On the Law of Iterated Logarithms for Brownian Motion on Compact Manifolds

by

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LIST OF ABBREVIATIONS

C^∞	Smooth functions on M
d	The dimension of the manifold
Δ	Laplace-Beltrami Operator
CLT	Central Limit Theorem
Dom	Domain
$\mathbf{E}_x$	Expected Value
$g(x,y)$	The Green Kernel
iid	Independent and Identically Distributed
L^p space	$L^p(\mathsf{P}) = \{\text{random variables } X\colon\ \mathbf{E}\left(\lvert X\rvert^p\right) < \infty\}$ for $p \geq 1$
LIL	Law of Iterated Logarithms
M	Compact C^∞ Riemannian Manifold
m	Volume element on the manifold
m_0	Entire volume of the manifold ($m_0 = m(M)$)
P	Probability
X_t	Brownian Motion at time t that starts at 0
X_t^x	Brownian Motion at time t that starts at x

SUMMARY

Let M be a compact C^∞ Riemannian manifold. Let X_t be a Brownian motion on the manifold for $t \geq 0$. We consider the volume element on the manifold and denote it by m. Let $m_0 = m(M)$, the entire volume of the manifold. By classic ergodic theory,

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t f(X_s) ds = \frac{1}{m_0} \int_M f(x) m(dx) \quad \text{almost surely.}$$

Hence, $\int_0^t f(X_s)\ ds$ goes to infinity with a rate of t and scalar $\frac{1}{m_0} \int_M f(x) m(dx)$. The second order leading term is given by

$$\int_0^t f(X_s) ds - \frac{t}{m_0} \int_M f(x) m(dx),$$

and the rate that this term converges to infinity is given by the law of iterated logarithms. Brosamler (1983), shows that

$$\limsup_{t \to +\infty} \frac{\int_0^t f(X_s) ds - \frac{t}{m_0} \int_M f(x) m(dx)}{\sqrt{2t \log \log t}}$$

is equal to $\sqrt{\frac{2}{m_0}(f, Gf)_{L^2}}$ for all continuous bounded functions f with probability one. G is a bounded linear functional such that G is, roughly speaking, the negative of the inverse Laplace Operator, and, as such, is also self-adjoint.

SUMMARY (Continued)

We consider the family of measures, $\mu_t^\omega(\cdot)$ (ω is usually suppressed in the notation) obtained by

$$\int_M f(x)\mu_t(dx) = \mu_t(f) = \frac{\int_0^t f(X_s)ds - \frac{t}{m_0}\int_M f(x)m(dx)}{\sqrt{2t\log\log t}}.$$

We study all of the accumulation points of μ_t in the mild sense; we say a sequence of measures μ_t converges mildly to another measure μ if and only if $\int_M f(x)\mu_t(dx) \to \int_M f(x)\mu(dx)$ for all smooth functions f. Define, as in Brosamler (1983), a bounded, linear, self-adjoint operator $G_{\frac{1}{2}}^{-1}$ in L^2; $G_{\frac{1}{2}}^{-1}$ is intuitively the square root of the negative of the Laplace Operator. We show that for any subsequence of times $\tilde{t}_n$, if $\mu_{\tilde{t}_n}$ converges mildly to μ, then μ is absolutely continuous with respect to the volume measure on the manifold and $\mu(M) = 0$. Moreover, denote by $q(x)$ the density of μ with respect to m, i.e. $\mu(dx) = q(x)m(dx)$; μ will have the characterization

$$\left\| G_{\frac{1}{2}}^{-1} q \right\|_{L^2} \leq \sqrt{\frac{2}{m_0}}.$$

We also show that if μ is a measure with density q, the characterization above, and $\mu(M) = 0$, then almost surely there exists a sequence of times t_n such that μ_{t_n} converges mildly to μ as n goes to infinity.

CHAPTER 1

INTRODUCTION

Baxter and Brosamler (2) and Brosamler (5) provide inspiration in finding a uniform law of iterated logarithms. In these papers, the authors are able to prove the law of iterated logarithms for Brownian Motion on a compact manifold for a fixed function f and for all f (simultaneously) that are smooth functions. We show the law of iterated logarithms, as stated in Brosamler (5), has accumulating points in a mild sense, and that these accumulating points have a specific characterization.

1.1 History

The initial statement of the law of iterated logarithms was first described by A.Y. Khinchine in 1924 (11); his work utilized Bernoulli trials. He was further able to extend this to "independent Poisson trials". Kolmogorov (11) extended this in 1929 to sums of independent random variables in the following way.

Theorem 1.1. *(Kolmogorov, as stated in (11))*

Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent random variables with finite variances. Take D to be a constant such that the series $\sum_{k=1}^{\infty} \frac{D\xi_k}{k^2}$ converges. If $s_n = \sum_{k=1}^{n} \xi_k, B_n = Ds_n$ and, as $n \to +\infty$

1. $B_n \to +\infty$,
2. $|\xi_n| \leq m_n = o\left(\sqrt{\frac{B_n}{\log\log B_k}}\right)$,

then

$$\mathsf{P}\left\{\limsup_{n\to+\infty}\frac{|s_n - Ms_n|}{2B_n\log\log B_n} = 1\right\} = 1.$$

Note: In the equation above, Gnedenko originally had in the denominator $2B_n \log\log B_k$*; we have corrected the version above. In addition,* D *and* ξ *were not explicitly stated in the section corresponding to the above items and final result in Gnedenko, and we have inferred their meaning through earlier notation in Gnedenko's paper.*

We can visually display this law, as in the example below.

Example 1.1. Suppose $X_i \overset{iid}{\sim} \mathsf{Ber}(0.5)$ random variables, for $i = 1, \ldots, 100000$. We take the law of iterated logarithms to be

$$\limsup_{n\to+\infty}\frac{S_n}{\sqrt{2n\log\log n}} = 1,$$

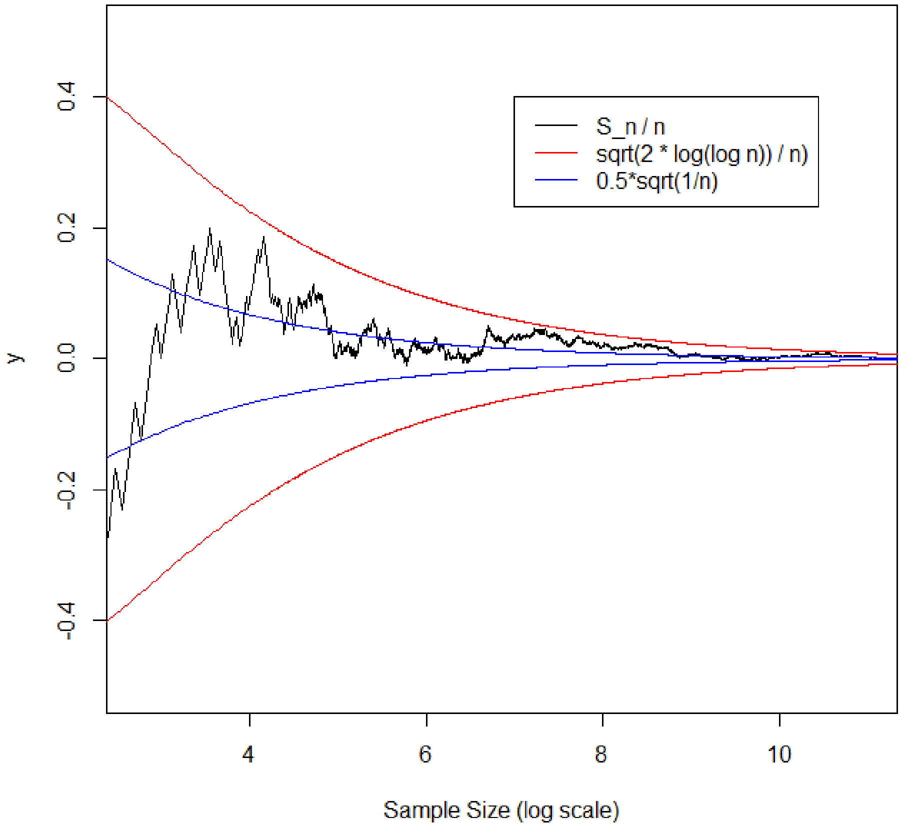
for $S_n = X_1 + \ldots + X_n$. If we remove lim sup, and proceed to manipulate each side of the equation, we can see that

$$S_n = \sqrt{2n\log\log n} \Rightarrow \frac{S_n}{n} = \sqrt{\frac{2\log\log n}{n}}.$$

Using `R`, we simulate this (the code appears in Appendix C) and compare S_n/n to $\sqrt{\frac{2\log\log n}{n}}$ and $0.5 \times \sqrt{\frac{1}{n}}$ (the standard deviation of S_n/n using the central limit theorem). During the simulation, to be able to see both positive and negative values, we took the values for each

Bernoulli random variable to be either 1 or -1 instead of 1 or 0. Using Figure 1, we can see that S_n/n is always in between $\pm\sqrt{\frac{2\log\log n}{n}}$.

Figure 1: Graph showing S_n/n (black) along with its standard deviation (blue) and the bounds given by the LIL (red). The $x-$axis is on a log scale, and the $y-$axis is in ones.



Following the initial statement of the law of iterated logarithms by Khinchine and its extended version by Kolmogorov, there have been many papers regarding the law of iterated logarithms and what occurs in different scenarios. Some selected works are summarized.

Hartman and Wintner in 1941 discuss what occurs when the second condition in Kolmogorov's Theorem is violated. In particular, they show that the law of iterated logarithms is also true for "unbounded but equal, or *nearly* equal, distributions is nevertheless correct" (13). Later that year, Hartman also shows that independent functions $x_n(t)$ defined on $0 \leq t \leq 1$ that are normally distributed with mean 0 and unknown variance will also follow the law of iterated logarithms (12).

In 1946, Feller was able to "characterize the upper and lower classes for all sequences satisfying

$$\int_{-\infty}^{\infty} x dV(x) = 0, \qquad \int_{-\infty}^{\infty} x^2 dV(x) = 1,\text{"}$$

where $\{X_n\}$ is a "sequence of mutually independent random variables having the same distribution function $V(x) = \mathsf{P}(X_n \leq x)$" (10).

Cassels showed in 1951 that there is an upper and lower bound of the law of iterated logarithms as n goes to infinity. In particular, this true almost always for fixed α and β (described in more detail below). However, there is "zero probability that"

$$\limsup_N \frac{R_N(\alpha, \beta)}{N^{\frac{1}{2}} \log \log^{\frac{1}{2}} N} = \omega(\beta - \alpha)$$
$$\liminf_N \frac{R_N(\alpha, \beta)}{N^{\frac{1}{2}} \log \log^{\frac{1}{2}} N} = -\omega(\beta - \alpha),$$

for all α, β "is false for *any* α, β [and] $\omega(y) = \{2y(1-y)\}^{\frac{1}{2}} (< 1)$" (7). In this scenario, "[l]et $x_1, x_2, \ldots, x_n, \ldots$ be a set of independent variables each with a uniform probability distribution in $0 \leq x \leq 1$. If $0 \leq \alpha < \beta \leq 1$, we denote by $F_N(\alpha, \beta)$ the number of $x_1, \ldots, x_N$ which satisfy $\alpha < x \leq \beta$, and put $R_N(\alpha, \beta) = F_N(\alpha, \beta) - N(\beta - \alpha)$" (7). According to Stackelberg, this was the first paper to "establish a uniform iterated logarithm law, namely for a family of sequences of independent Bernoulli variables" (29).

A relationship between the law of the iterated logarithm and the central limit theorem (CLT) were discussed by Petrov (1966). In particular, "a sequence of independent random variables $\{X_n\}$, $n = 1, 2, \ldots$, defined on a probability space $(\Omega, \mathscr{F}, \mathsf{P})$, and without, in general, a common distribution, with expectation equal to zero" may "satisfy CLT but fail to obey LIL" (26). Petrov proceeds to show that "if a somewhat stronger condition than the applicability of the CLT holds, then the sequence also obeys the LIL" (26).

The above work occurs for generic sequences of random variables. Strassen's 1964 paper on "An Invariance Principle for the Law of Iterated Logarithm" was applied the law of iterated logarithms to Brownian Motion on a "Banach space of continuous maps from $\langle 0, 1\rangle$ to R^k endowed with the supremum norm $\| \quad \|$, using the euclidean norm in R^k" (30). After Strassen, there were numerous papers that continued to extend results for the Law of Iterated Logarithms, not only for Brownian Motion, but on manifolds as well.

Kuelbs and Lepage (1973) extend Strassen's work to work for a "sequence of independent Gaussian random variables with values in a Banach space" (18). Further, Kuelbs (1975) works on a real separable Banach space B with the usual norm and rewrites the traditional form of the law of iterated logarithms to apply to "independent identically distributed B-valued random variables" such that they have a mean of zero, and the expected value of the norm squared is finite (17).

There has been a significant amount of work dealing with the law of iterated logarithms since this time, but now we focus on some of the papers that directly led to our work.

Philipp's 1969 paper focused on "sequences of random variables satisfying two kinds of mixing conditions" and that the law of iterated logarithms may be applied in this scenario (27). In 1971, Philipp further extended his results by relating the central limit theorem and the law of iterated logarithms with several mixing conditions. In addition, he extends the law of iterated logarithms so that "the 'uniform' law of the iterated logarithm holds if each random variable $x_n(\alpha)$ can be approximated rather closely by sums of random variables...provided that the subclasses A_t do not contain too many members", where A_t is a subset of $A = \{\langle x_n(\alpha), n = 1, 2, \ldots\rangle\}$, and A is a "family of sequences of random variables of $x_n(\alpha)$" (28). In 1976, Baxter and Brosamler show how energy relates to the law of iterated logarithms. In particular, they use a Green operator on a compact manifold to include the energy component (2). They use Philipp's (27) log-log law to show that

Theorem 1.2. *(Baxter and Brosamler (2))*

For all $x \in M$,

$$\mathsf{P}_x \left\{ \limsup_{t \to +\infty} \frac{\int_0^t f(X_s) ds}{(2t \log \log t)^{\frac{1}{2}}} = \sigma_f \right\} = 1.$$

$\{X_s, s \geq 0\}$ is a diffusion process on a smooth compact manifold M. f is a bounded Borel function on M, $f \not\equiv$ constant λ-a.e. and $\sigma_f = [2\langle f, f\rangle]^{\frac{1}{2}}$ such that $0 < \sigma_f < \infty$, and $\langle f, f\rangle = (fG, f)_{L^2}$, where G is the Green operator.

