Singular Loci of Restriction Varieties

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Summary

Restriction varieties are a fundamental class of subvarieties of orthogonal flag varieties. They parameterize isotropic partial flags satisfying certain rank conditions with respect to a flag that is not necessarily isotropic. Orthogonal Schubert varieties are examples of restriction varieties when the flag is isotropic. The intersection of a generic Type A Schubert variety with the orthogonal flag variety is also an example of a restriction variety. These two examples serve as two extremal cases; restriction varieties interpolate between these two examples. This thesis focuses on restriction varieties in the orthogonal Grassmannian $OG(k, n)$, we will refer to them as restriction varieties for brevity. The goal of this thesis is to study the singularities of restriction varieties.

We introduce a resolution of singularities for restriction varieties that is inspired by the Bott-Samelson/Zelevinsky resolution for Schubert varieties but is necessarily more complicated due to the richer geometry of restriction varieties. We use the resolution of singularities to study the singularities of a restriction variety. Our results rely on studying the exceptional locus of the resolution; we categorize the orbits in the image of the exceptional locus and we compute the dimension of the fibers of the resolution over each orbit.

Using a lemma that relates the image of the exceptional locus to the singularities of the restriction variety when the resolution is not a divisorial contraction, we show that certain components of the exceptional locus have images inside the singular locus. For the components that are excluded from these results, we study the tangent space to the restriction variety at a point. We find conditions for when the images of the components lie inside the singular locus. We conclude by illustrating how the results presented can be used to describe the singularities of a restriction variety.

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CHAPTER 1

Introduction

There are several ways of defining Schubert varieties in $G(k,n)$. Here we define them in a setting that is not common in the literature but that will generalize to restriction varieties in a straight-forward way: We use sequences whose steps correspond to rank conditions giving the Schubert variety. Let W be an n-dimensional vector space over the complex numbers $\mathbb{C}$ and consider $G(k,W) = G(k,n)$, the Grassmannian of k-planes on W. We define a Schubert variety Σ in $G(k,n)$ in terms of a fixed complete flag, that is, a nested sequence of subspaces

$$0 \subseteq W_1 \subseteq \cdots \subseteq W_{n-1} \subseteq W_n = W$$

with $\dim W_i = i$. Consider a subsequence $W_\bullet$ of length k:

$$W_{n_1} \subseteq \cdots \subseteq W_{n_k} .$$

The Schubert variety Σ associated to $W_\bullet$ is defined as the closure of the locus

$$\Sigma(W_\bullet)^0 = \{\Lambda \in G(k,n) \mid \dim(\Lambda \cap W_{n_i}) = i \text{ for all } 1 \leq i \leq k\} .$$

If there are steps in $W_\bullet$ with consecutively increasing dimensions, the number of independent rank conditions is less than the number of steps in the sequence. In this case, the Schubert variety $\Sigma(W_\bullet)$ can be defined in a more concise way by considering only the largest dimensional step in each group of steps with consecutively increasing dimensions.

EXAMPLE 1.1. *Let Σ be the Schubert variety in $G(5,17)$ associated to the sequence*

$$W_8 \subseteq W_9 \subseteq W_{10} \subseteq W_{11} \subseteq W_{12} \ .$$

Then Σ is defined as the closure of the locus

$$\Sigma^0 = \{\Lambda \in G(5,17) \mid \dim(\Lambda \cap W_{12}) = 5\} \ .$$

In other words, Σ is just the variety of 5-planes Λ contained in W_{12}; it is isomorphic to $G(5,12)$. Such Λ necessarily intersect W_{11} in dimension 4, W_{10} in dimension 3 and so on. In this example, the defining sequence gives only one independent rank condition.

EXAMPLE 1.2. *Let Σ be the Schubert variety in $G(5,17)$ associated to the sequence*

$$W_2 \subseteq W_3 \subseteq W_4 \subseteq W_{11} \subseteq W_{12} \ .$$

This means Σ is defined as the closure of the locus

$$\Sigma^0 = \{\Lambda \in G(5,17) \mid \dim(\Lambda \cap W_4) = 3 \textit{ and } \dim(\Lambda \cap W_{12}) = 5\} \ .$$

The rest of the steps are naturally satisfied for such k-planes Λ, so there are only two independent rank conditions defining Σ.

EXAMPLE 1.3. *Let Σ be the Schubert variety in $G(7,17)$ given by the sequence*

$$W_2 \subseteq W_6 \subseteq W_7 \subseteq W_{11} \subseteq W_{12} \subseteq W_{13} \subseteq W_{15} \ .$$

This variety is defined as the closure of the locus

$$\begin{aligned}\Sigma^0 = \{\Lambda \in G(7,17) \mid \ & \dim(\Lambda \cap W_2) = 1, \ \ \dim(\Lambda \cap W_7) = 3, \\ & \dim(\Lambda \cap W_{13}) = 6, \ \ \dim(\Lambda \cap W_{15}) = 7\} \ .\end{aligned}$$

Four independent rank conditions define Σ in this example.

In order to define Schubert varieties in $G(k,n)$ in a concise way by just noting the independent rank conditions, we introduce partitions. Define the partition $(n_{a_1}^{\alpha_1},\dots,n_{a_t}^{\alpha_t})$ associated to $W_\bullet:\ W_{n_1}\subseteq\cdots\subseteq W_{n_k}$ as

$$\alpha_l = \Big|\left\{n_i \text{ in } W_\bullet \;\middle|\; n_i \le n_{a_l},\ a_l - i = n_{a_l} - n_i\right\}\Big| \ \text{ for all } \ 1\le l\le t\ .$$

In other words, a_l marks the largest dimensional step in each group of steps with consecutively increasing dimensions and α_l counts the number of steps in the group. Note that we have $a_l = \sum_{i=1}^{l}\alpha_i$ and $a_t = k$. The Schubert variety Σ in $G(k,n)$ associated to the partition $(n_{a_1}^{\alpha_1},\dots,n_{a_t}^{\alpha_t})$ is given by t independent rank conditions and is defined as the closure of the locus

$$\Sigma^0 = \left\{\Lambda\in G(k,n)\;\middle|\;\dim\left(\Lambda\cap W_{n_{a_l}}\right) = a_l \ \text{ for all } \ 1\le l\le t\right\}\ .$$

Being homogeneous under the action of $GL(n)$, the open cell Σ^0 is smooth.

EXAMPLE 1.4. *The partition associated to the Schubert variety in* $G(5,17)$ *given by the sequence*

$$W_8\subseteq W_9\subseteq W_{10}\subseteq W_{11}\subseteq W_{12}$$

is (12^5).

EXAMPLE 1.5. *The partition associated to the Schubert variety in* $G(5,17)$ *given by the sequence*

$$W_2\subseteq W_3\subseteq W_4\subseteq W_{11}\subseteq W_{12}$$

is $(4^3,12^2)$.

EXAMPLE 1.6. *The partition associated to the Schubert variety in* $G(7,17)$ *given by the sequence*

$$W_2\subseteq W_6\subseteq W_7\subseteq W_{11}\subseteq W_{12}\subseteq W_{13}\subseteq W_{15}$$

is $(2^1,7^2,13^3,15^1)$.

The following proposition recalls the dimension of a Schubert variety in the sequence and the partition notations.

PROPOSITION 1.7. *The dimension of a Schubert variety Σ in $G(k,n)$ associated to the sequence $W_\bullet$: $W_{n_1} \subseteq \cdots \subseteq W_{n_k}$ or the partition $(n_{a_1}^{\alpha_1}, \ldots, n_{a_t}^{\alpha_t})$ is given by*

$$\dim \Sigma = \sum_{i=1}^{k} (n_i - i) = \sum_{l=1}^{t} \alpha_l (n_{a_l} - a_l) \ .$$

PROOF. The second equality is just the translation between the sequence notation and the partition notation. We prove the first equality using induction on k. If $k = 1$ then Σ is isomorphic to the projective space of dimension $n_1 - 1$ and the equality holds. Now suppose the proposition holds up to $k - 1$. Let Σ' be the Schubert variety of $(k-1)$-planes defined by the sequence obtained by omitting W_{n_k} from $W_\bullet$. Consider the map $f : \Sigma \to \Sigma'$ defined by $f : \Lambda \mapsto \Lambda \cap W_{n_{k-1}}$. The map f maps Σ onto Σ'. By the theorem on the dimension of the fibers of a morphism, we have $\dim \Sigma = \dim \Sigma' + \dim f^{-1}(L)$ for a general point L in Σ'. For general $L \in \Sigma'$, the inverse image $f^{-1}(L) = \{\Lambda \subseteq W_{n_k} \mid L \subseteq \Lambda\}$ is isomorphic to the Grassmannian $G(1, n_k - (k-1))$ and hence has dimension $n_k - k$. This proves the proposition. □

1. The Bott-Samelson/Zelevinsky Resolution

Schubert varieties in the Grassmannian admit a natural resolution $\pi : \widetilde{\Sigma} \to \Sigma$ such that the image of the exceptional locus of π is equal to the singular locus of Σ. Let Σ be given by the partition $(n_{a_1}^{\alpha_1}, \ldots, n_{a_t}^{\alpha_t})$ and let $\widetilde{\Sigma}$ be the Schubert variety in the flag variety $F(a_1, \ldots, a_t; n)$ defined by

$$\widetilde{\Sigma} = \left\{ (T^1, \ldots, T^t) \in F(a_1, \ldots, a_t; n) \;\middle|\; T^l \subseteq W_{n_{a_l}} \text{ for all } 1 \leq l \leq t \right\} \ .$$

Since $\widetilde{\Sigma}$ is an iterated tower of Grassmannians, it is smooth and irreducible. The natural projection $\pi : F(a_1, \ldots, a_t; n) \to G(k,n)$ given by $(T^1, \ldots, T^t) \mapsto T^t$ maps $\widetilde{\Sigma}$ onto Σ and the map is injective over the smooth open cell Σ^0. The inverse image

$\pi^{-1}(\Lambda)$ of a general point $\Lambda \in \Sigma^0$ is determined uniquely as

$$T^l = \Lambda \cap W_{n_{a_l}} \ , \ \ 1 \leq l \leq t \ .$$

By Zariski's Main Theorem, π is an isomorphism over Σ^0 and hence a resolution of singularities of Σ.

The map has positive dimensional fibers over the locus of k-planes Λ with the property that $\dim(\Lambda \cap W_{n_{a_l}}) > a_l$ for some $1 \leq l \leq t-1$. Let Σ_{s_l} be the closure of the locus

$$\Sigma^0_{s_l} = \left\{ \Lambda \in \Sigma \; \middle| \; \dim\left(\Lambda \cap W_{n_{a_l}}\right) = a_l + 1 \right\} \ .$$

The exceptional locus of π consists of the union of the inverse images of Σ_{s_l} for all $1 \leq l \leq t-1$. Let us study the codimension of the components of the exceptional locus of π. Over each Σ_{s_l}, the inverse image Σ_{s_l} is irreducible of codimension

$$\operatorname{codim}\left(\pi^{-1}(\Sigma_{s_l})\right) = \operatorname{codim}\left(\Sigma_{s_l}\right) - \dim\left(\pi^{-1}(\Lambda)\right)$$

for a general $\Lambda \in \Sigma_{s_l}$. By Proposition 1.7 we have

$$\begin{aligned} \operatorname{codim}\left(\Sigma_{s_l}\right) = \alpha_l(n_{a_l} - a_l) &+ \alpha_{l+1}(n_{a_{l+1}} - a_{l+1}) \\ &- (\alpha_l + 1)(n_{a_l} - a_l - 1) - (\alpha_{l+1} - 1)(n_{a_{l+1}} - a_{l+1}) \\ = n_{a_{l+1}} &- n_{a_l} - (a_{l+1} - a_l) + \alpha_l + 1 \ . \end{aligned}$$

On the other hand, for a general $\Lambda \in \Sigma_{s_l}$ we have

$$\begin{aligned} \pi^{-1}(\Lambda) = \{(T^1, \ldots, T^t) \mid T^g = \Lambda \cap W_{n_{a_g}} \ \text{ for all } \ 1 \leq g \leq t, \ \ g \neq l \\ \text{and } \ T^{l-1} \subseteq T^l \subseteq \Lambda \cap W_{n_{a_l}}\} \end{aligned}$$

So, for an element of $\pi^{-1}(\Lambda)$, the coordinate T^l is the only one that is not determined uniquely and it can be parameterized by $G(a_l - a_{l-1}, a_l + 1 - a_{l-1})$. This Grassmannian has dimension $a_l - a_{l-1} = \alpha_l$. Therefore we have

$$\operatorname{codim}\left(\pi^{-1}(\Sigma_{s_l})\right) = n_{a_{l+1}} - n_{a_l} - (a_{l+1} - a_l) + 1 \geq 2$$

since $n_{a_{l+1}} - n_{a_l} \geq a_{l+1} - a_l + 1$.

This shows that a component of the exceptional locus of π has codimension larger than 1. This observation with the following lemma determines the singular locus of a Schubert variety.

LEMMA 1.8. *([**7**], Lemma 2.3) Let $f : X \to Y$ be a birational morphism from a smooth, projective variety X onto a normal projective variety Y. Assume that f is an isomorphism in codimension one. Then $p \in Y$ is a singular point if and only if $f^{-1}(p)$ is positive dimensional.*

COROLLARY 1.9. *The image of the exceptional locus of the resolution of singularities $\pi : \widetilde{\Sigma} \to \Sigma$ is equal to the singular locus of Σ.*

EXAMPLE 1.10. *The Schubert variety Σ in $G(5, 17)$ associated to the partition (12^5) or the sequence*

$$W_8 \subseteq W_9 \subseteq W_{10} \subseteq W_{11} \subseteq W_{12}$$

is smooth, this is the Grassmannian $G(5, 12)$. In this case, the resolution of singularities has no positive dimensional locus. The variety $\widetilde{\Sigma}$ is given by

$$\widetilde{\Sigma} = \left\{T^1 \in G(5; 17) \;\middle|\; T^1 \subseteq W_{12}\right\}$$

and is identical to Σ.

EXAMPLE 1.11. *Consider the Schubert variety Σ in $G(5, 17)$ given by the partition $(4^3, 12^2)$. The variety $\widetilde{\Sigma}$ is given by*

$$\widetilde{\Sigma} = \left\{(T^1, T^2) \in F(3, 5; 17) \;\middle|\; T^1 \subseteq W_4 \text{ and } T^2 \subseteq W_{12}\right\} .$$

The projection $\pi : (T^1, T^2) \mapsto T^2$ *maps* $\widetilde{\Sigma}$ *onto* Σ. *The map* π *is positive dimensional over the locus*

$$\{\Lambda \in G(5,17) \mid \dim(\Lambda \cap W_4) > 3\}$$

which also equals the singular locus of Σ.

EXAMPLE 1.12. *For the Schubert variety given by the partition* $(2^1, 7^2, 13^3, 15^1)$, *the variety* $\widetilde{\Sigma}$ *is defined as*

$$\widetilde{\Sigma} = \{(T^1, T^2, T^3, T^4) \in F(1,3,6,7;17) \ \mid \ T^1 \subseteq W_2, \ \ T^2 \subseteq W_7, \\ T^3 \subseteq W_{13}, \ \ T^4 \subseteq W_{15}\} \ .$$

The projection $\pi : (T^1, T^2, T^3, T^4) \mapsto T^4$ *maps* $\widetilde{\Sigma}$ *onto* Σ. *The exceptional locus consists of the union of the inverse images of the closures of the following loci:*

$$\Sigma^0_{s_1} = \{\Lambda \in G(7,17) \mid \dim(\Lambda \cap W_2) = 2, \ \ \dim(\Lambda \cap W_7) = 3, \\ \dim(\Lambda \cap W_{13}) = 6, \ \ \dim(\Lambda \cap W_{15}) = 7\} \ .$$

$$\Sigma^0_{s_2} = \{\Lambda \in G(7,17) \mid \dim(\Lambda \cap W_2) = 1, \ \ \dim(\Lambda \cap W_7) = 4, \\ \dim(\Lambda \cap W_{13}) = 6, \ \ \dim(\Lambda \cap W_{15}) = 7\} \ .$$

$$\Sigma^0_{s_3} = \{\Lambda \in G(7,17) \mid \dim(\Lambda \cap W_2) = 1, \ \ \dim(\Lambda \cap W_7) = 3, \ \ \dim(\Lambda \cap W_{13}) = 7\} \ .$$

Consequently the singular locus of the Schubert variety Σ *is given by*

$$\Sigma^{sing} = \Sigma_{s_1} \cup \Sigma_{s_2} \cup \Sigma_{s_3} \ .$$

REMARK 1.13. *The subvarieties* Σ_{s_l} *of the Schubert variety* Σ *correspond to the hooks in the Young diagram of* Σ.

