equal to d, then P is differentially radically generated by Δ-polynomials of order and degree less than or equal to r.*
3. *For every d, there is $r = r(d, n, m)$ such that for every Δ-field k and all prime Δ-ideals $P, Q \subset k\{x\}$ with characteristic sets whose elements are of degree and order less than or equal to d, if every Δ-polynomial in P of degree and order less than or equal to r is in Q, then $P \subset Q$.*
4. *For every d, there is $r = r(d, n, m)$ such that for every Δ-field k, every set $S \subset k\{x\}$ of Δ-polynomials of degree and order less than or equal to d, and every pair P and Q of minimal prime Δ-ideals containing S, if every Δ-polynomial in P of degree and order less than or equal to r is in Q, then $P = Q$.*

5. *For every d, there is $r = r(d, n, m)$ such that for every Δ-field k, every set $S \subset k\{x\}$ of Δ-polynomials of degree and order less than or equal to d, and every $g \in k\{x\}$ of degree and order less than or equal to d, if $gf \notin \{s\}$ for all $f \in \{S\}$ of degree and order less than or equal to r, then g is not a zero divisor modulo $\{S\}$.*

Combining the previous theorem and Proposition 8.7.4, we obtain:

Proposition 8.7.6. *(In the ordinary case) Each of the conditions of the previous theorem are equivalent to the following:*

- *Let $\phi : V \to S$ be a morphism of differential algebraic varieties. The set*

$$\{s \mid \phi^{-1}(s) \text{ is generically irreducible, but not irreducible}\}$$

a constructible set in the Kolchin topology.

Remark 8.7.7. As we noted at the beginning of this section, we are working only in ordinary differential fields for this application; the results and approach discussed here do not seem to readily apply to the partial differential case. One might still apply the intersection theory developed above to reduce the question of generic irreducibility to the dimension zero case. However, difficulties still abound, because the Δ-*type* of the variety is likely to be greater than zero. In this case, one can not reduce to the algebraic category via the functor to algebraic D-varieties. In the partial case, one must use the prolongation sequences of (53).

CHAPTER 9

GENERICITY IN DIFFERENTIAL FIELDS

9.1 Introduction

We work over a fixed ordinary characteristic zero differential field k, with δ a derivation. As used in previous chapters, there are notions of generic points on differential varieties coming from both model theory and algebraic geometry adapted to the Kolchin topology. Recall, from the algebro-geometric perspective, the generic points on a differential variety are simply those not contained in any proper differential subvariety (34). For finite rank differential varieties, these topologically generic points are those $a \in V$ such that the field $k(a, \delta(a), \delta^2(a), \ldots)$ has transcendence degree equal to that of the differential function field of the variety (denoted $RD(V)$).

When dealing with infinite rank differential varieties, the condition on the transcendence degree does not make sense as stated, but the topological notion of genericity is still valid. Rather, in that case, one uses the Kolchin polynomial. When we work with affine differential varieties in one variable, $RD(V)$ is simply equal to the highest order derivative appearing in the polynomial f, such that $V = Z(f)$. Note that every differential algebraic subvariety of $\mathbb{A}^1$ arises in this way. For this result, various other notions of rank, and a careful development of the above ideas and more, see (45).

From the model theoretic perspective, there is another notion of a generic point on a variety, $a \in V$ is generic (over k) if the Morley rank of the type $tp(a/k)$ is equal to the Morley rank of the variety. The obvious question is:

Question 9.1.1. *To what extent do the model-theoretic and algebro-geometric notions of genericity agree?*

This question was investigated by various authors; the most relevant for the work we carry out is (4).

Again, we will only consider subvarieties of $\mathbb{A}^1$. Though this seems to be a big restriction, a result of (71) says that every ordinary differential variety embedded in projective space and of finite rank is isomorphic to a constructible set in $\mathbb{A}^1$ (that is, an open subset of a closed set in the Kolchin topology). If every finite transcendence degree differential variety has complete projective closure (there are no known counterexamples), then every finite rank differential variety is actually isomorphic to a finite rank closed subset of $\mathbb{A}^1$.

In fact, we proved a generalization of this theorem to the partial differential case 6.4.3. For order 1 differential varieties, these notions of genericity are identical since algebraic dependence in a differentially closed field is equivalent to algebraic dependence in the classical sense (for fields) assuming that the structures over which one considers dependence are actual differential fields (are definably closed).

