Bounding the Castelnuovo-Mumford Regularity of Algebraic Varieties

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THESIS

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To my parents, Jianguang Niu, Yuemei Wang,

my wife, Jin Tan, and

my daughter, Jinwen Niu.

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CHAPTER 1

INTRODUCTION

The purpose of this monograph is to study bounds for Castelnuovo-Mumford regularity of algebraic varieties.

The notion of Castelnuovo-Mumford regularity, or simply regularity, has attracted considerable attention in the past thirty years. It has several equivalent definitions in the context of algebraic geometry or commutative algebra. It was first introduced by Mumford in (1) to study families of curves on surfaces. Since then plenty of research has shown that the notion of regularity plays an important role in many classic areas such as linear systems, syzygies, vanishing theorems, moduli problems, and computational algebraic geometry.

We are interested in the study of regularity of algebraic varieties, in particular regularity bounds. There was a surprising result, obtained by Bertram, Ein and Lazarsfeld in (2), which initiated research on regularity bounds in its current form, and served as the starting point of this monograph. It says that if $X \subset \mathbb{P}^n$ is a nonsingular subvariety of codimension e cut out by hypersurfaces of degrees $d_1 \geq d_2 \cdots \geq d_t$ then its regularity is bounded by $d_1+d_2+\cdots+d_e-e+1$. This bound is reproved by Chardin and Ulrich (3) in the case that X has rational singularities. Recently deFernex and Ein (4) proved it again by assuming the pair $(\mathbb{P}^n, e{\cdot}X)$ is log canonical. In practical computation X is often defined by a homogeneous ideal I which may not be saturated. Thus to get a regularity bound for I will be of considerable practical importance. Fortunately, such a bound was obtained by Chardin and Ulrich in the aforementioned work. Our first main result, Theorem 3.3.2, was inspired by their work to give a computational regularity bound for I by assuming that X is a local complete intersection with log canonical singularities.

The approach leading to the result of Bertram, Ein and Lazarsfeld is to establish a vanishing theorem for the ideal sheaf defining the nonsingular variety concerned. In fact in their work they actually established a vanishing theorem for all powers of the ideal sheaf. Thus it is interesting to generalize this vanishing theorem for powers of an ideal sheaf which defines a singular variety, in particular in the light of the work of deFerenx and Ein as mentioned above. In this direction, we obtained such a result in Theorem 4.2.2 for a local complete intersection with log canonical singularities.

The study of the regularity of powers of an ideal sheaf leads to consideration of the asymptotic behavior of regularity. This interesting behavior was first observed by an approach of commutative algebra under the effort of Swanson (5), Cutkosky, Herzog and Trung (6) and Kodiyalam (7). They showed that the asymptotic regularity of an homogeneous ideal is actually a linear function of powers. On the geometric side Cutkosky, Ein and Lazarsfeld (8) give a very interesting asymptotic formula. Specifically, suppose that $\mathscr{I}$ is an ideal sheaf on a projective space, they show that $\lim\limits_{p\to\infty} \dfrac{\operatorname{reg} \mathscr{I}^p}{p} = s$. The constant s is an invariant to measure the local positivity of the ideal sheaf $\mathscr{I}$. Unfortunately, as opposed to homogeneous ideals, the asymptotic regularity of $\mathscr{I}$ is not a linear function in general. However, in Theorem 5.2.2 we prove that the asymptotic regularity of $\mathscr{I}$ can be bounded by linear functions, which is the best one can hope for.

There is a beautiful conjectured bound for the regularity of a variety. Suppose that X is a nondegenerate subvariety in a projective space. Eisenbud and Goto (9) has conjectured that $\operatorname{reg} X \leq \deg X - \operatorname{codim} X + 1$. This has been proved by Gruson, Lazarsfeld and Peskine (10) for integral curves and by Lazarsfeld (11) for nonsingular surfaces. Somewhat weaker results for nonsingular varieties of dimension up to six also proved by Kwak in (12) and (13). Except for the case of curves, very little is known about this conjecture for singular varieties. Collaborating with Lawrence Ein we are able to prove in Corollary 6.3.7 that the conjecture is true for normal surfaces with rational, Gorenstein elliptic and log canonical singularities.

There is a very interesting Mukai regularity theory, or M-regularity, on polarized abelian varieties developed by Pareschi and Popa in a series of papers (14), (15) and (16). It can be viewed as an analogue to the Castelnuovo-Mumford regularity theory. Many fundamental properties and theorems about the notion of M-regularity have been proved by Pareschi and Popa. Inspired by the work of Gruson, Lazarsfeld and Peskine on curves in projective spaces (10), collaborating with Luigi Lombardi, we consider M-regularity bounds for nonsingular curves in a polarized abelian variety. It is proven in Theorem 7.3.4 that there exist bounds that depend on the positivity of the polarization of the abelian variety.

CHAPTER 2

CASTELNUOVO-MUMFORD REGULARITY

Throughout this monograph, we will always work over the complex number field $k = \mathbb{C}$. A variety is always reduced and irreducible. A scheme is always of finite type over k. We follow the terminologies in (17) on algebraic geometry and the ones in (18) on commutative algebra.

In this chapter we briefly review the basic theory of Castelnuovo-Mumford regularity, most of which is well-known. Since we only need the stated results, we gather them without proofs mainly from (19), (20) and (21).

The following definition of regularity for any coherent sheaf on a projective space was originally introduced by Mumford in (1, Chapter 14).

Definition 2.0.1. A coherent sheaf $\mathscr{F}$ on $\mathbb{P}^n$ is *m-regular* if $H^i(\mathbb{P}^n, \mathscr{F}(m-i)) = 0$, for $i > 0$. The *Castelnuovo-Mumford regularity* $\operatorname{reg}(\mathscr{F})$ of $\mathscr{F}$ is the least integer m for which $\mathscr{F}$ is m-regular.

Remark 2.0.2. If the coherent sheaf $\mathscr{F}$ has support of dimension zero, then $\operatorname{reg}(\mathscr{F})$ will be $-\infty$. Otherwise $\operatorname{reg}\mathscr{F}$ is always a finite number. In particular, for any nonzero sheaf of ideals $\mathscr{I}$ its regularity is always finite.

We can also define the notion of regularity for any finitely generated graded module over a polynomial ring. More background on this can be found in (20).

Definition 2.0.3. Let $S = k[x_0, \cdots, x_n]$ be a polynomial ring over k and M a finitely generated graded S-module. Take a minimal graded free resolution of M as

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

with $F_i = \oplus_j S(-d_{i,j})$. The *Castelnuovo-Mumford regularity* $\operatorname{reg} M$ of M is the supremum of the numbers $d_{i,j} - i$ for all i and j.

The above two notions can be related to each other as follows. Suppose that M is a finitely generated graded module over $S = k[x_0, \cdots, x_n]$. Let $\mathscr{M}$ be the sheafification of M on the projective space $\mathbb{P}^n$. Denote by $\mathfrak{m} = (x_0, \ldots, x_n)$ the maximal ideal of S. Then there is an exact sequence of graded S-modules

$$0 \to H^0_{\mathfrak{m}}(M) \to M \to \oplus_d H^0(\mathbb{P}^n, \mathscr{M}(d)) \to H^1_{\mathfrak{m}}(M) \to 0,$$

and for any $i \geq 2$, $H^i_{\mathfrak{m}}(M) = \oplus_d H^{i-1}(\mathbb{P}^n, \mathscr{M}(d))$. From these, we can easily deduce the following proposition.

Proposition 2.0.4 ((20, Proposition 4.16))**.** *Let M be a finitely generated graded S-module, and let $\mathscr{M}$ be the coherent sheaf on $\mathbb{P}^n$ that it defines. Then the module M is d-regular if and only if*

1. *$\mathscr{M}$ is d-regular;*
2. *$H^0_{\mathfrak{m}}(M)_e = 0$ for every $e > d$; and*
3. *the canonical map $M_d \to H^0(\mathbb{P}^n, \mathscr{M}(d))$ is surjective.*

In particular, $\operatorname{reg} M \geq \operatorname{reg} \mathscr{M}$ always, with equality if $M = \oplus_d H^0(\mathbb{P}^n, \mathscr{M}(d))$.

Remark 2.0.5. Let us focus on a homogeneous ideal I of S. It has a unique saturation I^{sat}. If I defines a subscheme X of $\mathbb{P}^n$, then I^{sat} equals the unique defining homogeneous ideal I_X of X. The sheafificaton of I_X is the ideal sheaf $\mathscr{I}_X$ of X. Now it is easy to see the fundamental relation that $\operatorname{reg}\mathscr{I}_X = \operatorname{reg} I_X = \operatorname{reg} I^{sat} \leq \operatorname{reg} I$.

The following theorem shows the crucial point of the notion of regularity. It was first proved by Mumford in (1, Chapter 14).

Theorem 2.0.6. *Let $\mathscr{F}$ be an m-regular coherent sheaf on $\mathbb{P}^n$. Then for every $k \geq 0$:*

(1). $\mathscr{F}(m+k)$ is generated by its global sections.

(2). The natural maps $H^0(\mathbb{P}^n, \mathscr{F}(m)) \otimes H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(k)) \to H^0(\mathbb{P}^n, \mathscr{F}(m+k))$ are surjective.

(3). $\mathscr{F}$ is $(m+k)$-regular.

The following two useful propositions are very easy to prove by checking the definition directly from the corresponding long exact sequence.

Proposition 2.0.7. *Let $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ be an exact sequence of coherent sheaves on $\mathbb{P}^n$. Then*

1. $\operatorname{reg}\mathscr{F}_2 \leq \max\{\operatorname{reg}\mathscr{F}_1, \operatorname{reg}\mathscr{F}_3\}$*;*

2. $\operatorname{reg}\mathscr{F}_3 \leq \max\{\operatorname{reg}\mathscr{F}_1 - 1, \operatorname{reg}\mathscr{F}_2\}$.

Notice that in general it is very difficult to obtain the regularity of $\mathscr{F}_1$ only from the exact sequence in the proposition. In practice in order to determine $\operatorname{reg}\mathscr{F}_1$, we need more information on the morphism $H^0(\mathbb{P}^n, \mathscr{F}_2(k)) \to H^0(\mathbb{P}^n, \mathscr{F}_3(k))$. However such problem would not appear if we consider finitely generated graded S-modules instead of coherent sheaves on $\mathbb{P}^n$.

Proposition 2.0.8. *Suppose that* $0 \to M_1 \to M_2 \to M_3 \to 0$ *is an exact sequence of finitely generated graded* S*-modules. Then*

1. $\operatorname{reg} M_1 \leq \max\{\operatorname{reg} M_2, \operatorname{reg} M_3 + 1\}$;
2. $\operatorname{reg} M_2 \leq \max\{\operatorname{reg} M_1, \operatorname{reg} M_3\}$;
3. $\operatorname{reg} M_3 \leq \max\{\operatorname{reg} M_1 - 1, \operatorname{reg} M_2\}$.

It is worth mentioning the following theorem deduced from the work of Giusti and Galligo (see (21, Theorem 3.7) for details). It shows a doubly exponential bound for the regularity of an homogeneous ideal. Thus for any subvariety of a projective space there always exists a regularity bound determined by the degrees of its defining equations.

Theorem 2.0.9. *Let* $S = k[x_0, \cdots, x_n]$ *be a polynomial ring over* k *and* $I \subset S$ *be a homogeneous ideal. Let* $d(I)$ *be the maximum of the degrees of a minimal set of generators of* I*. Then* $\operatorname{reg} I \leq (2d(I))^{2^{n-1}}$

We conclude this section by citing the following two results which we shall use repeatedly.

Theorem 2.0.10. *Let* $f : Y \to X$ *be a morphism of schemes and* $\mathscr{F}$ *a coherent sheaf on* Y*. Then there is a spectral sequence* $E_2^{i,j} = H^i(X, R^j f_* \mathscr{F}) \Rightarrow H^{i+j}(Y, \mathscr{F})$.

Proposition 2.0.11 ((19, Lemma 4.3.10))**.** *Let* $f : Y \to X$ *be a morphism of projective varieties and let* A *be an ample line bundle on* X*. Suppose that* $\mathscr{F}$ *is a coherent sheaf on* Y *with the property that* $H^i(Y, \mathscr{F} \otimes f^* A^{\otimes m}) = 0$ *for all* $i > 0$ *and* $m \gg 0$*. Then* $R^i f_* \mathscr{F} = 0$ *for all* $i > 0$.

CHAPTER 3

REGULARITY BOUNDS FOR HOMOGENEOUS IDEALS

In this chapter, we study the regularity bound for a homogeneous ideal by imposing geometric conditions on the variety it defines. We mainly consider singularities of the variety. It turns out that under some mild conditions on singularities, one can expect to have a linear regularity bound in terms of the degrees of defining equations.

Turning to the details, let I be a homogeneous ideal in a polynomial ring $R = k[x_0, \ldots, x_n]$ generated by forms of degree $d_1 \geq d_2 \geq \cdots \geq d_t$. Let $X = \operatorname{Proj} R/I$ be the subscheme defined by I in $\mathbb{P}^n$ of codimension r. Note that in general the number t of defining equations is larger than the codimension r. We denote by I_X the saturation of I, i.e., the defining ideal of X. It is uniquely determined by X and always contains I. Furthermore, I_X is m-regular if and only if the ideal sheaf $\mathscr{I}_X$ of X is m-regular. In this chapter, we consider the regularity bounds for the coordinate ring R/I as a graded R-module, and it is easy to see that $\operatorname{reg} I = \operatorname{reg} R/I + 1$.

The result of Betram, Ein and Lazarsfeld (2) showed that if X is nonsingular then one has a linear bound $\operatorname{reg} R/I_X \leq \displaystyle\sum_{i=1}^{r} d_i - r$ in terms of d_i, (we call it as BEL-bound and we will come back to this result in the next chapter). Note that the sum in the bound is taken from one to the codimension r, i.e., the first r highest degrees of the defining equations. However, since $\operatorname{reg} R/I_X \leq \operatorname{reg} R/I$ this bound could not give any information on the regularity of R/I, which prevents further applications in practice. It was improved in the work of Chardin and Ulrich (3). They first generalized the BEL-bound of R/I_X to the case that X is a local complete intersection with rational singularities. Then they further gave a linear bound of R/I as $\operatorname{reg} R/I \leq \frac{(\dim X + 2)}{2}(\sum_{i=1}^{r} d_i - r)$, (which we call CU-bound). At this point, the story of bounding the regularity seems complete from either the geometric side or the algebraic side. This also strengthen the intuitions that mild singularities could lead to mild regularity bounds.

Recently BEL-bound of R/I_X was reproved by deFernex and Ein (4) in a much more general situation, namely that the pair $(\mathbb{P}^n, rX)$ is log canonical. Log canonical singularity basically arose from the minimal model program. It already includes a large range of singularities, e.g. local complete intersection with rational singularities considered by Chardin and Ulrich. A very interesting question is if one can establish a similar CU-bound for R/I under log canonical singularity. This is the main motivation of this chapter and we prove this bound in Theorem 3.3.2. Our approach is based on the generic linkage method used in the work of Chardin and Ulrich (3). The idea is to construct a generic link Y of X and then pass to the intersection divisor $Z = Y \cap X$ so that we can use induction on the dimension.

3.1 Flat family of log canonical singularities

Log canonical singularities arose from the minimal model program. In this section, we follow the approach in (22) to study this singularity in a flat family.

A normal variety X is said to be $\mathbb{Q}$-Gorenstein if its canonical Weil divisor K_X is $\mathbb{Q}$-Cartier, i.e., mK_X is a Cartier divisor for some integer m. We consider a pair (X, Y), where X is a normal $\mathbb{Q}$-Gorenstein variety and Y is a formal sum $Y = \sum_i q_i \cdot Y_i$ of proper closed subschemes Y_i of X with nonnegative rational coefficients q_i.

Let X' be a nonsingular variety which is proper and birational over X. If E is a prime divisor on X' then E defines a divisor *over* X. The image of E on X is called the *center* of E and denoted by $c_X(E)$.

Given a divisor E over X, we choose a proper birational morphism $\mu : X' \to X$ with X' nonsingular such that E is a divisor on X', and such that all the scheme-theoretic inverse images $\mu^{-1}(Y_i)$ are divisors. The *log discrepancy* $a(E; X, Y)$ is defined such that the coefficient of E in $K_{X'/X} - \sum_i q_i \cdot \mu^{-1}(Y_i)$ is $a(E; X, Y) - 1$. This number is independent of the choice of X'. If we further assume that the divisor $K_{X'/X} - \sum_i q_i \cdot \mu^{-1}(Y_i)$ has a simple normal crossing support then such morphism μ is called a *log resolution* of the pair (X, Y).

Let W be a nonempty closed subset of X. The *minimal log discrepancy* of the pair (X, Y) on W is defined by $\mathrm{mld}(W; X, Y) = \inf_{c_X(E) \subseteq W} \{a(E; X, Y)\}$, where the infimum is computed by considering all log resolution $\mu : X' \to X$ of the pair (X, Y) and all prime divisors E on any such X' with center $\mu(E)$ contained in W. If $\mathrm{mld}(p; X, Y) \geq 0$ for a closed point $p \in X$, we say that the pair (X, Y) is *log canonical* at p. If (X, Y) is log canonical at each closed point of X or equivalently $\mathrm{mld}(X; X, Y) \geq 0$, we then say that the pair (X, Y) is log canonical. If $\mathrm{mld}(X; X, Y) > 0$, we say that the pair (X, Y) is *Kawamata log terminal*, or klt for short. If $Y = 0$, we just write the pair (X, Y) as X.

One important theorem on minimal log discrepancy is Inversion of Adjunction. It is proved for local complete intersection varieties by Ein and Mustaţă.

Inversion of Adjunction ((23, Theorem1.1))**.** Let X be a normal, local complete intersection variety, and $Y = \sum_i q_i \cdot Y_i$, where $q_i \in \mathbb{R}_+$ and $Y_i \subset X$ are proper closed subschemes. If $D \subset X$

is a normal effective Cartier divisor such that $D \not\subseteq \cup_i Y_i$, then for every proper closed subset $W \subset D$, we have

$$\mathrm{mld}(W; X, D+Y) = \mathrm{mld}(W; D, Y|_D).$$

The local complete intersection log canonical singularities behave well in flat families. More specifically, consider a flat family over a local complete intersection log canonical scheme, where all fibers are also local complete intersection log canonical. Then we show that the total space itself is local complete intersection log canonical.

We start with the case where the flat family has a nonsingular base.

Proposition 3.1.1. *Let $f : Y \to X$ be a flat morphism of schemes of finite type over k. Assume that X is nonsingular and each fiber of f is local complete intersection log canonical. Then Y is local complete intersection log canonical.*

Proof. Since X and all fibers are normal and local complete intersections, by flatness of f, we see that Y is normal and a local complete intersection (24, Section 23). Choosing an irreducible component of Y and its image, we can assume that Y is a variety and f is surjective. Since the question is local we can further assume that $X = \operatorname{Spec} A$ is affine. Let $x \in X$ be a closed point defined by a maximal ideal m of A. Then $\mathscr{O}_{X,x}$ is a regular local ring with a maximal ideal $m_x = (t_1, t_2, \ldots, t_n)$ generated by a regular system of parameters $t_1, \cdots, t_n$ where $n = \dim X$. Shrinking X if necessary, we can extend t_i to X and therefore can assume that $m = (t_1, \ldots, t_n) \subset A$ generated by a regular sequence. Set $I_i = (t_1, \ldots, t_i)$. Note that $\mathscr{O}_{X,x}/(t_1, \ldots, t_i)$ is regular. By shrinking X again if necessary, we can further assume that A/I_i

is regular for each $i = 1, \dots, n$. Let $X_i = \operatorname{Spec} A/I_i$ be subschemes of X and consider the following fiber product

$$\begin{array}{ccc} Y_i & \longrightarrow & Y \\ {\scriptstyle f_i}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ X_i & \longrightarrow & X \end{array}$$

By the flatness of f_i and the assumption that each fiber of f_i is a local complete intersection and normal, we see that Y_i is a local complete intersection and normal for each $i = 1, \dots, n$.

Let y be a closed point on the fiber $Y_x = Y_n$. By the flatness of f, $(t_1, \dots, t_n)$ is a regular sequence in $\mathscr{O}_{Y,y}$ and therefore the t_i's define divisors $D_1, \dots, D_n$ around y in Y such that $Y_i = D_1 \cap D_2 \cap \cdots \cap D_i$, for $i = 1, \dots, n$. Now by Inversion of Adjunction, we have

$$\begin{aligned} \operatorname{mld}(y; Y_n) &= \operatorname{mld}(y; Y_{n-1}, D_n|_{Y_{n-1}}) \\ &= \operatorname{mld}(y; Y_{n-2}, D_n|_{Y_{n-2}} + D_{n-1}|_{Y_{n-2}}) \\ &= \dots \\ &= \operatorname{mld}(y; Y, D_1 + \cdots + D_n). \end{aligned}$$

From the assumption that $\operatorname{mld}(y; Y_n) \geq 0$, we get that $\operatorname{mld}(y; Y) \geq 0$, i.e. Y is log canonical at y, which proves the proposition. $\square$

Next let us consider the general case in which the flat family has a singular base. We first resolve the singularities of the base, and then base change to a new flat family over a nonsingular base. However, in this procedure, some extra divisors could be introduced on the new flat family. This means that we need to consider singularities of pairs on the new flat family.

Theorem 3.1.2. *Let $f : Y \to X$ be a flat morphism of schemes of finite type over k. Assume that X and all fibers of f are local complete intersection log canonical. Then Y is local complete intersection log canonical.*

Proof. As in the proof of Proposition 3.1.1, we may assume that X and Y are varieties and Y is normal and a local complete intersection. All we need to show is that Y is log canonical. Take a log resolution of X, $\mu : \widetilde{X} \to X$, and construct the fiber product $\widetilde{Y} = Y \times_X \widetilde{X}$:

$$\begin{array}{ccc} \widetilde{Y} & \longrightarrow & Y \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \widetilde{X} & \xrightarrow{\mu} & X \end{array}$$

By Proposition 3.1.1, $\widetilde{Y}$ is local complete intersection log canonical. Since X is log canonical, we can write the relative canonical divisor $K_{\widetilde{X}/X} = P - N$, where P and N are effective divisors supported in the exceptional locus of μ, so that $N = \sum E_i$ where E_i are prime divisors with simple normal crossings. By base change for relative canonical diviosrs, we have $K_{\widetilde{Y}/Y} = g^* K_{\widetilde{X}/X}$ and therefore $K_{\widetilde{Y}/Y} = g^*(P) - g^*(N)$.