2. Outline of Results

This thesis presents a resolution of singularities and gives a partial description of the singularities of restriction varieties in $OG(k, n)$. The Bott-Samelson/Zelevinsky resolution and the picture described in the previous section for Schubert varieties in $G(k, n)$ serve as a starting point.

One of the major differences of the restriction varieties from Schubert varieties in this study is that a component of the exceptional locus of the resolution of singularities introduced here does not have codimension larger than 1 in general. We describe components with this property and give examples in our discussion. Also, since restriction varieties are much more general than Schubert varieties, their combinatorial nature is more involved. This is reflected in the complicated statements of the results; we hope to remedy this by presenting lots of examples that unveil the intuition behind the general formulations.

In Chapter 2, we review restriction varieties. The definition and properties of restriction varieties are governed by basic facts about quadrics. We recall these properties and explain the conditions required to define restriction varieties. We introduce the partition notation for restriction varieties which is central in the statements of our results. We also introduce basis sequences which give a convenient point of view for studying the tangent space to a restriction variety.

In Chapter 3, we introduce a resolution of singularities for restriction varieties. We start by giving examples that illustrate the ideas behind the construction and then give the general definition. The definition of the resolution of singularities becomes more apparent when considered via a diagram; we explain this diagram throughout our discussions and emphasize that it can be used to define the resolution of singularities in general.

In Chapter 4, we study the exceptional locus and determine which components have codimension larger than 1. We prove a lemma that allows us to show that the image of a component of the exceptional locus with codimension larger than 1 lies in the singular locus of the restriction variety. This gives a partial description of the singular locus. We also observe that there are components of the exceptional locus with codimension 1 in general.

In Chapter 5, we study the components of the exceptional locus with codimension equal to 1. We present conditions under which the image of a component of the exceptional locus with codimension 1 is contained in the singular locus of the restriction variety. We study arcs contained in the restriction variety through a point to show singularity at a point, and hence along the orbit that contains the point.

In Chapter 6, we present examples where we describe the singular locus of several restriction varieties, presenting concrete cases of our previous observations. We consider orthogonal Schubert varieties and show the overlap between our notation and the permutation notation which the existing literature on Schubert varieties usually uses.

CHAPTER 2

Restriction Varieties in $OG(k,n)$

1. Preliminaries on Restriction Varieties

In this chapter, we define restriction varieties and review their basic properties. Let W be an n-dimensional vector space over the complex numbers $\mathbb{C}$ and Q a non-degenerate symmetric bilinear form on W. A linear space $\Lambda \subseteq W$ is called *isotropic* with respect to Q if $\gamma_1^T Q \gamma_2 = 0$ for all $\gamma_1, \gamma_2 \in \Lambda$. Let F_Q denote the quadratic polynomial associated to Q. A k-plane Λ is isotropic with respect to Q if and only if its projectivization is contained in the quadric hypersurface defined by F_Q. The *orthogonal Grassmannian* $OG(k,n)$ parameterizes k-dimensional subspaces of W that are isotropic with respect to Q. Equivalently this is the Fano variety of $(k-1)$-planes contained in a quadric hypersurface in $\mathbb{P}W$.

Let L_{n_j} be an isotropic linear space of vector space dimension n_j. In case $2n_j = n$ we denote isotropic linear spaces in different connected components as L_{n_j} and L'_{n_j}. Let $Q^{r_i}_{d_i}$ denote a subquadric of corank r_i cut out by a d_i-dimensional linear section of Q and denote this linear space by $\overline{Q^{r_i}_{d_i}}$. Let $F_{Q^r_d}$ denote the restriction of F to $\overline{Q^{r_i}_{d_i}}$ so that Q^r_d is given by the zero locus of $F_{Q^r_d}$. We denote the singular locus of $Q^{r_i}_{d_i}$ by $Q^{r_i,sing}_{d_i}$. We use the same notation for projectivizations contained in $\mathbb{P}W$. For convenience, let $r_0 = 0$ and $d_0 = n$.

We use sequences of the form

$$L_{n_1} \subseteq \ldots \subseteq L_{n_s} \subseteq Q^{r_{k-s}}_{d_{k-s}} \subseteq \ldots \subseteq Q^{r_1}_{d_1}$$

consisting of isotropic linear spaces L_{n_j} and sub-quadrics $Q^{r_i}_{d_i}$ of Q to define restriction varieties. The restriction variety V defined via this sequence parameterizes k-dimensional isotropic linear spaces that intersect L_{n_j} in a subspace of dimension j for all $1 \leq j \leq s$ and $Q^{r_i}_{d_i}$ in a subspace of dimension $k-i+1$ for all $1 \leq i \leq k-s$. We require the isotropic linear spaces and the singular loci of sub-quadrics to be in the most special position. This is expressed in the conditions

- $Q^{r_{i-1},sing}_{d_{i-1}} \subseteq Q^{r_i,sing}_{d_i}$ for every $1 \leq i \leq k-s$ and
- $\dim\left(L_{n_j} \cap Q^{r_i,sing}_{d_i}\right) = \min(n_j, r_i)$ for every $1 \leq j \leq s$ and $1 \leq i \leq k-s$.

This gives a motivation for counting the sub-quadrics $Q^{r_i}_{d_i}$ from the right; the singular loci form a nested sequence of subspaces $Q^{r_1,sing}_{d_1} \subseteq \ldots \subseteq Q^{r_{k-s},sing}_{d_{k-s}}$. Note that by the corank bound, $Q^{r_{i-1},sing}_{d_{i-1}} \subseteq Q^{r_i,sing}_{d_i}$ implies $r_{i+1} - r_i \leq d_i - d_{i+1}$. In particular, the corank of a sub-quadric in Q is bounded by its codimension. We note the effect on our sequence as

- $r_{i+1} + d_{i+1} \leq r_i + d_i$ for every $1 \leq i \leq k-s$.

This positioning of isotropic linear spaces and sub-quadrics has an effect on the k-planes V parameterizes as well. Let x_i be the number of isotropic linear spaces L_{n_j} of the sequence contained in $Q^{r_i,sing}_{d_i}$. We require the $(k-i+1)$-dimensional subspace of a k-plane Λ contained in $Q^{r_i}_{d_i}$ to intersect $Q^{r_i,sing}_{d_i}$ in a subspace of dimension x_i. The largest dimensional isotropic linear space with respect to a quadratic form Q^r_d has dimension $\left\lfloor \frac{d+r}{2} \right\rfloor$. Therefore a linear space of dimension $k-i+1$ intersects $Q^{r_i,sing}_{d_i}$ in a subspace of dimension at least $\max\left(0, k-i+1-\left\lfloor \frac{d-r}{2} \right\rfloor\right)$. Hence we get the condition

- For every $1 \leq i \leq k-s$,

$$x_i \geq k-i+1-\frac{d_i-r_i}{2}.$$

Another crucial requirement we make is the irreducibility of the sub-quadrics. A sub-quadric Q^r_d is irreducible if and only if its rank is at least 3. The following condition ensures that $Q^{r_{k-s}}_{d_{k-s}}$ and consequently every $Q^{r_i}_{d_i}$ is irreducible.

- $r_{k-s} \leq d_{k-s} - 3$.

The next condition concerns the variation of tangent spaces to a singular quadric. Let M be a codimension j linear subspace of a linear space L. Let Q^r_d be singular along M. Then the tangent spaces to Q^r_d along $L \setminus M$ vary at most in a $(j-1)$-dimensional family. In other words, the image of the Gauss map of Q^r_d restricted to the smooth points of L has dimension at most $j-1$. Therefore, if there is n_j, r_i in a sequence with $n_j = r_i + 1$, then $Q^{r_i,sing}_{d_i} \subseteq L_{n_j}$ is a codimension 1 linear subspace and the tangent spaces to $Q^{r_i}_{d_i}$ are constant along L_{n_j}. Hence the $(k-i+1)$-dimensional subspace contained in $Q^{r_i}_{d_i}$ are actually contained in $Q^{r_i+1}_{d_i-1}$ with singular locus L_{n_j}. Since the latter reflects the geometry of the k-planes in V better, we impose the following condition on our sequence.

- For any $1 \leq j \leq s$, there does not exist $1 \leq i \leq k-s$ such that $n_j - r_i = 1$.

The following technical condition puts a restriction on the singular loci of the sub-quadrics in the sequence; it disallows a sudden gap between $Q^{r_i,sing}_{d_i}$.

- For every $1 \leq i \leq k-s$ either $r_i = r_1 = x_1$ or $r_l - r_i \geq l - i - 1$ for every $l > i$. Furthermore, if $r_l = r_{l-1} > x_1$ for some l, then $d_i - d_{i+1} = r_{i+1} - r_i$ for all $i \geq l$ and $d_{l-1} - d_l = 1$.

We use sequences satisfying these conditions to define restriction varieties in order to make sure the resulting subvarieties of $OG(k,n)$ are geometrically meaningful. A sequence satisfying these conditions is called an *admissible sequence.*

DEFINITION 2.1. *Let $(L_\bullet, Q_\bullet)$ be an admissible sequence for $OG(k,n)$. A restriction variety $V(L_\bullet, Q_\bullet)$ is the subvariety of $OG(k,n)$ defined as the closure of*

$$V^0(L_\bullet, Q_\bullet) = \Big\{ \Lambda \in OG(k,n) \;\Big|\; \begin{aligned} &\dim\left(\Lambda \cap L_{n_j}\right) = j, \;\; 1 \le j \le s, \\ &\dim\left(\Lambda \cap Q_{d_i}^{r_i}\right) = k-i+1, \\ &\dim\left(\Lambda \cap Q_{d_i}^{r_i, sing}\right) = x_i, \;\; 1 \le i \le k-s \Big\}. \end{aligned}$$

EXAMPLE 2.2. *Schubert varieties in $OG(k,n)$ are restriction varieties defined via a sequence satisfying $d_i + r_i = n$ for all $1 \le i \le k-s$, that is, when the quadrics in the sequence are as singular as possible. The restriction of a general Schubert variety in $G(k,n)$ to $OG(k,n)$ is also a restriction variety associated to a sequence with $s = 0$ and $r_i = 0$ for all $1 \le i \le k-s$. Hence, restriction varieties interpolate between the restrictions of Schubert varieties in $G(k,n)$ to $OG(k,n)$ and Schubert varieties in $OG(k,n)$.*

When the inequality $x_i \ge k - i + 1 - \frac{d_i - r_i}{2}$ is an equality for an index i, then the $\frac{d_i+r_i}{2}$-dimensional linear spaces in $Q_{d_i}^{r_i}$ form two irreducible components.

EXAMPLE 2.3. *V defined by*

$$Q_3^0 \subseteq Q_4^0$$

in $OG(2,5)$ parameterizes lines on a smooth quadric surface Q_4^0 in $\mathbb{P}^3$ and consists of two irreducible components.

The $(k-i+1)$-dimensional subspaces contained in $Q_{d_i}^{r_i}$ may be distinguished by their parity of the dimension of their intersection with linear spaces in each of these components.

DEFINITION 2.4. *Let $(L_\bullet, Q_\bullet)$ be an admissible sequence. An index $1 \le i \le k-s$ such that*

$$x_i = k - i + 1 - \frac{d_i - r_i}{2}$$

is called a special index. For each special index, a marking $m_\bullet$ of $(L_\bullet, Q_\bullet)$ designates one of the irreducible components of $\frac{d_i+r_i}{2}$-dimensional linear spaces of $Q_{d_i}^{r_i}$ as even and the other one as odd, such that

- *If $d_{i_1} + r_{i_1} = d_{i_2} + r_{i_2}$ for two special indices $i_1 < i_2$ and the component containing a linear space Γ is designated even for i_2, then the component containing Γ is designated even for i_1 as well; and*
- *If $2n_s = d_i + r_i$ for a special index i, then the component to which L_{n_s} belongs is assigned the parity of s; and*
- *If $n = 2k$, $m_\bullet$ assigns the component containing l_k the parity that characterizes the component $OG(k, 2k)$. A marked restriction variety $V(L_\bullet, Q_\bullet, m_\bullet)$ is the Zariski closure of the subvariety of $V^0(L_\bullet, Q_\bullet)$ parameterizing k-dimensional isotropic subspaces W, where, for each special index i, W intersects subspaces of dimension $\frac{d_i+r_i}{2}$ of $Q_{d_i}^{r_i}$ designated even (respectively, odd) by $m_\bullet$ in a subspace of even (respectively, odd) dimension.*

We will use the next proposition when we compare dimensions of the restriction variety and its tangent space in various orbits in the next section.

PROPOSITION 2.5. ([**4**], Prop 4.16) *The marked restriction variety $V(L_\bullet, Q_\bullet, m_\bullet)$ associated to a marked admissible sequence is an irreducible variety of dimension*

$$\begin{aligned}\dim\left(V(L_\bullet, Q_\bullet, m_\bullet)\right) &= \sum_{j=1}^{s}(n_j - j) + \sum_{i=1}^{k-s}(d_i + x_i - 2s - 2i)\\ &= \sum_{j=1}^{s}(n_j - j) + \sum_{i=1}^{k-s}(d_i + x_i - 2(k-i+1))\end{aligned}$$

Note that this expression does not depend on the marking $m_\bullet$. The restriction variety $V(L_\bullet, Q_\bullet)$ has an irreducible component for every marking $m_\bullet$ and every irreducible component of $V(L_\bullet, Q_\bullet)$ has this dimension.

2. Basis Sequences

In this subsection we associate a sequence of vectors, brackets and braces to an admissible sequence. Examples of similar sequences can be found in [**4**], [**5**] and [**6**]. We will use basis sequences when we study an example illustrating future research ideas in the last chapter.