Already, for order 2 differential varieties the two notions of genericity are not identical. A generic point (in the Kolchin topological sense) on the differential variety

$$xx'' - x' = 0$$

has Morley rank 1 (see (45)). But, so does a generic solution to the proper differential subvariety $x' = 0$. Note that it is not necessary to specify which kind of generic point we speak of for the variety $x' = 0$, since both notions agree for order 1 differential subvarieties.

The above example shows that model theoretic generic points are not necessarily topologically generic. All of this was pointed out by Benoist (4) in which the following more specific question appeared:

Question 9.1.2. *For finite rank differential algebraic varieties, are Kolchin topological generic points always model theoretically generic?*

As the example above shows, irreducibility in the Kolchin topology does not imply that the variety has Morley degree 1. Before we give an example in which the topological generics are not model-theoretic generics, some situations where the notions agree will be noted so that we know where not to look. There are no new or deep results in section 1. Everything there was either proved by (4) or noted by (62) (in the latter case, sometimes without proof). Sections 2 and 3 contain a new example and a detailed algebraic analysis. The analysis is completely elementary differential algebra, inspired by the analysis of Poizat's example $xx'' - x' = 0$.

9.2 Comparing notions of genericity

In Differential algebraic groups, our two notions agree. The proof is an exercise in stable group theory which we will do below. Suppose G is a differential algebraic group. G is definable in DCF_0, so it is an ω-stable group. In ω-stable groups, there is a third notion of genericity. This notion is defined locally for formulas and (possibly incomplete) types. A formula $\phi(x)$ is *group generic* if finitely many translates of $\phi(x)$ by the action of the group via left multiplication cover the group. Call a type $p(x)$ group generic if each formula contain in $p(x)$ is group generic. In a ω-stable group the notions of group generic and RM-generic coincide (66). To the author's knowledge, the following proposition was first written down and proved in (4).

Proposition 9.2.1. *Suppose that G is an irreducible differential algebraic group. Then a type is RM-generic if and only if it is a topological generic.*

Proof. Suppose that $p(x)$ is a RM-generic but not a topological generic. Then finitely many left translates of any formula in $p(x)$ cover the group, but $p(x)$ is not topological generic, so the type is contained in a proper Kolchin closed subset of G. Take the formula witnessing this, $\phi(x)$. Now, finitely many left translates of $\phi(x)$ cover the group G, and each of these is clearly closed in the Kolchin topology (if a is a topological generic in $\phi(x)$ then $g\phi(x)$ is simply the zero set of the ideal of differential polynomials vanishing at ag). But, this is a problem. Now G is the finite union of proper closed subsets.

Now, assume that $p(x)$ is a type such that any realization a is topological generic. Then take any differential polynomial $P(x)$ vanishing at a. As a is topological generic, $P(x)$ vanishes everywhere in G. So, by quantifier elimination, then only possible non-group generic formula in

$p(x)$ is the negation of a differential polynomial equality. Suppose that $P(x) \neq 0$ is not group generic. Then $P(x) = 0$ is group generic, so finitely many translates cover G, which is again a contradiction if $Z(P) \cap G$ is a proper closed subset of G. Thus $P(x) \neq 0$ is group generic. □

Note that this argument also shows that for a differential algebraic group, irreducibility in the Kolchin topology implies that Morley degree is one. This is not true for general differential varieties. The last proposition is also true for homogeneous spaces in the sense of (35). The proof is essentially identical.

Since for finite rank differential algebraic varieties, the notions of genericity simply come from two different notions of rank, one might look for conditions purely on the ranks of a differential algebraic variety. The first natural condition in which we expect the notions of genericity to agree is for those varieties with RM = RD. This is true for linear differential algebraic varieties, but also holds for some nonlinear differential algebraic varieties (45).

Proposition 9.2.2. *Suppose that V is a differential variety such that $RM(V) = RD(V)$. Then RM-generic $\Leftrightarrow$ topological generic.*

Proof. Suppose that a is a RM-generic point of V. Then a must lie outside all order $RD(V)-1$ subvarieties since all of these have Morley rank at most $RD(V)-1$, since Morley rank is always bounded by RD. Conversely suppose that a is a topological generic point of V. Then there are infinitely many order $RD(V) - 1$ subvarieties of V with Morley rank $RD(V) - 1$, since $RM(V) = RD(V)$. But then a lies outside these infinitely many subvarieties by virtue of being topological generic. So, $RM(tp(a)) \geq RD(V)$. But of course, this is the maximum that Morley rank can possibly be by assumption. So, a is RM-generic. $\square$

Here is another simple situation where we will not find the example we are seeking.