They also provide several examples in the context of a Brownian Motion on a circle that gives the same constant for σ_f that is stated in Stackelberg (29). Brosamler (5) uses the result in Baxter and Brosamler (2) and furthers it so it works for all smooth functions f on the manifold. To arrive at his result, he incorporates the Green function developed in Baxter and Brosamler (2) as well as through functional analysis using the Laplace Operator. Our work follows from this particular result.

CHAPTER 2

PRELIMINARY RESULTS

2.1 Riemannian Manifolds

Definition 2.1. (O'Neill Definition 8.4 (25))

An *n-dimensional manifold* M is a set furnished with a collection $\mathscr{P}$ of *abstract patches* (one-to-one) functions $\mathbf{x}\colon D \to M$, D an open set in $\mathbb{R}^n$ satisfying

1. The covering property: The images of the patches in the collection $\mathscr{P}$ cover M.
2. The smooth overlap property: For any patches $\mathbf{x}, \mathbf{y}$ in $\mathscr{P}$, the composite functions $\mathbf{y}^{-1}\mathbf{x}$ and $\mathbf{x}^{-1}\mathbf{y}$ are Euclidean differentiable–and defined on open sets of $\mathbb{R}^n$.
3. The Hausdorff property: For any points $\mathbf{p} \neq \mathbf{q}$ in M there are disjoint patches $\mathbf{x}$ and $\mathbf{y}$ with $\mathbf{p}$ in $\mathbf{x}(D)$ and $\mathbf{q}$ in $\mathbf{y}(E)$.

Remark 2.1. Before we continue extending our definition of manifolds to a Riemannian manifold, we require additional background on manifolds.

Definition 2.2. (Lee (19))

We write local coordinates on any open subset $U \subset M$ as $\left(x^1, \ldots, x^n\right)$, $\left(x^i\right)$, or x, depending on context. ...[C]oordinates constitute a map from U to $\mathbb{R}^n$, it is more common to use a coordinate chart to *identify* U with its image in $\mathbb{R}^n$, and to identify a point in U with its coordinate representation $\left(x^i\right)$ in $\mathbb{R}^n$.

For any $p \in M$, the *tangent space* T_pM can be characterized either as the set of derivations of the algebra of germs at p of C^∞ functions [infinitely differentiable] on M (i.e., tangent vectors are "directional derivatives"), or as the set of equivalence classes of curves through p under a suitable equivalence relation (i.e., tangent vectors are "velocities"). Regardless of which characterization is taken as the definition, local coordinates (x^i) give a basis for T_pM consisting of the partial derivative operators $\partial/\partial x^i$. When there can be no confusion about which coordinates are meant, we usually abbreviate $\partial/\partial x^i$ by the notation ∂_i.

Remark 2.2. In our case, we will be considering compact C^∞ Riemannian manifolds. The term C^∞ was defined in Definition 2.2. We also require the manifold to be compact:

Definition 2.3. (Manetti Definition 4.35 (20))

A topological space is said to be **compact** if any open cover admits a finite subcover.

Remark 2.3. A Riemannian manifold has an additional condition in that it requires an inner product.

Definition 2.4. (Lee (19))

A *Riemannian metric* on a smooth manifold M is a 2-tensor field $g \in \mathscr{T}^2(M)$ that is *symmetric* (i.e., $g(X,Y) = g(Y,X)$) and *positive definite* (i.e., $g(X,X) > 0$ if $X \neq 0$). A Riemannian metric thus determines an inner product on each tangent space T_pM, which is typically written $\langle X, Y \rangle : = g(X,Y)$ for $X, Y \in T_pM$. A manifold together with a given Riemannian metric is called a *Riemannian manifold.*

Remark 2.4. We want to be able to integrate over the Riemannian manifold; we will integrate with respect to the volume measure on the manifold.

Lemma 2.1. *(Lee Lemma 3.2 (19))*

On any oriented Riemannian n-manifold (M, g), there is a unique $n-$form dV satisfying the property that $dV(E_1, \ldots, E_n) = 1$ whenever $(E_1, \ldots, E_n)$ is an oriented orthonormal basis for some tangent space T_pM.

Remark 2.5. (Lee (19))

This $n-$form dV (sometimes denoted dV_g for clarity) is called the (*Riemannian*) *volume element.*

2.2 Linear Algebra and a Semigroup on a Finite Space

2.2.1 Case 1: Non-Zero Eigenvalues

Let

$$A = \begin{pmatrix} -\lambda_1 & 0 & 0 & 0 \\ 0 & -\lambda_2 & 0 & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & -\lambda_n \end{pmatrix},$$

where $0 > -\lambda_1 \geq -\lambda_2 \geq -\lambda_3 \geq \cdots \geq -\lambda_n$ are the eigenvalues of A such that there are no eigenvalues that are identically zero. Take these eigenvectors to be those of the standard basis for $\mathbb{R}^n$, and denote them by $\mathbf{e}_i,\ i = 1, \ldots, n$. Let $\mathbf{x}$ and $\mathbf{y}$ be vectors. Then $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n$. By definition, $A\mathbf{v} = \lambda\mathbf{v}$, i.e.

$$\begin{aligned} A\mathbf{e}_1 &= -\lambda_1\mathbf{e}_1 \\ A\mathbf{e}_2 &= -\lambda_2\mathbf{e}_2 \\ &\vdots \\ A\mathbf{e}_n &= -\lambda_n\mathbf{e}_n. \end{aligned}$$

Recall that we can define e^x in terms of a sum in the following manner: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Let $e^{At} \colon \mathbb{R}^n \to \mathbb{R}^n$ be an operator defined by $e^{At} = \sum_{m=0}^{\infty} \frac{(At)^m}{m!}$, where t is a scalar. Clearly, e^{At} is well defined. In particular,

$$\begin{aligned} e^{At} &= \sum_{m=0}^{\infty} \frac{(At)^m}{m!} \\ &= \sum_{m=0}^{\infty} \frac{A^m t^m}{m!} \\ &= \sum_{m=0}^{\infty} \begin{pmatrix} (-\lambda_1)^m & 0 & 0 & 0 \\ 0 & (-\lambda_2)^m & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & (-\lambda_n)^m \end{pmatrix} \frac{t^m}{m!} \end{aligned}$$

$$= \begin{pmatrix} \sum_{m=0}^{\infty} \frac{(-\lambda_1 t)^m}{m!} & 0 & 0 & 0 \\ 0 & \sum_{m=0}^{\infty} \frac{(-\lambda_2 t)^m}{m!} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \sum_{m=0}^{\infty} \frac{(-\lambda_n t)^m}{m!} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{-\lambda_2 t} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & e^{-\lambda_n t} \end{pmatrix}.$$

Suppose we write $\mathbf{v}$ as a linear combination of unit vectors (in this case, these are the same as our basis): $\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{e}_i$. Then we can also rewrite $A\mathbf{v}$ as:

$$A\mathbf{v} = \begin{pmatrix} -\lambda_1 & 0 & 0 & 0 \\ 0 & -\lambda_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -\lambda_n \end{pmatrix} \sum_{i=1}^{n} a_i \mathbf{e}_i$$

$$= \begin{pmatrix} -\lambda_1 & 0 & 0 & 0 \\ 0 & -\lambda_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -\lambda_n \end{pmatrix} \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a_n \end{pmatrix}$$

$$= \begin{pmatrix} -\lambda_1 a_1 & 0 & 0 & 0 \\ 0 & -\lambda_2 a_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -\lambda_n a_n \end{pmatrix}$$

$$= -\lambda_1 a_1 \mathbf{e}_1 - \lambda_2 a_2 \mathbf{e}_2 - \cdots - \lambda_n a_n \mathbf{e}_n$$

$$= \sum_{i=1}^{n} -\lambda_i a_i \mathbf{e}_i.$$

Therefore, if we look at $e^{At}\mathbf{v}$, then

$$e^{At}\mathbf{v} = \begin{pmatrix} e^{-\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{-\lambda_2 t} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & e^{-\lambda_n t} \end{pmatrix} \sum_{i=1}^{n} a_i \mathbf{e}_i$$

$$= \sum_{i=1}^{n} e^{-\lambda_i t} a_i \mathbf{e}_i.$$

It is interesting to note that $e^{At}\mathbf{v}$ is a semigroup of linear transformations (hereafter known as a semigroup). Let s be a scalar. We show that e^{At} is operation preserving:

$$e^{As}\left(e^{At}\mathbf{v}\right) = e^{As}\left(\sum_{i=1}^{n} e^{-\lambda_i t} a_i \mathbf{e}_i\right);\ \text{by above work}$$

$$= e^{As} \cdot \sum_{i=1}^{n} e^{-\lambda_i t} a_i \mathbf{e}_i;\ \text{by the definition of our mapping}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} e^{-\lambda_i s} e^{-\lambda_i t} a_i \mathbf{e}_i; \text{ write as a linear combination of diagonal matrices} \\
&= \sum_{i=1}^{n} e^{-\lambda_i (s+t)} a_i \mathbf{e}_i \\
&= e^{A(s+t)} \cdot \sum_{i=1}^{n} a_i \mathbf{e}_i \\
&= e^{A(s+t)} \mathbf{v}.
\end{aligned}
$$

Note that this is usually written as $e^{As} \cdot e^{At} = e^{A(s+t)}$.

Now, we calculate derivatives:

$$
\begin{aligned}
\frac{d}{dt} e^{At} \mathbf{v} &= \frac{d}{dt} \sum_{i=1}^{n} e^{-\lambda_i t} a_i \mathbf{e}_i \\
&= \sum_{i=1}^{n} \frac{d}{dt} e^{-\lambda_i t} a_i \mathbf{e}_i \\
&= \sum_{i=1}^{n} -\lambda_i e^{-\lambda_i t} a_i \mathbf{e}_i \\
&= A \cdot e^{At} \mathbf{v} \\
&= A \left(e^{At} \mathbf{v} \right),
\end{aligned}
$$

where the last line indicates that A acts on $e^{At}\mathbf{v}$. Note that if derivatives were taken directly, on the left hand side, the same result would occur. Now, take derivatives when $t = 0$.

$$
\left. \frac{d}{dt} e^{At} \mathbf{v} \right|_{t=0} = A e^{At} \mathbf{v} \big|_{t=0} = A e^{0} \mathbf{v} = A \mathbf{v}.
$$

Hence, when we take the derivative of $e^{At}\mathbf{v}$ with respect to t and evaluate it at $t = 0$, we will recover the original matrix A. Therefore, e^{At} is generated by A. To recover A after a transformation given by e^{At}, we take derivatives.

Now, since A is a diagonal matrix such that the diagonal values are all non-zero, A has an inverse. In particular, this inverse is

$$A^{-1} = \begin{pmatrix} -\lambda_1^{-1} & 0 & 0 & 0 \\ 0 & -\lambda_2^{-1} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -\lambda_n^{-1} \end{pmatrix}.$$

Now, suppose we do not know A, but we know e^{At}. Let P_t be a semigroup of linear transformations. Then this family has the properties that, for fixed t:

1. $t, s \geq 0$.
2. $P_0 =$ Identity.
3. $P_t P_s = P_{t+s}$.
4. $P_t\mathbf{v} \to P_0\mathbf{v}$ as $t \to 0$ and $P_0\mathbf{v} = \mathbf{v}$.

Assume $P_t\mathbf{v} = e^{At}\mathbf{v}$. Let $G\mathbf{v} = \int_0^\infty P_t\mathbf{v}dt$. We claim that $G = -A^{-1}$.

Proof.

$$
\begin{aligned}
G\mathbf{v} &= \int_0^\infty P_t \mathbf{v} dt \\
&= \int_0^\infty e^{At}\mathbf{v} dt \\
&= \int_0^\infty \sum_{i=1}^n e^{-\lambda_i t} a_i \mathbf{e}_i dt \\
&= \sum_{i=1}^n \int_0^\infty e^{-\lambda_i t} a_i \mathbf{e}_i dt;\ \text{by Fubini's Theorem} \\
&= \sum_{i=1}^n a_i \mathbf{e}_i \int_0^\infty e^{-\lambda_i t} dt;\ \text{which is integrable since } \lambda_i > 0 \\
&= \sum_{i=1}^n a_i \mathbf{e}_i \left[\left. \frac{e^{-\lambda_i t}}{-\lambda_i} \right|_0^\infty \right] \\
&= \sum_{i=1}^n a_i \mathbf{e}_i \left[-\frac{1}{\lambda_i}(0) + \frac{1}{\lambda_i}(1) \right] \\
&= \sum_{i=1}^n \frac{1}{\lambda_i} a_i \mathbf{e}_i \\
&= -\sum_{i=1}^n -\frac{1}{\lambda_i} a_i \mathbf{e}_i \\
&= -A^{-1}\mathbf{v}.
\end{aligned}
$$

Therefore, $G = -A^{-1}$. □

2.2.2 Case 2: With One Zero Eigenvalue

Take $\lambda_1 = 0$ and use the same setup as above. Originally we defined $\mathbf{v} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$. The major difference in this case is that A does not have an inverse because one of its eigenvalues is zero. We want to restrict ourselves to a smaller space, say $W \subset \mathbb{R}^n$, such that A will have an inverse.

To be able to find an inverse of A on W, we must remove the $a_1\mathbf{e}_1$ term. To do this, we can consider the projection of $\mathbf{v}$ onto $\mathbf{e}_1$. When we project onto this eigenvector, there is no component on the $\mathbf{e}_1$ axis since $-\lambda_1 = 0$. Hence, for any $\mathbf{v}$, we only consider the projection P to the other dimensions, i.e., look at $P\mathbf{v}$ on the space spanned by $\{\mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_n\}$.

Suppose W is the space spanned by $\{\mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_n\}$. Then on W, $A\mathbf{v} = A(P\mathbf{v})$.

Proof.

$$\begin{aligned} A\mathbf{v} &= A\left(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n\right) \\ &= A\left(a_1\mathbf{e}_1 + P\mathbf{v}\right) \\ &= A\left(P\mathbf{v}\right), \end{aligned}$$

since $a_1\mathbf{e}_1$ and $P\mathbf{v}$ are orthogonal to each other since the $\mathbf{e}_i$'s form an orthonormal basis. □

On W, we will have

$$\mathbf{v} = a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n,$$

and

$$A = \begin{pmatrix} -\lambda_2 & 0 & 0 & 0 \\ 0 & -\lambda_3 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -\lambda_n \end{pmatrix}.$$

We can conclude from this that A will have an inverse on W.

The question that arises from this is how do we restrict ourselves to W and why is this a concern? This is an issue because $\mathbf{v}$ may not necessarily be in W. However, we *do* know that $P\mathbf{v} \in W$ and $W \ni P\mathbf{v} = (\mathbf{v} - P_1\mathbf{v})$, where $P_1\mathbf{v}$ is the projection onto $\mathbf{e}_1$. In other words, all we need to do is to take $\mathbf{v}$ and subtract out the component(s) that are associated with the zero eigenvalues. Now, we can define

$$G\mathbf{v} = \int_0^\infty e^{At}\left(\mathbf{v} - P_1\mathbf{v}\right) dt,$$

and with this construction,

$$G = -A^{-1}$$

on W.