Recall that we denote the quadratic polynomial corresponding to the symmetric bilinear form Q by F_Q and the smallest dimensional linear space containing a sub-quadric Q_d^r by $\overline{Q_d^r}$. We take F_Q to be

$$\sum_{i=1}^{m} x_i y_i \ \text{ if } \ n = 2m \ \text{ and } \ x_{m+1}^2 + \sum_{i=1}^{m} x_i y_i \ \text{ if } \ n = 2m+1 \ .$$

Similarly, the restrictions of the bilinear form $F_{Q_d^r}$ to $\overline{Q_d^r}$ are

$$\sum_{i=r+1}^{r+m} x_i y_i \ \text{ if } \ d-r = 2m \ \text{ and } \ x_{r+m+1}^2 + \sum_{i=r+1}^{r+m} x_i y_i \ \text{ if } \ d-r = 2m+1 \ .$$

Let the dual basis for x_i, y_i be e_i, f_i such that

$$x_i(e_j) = \delta_i^j, y_i(f_j) = \delta_i^j \ \text{ and } \ x_i(f_j) = y_i(e_j) = 0 \ .$$

Using e_i, f_i we give a basis for each L_{n_j} and $\overline{Q_{d_i}^{r_i}}$ as follows:

$$L_{n_j} = \left\langle e_1, \ldots, e_{n_j} \right\rangle$$

$$\overline{Q_{d_i}^{r_i}} = \left\langle e_1, \ldots, e_{r_i}, e_{r_i+1}, f_{r_i+1}, \ldots, e_{r_i+m}, f_{r_i+m} \right\rangle \quad \text{if} \quad d_i - r_i = 2m$$

$$\overline{Q_{d_i}^{r_i}} = \left\langle e_1, \ldots, e_{r_i}, e_{r_i+1}, f_{r_i+1}, \ldots, e_{r_i+m}, f_{r_i+m}, e_{r_i+m} - f_{r_i+m} + e_{r_i+m+1} \right\rangle \quad \text{if} \quad d_i - r_i = 2m+1$$

Given an admissible sequence

$$L_{n_1} \subseteq \ldots \subseteq L_{n_s} \subseteq Q_{d_{k-s}}^{r_{k-s}} \subseteq \ldots \subseteq Q_{d_1}^{r_1}$$

we form a sequence of vectors e_i, f_i, brackets and braces as follows: For each isotropic linear subspace L_{n_j}, we write down $e_{n_{j-1}+1}, \ \ldots, \ e_{n_j}$ followed by a bracket and for each sub-quadric $Q^{r_i}_{d_i}$, we write down the remaining vectors in the basis of $\overline{Q^{r_i}_{d_i}}$ followed by a brace.

In this sequence, the first n_j vectors span L_{n_j} and the first d_i vectors span $Q^{r_i}_{d_i}$.

EXAMPLE 2.6. *To $Q^0_5 \subseteq Q^0_6$ we associate the sequence*

$$e_1\ f_1\ e_2\ f_2\ (e_2 - f_2 + e_3)\ \big\}\ f_3 \big\}\ .$$

EXAMPLE 2.7. *To $L_2 \subseteq L_5 \subseteq Q^2_{12} \subseteq Q^0_{14}$ we associate the sequence*

$$e_1\ e_2\ \big]\ e_3\ e_4\ e_5\ \big]\ f_3\ f_4\ f_5\ e_6\ f_6\ e_7\ f_7\ \big\}\ f_1\ f_2 \big\}\ .$$

3. Partitions for Restriction Varieties

Restriction varieties can be parameterized by triads of partitions. These partitions will allow us to define restriction varieties using only the independent rank conditions, that is, the conditions that are not automatically satisfied as a result of the others. For an admissible sequence

$$L_{n_1} \subseteq \ldots \subseteq L_{n_s} \subseteq Q^{r_{k-s}}_{d_{k-s}} \subseteq \ldots \subseteq Q^{r_1}_{d_1}$$

write down the increasing sequences $(n_1, \ldots, n_s)$, $(d_{k-s}, \ldots, d_1)$ by grouping the consecutive integers as follows:

$$(n_1, \ldots, n_s) = (n^{\alpha_1}_{a_1}, \ldots, n^{\alpha_t}_{a_t}) \ \text{ and } \ (d_{k-s}, \ldots, d_1) = (d^{\beta_1}_{b_1}, \ldots, d^{\beta_u}_{b_u})$$

where

$$\alpha_l = \Big| \{n_j \text{ in the sequence} \mid n_j \leq n_{a_l}, a_l - j = n_{a_l} - n_j\} \Big| \ \text{ and}$$
$$\beta_l = \Big| \{d_i \text{ in the sequence} \mid d_i \leq d_{b_l}, i - b_l = d_{b_l} - d_i\} \Big|.$$

Here a_g (resp. b_h) is the largest dimensional isotropic linear subspace (resp. the largest dimensional sub-quadric) in each group and α_g (resp. β_h) counts the steps in the group for all $1 \leq g \leq t$ (resp. $1 \leq h \leq u$). Restriction varieties can be parameterized by partitions

$$(n_{a_1}^{\alpha_1}, \dots, n_{a_t}^{\alpha_t}), (d_{b_1}^{\beta_1}, \dots, d_{b_u}^{\beta_u}), (r_{b_1}, \dots, r_{b_u}).$$

The restriction variety given by the partitions $(n_{a_1}^{\alpha_1}, \dots, n_{a_t}^{\alpha_t}), (d_{b_1}^{\beta_1}, \dots, d_{b_u}^{\beta_u}), (r_{b_1}, \dots, r_{b_u})$ is defined as the closure of the locus

$$\begin{aligned} V^0 = \Big\{ \Lambda \in OG(k,n) \ \Big| \ & \dim\left(\Lambda \cap L_{n_{a_g}}\right) = a_g, \ \ 1 \leq g \leq t, \\ & \dim\left(\Lambda \cap Q_{d_{b_h}}^{r_{b_h}}\right) = k - b_h + 1, \\ & \dim\left(\Lambda \cap Q_{d_{b_h}}^{r_{b_h},sing}\right) = x_{b_h}, \ \ 1 \leq h \leq u \ \Big\}. \end{aligned}$$

EXAMPLE 2.8. *To the sequence* $L_2 \subseteq L_3 \subseteq L_6 \subseteq L_7 \subseteq L_8 \subseteq Q_{11}^{17} \subseteq Q_{12}^{17} \subseteq Q_{18}^{13}$ *we associate the partitions* $(3^2, 8^3), (12^2, 18^1), (17, 13)$.

We have $a_g = \sum_{l=1}^{g} \alpha_l$ and $k - b_h + 1 = s + \sum_{l=1}^{h} \beta_l$ for every $1 \leq g \leq t$ and $1 \leq h \leq u$. Note that $n_{a_t} = n_s$, $d_{b_u} = d_1$, $\sum_{l=1}^{t} \alpha_l = s$ and $\sum_{l=1}^{u} \beta_l = k - s$.

OBSERVATION 2.9. *In terms of these partitions Proposition 2.5 gives the dimension of a restriction variety by*

$$\begin{aligned} \dim\left(V(L_\bullet, Q_\bullet)\right) &= \sum_{g=1}^{t} \alpha_g \left(n_{a_g} - a_g\right) + \sum_{h=1}^{u} \sum_{t=1}^{\beta_h} \left(d_{b_h} + x_{b_h} - 2\left(k - b_h + 1\right) + (t-1)\right) \\ &= \sum_{g=1}^{t} \alpha_g \left(n_{a_g} - a_g\right) + \sum_{h=1}^{u} \beta_h \left(d_{b_h} + x_{b_h} - 2(k - b_h + 1) + \frac{\beta_h - 1}{2}\right) \end{aligned}$$

EXAMPLE 2.10. *The restriction variety* $\left[L_6 \subseteq L_7 \subseteq L_8\right]$ *is isomorphic to the Grassmannian* $G(3,8)$ *which parameterizes planes contained in a projective space of dimension 7. This variety is given by* $(8^3), (), ()$ *in terms of partitions and has dimension* $\alpha_1(n_{a_1} - a_1) = 3(8-3) = 15$.

EXAMPLE 2.11. *The restriction variety* $\left[Q^4_{11} \subseteq Q^3_{12} \subseteq Q^2_{13}\right]$ *is the Fano variety of planes contained in a quadric 11-fold in* $\mathbb{P}^{12}$ *singular along a line. In terms of partitions it is given by* $(), (13^3), (2)$ *and has dimension* $\beta_1(d_{b_1} + x_{b_1} - 2(3) + \frac{\beta_1 - 1}{2}) = 3(13 + 0 - 6 + 1) = 24$.

EXAMPLE 2.12. *The restriction variety* $\left[L_2 \subseteq L_3 \subseteq Q^7_{17} \subseteq Q^6_{18}\right]$ *parameterizes 3-dimensional projective linear spaces that are contained in a quadric hypersurface in* $\mathbb{P}^{17}$ *of corank 6 and that intersect a plane contained in the singular locus of the quadric along a line. In terms of partitions this variety is given by* $(3^2), (18^2), (6)$ *and has dimension* $\alpha_1(n_{a_1} - a_1) + \beta_1(d_{b_1} + x_{b_1} - 2(5) + \frac{\beta_1 - 1}{2}) = 2(3-2) + 2(18 + 2 - 8 + \frac{1}{2}) = 27$.

CHAPTER 3

The Resolution of Singularities

In this chapter, we present a resolution of singularities for restriction varieties. We first illustrate the resolution on a few examples and then introduce the general definition.

EXAMPLE 3.1. *Let V be the restriction variety in $OG(1,n)$ defined by the sequence Q^4_{11} of length 1. This variety is a singular quadric contained in a projective space of dimension 10 whose singular locus is isomorphic to the projective space of dimension 3. Consider the flag variety $\widetilde{V}$ defined by*

$$\widetilde{V} = \left\{(T,Z) \in OF(1,5;n) \mid Q^{4,sing}_{11} \subseteq Z \subseteq Q^4_{11}\right\} \subseteq OG(1,n) \times OG(5,n).$$

The second projection map $\pi_2 : (T,Z) \mapsto Z$ maps $\widetilde{V}$ onto $\left\{Z \in OG(5,n) \mid Q^{4,sing}_{11} \subseteq Z \subseteq Q^4_{11}\right\}$ which is isomorphic to $OG(1,7)$. Over such Z, the map has fibers $G(1,5)$ of dimension 4 so $\widetilde{V}$ is irreducible of dimension 9. The first projection map $\pi_1 : (T,Z) \mapsto T$ maps $\widetilde{V}$ onto V where the inverse image is determined uniquely over the smooth locus of V. By Zariski's theorem, $\pi_1 : \widetilde{V} \to V$ is a resolution of singularities for V where the image of the exceptional locus gives the singular locus of V.

EXAMPLE 3.2. *Let $V = \left[L_7 \subseteq Q^4_{11}\right]$, V parameterizes the lines in a singular quadric intersecting a fixed linear space that contains the singular locus of the quadric. Consider the variety defined by*

$$\widetilde{V} = \left\{(T^1,T^2,O,Z) \mid T^1 \subseteq T^2,\ Q^{4,sing}_{11} \subseteq O \subseteq Z,\ T^1 \subseteq O \subseteq L_7 \text{ and } T^2 \subseteq Z \subseteq Q^4_{11}\right\}$$

where $\dim T^j = j$, $\dim O = 5$ *and* $\dim Z = 6$. *The properties defining the variety* $\widetilde{V}$ *can be visualized by the following diagram:*

FIGURE 1. Definition of $\widetilde{V}$ for $V = \left[L_7 \subseteq Q^4_{11}\right]$

$$\begin{array}{ccccc} & & Q_{11}^{4,sing} & & \\ & & \rotatebox{90}{$\supseteq$} & & \\ T^1 & \subseteq & O & \subseteq & L_7 \\ \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} \\ T^2 & \subseteq & Z & \subseteq & Q^4_{11} \end{array}$$

Consider the following forgetful maps:

$$(T^1, T^2, O, Z) \mapsto (T^1, O, Z) \mapsto (T^1, O) \mapsto (O).$$

We show $\widetilde{V}$ *is an iterated tower of* $G(l, n)$ *and* $OG(l, n)$ *bundles via these maps. The linear space* O *satisfies* $Q_{11}^{4,sing} \subseteq O \subseteq L_7$ *and hence can be parameterized by* $G(5-4, 7-4) = G(1, 3)$. *For fixed* O, *the linear space* T^1 *satisfies* $T^1 \subseteq O$ *and hence can be parameterized by* $G(1, 5)$. *On the other hand,* Z *satisfies* $O \subseteq Z \subseteq Q^4_{11}$. *Since* Z *has to lie in the quadric cut out on* Q^4_{11} *by the linear space tangent to* Q^4_{11} *everywhere along* O, Z *is contained in a quadric of projective dimension 8 with a singular locus of projective dimension 4. Then* Z *can be parameterized by* $OG(1, 5)$. *Finally, the linear space* T^2 *satisfies* $T^1 \subseteq T^2 \subseteq Z$ *and hence can be parameterized by* $G(1, 5)$. *Thus* $\widetilde{V}$ *is a tower of the discussed* $G(1, 3)$, $G(1, 5)$, $OG(1, 5)$ *and* $G(1, 5)$ *bundles. This also shows that* $\widetilde{V}$ *is irreducible of dimension 13. The second projection map*

$$\pi : (T^1, T^2, O, Z) \mapsto T^2$$

maps $\widetilde{V}$ *onto* V *with fibers determined uniquely for a general point* Λ *contained in* V^0. *Therefore the map* $\pi : \widetilde{V} \to V$ *is a resolution of singularities by Zariski's theorem.*

EXAMPLE 3.3. *Let* $V = \left[L_5 \subseteq Q^7_{10} \subseteq Q^2_{20}\right]$. *For this restriction variety we consider* $\widetilde{V}$ *defined by*

$$\widetilde{V} = \big\{(T^1, T^2, T^3, O^1, O^2, Z^1, Z^2) \mid Q^{2,sing}_{20} \subseteq O^1 \subseteq O^2 \subseteq Z^2,\ Q^{7,sing}_{10} \subseteq Z^1,$$
$$T^1 \subseteq O^1 \subseteq L_5,\ T^2 \subseteq O^2 \subseteq Q^7_{10} \text{ and } T^3 \subseteq Z^2 \subseteq Q^2_{20}\big\}$$

where $\dim T^j = j$, $\dim O^1 = 3$, $\dim O^2 = 4$, $\dim Z^1 = 8$ *and* $\dim Z^2 = 5$. *The corresponding diagram is:*

FIGURE 2. Definition of $\widetilde{V}$ for $V = \left[L_5 \subseteq Q^7_{10} \subseteq Q^2_{20}\right]$

$$\begin{array}{ccccccc}
 & & Q^{2,sing}_{20} & & Q^{7,sing}_{10} & & \\
 & & \cap\!| & & \cap\!| & & \\
T^1 & \subseteq & O^1 & \subseteq & & & L_5 \\
\cap\!| & & \cap\!| & & & & \cap\!| \\
T^2 & \subseteq & O^2 & \subseteq & Z^1 & \subseteq & Q^7_{10} \\
\cap\!| & & \cap\!| & & & & \cap\!| \\
T^3 & \subseteq & Z^2 & \subseteq & & & Q^2_{20}
\end{array}$$

We consider the following forgetful maps:

$$(T^1, T^2, T^3, O^1, O^2, Z^1, Z^2) \mapsto (T^1, T^2, O^1, O^2, Z^1, Z^2) \mapsto (T^1, T^2, O^1, O^2, Z^1)$$
$$\mapsto (T^1, O^1, O^2, Z^1) \mapsto (T^1, O^1, Z^1) \mapsto (T^1, O^1) \mapsto (O^1).$$

The linear space O^1 *is parameterized by* $G(1,3)$ *and for fixed* O, T^1 *is parameterized by* $G(1,3)$. *The linear space* Z^1 *is parameterized by* $OG(1,3)$. *For fixed* Z^1, O^2 *satisfies* $O^1 \subseteq O^2 \subseteq Z^1$ *and hence can be parameterized by* $G(1,5)$. *Then* T^2 *is parameterized by* $G(1,3)$. *In the last row, as* $O^2 \subseteq Z^2 \subseteq Q^2_{20}$, Z^2 *is parameterized by* $OG(1,14)$. *Then* T^3 *is parameterized by* $G(1,3)$. *Thus* $\widetilde{V}$ *is a tower of the discussed* $G(1,3)$, $G(1,3)$, $OG(1,3)$, $G(1,5)$, $G(1,3)$, $OG(1,14)$ *and* $G(1,3)$ *bundles. Thus* $\widetilde{V}$ *is an irreducible smooth variety of dimension 25. The third projection map*

$$\pi : (T^1, T^2, T^3, O^1, O^2, Z^1, Z^2) \mapsto T^3$$

gives the resolution of singularities in this example.