Proposition 9.2.3. *Suppose that V is a differential algebraic variety such that $RD(V) = 2$. Then if $a \in V$ is topological generic, a is RM-generic.*

Proof. There are essentially two cases. Case 1: Assume that $RM(V) = 2$. In this case 9.2.2 applies. Case 2: Assume that $RM(V) = 1$. Any topological generic point on V is clearly not algebraic, and is thus of Morley rank at least 1. $\square$

In light propositions 9.2.1, 9.2.2, and 9.2.3, if we are seeking an example in which the topological generic points are not RM-generic, we must seek a variety in which:

- Is not a group (in particular nonlinear)
- Does not have $RD(V) = RM(V)$.
- Is the zero set of a differential equation of order at least 3

Thus, a minimal rank example in which the topological generics are RM-generic would be a third order differential variety, V, with only finitely many order two differential subvarieties of Morley rank two, $\{W_i\}_{i=1}^n$, such that the constructible set $V - \cup_{i=1}^n W_i$ has Morley rank 1. Verifying both of these conditions for a given example basically involves proving restrictions on the possible order 2 and order 1 subvarieties. For previous examples of this sort of technique, see the exposition of Poizat's example in (45).

From another perspective, the category of differential algebraic varieties of finite rank is closely connected to the category of algebraic varieties equipped with regular sections of their twisted tangent bundles (30). Assuming that we look for an example over the constants, the twisted tangent bundle is simply the tangent bundle. From that perspective, the example we are searching for would be one in which:

- V is an algebraic variety of dimension 3. s is a section of TV.
- There are finitely many two dimensional algebraic subvarieties W_i such that $s|_{W_i}$ is still a section of TW_i.
- For *any* proper subvariety U such that $s|_U$ is a regular section of TU, we have that $U \setminus \cup W_i$ is a finite collection of points.

We know of no previous such example in the literature.

For the remainder of the paper, we will let $f(x) = xx''' - x''$ and $V = Z(f)$. V has an order 2 subvariety, $Z(x'')$. In fact this is the only order 2 subvariety and $V - Z(x'')$ is strongly minimal. The following two sections are devoted to proving these facts by analyzing the subvarieties of V.

9.3 Order 2 Subvarieties

Throughout, we let $f(x) = xx''' - x''$ and $V = Z(f)$. This has the obvious order 2 subvariety $Z(x'')$. We will show that this is the only order 2 subvariety. So, let $g \in K[x, x, x'']$ be an order 2 differential polynomial. That is,

$$g = \sum_{n=0}^{N} a_n (x'')^n$$

where $a_n \in K[x, x']$, $N > 0$, and $a_N \neq 0$. In analyzing the order two differential subvarieties of V, it is only necessary to consider the zero sets of single differential polynomials, as every Kolchin closed subset of $\mathbb{A}^1$ is the zero set of a single differential polynomial. If $f \in I(g)$, so is any differential polynomial g_1 which differs from $xD(g)$ by a multiple of f. Or, if you like, think of f as a relation which holds on the differential polynomials in $I(g)$.

$$xD(g) = x\sum_{n=0}^{N}(a_n^D + \frac{\partial a_n}{\partial x}x' + \frac{\partial a_n}{\partial x'}x'')(x'')^n + x\sum_{n=0}^{N} na_n(x'')^{n-1}x'''$$

But, modulo f,

$$xD(g) \equiv_f g_1 := x\sum_{n=0}^{N}(a_n^D + \frac{\partial a_n}{\partial x}x' + \frac{\partial a_n}{\partial x'}x'')(x'')^n + \sum_{n=0}^{N} na_n(x'')^{n-1}x''$$

Now, unlike $xD(g)$, the new differential polynomial, g_1 is order 2. So, if it is to be in $I(g)$, then it must be the case that g divides g_1. The argument will proceed by considering x'' degree.