Denoting by F_j's the distinct irreducible components of those $g^*(E_i)$ (note that $g^*(E_i) = g^{-1}(E_i)$ as scheme-theoretical inverse image of E_i), we have $g^*(N) = \sum F_j$. This will be shown in detail at the beginning of the proof of Lemma 3.1.3 below.

Now we let $\pi : Y' \to \widetilde{Y}$ be a log resolution of $\widetilde{Y}$ such that

$$\begin{aligned} K_{Y'/Y} &= K_{Y'/\widetilde{Y}} + \pi^* K_{\widetilde{Y}/Y} \\ &= A - B + \pi^* g^* P - \sum \pi^* F_i \end{aligned}$$

where A is the positive part of $K_{Y'/\widetilde{Y}}$ and B is the negative part of $K_{Y'/\widetilde{Y}}$ and all prime divisors in the above formula are simple normal crossings. In order to show Y is log canonical, it is enough to show that the coefficient of each prime divisor in $B + \sum \pi^* F_i$ is 1. This is equivalent to showing that the pair $(\widetilde{Y}, g^{-1}N)$ is log canonical, which is shown in the following Lemma 3.1.3. □

Lemma 3.1.3. *Let $f : Y \to X$ be a flat morphism of varieties such that X is nonsingular and each fiber of f is local complete intersection log canonical. Assume that $E_1, \ldots, E_r$ are prime divisors on X with simple normal crossings. Then the pair $(Y, \sum_{i=1}^r f^{-1}(E_i))$ is log canonical, where f^{-1} means scheme-theoretical inverse image.*

Proof. From Proposition 3.1.1, Y is local complete intersection log canonical. Also for each divisor E_i, the scheme-theoretical inverse $f^{-1}(E_i)$ is local complete intersection log canonical. This implies that

$$\sum_{i=1}^{r} f^{-1}(E_i) = \sum_{j=1}^{s} F_j$$

where F_j are distinct irreducible components of the subschemes $f^{-1}(E_i)$. Note that since f is flat, each F_j only appears in one $f^{-1}(E_i)$, and if some F_j's are in the same $f^{-1}(E_i)$ then they are disjoint. Furthermore each F_j is a Cartier normal divisor on Y with local complete intersection log canonical singularities. We need to show the pair $(Y, \sum F_j)$ is log canonical.

We prove this by induction on the dimension of X. First assume that $\dim X = 1$. Then $E_1, \ldots, E_r$ are distinct points and $F_1, \ldots, F_s$ are pairwise disjoint. It is enough to show that for each j, $\mathrm{mld}(F_j; Y, F_j) \geq 0$. Choosing a closed point $p \in F_j$ of Y, by Inversion of Adjunction and the fact that F_j has log canonical singularities, we have $\mathrm{mld}(p; Y, F_j) = \mathrm{mld}(p; F_j) \geq 0$.

Assume X has any dimension. Since Y is log canonical, it is enough to show that for each j, $\mathrm{mld}(F_j; Y, \sum_{t=1}^{s} F_t) \geq 0$. Without loss of generality, we prove this for F_1 and assume that $F_1 \subseteq f^{-1}(E_1)$. Choosing any closed point $p \in F_1$ of Y, by Inversion of Adjunction, we have

$$\mathrm{mld}(p; Y, F_1 + \sum_{t=2}^{s} F_t) = \mathrm{mld}(p; F_1, \sum_{t=2}^{s} F_t|_{F_1}).$$

For $i = 2, \ldots, r$, we set $D_i = E_1 \cap E_i$ and note that $\sum_{t=2}^{s} F_t|_{F_1} = \sum_{i=2}^{r} f^{-1}(D_i)$, where f^{-1} means scheme-theoretical inverse image. Now we are in the situation $f : F_1 \to E_1$, where E_1 is nonsingular and $D_2, \ldots, D_r$ are divisors on E_1 with simple normal crossings. Then applying induction on F_1, we get that the pair $(F_1, \sum_{t=2}^{s} F_t|_{F_1})$ is log canonical and therefore $\mathrm{mld}(p; F_1, \sum_{t=2}^{s} F_t|_{F_1}) \geq 0$ which proves the lemma. $\square$

If $f : Y \to X$ is a surjective smooth morphism, then we can move singularities freely from Y to X. Using the notion of jet schemes, we have a quick proof for this as follows.

Given any scheme X, we can associate the m-th jet scheme X_m for any positive integer m. The properties of jet schemes are closely related to the singularities of X. We may use jet schemes to describe local complete intersection log canonical singularities. The work of Ein and Mustaţă shows that if X is a normal local complete intersection variety, then X has log canonical singularities if and only if X_m is equidimensional for every m. For more information on jet schemes and their application to singularities, we refer the reader to (22).

Proposition 3.1.4. *Let $f : Y \to X$ be a smooth surjective morphism of schemes of finite type over k. Then X is local complete intersection log canonical if and only if Y is local complete intersection log canonical.*

Proof. First note that since f is smooth, we have X is normal and a local complete intersection if and only if Y is normal and a local complete intersection. Since f is smooth and surjective, for every m we have an induced morphism between m-jet schemes $f_m : Y_m \to X_m$, which is smooth and surjective (22, Remark 2.10). Then Y_m is equidimensional if and only if X_m is equidimensional. Now by (23, Theorem 1.3), we get the proposition. □

Remark 3.1.5. In the proof, if f is smooth but not surjective, we can only get $f_m : Y_m \to X_m$ is smooth. Then equidimensionality of X_m will imply that Y_m is equidimensional. This means that for a smooth morphism $f : Y \to X$, if X is local complete intersection log canonical then Y is also local complete intersection log canonical singularities. This is a quick proof for a special case of Theorem 3.1.2.

3.2 Generic linkages

In this section, we study the log canonical singularities in a generic linkage. This could be compared to the work (3) on studying rational singularities in a generic linkage. The s-generic residual intersection theory can be found in (25). Throughout this section, all rings are assumed to be Noetherian k-algebras and a point on a scheme means a point locally defined by a prime ideal, not necessarily maximal. All fiber products are over the field k unless otherwise stated.

Let $S = \operatorname{Spec} R$ be an affine scheme and $X \subset S$ be a codimension r subscheme defined by an ideal $I = (z_1, \ldots, z_t)$. For an integer $s \geq 0$, let $M = (U_{ij})_{t \times s}$ be a $t \times s$ matrix of variables and $R' = R[U_{ij}]$ be the polynomial ring over R obtained by adjoining the variables of M. Define $S' = S \times \mathbb{A}^{t \times s} = \operatorname{Spec} R'$, which has a natural flat projection $\pi : S' \to S$. Let $X' = \pi^{-1}(X)$ be defined by the ideal IR'. Construct an ideal α in R' generated by $\alpha_1, \ldots, \alpha_s$ as follows:

$$\alpha = (\alpha_1, \ldots, \alpha_s) = (z_1, \ldots, z_t) \cdot M$$

and set $J = [\alpha : IR']$. The subscheme Y' of S' defined by J is called an *s-generic residual intersection* of X.

We define Z to be the scheme-theoretical intersection of X' and Y'. Its defining ideal is $I_Z = J + IR'$. We equip Z with a restricted projection morphism $\pi : Z \to X$ and call Z an *intersection divisor* of an s-generic residual intersection of X.

Note that if $s < r$, then α is generated by a regular sequence and therefore $J = \alpha$, $Z = X'$. The interesting case is when $s \geq r$. In particular, when $s = r$, Y' is called a *generic linkage* of X. Correspondingly, we call Z an *intersection divisor* of a generic linkage of X.

Under the assumption that X is a local complete intersection, the morphism $\pi : Z \to X$, and in particular its fibers, can be understood very well. This offers us an opportunity to pass singularities from X to Z.

We start with a lemma which describes the fibers of π when X is a complete intersection.

Lemma 3.2.1. *Let $S = \operatorname{Spec} R$ be a Gorenstein integral affine scheme and X be a complete intersection subscheme defined by a regular sequence $I = (z_1, \ldots, z_r)$ in R. For $s \geq 0$, let $M = (U_{ij})_{r \times s}$, $R' = R[U_{ij}]$, $\alpha = (\alpha_1, \ldots, \alpha_s) = (z_1, \ldots, z_r) \cdot M$ and $J = [\alpha : IR']$. Assume that Z is defined by $J + IR'$ and consider the natural morphism $\pi : Z \to X$. We have*

(1) If $s < r$, then $Z \cong X \times \mathbb{A}^{r \times s}$ and π is the projection to X.

(2) If $s \geq r$, then $\pi : Z \to X$ is a flat morphism and for any point $p \in X$,

$$\pi^{-1}(p) \cong k(p)[U_{ij}]/I_r(M)$$

where $I_r(M)$ is the $r \times r$ minors ideal of M.

(3) In particular, if $s = r$, then $\pi : Z \to X$ is a flat morphism such that each fiber is a local complete intersection with rational singularities.

Proof. (1) is trivial because in this case $J = \alpha$ and Z is defined by IR' so that $Z = \pi^{-1}(X) \cong X \times \mathbb{A}^{r \times s}$.

For (2) and (3), let $\mathfrak{q} \in X \subset S$ be a point and passing to the local ring $R_{\mathfrak{q}}$, we can assume R is local. By (25, Example 3.4), $J = (\alpha_1, \ldots, \alpha_s, I_r(M))$ and then Z is defined by $I_Z = J + IR' = (I, I_r(M))$.

Notice that $R[U_{ij}]/(I, I_r(M)) = R/I \otimes_R R[U_{ij}]/I_r(M)$. This means that the morphism $\pi : Z \to X$ can be constructed from the fiber product

$$\begin{array}{ccc} Z & \longrightarrow & \operatorname{Spec} R[U_{ij}]/I_r(M) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\theta} \\ X & \longrightarrow & S = \operatorname{Spec} R \end{array}$$

Since θ is flat, we obtain π is flat. The fiber of π at $p \in X$ is

$$\begin{aligned} F &= k(p) \otimes_{R/I} R[U_{ij}]/(I, I_r(M)) \\ &= k(p)[U_{ij}]/I_r(M). \end{aligned}$$

In particular, if $s = r$, we see that F is a local complete intersection with rational singularities.

□

Now we turn to the case where X is a local complete intersection.

Proposition 3.2.2. *Let $S = \operatorname{Spec} R$ be a Gorenstein integral affine scheme and X be a subscheme defined by an ideal $I = (z_1, \ldots, z_t)$ in R. For $s \geq 0$, let $M = (U_{ij})_{t \times s}$, $R' = R[U_{ij}]$, $\alpha = (\alpha_1, \ldots, \alpha_s) = (z_1, \ldots, z_t) \cdot M$, and $J = [\alpha : IR']$. Let Z be defined by $J + IR'$ and consider the natural morphism $\pi : Z \to X$. Let $\mathfrak{p} \in X$ be a point of S and assume that $I_\mathfrak{p}$ is generated by a regular sequence of length r. Then there is an affine neighborhood of $\mathfrak{p}$ over which π can be factored as follows*

$$\begin{tikzcd} Z \arrow[d, "\pi"] \arrow[dr, "\pi'"] & \\ X & P \arrow[l, "g"] \end{tikzcd}$$

such that $P = X \times \mathbb{A}^{(t-r) \times s}$ with g the projection to X and Z can be viewed as an intersection divisor of an s-generic residual intersection of P.

Note that the above diagram is local. More precisely, there is an affine neighborhood U of $\mathfrak{p}$ and the morphism $\pi : Z \to X$ in the above diagram really means the restriction of π over U, i.e. $\pi : \pi^{-1}(U) \cap Z \to U \cap X$.

Proof. By assumption, we may replace S by an affine neighborhood of $\mathfrak{p}$ such that I is generated by a regular sequence, say $z_1, \ldots, z_r$. Then

$$\begin{cases} z_{r+1} &= a_{1,r+1}z_1 + a_{2,r+1}z_2 + \ldots + a_{r,r+1}z_r \\ z_{r+2} &= a_{1,r+2}z_1 + a_{2,r+2}z_2 + \ldots + a_{r,r+2}z_r \\ & \ldots \\ z_t &= a_{1,t}z_1 + a_{2,t}z_2 + \ldots + a_{r,t}z_r \end{cases} \tag{3.1}$$

where $a_{ij} \in R$. Set $A = (a_{ij})_{r\times(t-r)}$. We can write $(z_{r+1}, \ldots, z_t) = (z_1, \ldots, z_r) \cdot A$. Denote $M = \begin{pmatrix} C \\ B \end{pmatrix}$, where

$$C = \begin{pmatrix} U_{11} & U_{12} & \cdots & U_{1s} \\ U_{21} & U_{22} & \cdots & U_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ U_{r1} & U_{r2} & \cdots & U_{rs} \end{pmatrix}, \quad B = \begin{pmatrix} U_{r+1,1} & U_{r+1,2} & \cdots & U_{r+1,s} \\ U_{r+2,1} & U_{r+2,2} & \cdots & U_{r+2,s} \\ \cdots & \cdots & \cdots & \cdots \\ U_{t1} & U_{t2} & \cdots & U_{ts} \end{pmatrix}.$$

Using the equations in (3.1), we can rewrite $(\alpha_1, \ldots, \alpha_s) = (z_1, \ldots, z_t) \cdot M$ as

$$(\alpha_1, \ldots, \alpha_s) = (z_1, \ldots, z_r) \cdot (A \cdot B + C).$$

Set $N = (V_{lm})_{r\times s} = A \cdot B + C$. Then the ring extension of R to R' can be obtained by extending twice: $R \to R_1 = R[U_{ij}|i > r] \to R' = R_1[V_{pq}]$. The first extension $R \to R_1$ gives the morphism $g : \operatorname{Spec} R_1 = S \times \mathbb{A}^{(t-r)\times s} \to S$. Let $P = g^{-1}(X) = X \times \mathbb{A}^{(t-r)\times s}$ defined by IR_1 which is the complete intersection generated by the regular sequence $(z_1, \ldots, z_r)$ in R_1. Restricting g to P, we get a projection $g : P \to X$. In the second extension, $R_1 \to R'$, we see that Z can be viewed as an intersection divisor of an s-generic residual intersection of P with morphism $\pi' : Z \to P$. $\square$

Since Z is a generic intersection divisor of X, the fibers of the morphism $\pi : Z \to X$ are local complete intersections with rational singularities and they are log canonical. So the morphism $\pi : Z \to X$ provides us a flat family of log canonical singularities, to which results of the previous section can be applied.

Proposition 3.2.3. *Let $S = \operatorname{Spec} R$ be a regular affine scheme and X be a subscheme defined by an ideal $I = (z_1, \ldots, z_t)$ with codimension r in S. Construct a generic linkage J of I as follows: let $M = (U_{ij})_{t\times r}$, $R' = R[U_{ij}]$, $\alpha = (\alpha_1, \ldots, \alpha_r) = (z_1, \ldots, z_t) \cdot M$, and $J = [\alpha : IR']$. Let Z be a subscheme of $\operatorname{Spec} R'$ defined by the ideal $J + IR'$ and consider the natural morphism $\pi : Z \to X$. If X is local complete intersection log canonical, then Z is also local complete intersection log canonical.*

Proof. Choose any point $\mathfrak{p} \in X$. By the assumption, $I_{\mathfrak{p}}$ is generated by a regular sequence with length $l \geq r$. By Proposition 3.2.2, there is an affine neighborhood of $\mathfrak{p}$, over which we can factor $\pi : Z \to X$ as follows

$$\begin{tikzcd} Z \arrow[d, "\pi"] \arrow[dr, "\pi'"] & \\ X & P \arrow[l, "g"] \end{tikzcd}$$

such that $P \cong X \times \mathbb{A}^{(t-l)\times r}$, which is defined by a regular sequence of length l in $S \times \mathbb{A}^{(t-l)\times r}$, and Z is an intersection divisor of a r-generic residual intersection of P. There are two possibilities as follows.

If $l = r$, then by Lemma 3.2.1 (3), $\pi' : Z \to P$ is a flat morphism whose fibers are locally complete intersection log canonical. Now by Proposition 3.1.4 and Theorem 3.1.2 we obtain that Z is local complete intersection log canonical.

If $l > r$, then by Lemma 3.2.1 (1), $Z \cong P \times \mathbb{A}^{l\times r}$. Using Proposition 3.1.4, we have that Z is local complete intersection log canonical.

□

We have passed the singularities from X to Z in above proposition. As we mentioned in the Introduction, we need to understand the generators of Z. Since Z is defined by $J + IR'$, basically, we need to know the generators of the generic linkage J. The method we will use here is quite standard in (3) and we shall be brief.

Lemma 3.2.4. *Let $X \subset \mathbb{P}^n$ be a equidimensional Gorenstein subscheme with log canonical singularities. Then* $\operatorname{reg} \omega_X = \dim X + 1$, *where ω_X is the canonical sheaf of X.*

Proof. By assumption, ω_X is a direct sum of the canonical sheaves of each irreducible component of X. We may assume that X is irreducible. Since X is log canonical, Kodaira vanishing holds for X (26, Corollary 1.3), i.e., $H^i(X, \omega_X(k)) = 0$, for all $k > 0$ and $i > 0$. Note that $H^{\dim X}(X, \omega_X) \neq 0$. Then we see $\operatorname{reg} \omega_X = \dim X + 1$. □

Proposition 3.2.5. *Let $X \subset \mathbb{P}^n$ be a equidimensional Gorenstein subscheme with log canonical singularities and codimension r. Assume that $Y \subset \mathbb{P}^n$ is direct linked with X by forms of degrees $d_1, \ldots, d_r$. Denote by J the defining ideal of Y and write $\sigma = \sum_{i=1}^{r}(d_i - 1)$. Then $J = (J)_{\leq \sigma}$.*

Proof. Let $I \subset R = k[x_0, \ldots, x_n]$ be the defining ideal of X and $d = \dim R/I$. Let $b = I \cap J$ be generated by forms in degrees $d_1, \ldots, d_r$ and ω be the canonical module of R/I. If $d = 2$, i.e., X is a nonsingular curve, then $(\omega)_{\leq d} = \omega$ by (3, Proposition 1.1). If $d > 2$, i.e., $\dim X > 1$, then $\operatorname{reg} \omega = \operatorname{reg} \omega_X = d$ by Lemma 3.2.4 and therefore we have $(\omega)_{\leq d} = \omega$.

Observe that $J/b = \operatorname{Hom}_R(R/I, R/b) = \operatorname{Ext}^r_R(R/I, R)[-d_1 - \cdots - d_r] = \omega[d - \sigma]$. Hence $(J/b)_{\leq \sigma} = (\omega[d-\sigma])_{\leq \sigma} = (\omega)_{\leq d}[d - \sigma] = \omega[d - \sigma]$. From the diagram

$$\begin{tikzcd}
0 \arrow[r] & (b)_{\leq \sigma} \arrow[r] \arrow[d, equal] & (J)_{\leq \sigma} \arrow[r] \arrow[d] & (J/b)_{\leq \sigma} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & b \arrow[r] & J \arrow[r] & J/b \arrow[r] & 0,
\end{tikzcd}$$

we see $(J)_{\leq \sigma} = J$. □

3.3 Proof of the main theorem

Applying the results we have established, we are able to give a bound for the Castelnuovo-Mumford regularity of a homogenous ideal which defines a local complete intersection log canonical scheme. This partially generalizes the work of Chardin and Ulrich (3) and gives a new geometric condition under which a reasonable bound can be obtained. For the convenience of the reader, we follow the construction from (3) and keep the same notations.

Proposition 3.3.1. *Let $R = k[x_0, \ldots, x_n]$ and $I \subset R$ be a homogeneous ideal of codimension r generated by forms $f_1, \ldots, f_t$ of degrees $d_1 \geq d_2 \geq \cdots \geq d_t \geq 1$. Let*

$$a_{ij} = \sum_{|\mu| = d_j - d_i} U_{ij\mu} x^{\mu}, \qquad \text{for } r+1 \leq i \leq t,\ 1 \leq j \leq r,$$

where $U_{ij\mu}$ are variables. Denote $A = (a_{ij})$, $K = k(U_{ij\mu})$, $R' = R \otimes_k K$ and define

$$(\alpha_1, \ldots, \alpha_r) = (f_1, \ldots, f_t) \begin{pmatrix} I_{r \times r} \\ A \end{pmatrix},$$

$J = [(\alpha_1, \ldots, \alpha_r)R' : IR']$. Assume that $X = \operatorname{Proj} R/I$ is local complete intersection log canonical. Then $Z = \operatorname{Proj} R'/IR' + J$ is local complete intersection log canonical.

Proof. Reduce the question to standard affine covers of $\mathbb{P}^n_k$. Without loss of generality, we focus on one affine cover $U = \operatorname{Spec} R_{(x_0)}$, where $R_{(x_0)}$ means the degree zero part of the homogeneous localization of R with respect to x_0, which is canonically isomorphic to $k[x_1/x_0, \ldots, x_n/x_0]$. Set $V = \pi^{-1}(U)$, where π is the natural morphism $\pi : \mathbb{P}^n_K \to \mathbb{P}^n_k$. Note that $V = \operatorname{Spec} R'_{(x_0)}$. For simplicity, we reset our notations as follows. Replace R by $R_{(x_0)}$, R' by $R'_{(x_0)}$, f_i by $f_i/x_0^{d_i}$, and I by $I_{(x_0)}$. Then on the affine open set U, X is generated by $I = (f_1, \ldots, f_t)$ in R. We redefine elements of the matrix A by setting $a_{ij} = \sum U_{ij\mu} x^{\mu}/x_0^{|\mu|}$. We can see that on V, Z is defined by the ideal $J + IR'$, where $J = [\alpha : IR']$ and $\alpha = (\alpha_1, \ldots, \alpha_r)$ is defined by the equations in the assumption. Note that a_{ij}'s now become variables over R and therefore A is a matrix of variables over R. We restrict to this affine case in the following proof.

Consider ring extensions $R[a_{ij}] \to R[U_{ij\mu}] \to R' = R \otimes_k K$. The first one is given by adjoining variables. The second one is the localization of $R[U_{ij\mu}]$ by the multiplicative set $W = k[U_{ij\mu}] \setminus \{0\}$. They give morphisms ϕ_1 and ϕ_2 respectively:

$$\operatorname{Spec} R' \xrightarrow{\phi_2} \operatorname{Spec} R[U_{ij\mu}] \xrightarrow{\phi_1} \operatorname{Spec} R[a_{ij}].$$

In $R[a_{ij}]$, set $J_1 = [\alpha : IR[a_{ij}]]$ and define a subscheme $Z_1 = \operatorname{Spec} R[a_{ij}]/(J_1 + IR[a_{ij}])$, so that $Z = (\phi_0 \circ \phi_1)^{-1}(Z_1)$. To show Z has the desired singularities, we just need to show Z_1 has the desired singularities. This is because ϕ_1 is smooth and it passes singularities from Z_1 to $\phi_1^{-1}(Z_1)$ by Proposition 3.1.4. Our singularities are preserved by localization and so ϕ_2 will continue passing singularities from $\phi_1^{-1}(Z_1)$ to Z. Hence all we need is to prove the proposition for Z_1 in $\operatorname{Spec} R[a_{ij}]$.