EXAMPLE 3.4. *As a final example, let us consider the restriction variety in $OG(10,70)$ given by the sequence*

$$L_2 \subseteq L_6 \subseteq L_{13} \subseteq L_{14} \subseteq L_{19} \subseteq Q_{30}^{17} \subseteq Q_{40}^{11} \subseteq Q_{45}^{8} \subseteq Q_{46}^{7} \subseteq Q_{50}^{3}.$$

In this case $\widetilde{V}$ satisfies the following diagram. The dimensions of the T, Z and O's are noted as subscripts.

FIGURE 3. Definition of $\widetilde{V}$ for $V = \Big[L_2 \subseteq L_6 \subseteq L_{13} \subseteq L_{14} \subseteq L_{19} \subseteq Q_{30}^{17} \subseteq Q_{40}^{11} \subseteq Q_{45}^{8} \subseteq Q_{46}^{7} \subseteq Q_{50}^{3}\Big]$

$$\begin{array}{ccccccccccc}
T_1^1 & \subseteq & L_2 & & & & & & & & \\
\rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & & & & & & & \\
 & & Q_{50}^{3,sing} & \subseteq & Q_{46}^{7,sing} & \subseteq & Q_{40}^{11,sing} & \subseteq & Q_{30}^{17,sing} & & \\
 & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & \\
T_2^2 & \subseteq & O_4^{4,n_{a_1}} & \subseteq & & & & & & & L_6 \\
\rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & & & & & & & \rotatebox{-90}{$\subseteq$} \\
T_4^3 & \subseteq & O_6^{4,n_{a_2}} & \subseteq & O_9^{3,n_{a_2}} & \subseteq & O_{13}^{2,n_{a_2}} & \subseteq & & & L_{14} \\
\rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & & & \rotatebox{-90}{$\subseteq$} \\
T_5^4 & \subseteq & O_7^{4,n_{a_3}} & \subseteq & O_{10}^{3,n_{a_3}} & \subseteq & O_{14}^{2,n_{a_3}} & \subseteq & O_{18}^{1,n_{a_3}} & \subseteq & L_{19} \\
\rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} \\
T_6^5 & \subseteq & O_8^{4,r_{b_1}} & \subseteq & O_{11}^{3,r_{b_1}} & \subseteq & O_{15}^{2,r_{b_1}} & \subseteq & Z_{19}^1 & \subseteq & Q_{30}^{17} \\
\rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & & & \rotatebox{-90}{$\subseteq$} \\
T_7^6 & \subseteq & O_9^{4,r_{b_2}} & \subseteq & O_{12}^{3,r_{b_2}} & \subseteq & Z_{16}^2 & \subseteq & & & Q_{40}^{11} \\
\rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & & & & & \rotatebox{-90}{$\subseteq$} \\
T_9^7 & \subseteq & O_{11}^{4,r_{b_3}} & \subseteq & Z_{14}^3 & \subseteq & & & & & Q_{46}^{7} \\
\rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & & & & & & & \rotatebox{-90}{$\subseteq$} \\
T_{10}^8 & \subseteq & Z_{12}^4 & \subseteq & & & & & & & Q_{50}^{3}
\end{array}$$

The variety $\widetilde{V}$ is a tower of $G(k,n)$ and $OG(k,n)$ bundles via 25 successive forgetful maps in this case. Starting with an element of $\widetilde{V}$, the forgetful maps trail each row from left to right going from the bottom row to the top row.

Let us fix terminology before giving the definition. In the following we say a sequence $A = \Big[A_1 \subseteq \ldots \subseteq A_k\Big]$ is contained in a sequence $B = \Big[B_1 \subseteq \ldots \subseteq B_k\Big]$ if

$A_i \subseteq B_i$ for all $1 \leq i \leq k$. We will denote by A both the sequence $\Big[A_1 \subseteq \ldots \subseteq A_k\Big]$ and the ordered set $(A_1, \ldots, A_k)$.

Let $V(L_\bullet, Q_\bullet)$ be a restriction variety defined by the sequence

$$L_{n_1} \subseteq \ldots \subseteq L_{n_s} \subseteq Q^{r_{k-s}}_{d_{k-s}} \subseteq \ldots \subseteq Q^{r_1}_{d_1},$$

or equivalently, by the partitions $(n^{\alpha_1}_{a_1}, \ldots, n^{\alpha_t}_{a_t}), (d^{\beta_1}_{b_1}, \ldots, d^{\beta_u}_{b_u}), (r_{b_1}, \ldots, r_{b_u})$. For each $Q^{r_{b_h}}_{d_{b_h}}$, let $V(Q^{r_{b_h}}_{d_{b_h}})$ be the subsequence consisting of isotropic linear subspaces $L_{n_{a_\theta}}$ and sub-quadrics $Q^{r_{b_\theta}}_{d_{b_\theta}}$ that strictly contain $Q^{r_{b_h},sing}_{d_{b_h}}$ and are strictly contained in $Q^{r_{b_h}}_{d_{b_h}}$. We introduce a subsequence $O(Q^{r_{b_h}}_{d_{b_h}})$ of the same length contained in $V(Q^{r_{b_h}}_{d_{b_h}})$ that consists of isotropic linear subspaces O.

$$\begin{array}{llllllllll}
V(Q^{r_{b_h}}_{d_{b_h}}): & \cdots & \subseteq & L_{n_{a_\theta}} & \subseteq & \cdots & \subseteq & Q^{r_{b_\theta}}_{d_{b_\theta}} & \subseteq & \cdots \\
\cup\!| & & & \cup\!| & & & & \cup\!| & & \\
O(Q^{r_{b_h}}_{d_{b_h}}): & \cdots & \subseteq & O^{h,n_{a_\theta}} & \subseteq & \cdots & \subseteq & O^{h,d_{b_\theta}} & \subseteq & \cdots
\end{array}$$

Also, the subsequence $\Big[L_1 \subseteq \ldots \subseteq Q^{r_{b_{u-1}}}_{d_{b_{u-1}}}\Big]$ obtained by omitting the last β_u sub-quadrics from the defining sequence will have a crucial role in the following definition.

Define:

$$\begin{aligned}
\widetilde{V}(L_\bullet, Q_\bullet) := \Big\{ \ \big(T^1, \ldots, T^{t+u},\ Z^1, \ldots, Z^u,\ O(Q^{r_{b_1}}_{d_{b_1}}), \ldots, O(Q^{r_{b_u}}_{d_{b_u}})\big) \ \Big| \\
& Q^{r_{b_h},sing}_{d_{b_h}} \subseteq O(Q^{r_{b_1}}_{d_{b_1}}) \subseteq Z^h \subseteq Q^{r_{b_h}}_{d_{b_h}}, \\
& O^{h,n_{a_\theta}} \subseteq L_{n_{a_\theta}} \text{ for all } L_{n_{a_\theta}} \text{ in } V(Q^{r_{b_h}}_{d_{b_h}}), \\
& O^{h,n_{a_\theta}} \subseteq O^{h+1,n_{a_\theta}} \text{ for all } L_{n_{a_\theta}} \text{ that lies in both } V(Q^{r_{b_h}}_{d_{b_h}}) \text{ and } V(Q^{r_{b_{h+1}}}_{d_{b_{h+1}}}), \\
& O^{h,r_{b_\theta}} \subseteq Q^{r_{b_\theta}}_{d_{b_\theta}} \text{ for all } Q^{r_{b_\theta}}_{d_{b_\theta}} \text{ in } V(Q^{r_{b_h}}_{d_{b_h}}), \\
& O^{h,r_{b_\theta}} \subseteq O^{h+1,r_{b_\theta}} \text{ for all } Q^{r_{b_\theta}}_{d_{b_\theta}} \text{ that lies in both } V(Q^{r_{b_h}}_{d_{b_h}}) \text{ and } V(Q^{r_{b_{h+1}}}_{d_{b_{h+1}}}), \\
& T^1 \subseteq \ldots \subseteq T^{t+u} \quad \text{for all } 1 \leq g \leq t \text{ and } 1 \leq h \leq u \\
& (\ T^1, \ldots, T^{t+u-1}\ ) \subseteq \Big[L_1 \subseteq \ldots \subseteq Q^{r_{b_{u-1}}}_{d_{b_{u-1}}}\Big] \text{ and } T^{t+u} \subseteq Z^u \ \Big\}
\end{aligned}$$

where $\dim T^g = a_g$, $\dim T^{t+h} = k - b_h + 1$, $\dim Z^h = r_{b_h} + (k - b_h + 1) - x_{b_h}$, $\dim O^{h,n_{a_\theta}} = r_{b_h} + a_\theta - x_{b_h}$ and $\dim O^{h,r_{b_\theta}} = r_{b_h} + (k - b_\theta + 1) - x_{b_h}$ for all $1 \leq g \leq t$ and $1 \leq h \leq u$.

Drawing a diagram, as in the examples above, puts this construction in a more approachable framework. Let $L_{n_{a_1}} \subseteq \ldots \subseteq L_{n_{a_\omega}}$ be the isotropic linear subspaces in the defining sequence contained in $Q_{d_{b_u}}^{r_{b_u},sing}$, thus contained in the singular locus of all the sub-quadrics. The defining properties of $\widetilde{V}$ are visualized in the following diagram. Here, the linear spaces $O^{h,\bullet}$ that lie in the column of $Q_{d_{b_h}}^{r_{b_h}}$ form the sequence $O(Q_{d_{b_h}}^{r_{b_h}})$ in the definition of $\widetilde{V}$ above.

FIGURE 4. Definition of $\widetilde{V}$ for general V

$$
\begin{array}{ccccccccccccccc}
T_1 & \subseteq & L_{n_{a_1}} & & & & & & & & & & & & \\
\rotatebox[origin=c]{-90}{$\subseteq$} & & \rotatebox[origin=c]{-90}{$\subseteq$} & & & & & & & & & & & & \\
\vdots & & \vdots & & & & & & & & & & & & \\
T_\omega & \subseteq & L_{n_{a_\omega}} & & & & & & & & & & & & \\
\rotatebox[origin=c]{-90}{$\subseteq$} & & \rotatebox[origin=c]{-90}{$\subseteq$} & & & & & & & & & & & & \\
 & & Q_{d_{b_u}}^{r_{b_u},sing} & \subseteq & Q_{d_{b_{u-1}}}^{r_{b_{u-1}},sing} & \subseteq & & \cdots & & \subseteq & Q_{d_{b_2}}^{r_{b_2},sing} & \subseteq & Q_{d_{b_1}}^{r_{b_1},sing} & & \\
 & & \rotatebox[origin=c]{-90}{$\subseteq$} & & & & & & & & & & & & \\
T_{\omega+1} & \subseteq & O^{u,n_{a_{\omega+1}}} & \subseteq & \cdots & & & & & & & & & & L_{n_{a_{\omega+1}}} \\
\rotatebox[origin=c]{-90}{$\subseteq$} & & \rotatebox[origin=c]{-90}{$\subseteq$} & & & & & & & & & & & & \rotatebox[origin=c]{-90}{$\subseteq$} \\
\vdots & & \vdots & & & & & & & & & & & & \vdots \\
\rotatebox[origin=c]{-90}{$\subseteq$} & & \rotatebox[origin=c]{-90}{$\subseteq$} & & & & & & & & & & & & \rotatebox[origin=c]{-90}{$\subseteq$} \\
T_t & \subseteq & O^{u,n_{a_t}} & \subseteq & \cdots & & & & & & & & & & L_{n_{a_t}} \\
\rotatebox[origin=c]{-90}{$\subseteq$} & & \rotatebox[origin=c]{-90}{$\subseteq$} & & & & & & & & & & & & \rotatebox[origin=c]{-90}{$\subseteq$} \\
T_{t+1} & \subseteq & O^{u,r_{b_1}} & \subseteq & O^{u-1,r_{b_1}} & \subseteq & \cdots & \subseteq & O^{3,r_{b_1}} & \subseteq & O^{2,r_{b_1}} & \subseteq & Z^1 & \subseteq & Q_{d_{b_1}}^{r_{b_1}} \\
\rotatebox[origin=c]{-90}{$\subseteq$} & & \rotatebox[origin=c]{-90}{$\subseteq$} & & \rotatebox[origin=c]{-90}{$\subseteq$} & & & & \rotatebox[origin=c]{-90}{$\subseteq$} & & \rotatebox[origin=c]{-90}{$\subseteq$} & & & & \rotatebox[origin=c]{-90}{$\subseteq$} \\
T_{t+2} & \subseteq & O^{u,r_{b_2}} & \subseteq & O^{u-1,r_{b_2}} & \subseteq & \cdots & \subseteq & O^{3,r_{b_2}} & \subseteq & Z^2 & \subseteq & & & Q_{d_{b_2}}^{r_{b_2}} \\
\rotatebox[origin=c]{-90}{$\subseteq$} & & \rotatebox[origin=c]{-90}{$\subseteq$} & & \rotatebox[origin=c]{-90}{$\subseteq$} & & & & \rotatebox[origin=c]{-90}{$\subseteq$} & & & & & & \rotatebox[origin=c]{-90}{$\subseteq$} \\
T_{t+3} & & & & & & & & Z^3 & \subseteq & & & & & Q_{d_{b_3}}^{r_{b_3}} \\
\rotatebox[origin=c]{-90}{$\subseteq$} & & \vdots & & \vdots & & & \iddots & & & & & & & \rotatebox[origin=c]{-90}{$\subseteq$} \\
\vdots & & & & & & & & & & & & & & \vdots \\
\rotatebox[origin=c]{-90}{$\subseteq$} & & \rotatebox[origin=c]{-90}{$\subseteq$} & & \rotatebox[origin=c]{-90}{$\subseteq$} & & & & & & & & & & \rotatebox[origin=c]{-90}{$\subseteq$} \\
T_{t+u-1} & \subseteq & O^{u,r_{b_{u-1}}} & \subseteq & Z^{u-1} & \subseteq & & & & & & & & & Q_{d_{b_{u-1}}}^{r_{b_{u-1}}} \\
\rotatebox[origin=c]{-90}{$\subseteq$} & & \rotatebox[origin=c]{-90}{$\subseteq$} & & & & & & & & & & & & \rotatebox[origin=c]{-90}{$\subseteq$} \\
T_{t+u} & \subseteq & Z^u & \subseteq & & & & & & & & & & & Q_{d_{b_u}}^{r_{b_u}}
\end{array}
$$

There is a natural projection from $\widetilde{V}(L_\bullet, Q_\bullet)$ to $V(L_\bullet, Q_\bullet)$ given by

$$\pi \ : \ \left(T^1, \ldots, T^{t+u}, \ Z^1, \ldots, Z^u, \ O(Q^{r_{b_1}}_{d_{b_1}}), \ldots, O(Q^{r_{b_u}}_{d_{b_u}})\right) \ \mapsto \ T^{t+u}.$$

PROPOSITION 3.5. *Let $V(L_\bullet, Q_\bullet, m_\bullet)$ be a marked restriction variety. The variety $\widetilde{V}(L_\bullet, Q_\bullet, m_\bullet)$ associated to $V(L_\bullet, Q_\bullet, m_\bullet)$ is a smooth irreducible variety of the same dimension as $V(L_\bullet, Q_\bullet, m_\bullet)$.*

PROOF. Consider the successive forgetful maps omitting one coordinate of $\widetilde{V}$ at a time, going from left to right in each row, starting at the bottom row and going up. The proof of this proposition is based on constructing a tower of $G(l, n)$ and $OG(l, n)$ bundles via these forgetful maps. In the following, we study the four possible types of rows in a diagram:

(1) For $L_{n_{a_g}} \subsetneq Q^{r_{b_u}}_{d_{b_u}}$, we have $T^{g-1} \subseteq T^g \subseteq L_{n_{a_g}}$. Hence T^g is parameterized by $G(a_g - a_{g-1}, n_{a_g} - a_{g-1})$ which has dimension $(a_g - a_{g-1})(n_{a_g} - a_g) = \alpha_g(n_{a_g} - a_g)$.