The leading term (with respect to x'') of g_1 is $x\frac{\partial a_N}{\partial x'}(x'')^{N+1}$. But the leading term of g is a_N, which has higher x' degree, so there is no chance that g divides g_1 unless $\frac{\partial a_N}{\partial x'} = 0$. So, $a_N \in K[x]$. Now, assuming that $\frac{\partial a_N}{\partial x'} = 0$, the leading term of g_1 is

$$(xa_N^D + xx'\frac{\partial a_n}{\partial x} + x\frac{\partial a_{N-1}}{\partial x'} + Na_N)(x'')^N \tag{9.1}$$

So, since the leading term of g is $a_N(x'')^N$ and the x'' degree of the polynomials is the same, using (Equation 9.1), one can see

$$g_1 = g(x\frac{a_N^D + x'\frac{\partial a_n}{\partial x} + \frac{\partial a_{N-1}}{\partial x'}}{a_N} + N) \tag{9.2}$$

Specifically, by previous work we know the following:

$$\begin{aligned} a_N &= \sum_{k=0}^{m} b_k x^k \\ \frac{\partial a_N}{\partial x} &= \sum_{k=0}^{m} k b_k x^{k-1} \\ a_N^D &= \sum_{k=0}^{m} D(b_k) x^k \end{aligned}$$

Now we can compare the $(x'')^0$ terms on either side of the equation (Equation 9.2).

$$LHS = x a_o^D + x x' \frac{\partial a_0}{\partial x} \tag{9.3}$$

$$RHS = a_0(x\frac{a_N^D + x'\frac{\partial a_n}{\partial x} + \frac{\partial a_{N-1}}{\partial x'}}{a_N} + N) \tag{9.4}$$

Now, by comparing the x' leading terms of (Equation 9.3) and (Equation 9.4) one can see

$$\frac{\partial a_{N-1}}{\partial x'} = c + x'd \tag{9.5}$$

where $c, d \in K[x]$. Now (Equation 9.2) becomes somewhat simpler,

$$g_1 = g(x\frac{a_N^D + x'\frac{\partial a_n}{\partial x} + c + x'd}{a_N} + N) \tag{9.6}$$

After regrouping some terms, this reduces (Equation 9.3) and (Equation 9.4) to

$$LHS = xa_0^D + xx'\frac{\partial a_0}{\partial x} \tag{9.7}$$

$$RHS = a_0(N + x\frac{a_N^D + c}{a_N} + xx'\frac{\sum_{k=0}^{m} kb_k x^{k-1} + d}{a_N}) \tag{9.8}$$

So, let

$$a_0 = \sum_{i=0}^{m_1} c_i (x')^i$$

where $c_i \in K[x]$. The x' leading term from (Equation 9.8) is

$$c_{m_1}(x')^{m_1+1}x\frac{\sum_{k=0}^{m} kb_k x^{k-1} + d}{a_N} \tag{9.9}$$

And the x' leading term from (Equation 9.7) is

$$x\frac{\partial c_{m_1}}{\partial x}(x')^{m_1+1} \tag{9.10}$$

So, if we suppose that

$$c_{m_1} = \sum_{j=1}^{m_2} d_j x^j$$

then from (Equation 9.9) and (Equation 9.10) we have the requirement:

$$d_{m_2}x^{m_2}x(\frac{\sum_{k=0}^{m} kb_kx^{k-1} + d}{a_N}) = m_2d_{m_2}x^{m_2}$$

Thus,

$$x\frac{\sum_{k=0}^{m} kb_kx^{k-1} + d}{a_N} = m_2 \tag{9.11}$$

So, we compare the lower order x terms, and see that $d_k = 0$ for $k < m_2$ But further, now we have

$$g_1 = g(N + x\frac{a_N^D + c}{a_N} + x'm_2). \tag{9.12}$$

Consider the $(x')^0$ term of the $(x'')^0$ term. Now, let

$$c_0 = \sum_{k=0}^{m_3} \alpha_k x^k.$$

Next, we may assume that $\alpha_{m_3} = 1$. If note, then divide the original polynomial by α_{m_3}; it still generates the same zero set. Comparing the left and right sides of (Equation 9.12).

$$LHS = xc_0^D \tag{9.13}$$

$$RHS = c_0(N + x\frac{a_N^D + c}{a_N}) \tag{9.14}$$

So,

$$xc_0^D = c_0(N + x\frac{a_N^D + c}{a_N}) \tag{9.15}$$

$$\sum_{k=0}^{m_3-1} \alpha_k^D x^{k+1} = \sum_{k=0}^{m_3} \alpha_k x^k (N + x\frac{a_N^D + c}{a_N}) \tag{9.16}$$

By comparing the x leading term of (Equation 9.16), we can see

$$x\frac{a_N^D + c}{a_N} \in K.$$

But, now the RHS of (Equation 9.16) has a nonzero x^j term where j is the minimum integer such that $\alpha_j \neq 0$. But the LHS of (Equation 9.16) has no x^j term. So, it must be the case that $c_0 = 0$.