2.3 Functional Analysis and a Semigroup on Infinite Dimensional Space

Assume we are working on a separable Hilbert space (an infinite space with a countable basis). Let $\phi_i,\ i = 1, 2, \ldots$, be an orthonormal basis and A be a self-adjoint operator with ϕ_i its eigenvectors and λ_i its corresponding eigenvalues. We also assume $0 > -\lambda_1 \geq -\lambda_2 \geq \cdots$.

Similar to the finite dimensional case, we can define e^{At} for each fixed t to be

$$e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}.$$

In addition, we still have the property that

$$\left(e^{At}\right) \circ \left(e^{As}\phi\right) = e^{A(t+s)}\phi,$$

where we interpret the right hand side as $e^{A(t+s)}$ is acting on ϕ.

Moreover, when taking the derivative of $e^{At}\phi$ at $t = 0$, we have that

$$\frac{d}{dt} e^{At}\phi \bigg|_{t=0} = A\phi,$$

for $\phi \in \mathrm{Dom}(A)$. We must have the domain restriction because A may not necessarily act on every ϕ. We want

$$A\phi = \sum_{i=1}^{\infty} a_i \left(-\lambda_i\right) \phi_i$$

to converge. Note that in finite dimensions, we did not need this restriction because we already knew that this would converge. In order to guarantee convergence, we require that the Hilbert norm of $\sum_{i=1}^{\infty} a_i (-\lambda_i) \phi_i$ be finite, i.e.

$$\sum_{i=1}^{\infty} a_i^2 (-\lambda_i)^2 < \infty.$$

In the case where $\lambda_1 > 0$, we do not need a subspace. We can define

$$G\phi = \int_0^{\infty} e^{At} \phi dt$$

and hence $G = -A^{-1}$. In the case where $\lambda_1 = 0$, we need to project ϕ down to a subspace, as in Section 2.2.2. Let W be this subspace. Then on W, define

$$G\phi = \int_0^{\infty} e^{At} (\phi - P_1 \phi) \, dt,$$

and thus $G = -A^{-1}$ on W.

2.4 The Laplace-Beltrami Operator on a Manifold

On a compact Riemannian manifold M, let Δ be the Laplace-Beltrami operator, and m be the volume measure on the manifold. Recall that the volume of the entire manifold is given by m_0. Note that Δ will act in the same way A did in the above work due to the spectral theorem for the Laplace-Beltrami operator.

Theorem 2.2. *(Buser Theorem 7.2.6 (6))*

Let M be a compact connected Riemannian manifold without boundary. The eigenvalue problem

$$\Delta\varphi = \lambda\varphi$$

has a complete orthonormal system of $C^\infty-$eigenfunctions $\varphi_0, \varphi_1, \ldots$ in $L^2(M)$ with corresponding eigenvalues $\lambda_0, \lambda_1, \ldots$. These have the following properties.

(i) $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to \infty$ *as* $n \to \infty$.

(ii) $p_M(x, y, t) = \sum_{n=0}^{\infty} e^{-\lambda_n t}\varphi_n(x)\varphi_n(y)$, *where the series converges uniformly on* $M \times M$ *for each* $t > 0$.

For additional references on the spectral theorem for the Laplace-Beltrami operator, see Dodziuk Section 2 (9) and Milgram and Rosenbloom pages 183-184 (22).

We can formally define the Laplace-Beltrami operator as follows:

Definition 2.5. (Hsu (15))

Let $X_i = \partial/\partial x^i$ and $g_{ij} = \langle X_i, X_j\rangle$. The matrix $g = \{g_{ij}\}$ is positive definite at each point. We take g^{ij} to be the inverse of g_{ij}. The Laplace-Beltrami operator may be defined in terms of local coordinates [see Definition 2.2] as

$$\Delta f = \frac{1}{\sqrt{\det(g)}}\frac{\partial}{\partial x^i}\left(\sqrt{\det(g)}g^{ij}\frac{\partial f}{\partial x^j}\right).$$

Δ is a nondegenerate second order elliptic operator.

The Hilbert space we are looking for/working with is $L^2(M,m)$. Let $f,g \in L^2(M)$ be two functions. Define their inner product as

$$(f,g)_{L^2} = (f,g) = \int_M f(x)g(x)m(dx).$$

As in Brosamler (1983) (5), under the inner product stated above, there exists $0 > -\lambda_1 > -\lambda_2 > \cdots$ such that the corresponding eigenvalues for Δ are $0, -\lambda_1, -\lambda_2, \ldots$ and correspond with the eigenvectors $\phi_0 = c, \phi_1, \phi_2, \ldots$ form an orthonormal basis of $L^2(M,m)$ (where c is a constant function). Moreover, $\phi_0, \phi_1, \phi_2, \ldots$ are all smooth functions on M. Hence, every function in $L^2(M,m)$ may be written as

$$f = \sum_{i=0}^{\infty} a_i \phi_i.$$

We can also calculate the constant value in the following manner:

$$\begin{aligned}
1 &= \|\phi_0\| \\
&= \sqrt{(\phi_0, \phi_0)} \\
&= \sqrt{\int_M c^2 m(dx)} \\
&= \sqrt{c^2 \int_M m(dx)} \\
&= \sqrt{c^2 m_0} \\
&= c\sqrt{m_0}.
\end{aligned}$$

Therefore, $c = m_0^{-1/2}$.

Because we have a zero eigenvalue, we need to project ϕ down to a subspace, as in Section 2.3. Let W be this subspace. Then on W, we will have that

$$G = -\Delta^{-1}.$$

2.5 Brownian Motion

Definition 2.6. (Hsu (14))

For Brownian motion on $\mathbb{R}^n$, its transition density function is the Gaussian heat kernel

$$p(t,x,y) = \left(\frac{1}{2\pi t}\right)^{n/2} e^{-|x-y|^2/2t},$$

and its infinitesimal generator is half of the Laplace Operator:

$$\frac{1}{2}\Delta = \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

Remark 2.6. Brownian motion at time t in $\mathbb{R}^n$, denoted by B_t, has the following properties (as stated in Øksendal (24)):

1. B_t is a Gaussian process.

2. B_t has independent increments, i.e.

$$B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_k} - B_{t_{k-1}}$$

are independent for all $0 \leq t_1 < t_2 < \cdots < t_k$.

3. $B_t(\omega)$ is continuous for almost all ω.

Remark 2.7. (Hsu (14))

It is useful to regard paths of Brownian motion [on $\mathbb{R}^n$] as the characteristic lines of the Laplace operator Δ. ... The counterpart of the Laplace operator on a Riemannian manifold M is the Laplace-Beltrami operator Δ_M, which will serve as the infinitesimal generator for Browmian motion on M. For a definition of the Laplace-Beltrami operator, see Definition 2.5.

Remark 2.8. In this work, we will denote the Laplace-Beltrami operator as Δ instead of Δ_M, as in the remark above.

Remark 2.9. Intuitively, we can understand Brownian motion on a compact Riemannian manifold as follows. Take a Brownian motion in $\mathbb{R}^2$. We take a sphere and roll it along the $\mathbb{R}^2$ surface, assuming that there is no slips while the ball is rolling. The path from $\mathbb{R}^2$ space is now imposed on the sphere; this can be considered as Brownian motion on the sphere. For a more comprehensive description, see McKean (21) page 122.

2.6 Relationship Between Functional Analysis and Probability

In this section, we relate the semigroup generated by the Laplace-Beltrami operator to probability through expected values. Let X_t be Brownian Motion on our manifold M that starts at x (denoted by X_t^x). Let $e^{\Delta t} = P_t$, then P_t is a semigroup of linear transformations. Then the semigroup $\left(e^{\Delta t} f\right)(x)$ can be written as

$$\left(e^{\Delta t} f\right)(x) = \left(P_t f\right)(x) = \mathbf{E}_x f\left(X_t\right),$$

which is the semigroup generated by the Laplacian.

Now, since we have ϕ_0, which is associated with a zero eigenvalue, we want to project down as before. Let this projection be denoted by $\mathbb{P}$. Similar to the other cases, we use the projection that removes ϕ_0:

$$f(x) - (\mathbb{P}_0 f)(x) = f(x) - \frac{1}{m_0} \int_M f(x) m(dx). \tag{2.1}$$

In this way, we can project f to a subspace W, where $W = \operatorname{span}\{\phi_1, \phi_2, \ldots\}$. Now, Δ will have an inverse in W. So

$$\begin{aligned}
(Gf)(x) &= \int_0^\infty e^{\Delta t} \left(f(x) - \mathbb{P}_0 f \right) dt \\
&= \int_0^\infty e^{\Delta t} \left(f(x) - \frac{1}{m_0} \int_M f(x) m(dx) \right) dt; \\
&\qquad \text{observe that the second term inside the integration is a number} \\
&= \int_0^\infty \mathbf{E}_x \left[f(X_t) - \frac{1}{m_0} \int_M f(x) m(dx) \right] dt.
\end{aligned}$$

Take y to be a variable in $p_t(x, y)$, the transition density function of X_t^x. Then we know

$$\mathsf{P}\{X_t^x \in A\} = \int_A p_t(x, y) m(dy).$$

Hence,

$$\mathbf{E}_x f(X_t) = \int_M f(y) p_t(x, y) m(dy).$$

Continuing our calculations,

$$
\begin{aligned}
(Gf)(x) &= \int_0^\infty \mathbf{E}_x \left[f\left(X_t\right) - \frac{1}{m_0} \int_M f(x)m(dx) \right] dt \\
&= \int_0^\infty \left\{ \int_M \left[f(y) - \frac{1}{m_0} \int_M f(x)m(dx) \right] p_t(x,y)m(dy) \right\} dt \\
&= \int_0^\infty \left\{ \int_M \left[f(y) - \frac{1}{m_0} \int_M f(y)m(dy) \right] p_t(x,y)m(dy) \right\} dt; \\
&\quad \text{a change in variable in the constant term is performed} \\
&= \int_0^\infty \left\{ \int_M f(y)p_t(x,y)m(dy) - \frac{1}{m_0} \int_M p_t(x,y) \left(\int_M f(y)m(dy) \right) m(dy) \right\} dt \\
&= \int_0^\infty \left\{ \int_M f(y)p_t(x,y)m(dy) - \frac{1}{m_0} \int_M p_t(x,y)m(dy) \int_M f(y)m(dy) \right\} dt \\
&= \int_0^\infty \left\{ \int_M f(y)p_t(x,y)m(dy) - \frac{1}{m_0} \int_M f(y)m(dy) \right\} dt; \text{ because } \int_M p_t(x,y)m(dy) = 1 \\
&= \int_0^\infty \left\{ \int_M \left[f(y)p_t(x,y) - \frac{1}{m_0} f(y) \right] m(dy) \right\} dt \\
&= \int_0^\infty \left\{ \int_M f(y) \left[p_t(x,y) - \frac{1}{m_0} \right] m(dy) \right\} dt \\
&= \int_M \left\{ \int_0^\infty f(y) \left[p_t(x,y) - \frac{1}{m_0} \right] dt \right\} m(dy) \\
&= \int_M f(y) \left\{ \int_0^\infty \left[p_t(x,y) - \frac{1}{m_0} \right] dt \right\} m(dy) \\
&= \int_M f(y)g(x,y)m(dy), \qquad (2.2)
\end{aligned}
$$

where we define $g(x,y)$ to be the Green Kernel

$$
g(x,y) = \int_0^\infty \left[p_t(x,y) - \frac{1}{m_0} \right] dt.
$$

Note that (Equation 2.2) "defines a bounded linear operator $G\colon L^2(M) \to L^2(M)$ which is nonnegative and symmetric" (Brosamler (5)).

2.7 Definitions

We require several definitions for our work. Definition 2.7 defines several new spaces. Definition 2.8 extends (Equation 2.2) and lists several properties of G. Definition 2.9 defines a new space for a special case of G.

Definition 2.7. (Brosamler (5))

$$\begin{aligned} L_0^2(M) &= \left\{ f(x) \in L^2(M); \int_M f(x)m(dx) = 0 \right\}, \\ C_0^\infty(M) &= \left\{ f(x) \in C^\infty(M); \int_M f(x)m(dx) = 0 \right\}, \\ B(M) &= \{ f\colon M \to \mathbb{R} \text{ measurable and bounded} \}. \end{aligned}$$

Definition 2.8. (Brosamler (5))

Let $p : (0, \infty) \times M \times M \to \mathbb{R}$ be the fundamental solution of $\frac{1}{2}\Delta_y p_t(x, y) = \frac{\partial}{\partial t} p_t(x, y)$ where Δ is the Laplace operator on M. Let

$$g_\alpha(x, y) = [\Gamma(\alpha)]^{-1} \int_0^\infty t^{\alpha-1} \left\{ p_t(x, y) - m_0^{-1} \right\} dt$$

for $x, y \in M$ and $x \neq y$, and where $p_t(x,y)$ is the transition density function of X_t^x. For real $\alpha > 0$, let

$$(G_\alpha f)(x) = \int_M g_\alpha(x,y) f(y) m(dy) \tag{2.3}$$

such that $G_\alpha : L_0^2 \to L_0^2$. Hence $\|G_\alpha f\|_{L^2} \leq \|f\|_L^2 \cdot \|g_{2\alpha}\|_{L^2}^{1/2}$. We list several properties of G_α.

1. $G_\alpha = G$ when $\alpha = 1$.
2. $G = -\Delta^{-1}$. (For details, see Section 2.4.)
3. G is a self-adjoint operator.
4. $G_\alpha : L^2 \to B$, where B is as in Definition 2.7, is a bounded linear operator if $\alpha > \frac{d}{4}$. (Brosamler Lemma 2.14 (5))

Remark 2.10. An explanation of how G_α will be used in terms of functional analysis and our work is given in Section 2.8.

Definition 2.9. (Brosamler Definition 2.12 (5))

Let $H_0^\alpha = G_{\frac{\alpha}{2}}\left(L_0^2\right)$ for real $\alpha > 0$. Let it have pointwise addition and pointwise multiplication by scalars and with the inner product

$$\langle G_{\frac{\alpha}{2}} f_1, G_{\frac{\alpha}{2}} f_2 \rangle_{H_0^\alpha} = 2^\alpha \left(f_1, f_2\right)_{L^2}.$$

H_0^α has several properties:

1. $H_0^\alpha \subseteq L_0^2$.