(2) Suppose for $L_{n_{a_g}}$, the sub-quadrics whose singular loci lie between $L_{n_{a_g}}$ and $L_{n_{a_{g-1}}}$ are $Q^{r_{b_{\eta+c}}}_{d_{b_{\eta+c}}}, \ldots, Q^{r_{b_\eta}}_{d_{b_\eta}}$ for some number c, that is,

$$Q^{r_{b_u}, sing}_{d_{b_u}} \subseteq \ldots \subseteq Q^{r_{b_{\eta+c+1}}, sing}_{d_{b_{\eta+c+1}}} \subsetneq L_{n_{a_{g-1}}} \subseteq Q^{r_{b_{\eta+c}}, sing}_{d_{b_{\eta+c}}} \subseteq \ldots \subseteq Q^{r_{b_\eta}, sing}_{d_{b_\eta}} \subsetneq L_{n_{a_g}}.$$

Note that $x_{b_\eta} = \ldots = x_{b_{\eta+c}} = a_{g-1}$ in this setting. The row consisting of T^g, $O^{\bullet, n_{a_g}}$, $L_{n_{a_g}}$ satisfies:

$$\begin{array}{cccccccccccccccc}
 & & Q^{r_{b_u}, sing}_{d_{b_u}} & \subseteq & \cdots & \subseteq & Q^{r_{b_{\eta+c+1}}, sing}_{d_{b_{\eta+c+1}}} & \subseteq & Q^{r_{b_{\eta+c}}, sing}_{d_{b_{\eta+c}}} & \subseteq & \cdots & \subseteq & Q^{r_{b_\eta}, sing}_{d_{b_\eta}} & \subseteq & \cdots & \\
 & & \rotatebox{-90}{$\subseteq$} & & & & \rotatebox{-90}{$\subseteq$} & & & & & & \rotatebox{-90}{$\subseteq$} & & & \\
\vdots & & \vdots & & & & \vdots & & & & & & & & & \vdots \\
\rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & & & \rotatebox{-90}{$\subseteq$} & & & & & & & & & \\
T^{g-1} & \subseteq & O^{u, n_{a_{g-1}}} & \subseteq & \cdots & \subseteq & O^{\eta+c+1, n_{a_{g-1}}} & \subseteq & & & & & & & & L_{n_{a_{g-1}}} \\
\rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & & & \rotatebox{-90}{$\subseteq$} & & & & & & & & & \rotatebox{-90}{$\subseteq$} \\
T^g & \subseteq & O^{u, n_{a_g}} & \subseteq & \cdots & \subseteq & O^{\eta+c+1, n_{a_g}} & \subseteq & O^{\eta+c, n_{a_g}} & \subseteq & \cdots & \subseteq & O^{\eta, n_{a_g}} & \subseteq & & L_{n_{a_g}} \\
\vdots & & \vdots & & & & \vdots & & \vdots & & & & \vdots & & & \vdots
\end{array}$$

We start by choosing $O^{\eta,n_{a_g}}$. The linear space $O^{\eta,n_{a_g}}$ satisfying $Q^{r_{b_\eta},sing}_{d_{b_\eta}} \subseteq O^{\eta,n_{a_g}} \subseteq L_{n_{a_g}}$ is parameterized by the Grassmannian $G((r_{b_\eta} + a_g - x_{b_\eta}) - r_{b_\eta}, n_{a_g} - r_{b_\eta})$. In a similar fashion, the parameterization of $T^g, O^{u,n_{a_g}}, \ldots, O^{\eta,n_{a_g}}$ are given by Grassmannians whose dimensions add up to $\alpha_g(n_{a_g} - a_g)$ as follows:

Coordinates of $\widetilde{V}$ in the g-th row:	*Dimensions of the corresponding Grassmannian:*
$Q^{r_{b_\eta},sing}_{d_{b_\eta}} \subseteq O^{\eta,n_{a_g}} \subseteq L_{n_{a_g}}$	$(a_g - x_{b_\eta})\big(n_{a_g} - a_g - (r_{b_\eta} - x_{b_\eta})\big)$
$Q^{r_{b_{\eta+1}},sing}_{d_{b_{\eta+1}}} \subseteq O^{\eta+1,n_{a_g}} \subseteq O^{\eta,n_{a_g}}$	$(a_g - x_{b_{\eta+1}})\big((r_{b_\eta} - x_{b_\eta}) - (r_{b_{\eta+1}} - x_{b_{\eta+1}})\big)$
$\vdots$	$\vdots$
$Q^{r_{b_{\eta+c}},sing}_{d_{b_{\eta+c}}} \subseteq O^{\eta+c,n_{a_g}} \subseteq O^{\eta+c-1,n_{a_g}}$	$(a_g - x_{b_{\eta+c}})\big((r_{b_{\eta+c-1}} - x_{b_{\eta+c-1}}) - ((r_{b_{\eta+c}} - x_{b_{\eta+c}})\big)$
$O^{\eta+c+1,n_{a_{g-1}}} \subseteq O^{\eta+c+1,n_{a_g}} \subseteq O^{\eta+c,n_{a_g}}$	$(a_g - a_{g-1})\big((r_{b_{\eta+c}} - x_{b_{\eta+c}}) - ((r_{b_{\eta+c+1}} - x_{b_{\eta+c+1}})\big)$
$\vdots$	$\vdots$
$O^{u,n_{a_{g-1}}} \subseteq O^{u,n_{a_g}} \subseteq O^{u-1,n_{a_g}}$	$(a_g - a_{g-1})\big((r_{b_{u-1}} - x_{b_{u-1}}) - ((r_{b_u} - x_{b_u})\big)$
$T^{g-1} \subseteq T^g \subseteq O^{u,n_{a_g}}$	$(a_g - a_{g-1})\big(r_{b_u} - x_{b_u}\big)$

(3) Consider the row that corresponds to $Q^{r_{b_1}}_{d_{b_1}}$. Depending on r_{b_1}, there are two possibilities for the diagram. If $r_{b_1} \geq n_{a_t}$ then Z^1 is determined by $Q^{r_{b_1},sing}_{d_{b_1}} \subseteq Z^1 \subseteq Q^{r_{b_1}}_{d_{b_1}}$. Explicitly, suppose $L_{n_{a_t}}$ is positioned as $Q^{r_{b_{c+1}},sing}_{d_{b_{c+1}}} \subsetneq L_{n_{a_t}} \subseteq Q^{r_{b_c},sing}_{d_{b_c}} \subseteq \ldots \subseteq Q^{r_{b_1},sing}_{d_{b_1}}$ for some number c. Note that $x_{b_1} = \ldots = x_{b_c} = t$ in this setting. The diagram is of the form:

$$
\begin{array}{ccccccccccccccccc}
 & & Q^{r_{b_u},sing}_{d_{b_u}} & \subseteq & \cdots & \subseteq & Q^{r_{b_{c+1}},sing}_{d_{b_{c+1}}} & \subseteq & Q^{r_{b_c},sing}_{d_{b_c}} & \subseteq & \cdots & \subseteq & Q^{r_{b_2},sing}_{d_{b_2}} & \subseteq & Q^{r_{b_1},sing}_{d_{b_1}} & & \\
 & & \rotatebox{-90}{$\subseteq$} & & & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & \\
\vdots & & \vdots & & & & \vdots & & & & & & & & & & \\
\rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & & & \rotatebox{-90}{$\subseteq$} & & & & & & & & & & \vdots \\
T^t & \subseteq & O^{u,n_{a_t}} & \subseteq & \cdots & \subseteq & O^{c+1,n_{a_t}} & & & & & & & & & & L_{n_{a_t}} \\
\rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & & & \rotatebox{-90}{$\subseteq$} & & & & & & & & & & \rotatebox{-90}{$\subseteq$} \\
T^{t+1} & \subseteq & O^{u,r_{b_1}} & \subseteq & \cdots & \subseteq & O^{c+1,r_{b_1}} & \subseteq & O^{c,r_{b_1}} & \subseteq & \cdots & \subseteq & O^{2,r_{b_1}} & \subseteq & Z^1 & \subseteq & Q^{r_{b_1}}_{d_{b_1}} \\
\vdots & & \vdots & & & & \vdots & & \vdots & & & & \vdots & & \vdots & & \vdots
\end{array}
$$

We start by choosing Z^1. The linear space Z^1 satisfies $Q^{r_{b_1},sing}_{d_{b_1}} \subseteq Z^1 \subseteq Q^{r_{b_1}}_{d_{b_1}}$ and $\dim Z^1 = r_{b_1} + (k - b_1 + 1) - x_{b_1} = r_{b_1} + \beta_1$. Hence Z^1 can be parameterized by $OG(\beta_1, d_{b_1} - r_{b_1})$. The linear spaces $T^{t+1}, O^{u,r_{b_1}}, \ldots, O^{2,r_{b_1}}$

can be parameterized by Grassmannians whose dimensions add up to $\beta_1(d_{b_1}+x_{b_1}-2(k-b_1+1)-\frac{\beta_1-1}{2})$ by the following. Note that $\dim OG(k,n)=k(n-2k+\frac{k-1}{2})$ (see [**4**] for a proof).

Coordinates of $\widetilde{V}$ in the $(t+1)$-st row:	*Dimensions of the corresponding Grassmannian:*
$Q^{r_{b_1},sing}_{d_{b_1}} \subseteq Z^1 \subseteq Q^{r_{b_1}}_{d_{b_1}}$	$\beta_1(d_{b_1}+x_{b_1}-2(k-b_1+1)-(r_{b_1}-x_{b_1})+\frac{\beta_1-1}{2})$
$Q^{r_{b_2},sing}_{d_{b_2}} \subseteq O^{2,r_{b_1}} \subseteq Z^1$	$(k-b_1+1-x_2)((r_{b_1}-x_{b_1})-(r_{b_2}-x_{b_2}))$
$Q^{r_{b_3},sing}_{d_{b_3}} \subseteq O^{3,r_{b_1}} \subseteq O^{2,r_{b_1}}$	$(k-b_1+1-x_3)((r_{b_2}-x_{b_2})-(r_{b_3}-x_{b_3}))$
$\vdots$	$\vdots$
$Q^{r_{b_c},sing}_{d_{b_c}} \subseteq O^{c,r_{b_1}} \subseteq O^{c-1,r_{b_1}}$	$(k-b_1+1-x_c)((r_{b_{c-1}}-x_{b_{c-1}})-(r_{b_c}-x_{b_c}))$
$O^{c+1,n_{a_t}} \subseteq O^{c+1,r_{b_1}} \subseteq O^{c,r_{b_1}}$	$(k-b_1+1-a_t)((r_{b_c}-x_{b_c})-(r_{b_{c+1}}-x_{b_{c+1}}))$
$\vdots$	$\vdots$
$O^{u,n_{a_t}} \subseteq O^{u,r_{b_1}} \subseteq O^{u-1,r_{b_1}}$	$(k-b_1+1-a_t)((r_{b_{u-1}}-x_{b_{u-1}})-(r_{b_u}-x_{b_u}))$
$T^t \subseteq T^{t+1} \subseteq O^{u,r_{b_1}}$	$(k-b_1+1-a_t)(r_{b_u}-x_{b_u})$

(4) As another case for the row that corresponds to $Q^{r_{b_1}}_{d_{b_1}}$, if $r_{b_1}<n_{a_t}$, then Z^1 is determined by $O^{1,n_{a_t}} \subseteq Z^1 \subseteq Q^{r_{b_1}}_{d_{b_1}}$. The linear space Z^1 has to be contained in the quadric cut out on $Q^{r_{b_1}}_{d_{b_1}}$ by the linear space everywhere tangent to $O^{1,n_{a_t}}$, that is, $Z^1 \subseteq Q^{r_{b_1}+(a_t-x_{b_1})}_{d_{b_1}-(a_t-x_{b_1})}$. Hence Z^1 can be parameterized by $OG(\beta_1, d_{b_1}-r_{b_1}-2(a_t-x_{b_1}))$. The parameterizations of $T^{t+1}, O^{u,r_{b_1}}, \dots, O^{2,r_{b_1}}$ are similar to the previous case, the total dimension is $\beta_1(d_{b_1}+x_{b_1}-2(k-b_1+1)-\frac{\beta_1-1}{2})$ as before. The diagram and the parameterizations in this case are as follows:

$$\begin{array}{ccccccccccc}
\vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\
T^t & \subseteq & O^{u,n_{a_t}} & \subseteq & \cdots & \subseteq & O^{2,n_{a_t}} & \subseteq & O^{1,n_{a_t}} & \subseteq & L_{n_{a_t}} \\
\cap\!| & & \cap\!| & & & & \cap\!| & & \cap\!| & & \cap\!| \\
T^{t+1} & \subseteq & O^{u,r_{b_1}} & \subseteq & \cdots & \subseteq & O^{2,r_{b_1}} & \subseteq & O^{1,r_{b_1}} & \subseteq & Q^{r_{b_1}}_{d_{b_1}} \\
\vdots & & \vdots & & & & \vdots & & \vdots & & \vdots
\end{array}$$

Coordinates of $\widetilde{V}$ in the $(t+1)$-st row:	*Dimensions of the corresponding Grassmannian:*
$O^{1,n_{a_t}} \subseteq Z^1 \subseteq Q^{r_{b_1}}_{d_{b_1}}$	$\beta_1(d_{b_1}+x_{b_1}-2(k-b_1+1)-(r_{b_1}-x_{b_1})+\frac{\beta_1-1}{2})$
$O^{2,n_{a_t}} \subseteq O^{2,r_{b_1}} \subseteq Z^1$	$(k-b_1+1-x_2)((r_{b_1}-x_{b_1})-(r_{b_2}-x_{b_2}))$
$\vdots$	$\vdots$
$T^t \subseteq T^{t+1} \subseteq O^{u,r_{b_1}}$	$(k-b_1+1-a_t)(r_{b_u}-x_{b_u})$

(5) Finally, the $(t+h)$-th row for some $h \geq 2$ is similar to the case above. The parameterizations are given by a tower of Grasmanninans contained in an orthogonal Grassmannian and the total dimension adds up to $d_{b_h} + x_{b_h} - 2(k - b_h + 1) - \frac{\beta_h - 1}{2}$. The diagram and the parameterizations are as follows:

$$\begin{array}{ccccccccccccc}
\vdots & & \vdots & & & & \vdots & & \vdots & & \vdots & & \vdots \\
T^{t+h-1} & \subseteq & O^{u,r_{b_{h-1}}} & \subseteq & \cdots & \subseteq & O^{h+1,r_{b_{h-1}}} & \subseteq & O^{h,r_{b_{h-1}}} & \subseteq & Z^{h-1} & \subseteq & Q^{r_{b_{h-1}}}_{d_{b_{h-1}}} \\
\cap\!\!| & & \cap\!\!| & & & & \cap\!\!| & & \cap\!\!| & & & & \cap\!\!| \\
T^{t+h} & \subseteq & O^{u,r_{b_u}} & \subseteq & \cdots & \subseteq & O^{h+1.r_{b_h}} & \subseteq & Z^h & \subseteq & & & Q^{r_{b_h}}_{d_{b_h}} \\
\vdots & & \vdots & & & & \vdots & & \vdots & & & & \vdots
\end{array}$$

Coordinates of $\widetilde{V}$ in the $(t+h)$-th row:	*Dimension of the corresponding Grassmannian:*
$O^{h,r_{b_{h-1}}} \subseteq Z^h \subseteq Q^{r_{b_h}}$	$\beta_h(d_{b_h} + x_{b_h} - 2(k - b_h + 1) - (r_{b_h} - x_{b_h}) + \frac{\beta_h - 1}{2})$
$O^{h+1,r_{b_{h-1}}} \subseteq O^{h+1,r_{b_h}} \subseteq Z^h$	$(b_{h-1} - b_h)((r_{b_h} - x_{b_h}) - (r_{b_{h+1}} - x_{b_{h+1}}))$
$\vdots$	$\vdots$
$T^{t+h-1} \subseteq T^{t+h} \subseteq O^{u,r_{b_h}}$	$(b_{h-1} - b_h)(r_{b_u} - x_{b_u})$