To this end, we introduce a new matrix of variables $B = (b_{lm})_{r\times r}$ and set

$$C = \begin{pmatrix} B \\ A \cdot B \end{pmatrix} = (c_{uv})_{t\times s},$$

which is also a matrix of variables over R. In the ring $R[c_{uv}]$, we construct an intersection divisor Z' of X as follows: let $\alpha' = (\alpha'_1, \dots, \alpha'_r) = (f_1, \dots, f_t) \cdot C$, $J' = (\alpha' : IR[c_{uv}])$ and define $Z' = \operatorname{Spec} R[c_{uv}]/(J' + IR[c_{uv}])$. Then consider the diagram

$$\begin{array}{ccc} \operatorname{Spec} R[a_{ij}] & \xleftarrow{q} & \operatorname{Spec} R[a_{ij}] \otimes_k k(b_{lm}) \\ \downarrow & & \downarrow p \\ \operatorname{Spec} R & \longleftarrow & \operatorname{Spec} R[c_{uv}] \end{array}$$

where q is induced by the base field extension $R[a_{ij}] \to R[a_{ij}] \otimes_k k(b_{lm})$, and p is induced by $R[c_{uv}] \to R[a_{ij}] \otimes_k k(b_{lm})$, which is the localization of $R[c_{uv}]$ with respect to the multiplicative set $k[b_{lm}] \setminus \{0\}$. We note that $p^{-1}(Z') = q^{-1}(Z_1) = Z_1 \otimes_k k(b_{lm})$. By Proposition 3.2.3, Z' is local complete intersection log canonical. Since p is induced by localization, we obtain that $p^{-1}(Z')$ is also local complete intersection log canonical. Finally because q is the base field change of Z_1 from k to $k(b_{lm})$, it is easy to see that Z_1 is local complete intersection log canonical if and only if $q^{-1}(Z_1) = Z_1 \otimes_k k(b_{lm})$ is local complete intersection log canonical. This proves the proposition. □

Theorem 3.3.2. *Let $R = k[x_0, \ldots, x_n]$ and $I = (f_1, \ldots, f_t)$ be a homogeneous ideal, not a complete intersection, generated in degrees $d_1 \geq d_2 \geq \cdots \geq d_t \geq 1$ of codimension r. Assume that $X = \operatorname{Proj} R/I$ is local complete intersection log canonical and $\dim X \geq 1$. Then*

$$\operatorname{reg} R/I \leq \frac{(\dim X + 2)!}{2}\left(\sum_{i=1}^{r} d_i - r - 1\right),$$

unless $R = k[x_0, x_1, x_2]$ and $I = lH$ with l a linear form and H a complete intersection of 3 forms of degree $d_1 - 1$, in which case $\operatorname{reg} R/I = 3d_1 - 5$.

Proof. We construct R', $\alpha = (\alpha_1, \ldots, \alpha_r)$, J and Z as in Proposition 3.3.1 and write $\sigma = \sum_{i=1}^{r}(d_i - 1)$ and $d = \dim R/I$.

By the assumption that I is not a complete intersection, we may assume that $d_2 \geq 2$. Also we note that if $\sigma = 1$, then $\operatorname{ht} I = 1$ and there is a linear form l and a homogeneous ideal H such that $f_i = lh_i$ and $H = (h_1, \ldots, h_t)$, where h_i are all linear forms, so we get $\operatorname{reg} R/I = \operatorname{reg} R/(l) + \operatorname{reg} R/H = 0$. Then we may assume in the following proof that $\sigma \geq 2$.

We consider the codimension r in two cases.

Case of $r \geq 2$. We proceed by induction on d. For $d = 2$, we have $n \geq 3$. Applying (3, Proposition 2.2), we have $\operatorname{reg} R/I \leq \dfrac{(\dim X + 2)!}{2}(\sigma - 1)$.

Assume that $d \geq 3$. Let $X' = \operatorname{Proj} R'/IR'$ which is local complete intersection log canonical. Let $(IR')^{top}$ be the unmixed part of IR'; it defines an equidimensional subscheme X'^{top} which is local complete intersection log canonical and J is directly linked with $(IR')^{top}$ by α. By Proposition 3.2.5, $J = (J)_\sigma$. Set $Z' = \operatorname{Proj} R'/(IR')^{top} + (J)_\sigma$ which is a Cartier divisor on X'^{top}, then in the ring $R'/(IR')^{top}$, $\overline{J}$ is generated by d forms $\overline{\beta_1}, \ldots, \overline{\beta_d}$ of degrees at most σ, which give forms $\beta_1, \ldots, \beta_d$ in J of degrees at most σ such that $Z' = \operatorname{Proj} R'/(IR')^{top} + (\beta_1, \ldots, \beta_d)$, and therefore we obtain $Z = \operatorname{Proj} R'/IR' + (\beta_1, \ldots, \beta_d)$. Let $J' = (\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_d)$. We have an exact sequence

$$0 \to R'/IR' \cap J' \to R'/IR' \oplus R'/J' \to R'/IR' + J' \to 0.$$

From this, we get

$$\operatorname{reg} R/I = \operatorname{reg} R'/IR' \leq \max\{\operatorname{reg}(R'/IR' \cap J'), \operatorname{reg}(R'/IR' + J')\}.$$

Since $IR' \cap J' = (\alpha_1, \ldots, \alpha_r)$ is a complete intersection, $\operatorname{reg}(R'/IR' \cap J') = \sigma$. We just need to bound $\operatorname{reg}(R'/IR' + J')$. Note that $IR' + J' = (f_1, \ldots, f_t, \beta_1, \ldots, \beta_d)$ and $\operatorname{ht}(IR' + J') = r + 1$. By assumption of $d_1 \geq 2$, we have $\sigma \geq d_{r+1}$.

If $IR' + J'$ is a complete intersection, then some $r+1$ generators will be a regular sequence. Assume that $f_{i_1}, \dots, f_{i_p}, \beta_{j_1}, \dots, \beta_{j_q}$ are such generators where $p+q = r+1$. Then

$$\operatorname{reg} R'/IR' + J' = \sum_{\eta=1}^{p} (\deg f_{i_\eta} - 1) + \sum_{\mu=1}^{q} (\deg \beta_{i_\mu} - 1).$$

If $p \leq r$, then we can get $\operatorname{reg} R'/IR' + J' \leq \sigma + d(\sigma - 1) \leq \dfrac{(d+1)!}{2}(\sigma - 1)$. Otherwise $p = r+1$, then we still have $\operatorname{reg} R'/IR' + J' \leq \sigma + \sigma - 1 \leq \dfrac{(d+1)!}{2}(\sigma - 1)$.

If $IR' + J'$ is not a complete intersection, then let $f_{i_1}, \dots, f_{i_p}, \beta_{j_1}, \dots, \beta_{j_q}$ be $r+1$ highest degree generators. By Proposition 3.3.1, $Z = \operatorname{Proj} R'/IR' + J'$ is local complete intersection log canonical, then we use induction for $IR' + J'$ to get

$$\operatorname{reg} R'/IR' + J' \leq \frac{d!}{2} \left(\sum_{\eta=1}^{p} (\deg f_{i_\eta} - 1) + \sum_{\mu=1}^{q} (\deg \beta_{i_\mu} - 1) - 1 \right).$$

If $p \leq r$, then the left part of the equality is $\leq \dfrac{d!}{2}(\sigma + d(\sigma - 1) - 1) \leq \dfrac{(d+1)!}{2}(\sigma - 1)$. If $p = r+1$, then the left part is $\leq \dfrac{d!}{2}(\sigma + d_{r+1} - 1 - 1) \leq \dfrac{d!}{2}(\sigma + \sigma - 1 - 1) \leq \dfrac{(d+1)!}{2}(\sigma - 1)$. Hence we still obtain $\operatorname{reg} R'/IR' + J' \leq \dfrac{(d+1)!}{2}(\sigma - 1)$. This proves the result for $r \geq 2$.

Case of $r = 1$. There is an homogeneous form l and an homogeneous ideal H such that $f_i = lh_i$, $I = lH$ and $H = (h_1, \dots, h_t) = [I : l]$. Since X is a local complete intersection and normal, $\operatorname{ht} H \geq n$. Also by assumption of $d \geq 2$ we have $n \geq 2$. We consider the following two cases for n.

$n = 2$, then $R = k[x_0, x_1, x_2]$, $\operatorname{ht} I = 1$. Applying (3, Proposition 2.2), we get $\operatorname{reg} R/I \leq 3(\sigma - 1)$, unless $R = k[x_0, x_1, x_2]$, l is a linear form and H a complete intersection of 3 forms of degree $d_1 - 1$, in which case $\operatorname{reg} R/I = 3d_1 - 5$.

$n \geq 3$, then $d = n$. We first note that we have the inequality

$$\deg l + \sum_{i=1}^{n+1}(\deg h_i - 1) \leq \frac{(n+1)!}{2}(\sigma - 1).$$

If $\operatorname{ht} H = n+1$, then $\dim R/H = 0$, and thus we have $\operatorname{reg} R/H \leq \sum_{i=1}^{n+1}(\deg h_i - 1)$, from which we get $\operatorname{reg} R/I = \operatorname{reg} R/(l) + \operatorname{reg} R/H \leq \frac{(n+1)!}{2}(\sigma - 1)$. If $\operatorname{ht} H = n$ and H is a complete intersection, it is easy to see $\operatorname{reg} R/I \leq \frac{(n+1)!}{2}(\sigma - 1)$. If $\operatorname{ht} H = n$ and H is not a complete intersection, then by (3, Proposition 2.1.), $\operatorname{reg} R/H \leq \sum_{i=1}^{n+1}(\deg h_i - 1)$. So we still obtain $\operatorname{reg} R/I \leq \frac{(n+1)!}{2}(\sigma - 1)$. □

Remark 3.3.3. It is well known that if I is a complete intersection, then $\operatorname{reg} R/I \leq \sigma$. Including the situation of a complete intersection in the theorem above, we get Theorem 1.1 in the Introduction.

CHAPTER 4

VANISHING THEOREMS FOR IDEAL SHEAVES

In this chapter we study the cohomology groups of the ideal sheaf of a subvariety of a projective space. The vanishing of these groups usually leads to a regularity bound for the variety. The establishment of some appropriate vanishing theorems is one of the most efficient ways to obtain regularity bounds.

As we mentioned in the previous chapter, in (2) Bertram, Ein and Lazarsfeld established a vanishing theorem for powers of an ideal sheaf, which initiated the study of the regularity bounds in its current form, as far as we know. The theorem says that if $\mathscr{I}$ defines a nonsingular subvariety X of codimension e in $\mathbb{P}^n$ cut out scheme-theoretically by hypersurfaces of degrees $d_1 \geq d_2 \cdots \geq d_t$, then $H^i(\mathbb{P}^n, \mathscr{I}^p(k)) = 0$ for $i > 0$ and $k \geq pd_1 + d_2 + \cdots + d_e - n$. From this theorem we immediately obtain the regularity bound $\operatorname{reg} \mathscr{I}^p \leq pd_1 + d_2 + \cdots + d_e - e + 1$.

In 2008 deFernex and Ein (4) established a similar vanishing theorem for an ideal sheaf without considering powers, i.e., the case of $p = 1$ as above. By assuming that the pair $(\mathbb{P}^n; eX)$ is log canonical, they followed a novel approach based on results on log canonical singularities and multiplier ideal sheaves, which offered a new insight into the regularity problem of varieties.

Comparing to Bertram-Ein-Lazarsfeld theorem, and motivated by deFernex-Ein theorem, in this chapter we will establish a vanishing theorem in Theorem 4.2.2 for powers of the ideal sheaf defining a local complete intersection subvariety with log canonical singularities.

Mainly following the approach of deFernex and Ein in (4), we construct a formal sum $Z = (1-\delta)B + \delta eV + (p-1)V$, for $0 < \delta \ll 1$ and $p \geq 1$, where B is the base scheme of some linear series. Then in a neighborhood of V, the associated multiplier ideal sheaf $\mathcal{J}(X, Z)$ is equal to $\mathscr{I}^p$. This gives us a chance to apply Nadel's vanishing theorem to the multiplier ideal sheaf $\mathcal{J}(X, Z)$, from which we are able to deduce the vanishing theorem of $\mathscr{I}^p$.

Having the above vanishing theorem in hand and applying it to a subvariety of $\mathbb{P}^n$, we obtain a regularity bound for the powers of an ideal sheaf in terms of its generating degrees as in Corollary 4.2.3.

4.1 Multiplier ideal sheaves

In this section, we briefly review the definition of multiplier ideal sheaves and give some quick applications to the problem of bounding regularity. We follow the terminologies on log canonical singularity defined in the beginning of Section 3.1, and the approach in (27), mainly Chapter 9, to the theory of multiplier ideal sheaves.

Consider a pair (X, Z), where X is a normal, $\mathbb{Q}$-Gorenstein variety and Z is a formal finite sum $Z = \sum_j q_j Z_j$ of proper closed subschemes Z_j of X with nonnegative rational coefficients q_j.

Definition 4.1.1. Let X be a nonsingular variety and let $f : X' \to X$ be a log resolution of the pair (X, Z). We define the *multiplier ideal sheaf* $\mathcal{J}(X, Z)$ associated to (X, Z) as $\mathcal{J}(X, Z) = f_* \mathscr{O}_{X'}(K_{X'/X} - \lfloor \sum q_j f^{-1}(Z_j) \rfloor) \subseteq \mathscr{O}_X$.

Notation 4.1.2. If X is clear from the context, then we also write the multiplier ideal sheaf $\mathcal{J}(X, Z)$ simply as $\mathcal{J}(Z)$.

In the definition above the symbol $\lfloor \sum q_j f^{-1}(Z_j) \rfloor$ means the round-down or integer part of the $\mathbb{Q}$-divisor $\sum q_j f^{-1}(Z_j)$. Notice that since X is nonsingular we see that the pushforward of the relative canonical divisor $\mathscr{O}_{X'}(K_{X'/X})$ is trivial (27, Lemma 9.2.19.) and therefore the coherent sheaf $\mathcal{J}(X, Z)$ is an ideal sheaf. Also the definition of the multiplier ideal sheaf $\mathcal{J}(X, Z)$ is independent on the choice of the log resolution f, cf. (27, Section 9.2.).

In order to use multiplier ideal sheaves in the study of the regularity problem of an ideal sheaf , in most case we need to construct an appropriated multiplier ideal sheaf such that it at least locally equals to the ideal sheaf we considered. In fact, there is a more general question to ask when an ideal sheaf is a multiplier ideal sheaf. We shall prove some results in this direction.

For nonsingular varieties it is easy to prove the following result by blowing-up along nonsingular subvarieties.

Proposition 4.1.3. *Let X be a nonsingular variety and $V \subset X$ be a nonsingular subvariety of codimension e defined by $\mathscr{I}_V$. Then we have $\mathcal{J}(cV) = \mathscr{O}_X$ if $c < e$, and $\mathcal{J}(cV) = \mathscr{I}_V^{[c]-e+1}$ if $c \geq e$. In particular, $\mathcal{J}(eV) = \mathscr{I}_V$.*

Proposition 4.1.4. *Let X be a nonsingular variety and $V \subset X$ be a subscheme of codimension e defined by $\mathscr{I}_V$. Then $\mathscr{I}_V \subseteq \mathcal{J}(eV)$ if and only if $\mathcal{J}((e-1)V)$ is trivial.*

Proof. If $e = 1$, then $\mathcal{J}((e-1)V) = \mathscr{O}_X$ is trivial and the result is obvious. In the sequel, we assume that codimension $e > 1$.

We prove "only if" part first, i.e., assume that $\mathscr{I}_V \subseteq \mathcal{J}(eV)$. Take a log resolution $\mu : X' \to X$ of the pair (X, V). It is enough to show that for any prime divisor E on X', we have the inequality $\mathrm{ord}_E K_{X'/X} - e \cdot \mathrm{ord}_E \mathscr{I}_V \geq -\mathrm{ord}_E \mathscr{I}_V$. This is a local question. Let η be the generic point of E such that $\mu(\eta) = p \in X$. Then for any local equation $f \in \mathscr{I}_{V,p}$, since $\mathscr{I}_V \subseteq \mathcal{J}(eV)$, we have locally, (i.e., around η), $\mathrm{div}(f) + K_{X'/X} - e \cdot \mu^{-1}(V) \geq 0$. This implies $\mathrm{ord}_E K_{X'/X} - e \cdot \mathrm{ord}_E \mathscr{I}_V \geq -\mathrm{ord}_E f$, and therefore $\mathrm{ord}_E K_{X'/X} - e \cdot \mathrm{ord}_E \mathscr{I}_V \geq -\mathrm{ord}_E \mathscr{I}_V$.

Next we prove "if" part by assuming $\mathcal{J}((e-1)V)$ is trivial. Still take a log resolution $\mu : X' \to X$ for (X, V). Then it enough to show for any prime divisor E on X' we have inequality $\mathrm{ord}_E(K_{X'/X}) - \mathrm{ord}_E \mathscr{I}_V^e \geq -\mathrm{ord}_E \mathscr{I}_V$, that is $\mathrm{ord}_E(K_{X'/X}) - (e-1)\mathrm{ord}_E \mathscr{I}_V \geq 0$.

Note that $e > 1$, every divisor in $K_{X'/X} - (e-1)\mu^{-1}(V)$ is exceptional. By assumption that $\mathcal{J}((e-1) \cdot V) = \mu_* \mathscr{O}_{X'}(K_{X'/X} - (e-1)\mu^{-1}(V)) = \mathscr{O}_X$, we see that $K_{X'/X} - (e-1)\mu^{-1}(V)$ is effective and then $K_{X'/X} - (e-1)\mu^{-1}(V) \geq 0$, which proves the result. $\square$

The proof above also leads to the following corollary, which gives a slightly more general result.

Corollary 4.1.5. *Let X be a nonsingular variety and $V \subset X$ be a subscheme of codimension $e > 1$ defined by $\mathscr{I}_V$. Then for any integer $a > 0$, $\mathscr{I}_V \subseteq \mathcal{J}(a \cdot V)$ if and only if $\mathcal{J}((a-1) \cdot V)$ is trivial.*

Lemma 4.1.6. *Let X be a nonsingular variety and $V \subset X$ be a reduced equidimensional subscheme of codimension e defined by $\mathscr{I}_V$. Then $\mathcal{J}(e \cdot V) \subseteq \mathscr{I}_V$.*

Proof. Let $\mathscr{I}_V = \cap \mathscr{I}_{V_i}$ be a primary decomposition of $\mathscr{I}_V$ such that each $\mathscr{I}_{V_i}$ defines a component V_i of V with the generic point η_i. By assumption, V is generic smooth, and therefore for each η_i, there is an open set U such that $\mathscr{I}_V|_U = \mathscr{I}_{V_i}|_U$ and V is nonsingular on U. By

Proposition 4.1.3 we have $\mathcal{J}(e \cdot V)|_U = \mathscr{I}_{V_i}|_U$ which means $\mathcal{J}(e \cdot V)_{\eta_i} = \mathscr{I}_{V_i,\eta_i}$. Then we obtain the inclusion $\mathcal{J}(e \cdot V) \subseteq \mathscr{I}_{V_i}$ since $\mathscr{I}_{V_i}$ is primary. Hence we get $\mathcal{J}(e \cdot V) \subseteq \mathscr{I}_V$. □

Theorem 4.1.7. *Let X be a nonsingular variety and $V \subset X$ be a reduced equidimensional subscheme of codimension e defined by $\mathscr{I}_V$ of codimension e. Then $\mathcal{J}(e \cdot V) = \mathscr{I}_V$ if and only if $\mathcal{J}((e-1) \cdot V)$ is trivial.*

Proof. By Lemma 4.1.6 we always have $\mathcal{J}(e \cdot V) \subseteq \mathscr{I}_V$. Thus we just need to show $\mathscr{I}_V \subseteq \mathcal{J}(e \cdot V)$ if and only if $\mathcal{J}((e-1) \cdot V)$ which follows from Proposition 4.1.4. □

From Lemma 4.1.9 we immediately have the following corollary.

Corollary 4.1.8. *Let X be a nonsingular variety and $V \subset X$ be a reduced equidimensional subscheme of codimension e defined by $\mathscr{I}_V$. Assume that the pair $(X, e \cdot V)$ is log canonical, then $\mathcal{J}(e \cdot V) = \mathscr{I}_V$.*

Lemma 4.1.9. *Let X be a nonsingular variety and $V \subset X$ be a subscheme of codimension e. Assume that the pair $(X, e \cdot V)$ is log canonical, then $\mathcal{J}((e-1) \cdot V)$ is trivial.*

Proof. If $e = 1$, then there is nothing to prove. Assume that $e > 1$. Take a log resolution $\mu : X' \to X$ of the pair $(X, e \cdot V)$. For any prime divisor E over X on X', we have the inequality $\mathrm{ord}_E(K_{X'/X}) - \mathrm{ord}_E \mathscr{I}_V^e \geq -1$ since $(X, e \cdot V)$ is log canonical. Then it is easy to see that if $\mathrm{ord}_E \mathscr{I}_V = 0$, then $\mathrm{ord}_E(K_{X'/X}) \geq 0 = -\mathrm{ord}_E \mathscr{I}_V$. If $\mathrm{ord}_E \mathscr{I}_V \neq 0$ then $\mathrm{ord}_E(K_{X'/X}) - \mathrm{ord}_E \mathscr{I}_V^e \geq -1 \geq -\mathrm{ord}_E \mathscr{I}_V$. So in any case we have $\mathrm{ord}_E(K_{X'/X}) - \mathrm{ord}_E \mathscr{I}_V^e \geq -\mathrm{ord}_E \mathscr{I}_V$ which implies that $\mathrm{ord}_E(K_{X'/X}) - \mathrm{ord}_E \mathscr{I}_V^{(e-1)}$ is effective and gives the result. □

There is an important vanishing theorem, which is usually called Nadel's vanishing theorem and is a variant of Kawamata-Viehweg vanishing theorem in the context of multiplier ideal sheaves. Many applications of multiplier ideal sheaves in regularity problems will depend on this theorem. For the proof of the theorem see (27, Section 9.4.B.) for details.