Now we proceed by induction on the number of c_i which are zero (we have just proved the base case of the induction). So, suppose the $c_0, c_1, \ldots, c_l$ are all zero. Consider the $(x')^{l+1}$ terms in the $(x'')^0$ term,

$$g_1 = g(N + x\frac{a_n^D + c}{a_N} + x'm_2) \tag{9.17}$$

On the LHS (Equation 9.17) (recall that $c_l = 0$ so $\frac{\partial a_0}{\partial x} xx'$ contributes no $(x')^{l+1}$ terms):

$$xc_{l+1}^D.$$

On the RHS of (Equation 9.17) ($c_l = 0$ so $x'M_2$ does not give any $(x')^{l+1}$ terms):

$$c_{l+1}(N + x\frac{a_n^D + c}{a_N}).$$

But, this is precisely the condition that we showed was impossible unless $c_l = 0$ (the statement is literally the same but now with c_{l+1} instead of c_0). But, now c_i must be zero for $i = 1, \ldots, m_1$, that is x'' divides g. But, since we assumed that g is irreducible, $g = x''$. Thus, $x'' = 0$ is the unique irreducible order two subvariety of V.

Note, at this point we know that $RU(V) = RM(V) = 2$. To see this, recall that $\{Z(x' = c)\}_{c \in C_K}$ is a uniformly definable family of order 1 subvarieties of $Z(x'')$. On the other hand $RH(V) = RD(V) = 3$. The only remaining question about ranks associated with this differential variety is the rank of the Kolchin open subvariety $V - Z(x'')$. At this point, it might be the case that there is a uniformly definable family of order 1 subvarieties, making topological generic points of V Lascar and Morley Rank 2 (it might, a priori, be the case that the Lascar rank and Morley rank differ). So, the remaining questions about the rank of this variety can be answered by understanding the order 1 subvarieties which are outside of $Z(x'')$. Actually, in some sense any answer would be interesting. If there were infinitely many order 1 subvarieties of this open subvariety, but not an infinite uniformly definable family, then it would be an example of a definable set for which Morley rank and Lascar rank differ (previous examples have had RM at least 5 (31)). If Morley rank and Lascar rank were both two; perhaps the situation is not quite so interesting, but it is another example of an irreducible variety (in the Kolchin topology) having Morley degree two. Next, we analyze the order one subvarieties of V.

9.4 Order 1 Subvarieties

Now consider the potential order 1 subvarieties of $V = Z(xx''' - x'')$. Throughout, let $f(x) = xx''' - x''$.

Any such subvariety is the zero set of an irreducible differential polynomial $g(x, x') \in K[x, x']$. So, let

$$g(x, x') = \sum_{n=0}^{N} a_n (x')^n,$$

where $a_n \in K[x]$ and $a_N \neq 0$. Now , we wish to restrict the types of differential polynomials which might occur as order one subvarieties, so the general technique will be to differentiate twice, and apply the third order relation which holds on V.

$$D(g) = \sum_{n=0}^{N} n a_n (x')^{n-1} x'' + \sum_{n=0}^{N} \frac{\partial a_n}{\partial x} (x')^{n+1} + \sum_{n=0}^{N} a_n^D (x')^n$$

$$\begin{aligned} D^2(g) = &\sum_{n=0}^{N} n a_n (x')^{n-1} x''' + \sum_{n=0}^{N} n \frac{\partial a_n}{\partial x} (x')^n x'' + \sum_{n=0}^{N} n(n-1) a_n (x')^{n-2} (x'')^2 \\ &+ \sum_{n=0}^{N} n a_n^D (x')^{n-1} x'' + \sum_{n=0}^{N} \frac{\partial a_n^D}{\partial x} (x')^{n+1} + \sum_{n=0}^{N} \frac{\partial^2 a_n}{\partial x^2} (x')^{n+2} \\ &+ \sum_{n=0}^{N} (n+1) \frac{\partial a_n}{\partial x} (x')^n x'' + \sum_{n=0}^{N} a_n^{D^2} (x')^n + \sum_{n=0}^{N} \frac{\partial a_n^D}{\partial x} (x')^{n+1} \\ &+ \sum_{n=0}^{N} n a_n^D (x')^{n-1} x'' \end{aligned}$$