2. $\|\cdot\|_{H_0^\alpha}$ is the norm induced by $\langle\cdot,\cdot\rangle_{H_0^\alpha}$.

3. $C_0^\infty \subseteq H_0^\alpha$ for real $\alpha > 0$.

4. C_0^∞ is dense in H_0^α for $\alpha > 0$ so that the spaces H_0^α are completions of C_0^∞ with the norm $\|\cdot\|_{H_0^\alpha}$.

5. (Equation 2.3) is also a semigroup of bounded linear operators on each H_0^β for $\beta > 0$.

2.8 The Selection of Functions and the Operator G_α

We cannot guarantee that operators in L^2 are invertible because one or more of their eigenvalues may be the zero eigenvalue. If we take $L_0^2 = L^2 \cap W$ (where W is as defined in Section 2.4), any zero eigenvalue is removed; therefore, there exists an inverse of an operator in L_0^2.

Consider the measure

$$\mu_t(f) = \frac{\int_0^t f\left(X_s\right) ds - \frac{t}{m_0}\int_M f(x)m(dx)}{\sqrt{2t\log\log t}},$$

where $f \in L^2$. By Definition 2.7, if $f \in L_0^2$, then $\int_M f(x)m(dx) = 0$; this is directly related to (Equation 2.1) from Section 2.6. This means that our measure is

$$\mu_t(f) = \frac{\int_0^t f\left(X_s\right) ds}{\sqrt{2t\log\log t}}, \quad f \in L_0^2.$$

Therefore, we only want to consider functions from W.

In particular, recall that $G = -\Delta^{-1}$. Assume $f \in W$. Then

$$\Delta f = \sum_{i=1}^{\infty} \frac{1}{2} \left(-\lambda_i\right) a_i \phi_i$$

$$Gf \overset{\text{(Definition 2.8)}}{=} -\Delta^{-1} f = \sum_{i=1}^{\infty} 2 \cdot \frac{1}{\lambda_i} a_i \phi_i. \tag{2.4}$$

However, λ_i^{-1} is not "nice enough"; we want the length of (Equation 2.4) to be as small as possible. To accomplish this, take

$$G_\alpha f = \sum_{i=1}^{\infty} 2^\alpha \cdot \frac{1}{\lambda_i^\alpha} a_i \phi_i.$$

This changes the multiplicity of λ_i. In response, the a_i's become even smaller so that the length of $\sum_{i=1}^{\infty} 2^\alpha \cdot \frac{1}{\lambda_i^\alpha} a_i \phi_i$ is

$$\sum_{i=1}^{\infty} 2^{2\alpha} \cdot \frac{1}{\lambda_i^{2\alpha}} a_i^2 < \infty.$$

Therefore, for $f \in L_0^2$, $G_\alpha f$ is a "better" function because the functions will converge at a faster rate. Also note that in this configuration,

$$G_\alpha = \left(-\Delta^{-1}\right)^\alpha.$$

Using Definition 2.9, the space of this family of functions will be given by H_0^α, and

$$G_{\frac{\alpha}{2}} = \left(-\Delta^{-1}\right)^{\frac{\alpha}{2}}. \tag{2.5}$$

CHAPTER 3

STATEMENT OF THE PROBLEM

Let M be a compact C^∞ Riemannian manifold. Let Ω be the sample space. Let $\mathscr{B}(M)$ be the Borel σ-field on M such that $\mathscr{B} \subseteq M$. Let X_t be a Brownian motion on the manifold for $t \geq 0$. We consider the volume element on the manifold and denote it by m. Let $m_0 = m(M)$, the entire volume of the manifold. Let d be the dimension of M.

3.1 Ergodic Theory

Birkhoff (3) and von Neumann (31) "initiated a new field of mathematical-research called ergodic theory" (Moore (23)).

Remark 3.1. (Baxter and Brosamler (2))

If

$$A_t = \int_0^t \chi_A (X_s)\, ds$$

is the total time up to time t which the path spends in a Borel set $A \subseteq M$, we know from the ergodic theorem that for all $x \in M$,

$$\mathsf{P}\left\{ \lim_{t \to +\infty} t^{-1} A_t = \lambda(A) \right\} = 1,$$

where λ is the invariant probability measure on M, associated with the diffusion. More generally, if

$$A_t = \int_0^t f(X_s)\, ds,$$

for $f \in L_\infty(d\lambda)$, one has for all $x \in M$:

$$\mathsf{P}\left\{ \lim_{t \to +\infty} t^{-1} A_t = \int_M f(x) d\lambda(x) \right\} = 1, \tag{3.1}$$

which may be considered as a law of large numbers for the family of random variables $\{f(X_s), s \geqq 0\}$.

Remark 3.2. The above remark from Baxter and Brosamler was also presented in Brosamler (5) as Equation 1.2 with a modification in the integral over the manifold, as seen in the next theorem.

Theorem 3.1. *(Brosamler Equation 1.2 (5))*
For all $f \in L^1(M)$, all $x \in M$

$$\mathsf{P}\left\{ \omega; \lim_{t \to +\infty} \frac{1}{t} \int_0^t f(X_s)\, ds = m_0^{-1} \int_M f dm \right\} = 1, \tag{3.2}$$

where $m_0 = m(M)$.

Remark 3.3. The reason for the difference between (Equation 3.1) and (Equation 3.2) is because "$\frac{m}{m(M)}$ is the invariant probability measure for Brownian motion on M" (5). Brosamler (5) mentions that one consequence of Theorem 3.1 is the following.

Theorem 3.2. *(Brosamler Equation 1.3 (5))*

For all $x \in M$*,*

$$\mathsf{P}\left\{\omega;\lim_{t\to+\infty}\frac{1}{t}\int_0^1 f\left(X_s\right)ds = m_0^{-1}\int_M f dm,\ \ \textit{all}\ \ f\in C(M)\right\} = 1.$$

Remark 3.4. In the theorem above, $\int_0^t f(X_s)ds$ goes to infinity with a rate of t and scalar $\frac{1}{m_0}\int_M f(x)m(dx)$. This is the result from classic ergodic theory.

3.2 Log-Log Results from Baxter and Brosamler (1976) and Brosamler (1983)

Using Definitions 2.7, 2.8, and 2.9, Baxter and Brosamler (2) provide a version of a $\log_2$-law for Brownian motion on compact manifolds.

Theorem 3.3. *(Baxter and Brosamler (2), as stated in Brosamler (5))*

For bounded measurable $f : M \to \mathbb{R}$*:*

$$\mathsf{P}\left\{\limsup_{t\to+\infty}\frac{\int_0^t f(X_s)ds - \frac{t}{m_0}\int_M f(x)m(dx)}{\sqrt{2t\log\log t}} = \sqrt{\frac{2}{m_0}\left(f,Gf\right)}\right\} = 1, \quad \forall x\in M. \tag{3.3}$$

Here

$$(Gf)(x) = \int_M g(x,y)f(y)m(dy),$$

where the kernel g *is unique and*

$$\int_M g(x,y)m(dy) = 0, \quad x\in M.$$

Note that the derivation of $(Gf)(x)$ in the above theorem may be found in Section 2.6. Brosamler (5) extends this to all $f \in C^{\infty}(M)$:

Theorem 3.4. *(Brosamler Theorem 1.7 (5))*

For any compact C^{∞} Riemannian manifold (M, g) and associated Brownian motion X we have for all $x \in M$

$$\mathsf{P}\left\{\limsup_{t\to+\infty} \frac{\int_0^t f(X_s)ds - \frac{t}{m_0}\int_M f(x)m(dx)}{\sqrt{2t\log\log t}} = \sqrt{\frac{2}{m_0}\,(f, Gf)},\ \ \forall f \in C^{\infty}\right\} = 1, \tag{3.4}$$

where $m_0 = m(M)$.

(Equation 3.4) is shown immediately from the following theorem:

Theorem 3.5. *(Brosamler Theorem 3.16 (5))*

For any compact C^{∞} Riemannian manifold (M, g) of dimension $d \geqq 1$ and associated Brownian motion X, we have

$$\mathsf{P}\left\{\omega;\, \underset{t\to+\infty}{\text{cluster set}}\, \frac{\int_0^t f(X_s(\omega))ds}{\sqrt{2t\log\log t}} = \left[-\sqrt{\frac{2}{m_0}(f, Gf)_{L^2}}, \sqrt{\frac{2}{m_0}(f, Gf)_{L^2}}\right]\ \text{for all } f \in H_0^{\alpha}\right\} = 1,$$

$x \in M$, if $\alpha > \max\left(d - \frac{3}{2}, \frac{d}{2}\right)$.

3.3 Goals

View the left hand side of (Equation 3.4) as a collection of random measures. We make a new definition of convergence similar to weak convergence, but works for all smooth functions.

Definition 3.1. The signed measure μ_t converges mildly to another signed measure μ, written as $\mu_t \xrightarrow{\text{mildly}} \mu$, if and only if $\int f d\mu_t \to \int f d\mu,\ \forall f \in C^\infty(M)$. This is a type of convergence under the weak topology of signed measures.

Remark 3.5. Intuitively, weak convergence implies mild convergence. However, the reverse is not necessarily true.

We will prove the following:

Theorem 3.6. *Let (M, g) be a compact C^∞ Riemannian manifold with associated Brownian motion X_t. Let $\omega \in \Omega$. We define a signed measure μ_t^ω for fixed t and fixed ω by*

$$\mu_t^\omega(f) = \frac{\int_0^t f(X_s)ds - \frac{t}{m_0}\int_M f(x)m(dx)}{\sqrt{2t\log\log t}}. \tag{3.5}$$

Consider the family of measures $\{\mu_t^\omega, t \geq 0\}$. For any subsequence of times $\tilde{t}_n$, if $\mu_{\tilde{t}_n}$ converges mildly to μ then μ is a measure such that it is absolutely continuous with respect to the volume measure on the manifold. Take $G_{\frac{\alpha}{2}}$ as in Definition 2.8. μ is characterized by $\left\| G_{\frac{1}{2}}^{-1} q \right\|_{L^2} \leq \sqrt{\dfrac{2}{m_0}}$, where q is the density of $\mu(dx)$ such that $q(x) \in L^2$, and $\mu(M) = 0$.

Remark 3.6. Usually a measure would be written as $\mu(\omega)$, but for the sake of notation, we put ω as a superscript of the measure because we also require a function for our measure; within the definition, the notation (ω) is suppressed (as for $X_s(\omega)$).

Remark 3.7. Both $\int_0^t f(X_s)ds$ and $\frac{t}{m_0}\int_M f(x)m(dx)$ from (Equation 3.5) are measures. Some proofs for the case when $f = \text{Indicator}(A)$ may be found in Appendix A.

Theorem 3.7. *Suppose there exists a measure μ, characterized by $\left\| G_{\frac{1}{2}}^{-1} q \right\|_{L^2} \leq \sqrt{\frac{2}{m_0}}$ with $\mu(M) = 0$. Then there exists a sequence of times t_n such that μ_{t_n} converges mildly to μ for some ω.*

Theorem 3.8. *Assume μ is a measure with density q, characterized by $\left\| G_{\frac{1}{2}}^{-1} q \right\|_{L^2} \leq \sqrt{\frac{2}{m_0}}$, where $G_{\frac{\alpha}{2}}^{-1}$ is as in Definition 2.8 and $\mu(M) = 0$. Then almost surely there exists a sequence of times t_n such that μ_{t_n} converges mildly to μ as $n \to +\infty$.*

Remark 3.8. This work commenced when we asked the question of whether a characterization of μ could be found (Theorem 3.6). Once we found a characterization, we needed to confirm that the limiting measure existed (Theorem 3.7). For a more complete result, we then wanted to show that if we started with the characterization of μ, we could find a sequence of times t_n of t so that μ_t would converge to μ for all ω.

The rest of this work is structured as follows. The proof of Theorem 3.6 is in Chapter 4. The proof of Theorem 3.7 is in Chapter 5. The proof of Theorem 3.8 is in Chapter 6.

CHAPTER 4

PROOF OF THEOREM 3.6

To prove Theorem 3.6, we state and prove several lemmas. In combining all of these lemmas, we give the conclusion of the proof in Section 4.4.

4.1 Bounds for $\mu(f)$

Lemma 4.1. $\left|\int_M f(x)\mu(dx)\right| \leq \sqrt{\frac{2}{m_0}}\|h\|_{L^2}$, *where* $h = G_{\frac{1}{2}} f$.

Proof. Suppose μ is an accumulating point of μ_t, in the mild sense. If $f \in C^\infty$, f is bounded and measurable and $f \in H_0^\alpha$ for all $\alpha > \max\left(d - \frac{3}{2}, \frac{d}{2}\right)$. Then we can write

$$\int_M f(x)\mu_t(dx) = \mu_t(f) \overset{\text{(Equation 3.5)}}{=} \frac{\int_0^t f(X_s)ds - \frac{t}{m_0}\int_M f(x)m(dx)}{\sqrt{2t\log\log t}}.$$

We know by (Equation 3.4) that

$$\mu_t(f) \leq \limsup_{t\to+\infty} \mu_t(f) = \limsup_{t\to+\infty} \frac{\int_0^t f(X_s)ds - \frac{t}{m_0}\int_M f(x)m(dx)}{\sqrt{2t\log\log t}} \leq \sqrt{\frac{2}{m_0}(f, Gf)_{L^2}}.$$

Since μ is a collection of accumulating points of μ_t, then there is a sequence t_n such that

$$\int_M f(x)\mu_{t_n}(dx) \to \int_M f(x)\mu(dx).$$

Moreover, $\left|\int_M f(x)\mu_t(dx)\right|$ is bounded, as in (Equation 3.4) so

$$-\sqrt{\frac{2}{m_0}(f,Gf)_{L^2}} \leq \int_M f(x)\mu(dx) \leq \sqrt{\frac{2}{m_0}(f,Gf)_{L^2}}.$$

Hence

$$\begin{aligned}
\left|\int_M f(x)\mu(dx)\right| &\leq \sqrt{\frac{2}{m_0}(f,Gf)_{L^2}} \\
&= \sqrt{\frac{2}{m_0}}\sqrt{(f,Gf)_{L^2}} \\
&= \sqrt{\frac{2}{m_0}}\sqrt{(f,G_1 f)_{L^2}};\ \text{by Definition 2.8} \\
&= \sqrt{\frac{2}{m_0}}\sqrt{(f,G_{\frac{1}{2}}G_{\frac{1}{2}} f)_{L^2}};\ \text{since } G_{\alpha+\beta} = G_\alpha \circ G_\beta \\
&= \sqrt{\frac{2}{m_0}}\sqrt{(G_{\frac{1}{2}} f, G_{\frac{1}{2}} f)_{L^2}} \\
&= \sqrt{\frac{2}{m_0}}\left\|G_{\frac{1}{2}} f\right\|_{L^2} \\
&= \sqrt{\frac{2}{m_0}}\|h\|_{L^2};\ \text{where } h = G_{\frac{1}{2}} f. \qquad (4.1)
\end{aligned}$$