Theorem 4.1.10. *Suppose that X is a nonsingular projective variety in the pair (X, Z). Let L_j and A be Cartier divisors on X such that $\mathscr{I}_{Z_j} \otimes L_j$ is globally generated for each j and $A - \sum q_j L_j$ is big and nef. Then*

$$H^i(X, \omega_X \otimes A \otimes \mathfrak{J}(X, Z)) = 0 \qquad \textit{for} \;\; i > 0,$$

where ω_X is the canonical sheaf of X.

Directly applying this theorem we can immediately get regularity bounds for multiplier ideal sheaves.

Corollary 4.1.11. *Let $V \subset \mathbb{P}^n$ be a subscheme defined by $\mathscr{I}_V$ of codimension e. Assume that $\mathscr{I}_V(d)$ is globally generated for some $d > 0$. Then for any positive number a, we have*

$$H^i(\mathbb{P}^n, \mathfrak{J}(a \cdot V)(k + ad - n)) = 0 \qquad \textit{for } i > 0, k \geq 0.$$

In particular $\operatorname{reg} \mathfrak{J}(a \cdot V) \leq ad - e + 1$.

Proof. Use Theorem 4.1.10 in the following form: let X be a nonsingular projective variety and $V \subset X$ be a subscheme defined by $\mathscr{I}_V$. If for an integer $a > 0$ there is a line bundle L and A such that $L \otimes \mathscr{I}_V$ is globally generated and $A \otimes L^{\otimes a}$ is nef and big. Then we have

$$H^i(X, \omega_X \otimes A \otimes \mathfrak{J}(a \cdot V)) = 0, \text{ for all } i > 0.$$

Now applying it to the situation that $X = \mathbb{P}^n$, $L = \mathscr{O}_{\mathbb{P}^n}(1)$ and $A = \mathscr{O}_{\mathbb{P}^n}(1)$ we obtain the result. $\square$

Corollary 4.1.12. *Let $V \subset \mathbb{P}^n$ be a reduced equidimensional subscheme of codimension e defined by $\mathscr{I}_V$ and assume that the pair $(\mathbb{P}^n, eV)$ is log canonical. Suppose that $\mathscr{I}_V(d)$ is globally generated for some $d > 0$. Then one has* $\operatorname{reg} \mathscr{I}_V \le ed - e + 1$.

Proof. Applying Corollary 4.1.11 to the multiplier ideal sheaf $\mathcal{J}(e \cdot V)$ and noticing the equality $\mathcal{J}(e \cdot V) = \mathscr{I}$ by Corollary 4.1.8 we obtain the result. $\square$

Keep the notations as in above corollary, we can allow the pair $(\mathbb{P}^n, e \cdot V)$ has isolated non log canonical points. Specifically we define $\mathrm{Nlc}(\mathbb{P}^n, e \cdot V)$ to be a closed subset of $\mathbb{P}^n$ such that

$$\mathrm{Nlc}(\mathbb{P}^n, V) = \{p \in \mathbb{P}^n \,|\, \mathrm{mld}(p; \mathbb{P}^n, e \cdot V) < 0\},$$

i.e., the closed set containing the points where the pair $(\mathbb{P}^n, e \cdot V)$ fails to be log canonical. Since $\mathbb{P}^n$ is nonsingular the set $\mathrm{Nlc}(\mathbb{P}^n, e \cdot V)$ must be a closed subset of V. In particular if the pair $(\mathbb{P}^n, e \cdot V)$ is log canonical then $\mathrm{Nlc}(\mathbb{P}^n, e \cdot V)$ is empty and we set $\dim \mathrm{Nlc}(\mathbb{P}^n, e \cdot V) = -1$.

Corollary 4.1.13. *Let $V \subset \mathbb{P}^n$ be a reduced equidimensional subscheme of codimension e defined by $\mathscr{I}_V$ and assume that the pair $(\mathbb{P}^n, e \cdot V)$ is log canonical outside* $\mathrm{Nlc}(\mathbb{P}^n, e \cdot V)$. *Suppose that $\mathscr{I}_V(d)$ is globally generated for some $d > 0$ and* $\dim \mathrm{Nlc}(\mathbb{P}^n, e \cdot V) \le 0$. *Then one has* $\operatorname{reg} \mathscr{I}_V \le ed - e + 1$.

Proof. Using a short exact sequence $0 \to \mathcal{J}(e \cdot V) \to \mathscr{I}_V \to \mathscr{Q} \to 0$ where the inclusion $\mathcal{J}(e \cdot V) \subseteq \mathscr{I}_V$ is from Lemma 4.1.6 and $\mathscr{Q}$ is the quotient. Since $\mathrm{Nlc}(\mathbb{P}^n, e \cdot V)$ and by Corollary

4.1.8 we see $\dim \operatorname{Supp} \mathscr{Q} \leq 0$ and then apply Theorem 4.1.10 to deduce $H^i(\mathbb{P}^n, \mathscr{I}_V(k)) = 0$ for $i > 0$, $k > ed - n$, from which the result follows. $\square$

4.2 Vanishing theorems

In this section, we prove our main vanishing theorem for the powers of an ideal sheaf. We basically follow the idea of Ein and deFernex (4). The crucial point is to construct appropriated multiplier ideal sheaves locally equal to the powers of the ideal sheaf.

Lemma 4.2.1. *Let X be a nonsingular projective variety and $V \subset X$ be a local complete interesection subvariety of codimension e. Suppose that V is scheme-theoretically given by $V = H_1 \cap \cdots \cap H_t$, for some $H_i \in |L^{\otimes d_i}|$, where L is a globally generated line bundle on X and $d_1 \geq \cdots \geq d_t$. Then V is log canonical if and only if the pair (X, eV) is log canonical.*

Proof. Using (4, Proposition 3.1.) we see that for any point $p \in X$ there are sufficiently general divisors $D_i \in |L^{\otimes d_i} \otimes \mathscr{I}_V|$, for $i = 1, \ldots, e$, such that $\operatorname{mld}(p; X, eV) = \operatorname{mld}(p; X, D_1 + \cdots + D_e)$. Then applying (4, Proposition 3.5.) we see that $\operatorname{mld}(p; V, 0) = \operatorname{mld}(p; X, D_1 + \cdots + D_e)$ since by assumption V is a local complete intersection. Thus for any point $p \in X$, we have $\operatorname{mld}(p; X, eV) = \operatorname{mld}(p; V, 0)$. V is log canonical if and only if the pair (X, eV) is log canonical. $\square$

Theorem 4.2.2. *Let X be a nonsingular projective variety and $V \subset X$ be a local complete intersection subvariety with log canonical singularities. Suppose that V is scheme-theoretically given by $V = H_1 \cap \cdots \cap H_t$, for some $H_i \in |L^{\otimes d_i}|$, where L is a globally generated line bundle on X and $d_1 \geq \cdots \geq d_t$. Set $e = \operatorname{codim}_X V$, then we have*

$$H^i(X, \omega_X \otimes L^{\otimes k} \otimes A \otimes \mathscr{I}_V^p) = 0, \ \textit{for } i > 0, \ k \geq pd_1 + d_2 + \cdots + d_e,$$

where $p \geq 1$ *and* A *is a nef and big line bundle on* X*.*

Proof. First note that by the assumption of V being a local complete intersection, V is log canonical if and only if the pair (X, eV) is log canonical from Lemma 4.2.1.

Consider the base locus subscheme $B \subset X$ of the linear series $|L^{\otimes(d_1+\cdots+d_e)} \otimes \mathscr{I}_V^e|$. For each $p \in V$ using (4, Corollary 3.5 or Proposition 3.1), we see that there is a divisor $D \in |L^{\otimes(d_1+\cdots+d_e)} \otimes \mathscr{I}_V^e|$ such that (X, D) is log canonical at p. This implies that the pair (X, B) is also log canonical at p and therefore is log canonical in a neighborhood of V.

Take a log resolution $\mu : X' \to X$ of B and V such that the scheme-theoretical inverse images $\mu^{-1}(B)$ and $\mu^{-1}(V)$ and the exceptional locus of μ are divisors with simple normal crossing supports. Then μ factors through the blowing-up of X along V. Let F be the unique exceptional divisor on X' dominating V and coming from this blowing-up. We have the following two observations: (1) for any divisor E on X', we have $\mathrm{Val}_E B \geq e \, \mathrm{Val}_E V$, since $\mathscr{I}_B \subseteq \mathscr{I}_V^e$ by the definition of B; (2) in particular, for the divisor F, we have $\mathrm{Val}_F B = e \, \mathrm{Val}_F V = e$.

Now, for $0 < \delta \ll 1$, we construct a formal sum

$$Z = (1-\delta)B + \delta eV + (p-1)V, \quad \text{for } p \geq 1,$$

and associate to Z the multiplier ideal sheaf $\mathcal{J}(X, Z)$. We compare $\mathcal{J}(X, Z)$ with $\mathscr{I}_V^p$ locally around V. For this, let U be a neighborhood of V such that all prime divisors in

$$K_{X'/X} - (1-\delta)\mu^{-1}(B) + \delta e \mu^{-1}(V) + (p-1)\mu^{-1}(V)$$

over U have centers intersecting V and the pair (X, B) is log canonical in U. Picking a such prime divisor E on X', there are two possibilities for its center.

First assume that $C_X(E) \subseteq V$. Then $\mathrm{Val}_E V \geq 1$. Since the pair (X, B) is log canonical around V, we have $\mathrm{Val}_E B - \mathrm{ord}_E K_{X'/X} \leq 1$, and therefore $\mathrm{Val}_E B - \mathrm{ord}_E K_{X'/X} \leq \mathrm{Val}_E V$. Thus

$$\begin{aligned} & \mathrm{Val}_E((1-\delta)B + \delta e V + (p-1)V) - \mathrm{ord}_E K_{X'/X} \\ \leq\ & \mathrm{Val}_E B - \mathrm{ord}_E K_{X'/X} + \mathrm{Val}_E (p-1)V \\ \leq\ & \mathrm{Val}_E pV. \end{aligned}$$

Then we have $\mathrm{ord}_E K_{X'/X} - \mathrm{Val}_E Z \geq -\mathrm{Val}_E \mathscr{I}_V^p$.

Second assume that $C_X(E) \cap V$ is not empty but $C_X(E) \nsubseteq V$. Then $\mathrm{Val}_E V = 0$. We see that

$$\begin{aligned} & \mathrm{Val}_E((1-\delta)B + \delta e V + (p-1)V) - \mathrm{ord}_E K_{X'/X} \\ =\ & \mathrm{Val}_E((1-\delta)B) - \mathrm{ord}_E K_{X'/X} < 1. \end{aligned}$$

The last inequality is because the pair (X, B) is log canonical in U and therefore the pair $(X, (1-\delta)B)$ is Kawamata log terminal in U. Hence we obtain $\mathrm{ord}_E K_{X'/X} - \mathrm{Val}_E Z > -1$.

Combining above two possibilities, we obtain that for any divisor E over U,

$$\mathrm{ord}_E K_{X'/X} - \lfloor \mathrm{Val}_E Z \rfloor \geq -\mathrm{Val}_E \mathscr{I}_V^p.$$

This implies that on U, we have the inclusion $\mathscr{I}_V^p|_U \subseteq \mathcal{J}(X,Z)|_U$.

Next we prove globally on X, $\mathcal{J}(X,Z) \subseteq \mathscr{I}_V^p$. From the definition of multiplier ideal sheaves and the fact that $\mathscr{I}_B \subseteq \mathscr{I}_V^e$, we have $\mathcal{J}(X,Z) \subseteq \mathcal{J}(X, eV + (p-1)V)$.

Let η be the generic point of V. Take a neighborhood U' of η in X such that $V|_{U'}$ is nonsingular. The blowing-up of U' along $V|_{U'}$ gives a log resolution of U' and $V|_{U'}$. Computing $\mathcal{J}(X, eV + (p-1)V)$ on this blowing-up, we see that at the point η,

$$\mathcal{J}(X, eV + (p-1)V)_\eta = \mathscr{I}_{V,\eta}^p.$$

Thus globally on X, $\mathcal{J}(X, eV + (p-1)V) \subseteq \mathscr{I}_V^{<p>}$. Since V is a local complete intersection, $\mathscr{I}_V^p = \mathscr{I}_V^{<p>}$ and therefore on X, we have $\mathcal{J}(X,Z) \subseteq \mathscr{I}_V^p$.

From above, in the open neighborhood U of V, we have the equality $\mathcal{J}(X,Z)|_U = \mathscr{I}_V^p|_U$ and therefore $\mathcal{J}(X,Z) = \mathscr{I}_V^p \cap \mathscr{I}_W$ for some subscheme W of X disjoint from V.

Applying Nadel's vanishing theorem to $\mathcal{J}(X,Z) = \mathscr{I}_V^p \cap \mathscr{I}_W$ and using (4, Lemma 4.3.), we have the vanishing

$$H^i(X, \omega_X \otimes L^{\otimes k} \otimes A \otimes \mathscr{I}_V^p) = 0, \text{ for } i > 0, \ k \geq pd_1 + d_2 + \cdots + d_e,$$

where $p \geq 1$ and A is a nef and big line bundle on X. $\square$

Corollary 4.2.3. *Let $V \subset \mathbb{P}^n$ be a subvariety of codimension e such that V is a local complete intersection with log canonical singularities. Assume that V is cut out scheme-theoretically by hypersurfaces of degrees $d_1 \geq \cdots \geq d_t$. Then $H^i(\mathbb{P}^n, \mathscr{I}_V^p(k)) = 0$, for $i > 0$, $k \geq pd_1 + d_2 + \cdots + d_e - n$. In particular, one has* $\operatorname{reg} \mathscr{I}^p \leq pd_1 + d_2 + \cdots + d_e - e + 1$.

4.3 Application to Multi-regularity

In this section we apply the results from previous sections to the study of multi-regularity on biprojective spaces, a variant of the notion of regularity on projective spaces. In (28), authors gave a general definition of multigraded regularity of coherent sheaf on toric varieties. Here, we only consider the special case of biprojective space.

Definition 4.3.1. Let $\mathscr{F}$ be a coherent sheaf on a biprojective space $Y = \mathbb{P}^a \times \mathbb{P}^b$. We say that $\mathscr{F}$ is $\mathbf{m} = (m_1, m_2)$-regular if $H^i(Y, \mathscr{F}(m_1 - u, m_2 - v)) = 0$ for all $i > 0$ and $u + v = i$, $(u, v) \in \mathbb{N}^2$ (we assume $0 \in \mathbb{N}$).

Denote by $\operatorname{reg}\mathscr{F}$ the set of the pair $\mathbf{m}$ such that $\mathscr{F}$ is $\mathbf{m}$-regular. An important property of this biregualrity is that if $\mathscr{F}$ is $\mathbf{m}$-regular, then it is $\mathbf{m} + \mathbb{N}^2$-regular.

We are interested in the multigraded regularity of ideal sheaves of subschemes in biprojective spaces. The following version of Nadel vanishing theorem can be easily deduced from Theorem 4.1.10.

Proposition 4.3.2. *Let X be a nonsingular projective variety over and L_1 and L_2 be globally generated line bundles on X. Assume that $V \subset X$ is a subscheme of codimension e defined by $\mathscr{I}_V$ such that $\mathscr{I}_V \otimes L_1^c \otimes L_2^d$ is globally generated for some positive integer c and d. If there is a line bundle A such that $A \otimes L_1^{ec} \otimes L_2^{ed}$ is nef and big. Then*

$$H^i(X, \omega_X \otimes A \otimes L_1^{k_1} \otimes L_2^{k_2} \otimes \mathfrak{J}(e \cdot V)) = 0,$$

for all $i > 0$ and all $k_1 \geq 0$, $k_2 \geq 0$.

Applying this vanishing theorem to the case of biprojective space we obtain multigraded regularity of multiplier ideal sheaf $\mathcal{J}(e \cdot V)$. Under the condition that the pair $(X, e \cdot V)$ is log canonical, we can compare $\mathscr{I}_V$ to the multiplier ideal sheaf and therefore have a chance to get its multigraded regularity.

Lemma 4.3.3. *Let $V \subset X = \mathbb{P}^m \times \mathbb{P}^n$ be a subscheme of dimension r. Then for any pair $(p, q) \in \mathbb{N}^2$, one has $H^i(X, \mathscr{I}_V(p-u, q-v)) = 0$, for all $i \geq r+2$ and all $(u, v) \in \mathbb{N}^2$ with $u + v = i$.*

Proof. Since $\mathscr{O}_X$ is $(0,0)$-regular, it is (p, q)-regular if $(p, q) \in \mathbb{N}^2$. So we have $H^i(X, \mathscr{O}_X(p-u, q-v)) = 0$, for all $i > 0$ and all $(u, v) \in \mathbb{N}^2$ with $u + v = i$.

Considering the short exact sequence

$$0 \to \mathscr{I}_V(p-u, q-v) \to \mathscr{O}_X(p-u, q-v) \to \mathscr{O}_V(p-u, q-v) \to 0,$$

it is easy to get the result. $\square$

Corollary 4.3.4. *Let $X = \mathbb{P}^m \times \mathbb{P}^n$ and $V \subset X$ be a subscheme with codimension e defined by $\mathscr{I}_V$ such that $\mathscr{I}_V(c, d)$ is globally generated for some nonnegative integers c and d. Assume that V is equidimensional without embedded components and the pair $(X, e \cdot V)$ is log canonical, then $\mathscr{I}_V$ is $(ec - e + n + 1, ed - e + m + 1)$-regular.*

Next we consider the case that a subvariety of $\mathbb{P}^m \times \mathbb{P}^n$ is cut out by given bidegree equations. We say two pairs of integers $(a, b) \geq (c, d)$ if and only if $a \geq c$ and $b \geq d$.

Theorem 4.3.5. *Let X be a locally complete intersection projective variety with rational singularities and let $V \subset X$ be a pure-dimensional proper subscheme of codimension e with no* *embedded components. Suppose that V is scheme-theoretically given by $V = H_1 \cap H_2 \cap \cdots \cap H_t$ for some divisors $H_i \in |L_i|$, where L_i's are line bundles and satisfy the condition that for each $i = 1, ..., t-1$, $L_i \otimes L_{i+1}^{-1}$ is generated by global sections and L_t is also generated by global sections. Assume that the pair $(X, e \cdot V)$ is log canonical. Then one has*

$$H^i(X, \omega_X \otimes L \otimes A \otimes \mathscr{I}_V) = 0,$$

for all $i > 0$, where A is a nef and big line bundle and L is a line bundle such that $L \otimes L_1^{-1} \otimes L_2^{-1} \otimes \cdots \otimes L_e^{-1}$ is generated by global sections.

Corollary 4.3.6. *Let $V \subset \mathbb{P}^n \times \mathbb{P}^n$ be a pure-dimensional proper subscheme of codimension e with no embedded components. Assume that V is given by equations of bidegree $(c_1, d_1) \geq (c_2, d_2) \geq \cdots \geq (c_t, d_t)$ and the pair $(\mathbb{P}^m \times \mathbb{P}^n, e \cdot V)$ is log canonical. Then the ideal sheaf $\mathscr{I}_V$ of V is $(\sum_{i=1}^{e} c_i - e + 1 + n, \sum_{i=1}^{e} d_i - e + 1 + m)$-regular.*

Proof. We use Theorem 4.3.5 directly. Write $X = \mathbb{P}^m \times \mathbb{P}^n$. Let $L_i = \mathscr{O}_X(c_i, d_i)$ for $i = 1, 2, \cdots, t$ and let $L = \mathscr{O}_X(\sum_{i=1}^{e} c_i + k_1, \sum_{i=1}^{e} d_i + k_2)$ for $k_1 \geq 0$, $k_2 \geq 0$. Also let $A = \mathscr{O}_X(1,1)$ be a ample line bundle. Notice that $\omega_X = \mathscr{O}_X(-m-1, -n-1)$, the canonical line bundle of X. Then we have vanishing $H^i(X, \mathscr{I}_V(\sum_{i=1}^{e} c_i - m + k_1, \sum_{i=1}^{e} d_i - n + k_2)) = 0$ for all $i > 0$, $k_1 \geq 0$ and $k_2 \geq 0$. This implies that for $0 < i \leq r+1$, and $(u, v) \in \mathbb{N}^2$ with $u + v = i$, we have $H^i(X, \mathscr{I}_V(\sum_{i=1}^{e} c_i - m + r + 1 - u, \sum_{i=1}^{e} d_i - n + r + 1 - v)) = 0$. Then plus Lemma 4.3.3 (notice that $(\sum_{i=1}^{e} c_i - m + r + 1, \sum_{i=1}^{e} d_i - n + r + 1) \in \mathbb{N}^2$) we obtain the result. $\square$

Remark 4.3.7. If we assume that $\mathscr{I}_V(c,d)$ is generate by global sections, i.e., V is given by equations of bidegrees $(c_1,d_1) = (c_2,d_2) = \cdots = (c_t,d_t) = (c,d)$, then from Corollary 4.3.6 above we immediately get Corollary 4.3.4.

CHAPTER 5

ASYMPTOTIC REGULARITY OF IDEAL SHEAVES

In this chapter, we study the asymptotic behavior of the Castelnuovo-Mumford regularity of an ideal sheaf on a projective space. In general it is very difficult to bound the regularity efficiently in terms of invariants of an ideal, for example, the degrees of generators. However, if one considers the regularity of sufficiently high powers of the ideal, i.e., the asymptotic regularity, then the picture seems tractable.

The phenomenon that the asymptotic regularity can be bounded by a linear function was first observed by Swanson (5) by means of commutative algebra. She showed that for any homogeneous ideal I, there exist constants d and e such that $\operatorname{reg} I^p \leq dp + e$ for all $p \geq 1$. Later Cutkosky, Herzog and Trung (6) and Kodiyalam (7) established an effective result which shows that the asymptotic regularity can indeed be a linear function, i.e., for p large enough $\operatorname{reg} I^p = dp + e$ for some constants d and e. A very concrete algebraic meaning of the rate d was also given in their work. However, the constant e seems mysterious in their theory and very little is currently known.

Turning to a geometric setting, one considers an ideal sheaf $\mathscr{I}$ on the projective space $\mathbb{P}^n$, and expects to establish a theory in which the asymptotic regularity shall be related to geometric invariants of the ideal sheaf. The first result in this direction is due to Cutkosky, Ein and Lazarsfeld (8). They established the equality $\lim\limits_{p\to\infty} \dfrac{\operatorname{reg} \mathscr{I}^p}{p} = s$, where s is an invariant of $\mathscr{I}$ measuring the local positivity of $\mathscr{I}$ (see Section 5.1 for details). Thus the asymptotic regularity is a linear-like function of the slope s. However, several examples (e.g. (6), (8)) have shown that s could be an irrational number and therefore $\operatorname{reg} \mathscr{I}^p$ is in general far from being a linear function for p sufficiently large. Thus the best one can hope for is that the asymptotic regularity is bounded by linear functions or the difference $|\operatorname{reg} \mathscr{I}^p - sp|$ is bounded by a constant. This question was raised by Cutkosky and Kurano in (29) and is the motivation of this chapter.