The variety $\widetilde{V}$ is smooth as it is an iterated tower of the ordinary and the orthogonal Grassmannian bundles observed above. The inverse image $\pi^{-1}(\Lambda)$ of a point Λ in V is irreducible by the same observations, hence $\widetilde{V}$ is irreducible for a marked restriction variety. Furthermore, combining the results from each row of the diagram, $\dim \widetilde{V}$ is given by

$$\dim \widetilde{V} = \sum_{g=1}^{t} \alpha_g \left(n_{a_g} - a_g\right) + \sum_{h=1}^{u} \beta_h \left(d_{b_h} + x_{b_h} - 2(k - b_h + 1) - \frac{\beta_h - 1}{2}\right) = \dim V$$

which concludes the proof. □

Over $V^0(L_\bullet, Q_\bullet)$, the inverse image of a point $\pi^{-1}(\Lambda)$ is determined uniquely by

$$T^g = \Lambda \cap L_{n_{a_g}}, \quad T^{t+h} = \Lambda \cap Q^{r_{b_h}}_{d_{b_h}},$$

$$O^{h,r_{b_\theta}} = \overline{Q^{r_{b_h},sing}_{d_{b_h}}, \ \Lambda \cap Q^{r_{b_\theta}}_{d_{b_\theta}}}, \quad O^{h,n_{a_\theta}} = \overline{Q^{r_{b_h},sing}_{d_{b_h}}, \ \Lambda \cap L_{n_{a_\theta}}} \quad \text{and}$$

$$Z^h = \overline{Q^{r_{b_h},sing}_{d_{b_h}}, \ \Lambda \cap Q^{r_{b_h}}_{d_{b_h}}} \qquad \text{for all } 1 \leq g \leq t, \ \ 1 \leq h \leq u.$$

$V^0(L_\bullet, Q_\bullet)$ is in the smooth locus of $V(L_\bullet, Q_\bullet)$ since it is homogeneous under the action of $SO(n)$. Then, Zariski's main theorem shows that π is an isomorphism over $V^0(L_\bullet, Q_\bullet)$. Therefore we have

THEOREM 3.6. *The map* $\pi : \widetilde{V}(L_\bullet, Q_\bullet) \to V(L_\bullet, Q_\bullet)$ *is a resolution of singularities.*

CHAPTER 4

The Exceptional Locus

We now study the exceptional locus of π. More specifically, we are interested in the codimension of the components of the exceptional locus.

Corresponding to the three types of conditions in Definition 2.1, namely,

$$\dim(\Lambda \cap Q_{d_i}^{r_i,sing}) = x_i, \ \ \dim(\Lambda \cap L_{n_j}) = j, \ \ \text{and} \ \ \dim(\Lambda \cap Q_{d_i}^{r_i}) = k - i + 1,$$

we consider three types of orbits where π has positive dimensional fibers. The following loci Σ categorize the closures of these orbits. The central orbits of any two Σ are disjoint if one is not contained in the other. This ensures that the fibers of π have the same dimension throughout each central orbit. The image of the exceptional locus of π is equal to the union of Σ's.

I: $\Sigma_{r_{b_h}}$: The closure of the locus of k-planes Λ such that $\dim(\Lambda \cap Q_{d_{b_h}}^{r_{b_h},sing}) = x_{b_h} + 1$ for some $1 \leq h \leq u$ and all the remaining conditions of V^0 are unchanged.

II: $\Sigma_{n_{a_g}}$: The closure of the locus of k-planes Λ such that $\dim(\Lambda \cap L_{n_{a_g}}) = a_g + 1$ for some $1 \leq g \leq t$ and all the remaining conditions of V^0 are unchanged.

III: $\Sigma_{d_{b_h}}$: The closure of the locus of k-planes Λ such that $\dim(\Lambda \cap Q_{d_{b_h}}^{r_{b_h}}) = k - b_h + 2$ for some $1 \leq h \leq u-1$ and all the remaining conditions of V^0 are unchanged.

Note that these loci do not always exist. There are natural numerical restrictions resulting from the rank conditions defining a restriction variety.

EXAMPLE 4.1. *The locus $\Sigma_{r_{b_1}}$ does not make sense for the restriction variety given by $\left[Q_8^0 \subseteq Q_9^0\right]$ since $Q_9^{0,sing}$ is empty. Similarly the locus $\Sigma_{r_{b_1}}$ does not exist* *for the restriction variety given by* $\left[L_1 \subseteq Q_7^1\right]$ *since* $x_1 = 1$ *and it is not possible to intersect* $Q_7^{1,sing}$ *in a higher dimension.*

EXAMPLE 4.2. *The locus* $\Sigma_{n_{a_1}}$ *does not exist for the restriction variety given by* $\left[L_1 \subseteq L_7 \subseteq L_8\right]$*; lines contained in* L_8 *containing* L_1 *cannot intersect* L_8 *or* L_1 *in higher dimension. Similarly,* $\Sigma_{r_{b_1}}$ *does not exist for the restriction variety given by* $\left[Q_7^2 \subseteq Q_8^1\right]$*.*

A special case for the existence of $\Sigma_{n_{a_t}}$ is when the restriction variety V lies in $OG(k, 2k)$. The orthogonal Grassmannian $OG(k, 2k)$ has two connected components and two linear spaces belong to the same connected component if and only if their intersection is equal to k mod 2. Thus, when defining $\Sigma_{n_{a_t}}$, it must be checked that the linear spaces in $\Sigma_{n_{a_t}}$ lie in the same component of the restriction variety.

EXAMPLE 4.3. *Let* $V = \left[L_2 \subseteq Q_4^0\right]$*, the variety of lines contained in a smooth quadric surface intersecting a fixed line on the surface. This is one of the components of the lines on the quadric surface. The fixed line in the partial flag, namely* L_2*, or equivalently the restriction variety given by the sequence* $\left[L_1 \subseteq L_2\right]$*, would be the locus defined as* $\Sigma_{n_{a_1}}$ *in this case and this does not lie in* V*. The variety* V *is actually isomorphic to* $\mathbb{P}^1$ *and hence smooth.*

EXAMPLE 4.4. *Let* V *be the restriction variety in* $OG(4, 8)$ *given by* $\left[L_1 \subseteq L_3 \subseteq L_4 \subseteq Q_7^1\right]$*. A general element* Λ *of* V *satisfies* $\dim(\Lambda \cap L_4) = 3$*, therefore* L_4 *and* Λ *lie in different components of* $OG(4, 8)$*. This shows that the restriction variety given by the sequence* $\left[L_1 \subseteq L_2 \subseteq L_3 \subseteq L_4\right]$*, which is a single point in* $OG(4, 8)$*, namely* L_4 *itself, does not lie in the closure of* V*. Thus the locus* $\Sigma_{n_{a_2}}$ *is not in the image of the exceptional locus of* π *in this case.*

EXAMPLE 4.5. *Let* V *be given by* $\left[L_3 \subseteq L_4 \subseteq Q_7^1 \subseteq Q_8^0\right]$*. A general element* Λ *of* V *satisfies* $\dim(\Lambda \cap L_4) = 2$*, and hence* L_4 *lies in the same component of* $OG(k, 2k)$ *as* V*. Using the same observation, since we have* $\dim(\Lambda \cap L_4) = 4$ *mod 2 for linear* *spaces* Λ *in the same component as* L_4*, we conclude* $\dim(\Lambda \cap L_4)$ *must be either 2 or 4. Therefore, in this case we have* $\Sigma_{n_{a_1}} = \Big[L_1 \subseteq L_2 \subseteq L_3 \subseteq L_4\Big]$.

More generally, the same consideration applies to $\Sigma_{n_{a_t}}$ when r_{b_1} is a *special index* (Definition 2.4).

EXAMPLE 4.6. *Let* $V = \Big[L_3 \subseteq L_4 \subseteq Q^1_7\Big]$. *Then the locus* $\Sigma_{n_{a_t}}$ *does not exist foe* V *since* L_4 *and the span* $\overline{\Lambda, Q_7^{1,sing}}$ *of a general element* Λ *in* V *with the singular locus of* Q^1_7 *lie in different components of the 4-dimensional linear spaces contained in* Q^1_7.

The following remark combines the observations we have made above about the definition of each type of locus Σ.

REMARK 4.7. *The numerical conditions for the definition of each type of locus* Σ *in the image of the exceptional locus of* π *can be given as:*

I: *The locus* $\Sigma_{r_{b_h}}$*, for some* $1 \leq h \leq u$*, exists if* $r_{b_h} > x_{b_h}$.

II: *The locus* $\Sigma_{n_{a_g}}$*, for some* $1 \leq g \leq t$*, exists if* $n_{a_g} > a_g$*. Moreover, if* $d_{b_1} + r_{b_1} = 2n_{a_t}$ *and* b_1 *is a special index, then* $\Sigma_{n_{a_t}}$ *exists if* $n_{a_t} > a_t + 1$ *and* $k > a_t + 1$.

III: *The locus* $\Sigma_{d_{b_h}}$*, for some* $1 \leq h \leq u-1$*, exists if* $u > 1$.

Over each Σ, $\pi^{-1}(\Sigma)$ is irreducible of codimension

$$\operatorname{codim}(\pi^{-1}(\Sigma)) = \operatorname{codim}(\Sigma) - \dim(\pi^{-1}(\Lambda))$$

for a general point Λ in Σ. We now consider each Σ separately and study $\operatorname{codim}(\pi^{-1}(\Sigma))$ in each case. We summarize our computations in Observation 4.24.

I: $\Sigma_{r_{b_h}}$: $\dim(\Lambda \cap Q^{r_{b_h},sing}_{d_{b_h}}) = x_{b_h} + 1$ for some $1 \leq h \leq u$

Given the corank r_{b_h}, we divide this case into sub-cases depending on the relation between r_{b_h} and the dimensions n_{a_g} of the isotropic linear spaces appearing in the sequence defining V. The sub-cases we consider in the following are:

I.A: $r_{b_h} > n_s$

I.B: $r_{b_h} < n_s$ and $r_{b_h} \neq n_j$ for all j

I.C: $r_{b_h} = n_j$ for some $n_j < n_s$

I.D: $r_{b_1} = n_s$

I.A: Suppose $r_{b_h} > n_s$. A general element of $\Sigma_{r_{b_h}}$ is obtained by specializing $\Lambda \in V^0$ so that it intersects $Q_{d_{b_h}}^{r_{b_h},sing}$ in one more dimension. Equivalently, this is the restriction variety associated to the sequence obtained by putting $L_{r_{b_h}}$ to the right of L_{n_s}, in the place of $Q_{d_{k-s}}^{r_{k-s}}$. Note that $\Sigma_{r_{b_{h-1}}}$ contains $\Sigma_{r_{b_h}}$, so all $\Sigma_{r_{b_h}}$ with $r_{b_h} > n_s$ are contained in $\Sigma_{r_{b_1}}$. Therefore it is sufficient to consider $\Sigma_{r_{b_1}}$.

EXAMPLE 4.8. *Let V be the restriction variety given by the sequence $\left[L_3 \subseteq Q_{10}^7 \subseteq Q_{20}^5\right]$. The loci $\Sigma_{r_{b_1}}$ and $\Sigma_{r_{b_2}}$ are defined as the closures of the following loci:*

$$\Sigma_{r_{b_1}}^0 := \left\{\Lambda \in V \mid \dim(\Lambda \cap Q_{10}^{7,sing}) = 2 \textit{ with other conditions of } V^0 \textit{ unchanged}\right\}$$

$$\Sigma_{r_{b_2}}^0 := \left\{\Lambda \in V \mid \dim(\Lambda \cap Q_{20}^{5,sing}) = 2 \textit{ with other conditions of } V^0 \textit{ unchanged}\right\}$$

Equivalently, they are given by the sequences $\Sigma_{r_{b_2}} = \left[L_3 \subseteq L_5 \subseteq Q_{18}^7\right]$ and $\Sigma_{r_{b_1}} = \left[L_3 \subseteq L_7 \subseteq Q_{20}^5\right]$. Since $\Sigma_{r_{b_2}}$ is contained in $\Sigma_{r_{b_1}}$, it is sufficient to consider $\operatorname{codim}(\pi^{-1}(\Sigma_{r_{b_1}}))$.

As a result of the specialization, x_i increases by 1 for $\beta_1 - 1$ sub-quadrics, namely, for $Q_{d_i}^{r_i}$ with $b_1 \leq i < k-s$. These are the sub-quadrics that are in the same group as $Q_{d_{b_1}}^{r_{b_1}}$; the newly introduced $L_{r_{b_h}}$ is the isotropic linear space in the modified sequence that is contained in $Q_{d_i}^{r_i}$ for $b_1 \leq i < k-s$. The difference between the dimensions of the varieties obtained by the original and the modified sequence can be calculated using Observation 2.9.

$$\begin{aligned}\operatorname{codim}(\Sigma_{r_{b_1}}) &= \big(d_{k-s} + x_{k-s} - 2(s+1)\big) - \big((r_{b_1} - (s+1)) - (\beta_1 - 1)\big)\\ &= d_{k-s} - r_{b_1} - \beta_1\end{aligned}$$

since $x_{k-s} = s$ by our assumption that $r_{b_1} > n_s$.

Now we study the inverse image $\pi^{-1}(\Lambda)$ of a general point Λ in $\Sigma_{r_{b_1}}$. By assumption there is no O containing $Q^{r_{b_1},sing}_{d_{b_1}}$ and O's contained in $Q^{r_{b_1},sing}_{d_{b_1}}$ are determined uniquely by Λ. We have $\overline{T^{t+1}, Q^{r_{b_1},sing}_{d_{b_1}}} \subseteq Z_1 \subseteq Q^{r_{b_1}}_{d_{b_1}}$ where $\dim(\overline{T^{t+1}, Q^{r_{b_1},sing}_{d_{b_1}}}) = r_{b_1} + (k - b_1 + 1) - (x_{b_1} + 1)$ and $\dim Z_1 = \dim(\overline{T^{t+1}, Q^{r_{b_1},sing}_{d_{b_1}}}) + 1$. Since Z^1 has to lie in the orthogonal complement of $\overline{T^{t+1}, Q^{r_{b_1},sing}_{d_{b_1}}}$, we have $\overline{T^{t+1}, Q^{r_{b_1},sing}_{d_{b_1}}} \subseteq Z_1 \subseteq Q^{r_{b_1}+(k-b_1+1-x_{b_1}-1)}_{d_{b_1}-(k-b_1+1-x_{b_1}-1)}$. Such Z_1 can be parameterized by $OG(1, d_{b_1} - r_{b_1} - 2(k - b_1 + 1 - x_{b_1} - 1))$. Therefore

$$\begin{aligned}\operatorname{codim}(\pi^{-1}(\Sigma_{r_{b_1}})) &= d_{k-s} - r_{b_1} - \beta_1 - \Big(d_{b_1} - r_{b_1} - 2(k - b_1 + 1 - x_{b_1} - 1) - 2\Big)\\ &= d_{k-s} - d_{b_1} + 2(k - b_1 + 1 - x_{b_1}) - \beta_1\\ &= 1\end{aligned}$$

since $d_{b_1} - d_{k-s} = \beta_1 - 1$ and $k - b_1 + 1 - s = \beta_1$.

EXAMPLE 4.9. *Let* $V = \Big[L_3 \subseteq Q^7_{10} \subseteq Q^5_{20}\Big]$, *then*

$$\begin{aligned}\widetilde{V} = \{(T^1, T^2, T^3, Z^1, Z^2, O^{2,r_{b_1}}) \mid Q^{5,sing}_{20} \subseteq O^{2,r_{b_1}} \subseteq Z^2,\ \ Q^{7,sing}_{10} \subseteq Z^1,\\ T^1 \subseteq L_3,\ \ T_2 \subseteq O^{2,r_{b_1}} \subseteq Z^1 \subseteq Q^7_{10},\ \ T^3 \subseteq Z^2 \subseteq Q^5_{20}\ ,\}\end{aligned}$$

equivalently, the diagram is the following.