□

4.2 A Representation for $\int_M f(x)\mu(dx)$

Lemma 4.2. *$\int_M f(x)\mu(dx)$ is a bounded linear functional on $L^2(M,m)$.*

Proof. First, we know that G is a bounded linear functional in L^2 by Definition 2.8. Hence

$$\left|\int_M f(x)\mu(dx)\right| \leq \sqrt{\frac{2}{m_0}(f,Gf)};\ \text{by Brosamler (5)}$$

$$\leq \sqrt{\frac{2}{m_0}\|f\|_{L^2}\|Gf\|_{L^2}}; \text{ by Cauchy-Schwarz Inequality}$$
$$\leq \sqrt{\frac{2}{m_0}c_1\|f\|^2_{L^2}}; \text{ because } G \text{ is bounded,}$$
$$\|Gf\|_{L^2} \leq c_1\|f\|_{L^2} \text{ for some constant } c_1 < \infty$$
$$\leq \sqrt{\frac{2c_1}{m_0}}\|f\|_{L^2}$$
$$< \infty.$$

This means that $\int_M f(x)\mu(dx)$ is a bounded linear functional and on L^2 for all $f \in C^\infty$. Hence, we can extend $\int_M f(x)\mu(dx)$ to the entire L^2 space. $\square$

We now state a general version of Riesz Representation Theorem, and a special case of it for an L^2 space:

Theorem 4.3. *(General Case, as stated in Bogachev (4))*
Let f be a continuous linear function on a Hilbert space H. Then, there exists a unique vector v such that

$$f(x) = (x, v) \textit{ for all } x \in H.$$

Theorem 4.4. *(Special Case when $p = 2$, as stated in Athreya and Lahiri (1))*
Let $1 \leq p < \infty$. Let $q = \frac{p}{p-1}$ for $1 < p < \infty$ and $q = \infty$ if $p = 1$. [In particular, take $p = q = 2$.] Let $T : L^2(\mu) \to \mathbb{R}$ be linear and continuous. Then there exists a $g \in L^2(\mu)$ such that $T = T_g$, i.e.

$$T(f) = T_g(f) \equiv \int fg d\mu$$

for all $f \in L^2(\mu)$.

Lemma 4.5. *$\mu(dx)$ can be written in terms of a density function, $q(x) \in L^2$.*

Proof. By Theorems 4.3 and 4.4, we can represent our bounded linear functional $\mu(dx)$ as $q(x)m(dx)$ such that $q(x)$ is the density of $\mu(dx)$, and in particular, $q(x) \in L^2$ by Theorem 4.4. $\square$

Remark 4.1. Rewriting our integral, we also can obtain the bounded linear functional,

$$\int_M f(x)\mu(dx) = \int_M f(x)q(x)m(dx). \tag{4.2}$$

To characterize our measure, we need to rewrite $\int_M f(x)q(x)m(dx)$ in terms of $h = G_{\frac{1}{2}}f$. First, we find an upper bound in terms of h:

$$\begin{aligned}
\int_M f(x)q(x)m(dx) &= \left|\int_M f(x)\mu(dx)\right| \\
&\leq \sqrt{\frac{2}{m_0}(f, Gf)_{L^2}} \\
&= \sqrt{\frac{2}{m_0}}\left\|G_{\frac{1}{2}}f\right\|_{L^2} \\
&= \sqrt{\frac{2}{m_0}}\|h\|_{L^2} \text{ by (Equation 4.1).}
\end{aligned} \tag{4.3}$$

Remark 4.2. Since the upper bound is in terms of the norm of h, we must also try to rewrite $\int_M f(x)q(x)m(dx)$ in terms of $\|h\|_{L^2}$. However, we must first show the existence of $G_{\frac{\alpha}{2}}^{-1}$ and whether or not this may be applied to $q(x)$, our density function.

4.3 Existence of $G_{\alpha/2}^{-1}$, the Self-Adjoint Property, and Application to $q(x)$

4.3.1 Existence and the Self-Adjoint Property

Earlier, in Definition 2.8, we observed that $G = -\Delta^{-1}$. This means that $G^{-1} = -\Delta$. This exists because the Laplacian exists. We also know that G is self-adjoint. The meaning of an adjoint and a self-adjoint operator are given in Definitions 4.1 and 4.2, respectively.

Definition 4.1. (Kreyszig Definition 3.9-1 (16))

Let $T\colon H_1 \to H_2$ be a linear operator, where H_1 and H_2 are Hilbert spaces. Then the *Hilbert-adjoint operator* T^* of T is the operator

$$T^*\colon H_2 \to H_1$$

such that for all $x \in H_1$ and $y \in H_2$,

$$\langle Tx, y\rangle = \langle x, T^*y\rangle.$$

Definition 4.2. (Kreyszig Definition 3.10-1 (16))

A linear operator $T\colon H \to H$ on a Hilbert space H is said to be *self-adjoint* or *Hermitian* if $T^* = T$. If T is self-adjoint, we have

$$\langle Tx, y\rangle = \langle x, Ty\rangle.$$

Lemma 4.6. *$G_{\frac{\alpha}{2}}^{-1}$ is self-adjoint.*

Proof. We need the following definition and remark.

Definition 4.3. (Chen (8))

A family $\{E(\lambda); -\infty < \lambda < \infty\}$ of projection operators on H, a separable real Hilbert space, is called a *resolution of identity*, if

1. $E(\lambda) \circ E(\gamma) = E\left(\min\{\lambda, \gamma\}\right)$

2. $E(-\infty)$ is zero operator, $E(\infty)$ is identity operator, and $E(\lambda+0) = E(\lambda)$ for every $\lambda \in \mathbb{R}$, where $E(-\infty), E(\infty)$ and $E(\lambda+0)$ are the linear operators defined as

$$E\left(\pm\infty\right) = \lim_{\lambda \to \pm\infty} E(\lambda)(x),$$
$$E(\lambda+0)(x) = \lim_{\gamma \to \lambda^+} E(\gamma)(x),$$

for all $x \in H$.

Remark 4.3. (Chen (8))

In the light casted by the representation

$$\xi(A) = \sum_{k=1}^{N} \xi\left(\lambda_k\right)\left(E(k) - E(k-1)\right),$$

given a self-adjoint operator A in the form of

$$A = \int_{-\infty}^{\infty} \lambda E(d\lambda)$$

the function of A is defined as the self-adjoint operator

$$\xi(A) = \int_{-\infty}^{\infty} \xi(\lambda) E(d\lambda),$$

where $\xi(\cdot)$ is a Borel function on $\mathbb{R}$.

From the above remark, this means that any function of a self-adjoint operator will also be self-adjoint. In equation (Equation 2.5), we stated that $G_{\frac{\alpha}{2}} = \left(-\Delta^{-1}\right)^{\frac{\alpha}{2}}$, and this is a function of Δ, the Laplacian. In fact, the Laplacian is self-adjoint (see Appendix B for an illustration involving matrices). $G_{\frac{\alpha}{2}}$ is a function of a self-adjoint operator, and hence must also be self-adjoint. Since $G_{\frac{\alpha}{2}} = \left(-\Delta^{-1}\right)^{\frac{\alpha}{2}}$, this means that $G_{\frac{\alpha}{2}}^{-1} = (-\Delta)^{\frac{\alpha}{2}}$. Because $G_{\frac{\alpha}{2}}^{-1}$ is also a function of the Laplacian, this means that $G_{\frac{\alpha}{2}}^{-1}$ must also be a self-adjoint operator. $\square$

4.3.2 Application to $q(x)$

Lemma 4.7. *$\int_M f(x)q(x)m(dx) = \left(h, G_{\frac{1}{2}}^{-1} q\right)_{L^2}$, where $h = G_{\frac{1}{2}} f$.*

Remark 4.4. To show that Lemma 4.7 is true, we first show Lemma 4.8 holds.

Lemma 4.8. *$q(x) \in \mathrm{Dom}\left(G_{\frac{1}{2}}^{-1}\right)$ so that $\left(G_{\frac{1}{2}}^{-1} h, q\right)_{L^2} = \left(h, G_{\frac{1}{2}}^{-1} q\right)_{L^2}$.*

Proof. We show this is true for $G_{\frac{\alpha}{2}}^{-1}$, for any real $\alpha > 0$. First, note that

$$\left(G_{\frac{\alpha}{2}}^{-1} h, q\right)_{L^2} = (h, q^*)_{L^2}\,; \quad q^* \in \mathrm{Dom}\left(G_{\frac{\alpha}{2}}^{-1}\right)^*$$

by the properties of an adjoint operator (see Definition 4.1) where * denotes an adjoint operator, and $q^* = \left(G_{\frac{\alpha}{2}}^{-1}\right)^* q$. Because $G_{\frac{\alpha}{2}}^{-1}$ is self-adjoint, this means that

$$\mathrm{Dom}\left(G_{\frac{\alpha}{2}}^{-1}\right)^* = \mathrm{Dom}\left(G_{\frac{\alpha}{2}}^{-1}\right)$$

which implies that

$$q^* = \left(G_{\frac{\alpha}{2}}^{-1}\right)^* q = \left(G_{\frac{\alpha}{2}}^{-1}\right) q.$$

$\square$

Proof of Lemma 4.7. Start with (Equation 4.3). Take $\alpha = 1$.

$$\begin{aligned}
\int_M f(x)q(x)m(dx) &= \int_M G_{\frac{1}{2}}^{-1} G_{\frac{1}{2}} f(x)q(x)m(dx) \\
&= \int_M G_{\frac{1}{2}}^{-1} h(x)q(x)m(dx); \text{ where } h = G_{\frac{1}{2}} f \\
&= \left(G_{\frac{1}{2}}^{-1} h, q\right)_{L^2}, \qquad (4.4) \\
&= (h, q^*)_{L^2}; \text{ where } q^* = \left(G_{\frac{1}{2}}^{-1}\right)^* q \\
&\overset{\text{Lemma 4.8}}{=} \left(h, G_{\frac{1}{2}}^{-1} q\right)_{L^2}. \qquad (4.5)
\end{aligned}$$

$\square$

4.4 Characterization

Remark 4.5. (Conclusion of the Proof of Theorem 3.6)

We want to try and rewrite $\int_M f(x)q(x)m(dx)$ in terms of the L^2 norm and h. Recall that $q(x)$ is the density of our measure μ. Then

$$\begin{aligned}
\left(G_{\frac{1}{2}}^{-1}q, h\right)_{L^2} &\overset{\text{(Equation 4.5)}}{=} \left(G_{\frac{1}{2}}^{-1}h, q\right)_{L^2} \\
&\overset{\text{(Equation 4.4)}}{=} \int_M f(x)q(x)m(dx) \\
&\overset{\text{(Equation 4.2)}}{=} \left|\int_M f(x)\mu(dx)\right| \\
&\overset{\text{(Equation 4.3)}}{\leq} \sqrt{\frac{2}{m_0}}\|h\|_{L^2}.
\end{aligned}$$

Therefore, it must be the case that

$$\left\|G_{\frac{1}{2}}^{-1}q\right\|_{L^2} \leq \sqrt{\frac{2}{m_0}}$$

completely characterizes the measure.

CHAPTER 5

PROOF OF THEOREM 3.7

To prove Theorem 3.7, we first find an upper bound for $|\mu_t(f)|$. We then state several lemmas. Lemma 5.1 is a basic proof of Theorem 3.7, but must be extended to work for all functions f. Lemma 5.2 extends Lemma 5.1 for $f \in H_0^\alpha$, and Lemma 5.3 extends Lemma 5.2 for $f \in L^2$ and defines the particular measure we are looking for.

5.1 Bounds for $\mu_t(f)$

Assume $f : M \to \mathbb{R}$ is bounded and measurable such that $f \in C^\infty$. In particular, we let $f \in C_0^\infty$ to remove the constant term (similar to Theorem 3.5). For this f,

$$\mu_t^\omega(f) = \frac{\int_0^t f(X_s)ds}{\sqrt{2t \log\log t}}.$$

From (Equation 3.4), we know that

$$\mu_t^\omega(f) \leq \limsup_{t\to+\infty} \mu_t^\omega(f) = \limsup_{t\to+\infty} \frac{\int_0^t f(X_s)ds}{\sqrt{2t \log\log t}} \leq \sqrt{\frac{2}{m_0}(f, Gf)_{L^2}}.$$

Similarly, by replacing f by $-f$,

$$-\sqrt{\frac{2}{m_0}(f, Gf)_{L^2}} \leq \liminf_{t\to+\infty} \frac{\int_0^t f(X_s)ds}{\sqrt{2t \log\log t}} = \liminf_{t\to+\infty} \mu_t^\omega(f).$$

We still have the property that for $f \in C_0^\infty$,

$$\left|\int_M f(x)\mu_t(dx)\right| \leq \sqrt{\frac{2}{m_0}\,(f, Gf)_{L^2}}.$$

5.2 Accumulating Points for Smooth f

Lemma 5.1. *For any subsequence of times $\tilde{t}_k$ of t, $\mu_{\tilde{t}_k}(f_k) \to a_{f_k}$ for all k, where $\{f_k\}$ is a countable dense subset of H_0^α for all k.*

Proof. To show that accumulating points exist, we use a diagonalization argument. First, $\limsup_{t\to+\infty} \mu_t^\omega(f)$ is bounded above, so

$$a_f(\omega) = \limsup_{t\to+\infty} \mu_t^\omega(f) \leq \sqrt{\frac{2}{m_0}(f, Gf)_{L^2}},$$

for a_f a constant number depending on f, and fixed ω. We will omit ω in the remainder of this chapter for simplicity of notation. If we change our function from f to f_1, we do not know if we will have convergence for $\mu_t(f_1)$. However, we can find a subsequence t_n of t such that we will have convergence. Let t_n be any subsequence of t such that

$$\mu_{t_n}(f_1) \to a_{f_1}, \quad f_1 \in C_0^\infty,$$

where a_{f_1} is a constant number that depends on f_1. Because this limit exists, we will have that

$$\lim_{t_n\to+\infty} \mu_{t_n}(f_1) = a_{f_1}.$$

However, if we change our function from f_1 to f_2, we do not know if we will have convergence for $\mu_{t_n}(f_2)$. We can find a subsequence t_{n_i} of t_n such that we will have convergence. Let t_{n_i} be any subsequence of t_n such that

$$\mu_{t_{n_i}}(f_2) \to a_{f_2}, \quad a_{f_2} \in C_0^{\infty},$$

where a_{f_2} is a constant number that depends on f_2. Because this limit exists, we will have that

$$\lim_{t_{n_i} \to +\infty} \mu_{t_{n_i}}(f_2) = a_{f_2}.$$

We can continue repeating this argument.