The main result we establish in this chapter is Theorem 5.2.2 which states that the asymptotic regularity of an ideal sheaf is bounded from below and above by linear functions with the slope as its s-invariant. Starting from this result, we are able to establish asymptotic regularity bounds for symbolic powers of an ideal sheaf in Theorem 5.3.2.

5.1 The s-invariant of an ideal sheaf

Let $\mathscr{I}$ be an ideal sheaf on the projective space $\mathbb{P}^n$. Construct the blowing-up $\mu : W = \operatorname{Bl}_{\mathscr{I}} \mathbb{P}^n \to \mathbb{P}^n$ of $\mathbb{P}^n$ along $\mathscr{I}$ with the exceptional Cartier divisor E on W such that $\mathscr{I} \cdot \mathscr{O}_W = \mathscr{O}_W(-E)$. Let H be the hyperplane divisor of $\mathbb{P}^n$. Notice that for m sufficiently large $m\mu^*H - E$ is ample on W since $\mathscr{O}_W(-E)$ is μ-ample. Following (8) we associate $\mathscr{I}$ an invariant to measure its positivity, which is defined as follows.

Definition 5.1.1. The *s-invariant* of $\mathscr{I}$ (with respect to the divisor H) is the positive real number $s(\mathscr{I}) = \min\{\ s \mid s\mu^*H - E \ \text{ is nef}\ \}$. Here $s\mu^*H - E$ is considered as an $\mathbb{R}$-divisor on W.

In general, on any projective nonsingular variety X with a fixed ample line bundle H we can define the s-invariant for an ideal sheaf $\mathscr{I}$ with respect to H in a similar fashion. However if we assume $X = \mathbb{P}^n$ then we always assume that the s-invariant is defined with respect to the hyperplane divisor. Usually the general case will follow easily from the case of $X = \mathbb{P}^n$.

Another important invariant associated to $\mathscr{I}$ is $d(\mathscr{I})$, the minimal number such that $\mathscr{I}(d)$ is generated by its global sections. There is a fundamental inequality of s-invariant $s(\mathscr{I})$ and $d(\mathscr{I})$ as follows.

Proposition 5.1.2 ((8, Lemma 1.4.)). *One has the inequality $s(\mathscr{I}) \leq d(\mathscr{I})$. More generally, $s(\mathscr{I}) \leq \dfrac{d(\mathscr{I}^p)}{p}$ for every integer $p \geq 1$.*

The following proposition plays an essential role in the approach of using blowing-up to study asymptotic regularity. It shows that the sufficiently high powers of an ideal sheaf actually equal the pushfoward the multiples of the exceptional divisor on the blowing-up.

Proposition 5.1.3 ((8, Lemma 3.3.)). *Let $\mathscr{I} \subset \mathscr{O}_X$ be an ideal sheaf on a nonsingular projective variety X, and $\mu : W = \mathrm{Bl}_{\mathscr{I}}(X) \to X$ the blowing-up of $\mathscr{I}$, with exceptional divisor E. There exists an integer $p_0 > 0$ with the property that if $p \geq p_0$, then $\mu_*(\mathscr{O}_W(-pE)) = \mathscr{I}^p$, and for any divisor D on X, $H^i(X, \mathscr{I}^p(D)) = H^i(W, \mathscr{O}_W(\mu^* D - pE))$ for all $i \geq 0$.*

5.2 The asymptotic regularity of an ideal sheaf

We prove our main result of this chapter that the asymptotic regularity is bounded by linear functions. We follow the approach in (8) by using the following Fujita's vanishing theorem on the blowing-up of an ideal sheaf.

Theorem 5.2.1 ((19, Theorem 1.4.35.)). *Let V be a projective variety. Fix A an ample divisor and $\mathscr{F}$ a coherent sheaf. There is a number $m_0 = m_0(A, \mathscr{F})$ such that for any nef divisor B, $H^i(V, \mathscr{F}(mA + B)) = 0$, for $i > 0, m \geq m_0$.*

Notice that the crucial point in above theorem is that the number m_0 only depends on the ample divisor A and the coherent sheaf $\mathscr{F}$ not on the nef divisor B.

Theorem 5.2.2. *Let $\mathscr{I}$ be an ideal sheaf on $\mathbb{P}^n$ and let $s = s(\mathscr{I})$ be the s-invariant. Then there exists a constant e such that for all $p \geq 1$, we have $sp \leq \operatorname{reg} \mathscr{I}^p \leq sp + e$.*

Proof. We first prove the upper bound of $\operatorname{reg} \mathscr{I}^p$. For this, it suffices to show that there exists a constant e such that for all $p \geq 1$, we have $\operatorname{reg} \mathscr{I}^p \leq \lceil sp \rceil + e$, where $\lceil sp \rceil$ means the least integer greater than sp.

Consider the blowing-up $\mu : W = \operatorname{Bl}_{\mathscr{I}}(\mathbb{P}^n) \to \mathbb{P}^n$ of $\mathbb{P}^n$ along $\mathscr{I}$, with the exceptional divisor E. Let $H = \mathscr{O}_{\mathbb{P}^n}(1)$ be the ample hyperplane divisor.

We choose a rational number ϵ such that $(\lceil s \rceil + \epsilon)\mu^*H - E$ is ample. By considering the sheaf $\mathscr{F} = \mathscr{O}_W$ in Theorem 5.2.1, we can have a large integer n_0 such that $n_0\epsilon$ is an integer number and such that the ample divisor $A := n_0(\lceil s \rceil + \epsilon)\mu^*H - n_0E$ satisfies the vanishing result of Theorem 5.2.1 for any nef line bundle. Notice that we can write $n_0(\lceil s \rceil + \epsilon) = \lceil n_0 s \rceil + e_0$ for some non-negative integer e_0 and therefore $A = (\lceil n_0 s \rceil + e_0)\mu^*H - n_0E$. We fix such ϵ, n_0 and e_0 and the ample divisor A in the sequel.

Now for an integer p large enough, say larger than n_0, we consider a divisor B_p defined as $B_p := \lceil (p - n_0)s \rceil \mu^*H - (p - n_0)E$. Then B_p is nef because of the definition of s and the inequality $\dfrac{\lceil (p-n_0)s \rceil}{p-n_0} \geq \dfrac{(p-n_0)s}{p-n_0} = s$. Now we add this nef divisor B_p to the ample divisor A constructed above. The resulting divisor is $A + B_p = (\lceil n_0 s \rceil + \lceil (p-n_0)s \rceil + e_0)\mu^* H - pE$.

Notice that the divisor $A + B_p$ has no higher cohomology by the choice of A and Theorem 5.2.1. It is an easy fact that for any positive real number a and b, $\lceil a \rceil + \lceil b \rceil = \lceil a + b \rceil + c$ where $c = 0$ or 1. Thus we can write $\lceil n_0 s \rceil + \lceil (p-n_0)s \rceil = \lceil n_0 s + (p-n_0)s \rceil + c = \lceil sp \rceil + c$ where $c = 0$ or 1 and then the divisor $A + B_p = (\lceil sp \rceil + e_0 + c)\mu^* H - pE$. Finally by adding an additional $\mu^* H$ to $A + B_p$ when $c = 0$ we obtain a divisor $R_p = A + B_p + (1-c)\mu^* H$ (this possible extra $\mu^* H$ is just for canceling the awkward number c). Since $\mu^* H$ is nef the divisor R_p does not have any higher cohomology by the choice of A and Theorem 5.2.1. That means we get

$$H^i(W, \mathscr{O}_W((\lceil sp \rceil + e_0 + 1)\mu^* H - pE)) = 0 \qquad \text{for } i > 0 \quad \text{and } p \gg 0.$$

Thus by Proposition 5.1.3 there is a number p_0 such that for $p > p_0$, we have

$$H^i(\mathbb{P}^n, \mathscr{I}^p(\lceil sp \rceil + e_0 + 1)) = 0 \quad \text{for } i > 0.$$

Therefore $\mathscr{I}^p$ is $(\lceil sp \rceil + e_0 + 1 + n)$-regular for $p > p_0$. Taking into account the finitely many cases where $p \leq p_0$, we can have a constant e such that $\operatorname{reg} \mathscr{I}^p \leq \lceil sp \rceil + e$ for all $p \geq 1$.

Next, we prove the lower bound of $\operatorname{reg} \mathscr{I}^p$. For $p \geq 1$, suppose $a_p = \operatorname{reg} \mathscr{I}^p$. Then $\mathscr{I}^p(a_p)$ is generated by its global sections. Thus the invertible sheaf $\mathscr{O}_W(a_p \mu^* H - pE)$ is also generated by its global sections and in particular is nef. Hence by the definition of s, we get $\dfrac{a_p}{p} \geq s$, that is $a_p \geq sp$. So we get the lower bound $\operatorname{reg} \mathscr{I}^p \geq sp$.

Combining arguments together we can find a constant e such that $sp \leq \operatorname{reg} \mathscr{I}^p \leq \lceil sp \rceil + e$ from which the theorem follows. $\square$

Taking limit on both side of the inequalities in the theorem we recover the following result of Cutkosky, Ein and Lazarsfeld mentioned at the beginning of this chapter.

Corollary 5.2.3 ((8, Theorem B.)). *Keeping notation as in the theorem above, one has the equality* $\lim_{p\to\infty} \frac{\operatorname{reg} \mathscr{I}^p}{p} = s$.

Recall that if d is an integer such that $\mathscr{I}(d)$ is generated by its global sections, then we always have that $s(\mathscr{I}) \leq d$ by the inequality in Proposition 5.1.2. Applying above theorem we immediately have the following corollary which can be viewed as an analogue of Swanson's result (5, Theorem 3.6.) of homogeneous ideals.

Corollary 5.2.4. *Let d be an integer such that $\mathscr{I}(d)$ is generated by its global sections. Then there is a constant e such that for all $p \geq 1$, one has* $\operatorname{reg} \mathscr{I}^p \leq dp + e$.

Following notation in (29), for an ideal sheaf we define a function $\sigma_{\mathscr{I}} : \mathbb{N} \to \mathbb{Z}$ such that

$$\operatorname{reg} \mathscr{I}^p = \lfloor ps(\mathscr{I}) \rfloor + \sigma_{\mathscr{I}}(p).$$

This function $\sigma_{\mathscr{I}}$ has been studied by Cutkosky and Kurano in (29) and there they proposed the question about the boundedness of the function $\sigma_{\mathscr{I}}$. From the proof of Theorem 5.2.2, we immediately have the following corollary which provides an affirmative answer that the function $\sigma_{\mathscr{I}}$ is always bounded.

Corollary 5.2.5. *For any ideal sheaf $\mathscr{I}$, the function $\sigma_{\mathscr{I}}$ defined as above is bounded. More precisely there is a constant e such that* $0 \leq \sigma_{\mathscr{I}}(p) \leq e$.

Thus we see that the function $\sigma_{\mathscr{I}}(p)$ is always positive. In the work (29) and (6) the function $\sigma_{\mathscr{I}}(p)$ is showed to be positive by careful calculation for specific examples.

We conclude this section by giving a similar result on the asymptotic regularity of the integral closure of an ideal sheaf. Let $\mathscr{I}$ be an ideal sheaf on $\mathbb{P}^n$. Consider the blowing-up $\mu : W = \mathrm{Bl}_{\mathscr{I}}(\mathbb{P}^n) \to \mathbb{P}^n$ of $\mathbb{P}^n$ along $\mathscr{I}$, with the exceptional divisor E. Let $H = \mathscr{O}_{\mathbb{P}^n}(1)$ be the ample hyperplane divisor. Let $f : W^+ \to W$ be the normalization of W and let $\nu : W^+ \to \mathbb{P}^n$ be the composition of $\mu \circ f$ and denote by F the exceptional divisor on W^+ such that $\mathscr{I} \cdot \mathscr{O}_{W^+} = \mathscr{O}_{W^+}(-F)$. The *integral closure* $\overline{\mathscr{I}}$ of $\mathscr{I}$ is defined by the ideal $\nu_*\mathscr{O}_{X^+}(-F)$. For any $p \geq 1$, the integral closure $\overline{\mathscr{I}^p}$ of $\mathscr{I}^p$ is then equal to $\nu_*\mathscr{O}_{X^+}(-pF)$. Note that since f is finite and $\mathscr{O}_W(-E)$ is μ-ample, $\mathscr{O}_{W^+}(-F) = f^*\mathscr{O}_W(-E)$ is ν-ample and for any real number ϵ, $\epsilon\nu^*H - F$ is ample on W^+ if and only if $\epsilon\mu^*H - E$ is ample on W. This implies that $s(\mathscr{I}) = s(\overline{\mathscr{I}})$. Thus the proof of Theorem 5.2.2 works for the integral closure $\overline{\mathscr{I}^p}$ directly, and we have the following proposition.

Proposition 5.2.6. *There exists a constant e such that $sp \leq \operatorname{reg} \overline{\mathscr{I}^p} \leq sp + e$. Furthermore one has the equality* $\displaystyle\lim_{p\to\infty} \frac{\operatorname{reg} \overline{\mathscr{I}^p}}{p} = s$.

5.3 The regularity of symbolic powers of an ideal sheaf

In this section, we apply the results established in the preceding section to study the asymptotic regularity of symbolic powers of an ideal sheaf. Assume in the sequel that $\mathscr{I}$ is an ideal sheaf on a nonsingular variety X (not necessarily projective) and it defines a reduced subscheme Z of X. We recall the definition of symbolic powers of $\mathscr{I}$.

Definition 5.3.1. The p-th symbolic power of $\mathscr{I}$ is the ideal sheaf consisting of germs of functions that have multiplicity $\geq p$ at each generic point of Z, i.e.,

$$\mathscr{I}^{(p)} = \{f \in \mathscr{O}_X \mid f \in \mathfrak{m}_\eta^p \quad \text{for each generic point } \eta \text{ of } Z\},$$

where $\mathfrak{m}_\eta$ means the maximal ideal of the local ring $\mathscr{O}_{X,\eta}$.

If symbolic powers are almost the same as ordinary powers, then we can obtain regularity bounds for symbolic powers easily.

Theorem 5.3.2. *Let $\mathscr{I}$ be an ideal sheaf on $\mathbb{P}^n$ and let $s = s(\mathscr{I})$ be the s-invariant. Suppose that except at an isolated set of points the symbolic power $\mathscr{I}^{(p)}$ agree with the ordinary power $\mathscr{I}^p$ for all $p \geq 1$. Then there exists a constant e such that for all $p \geq 1$, one has $\operatorname{reg} \mathscr{I}^{(p)} \leq sp + e$.*

Proof. Consider a short exact sequence

$$0 \to \mathscr{I}^p \to \mathscr{I}^{(p)} \to Q \to 0.$$

By assumption we see that the quotient Q has $\dim \operatorname{Supp} Q \leq 0$. Thus Q has no higher cohomology groups. Then we have $\operatorname{reg} \mathscr{I}^{(p)} \leq \operatorname{reg} \mathscr{I}^p$, and the result follows from Theorem 5.2.2. $\square$

In order to see when an ideal sheaf satisfies the condition in Theorem 5.3.2, we consider an algebraic set of X,

$$\operatorname{Nlci}(\mathscr{I}) = \{x \in X | \mathscr{I} \text{ is not a local complete intersection at } x\}.$$

We make a convention that if $\mathscr{I}$ is trivial at x then $\mathscr{I}$ is a local complete intersection at x. This algebraic set will be used to control the set where ordinary powers are not equal to symbolic powers. The main criterion for comparing ordinary and symbolic powers we will use is established in the work of Li and Swanson (30), which generalizes the early work of Hochster (31). We cite this criterion here in the form used later.

Lemma 5.3.3 ((30, Corollary 3.8.))**.** *Assume that an ideal sheaf $\mathscr{I}$ on a nonsingular variety X defines a reduced subscheme. For any point $x \in X$ such that x is not in* $\mathrm{Nlci}(\mathscr{I})$*, we have*

$$\mathscr{I}_x^p = \mathscr{I}_x^{(p)}, \quad \textit{for all } p \geq 1.$$

Notice that from this lemma, we see that the set $\mathrm{Nlci}(\mathscr{I})$ covers the points where $\mathscr{I}^p$ is not equal to $\mathscr{I}^{(p)}$ for some $p \geq 1$. Now we can easily get the following corollaries.

Corollary 5.3.4. *Let $\mathscr{I}$ be an ideal sheaf on $\mathbb{P}^n$ and let $s = s(\mathscr{I})$ be the s-invariant. Assume that $\mathscr{I}$ defines a reduced subscheme and* $\dim \mathrm{Nlci}(\mathscr{I}) \leq 0$*. Then there exists a constant e such that for all $p \geq 1$, one has* $\operatorname{reg} \mathscr{I}^{(p)} \leq sp + e$.

Corollary 5.3.5. *Let $\mathscr{I}$ be an ideal sheaf on $\mathbb{P}^n$ and let $s = s(\mathscr{I})$ be the s-invariant. Assume that $\mathscr{I}$ defines a reduced subscheme of dimension ≤ 1. Then there exists a constant e such that for all $p \geq 1$, one has* $\operatorname{reg} \mathscr{I}^{(p)} \leq sp + e$.

Proof. Let Z be the subscheme defined by $\mathscr{I}$. Then the irreducible components of Z are distinct points or reduced irreducible curves. Thus from Lemma 5.3.3 except for those finitely many points which are singular points of each dimension 1 component and intersection points of two dimension 1 components, $\mathscr{I}^p$ is equal to $\mathscr{I}^{(p)}$ for all $p \geq 1$. Then the result follows from Theorem 5.3.2. □

Remark 5.3.6. Typical low dimensional varieties satisfying the hypothesis of Theorem 5.3.2 are integral curves, normal surfaces and terminal threefolds. It would be very interesting to know if the bound in Theorem 5.2.2 works for any ideal sheaf. We need some new ideas to solve this problem. However, we propose a conjecture in this direction.

Conjecture 5.3.7. *Let $\mathscr{I}$ be an ideal sheaf defining a reduced subscheme of $\mathbb{P}^n$ and let $s = s(\mathscr{I})$ be the s-invariant. Then there is a number e such that for all $p \geq 1$, one has* $\operatorname{reg} \mathscr{I}^{(p)} \leq sp + e$.

CHAPTER 6

REGULARITY BOUNDS FOR NORMAL SURFACES

This chapter is joint work with Lawrence Ein. We focus on geometric aspects of the problem of bounding regularity of algebraic varieties. This problem has attracted considerable attention in recent years, partially because it relates to many classical geometric problems.

Given a nondegenerate projective variety X in the projective space $\mathbb{P}^N$ there are several questions that arise naturally. The first one is if one can write down its defining equations with degrees as small as possible. It is well-know that set-theoretically X is cut out by hypersurfaces of degree no more than $\deg X$. Furthermore if X is nonsingular then it can be cut out scheme-theoretically by hypersurfaces of degree $\deg X$ (32). Thus, for a nonsingular variety it can always be defined by equations of degrees no more than its geometric degree.

If we think of hypersurfaces as linear systems, then from this point of view a natural question is to understand at which degree the hypersurfaces will cut a complete linear system on the variety X. Of course, this is always the case for sufficiently high degrees, which is an easy conclusion of Serre's vanishing theorem.

If we turn to consider the syzygy problem of X, i.e., studying a minimal resolution of the coordinate ring of X, or equivalently the ideal sheaf of X, then the central point is to understand the degrees of syzygy modules. In general, to describe those degrees explicitly is very difficult, thus it is reasonable to at least give bounds for them.

It turns out that all those questions are closely related to the problem of regularity bounds for X. In fact this is perhaps the major reason why regularity has recently attracted considerable attention.

Suppose that X is a curve (not necessarily nonsingular). Gruson, Lazarsfeld and Peskine (10) gave a sharp bound for the regularity of X. They showed that if $X \subset \mathbb{P}^N$ $(N \geq 3)$ is a reduced irreducible nondegenerate curve of degree d, then $\operatorname{reg} X \leq d - N + 2$ and this bound is sharp. Later in (11) Lazarsfeld proved a sharp regularity bound for nonsingular surface. He showed that if $X \subset \mathbb{P}^N$ $(N \geq 4)$ is a nondegenerate nonsingular surface of degree d then $\operatorname{reg} X \leq d - N + 3$. For arbitrary varieties, in (21) Mumford has shown that if $X \subset \mathbb{P}^N$ is nondegenerate, nonsingular and of degree d, then $\operatorname{reg} X \leq (\dim X + 1)(\deg X - 2) + 2$. This bound has been improved by Bertram, Ein and Lazarsfeld in (2) to $\operatorname{reg} X \leq \min(\operatorname{codim} X, \dim X + 1)(\deg X - 1) - \operatorname{codim} X + 1$

It was conjectured by Eisenbud and Goto (9) that $\operatorname{reg} X \leq \deg X - \operatorname{codim} X + 1$ for any nondegenerate subvariety X in $\mathbb{P}^N$. Aforementioned results show that the conjecture is true for integral curves and nonsingular surfaces. Also in his work (12) and (13) Kwak has established somewhat weaker results for nonsingular varieties of dimension no more than six. The motivation of this chapter is to verify Eisenbud-Goto conjecture for singular surfaces (see Corollary 6.3.7). Our approach is based on generic projections and duality theory.

6.1 Singularities of normal surfaces

In this section, we briefly review the singularity theory of normal surfaces. We are interested in the following singularities: rational, Gorenstein elliptic and log canonical. The classification of rational singularity and Gorenstein elliptic singularity can be found in (33) and the classification of log canonical singularity can be found in (34) and (35).

Log canonical singularity is defined in Chapter 3, that is also applied to the surface case here. We shall give the definitions of rational and Gorenstein singularities. Since the singularity we concerned is essentially a local property it is no harm to assume that the surface X is affine and a closed point P is the only singular point of X.

Definition 6.1.1. Let $f : Y \to X$ be a resolution of a normal surface singularity $P \in X$. Then $P \in X$ is a rational singularity if $R^1 f_* \mathscr{O}_Y = 0$. It is an elliptic singularity if $R^1 f_* \mathscr{O}_Y$ is 1-dimensional as a vector space. It is Gorenstein elliptic if it is elliptic and the local ring $\mathscr{O}_{X,P}$ is Gorenstein.