Now, multiply both sides of the equation by x, and note that since we assume that $f \in I(g)$, we know that we might, given a differential polynomial in $I(g)$, we might replace any instance of xx''' with x'' and we would still have a differential polynomial in $I(g)$. So, multiply the above expression for $D^2(g)$ by x and replace the instance of xx''' by x''. Now we have some other differential polynomial, call it $g_1(x)$ which is still in $I(g)$.

$$
\begin{aligned}
g_1(x) = & \sum_{n=0}^{N} na_n (x')^{n-1} x'' + x\sum_{n=0}^{N} n\frac{\partial a_n}{\partial x}(x')^n x'' + x\sum_{n=0}^{N} n(n-1)a_n (x')^{n-2}(x'')^2 && (9.18)\\
& + x\sum_{n=0}^{N} na_n^D (x')^{n-1} x'' + x\sum_{n=0}^{N} \frac{\partial a_n^D}{\partial x}(x')^{n+1} + x\sum_{n=0}^{N} \frac{\partial^2 a_n}{\partial x^2}(x')^{n+2} && (9.19)\\
& + x\sum_{n=0}^{N} (n+1)\frac{\partial a_n}{\partial x}(x')^n x'' + x\sum_{n=0}^{N} a_n^{D^2}(x')^n + x\sum_{n=0}^{N} \frac{\partial a_n^D}{\partial x}(x')^{n+1} && (9.20)\\
& + x\sum_{n=0}^{N} na_n^D (x')^{n-1} x'' && (9.21)
\end{aligned}
$$

Now both g_1 and $D(g)$ are differential polynomials in $I(g)$. So, we could replace instances of

$$\sum_{n=0}^{N} na_n (x')^{n-1} x''$$

in $g_1(x)$ with

$$-\sum_{n=0}^{N} \frac{\partial a_n}{\partial x}(x')^{n+1} - \sum_{n=0}^{N} a_n^D (x')^n$$

and get another differential polynomial, call it g_2, in $I(g)$ (since on the variety $Z(g)$, this relation holds).

$$g_2(x) = -\sum_{n=0}^{N} \frac{\partial a_n}{\partial x}(x')^{n+1} - \sum_{n=0}^{N} a_n^D (x')^n \tag{9.22}$$

$$+ xx'(-\sum_{n=0}^{N} \frac{\partial^2 a_n}{\partial x^2}(x')^{n+1} - \sum_{n=0}^{N} \frac{\partial a^D}{\partial x}_n (x')^n) \tag{9.23}$$

$$+ x(-\sum_{n=0}^{N}(n+1)\frac{\partial a_n}{\partial x}(x')^n x'' - \sum_{n=0}^{N} n a_n^D (x')^{n-1} x'') \tag{9.24}$$

$$+ x(-\sum_{n=0}^{N} \frac{\partial a_n^D}{\partial x}(x')^{n+1} - \sum_{n=0}^{N} a_n^{D^2}(x')^n) \tag{9.25}$$

$$+ x\sum_{n=0}^{N} \frac{\partial a_n^D}{\partial x}(x')^{n+1} + x\sum_{n=0}^{N} \frac{\partial^2 a_n}{\partial x^2}(x')^{n+2} \tag{9.26}$$

$$+ x(-\sum_{n=0}^{N} \frac{\partial^2 a_n}{\partial x^2}(x')^{n+2} - \sum_{n=0}^{N} \frac{\partial a_n^D}{\partial x}(x')^{n+1}) \tag{9.27}$$

$$+ x\sum_{n=0}^{N} a_n^{D^2}(x')^n + x\sum_{n=0}^{N} \frac{\partial a_n^D}{\partial x}(x')^{n+1} \tag{9.28}$$

$$+ x(-\sum_{n=0}^{N} \frac{\partial a_n^D}{\partial x}(x')^{n+1} - \sum_{n=0}^{N} a_n^{D^2}(x')^n) \tag{9.29}$$

Note that $x'g(x) \in I(g)$ so $D(x'g(x)) \in I(g)$ so the following relation holds on $Z(g)$,

$$\sum_{n=0}^{N}(n+1)a_n(x')^n x'' = -\sum_{n=0}^{N} \frac{\partial a_n}{\partial x}(x')^{n+2} - \sum_{n=0}^{N} a_n^D (x')^{n+1}.$$

Taking the partial derivative with respect to x yields an identity used in line 3 of the equation for g_1.