Next, pick $\tilde{t}_1 \in \{t_n\}$, $\tilde{t}_2 \in \{t_{n_i}\}$, $\tilde{t}_3 \in \left\{t_{n_{i_j}}\right\}$, $\ldots$, $\tilde{t}_k \in \left\{ t_{n_{\ddots_k}} \right\}$ such that $\tilde{t}_1 < \tilde{t}_2 < \tilde{t}_3 < \cdots < \tilde{t}_k$ and $\tilde{t}_k \uparrow \infty$. We know by Definition 2.9 that $C_0^{\infty} \subseteq H_0^{\alpha}$ and that H_0^{α} is the completion of C_0^{∞}. Because of this, H_0^{α} contains a countable dense set. In particular, pick $\{f_k\}$ such that it is dense in H_0^{α}. Therefore, $\{\tilde{t}_k\}$ will be a sequence of times such that $\tilde{t}_1 < \tilde{t}_2 < \cdots < \tilde{t}_k \uparrow \infty$, then

$$\mu_{\tilde{t}_k}(f_k) \to a_{f_k}, \quad \forall k. \tag{5.1}$$

A depiction of this selection is shown below:

$$\begin{array}{rccccccccc}
\text{Subsequences:} & \{t_n\} & & \{t_{n_i}\} & & \left\{t_{n_{i_j}}\right\} & & \cdots & & \left\{t_{n_{\ddots_k}}\right\} \\
\text{Element:} & \tilde{t}_1 & < & \tilde{t}_2 & < & \tilde{t}_3 & < & \cdots & < & \tilde{t}_k \\
\text{Function Acted On:} & f_1 & , & f_2 & , & f_3 & , & \cdots & , & f_k
\end{array}$$

The collection of all of these subsequences will be given by $\tilde{t}_k$. □

Remark 5.1. If we take different ω's, we can choose a different subsequence such that this construction holds.

Example 5.1. Let $t_n = \{1, 2, 3, 4, \ldots\}$. We can find a subsequence of this: $\{2, 4, 6, 8, \ldots\} = t_{n_i}$. For this subsequence, there is a limit point: $\mu_{t_{n_i}}(f_2) \to a_{f_2}$. We can then find a further subsequence: $\{4, 8, 12, \ldots\} = t_{n_{i_j}}$ such that it has a limit point: $\mu_{t_{n_{i_j}}}(f_3) \to a_{f_3}$. We can continue this process such that

$$\mu_{t_{n_{\ddots_k}}}(f_k) \to a_{f_k}.$$

We then pick

$$\left.\begin{array}{l}
\tilde{t}_1 = 1 \in \{1, 2, 3, 4, \ldots\} = t_n \\
\tilde{t}_2 = 2 \in \{2, 4, 6, 8, \ldots\} = t_{n_i} \\
\tilde{t}_3 = 4 \in \{4, 8, 12, \ldots\} = t_{n_{i_j}} \\
\qquad\qquad \vdots
\end{array}\right\downarrow \text{further subsequences}$$

such that $\tilde{t}_k \uparrow \infty$.

5.3 Extension to H_0^α

Lemma 5.2. *$\mu_{\tilde{t}_k}(f) \to a_f, \ \forall f \in H_0^\alpha$, i.e. this sequence converges in H_0^α for any f.*

Proof. Suppose $\mu_{\tilde{t}_k}(f) \not\to a_f$ for all $f \in H_0^\alpha$, and fixed ω. Then

$$i_f = \liminf_{\tilde{t}_k \to +\infty} \mu_{\tilde{t}_k}(f) < \limsup_{\tilde{t}_k \to +\infty} \mu_{\tilde{t}_k}(f) = s_f.$$

This means that there is some notion of distance between i_f and s_f, where i and s are constants that depend on f.

Now, because H_0^α contains a countable dense set, then for any $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that $0 < \|f - f_k\|_{H_0^\alpha} < \delta(\epsilon)$ for $f_k \in H_0^\alpha$. $\mu_t(f)$ is a bounded linear functional by Lemma 4.2.

Since this is true, then any subsequence $\tilde{t}_k$ of t will also have this same property. Hence, we know that $\mu_{\tilde{t}_k}(f)$ is a bounded linear functional. We then have

$$\begin{aligned} \left|\mu_{\tilde{t}_k}(f) - \mu_{\tilde{t}_k}(f_k)\right| &= \left|\mu_{\tilde{t}_k}(f - f_k)\right|; \text{ since } \mu_{\tilde{t}_k} \text{ is linear} \\ &\leq \|f - f_k\|_{H_0^\alpha} \cdot \text{ constant} \\ &< \delta(\epsilon) \cdot \text{ constant.} \end{aligned}$$

This means that $\left|\mu_{\tilde{t}_k}(f) - \mu_{\tilde{t}_k}(f_k)\right|$ is also bounded. Hence, we can conclude that

$$\left|\mu_{\tilde{t}_k}(f) - \mu_{\tilde{t}_k}(f_k)\right| < \epsilon.$$

By (Equation 5.1), we know that $\mu_{\tilde{t}_k}(f_k) \to a_{f_k}$ so that

$$\left|\mu_{\tilde{t}_k}(f) - a_{f_k}\right| < \epsilon.$$

As $\epsilon \downarrow 0$, then $\mu_{\tilde{t}_k}(f)$ and a_{f_k} are "close" and approximately the same as $\tilde{t}_k$ goes to infinity. This means that $\mu_{\tilde{t}_k}(f)$ has a limit point. If this is the case, then $\liminf_{\tilde{t}_k \to +\infty} \mu_{\tilde{t}_k}(f)$ and $\limsup_{\tilde{t}_k \to +\infty} \mu_{\tilde{t}_k}(f)$ should also be "close", but because we assumed $\mu_{\tilde{t}_k}(f) \not\to a_f$, this is a contradiction! Therefore, $\mu_{\tilde{t}_k}(f) \to a_f$ for fixed ω and $f \in H_0^\alpha$. □

5.4 Extension to L^2

Lemma 5.3. *There exists a measure μ_∞ such that $\mu_\infty(f) = a_f$.*

Proof. We drop the subscript '∞' on μ_∞ for convenience of notation. Define μ such that $\mu\colon f \to a_f$ is a linear and bounded map. Combining the result from Section 5.3 along with our definition for μ, we have

$$\mu_{\tilde{t}_k}(f) \to \mu(f).$$

Now we must show that $\mu(f)$ is a measure.

In Section 4.2, we showed that $|\mu_t(f)|$ is a bounded linear functional. In particular, $\mu_t(f)$ is a bounded linear functional for $f \in H_0^\alpha$, which is a dense set. A bounded linear functional on a dense set can be extended to L^2 and hence $H_0^\alpha \subseteq L_0^2 \subseteq L^2$ (as in Definition 2.9). If $|\mu_t(f)|$ is a bounded linear functional, than any subsequence of t will also have this same property. Hence $\left|\mu_{\tilde{t}_k}(f)\right|$ is a bounded linear functional. As $\tilde{t}_k \to +\infty$, we can see that

$$\begin{aligned}
\lim_{\tilde{t}_k \to +\infty} \mu_{\tilde{t}_k}(f) &= \lim_{\tilde{t}_k \to +\infty} \int_M f(x)\mu_{\tilde{t}_k}(dx) \\
&= \int_M f(x) \lim_{\tilde{t}_k \to +\infty} \mu_{\tilde{t}_k}(dx) \\
&= \int_M f(x)\mu(dx) \\
&= \mu(f).
\end{aligned}$$

From this, it is easy to see that μ is a measure. In particular, this is the same measure that we found in our characterization. In addition, we know that from Section 4.2 that $\mu(f)$ is a bounded linear functional. This means that we can apply the Riesz Representation Theorem and achieve the same result as in (Equation 4.2) and therefore $\mu(dx) = q(x)m(dx)$, for some density function $q(x)$ so that

$$\mu\left(f\right) = \int_M f(x)q(x)m(dx), \quad \forall f \in L^2.$$

□

Remark 5.2. Because we were able to extend this result to L^2, we could also take $f =$ Indicator(A) as in Appendix A.

CHAPTER 6

PROOF OF THEOREM 3.8

Assume $f : M \to \mathbb{R}$ is bounded and measurable such that $f \in C^{\infty}$. In particular, we let $f \in C_0^{\infty}$. Suppose that $\mu(f)$ is characterized by

$$\left\| G_{\frac{1}{2}}^{-1} q \right\|_{L^2} \leq \sqrt{\frac{2}{m_0}},$$

where $G_{\frac{1}{2}}^{-1}$, q, and m_0 are as previously stated. We require the following definition and theorems from Brosamler (5).

Definition 6.1. (Brosamler Equation 3.5 (5))

For Brownian motion on a compact C^{∞} Riemannian manifold M, define for bounded measurable $f : M \to \mathbb{R}^1$

$$L_t(f, \omega) = \int_0^t f\left(X_s(\omega)\right) ds, \quad t \geqq 0.$$

Definition 6.2. (Brosamler (5))

For $f_1, \ldots, f_n \in L_0^2$, the matrix $\big((f_i, Gf_j)_{L^2}, i, j = 1, \ldots, n \big)$ is nonnegative definite. It is positive definite if and only if $f_1, \ldots, f_n$ are linearly independent. For linearly independent $f_1, \ldots, f_n \in L_0^2$ we define the ellipsoid $E_{f_1,\ldots,f_n}$ by

$$E_{f_1,\ldots,f_n} = \left\{ (\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^n, \quad \sum_{i,j=1}^{n} a_{ij} \zeta_i \zeta_j \leqq 1 \right\},$$

where $\left(\frac{m_0}{2}a_{ij}\right)$ is the inverse matrix of $\left((f_i, Gf_j)_{L^2}, i,j = 1,\ldots,n\right)$.

Theorem 6.1. *(Brosamler Theorem 4.1 (5))*

For all $n \geqq 1$, all linearly independent bounded measurable functions $f_1,\ldots,f_n : M \to \mathbb{R}$ we have for all $x \in M$

$$\mathsf{P}_x\left\{\mathbb{R}^n - \underset{t\to+\infty}{\mathit{cluster\ set}}\,\frac{(L_t(f_1),\ldots,L_t(f_n))}{\sqrt{2t\log\log t}} = E_{f_1,\ldots,f_n}\right\} = 1.$$

The next theorem is the universal law for vector functions.

Theorem 6.2. *(Brosamler Theorem 4.6 (5))*

If $\alpha > \max\left(d - \frac{3}{2}, \frac{d}{2}\right)$, then for all $x \in M$

$$\mathsf{P}_x\Bigg\{\mathbb{R}^n - \underset{t\to+\infty}{\mathit{cluster\ set}}\,\frac{(L_t(f_1),\ldots,L_t(f_n))}{\sqrt{2t\log\log t}} = E_{f_1,\ldots,f_n}$$
$$\mathit{all}\ n \geqq 1,\ \ \mathit{all\ linearly\ independent}\ f_1,\ldots,f_n \in H_0^\alpha\Big\} = 1.$$

From Theorem 6.2, we can see that

$$\begin{aligned}
E_{f_1,\ldots,f_n} &= \underset{t\to+\infty}{\text{cluster set}}\,\frac{(L_t(f_1),\ldots,L_t(f_n))}{\sqrt{2t\log\log t}} \\
&= \underset{t\to+\infty}{\text{cluster set}}\left(\frac{L_t(f_1)}{\sqrt{2t\log\log t}},\ldots,\frac{L_t(f_n)}{\sqrt{2t\log\log t}}\right) \\
&= \underset{t\to+\infty}{\text{cluster set}}\,(\mu_t(f_1),\ldots,\mu_t(f_n))
\end{aligned}$$

almost surely.

Assume that for each fixed t,

$$V_t^n = \frac{(L_t(f_1), L_t(f_2), \ldots, L_t(f_n))}{\sqrt{2t \log\log t}} = (\mu_t(f_1), \ldots, \mu_t(f_n)) \in \mathbb{R}^n,$$

and

$$V^n = (\mu(f_1), \ldots, \mu(f_n)) \,.$$

Suppose $f_1, \ldots, f_n, \ldots$ is a dense and linearly independent set of functions in H_0^α. To prove Theorem 3.8, we need to show several things:

Claim 1. For any dense and linearly independent set of functions $f_1, \ldots, f_n \in H_0^\alpha$ and $t \geq 0$,

$$V^n \in E_{f_1,\ldots,f_n} \overset{a.s.}{=} \underset{t\to+\infty}{\text{cluster set}}\, V_t^n.$$

Claim 2. Assume the above claim is true so that for any V^n we can find a sequence of times t_m^n such that $V_{t_m^n}^n \to V^n$. We can find a sequence of times $\bar{t}_n$ such that $V_{\bar{t}_n}^N \to V^N$ for all N.

Claim 3. We can extend the above so that it works for all functions $f \in H_0^\alpha$ instead of only for a sequence of functions $f_1, f_2, \ldots$ in H_0^α.

6.1 Proof of Claim 1

For any given f,

$$
\begin{aligned}
\mu(f) &= \int_M f(x)\mu(dx) \\
&= \int_M f(x)q(x)m(dx);\ \text{because } q(x) \text{ is the density of } \mu \text{ by assumption} \\
&= \int_M f(x)G_{\frac{1}{2}}G_{\frac{1}{2}}^{-1}q(x)m(dx) \\
&= \left(f, G_{\frac{1}{2}}G_{\frac{1}{2}}^{-1}q\right)_{L^2} \\
&= \left(G_{\frac{1}{2}}f, G_{\frac{1}{2}}^{-1}q\right)_{L^2};\ \text{since } f \in \text{Dom}\left(G_{\frac{1}{2}}\right) \\
&= \int_M G_{\frac{1}{2}}f(x)G_{\frac{1}{2}}^{-1}q(x)m(dx) \\
&\leq \left\|G_{\frac{1}{2}}f\right\|_{L^2}\left\|G_{\frac{1}{2}}^{-1}q\right\|_{L^2};\ \text{Cauchy-Schwarz Inequality} \\
&\leq \sqrt{\frac{2}{m_0}}\left\|G_{\frac{1}{2}}f\right\|_{L^2};\ \text{by the characterization of } \mu(f) \\
&= \sqrt{\frac{2}{m_0}\left(G_{\frac{1}{2}}f, G_{\frac{1}{2}}f\right)_{L^2}} \\
&= \sqrt{\frac{2}{m_0}\left(Gf, f\right)_{L^2}}.
\end{aligned}
$$

Remark 6.1. Brosamler (5) proves Theorem 6.1 for a special case when

$$
E_{f_1,\ldots,f_n} = \left\{(\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^n,\ \ \sum_{i=1}^n \zeta_i^2 \leqq \frac{2}{m_0}\right\}. \tag{6.1}
$$

In addition, he uses the fact that for some f

$$(f_i, Gf_j)_{L^2} = \delta_{ij} = \begin{cases} 1, & i = j \\ \\ 0, & i \neq j, \end{cases} \quad i, j = 1, \ldots n. \tag{6.2}$$

Lemma 6.3. *For $n = 2$, $(\mu(f_1), \mu(f_2)) \in E_{f_1, f_2}$.*

Remark 6.2. To show that Lemma 6.3 is true, we first show Lemma 6.4 holds.