There are two important local invariants associated to a singular point $P \in X$: the multiplicity $\operatorname{mult}_P X$ and the embedding dimension $\operatorname{embdim}_P X$. According to the singularity theory these two invariants can be computed under different singularity conditions. The way to do so is to use the fundamental cycle of a singularity. Precisely, suppose that $f : Y \to X$ is a resolution of normal surfaces singularity $P \in X$. Let $\Gamma = \{\Gamma_i\}_{i=1}^k$ be the set of the irreducible components of the fiber $f^{-1}(P)$. They are a connected bunch of curves (not necessarily nonsingular) on Y and have a negative definite intersection matrix $(\Gamma_i \cdot \Gamma_j)$. Then there is a unique smallest positive cycle Z with supports in the set Γ such that $-Z$ is f-nef, i.e., satisfying the condition that $Z \cdot \Gamma_i \leq 0$ for all i. This cycle Z is called the fundamental cycle of X (this is defined in (36), but in (33) it is called the numerical cycle of X).

Using the fundamental cycle Z, one can give a precise value for $\operatorname{mult}_P X$ and $\operatorname{embdim}_P X$ if the point P is a rational or Gorenstein elliptic singularity

Theorem 6.1.2 ((36, Corollary 6.)). *If $P \in X$ is a rational singularity on a normal surface X and Z is the fundamental cycle then* $\operatorname{mult}_P = -Z^2$ *and* $\operatorname{embdim}_P X = -Z^2 + 1$.

Theorem 6.1.3 ((33, Chapter 4.)). *Let $P \in X$ be a Gorenstein elliptic singular point on a normal surface X and let $d = -Z^2$ be the degree of P. Then*

(1) if $d = 1, 2$, then $\operatorname{mult}_P X = 2$ *and* $\operatorname{embdim}_P X = 3$*;*

(2) if $d \geq 3$, then $\operatorname{mult}_P X = d$ *and* $\operatorname{embdim}_P X = d$.

Log canonical singularity arose in the minimal model theory of surfaces. The classification theory shows that log canonical singularity is either rational or Gorenstein elliptic. Recall that in the definition of log canonical singularity we always assume that the surface is $\mathbb{Q}$-Gorenstein. The following two propositions can be found in (35).

Theorem 6.1.4. *Let $P \in X$ be a log terminal singular point on a normal surface X. Then it is rational singular.*

Theorem 6.1.5. *Let $P \in X$ be a log canonical singularity which is not log terminal. Let r be the index of K_X at P.*

(1) If $r = 1$, then P is elliptic singular.

(2) If $r \geq 2$, then P is rational singular.

Now we conclude this section by summarizing all above results in the form we shall use later.

Proposition 6.1.6. *Let $P \in X$ be a singularity of a normal surface. Suppose that P is one the following singularity: rational, Gorenstein elliptic and log canonical. Then one has* $\operatorname{mult}_P X \leq \operatorname{embdim}_P X$*.*

6.2 Regularity of dimension zero subschemes

In this section, we study the regularity of dimension zero subschemes of $\mathbb{P}^N$ and give a vanishing theorem for their defining ideal sheaves. Classically, the regularity bound of such schemes is measured by their degrees, i.e., the length of their structure sheaves. The bound obtained by this method is somewhat too weak to use, in particular if we consider nonreduced schemes. Thus in order to get better bounds, nilpotent elements must be involved.

We shall use normality instead of regularity in this section to avoid shifting index in arguments. A subscheme $X \subset \mathbb{P}^N$ is said to be k-normal if the morphism $H^0(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(i)) \to H^0(X, \mathscr{O}_X(i))$ is surjective for all $i \geq k$. Equivalently k-normal means $H^1(\mathbb{P}^N, \mathscr{I}_X(i)) = 0$ for $i \geq k$ where $\mathscr{I}_X$ is the ideal sheaf of X. Notice that X is k normal if and only if it is $k+1$ regular.

We first introduce an invariant for a local Artinian ring to measure the nilpotent elements in the ring.

Definition 6.2.1. Let (A, m) be a local Artinian ring. We define a μ-number associated to A as $\mu_A := \max\limits_i \{i \mid m^i \neq 0\}$. If $m = 0$, i.e., A is a field, then write $\mu_A = 0$. We also write μ instead of μ_A if no confusion arise.

Now let $X \subset \mathbb{P}^N$ be a dimension zero subscheme with the support of distinct point $\{p_1, ..., p_t\}$. If X is reduced then it is well-known that X is t-regular (note that we do not assume points p_i's are in general linear position). If X is nonreduced then the classical way to bound the regularity is to use the length l_X of X , which is the length of structure sheaf $\mathscr{O}_X$. It is easy to see that X is l_X-regular. However in general the length of X could be very large even if X is only supported at one point. Thus in the following theorem we give a bound by considering the number of supporting points and the μ-number of the local ring at each point, which is smaller than the length of X.

Theorem 6.2.2. *Let $X \subset \mathbb{P}^N$ be a zero dimensional subscheme supported at distinct points $\{p_1, ..., p_t\}$ such that for each $1 \leq i \leq t$, $\mu_i = \mu_{\mathscr{O}_{X,p_i}}$. Then one has*

$$H^1(\mathbb{P}^N, \mathscr{I}_X(k)) = 0 \quad \text{for} \;\; k \geq \mu_1 + \mu_2 + \cdots + \mu_t + t - 1,$$

i.e., X is $\mu_1 + \mu_2 + \cdots + \mu_t + t$ regular.

Proof. Assume that $\mu_1 \geq \mu_2 \geq \cdots \mu_t \geq 0$. We prove by descending induction on μ_i.

If $\mu_1 = \cdots \mu_t = 0$, i.e., X is reduced, then the result is classical. For general case, let $(A_i, m_i) = (\mathscr{O}_{X,p_i}, m_i)$ be the local ring of $\mathscr{O}_X$ at point p_i with the maximal ideal m_i. Then by definition $\mu_i = \mu_{A_i}$. Let $j = \max\{i | \mu_i \neq 0\}$. Then for $i > j$, $A_i = k$ and then

$$X = \operatorname{Spec} A_1 \oplus \cdots \oplus \operatorname{Spec} A_j \oplus \operatorname{Spec} k \oplus \cdots \oplus \operatorname{Spec} k.$$

We consider a subscheme of X defined as

$$X_j = \operatorname{Spec} A_1 \oplus \cdots \oplus \operatorname{Spec} A_j / m_j^{\mu_j} \oplus \operatorname{Spec} k \oplus \cdots \oplus \operatorname{Spec} k.$$

By induction, X_j is $a := \mu_1 + \cdots + (\mu_j - 1) + t - 1$ normal. Considering an short exact sequence $0 \to \mathscr{I}_X \to \mathscr{I}_{X_j} \to m_j^{\mu_j} \to 0$, and noticing that $\mathscr{I}_{X_j}$ is a normal, we have an exact sequence

$$H^0(\mathbb{P}^N, \mathscr{I}_{X_j}(a+1)) \xrightarrow{\theta_j} m_j^{\mu_j} \longrightarrow H^1(\mathbb{P}^N, \mathscr{I}_X(a+1)) \longrightarrow 0.$$

All we need is to show that the morphism θ_j is surjective, which we prove as follows.

From the exact sequence $0 \to \mathscr{I}_{X_j} \to \mathscr{O}_{\mathbb{P}^N} \to \mathscr{O}_{X_j} \to 0$, we have an exact sequence

$$0 \longrightarrow H^0(\mathscr{I}_{X_j}(a+1)) \longrightarrow H^0(\mathscr{O}_{\mathbb{P}^N}(a+1)) \xrightarrow{\phi} \mathscr{O}_{X_j} \longrightarrow 0.$$

Assume that m_j is generated by the sections $s_1, \cdots, s_e$ of $H^0(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^n}(1))$, where $1 \leq e \leq N$. Then $m_j^{\mu_j}$ will be generated by the sections of $H^0(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(\mu_j))$ of the form

$$\sigma_{i_1 \cdots i_{\mu_j}} = s_{i_1} \cdots s_{i_{\mu_j}}, \text{ where } 1 \leq i_1 \leq \cdots \leq i_{\mu_j} \leq e.$$

Also for each $i \neq j$ there is a section $l_i \in H^0(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(1))$ such that $l_i \in m_i$ but $l_i \notin m_j$ because of the base point freeness of $\mathscr{O}_{\mathbb{P}^N}(1)$. Then we see that the sections in $H^0(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(a+1))$ of the form

$$s = \sigma_{i_1 \cdots i_{\mu_j}} l_1^{\mu_1+1} \cdots l_t^{\mu_t+1}$$

will satisfy $\phi(s) = 0$ and therefore $s \in H^0(\mathbb{P}^N, \mathscr{I}_{X_j}(a+1))$. Thus those sections will give the surjective morphism $\theta_j : H^0(\mathbb{P}^N, \mathscr{I}_{X_j}(a+1)) \to m_j^{\mu_j} \to 0$. This proves that X is $a+1$ normal. $\square$

The theorem will have a very simple form if X is only supported at one point, which we give in the following corollary.

Corollary 6.2.3. *Let $X \subset \mathbb{P}^N$ be a subscheme supported at one point x with $\mu \geq 0$. Then we have $H^1(\mathbb{P}^N, \mathscr{I}_X(k)) = 0$ for $k \geq \mu$, i.e., X is $\mu + 1$ regular.*

6.3 Eisenbud-Goto conjecture

We prove Eisenbud-Goto conjecture for normal surfaces in this section. The approach we follow is to use generic projections and the duality theory. The generic projection method has been used by Lazarsfeld in (11) and Kwak in (12) and (13). Let X be a nondegenerate surface in $\mathbb{P}^N$ $(N \geq 4)$. Take a general linear space Λ of $\mathbb{P}^N$ of codimension 4 disjoint from X. Blowing up $\mathbb{P}^N$ along the center Λ and then projecting to $\mathbb{P}^3$, we obtain the following diagram

$$\begin{array}{ccc} \mathrm{Bl}_\Lambda \mathbb{P}^N & \xrightarrow{q} & \mathbb{P}^3 \\ {\scriptstyle p}\downarrow & & \\ \mathbb{P}^N. & & \end{array}$$

Denote by $f : X \to \mathbb{P}^3$ the corresponding linear projection of X determined by the center Λ. Considering the morphism $q_*(p^*\mathscr{O}_{\mathbb{P}^N}(2)) \to q_*(p^*\mathscr{O}_X(2))$ induced by $\mathscr{O}_{\mathbb{P}^N}(2) \to \mathscr{O}_X(2)$ and noticing that $q_*(p^*\mathscr{O}_X(2)) = f_*\mathscr{O}_X(2)$, we then get a morphism

$$w_2 : q_*(p^*\mathscr{O}_{\mathbb{P}^N}(2)) \to f_*\mathscr{O}_X(2).$$

If we choose the coordinates of $\mathbb{P}^N$ as $T_0, \cdots, T_N$ such that Λ is defined by the linear forms $T_0 = T_1 = T_2 = T_3 = 0$. Denote by $V =< T_4, \cdots, T_N >$ the vector subspace of $H^0(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(1))$, then we can identify

$$q_*(p^*\mathscr{O}_{\mathbb{P}^N}(2)) = S^2V \otimes \mathscr{O}_{\mathbb{P}^3} \oplus V \otimes \mathscr{O}_{\mathbb{P}^3}(1) \oplus \mathscr{O}_{\mathbb{P}^3}(2),$$

where S^2V is the second symmetric power of V.

Now suppose that X is a normal surface, then X has only finite many singular points. We can choose a general center Λ in such a way that each singular point is the only point in the fiber of the projection f. This can be done because the secant variety through each singular point is at most dimension 3 but Λ has codimension 4. This choice of the center Λ is crucial for our work so that in the sequel we always assume the projection f is determined by a such general center Λ.

We want to give a condition on the singular points of X under which the morphism w_2 is surjective. For this let $y \in \mathbb{P}^3$ be a closed point and let $L_y = q^{-1}(y)$ be the fiber of q over y. Since the point y is cut out by some linear forms l_1, l_2, l_3 in $\mathbb{P}^3$ then the linear space L_y is also cut out by the same forms l_1, l_2, l_3 in $\mathbb{P}^N$. Denote by $X_y = X \cap L_y$ then we see that $f^{-1}(y) = X_y$, the fiber of f over the point y. Notice that X_y is either empty or a dimension zero subscheme of L_y. Restricting the morphism w_2 to the point y we get a following morphism by base change

$$w_2(y) : H^0(L_y, \mathscr{O}_{L_y}(2)) \to H^0(L_y, \mathscr{O}_{X_y}(2)).$$

Thus w_2 is surjective if and only if $w_2(y)$ is surjective for each $y \in \mathbb{P}^3$. If y is not the image of a singular point of X then by the choice of the projection center we see that X_y is cut out by L_y at the nonsingular loci of X. Thus from the classical generic projection theory of nonsingular surface (see (37)) the length of X_y is at most 3 and therefore the morphism $w_2(y)$ is surjective at y. All we need to check is that $w_2(y)$ is surjective if y is the image of a singular point of X. The following is an easy criterion to detect such surjectivity by the local invariants of singular points.

Lemma 6.3.1. *Suppose that $P \in X$ is a singular point such that* $\operatorname{mult}_P X \le \operatorname{embdim}_P X$*, then w_2 is surjective at $f(P)$.*

Proof. Let $y = f(P) \in \mathbb{P}^3$ and let $L_y = q^{-1}(y)$ be the fiber of q over y. Suppose the point y is cut out by linear forms l_1, l_2, l_3 in $\mathbb{P}^3$, then the linear space L_y is also cut out by the forms l_1, l_2, l_3 in $\mathbb{P}^N$. Denote by $X_y = X \cap L_y$. According to the choice of the projection center we have $\operatorname{Supp} X_y = P$. From the base change, it is enough to show that the morphism $H^0(L_y, \mathscr{O}_{L_y}(2)) \to H^0(L_y, \mathscr{O}_{X_y}(2))$ is surjective. Denote by $A = \mathscr{O}_{X,x}$ the local ring of P on X. Locally at the point P, (l_1, l_2) is a regular sequence of A and is in the cotangent space since our projection is general. Thus if denote by $B = A/(l_1, l_2)$, we have $\operatorname{mult}_P X = l(B)$ and $\operatorname{embdim}(B) = \operatorname{embdim}_P X - 2$. By assumption that $\operatorname{mult}_P X \le \operatorname{embdim}_P X$, we see that $l(B) \le \operatorname{embdim} B + 2$ which implies that the μ-number μ_B associated to the artinian ring B satisfies the inequality $\mu_B \le 2$. Thus the μ-number μ of the local ring $\mathscr{O}_{X_y,P}$ is less than 2 since $\mathscr{O}_{X_y,P} = B/(l_3)$. Then from the exact sequence $0 \to \mathscr{I}_{X_y/L_y} \to \mathscr{O}_{L_y} \to \mathscr{O}_{X_y} \to 0$ and the Corollary 6.2.3, we obtain immediately the surjectivity of the morphism $H^0(L_y, \mathscr{O}_{L_y}(2)) \to H^0(L_y, \mathscr{O}_{X_y}(2))$, which proves the lemma. $\square$

Corollary 6.3.2. *Suppose that X has one of the following singularities: rational, Gorenstein elliptic and log canonical. Then the morphism w_2 is surjective.*

Proof. This is immediately by Proposition 6.1.6. $\square$

Remark 6.3.3. In fact we can get a general condition to guarantee the surjectivity of w_2. Let $P \in X$ be a singular point. Let $(A, m) = (\mathscr{O}_{X,P}, m_P)$ be the local ring at P. The surjectivity of

w_2 at the point $y = f(P)$ can be obtained under the following condition: for a general regular sequence l_1, l_2, l_3 in A let $B = A/(l_1, l_2, l_3)$ then the μ-number of B is less than 2.

The Serre duality theory will play an important role in our proof of the main theorem in this section. Suppose that $V \subset \mathbb{P}$ is a Cohen-Macaulay subvariety in a projective space $\mathbb{P}$ of codimension c and dimension n. Its dualizing sheaf ω_V is defined by the formula $\omega_V = \mathscr{E}\mathrm{xt}^c_{\mathscr{O}_{\mathbb{P}}}(\mathscr{O}_V, \omega_{\mathbb{P}})$ where $\omega_{\mathbb{P}}$ is the dualizing sheaf of $\mathbb{P}$. The Serre duality theorem says that for any locally free sheaf $\mathscr{E}$ on X one has $H^i(X, \mathscr{E}) \simeq H^{n-i}(X, \mathscr{E}^\vee \otimes \omega_V)^\vee$ for all $i \geq 0$. Dualizing sheaves can be transferred by a finite morphism. Specifically, suppose that $f : V \to W$ be a finite morphism of Cohen-Macaulay projective varieties such that $\operatorname{codim}_W f(V) = c$, then one has $f_*\omega_V = \mathscr{E}\mathrm{xt}^c_{\mathscr{O}_W}(f_*\mathscr{O}_V, \omega_W)$.

The following lemma is a Kodaira type vanishing theorem for a normal surface, which may be known to experts.

Lemma 6.3.4. *Let X be a projective normal surface and L a very ample line bundle on X. Then $H^0(X, L^{-1}) = H^1(X, L^{-1}) = 0$.*

Proof. Since L is very ample, we have $H^0(X, L^{-1}) = 0$. Let $f : \widetilde{X} \to X$ be a resolution of singularities. Since X is normal, $f_*\mathscr{O}_{\widetilde{X}} = \mathscr{O}_X$. From the exact sequence associated to the spectral sequence of higher direct image of f^*L^{-1}, we have

$$0 \to H^1(X, L^{-1}) \to H^1(\widetilde{X}, f^*L^{-1}) \to H^0(X, R^1 f_* f^* L^{-1}).$$

By Kodaira vanishing theorem that $H^1(\widetilde{X}, f^*L^{-1}) = 0$, we see that $H^1(X, L^{-1}) = 0$. $\square$

Corollary 6.3.5. *Let ω_X be the dualizing sheaf of X, then $H^1(X, \omega_X \otimes L) = H^2(X, \omega_X \otimes L) = 0$.*

Now we come to our main theorem to prove Eisenbud-Goto conjecture for a normal surfaces by assuming the morphism w_2 is surjective. Our proof relies on the duality theory.

Theorem 6.3.6. *Let $X \subset \mathbb{P}^N$ be a nondegenerate normal surface and suppose that w_2 is surjective, then* $\operatorname{reg} X \leq \deg X - \operatorname{codim} X + 1$.

Proof. Recall that we choose coordinates of $\mathbb{P}^N$ such that Λ is defined by $T_0 = T_1 = T_2 = T_3 = 0$ and denote by $V =< T_4, ..., T_N >$ the vector subspace of $H^0(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(1))$, then

$$q_*(p^*\mathscr{O}_{\mathbb{P}^N}(2)) = S^2V \otimes \mathscr{O}_{\mathbb{P}^3} \oplus V \otimes \mathscr{O}_{\mathbb{P}^3}(1) \oplus \mathscr{O}_{\mathbb{P}^3}(2),$$

where S^2V is the second symmetric power of V. Twisting w_2 by $\mathscr{O}_{\mathbb{P}^3}(-2)$ first and let E be the kernel, then we have an exact sequence

$$0 \longrightarrow E \longrightarrow S^2V \otimes \mathscr{O}_{\mathbb{P}^3}(-2) \oplus V \otimes \mathscr{O}_{\mathbb{P}^3}(-1) \oplus \mathscr{O}_{\mathbb{P}^3} \longrightarrow f_*\mathscr{O}_X \longrightarrow 0. \tag{6.1}$$

Since X is Cohen-Macaulay and f is finite, $f_*\mathscr{O}_X$ is a sheaf of codimension one Cohen-Macaulay $\mathscr{O}_{\mathbb{P}^3}$-module and therefore E is a locally free sheaf of rank

$$r = \operatorname{rank} S^2V \otimes \mathscr{O}_{\mathbb{P}^3}(-2) \oplus V \otimes \mathscr{O}_{\mathbb{P}^3}(-1) \oplus \mathscr{O}_{\mathbb{P}^3} = \frac{(N-2)(N-1)}{2}.$$

We claim that $E^\vee$ is (-2)-regular. To see this, let ω_X be the dualizing sheaf of X. Applying $\mathscr{H}\mathrm{om}(\_\,, \omega_{\mathbb{P}^3})$ to the exact sequence of (6.1), we get an exact sequence

$$0 \longrightarrow S^2V \otimes \omega_{\mathbb{P}^3}(2) \oplus V \otimes \omega_{\mathbb{P}^3}(1) \oplus \omega_{\mathbb{P}^3} \longrightarrow E^\vee(-4) \longrightarrow f_*\omega_X \longrightarrow 0. \tag{6.2}$$

Twisting it by $\mathscr{O}_{\mathbb{P}^3}(1)$ and taking H^1 cohomology, we see that $H^1(\mathbb{P}^3, E^\vee(-3)) = 0$. Then taking H^2 cohomology of the exact sequence (6.2), we have

$$0 \longrightarrow H^2(\mathbb{P}^3, E^\vee(-4)) \longrightarrow H^2(\mathbb{P}^3, f_*\omega_X) \longrightarrow H^3(\mathbb{P}^3, \omega_{\mathbb{P}^3}) \longrightarrow \cdots .$$

Since $H^2(\mathbb{P}^3, f_*\omega_X) = H^2(X, \omega_X) = H^3(\mathbb{P}^3, \omega_{\mathbb{P}^3}) = k$ we obtain $H^2(\mathbb{P}^3, E^\vee(-4)) = 0$. For H^3 cohomology of $E^\vee$, twist the exact sequence (6.2) by $\mathscr{O}_{\mathbb{P}^3}(-1)$ and then take H^3 cohomology to get an exact sequence

$$H^2(f_*\omega_X(-1)) \xrightarrow{\theta} H^3(\mathbb{P}^3, V \otimes \omega_{\mathbb{P}^3} \oplus \omega_{\mathbb{P}^3}(-1)) \longrightarrow H^3(\mathbb{P}^3, E^\vee(-5)) \longrightarrow 0.$$

We shall show the morphism θ is surjective. By duality, it is the same as

$$H^0(X, \mathscr{O}_X(1))^\vee \longrightarrow H^0(\mathbb{P}^3, V \otimes \mathscr{O}_{\mathbb{P}^3} \oplus \mathscr{O}_{\mathbb{P}^3}(1))^\vee$$

which is the dual of the morphism

$$H^0(\mathbb{P}^3, V \otimes \mathscr{O}_{\mathbb{P}^3} \oplus \mathscr{O}_{\mathbb{P}^3}(1)) \longrightarrow H^0(X, \mathscr{O}_X(1)).$$

Note that $H^0(\mathbb{P}^3, V \otimes \mathscr{O}_{\mathbb{P}^3} \oplus \mathscr{O}_{\mathbb{P}^3}(1)) = H^0(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(1))$. Since X is nondegenerate in $\mathbb{P}^N$ the morphism $H^0(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(1)) \longrightarrow H^0(X, \mathscr{O}_X(1))$ is injective and therefore θ is surjective. Thus we obtain $H^3(\mathbb{P}^3, E^\vee(-5)) = 0$ and conclude that $E^\vee$ is (-2)-regular.