Also note that

$$\frac{\partial}{\partial x'}(\sum_{n=0}^{N} n a_n (x')^{n-1}(x'')^2) = \sum_{n=0}^{N} n(n-1)a_n(x')^{n-2}(x'')^2.$$

This means that on the variety $Z(g)$ the following relation (obtained by differentiating both sides of (**) with respect to x' and multiplying by xx") holds:

$$x\sum_{n=0}^{N} n(n-1)a_n(x')^{n-2}(x'')^2 = x(-\sum_{n=0}^{N}(n+1)\frac{\partial a_n}{\partial x}(x')^n x'' - \sum_{n=0}^{N} na_n^D(x')^{n-1}x'').$$

This explains line (3) in the expression for $g_2(x)$ above.

Now, there are still two instances of x" in $g_2(x)$, namely, from line (3) of the expression for g_2:

$$x(-\sum_{n=0}^{N}(n+1)\frac{\partial a_n}{\partial x}(x')^n x'' - \sum_{n=0}^{N} na_n^D(x')^{n-1}x'').$$

Using the same technique as above, get rid of these instances via a relation which holds on $Z(g)$ to obtain the following differential polynomial, which is in $I(g)$:

$$
\begin{aligned}
g_3(x) = &-\sum_{n=0}^{N} \frac{\partial a_n}{\partial x}(x')^{n+1} - \sum_{n=0}^{N} a_n^D (x')^n \\
&+ xx'(-\sum_{n=0}^{N} \frac{\partial^2 a_n}{\partial x^2}(x')^{n+1} - \sum_{n=0}^{N} \frac{\partial a^D}{\partial x}{}_n (x')^n) \\
&+ x(\sum_{n=0}^{N} \frac{\partial^2 a_n}{\partial x^2}(x')^{n+2} + \sum_{n=0}^{N} \frac{\partial a_n^D}{\partial x}(x')^{n+1}) \\
&+ x(\sum_{n=0}^{N} \frac{\partial a_n^D}{\partial x}(x')^{n+1} + \sum_{n=0}^{N} a_n^{D^2}(x')^n) \\
&+ x(-\sum_{n=0}^{N} \frac{\partial a_n^D}{\partial x}(x')^{n+1} - \sum_{n=0}^{N} a_n^{D^2}(x')^n) \\
&+ x\sum_{n=0}^{N} \frac{\partial a_n^D}{\partial x}(x')^{n+1} + x\sum_{n=0}^{N} \frac{\partial^2 a_n}{\partial x^2}(x')^{n+2} \\
&+ x(-\sum_{n=0}^{N} \frac{\partial^2 a_n}{\partial x^2}(x')^{n+2} - \sum_{n=0}^{N} \frac{\partial a_n^D}{\partial x}(x')^{n+1}) \\
&+ x\sum_{n=0}^{N} a_n^{D^2}(x')^n + x\sum_{n=0}^{N} \frac{\partial a_n^D}{\partial x}(x')^{n+1} \\
&+ x(-\sum_{n=0}^{N} \frac{\partial a_n^D}{\partial x}(x')^{n+1} - \sum_{n=0}^{N} a_n^{D^2}(x')^n)
\end{aligned}
$$

Combining like terms, notice that all of the lines in pairs after the first. Then multiplying by minus 1:

$$
g_4(x) = \sum_{n=0}^{N} \frac{\partial a_n}{\partial x}(x')^{n+1} + \sum_{n=0}^{N} a_n^D (x')^n
$$

Now g_4 is an order 1 differential polynomial contained in $I(g)$, and we notice that

$$g_5 = D(g) - g_4 = (\sum_{n=0}^{N} na_n(x')^{n-1})x''.$$

Of course g_5 is still in $I(g)$, but then either $x'' \in I(g)$ or

$$\sum_{n=0}^{N} na_n(x')^{n-1} \in I(g).$$

But, the latter is impossible since the differential polynomial is degree 1 and thus the only chance for it to be in $I(g)$ is by virtue of being divisible by g. By x' degree it is impossible that g divides $\sum_{n=0}^{N} na_n(x')^{n-1} \in I(g)$. Now we know $x'' \in I(g)$. So, the only order 1 subvarieties of V are actually subvarieties of $x'' = 0$.