FIGURE 5. Definition of $\widetilde{V}$ for $V = \left[L_3 \subseteq Q^7_{10} \subseteq Q^5_{20}\right]$

$$
\begin{array}{ccccccc}
T^1 & \subseteq & L_3 & & & & \\
\rotatebox[origin=c]{90}{$\subseteq$} & & \rotatebox[origin=c]{90}{$\subseteq$} & & & & \\
 & & Q_{20}^{5,sing} & \subseteq & Q_{10}^{7,sing} & & \\
 & & \rotatebox[origin=c]{90}{$\subseteq$} & & \rotatebox[origin=c]{90}{$\subseteq$} & & \\
T^2 & \subseteq & O^{2,r_{b_1}} & \subseteq & Z^1 & \subseteq & Q^7_{10} \\
\rotatebox[origin=c]{90}{$\subseteq$} & & \rotatebox[origin=c]{90}{$\subseteq$} & & & & \rotatebox[origin=c]{90}{$\subseteq$} \\
T^3 & \subseteq & Z^2 & \subseteq & & & Q^5_{20}
\end{array}
$$

The subvariety $\Sigma_{r_{b_1}} = \left[L_3 \subseteq L_7 \subseteq Q^5_{20}\right] \subseteq V$ *has codimension 2. In the inverse image* $\pi^{-1}(\Lambda)$ *of a general point* Λ *in* $\Sigma_{r_{b_1}}$*, we have* $T^3 = \Lambda$, $T^2 = \Lambda \cap Q^7_{10} = \Lambda \cap L_7$, $T^1 = \Lambda \cap L_3$, $O^{2,r_{b_1}} = \overline{Q_{20}^{5,sing}, \Lambda \cap Q^7_{10}}$, $Z^2 = \overline{Q_{20}^{5,sing}, \Lambda}$ *and* $Q_{10}^{7,sing} \subseteq Z^1 \subseteq Q^7_{20}$ *where* $\dim Z^1 = 8$*. The linear space* Z^1 *is parameterized by a smooth plane quadric, or equivalently,* $OG(1,3)$*. Thus* $\dim(\pi^{-1}(\Lambda)) = 1$ *and* $\operatorname{codim}(\pi^{-1}(\Sigma_{r_{b_1}})) = 1$.

EXAMPLE 4.10. *Let* $V = \left[L_1 \subseteq Q^3_6 \subseteq Q^1_8\right]$*, an orthogonal Schubert variety in* $OG(3,9)$*. The following diagram defines* $\widetilde{V}$*.*

FIGURE 6. Definition of $\widetilde{V}$ for $V = \left[L_1 \subseteq Q^3_6 \subseteq Q^1_8\right]$

$$
\begin{array}{ccccc}
T^1 & \subseteq & L_1 & & \\
\rotatebox[origin=c]{90}{$\subseteq$} & & \rotatebox[origin=c]{90}{$\subseteq$} & & \\
 & & Q_6^{3,sing} & & \\
 & & \rotatebox[origin=c]{90}{$\subseteq$} & & \\
T^2 & \subseteq & Z^1 & \subseteq & Q^3_6 \\
\rotatebox[origin=c]{90}{$\subseteq$} & & & & \rotatebox[origin=c]{90}{$\subseteq$} \\
T^3 & \subseteq & & \subseteq & Q^1_8
\end{array}
$$

The subvariety $\Sigma_{r_{b_1}} = \left[L_1 \subseteq L_3 \subseteq Q^1_8\right]$ *has codimension 2. In the inverse image* $\pi^{-1}(\Lambda)$ *of a general point* Λ *in* $\Sigma_{r_{b_1}}$*, only* Z^1 *is not determined uniquely. We have* $\dim Z^1 = 4$ *and* $Q_6^{3,sing} \subseteq Z^1 \subseteq Q^3_6$*, from which we conclude* Z^1 *is parameterized by* $OG(1,3)$*. Thus* $\dim(\pi^{-1}(\Lambda)) = 1$ *and* $\operatorname{codim}(\pi^{-1}(\Sigma_{r_{b_1}})) = 1$.

I.B: Next we consider $\Sigma_{r_{b_h}}$ such that there are L_{n_j} in the sequence with $r_{b_h} < n_j$ but no L_{n_j} with $n_j = r_{b_h}$. Let $n_{j_\sharp} := \min\{n_j \mid r_{b_h} < n_j\}$. If $r_{b_{h-1}}$ satisfies $r_{b_h} < r_{b_{h-1}} < n_{j_\sharp}$ then $\Sigma_{r_{b_{h-1}}}$ contains $\Sigma_{r_{b_h}}$. Therefore it is sufficient to consider r_{b_h} such that $r_{b_h} < n_{j_\sharp} < r_{b_{h-1}}$. Note that in this case $\max\{r_i \mid r_i < n_{j_\sharp}\} = r_{b_h} - (\beta_h - 1) = r_{b_{h-1}+1}$.

Specializing Λ so that it intersects $Q^{r_{b_h},sing}_{d_{b_h}}$ in one more dimension is equivalent to making two changes: The first one is changing $L_{n_{j_\sharp}}$ to $L_{r_{b_h}}$, an isotropic linear subspace of dimension r_{b_h}, so that the condition for $\Sigma_{r_{b_h}}$ is satisfied, that is, $\dim(\Lambda \cap Q^{r_{b_h}}_{d_{b_h}})$ increases by one. The second one is changing $Q^{r_{b_{h-1}+1}}_{d_{b_{h-1}+1}}$ to $Q^{n_{j_\sharp}}_{d_{b_{h-1}+1}-(n_{j_\sharp}-r_{b_{h-1}+1})}$ induced by the other conditions of V^0. The linear space $L_{r_{b_h}}$ is an additional isotropic linear space in the modified sequence that is contained in $Q^{n_{j_\sharp},sing}_{d_{b_{h-1}+1}-(n_{j_\sharp}-r_{b_{h-1}+1})}$; this increases $x_{b_{h-1}+1}$ by 1. Comparing the modified sequence's dimension with the original one's, we have

$$\operatorname{codim}(\Sigma_{r_{b_h}}) = n_{j_\sharp} - r_{b_h} + n_{j_\sharp} - r_{b_{h-1}+1} - 1.$$

EXAMPLE 4.11. *Let* $V = \left[L_7 \subseteq Q^5_{15} \subseteq Q^2_{25}\right]$, *then* $\Sigma_{r_{b_1}} = \left[L_5 \subseteq Q^7_{13} \subseteq Q^2_{25}\right]$. *Specializing a general element* Λ *of* V *so that it intersects* L_5 *increases* x_2 *by 1. In this example,* $\operatorname{codim}(\Sigma_{r_{b_1}}) = 2 + 2 - 1 = 3$.

REMARK 4.12. *Changing* $Q^{r_{b_{h-1}+1}}_{d_{b_{h-1}+1}}$ *to* $Q^{n_{j_\sharp}}_{d_{b_{h-1}+1}-(n_{j_\sharp}-r_{b_{h-1}+1})}$ *ensures that the rest of the conditions of* V^0 *remain unchanged in* $\Sigma_{r_{b_h}}$. *In the previous example, in the sequence of* $\Sigma_{r_{b_1}}$, *we have* Q^7_{13} *instead of* Q^5_{15} *to ensure that for general* Λ, $\dim(\Lambda \cap L_7) = 1$ *which is one of the conditions of* V^0 *that remains unchanged for a general element in* $\Sigma_{r_{b_1}}$.

Note that the linear space $L_{n_{j_\sharp}}$ may not be among $L_{n_{a_g}}$, that is, the largest dimensional isotropic linear space in a group with consecutively increasing dimensions. Let $L_{n_{a_{g_\sharp}}}$ be the smallest $L_{n_{a_g}}$ containing $L_{n_{j_\sharp}}$. In the inverse image $\pi^{-1}(\Lambda)$ of a general point Λ in $\Sigma_{r_{b_h}}$, all coordinates are determined uniquely except for

$O^{h,n_{a_{g_\sharp}}}$ and Z^h. We have $\overline{Q_{d_{b_h}}^{r_{b_h},sing}, \Lambda \cap L_{n_{a_{g_\sharp}}}} \subseteq O^{h,n_{a_{g_\sharp}}} \subseteq L_{n_{a_{g_\sharp}}}$ thus $O^{h,n_{a_{g_\sharp}}}$ can be parameterized by $G(1, n_{a_{g_\sharp}} - (r_{b_h} + a_{g_\sharp} - x_{b_h}) + 1)$. Then Z^h is determined uniquely as $\overline{O^{h,n_{a_{g_\sharp}}}, \Lambda \cap Q_{d_{b_h}}^{r_{b_h}}}$. Thus $\dim(\pi^{-1}(\Lambda)) = n_{a_{g_\sharp}} - (r_{b_h} + a_{g_\sharp} - x_{b_h})$ and

$$\begin{aligned}
\operatorname{codim}(\pi^{-1}(\Sigma_{r_{b_h}})) &= n_{j_\sharp} - r_{b_h} + \beta_h(n_{j_\sharp} - r_{b_{h-1}+1}) - \beta_h - (n_{a_{g_\sharp}} - (r_{b_h} + a_{g_\sharp} - x_{b_h})) \\
&= \Big((n_{j_\sharp} - r_{b_h}) - (n_{a_{g_\sharp}} - (r_{b_h} + a_{g_\sharp} - x_{b_h}))\Big) + \beta_h(n_{j_\sharp} - r_{b_{h-1}+1} - 1) \\
&\geq 2
\end{aligned}$$

since $\Big((n_{j_\sharp} - r_{b_h}) - (n_{a_{g_\sharp}} - (r_{b_h} + a_{g_\sharp} - x_{b_h}))\Big) \geq 1$ and there is no $n_j,\ r_i$ such that $n_j - r_i = 1$ by the condition on variation of tangent spaces.

EXAMPLE 4.13. *Let* $V = \Big[L_6 \subseteq L_7 \subseteq Q_{15}^2\Big]$, *then* $\widetilde{V}$ *is given by the following diagram.*

FIGURE 7. Definition of $\widetilde{V}$ for $V = \Big[L_6 \subseteq L_7 \subseteq Q_{15}^2\Big]$

$$\begin{array}{ccccc}
 & & Q_{15}^{2,sing} & & \\
 & & \rotatebox{90}{$\supseteq$} & & \\
T^1 & \subseteq & O^{1,n_{a_1}} & \subseteq & L_7 \\
\rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} \\
T^2 & \subseteq & Z^1 & \subseteq & Q_{15}^2
\end{array}$$

The subvariety $\Sigma_{r_{b_1}} = \Big[L_2 \subseteq L_7 \subseteq Q_{15}^2\Big]$ *has codimension 7. In the inverse image* $\pi^{-1}(\Lambda)$ *of a general point* Λ *in* $\Sigma_{r_{b_1}}$, *we have* $T^2 = \Lambda, T^1 = \Lambda \cap L_7$. *As above,* Z^1 *is determined as* $Z^1 = \overline{O^{1,n_{a_1}}, \Lambda}$ *so the nontrivial part is the parametrization of* $O^{1,n_{a_1}}$. *We have* $\overline{Q_{15}^{2,sing}, \Lambda \cap L_7} \subseteq O^{1,n_{a_1}} \subseteq L_7$ *which is parameterized by* $G(1,4)$. *Thus* $\dim(\pi^{-1}(\Lambda)) = 3$ *and* $\operatorname{codim}\pi^{-1}(\Sigma_{r_{b_1}}) = 7 - 3 = 4$.

EXAMPLE 4.14. *Let* $V = \Big[L_7 \subseteq Q_{15}^5 \subseteq Q_{25}^2\Big]$, *then* $\widetilde{V}$ *is given by the following diagram.*

FIGURE 8. Definition of $\widetilde{V}$ for $V = \left[L_7 \subseteq Q_{15}^5 \subseteq Q_{25}^2\right]$

$$
\begin{array}{ccccccc}
 & & Q_{25}^{2,sing} & & Q_{15}^{5,sing} & & \\
 & & \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & \\
T^1 & \subseteq & O^{2,n_{a_1}} & \subseteq & O^{1,n_{a_1}} & \subseteq & L_7 \\
\rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} \\
T^2 & \subseteq & O^{2,r_{b_1}} & \subseteq & Z^1 & \subseteq & Q_{15}^5 \\
\rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & & & \rotatebox{90}{$\supseteq$} \\
T^3 & \subseteq & Z^2 & \subseteq & & & Q_{25}^2
\end{array}
$$

The subvariety $\Sigma_{r_{b_1}} = \left[L_5 \subseteq Q_{13}^7 \subseteq Q_{25}^2\right]$ *has codimension 3. In the inverse image* $\pi^{-1}(\Lambda)$ *of a general point* Λ *in* $\Sigma_{r_{b_1}}$*, we have* $T^3 = \Lambda$*,* $T^2 = \Lambda \cap Q_{15}^5 = \Lambda \cap Q_{13}^7$*,* $T^1 = \Lambda \cap L_7 = \Lambda \cap L_5$*,* $O^{2,n_{a_1}} = \overline{Q_{25}^{2,sing}, \Lambda \cap L_7}$*,* $O^{2,r_{b_1}} = \overline{Q_{25}^{2,sing}, \Lambda \cap Q_{15}^5}$*,* $Z^2 = \overline{Q_{25}^{2,sing}, \Lambda}$*. The linear space* $O^{1,n_{a_1}}$ *satisfies* $Q_{15}^{5,sing} \subseteq Q_{15}^5 \subseteq L_7$ *and hence can be parameterized by* $G(1,2)$*. Then* Z^1 *is determined uniquely as* $Z^1 = \overline{O^{1,n_{a_1}}, \Lambda \cap Q_{15}^5}$*. Thus* $\dim(\pi^{-1}(\Lambda)) = 1$ *and* $\operatorname{codim}(\pi^{-1}(\Sigma_{r_{b_1}})) = 2$.

I.C: Now we consider r_{b_h} such that there is L_{n_j} with $n_j < n_s$ in the defining sequence satisfying $n_j = r_{b_h}$. Since there is no L_{n_j} in the sequence with $n_j = r_{b_h}+1$, we have $r_{b_h} = n_{a_g}$ for some $1 \leq g \leq t-1$. Specializing Λ so that it intersects $Q_{d_{b_h}}^{r_{b_h},sing} = L_{n_j}$ in one more dimension is equivalent to making two changes: The first one is moving the group of α_g isotropic linear spaces $L_{n_{a_g}-\alpha_g+1}, L_{n_{a_g}-\alpha_g+2}, \ldots, L_{n_{a_g}}$ one position to the right and putting $L_{n_{a_g}-\alpha_g}$ in the sequence to the left of these linear spaces. The second one is changing $Q_{d_i}^{r_i}$ to $Q_{d_i-(n_{a_g+1}-r_{b_{h-1}}+1)}^{r_i+(n_{a_g+1}-r_{b_{h-1}}+1)}$ for all i such that $b_h \leq i < b_{h-1}$. This increases x_i by 1 for $b_h \leq i < b_{h-1}$ as $L_{n_{a_g}-\alpha_g}$ is an additional isotropic linear space in the modified sequence that is contained in the singular locus of each $Q_{d_i-(n_{a_g+1}-r_{b_{h-1}}+1)}^{r_i+(n_{a_g+1}-r_{b_{h-1}}+1)}$ with $b_h \leq i < b_{h-1}$.