Back to the exact sequence (6.1) and let $d = \deg X$. Since $\operatorname{Supp} f_*\mathscr{O}_X$ is a degree d hypersurface of $\mathbb{P}^3$ we have

$$\begin{aligned} c_1(E) &= -d + c_1(S^2V \otimes \mathscr{O}_{\mathbb{P}^3}(-2) \oplus V \otimes \mathscr{O}_{\mathbb{P}^3}(-1) \oplus \mathscr{O}_{\mathbb{P}^3}) \\ &= -d - (N-1)(N-3), \end{aligned}$$

and therefore $\det E = \mathscr{O}_{\mathbb{P}^3}(-d-(N-1)(N-3))$. Now from the canonical identity that $E = (\wedge^{r-1}E)^\vee \otimes \det E$, we have that E is $(-2)(r-1)+d+(N-1)(N-3)$-regular, i.e. $(d-N+3)$-regular. From the exact sequence (6.1), we also get that $f_*\mathscr{O}_X$ and hence $\mathscr{O}_X$ is $(d-N+3)$-regular. Finally, using (11, Lemma 1.5.), we conclude that $\operatorname{reg} X = (d-N+3)$. $\square$

Now combining with Corollary 6.3.2, we obtain our main result of this section.

Corollary 6.3.7. *Suppose that a normal surface $X \subset \mathbb{P}^N$ has the following singularities: rational, Gorenstein elliptic and log canonical. Then one has* $\operatorname{reg} X \leq \deg X - \operatorname{codim} X + 1$.

CHAPTER 7

M-REGULARITY OF CURVES IN ABELIAN VARITIES

This chapter is joint work with Luigi Lombardi. An abelian variety is a variety equipped with a group structure. A polarized abelian variety is an abelian variety with an assigned ample line bundle. The geometry of polarized abelian varieties are very similar to the geometry of projective spaces. The interesting and surprising parallel between these two objects has been shown in a beautiful lecture: Projective vs. abelian Geometry, given by Popa in University of Illinois at Chicago 2010. According to this philosophy, we are interested in a parallel theory to the Castelnuovo-Mumford regularity theory on projective spaces.

In the last ten years a Mukai regularity, or M-regularity, theory bas been developed systematically by Pareschi and Popa in a series of paper (14), (15) and (16). It turns out that this theory is a good analogue to the Castelnuovo-Mumford regularity theory. In their work they introduced the notion of M-regularity of a coherent sheaf on an abelian variety and establish fundamental properties of this concept.

The motivation of this chapter is to explore geometric aspects of M-regularity theory. As a beginning of such an attempt in this direction, we consider a problem of M-regularity of nonsingular curves in abelian varieties. Recall that at the beginning of Chapter 6, we mentioned a classical result proved by Gruson, Lazarsfeld and Peskine that if $X \subset \mathbb{P}^N$ $(N \geq 3)$ is a reduced irreducible nondegenerate curve of degree d, then $\operatorname{reg} X \leq d - N + 2$. This result can also be written as $\operatorname{reg} X \leq \deg X - \operatorname{codim} X + 1$. Analogously, we consider a polarized abelian variety

(A, L) and let $X \subset A$ be nonsingular curve. Twisting $\mathscr{I}_X$ by L^n we see that for n sufficiently large $\mathscr{I}_X \otimes L^n$ is always M-regular. Thus an interesting question is if we can find such an n as small as possible, or equivalently find a bound for the M-regularity of X (for more precise description see Definition 7.3.1).

7.1 M-regularity on abelian varieties

In this section, we briefly review the M-regularity theory on abelian varieties, developed systematically by Pareschi and Popa in (14), (15) and (16). It can be viewed as an analogue to the one of Castelnuovo-Mumford regularity on projective spaces.

Let X be an abelian variety of dimension r. Denote by $\widehat{X}$ its dual variety $\mathrm{Pic}^0(X)$. Consider the fiber product diagram

$$\begin{array}{ccc} X \times \widehat{X} & \xrightarrow{p_{\widehat{X}}} & \widehat{X} \\ {\scriptstyle p_X}\big\downarrow & & \\ X & & \end{array}$$

We fix a Poincare line bundle $\mathcal{P}$ on $X \times \widehat{X}$, properly normalized, that is for any closed point $\alpha \in \widehat{X}$, restricting $\mathcal{P}$ to the fiber X_α over α we get $\mathcal{P}_\alpha = \alpha$ and restricting $\mathcal{P}$ to the fiber $\widehat{X}_0$ over $0 \in X$ we get $\mathscr{O}_{\widehat{X}}$. For more detailed discussion on the dual variety $\widehat{X}$ and Poincare line bundles see (38, Section 13.). For a coherent sheaf $\mathscr{F}$ on X, we define its Fourier-Mukai transformation as

$$\widehat{S}(\mathscr{F}) := Rp_{\widehat{X}*}(p_X^* \mathscr{F} \otimes \mathcal{P}).$$

Note that in the transformation above the operations in $p_X^* \mathscr{F} \otimes \mathcal{P}$ are the usual pullback and tensor product since p_X is flat and $\mathcal{P}$ is flat over X. However, the push-forward $Rp_{\widehat{X}*}$ is a derived push-forward in derived categories. Thus $\widehat{S}(\mathscr{F})$ is actually a complex in the derived category of coherent sheaves on $\widehat{X}$.

For a complex $\mathscr{C}_\bullet$, we define $R^i\mathscr{C}_\bullet$ as its i-th cohomology $\mathscr{H}^i(\mathscr{C}_\bullet)$. For the complex $\widehat{S}(\mathscr{F})$ we have $R^i\widehat{S}(\mathscr{F}) := \mathscr{H}^i(\widehat{S}(\mathscr{F})) = R^i p_{\widehat{X}*}(p_X^*\mathscr{F} \otimes \mathcal{P})$, i.e., the i-th higher direct image of the sheaf $p_X^*\mathscr{F} \otimes \mathcal{P}$ on $\widehat{X}$ under the projection $p_{\widehat{X}}$. We denote by $S^i(\mathscr{F}) := \operatorname{Supp} R^i p_{\widehat{X}*}(p_X^*\mathscr{F} \otimes \mathcal{P})$.

Now we give the definition of M-regular for a coherent sheaf on a abelian variety.

Definition 7.1.1 (M-regular, (14, Definition 2.1.))**.** A coherent sheaf $\mathscr{F}$ on X is called Mukai regular or M-regular, if $\operatorname{codim} S^i(\mathscr{F}) > i$ for $i = 1, \ldots, g$, where for $i = g$, this means that $S^g(\mathscr{F})$ is empty.

In practise it is very hard to check directly from the definition the M-regularity of a coherent sheaf. If the coherent sheaf satisfies some stronger vanishing properties then it is possible to check its M-regularity. For this we need the following definition.

Definition 7.1.2. A coherent sheaf $\mathscr{F}$ satisfies the index theorem with index i (I.T.i, for short) if $h^j(X, \mathscr{F} \otimes \alpha) = 0$, for all $\alpha \in \operatorname{Pic}^0(X)$ and all $j \neq i$.

In this chapter, we are mainly interested in the coherent sheaves which satisfies I.T.0. The reason for this is given in the following two easy lemmas. They show that I.T.0 implies M-regularity and any ample line bundle satisfies I.T.0 automatically.

Lemma 7.1.3. *If a coherent sheaf $\mathscr{F}$ satisfies I.T.0 then it is M-regular.*

Proof. Notice that $p_X^*\mathscr{F} \otimes \mathcal{P}$ is flat over $\widehat{X}$ because $\mathscr{F}$ is flat over k and $\mathcal{P}$ is flat over $\widehat{X}$. Then for any $\alpha \in \widehat{X} = \operatorname{Pic}^0(X)$ and each $i = 1, \ldots, r$ the natural morphism

$$R^i\widehat{S}(\mathscr{F})\otimes k(\alpha)\longrightarrow H^i(X,\mathscr{F}\otimes\alpha)=0,$$

is surjective and therefore $R^i\widehat{S}(\mathscr{F})\otimes k(\alpha)=0$ by (17, Proposition III.12.11.). This immediately implies that $R^i\widehat{S}(\mathscr{F})=0$ for $i>0$. □

Lemma 7.1.4. *If L is an ample line bundle on X then L satisfies I.T.0 and L is M-regular.*

Proof. Since for any $\alpha\in\mathrm{Pic}^0(X)$, $L\otimes\alpha$ is ample and then the result follows from Kodaira vanishing theorem. □

In developing M-regularity theory Pareschi and Popa introduced the notion of globally continuously generated for a coherent sheaf. It turned out to be an important and useful concept in the M-regularity theory. The concept of globally continuously generated coherent sheaves can be defined for any irregular variety Y, i.e., the variety such that $H^1(\mathscr{O}_Y)\neq 0$ and therefore $\mathrm{Pic}^0(Y)$ is defined.

Definition 7.1.5. Let Y be an irregular variety. We define a sheaf $\mathscr{F}$ on Y to be *continuously globally generated* if for any non-empty open subset $Y\subset\mathrm{Pic}^0(Y)$ the sum of evaluation maps

$$\bigoplus_{\alpha\in U}H^0(\mathscr{F}\otimes\alpha)\otimes\alpha^\vee\to\mathscr{F}$$

is surjective.

Remark 7.1.6. In practise we shall use a simplified equivalent definition of continuously globally generated coherent sheaves (for more details see (15, Remark 2.2.)). Specifically, suppose that $\mathscr{F}$ is continuously globally generated, then for any point $x\in X$, there are finitely many sections $s_j\in H^0(X,\mathscr{F}\otimes\alpha_i)$ such that s_j's generate $\mathscr{F}\otimes\alpha_i$ at x and therefore they generate

$\mathscr{F} \otimes \alpha_i$ at a neighborhood of x. Covering X by finitely many such open sets, we then see that there is a positive integer N such that for general α_1,...,$\alpha_N \in \mathrm{Pic}^0(X)$, the map

$$\bigoplus_{i=1}^{N} H^0(\mathscr{F} \otimes \alpha_i) \otimes \alpha_i^{\vee} \to \mathscr{F}$$

is surjective. Notice that the number N only depends on $\mathscr{F}$. This can be taken as an equivalent definition of continuously globally generated, which we shall use frequently in the sequel.

The property of continuously globally generated is preserved by tensor products in the following proposition.

Proposition 7.1.7 ((14, Proposition 2.12.))**.** *Let X be an irregular variety and $\mathscr{F}$ a coherent sheaf and L a line bundle on X, both continuously globally generated. Then $\mathscr{F} \otimes L$ is globally generated.*

The crucial point we will use the concept of continuously globally generated is that an M-regular sheaf is always continuously globally generated.

Proposition 7.1.8 ((14, Proposition 2.13.))**.** *If $\mathscr{F}$ is M-regular, then there is a positive integer N such that for general $\alpha_1, ..., \alpha_N \in Pic^0(X)$, the sum of twisted evaluation maps*

$$\bigoplus_{i=1}^{N} H^0(\mathscr{F} \otimes \alpha_i) \otimes \alpha_i^{\vee} \to \mathscr{F}$$

is surjective.

We already know in Lemma 7.1.3 that I.T.0 implies M-regular. But in general the converse is not true. However the following proposition states that after tesoring a I.T.0 locally free sheaf, we may turn M-regular into I.T.0. In this sense we see that being M-regular is not so far from being I.T.0.

Proposition 7.1.9 ((14, Proposition 2.9.))**.** *Let $\mathscr{F}$ be an M-regular coherent sheaf on X and E a locally free sheaf satisfying I.T.0 Then $\mathscr{F} \otimes E$ satisfies I.T.0.*

In particular, we see an I.T.0 coherent sheaf tensoring with an I.T.0 locally free sheaf still gets an I.T.0. coherent sheaf.

Corollary 7.1.10. *Let $\mathscr{F}$ be an coherent sheaf and E a locally free sheaf on X. Assume that they both satisfy I.T.0. Then $\mathscr{F} \otimes E$ satisfies I.T.0.*

We also can obtain a type of Kodaira vanishing theorem for an M-regular coherent sheaf.

Corollary 7.1.11. *Let $\mathscr{F}$ be an M-regular and L an ample line bundle. Then $\mathscr{F} \otimes L$ is I.T.0, i.e. $H^i(\mathscr{F} \otimes L \otimes \alpha) = 0$ for all $\alpha \in Pic^0(X)$ and all $i > 0$.*

7.2 Vanishing theorem of the graph

Throughout the rest of this chapter we fix (A, L) a polarized abelian variety of dimension r where L is an ample line bundle. The abelian variety A has a unique origin 0: it is a closed point corresponding to 0 in the underlying abelian group structure of A. We denote by $\mathscr{I}_0$ the defining ideal sheaf of the origin 0 in A. There is also a difference morphism $\delta : A \times A \to A$ defined as $\delta(x, y) = x - y$. Notice that the pullback of $\mathscr{I}_0$ under δ will give the defining ideal sheaf of the diagonal subvariety of $A \times A$.

One way to measure the positivity of L is to use the Seshadri constant $\varepsilon_0(A, L)$ at the point 0. It is defined by the reciprocal of the s-invariant of $\mathscr{I}_0$ introduced in Section 5.1 Definition

5.1.1. Since A is an abelian variety the group action of A to itself will show that $\varepsilon_0(A, L)$ is equal to the Seshadri constant $\varepsilon_p(A, L)$ at any other closed point p of A. Thus we get a unique number $\varepsilon = \varepsilon_0(A, L)$ associated to L and call it as the Seshadri constant of L. From (19, Example 5.1.10.) one has $\varepsilon \geq 1$.

The purpose of this section is to establish a vanishing theorem for the ideal sheaf of the graph of any nonsingular subvariety of A, involving the line bundle L. The crucial point is that such ideal sheaf actually can be realized as a multiplier ideal sheaf for some effective $\mathbb{Q}$-divisor and then Nadel's vanishing theorem can be applied. This method has been used by Lazarsfeld, Pareschi and Popa (39) in the studying of syzygies of abelian varieties. Our approach is inspired by their work.

The first observation is that this ideal sheaf $\mathscr{I}_0$ can be realized as a multiplier ideal sheaf associated to an effective $\mathbb{Q}$-divisor on A.

Proposition 7.2.1. *Let $c > 1/\varepsilon$ be a rational number. Then there exists an effective $\mathbb{Q}$-divisor F_0 numerically equivalent to the $\mathbb{Q}$-divisor crL such that the associated multiplier ideal sheaf $\mathscr{I}(A, F_0) = \mathscr{I}_0$.*

Proof. Let $\mu : A' = \mathrm{Bl}_0 A \to A$ be the blowing-up of A along 0 with an exceptional divisor E. Notice that the relative canonical divisor $K_{A'/A} = (r-1)E$. The crucial point is to construct an effective $\mathbb{Q}$-divisor F_0 such that it has a round-down divisor $\lfloor \mu^* F_0 \rfloor = rE$.

From the definition of the Seshadri constant ε and assumption that $c > 1/\varepsilon$, we see that the $\mathbb{Q}$-divisor $cr\mu^*L - rE$ is ample. Thus we can take m a sufficiently large positive integer to get a very ample integral divisor $m(cr\mu^*L - rE)$. Then take a general smooth irreducible element

$D' \in |m(cr\mu^*L - rE)|$ and let $D = \mu(D')$ be the image of D' on A which is a prime divisor on A, we see that D' is the strict transformation of D on A'.

We compute the multiplicity $\operatorname{mult}_0 D$ of D at the point 0 by using the formula $\operatorname{mult}_0 D = (-1)^r D' \cdot E^{r-1}$. Notice that D' is linear equivalent to the divisor $mcr\mu^*L - mrE$ and that the value $\mu^*L \cdot E^{r-1} = 0$. We then get $\operatorname{mult}_0 D = mr(-1)^{r+1}E^r$. But $(-1)^{r+1}E^r = \operatorname{mult}_0 A = 1$ since A is nonsingular at 0 and therefore $\operatorname{mult}_0 D = mr$. Thus we see $\mu^*D = D' + mrE$ and this implies D is numerically equivalent to $mcrL$ since D' is linearly equivalent to $m(cr\mu^*L - rE)$.

Finally we define an effective $\mathbb{Q}$-divisor $F_0 = \dfrac{1}{m}D$. Then F_0 is numerically equivalent to the divisor crL. Also the pullback of F_0 by μ is $\mu^*F_0 = \dfrac{1}{m}D' + rE$. So we get $K_{A'/A} - \lfloor \mu^*F_0 \rfloor = -E$ and therefore we have $\mathcal{J}(A, F_0) = \mu_*(-E) = \mathscr{I}_0$. □

Lemma 7.2.2. *Let X be a nonsingular subvariety of A (could be A itself). Let $\sigma : X \times A \to A$ be the composition of the morphisms $X \times A \hookrightarrow A \times A \xrightarrow{\delta} A$. Then σ is a smooth morphism.*

Proof. This is because by generic smoothness and action of A on $X \times A$ which is compatible with σ. □

Notation 7.2.3. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be coherent sheaves on varieties V_1 and V_2 respectively. We write $\mathscr{F}_1 \boxtimes \mathscr{F}_2 = p_1^*\mathscr{F}_2 \otimes p_2^*\mathscr{F}_2$ where p_1 and p_2 are projections from the fiber product $V_1 \times V_2$ to V_1 and V_2 respectively.

Proposition 7.2.4. *Let X be a nonsingular subvariety of A (could be A itself) and let Γ be the graph of X in $X \times A$ defined by an ideal sheaf $\mathscr{I}_\Gamma$. Then there is an effective $\mathbb{Q}$-divisor F_0 on A, numerically equivalent to the $\mathbb{Q}$-divisor crL with $c > 1/\varepsilon$ a rational number, such that the* *multiplier ideal sheaf* $\mathcal{J}(X \times A, \sigma^* F_0)$ *associated to* $\sigma^* F_0$ *is equal to* $\mathscr{I}_\Gamma$ *where* $\sigma : X \times A \to A$ *is the composition of the morphisms* $X \times A \hookrightarrow A \times A \xrightarrow{\delta} A$.

Proof. Let F_0 be an effective $\mathbb{Q}$-divisor given by Proposition 7.2.1. Since the morphism σ is smooth by Lemma 7.2.2, we then have $\sigma^*\mathcal{J}(A, F_0) = \mathcal{J}(X \times A, \sigma^* F_0)$. Let $\mathscr{I}_0$ be the defining ideal sheaf of the origin 0 on A. Notice that $\sigma^*\mathscr{I}_0 = \mathscr{I}_\Gamma$. Now using Proposition 7.2.1 again we obtain immediately $\mathcal{J}(X \times A, \sigma^* F_0) = \mathscr{I}_\Gamma$. □

Proposition 7.2.5. *Keep notation as in Proposition 7.2.4. One has*

$$H^i(X \times A, \omega_{X\times A} \otimes M \otimes \mathscr{I}_\Gamma) = 0, \qquad \textit{for all } i > 0,$$

where M *is a line bundle on* $X \times A$ *such that* $M - 2cr(L_X \boxtimes L)$ *is nef and big and* $\omega_{X\times A}$ *is the canonical line bundle of* $X \times A$.

Proof. Note that on $A \times A$, we have $L^2 \boxtimes L^2 = \delta^* L \otimes N$ for some nef divisor N. Denote by L_X as the restriction of L to X. Then on $X \times A$, $L_X^2 \boxtimes L^2 = \sigma^* L \otimes P$ where P is an nef divisor on $X \times A$ (in fact P is the restriction of N). Thus $\sigma^* L = L_X^2 \boxtimes L^2 \otimes P^{-1}$. Note also that $\mathscr{I}_\Gamma = \mathcal{J}(X \times A, cr\sigma^* L)$ by Proposition 7.2.4. Thus if $M - crL_C^2 \boxtimes L^2$ is nef and big, then $M - crL_C^2 \boxtimes L^2 + crP$ is nef and big and therefore $M - cr\sigma^* L$ is nef and big. Then from Nadel's vanishing theorem, we get the result. □

Theorem 7.2.6. *Let* $X \subset A$ *be a smooth subvariety of* A *(could be* A*) and* L *be an ample divisor on* A *with the Seshadri constant* ε*. Let* Γ *be the graph of* X *in* $X \times A$ *defined by an ideal sheaf* $\mathscr{I}_\Gamma$*. Then one has*

$$H^i(X \times A, (\omega_X \otimes B_1 \otimes L_X^{\lfloor \frac{2r}{\varepsilon} \rfloor + 1}) \boxtimes (B_2 \otimes L^{\lfloor \frac{2r}{\varepsilon} \rfloor + 1}) \otimes \mathscr{I}_\Gamma) = 0 \quad \textit{for } i > 0,$$

where ω_X is the canonical line bundle of X, B_1 is a nef line bundle on X and B_2 is a nef line bundle on A.

Proof. In Proposition 7.2.5, we can choose the rational number $c > \dfrac{1}{\varepsilon}$ so that $2cr - \dfrac{2r}{\varepsilon} \ll 1$ and $\lfloor \dfrac{2r}{\varepsilon} \rfloor + 1 > 2cr$. Since the divisor $(\lfloor \dfrac{2r}{\varepsilon} \rfloor + 1)(L \boxtimes L)$ is big and nef on $A \times A$ then the divisor $(\lfloor \dfrac{2r}{\varepsilon} \rfloor + 1)(L_X \boxtimes L)$ is big and nef on $X \times A$. Let $M = (B_1 \boxtimes B_2) + (\lfloor \dfrac{2r}{\varepsilon} \rfloor + 1)(L_X \boxtimes L)$. Then we see $M - 2cr(L_X \boxtimes L)$ is big and nef. Also notice that $\omega_{X \times A} = \omega_X \boxtimes \mathscr{O}_A$. Then the result follows immediately from Proposition 7.2.5. $\square$

Corollary 7.2.7. *Keep notation as in Theorem 7.2.6. Assume that $\varepsilon > 2r$. Then one has*

$$H^i(X \times A, (\omega_X \otimes B_1 \otimes L_X) \boxtimes (B_2 \otimes L) \otimes \mathscr{I}_\Gamma) = 0 \quad \textit{for } i > 0,$$

where ω_X is the canonical line bundle of X, B_1 is a nef line bundle on X and B_2 is a nef line bundle on A.