Lemma 6.4. *Let $\mathbf{u} = (a_1, a_2)^\top$ be a unit vector such that $a_1^2 + a_2^2 = 1$, where a_1 and a_2 are constants. Then*

$$\langle V^2, \mathbf{u} \rangle \leq \sqrt{\frac{2}{m_0}}$$

for any projection, where $\langle \cdot, \cdot \rangle$ denotes the inner product.

Proof.

$$\begin{aligned} \langle V^2, \mathbf{u} \rangle &= a_1 \mu(f_1) + a_2 \mu(f_2) \\ &= \mu(a_1 f_1) + \mu(a_2 f_2); \text{ by the definition of a measure} \\ &= \mu(a_1 f_1 + a_2 f_2); \ \mu \text{ is linear} \\ &\leq \sqrt{\frac{2}{m_0} \left(G(a_1 f_1 + a_2 f_2), a_1 f_1 + a_2 f_2 \right)_{L^2}} \\ &= \sqrt{\frac{2}{m_0} \left(a_1 G f_1 + a_2 G f_2, a_1 f_1 + a_2 f_2 \right)_{L^2}} \end{aligned}$$

$$= \sqrt{\frac{2}{m_0}\left[a_1^2\underbrace{(Gf_1,f_1)_{L^2}}_{=\delta_{1,1}=1}+a_2^2\underbrace{(Gf_2,f_2)_{L^2}}_{=\delta_{2,2}=1}+a_1a_2\underbrace{(Gf_1,f_2)_{L^2}}_{=\delta_{1,2}=0}+a_2a_1\underbrace{(Gf_2,f_1)_{L^2}}_{=\delta_{2,1}=0}\right]}$$
$$= \sqrt{\frac{2}{m_0}\left[a_1^2+a_2^2\right]};\ \text{by (Equation 6.2)}$$
$$= \sqrt{\frac{2}{m_0}};\ \text{since we have the restriction that } a_1^2+a_2^2=1.$$

□

Proof of Lemma 6.3. Let a_1 and a_2 be constants. We want to show $V^2 \in E_{f_1,f_2}$. Let $\zeta_1 = \mu(f_1)$ and $\zeta_2 = \mu(f_2)$ from (Equation 6.1). Now, since we have $n = 2$, then E_{f_1,f_2} is the same as considering a ball of radius $\sqrt{2/m_0}$. Therefore, we need to show V^2 is contained in the ball of radius $\sqrt{2/m_0}$. However, this is also the same as showing that V^2 has length $\leq \sqrt{2/m_0}$. Considering the vector V^2, we take the projection of V^2 to the line that has the same angle as V^2; when we do this, we obtain exactly the length of V^2. Let $\mathbf{u} = (a_1, a_2)^\top$ such that $a_1^2 + a_2^2 = 1$; this guarantees that $\mathbf{u}$ is a unit vector. Instead of showing that V^2 has length $\leq \sqrt{2/m_0}$, we can show that

$$\langle V^2, \mathbf{u}\rangle \leq \sqrt{\frac{2}{m_0}},$$

for any projection, where $\langle\cdot,\cdot\rangle$ denotes the inner product. By Lemma 6.4, Lemma 6.3 holds. □

Remark 6.3. The previous proof can be generalized to n dimensions. When we consider the unit vector $\mathbf{u}$, we can take it such that $\mathbf{u} = (a_1, a_2, \ldots, a_n)^\top$ for $\sum_{i=1}^n a_i^2 = 1$. For $(a_1Gf_1 + \cdots + a_nGf_n, a_1f_1 + \cdots + a_nf_n)_{L^2}$, all the cross-terms will be removed because $\delta_{ij} = 0$ for $i \neq j$. Since $(Gf_i, f_i)_{L^2} = \delta_{ii} = 1$, all that remains are the coefficients a_i^2. Therefore, under the square root, we will have $(2/m_0)\sum_{i=1}^n a_i^2 = 2/m_0$, which completes the proof for n-dimensions.

6.2 Proof of Claim 2

By assumption, we know that for each fixed n, $V^n_{t^n_m} \to V^n$ as $m \to +\infty$. Start with $n = 1$. Because $V^1_{t^1_m} \to V^1$, then

$$\left|V^1_{t^1_m} - V^1\right| \to 0, \quad m \to +\infty.$$

In particular, can find $\bar{t}_1 \in \{t^1_m\}$ such that

$$\left|V^1_{\bar{t}_1} - V^1\right| < 1.$$

For $n = 2$, because $V^2_{t^2_m} \to V^2$, then

$$\left|V^2_{t^2_m} - V^2\right| \to 0, \quad m \to +\infty.$$

In particular, we can find $\bar{t}_2 \in \{t^2_m\}$ such that

$$\left|V^2_{\bar{t}_2} - V^2\right| < \frac{1}{2},$$

and

$$\bar{t}_1 < \bar{t}_2.$$

For $n = 3$, because $V^3_{t^3_m} \to V^3$, then

$$\left|V^3_{t^3_m} - V^3\right| \to 0, \quad m \to +\infty.$$

In particular, we can find $\bar{t}_3 \in \{t^3_m\}$ such that

$$\left|V^3_{\bar{t}_3} - V^3\right| < \frac{1}{3},$$

and

$$\bar{t}_1 < \bar{t}_2 < \bar{t}_3.$$

We repeat this process for all n with $V^n_{t^n_m} \to V^n$ and choose $\bar{t}_n \in \{t^n_m\}$ such that

$$\left|V^n_{t^n_m} - V^n\right| < \frac{1}{n}, \tag{6.3}$$

and

$$\bar{t}_1 < \bar{t}_2 < \bar{t}_3 < \cdots < \bar{t}_n.$$

By this construction, we have found a sequence of times $\{\bar{t}_1, \bar{t}_2, \bar{t}_3, \ldots, \bar{t}_n\}$, listed in ascending order.

For fixed N, we want to show that $V_{\bar{t}_n}^N \to V^N$. Now, we know that $V_{\bar{t}_n}^N$ contains the first N components of $V_{\bar{t}_n}^n$ for $n > N$ and V^N contains the first N components of V^n for $n > N$. By our construction,

$$\left|V_{\bar{t}_n}^N - V^N\right| \leq \left|V_{\bar{t}_n}^n - V^n\right| < \frac{1}{n},$$

and hence $V_{\bar{t}_n}^N \to V^N$ as $n \to +\infty$. Therefore, we have found a sequence of times $\bar{t}_n$ such that $V_{\bar{t}_n}^N \to V^N$, for all N.

Example 6.1. $V_{t_m^n}^n$ is a vector such that it converges to another vector V^n. We have

$$\begin{aligned}
V_{t_m^1}^1 &= \begin{pmatrix} z_{t_m^1}^1 \end{pmatrix} \to V^1 = \begin{pmatrix} z^1 \end{pmatrix}; \quad \text{1 dimension} \\
V_{t_m^2}^2 &= \begin{pmatrix} z_{t_m^2}^1 \\ z_{t_m^2}^2 \end{pmatrix} \to V^2 = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}; \quad \text{2 dimensions} \\
V_{t_m^3}^3 &= \begin{pmatrix} z_{t_m^3}^1 \\ z_{t_m^3}^2 \\ z_{t_m^3}^3 \end{pmatrix} \to V^3 = \begin{pmatrix} z^1 \\ z^2 \\ z^3 \end{pmatrix}; \quad \text{3 dimensions} \\
&\quad\ \vdots \qquad\qquad\quad \vdots
\end{aligned}$$

where $z_{t_m^n}$ are constants that depend on t_m^n; the superscript denotes which component of $V_{t_m^n}^n$ is being considered. $z^1, z^2, \ldots$ are constants. In each vector on the right side, z^1 is the same; z^2 is the same, etc.

For $n=1$, we choose $\bar{t}_1$ so that $z^1_{t^1_m}$ is close to z^1. Note that $V^1_{\bar{t}_1}$ is one vector in $V^1_{t^1_m}$. For $n=2$, we choose $\bar{t}_2$ so that $z^1_{t^2_m}$ is close to z^1 and$z^2_{t^2_m}$ is close to z^2 with $\bar{t}_1<\bar{t}_2$. Note that $V^2_{\bar{t}_2}$ is one vector in $V^2_{t^2_m}$. For $n=3$, we choose $\bar{t}_3$ so that $z^1_{t^3_m}$ is close to z^1, $z^2_{t^3_m}$ is close to z^2, and $z^3_{t^3_m}$ is close to z^3 with $\bar{t}_1<\bar{t}_2<\bar{t}_3$. Note that $V^3_{\bar{t}_3}$ is one vector in $V^3_{t^3_m}$.

Now, for example, take $n=3$ and $N=2$ so that $n>N$. Notice that $V^2_{\bar{t}_3}$ contains the first two components of $V^3_{\bar{t}_3}$, and V^2 contains the first two components of V^3. We have $\left|V^2_{\bar{t}_3}-V^2\right| \leq \left|V^3_{\bar{t}_3}-V^3\right| < \frac{1}{3}$.

6.3 Proof of Claim 3

What we have shown with Claims 1 and 2 is that if μ has the specified characterization, then $\mu_t(f_n) \xrightarrow{\text{mildly}} \mu(f_n)$ for a sequence of functions $f_1, f_2, \ldots$ that is dense in H^α_0. We need to show that $\mu_t(f) \xrightarrow{\text{mildly}} \mu(f)$ for any $f \in H^\alpha_0$. Note the following:

Definition 6.3. (Brosamler Equation 3.6 (5))

Define for $\alpha > \frac{d}{2}$ the H^α_0-valued process $L^\alpha(t,\omega)$ by

$$L^\alpha(t,\omega) = 2^{-\alpha} G_{\frac{\alpha}{2}} \left\{ \int_0^t g_{\frac{\alpha}{2}}\left(\cdot, X_s\right) ds \right\} = 2^{-\alpha} \int_0^t g_\alpha\left(x, X_s\right) ds. \tag{6.4}$$

Theorem 6.5. *(Brosamler Theorem 3.8 (5))*

For any compact C^∞ Riemannian manifold M of dimension $d \geqq 1$ and associated Brownian motion X, let the H^α_0-valued process $L^\alpha(t)$ be defined by [(Equation 6.4)*] for $\alpha > \frac{d}{2}$. If $\alpha >$* $\max\left(d-\frac{3}{2}, \frac{d}{2}\right)$*, then for all $x \in M$, P_x-a.a. ω the random set $\left\{ \frac{L^\alpha(t,\omega)}{\sqrt{2t\log\log t}},\ t \geqq 3 \right\}$ in H^α_0 is conditionally norm-compact.*

Lemma 6.6. *(Brosamler (5))*

By Theorem 6.5 we have for all $x \in M$, P_x-a.a. ω

$$C(\omega) = \sup_{t \geqq 3} \frac{\|L^\alpha(t)\|_{H_0^\alpha}}{\sqrt{2t \log\log t}} < \infty.$$

We conclude from $\left|\langle L^\alpha(t), f\rangle_{H_0^\alpha}\right| \leqq \|f\|_{H_0^\alpha} \|L^\alpha(t)\|_{H_0^\alpha}$ *that for all $x \in M$, P_x-a.a. ω*

$$\left| \frac{\int_0^t f(X_s)\, ds}{\sqrt{2t \log\log t}} \right| \leqq \|f\|_{H_0^\alpha} C(\omega), \quad t \geqq 3, \ \ f \in H_0^\alpha. \tag{6.5}$$

Suppose $f \in H_0^\alpha$. By (Equation 6.5), we know that $\mu_t(f_n) \to \mu_t(f)$ uniformly in t if $\|f\|_{H_0^\alpha} C(\omega)$ is "small". Next, we show that $\mu(f_n) \to \mu(f)$.

Similar to the proof of Lemma 5.2, H_0^α contains a countable dense set. Thus for any $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that $0 < \|f - f_n\|_{H_0^\alpha} < \delta(\epsilon)$ for $f_n \in H_0^\alpha$. This means that f_n and f are "close". Recall that if $f_n, f \in H_0^\alpha$, then $f_n, f \in L^2$ since by Definition 2.9, $H_0^\alpha \subseteq L_0^2 \subseteq L^2$. We also know that μ is continuous on H_0^α. Then

$$\begin{aligned}
|\mu(f) - \mu(f_n)| &= |\mu(f - f_n)|; \text{ since } \mu \text{ is linear} \\
&= \left| \int_M (f(x) - f_n(x))\, \mu(dx) \right| \\
&= \left| \int_M (f(x) - f_n(x))\, q(x) m(dx) \right|; \text{ by our characterization} \\
&\leq \int_M |f(x) - f_n(x)| \cdot |q(x)| m(dx); \text{ by Jensen's Inequality}
\end{aligned}$$

$$\leq \|q\|_{L^2} \cdot \|f - f_n\|_{L^2}; \text{ by Hölder's Inequality and } q(x) \in L^2$$

$$< \|q\|_{L^2} \cdot \delta(\epsilon)$$

because f and f_n are close in the L^2-norm since $H_0^\alpha \subseteq L^2$ and $\|f - f_n\|_{H_0^\alpha} < \delta(\epsilon)$. We can conclude that $|\mu(f) - \mu(f_n)| < \epsilon$, and

$$\mu(f_n) \to \mu(f).$$

Hence, we can say that

$$\begin{array}{ccc} \mu_t\left(f_n\right) & \rightarrow & \mu\left(f_n\right) \quad \text{by Question 2} \\ \downarrow & & \downarrow \\ \mu_t\left(f\right) & & \mu\left(f\right). \end{array}$$

Therefore, we must have that $\mu_t(f) \to \mu(f)$ for all $f \in H_0^\alpha$ and we have completed the proof of Theorem 3.8.

APPENDICES

Appendix A

ADDITIONAL PROOFS

It is possible to show that μ_t is a (possibly infinite) signed measure. For an example, we consider the case for $f = \text{Indicator}(A)$ in $\mu_t^\omega(f)$. We show that μ_t is a measure by showing that each part of the numerator of $\mu_t(f)$ are measures.