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Spring 2011 **Midwest Model Theory**, *The Ohio State University*, Indecomposability in Partial Differential Fields.

Spring 2010 **Graduate Student Conference in Logic**, *University of Wisconsin, Madison*, Genericity in differential fields.

Fall 2010 **Southern Wisconsin Logic Colloquium**, *University of Wisconsin*, The axioms of differentially closed fields.

Fall 2010 **Definability in Number Theory**, *University of Ghent*, Difference Field Extensions.

Fall 2010 **Kolchin Seminar in Differential Algebra**, *City University of New York*, Definability of rank in the differential parameter space.

Fall 2010 **Model Theory Seminar**, *City University of New York*, The axioms of differential fields.

Spring 2010 **Kolchin Seminar in Differential Algebra**, *City University of New York*, Generic Points are not necessarily generic.

Talks in Chicago

Spring 2012 **Graduate Student Colloquium**, *University of Illinois at Chicago*, Differential algebraic geometry.

Spring 2012 **Louise Hay Logic Seminar**, *University of Illinois at Chicago*, Some conjectures and open problems in differential algebraic geometry.

Fall 2011 **Louise Hay Logic Seminar**, *University of Illinois at Chicago*, Galois cohomology and model theory.

Fall 2011 **Logic Seminar**, *University of Illinois at Chicago*, Stability theory without model theory.

Spring 2011 **Logic Seminar**, *University of Illinois at Chicago*, Isogeny in Superstable Groups.

Spring 2011 **Graduate Student Number Theory Seminar**, *University of Illinois at Chicago*, Varieties over separably closed fields.

Fall 2010 **Louise Hay Logic Seminar**, *University of Illinois at Chicago*, Regular Types, orthogonality, and domination.

Fall 2010 **Louise Hay Logic Seminar**, *University of Illinois at Chicago*, Nontrivial types of Lascar rank one.

Fall 2010 **Logic Seminar**, *University of Illinois at Chicago*, An invariant of difference field extensions.

Fall 2010 **Graduate Student Number Theory Seminar**, *University of Illinois at Chicago*, The Frobenius for almost every p.

Spring 2010 **Graduate Student Number Theory Seminar**, *University of Illinois at Chicago*, Diophantine geometry and model theory.

Spring 2010 **Logic Seminar**, *University of Illinois at Chicago*, Generic Points are not necessarily generic.

Spring 2010 **Louise Hay Logic Seminar**, *University of Illinois at Chicago*, Differential fields.

Fall 2009 **Louise Hay Logic Seminar**, *University of Illinois at Chicago*, Hrushovski constructions.

Summer 2009 **Conference on Set Theory and AECs**, *University of Illinois at Chicago*, Introduction to weak diamond.

Spring 2009 **Chicago Joint Logic Seminar**, *University of Chicago*, Fraisse limits and Hrushovski constructions.

Fall 2008 **Chicago Joint Logic Seminar**, *University of Chicago*, Slender modules and measurable cardinals.

Spring 2008 **Chicago Joint Logic Seminar**, *University of Chicago*, Introduction to O-minimality.

Fall 2007 **Graduate Student Logic Semainar**, *University of Illinois at Chicago*, Spectra.

Research Papers

2011 **Indecomposability in partial differential fields**, *http://arxiv.org/abs/1106.0695*, Under Review, Journal of Pure and Applied Algebra.

2011 **Completeness in partial differential fields**, *http://arxiv.org/abs/1106.0703*, Under Review, Journal of Algebra.

2011 **Isogeny in superstable groups**, *http://arxiv.org/abs/1110.1766*, Under Review, Archive for Mathematical Logic.

2011 **Genericity in differential fields**, *http://www.math.uic.edu/ freitag/GenericPoints.pdf*, Preprint.

2011 **Differential algebraic groups of small typical dimension**, *http://www.math.uic.edu/ freitag/SmallTypical.pdf*, Preprint.

Professional Service

Spring 2012 **Organizer**, *Graduate Student Mathematics Colloquim*, University of Illinois at Chicago.

2009–2010 **Organizer**, *Louise Hay Logic Seminar*, University of Illinois at Chicago and University of Chicago.

Spring 2011 **Organizer**, *Graduate Student Logic Conference*, University of Illinois at Chicago.

Spring 2011 **Memeber**, *Faculty Mentoring Award Nomination Committee*, University of Illinois at Chicago.

Membership in Professional Organizations

Association of Symbolic Logic.

American Mathematical Society.