Corollary 7.2.8. *Assume that $\varepsilon > 2r$ and let Γ be the diagonal of $A \times A$ defined by an ideal sheaf $\mathscr{I}_\Gamma$. Then one has*

$$H^i(A \times A, (B_1 \otimes L) \boxtimes (B_2 \otimes L) \otimes \mathscr{I}_\Gamma) = 0 \quad \textit{for } i > 0,$$

where B_1 and B_2 are nef line bundles on A.

7.3 M-regularity of curves

In this section, using the vanishing theorem of graph established in previous section, we study the M-regularity of curves in an abelian variety. Let us first make a precise definition on the M-regularity of an subvariety.

Definition 7.3.1. Let (A, L) be a polarized abelian variety and $\mathscr{F}$ be a coherent sheaf on A. For an integer n we say $\mathscr{F}$ is n M-regular (with respect to L) if $\mathscr{F} \otimes L^n$ is M-regular. For a subvariety X of A we say X is n M-regular if its defining ideal sheaf $\mathscr{I}_X$ is n M-regular.

It is easy to see that any coherent sheaf $\mathscr{F}$ is always n M-regular for n is sufficiently large. Furthermore from Corollary 7.1.11, we see if $\mathscr{F}$ is n M-regular then it is $n+1$ M-regular. Thus we hope to find such n as smaller as possible. The first nontrivial situation we are interested in is to study the M-regularity of a nonsingular curve in A. This problem can be compared with the work of Gruson, Lazarsfeld and Peskine (10) on the study of Castelnuovo-Mumford regularity bounds for curves in projective spaces.

Recall that (A, L) is a polarized abelian variety of dimension r where L is an ample line bundle with the Seshadri constant ε. Let C be a nonsingular curve in A. We denote by Γ the graph of C in the product $C \times A$ defined by an ideal sheaf $\mathscr{I}_\Gamma$. Fix the fiber product diagram

$$\begin{array}{ccc} C \times A & \xrightarrow{q} & A \\ \big\downarrow{\scriptstyle p} & & \\ C. & & \end{array}$$

We denote by L_C the restriction of L on C. Let $B = \omega_C \otimes N \otimes L_C^{\lfloor \frac{2r}{\varepsilon} \rfloor + 1}$ be a line bundle on C, where N is a nef line bundle on C. We will eventually choose an appropriated nef line bundle N. Before that just think of N as a variant nef line bundle in B.

Lemma 7.3.2. *One has $R^i q_*(p^*B \otimes \mathscr{I}_\Gamma) = 0$, for all $i > 0$.*

Proof. All we need to show is that $R^1 q_*(p^*B \otimes \mathscr{I}_\Gamma) = 0$ since q has fibers of dimension at most one. In fact by Theorem 7.2.6 we have $H^i(C \times A, (\omega_C \otimes N \otimes L_C^{\lfloor \frac{2r}{\varepsilon} \rfloor + 1}) \boxtimes L^m \otimes \mathscr{I}_\Gamma) = 0$ for all $i > 0$ and m sufficiently large. Then by Proposition 2.0.11, we obtain that $R^i q_*(p^*B \otimes \mathscr{I}_\Gamma) = 0$, for all $i > 0$. $\square$

Tensoring p^*B to a short exact sequence $0 \to \mathscr{I}_\Gamma \to \mathscr{O}_{C \times A} \to \mathscr{O}_\Gamma \to 0$ we have a short exact sequence $0 \to p^*B \otimes \mathscr{I}_\Gamma \to p^*B \to B \to 0$ on $C \times A$. Pushing it forward to A by the morphism q and using $R^1 q_*(p^*B \otimes \mathscr{I}_\Gamma) = 0$ by Lemma 7.3.2, we get a short exact sequence on A

$$0 \to q_*(p^*B \otimes \mathscr{I}_\Gamma) \to H^0(C, B) \otimes \mathscr{O}_A \to B \to 0. \tag{7.1}$$

Lemma 7.3.3. *Keeping notation as above, one has the sheaf $q_*(p^*B \otimes \mathscr{I}_\Gamma) \otimes L^{\lfloor \frac{2r}{\varepsilon} \rfloor + 1}$ is continuously globally generated.*

Proof. First, we notice that for any integer $a \geq 0$, by Kodaira vanishing theorem $B \otimes L^a$ satisfies I.T.0, i.e., for any $\alpha \in \mathrm{Pic}^0(A)$, one has $H^i(A, B \otimes L^a \otimes \alpha) = 0$ for $i > 0$. Second, for any integer $a \geq 1$, by the ampleness of L, one has $H^0(X, B) \otimes L^a$ also satisfies I.T.0.

Now for any $\alpha \in \mathrm{Pic}^0(A)$ we consider $q_*(p^*B \otimes \mathscr{I}_\Gamma) \otimes L^a \otimes \alpha$. By Theorem 2.0.10, there is a spectral sequence

$$E_2^{ij} = H^i(A, R^j q_*(B \boxtimes (L^a \otimes \alpha) \otimes \mathscr{I}_\Gamma)) \Rightarrow H^{i+j}(C \times A, B \boxtimes (L^a \otimes \alpha) \otimes \mathscr{I}_\Gamma),$$

Notice that $R^j q_*(B \boxtimes (L^a \otimes \alpha) \otimes \mathscr{I}_\Gamma) = R^j q_*(p^* B \otimes \mathscr{I}_\Gamma) \otimes L^a \otimes \alpha$ by projection formula. Thus from the vanishing $R^j q_*(p^* B \otimes \mathscr{I}_\Gamma) = 0$ for $j > 0$ by Lemma 7.3.2 , we see that the spectral sequence degenerates and therefore we have

$$H^i(A, q_*(p^* B \otimes \mathscr{I}_\Gamma) \otimes L^a \otimes \alpha) = H^i(C \times A, B \boxtimes (L^a \otimes \alpha) \otimes \mathscr{I}_\Gamma).$$

Thus with reference to Theorem 7.2.6, we choose $a = \lfloor \frac{2r}{\varepsilon} \rfloor + 1$. Then the sheaf $q_*(p^* B \otimes \mathscr{I}_\Gamma) \otimes L^{\lfloor \frac{2r}{\varepsilon} \rfloor + 1}$ satisfies I.T.0 and therefore is M-regular by Lemma 7.1.3. Thus it is continuously globally generated by Proposition 7.1.8. $\square$

Now we come to our main theorem of this chapter to give a bound for the M-regularity of the nonsingular curve C.

Theorem 7.3.4. *With respect to L the curve C is*

$$[(\lfloor \frac{2r}{\varepsilon} \rfloor + 1) \deg_L C + \operatorname{codim} C + g - 1](\lfloor \frac{2r}{\varepsilon} \rfloor + 1) + 1$$

M-regular.

Proof. Write $\mathscr{F} = q_*(p^* B \otimes \mathscr{I}_\Gamma) \otimes L^{\lfloor \frac{2r}{\varepsilon} \rfloor + 1}$, which is continuously globally generated by Lemma 7.3.3. Then there is a positive integer N such that for general $\alpha_1, \ldots, \alpha_N \in \operatorname{Pic}^0(X)$, the sum of twisted evaluation maps

$$\bigoplus_{i=1}^{N} H^0(\mathscr{F} \otimes \alpha_i) \otimes \alpha_i^\vee \to \mathscr{F}$$

is surjective. We write $W_i = H^0(\mathscr{F} \otimes \alpha_i)$, $V = H^0(C, B)$, $n_0 = \dim V$ and $D = L^{\lfloor \frac{2r}{\varepsilon} \rfloor + 1}$. Then from a short exact sequence $0 \to \mathscr{F} \to H^0(X, B) \otimes D \to B \otimes D \to 0$ obtained by tensoring D to the short exact sequence in 7.1, we have an exact sequence

$$\bigoplus_{i=1}^N W_i \otimes \alpha_i^\vee \to V \otimes D \to B \otimes D \to 0.$$

Twist it by D^{-1}, we get

$$\bigoplus_{i=1}^N W_i \otimes \alpha_i^\vee \otimes D^{-1} \to V \otimes \mathscr{O}_A \to B \to 0.$$

We denote by $E = \bigoplus_{i=1}^N W_i \otimes \alpha_i^\vee \otimes D^{-1}$. Notice that the 0-th Fitting ideal of B is $\mathscr{I}_C$. Thus applying Eagon-Northcott complex to B, we have a complex

$$\cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow \mathscr{I}_C \to 0.$$

Let us look at the term E_j in the above complex. E_j has the form of direct sum of $\wedge^{n_0+j} E$. We may write $\wedge^{n_0+j} E$ in the form $\oplus_{m_j} D^{-n_0-j} \otimes \alpha_{m_j}$ where $\alpha_{m_j} \in \{\alpha_1, \cdots, \alpha_N\}$.

This complex is exact away from C. Chasing from it, we see that $\mathscr{I}_C \otimes D^{n_0+r-1} \otimes L$ is I.T.0. Thus $\mathscr{I}_C \otimes L^{(n_0+r-1)(\lfloor \frac{2r}{\varepsilon} \rfloor + 1)+1}$ is M-regular. Recall that $B = \omega_C \otimes N \otimes L_C^{\lfloor \frac{2r}{\varepsilon} \rfloor + 1}$, and we choose $N = \mathscr{O}_C$, then $B = \omega_C \otimes L_C^{\lfloor \frac{2r}{\varepsilon} \rfloor + 1}$. Thus by Rieman-Roch Theorem, we have

$$\begin{aligned} n_0 = h^0(C, \omega_C \otimes L_C^{\lfloor \frac{2r}{\varepsilon} \rfloor + 1}) &= (\lfloor \frac{2r}{\varepsilon} \rfloor + 1) \deg_L C + 2g - 2 + 1 - g \\ &= (\lfloor \frac{2r}{\varepsilon} \rfloor + 1) \deg_L C + g - 1. \end{aligned}$$

Thus, the M-regular of $\mathscr{I}_C$ with respect to L is

$$\begin{aligned}(n_0+r-1)(\lfloor\frac{2r}{\varepsilon}\rfloor+1)+1 &= [(\lfloor\frac{2r}{\varepsilon}\rfloor+1)\deg_L C+g-1+r-1](\lfloor\frac{2r}{\varepsilon}\rfloor+1)+1\\ &= [(\lfloor\frac{2r}{\varepsilon}\rfloor+1)\deg_L C+\operatorname{codim} C+g-1](\lfloor\frac{2r}{\varepsilon}\rfloor+1)+1.\end{aligned}$$

This finishes the proof. □

Recall that the Seshadri constant $\varepsilon \geq 1$, then $(\lfloor\frac{2r}{\varepsilon}\rfloor+1) \leq 2r+1$. We then have

Corollary 7.3.5. *With respect to L the curve C is $[(2r+1)\deg_L C+\operatorname{codim} C+g-1](2r+1)+1$ M-regular.*

If the Seshadri constant ε large enough then we could get a better bound for M-regularity as follows.

Corollary 7.3.6. *Suppose that $\varepsilon > 2r$, then C is $(\deg_L C+\operatorname{codim} C+g)$ M-regular with respect to L.*

Proof. Since $\varepsilon > 2r$ we see that $\lfloor\frac{2r}{\varepsilon}\rfloor+1=1$. Then the result follows from Theorem 7.3.4. □

CITED LITERATURE

1. Mumford, D.: Lectures on curves on an algebraic surface. With a section by G. M. Bergman. Annals of Mathematics Studies, No. 59. Princeton, N.J., Princeton University Press, 1966.

2. Bertram, A., Ein, L., and Lazarsfeld, R.: Vanishing theorems, a theorem of Severi, and the equations defining projective varieties. J. Amer. Math. Soc., 4(3):587–602, 1991.

3. Chardin, M. and Ulrich, B.: Liaison and Castelnuovo-Mumford regularity. Amer. J. Math., 124(6):1103–1124, 2002.

4. de Fernex, T. and Ein, L.: A vanishing theorem for log canonical pairs. Amer. J. Math., 132(5):1205–1221, 2010.

5. Swanson, I.: Powers of ideals. Primary decompositions, Artin-Rees lemma and regularity. Math. Ann., 307(2):299–313, 1997.

6. Cutkosky, S. D., Herzog, J., and Trung, N. V.: Asymptotic behaviour of the Castelnuovo-Mumford regularity. Compositio Math., 118(3):243–261, 1999.

7. Kodiyalam, V.: Asymptotic behaviour of Castelnuovo-Mumford regularity. Proc. Amer. Math. Soc., 128(2):407–411, 2000.

8. Cutkosky, S. D., Ein, L., and Lazarsfeld, R.: Positivity and complexity of ideal sheaves. Math. Ann., 321(2):213–234, 2001.

9. Eisenbud, D. and Goto, S.: Linear free resolutions and minimal multiplicity. J. Algebra, 88(1):89–133, 1984.

10. Gruson, L., Lazarsfeld, R., and Peskine, C.: On a theorem of Castelnuovo, and the equations defining space curves. Invent. Math., 72(3):491–506, 1983.

11. Lazarsfeld, R.: A sharp Castelnuovo bound for smooth surfaces. Duke Math. J., 55(2):423–429, 1987.

12. Kwak, S.: Castelnuovo regularity for smooth subvarieties of dimensions 3 and 4. J. Algebraic Geom., 7(1):195–206, 1998.

13. Kwak, S.: Generic projections, the equations defining projective varieties and Castelnuovo regularity. Math. Z., 234(3):413–434, 2000.

14. Pareschi, G. and Popa, M.: Regularity on abelian varieties. I. J. Amer. Math. Soc., 16(2):285–302 (electronic), 2003.

15. Pareschi, G. and Popa, M.: Regularity on abelian varieties. II. Basic results on linear series and defining equations. J. Algebraic Geom., 13(1):167–193, 2004.

16. Pareschi, G. and Popa, M.: Regularity on abelian varieties III: relationship with Generic Vanishing and applications. ArXiv e-prints, February 2008.

17. Hartshorne, R.: Algebraic geometry. New York, Springer-Verlag, 1977. Graduate Texts in Mathematics, No. 52.

18. Eisenbud, D.: Commutative algebra, volume 150 of Graduate Texts in Mathematics. New York, Springer-Verlag, 1995. With a view toward algebraic geometry.

19. Lazarsfeld, R.: Positivity in algebraic geometry. I, volume 48 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Berlin, Springer-Verlag, 2004. Classical setting: line bundles and linear series.

20. Eisenbud, D.: The geometry of syzygies, volume 229 of Graduate Texts in Mathematics. New York, Springer-Verlag, 2005. A second course in commutative algebra and algebraic geometry.

21. Bayer, D. and Mumford, D.: What can be computed in algebraic geometry? In Computational algebraic geometry and commutative algebra (Cortona, 1991), Sympos. Math., XXXIV, pages 1–48. Cambridge, Cambridge Univ. Press, 1993.

22. Ein, L. and Mustaţă, M.: Jet schemes and singularities. In Algebraic geometry—Seattle 2005. Part 2, volume 80 of Proc. Sympos. Pure Math., pages 505–546. Providence, RI, Amer. Math. Soc., 2009.

23. Ein, L. and Mustaţă, M.: Inversion of adjunction for local complete intersection varieties. Amer. J. Math., 126(6):1355–1365, 2004.

24. Matsumura, H.: Commutative ring theory, volume 8 of Cambridge Studies in Advanced Mathematics. Cambridge, Cambridge University Press, 1986. Translated from the Japanese by M. Reid.

25. Huneke, C. and Ulrich, B.: Residual intersections. J. Reine Angew. Math., 390:1–20, 1988.

26. Kovacs, S. J., Schwede, K. E., and Smith, K. E.: The canonical sheaf of du bois singularities, 2008.

27. Lazarsfeld, R.: Positivity in algebraic geometry. II, volume 49 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Berlin, Springer-Verlag, 2004. Positivity for vector bundles, and multiplier ideals.

28. Maclagan, D. and Smith, G. G.: Uniform bounds on multigraded regularity. J. Algebraic Geom., 14(1):137–164, 2005.

29. Cutkosky, S. D. and Kurano, K.: Asymptotic regularity of powers of ideals of points in a weighted projective plane. ArXiv e-prints, October 2009.

30. Li, A. and Swanson, I.: Symbolic powers of radical ideals. Rocky Mountain J. Math., 36(3):997–1009, 2006.

31. Hochster, M.: Criteria for equality of ordinary and symbolic powers of primes. Math. Z., 133:53–65, 1973.

32. Mumford, D.: Varieties defined by quadratic equations. In Questions on Algebraic Varieties (C.I.M.E., III Ciclo, Varenna, 1969), pages 29–100. Rome, Edizioni Cremonese, 1970.

33. Reid, M.: Chapters on algebraic surfaces. In Complex algebraic geometry (Park City, UT, 1993), volume 3 of IAS/Park City Math. Ser., pages 3–159. Providence, RI, Amer. Math. Soc., 1997.

34. Kollár, J. and Mori, S.: Birational geometry of algebraic varieties, volume 134 of Cambridge Tracts in Mathematics. Cambridge, Cambridge University Press, 1998.

With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

35. Matsuki, K.: Introduction to the Mori program. Universitext. New York, Springer-Verlag, 2002.

36. Artin, M.: On isolated rational singularities of surfaces. Amer. J. Math., 88:129–136, 1966.

37. Griffiths, P. and Harris, J.: Principles of algebraic geometry. New York, Wiley-Interscience [John Wiley & Sons], 1978. Pure and Applied Mathematics.

38. Mumford, D.: Abelian varieties. Tata Institute of Fundamental Research Studies in Mathematics, No. 5. Published for the Tata Institute of Fundamental Research, Bombay, 1970.

39. Robert Lazarsfeld, Giuseppe Pareschi, M. P.: Local positivity, multiplier ideals, and syzygies of abelian varieties, 2010.

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Research Interests

Algebraic Geometry and Commutative Algebra.

Education

- Ph.D. in Mathematics, University of Illinois at Chicago, 2011
  Thesis:*Bounding the Castelnuovo-Mumford regularity of algebraic varieties*
  Thesis Advisor: Lawrence Ein
- M.S. in Mathematics, Fudan University, China, 2005
- B.S. in Mathematics, Fudan University, China, 2001

Seminar Talks

- Joint Mathematics Meetings, New Orleans, January 9, 2011,
  AMS Session on Commutative Rings and Fields,
  Asymptotic Regularity of Powers of Ideal Sheaves.
- University of Wisconsin-Madison, October 23, 2010,
  Midwest Algebraic Geometry Graduate Conference,
  A Vanishing Theorem of Powers of Ideal Sheaves.
- University of Notre Dame, March 24, 2010,
  Algebra Seminar,
  Bounds on Regularity of Projective Varieties.
- Purdue University, March 23, 2009,
  Graduate Student Commutative Algebra Seminar,
  Castelnuovo-Mumford Regularity of Log Canonical Subvarieties in $\mathbb{P}^n$.
- University of Illinois at Chicago, 2007, 2008, 2009, 2010,
  Graduate Algebraic Geometry Seminar.

Publications and Preprints

- *A Vanishing Theorem and Asymptotic Regularity of Powers of Ideal Sheaves.*
  arXiv:1009.0314.(preprint)
- *A Bound for the Castelnuovo-Mumford Regularity of Log Canonical Varieties.*
  J. Pure Appl. Algebra 215 (2011), 2180-2189 .

Conferences and Workshops Attended

- Midwest Commutative Algebra and Geometry Conference
  Purdue University, May 22-26, 2011.
- Workshop in Algebraic Geometry: on the occasion of the retirement of Igor Dolgachev
  University of Michigan, April 2-3, 2011.
- Joint Mathematics Meetings 2011
  New Orleans, January 6-9, 2011.
  Service: Chair of AMS Session on Commutative Rings and Fields.
- AMS Sectional Meeting
  University of Notre Dame, November 5 - 7, 2010.
- Midwest Algebraic Geometry Graduate Conference
  University of Wisconsin-Madison, October 23 - 24, 2010.
- Ohio State-Michigan-UIC Algebraic Geometry Workshop
  University of Illinois at Chicago, October 2 - 3, 2010.
- Mathematics Research Community (MRC) on Commutative Algebra
  Snowbird, Utah, June 26 - July 2, 2010.
- Midwest Algebra, Geometry and their Interactions Conference (MAGIC'10)
  University of Notre Dame, April 23 - 25, 2010.
- UM/UIC/OSU Weekend Algebraic Geometry Workshop
  Ohio State University, March 13 - 14, 2010.
- Algebraic Geometry: A Conference in Honor of Anatoly Libgober's 60th Birthday
  University of Illinois at Chicago, October 16 - 18, 2009.
- Fourier-Mukai Functors, Regularity on Abelian Varieties and Generic Vanishing Theorems
  University of Michigan, May 4 - 8 2009.
- 2nd Bluegrass Algebra Conference
  University of Kentucky, March 6 - 8 2009.

- Multiplier Ideals and Their Applications
  Ohio State University, February 23 - 27, 2009.
- Classical Algebraic Geometry Today
  MSRI, Berkeley, January 26 - 30, 2009.
- University of Michigan/Ohio State/UIC Algebraic Geometry Workshop
  University of Michigan, October 18 - 19, 2008.
- Algebraic Geometry and Commutative Algebra:
  A conference to celebrate Robin Hartshorne's 70th birthday
  University of Illinois at Chicago, April 11 - 13, 2008.
- University of Michigan/Ohio State/UIC Algebraic Geometry Workshop
  University of Michigan, March, 2007.
- UIC-Purdue Worshop on Algebraic Geometry and Commutative Algebra
  Purdue University, December 2 - 3, 2006.

Teaching Experience

- Instructor
  - Multivariable Calculus, Summer 2010, University of Illinois at Chicago (UIC)
  - Linear Algebra, 2004-2005, Shanghai University of Finance and Economics
- Teaching Assistant
  - Multivariable Calculus (Labs), Spring 2010, UIC
  - Precalculus, Summer 2009, Fall 2007, Fall 2006, UIC
  - Beginning Algebra, Spring 2009, UIC
  - Calculus, Fall 2008, UIC
  - Intermediate Algebra, Summer 2008, Spring and Summer 2006, Fall 2005, UIC
  - Calculus for Business, Spring 2007, UIC
  - Abstract algebra, Calculus, 2003 - 2005, Fudan University

Other experience

- Programmer, Shanghai Yulong Software Engineering Co., Ltd., China, 2000 - 2002
  Designing algorithms to simulate computer numerical control machine on PC.
  Programming languages: C, C^{++}, Visual Basic, Java, OpenGL.

Reference

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Mihnea Popa, University of Illinois at Chicago.

Bernd Ulrich, Purdue University.

John Wood, University of Illinois at Chicago (teaching).