$\int_0^t 1_A(X_s)ds$ is a measure.

Denote $\int_0^t 1_A(x_s)ds$ by γ.

1. $\gamma(\varnothing) = 0$

   Proof. We can see that

$$\gamma(\varnothing) = \int_0^t 1_\varnothing (X_s)\, ds = \int_0^t 0 \; ds = 0.$$

   $\square$

2. $\gamma(A) \geq 0$ for all $A \in \mathscr{B}(M)$.

   Proof. Assume $1_A(X_s) = 0$. Then $\gamma(A) = \int_0^t 0 \; ds = 0$. Assume $1_A(X_s) = 1$. Then $\gamma(A) = \int_0^t 1 \; ds = t \geq 0$. Therefore, $\gamma(A) \geq 0$. $\square$

Appendix A (Continued)

3. If $A_1, A_2, \ldots, A_n, \ldots \in \mathscr{B}(M)$ and disjoint, then $\gamma\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \gamma(A_n)$.

Proof. Assume $A_1, A_2, \ldots, A_n, \ldots \in \mathscr{B}(M)$ and disjoint. Then

$$\gamma\left(\cup_{n=1}^{\infty} A_n\right) = \int_0^t 1_{\cup_{n=1}^{\infty} A_n}(X_s)\ ds = \int_0^t 1_{A_1 \cup A_2 \cup \cdots \cup A_n \cup \cdots}(X_s)\ ds.$$

As long as $X_s \in A_n$ for some n, then $1_{\cup_{n=1}^{\infty} A_n}(X_s) = 1$. Since the A_n's are disjoint, if X_s is in one of them, it cannot be in another one, so the other indicators are equal to 0. This means that

$$\begin{aligned}
\gamma\left(\cup_{n=1}^{\infty} A_n\right) &= \int_0^t \max\left\{1_{A_1}(X_s), 1_{A_2}(X_s), \ldots, 1_{A_n}(X_s), \ldots\right\} ds \\
&= \int_0^t \left[1_{A_1}(X_s) + \cdots + 1_{A_n}(X_s) + \cdots\right] ds \\
&= \int_0^t \left[\sum_{n=1}^{\infty} 1_{A_n}(X_s)\right] ds \\
&= \sum_{n=1}^{\infty} \int_0^t 1_{A_n}(X_s) ds; \text{ by the Monotone Convergence Theorem} \\
&= \sum_{n=1}^{\infty} \gamma(A_n).
\end{aligned}$$

□

Appendix A (Continued)

$\frac{t}{m_0}\int_M 1_A(x)m(dx)$ is a measure.

Denote $\frac{t}{m_0}\int_M 1_A(x)m(dx)$ by γ.

1. $\gamma(\varnothing) = 0$

   Proof.

$$\gamma(\varnothing) = \frac{t}{m_0}\int_M 1_{\varnothing}(x)m(dx) = \frac{t}{m_0}\int_M 0\ m(dx) = 0.$$

□

2. $\gamma(A) \geq 0$

   Proof. Assume $x \notin A$. Then

$$\gamma(A) = \frac{t}{m_0}\int_M 1_A(x)m(dx) = \frac{t}{m_0}\int_M 0\ m(dx) = 0.$$

Assume $x \in A$. Then

$$\gamma(A) = \frac{t}{m_0}\int_M 1_A(x)m(dx) = \frac{t}{m_0}\int_M 1\ m(dx) = \frac{t}{m_0}\cdot m_0 = t \geq 0.$$

□

Appendix A (Continued)

3. If $A_1, A_2, \ldots, A_n, \ldots \in \mathscr{B}(M)$ and disjoint, then $\gamma((\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \gamma(A_n)$.

Proof. Assume $A_1, A_2, \ldots, A_n, \ldots \in \mathscr{B}(M)$ and disjoint. Then by the same reasoning as in the proof for the first measure,

$$\begin{aligned}
\gamma\left(\cup_{n=1}^{\infty} A_n\right) &= \frac{t}{m_0} \int_M 1_{\cup_{n=1}^{\infty} A_n}(x) m(dx) \\
&= \frac{t}{m_0} \int_M \max\left\{1_{A_1}(x), 1_{A_2}(x), \ldots, 1_{A_n}(x), \ldots\right\} m(dx) \\
&= \frac{t}{m_0} \int_M \sum_{n=1}^{\infty} 1_{A_n}(x) m(dx) \\
&= \sum_{n=1}^{\infty} \frac{t}{m_0} \int_M 1_{A_n}(x) m(dx); \text{ by the Monotone Convergence Theorem} \\
&= \sum_{n=1}^{\infty} \gamma(A_n).
\end{aligned}$$

□

Property: $\mu_t(1_M) = 0$ for fixed t.

Proof.

$$\begin{aligned}
\mu_t(1_M) &= \frac{\int_0^t 1_M(X_s)ds - \frac{t}{m_0}\int_M 1_M(x)m(dx)}{\sqrt{2t\log\log t}} = \frac{\int_0^t 1\ ds - \frac{t}{m_0}\int_M 1\ m(dx)}{\sqrt{2t\log\log t}} \\
&= \frac{t - \frac{t}{m_0} m_0}{\sqrt{2t\log\log t}} = \frac{t-t}{\sqrt{2t\log\log t}} = 0.
\end{aligned}$$

□

Appendix B

MATRIX REPRESENTATION OF A SELF-ADJOINT OPERATOR

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ and have eigenvalues $0 < \lambda_1 \leq \lambda_2, \leq \cdots \leq \lambda_n$. We can write its spectral decomposition as $TDT^\top$, where $D = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. We can also rewrite this:

Remark B.1. (Chen (8))

This representation can be rewritten as

$$A = \sum_{k=1}^{n} \lambda_k \left(E(k) - E(k-1)\right)$$

where $E(0) = 0_{n\times n}$, $E(k) = TD_kT^\top$, where D_k is an $n \times n$ diagonal matrix whose first k elements on the diagonal are 1, and the rest of the elements are 0.

We illustrate a short example.

Example B.1. Take $n = 3$. Then $A_{3\times 3} = TDT^\top$. In particular,

$$\begin{aligned}
A &= \sum_{k=1}^{3} \lambda_k \left(E(k) - E(k-1)\right) \\
&= \lambda_1 \left(E(1) - E(0)\right) + \lambda_2 \left(E(2) - E(1)\right) + \lambda_3 \left(E(3) - E(2)\right) \\
&= \lambda_1 TD_1T^\top + \lambda_2 T \left(D_2 - D_1\right) T^\top + \lambda_3 T \left(D_3 - D_2\right) T^\top \\
&= T \left(\lambda_1 D_1 + \lambda_2(D_2 - D_1) + \lambda_3(D_3 - D_2)\right) T^\top. \qquad \text{(B.1)}
\end{aligned}$$

We have

$$
\begin{aligned}
D_1 &= \operatorname{diag}\{1,0,0\} \\
D_2 - D_1 &= \operatorname{diag}\{1,1,0\} - \operatorname{diag}\{1,0,0\} = \operatorname{diag}\{0,1,0\} \\
D_3 - D_2 &= \operatorname{diag}\{1,1,1\} - \operatorname{diag}\{1,1,0\} = \operatorname{diag}\{0,0,1\}.
\end{aligned}
$$

Then (Equation B.1) will become

$$
\begin{aligned}
\text{(Equation B.1)} &= T\left(\lambda_1 \cdot \operatorname{diag}\{1,0,0\} + \lambda_2 \cdot \operatorname{diag}\{0,1,0\} + \lambda_3 \cdot \operatorname{diag}\{0,0,1\}\right) T^{\top} \\
&= T \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} T^{\top}.
\end{aligned}
$$

Now, for each self-adjoint operator, we can write this as a function of the resolution of identity:

Theorem B.1. *(Chen (8))*

Given a self-adjoint operator A, there is a unique resolution of identity $\{E(\lambda); -\infty < \lambda < \infty\}$, such that

$$
\begin{aligned}
A &= \int_{-\infty}^{\infty} \lambda E\left(d\lambda\right), \qquad \text{(B.2)} \\
\operatorname{Dom}(A) &= \left\{ x \in H; \int_{-\infty}^{\infty} |\lambda|^2 \mu_x\left(d\lambda\right) < \infty \right\}.
\end{aligned}
$$

In fact, this representation is unique:

Theorem B.2. *(Yosida (32))*

A self-adjoint operator H in a Hilbert space X admits a uniquely determined spectral resolution.

We can also write the resolution of identity in terms of an integral or of a finite sum:

Remark B.2. (Chen (8))

When ξ is a step function supported on a finite interval $[a, b]$ and is piece-wisely defined with respect to a partition $a = \lambda_0 < \lambda_1 < \cdots < \lambda_n = b$, we define

$$\int_{-\infty}^{\infty} \xi(\lambda) E(d\lambda) = \sum_{k=1}^{n} c_k \left(E(\lambda_k) - E\left(\lambda_{k-1}\right)\right)$$

where c_k is the value of the function $\xi(\cdot)$ on the k^{th} sub-interval $(\lambda_{k-1}, \lambda_k]$.

In our work above, we can see that not only is the representation is unique, but for $\xi(\lambda) = \lambda$,

$$\begin{aligned} A &= \int_{-\infty}^{\infty} \lambda E(d\lambda) \\ &= \sum_{k=1}^{n} \lambda_k \left(E\left(\lambda_k\right) - E\left(\lambda_{k-1}\right)\right), \end{aligned}$$

where the second equality is similar to the remark above, and

$$= \sum_{k=1}^{n} \lambda_k \left(E(k) - E(k-1)\right),$$

when A is a matrix.

Appendix B (Continued)

In our example above, when A was a matrix, D_k was a diagonal matrix made up of the eigenvalues. In our scenario, the eigenvalues of $-\Delta$ will be the same as for A on L_0^2. Therefore, applying the same process to $-\Delta$, we will have that

$$-\Delta = \sum_{k=1}^{n} \lambda_k \left(E(\lambda_k) - E(\lambda_{k-1})\right) = \int_{-\infty}^{\infty} \lambda E(d\lambda),$$

where the first equality holds for a specific partition. In fact, because we only consider positive eigenvalues for $-\Delta$, the integration will run from 0 to ∞. Because we can represent $-\Delta$ in this form, it must be a self-adjoint operator because the representation of self-adjoint operators is unique by Theorem B.2. We will also have

$$\Delta = \int_{-\infty}^{\infty} \lambda E(d\lambda).$$

Appendix C

R CODE FOR EXAMPLE 1

```
## Set Values

n <- 100000

p <- 0.5

seed <- 16

## Create Blank Matrices

S_matrix <- matrix(data = NA, nrow = 1, ncol = n, byrow = TRUE)

Xbar_matrix <- matrix(data = NA, nrow = 1, ncol = n, byrow = TRUE)

sqrt1 <- matrix(data = NA, nrow = 1, ncol = n, byrow = TRUE)

sqrt2 <- matrix(data = NA, nrow = 1, ncol = n, byrow = TRUE)

## Simulate from a binomial.  When binomial = 0, change to -1.

set.seed(seed)

X_matrix <- t(as.matrix(rbinom(n, size = 1, prob = p)))

for(i in 1:n){ if(X_matrix[,i] == 0){X_matrix[,i] = -1} }

## Generate S_n matrix

for(i in 1:n){ S_matrix[,i] <- sum(X_matrix[,1:i]) }

## Generate S_n / n matrix

for(i in 1:n){ Xbar_matrix[,i] <- S_matrix[,i] / i }
```

Appendix C (Continued)

```
## Generate boundary points

## LIL

for(i in 3:n){ sqrt1[,i] <- sqrt(2 * log(log(i)) / i) }

## Standard Deviation

for(i in 1:n){ sqrt2[,i] <- 0.5 * sqrt(1/i) }


## Plot Graph

plot(x = log(1:n), y = Xbar_matrix, type = "l",

    xlab = "Sample Size (log scale)",

    ylab = "y", ylim = c(-0.5,0.5), xlim = c(exp(1),11))

lines(x = log(1:n), y = sqrt2, col = "blue")

lines(x = log(3:n), y = sqrt1[,3:n], col = "red")

lines(x = log(1:n), y = -sqrt2, col = "blue")

lines(x = log(3:n), y = -sqrt1[,3:n], col = "red")

legend(x = 7, y = 0.4,

    c("S_n / n", "sqrt(2 * log(log n)) / n)", "0.5*sqrt(1/n)"),

    lty = c(1,1,1), col = c("black","red","blue"))
```

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Bachelor of Science in Applied Mathematics and Statistics; Economics **May 2007**

Southeast Missouri State University; Summa Cum Laude, Honors College Graduate

Research Interests

Random walks, stochastic processes, Brownian motion, law of iterated logarithms, and applications to ecology.

Teaching Experience

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Teaching Assistant

- STAT 381 - Applied Statistical Methods I (Spring 2016) – Lecturer
- MATH 170 - Calculus for the Life Sciences (Fall 2015)
- MATH 121 - Precalculus Mathematics (Spring 2015)
- MATH 090 - Intermediate Algebra (Fall 2014)
- STAT 101 - Introduction to Statistics (Fall 2013, Spring 2014)
- MATH 165 - Business Calculus (Spring 2012)

Southeast Missouri State University — August 2007 - May 2009

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- MA095 - Intermediate Algebra (Fall 2007, Spring 2008, Fall 2008, Spring 2009)

Work Experience

Senior Analyst, AON, Chicago — August 2009 - November 2011

Preprints and Publications

1. Pajda-De La O, J., Singh, P., and Scheibe, J.: Persistence of a swamp rabbit metapopulation: the incidence function model approach. International Journal of Ecology and Development, 25(2), 2013.

Conference Presentations and Seminar Talks

7. Colloquium, Roosevelt University — October 30, 2015
6. MAA Indiana Section Meeting, Purdue North Central — October 17, 2015
5. Summer School on Stochastic Analysis and Geometry — August 30, 2014
4. Statistics Seminar, University of Illinois at Chicago — February 29, 2012
3. Catastrophe Analysis, presented at an AON client meeting — September 2010, 2011

   Presented completed book of results to executives of an insurance company
2. MAA Missouri Section Meeting, Truman State University — April 18, 2009
1. Math Club, Southeast Missouri State University — December 2008, February 2009

Memberships/Honor Societies

American Statistical Association — April 2015 - present

Mathematical Association of America — November 2009 - present

Phi Kappa Phi — inducted 2006, lifetime member

Honor Society for all academic disciplines

Activities

Statistics Graduate Student Committee (Chair) — August 2014 - present

Awards

2014-2015 Graduate Student Service Award (Statistics)