Note that even if there is a sub-quadric $Q_{d_\xi}^{r_\xi}$ in the sequence with a singular locus of dimension $n_{a_g} - \alpha_g - 1$, this change turns it into $Q_{d_\xi-(\alpha_g+1)}^{r_\xi+(\alpha_g+1)}$. Then x_ξ increases by (α_g+1) and the dimension of $\Sigma_{r_{b_h}}$ does not change.

Observation 2.9 can be used to calculate the dimensions of both sequences, we have

$$\begin{aligned}\operatorname{codim}(\Sigma_{r_{b_1}}) = {} & \alpha_g \left(n_{a_g} - a_g\right) + \alpha_{g+1}\left(n_{a_{g+1}} - a_{g+1}\right) \\ & - (\alpha_g + 1)\left(n_{a_g} - a_g - 1\right) - (\alpha_{g+1} - 1)\left(n_{a_{g+1}} - a_{g+1}\right) \\ & + \beta_h(n_{a_g+1} - r_{b_{h-1}+1}) - \beta_h.\end{aligned}$$

In the inverse image $\pi^{-1}(\Lambda)$ of a general point Λ in $\Sigma_{r_{b_h}}$, all coordinates are determined uniquely except for $O^{h,n_{a_{g+1}}}$, Z^h and the coordinates in the g-th row. We have $\overline{Q^{r_{b_h},sing}_{d_{b_h}}, \Lambda \cap L_{n_{a_{g+1}}}} \subseteq O^{h,n_{a_{g+1}}} \subseteq L_{n_{a_{g+1}}}$ thus $O^{h,n_{a_{g+1}}}$ can be parameterized by $G(1, n_{a_{g+1}} - (r_{b_h} + a_{g+1} - x_{b_h}) + 1) = G(1, n_{a_{g+1}} - n_{a_g} - \alpha_{g+1} + 1)$. Then Z^h is determined uniquely as $\overline{O^{h,n_{a_{g+1}}}, \Lambda \cap Q^{r_{b_h}}_{d_{b_h}}}$. On the other hand, the g-th row is determined uniquely once T^g is determined. The linear space T^g satisfies $T^{g-1} \subseteq T^g \subseteq \Lambda \cap L_{n_{a_g}}$ and hence can be parameterized by $G(\alpha_g, \alpha_g + 1)$. Thus $\dim(\pi^{-1}(\Lambda)) = n_{a_{g+1}} - n_{a_g} - \alpha_{g+1} + \alpha_g$ and

$$\operatorname{codim}(\pi^{-1}(\Sigma_{r_{b_h}})) = \beta_h(n_{a_g+1} - r_{b_{h-1}+1} - 1) + 1$$

which is greater than 1 as there is no n_j, r_i such that $n_j - r_i = 1$ in the defining sequence.

EXAMPLE 4.15. *Let $V = \left[L_2 \subseteq L_4 \subseteq Q^2_7\right]$, an orthogonal Schubert variety in $OG(3,9)$. The definition of $\widetilde{V}$ is given by the following diagram.*

FIGURE 9. Definition of $\widetilde{V}$ for $V = \left[L_2 \subseteq L_4 \subseteq Q^2_7\right]$

$$\begin{array}{ccccc} T^1 & \subseteq & L_2 & & \\ \cap\vert & & \Vert & & \\ & & Q^{2,sing}_7 & & \\ & & \cap\vert & & \\ T^2 & \subseteq & O^1 & \subseteq & L_4 \\ \cap\vert & & \cap\vert & & \cap\vert \\ T^3 & \subseteq & Z^1 & \subseteq & Q^2_7 \end{array}$$

The subvariety $\Sigma_{r_{b_1}}$ is given by the sequence $\Big[L_1 \subseteq L_2 \subseteq L_4\Big]$ as Q^2_7 becomes L_4 if its corank is increased by 2. The variety $\Sigma_{r_{b_1}}$ has codimension 4. In the inverse image $\pi^{-1}(\Lambda)$ of a general point Λ in $\Sigma_{r_{b_1}}$, the coordinates T^3 and O^1 are determined uniquely as $T^3 = O^1 = \Lambda$ and $T^2 = L_2$. The coordinate T^1 satisfies $T^1 \subseteq L_2$ and is parameterized by $G(1,2)$. The coordinate Z^1 satisfies $O^1 \subseteq Z^1 \subseteq Q^3_7$ and is parameterized by $OG(1,3)$. Thus $\dim(\pi^{-1}(\Lambda)) = 2$ *and* $\operatorname{codim}(\pi^{-1}(\Sigma_{r_{b_1}})) = 2$.

EXAMPLE 4.16. *Let $V = \Big[L_5 \subseteq L_{10} \subseteq Q^6_{19} \subseteq Q^5_{20} \subseteq Q^2_{30}\Big]$, then $\widetilde{V}$ is given by the following diagram.*

FIGURE 10. Definition of $\widetilde{V}$ for $V = \Big[L_5 \subseteq L_{10} \subseteq Q^6_{19} \subseteq Q^5_{20} \subseteq Q^2_{30}\Big]$

$$
\begin{array}{ccccccc}
 & & Q^{2,sing}_{30} & \subseteq & Q^{5,sing}_{20} & & \\
 & & \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & \\
T^1 & \subseteq & O^{2,n_{a_1}} & \subseteq & & & L_5 \\
\rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & & & \rotatebox{90}{$\supseteq$} \\
T^2 & \subseteq & O^{2,n_{a_2}} & \subseteq & O^{1,n_{a_2}} & \subseteq & L_{10} \\
\rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} \\
T^3 & \subseteq & O^{2,r_{b_1}} & \subseteq & Z^1 & \subseteq & Q^5_{20} \\
\rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} & & & & \rotatebox{90}{$\supseteq$} \\
T^4 & \subseteq & Z^2 & \subseteq & & & Q^2_{30}
\end{array}
$$

The subvariety $\Sigma_{r_{b_1}} = \Big[L_4 \subseteq L_5 \subseteq Q^{10}_{15} \subseteq Q^9_{16} \subseteq Q^2_{30}\Big]$ has codimension 12. In the inverse image $\pi^{-1}(\Lambda)$ of a general point Λ in $\Sigma_{r_{b_1}}$, we have $T^4 = \Lambda$, $T^3 = \Lambda \cap Q^9_{16} = \Lambda \cap Q^5_{20}$, $T^2 = \Lambda \cap L_5 = \Lambda \cap L_{10}$, $O^{2,n_{a_2}} = \overline{Q^{2,sing}_{30}, \Lambda \cap L_{10}}$, $O^{2,r_{b_1}} = \overline{Q^{2,sing}_{30}, \Lambda \cap Q^5_{20}}$, $Z^2 = \overline{Q^{2,sing}_{30}, \Lambda}$. The linear space $O^{1,n_{a_2}}$ satisfies $\overline{Q^{5,sing}_{20}, \Lambda \cap L_{10}} \subseteq O^{1,n_{a_2}} \subseteq L_{10}$ and hence can be parameterized by $G(1,5)$. Then Z^1 is determined uniquely as $Z^1 = \overline{O^{1,n_{a_2}}, \Lambda \cap Q^5_{20}}$. On the other hand, T^1 satisfies $T^1 \subseteq \Lambda \cap L_5$ and hence can be parameterized by $G(1,2)$. Then $O^{2,n_{a_1}} = \overline{Q^{2,sing}_{30}, T^1}$. Thus $\dim(\pi^{-1}(\Lambda)) = 5$ *and* $\operatorname{codim}(\pi^{-1}(\Sigma_{r_{b_1}})) = 12 - 5 = 7$.

I.D: The only remaining case is when $r_{b_1} = n_{a_t}$. Specializing Λ so that it intersects $Q^{r_{b_h},sing}_{d_{b_h}} = L_{n_{a_t}}$ in one more dimension is equivalent to making two changes: The first one is moving the group of α_t isotropic linear spaces $L_{n_{a_t}-\alpha_t+1}$, $L_{n_{a_t}-\alpha_t+2}$, … , $L_{n_{a_t}}$ one position to the right and putting $L_{n_{a_t}-\alpha_t}$ in the sequence to the left of these linear spaces. The second one is omitting $Q^{r_{k-s}}_{d_{k-s}}$ from the sequence. This increases x_i by 1 for $b_1 \leq i < k-s$ as $L_{n_{a_t}-\alpha_t}$ is an additional isotropic linear space in the modified sequence that is contained in the singular locus of each $Q^{r_i}_{d_i}$ with $b_1 \leq i < k-s$. We have

$$\begin{aligned}\operatorname{codim}(\Sigma_{r_{b_1}}) &= \alpha_t\,(n_{a_t} - a_t) + \sum_{t=1}^{\beta_1} (d_{b_1} + x_{b_1} - 2(s+\beta_1) + t - 1) \\ &\quad - (\alpha_t + 1)\,(n_{a_t} - a_t - 1) - \sum_{t=1}^{\beta_1 - 1} (d_{b_1} + x_{b_1} - 2(s+\beta_1) + t - 1) - (\beta_1 - 1) \\ &= \alpha_t + d_{b_1} - n_s - 2\beta_1 + 1\end{aligned}$$

Note that by assumption there is no O containing $Q^{r_{b_1}}_{d_{b_1}}$ and other O's are determined uniquely as there is no change in the relevant rank conditions. The only nontrivial parameterizations are observed for Z^1 and the coordinates in the t-th row. As in (I.A), we have $\overline{T^{t+1}, Q^{r_{b_1},sing}_{d_{b_1}}} \subseteq Z_1 \subseteq Q^{r_{b_1}}_{d_{b_1}}$ where $\dim(\overline{T^{t+1}, Q^{r_{b_1},sing}_{d_{b_1}}}) = r_{b_1} + (k - b_1 + 1) - (x_{b_1} + 1)$ and $\dim(Z_1) = \dim(\overline{T^{t+1}, Q^{r_{b_1},sing}_{d_{b_1}}}) + 1$. Since Z^1 has to lie in the orthogonal complement of $\overline{T^{t+1}, Q^{r_{b_1},sing}_{d_{b_1}}}$, we have $\overline{T^{t+1}, Q^{r_{b_1},sing}_{d_{b_1}}} \subseteq Z_1 \subseteq Q^{r_{b_1}+(k-b_1+1-x_{b_1}-1)}_{d_{b_1}-(k-b_1+1-x_{b_1}-1)}$. Such Z_1 can be parameterized by $OG(1, d_{b_1} - r_{b_1} - 2(k - b_1 + 1 - x_{b_1} - 1))$. On the other hand, the t-th row can be determined once T^1 is determined. The linear space T^1 satisfies $T^t \subseteq \Lambda \cap L_{n_{a_t}}$ and hence can be parameterized by $G(\alpha_t, \alpha_t + 1)$. Thus $\dim(\pi^{-1}(\Lambda)) =$ $d_{b_1} - r_{b_1} - 2(k - b_1 + 1 - x_{b_1} - 1) - 2 + \alpha_t$ and we have

$$\operatorname{codim}(\pi^{-1}(\Sigma_{r_{b_1}})) = 1.$$

EXAMPLE 4.17. *Let* $V = \left[L_2 \subseteq L_3 \subseteq Q^3_7\right]$, *an orthogonal Schubert variety in* $OG(3,9)$. *The following diagram defines* $\widetilde{V}$.

FIGURE 11. Definition of $\widetilde{V}$ for $V = \left[L_2 \subseteq L_3 \subseteq Q_7^3\right]$

$$
\begin{array}{ccccc}
T^1 & \subseteq & L_3 & & \\
\rotatebox{-90}{$\subseteq$} & & \| & & \\
 & & Q_6^{3,sing} & & \\
 & & \rotatebox{-90}{$\subseteq$} & & \\
T^2 & \subseteq & Z^1 & \subseteq & Q_6^3
\end{array}
$$

The subvariety $\Sigma_{r_{b_1}} = \left[L_1 \subseteq L_2 \subseteq L_3\right]$*, which consists of a single point, has codimension 4. In the inverse image* $\pi^{-1}(\Lambda)$ *of a general point* Λ *in* $\Sigma_{r_{b_1}}$*, we have* $T^1 \subseteq L_3$ *whih is parameterized by* $G(2,3)$*. Also,* $\dim(Z^1) = 4$ *with* $Q_6^{3,sing} \subseteq Z^1 \subseteq Q_6^3$*, so* Z^1 *is parameterized by* $OG(1,3)$ *which has dimension 1. Thus* $\dim(\pi^{-1}(\Lambda)) = 3$ *and* $\operatorname{codim}(\pi^{-1}(\Sigma_{r_{b_1}})) = 1$*.*

EXAMPLE 4.18. *Let* $V = \left[L_5 \subseteq Q_{10}^5 \subseteq Q_{30}^2\right]$*, then* $\widetilde{V}$ *is given by the following diagram.*

FIGURE 12. Definition of $\widetilde{V}$ for $V = \left[L_5 \subseteq Q_{10}^5 \subseteq Q_{30}^2\right]$

$$
\begin{array}{ccccccc}
 & & Q_{30}^{2,sing} & \subseteq & Q_{10}^{5,sing} & & \\
 & & \rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & \\
T^1 & \subseteq & O^{2,n_{a_1}} & \subseteq & & & L_5 \\
\rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & & & \rotatebox{-90}{$\subseteq$} \\
T^2 & \subseteq & O^{2,r_{b_1}} & \subseteq & Z^1 & \subseteq & Q_{10}^5 \\
\rotatebox{-90}{$\subseteq$} & & \rotatebox{-90}{$\subseteq$} & & & & \rotatebox{-90}{$\subseteq$} \\
T^3 & \subseteq & Z^2 & \subseteq & & & Q_{30}^2
\end{array}
$$

The subvariety $\Sigma_{r_{b_1}} = \left[L_4 \subseteq L_5 \subseteq Q_{30}^2\right]$ *has codimension 7. In the inverse image* $\pi^{-1}(\Lambda)$ *of a general point* Λ *in* $\Sigma_{r_{b_1}}$*, we have* $T^3 = \Lambda$*,* $T^2 = \Lambda \cap L_5 = \Lambda \cap Q_{10}^5$*,* $O^{2,r_{b_1}} = \overline{Q_{30}^{2,sing}, \Lambda \cap Q_{10}^5}$*,* $Z^2 = \overline{Q_{30}^{2,sing}, \Lambda}$*. We have* $Q_{30}^{2,sing} \subseteq O^{2,n_{a_1}} \subseteq L_5$ *which can be parameterized by* $G(1,3)$*. Then the linear space* T^1 *which satisfies* $T^1 \subseteq O^{2,n_{a_1}}$ *can be parameterized by* $G(1,3)$*. On the other hand,* Z^1 *satisfies* $Q_{10}^{5,sing} \subseteq Z^1 \subseteq Q_{10}^5$ *and hence can be parameterized by* $OG(1,5)$*. Thus* $\dim(\pi^{-1}(\Lambda)) = 6$ *and* $\operatorname{codim}(\pi^{-1}(\Sigma_{r_{b_1}})) = 1$*.*

II: $\Sigma_{n_{a_g}}$: $\dim(\Lambda \cap L_{n_{a_g}}) = a_g + 1$ for some $1 \leq g \leq t$

Depending on n_{a_g}, we divide this case into the following two subcases:

II.A: $g = t$

II.B: $g < t$

II.A: $\Sigma_{n_{a_t}}$: $\dim(\Lambda \cap L_{n_{a_t}}) = a_t + 1$ (or equivalently, $\dim(\Lambda \cap L_{n_s}) = s + 1$)

If $r_{b_1} = n_{a_t}$, then $\Sigma_{n_{a_t}}$ corresponds to $\Sigma_{r_{b_1}}$. If $r_{b_1} > n_{a_t}$ then $\Sigma_{r_{b_1}}$ contains $\Sigma_{n_{a_t}}$. So we assume $r_{b_1} < n_{a_t}$ in the following. Specializing Λ is equivalent to moving the group of α_t isotropic linear spaces $L_{n_{a_t}-\alpha_t+1}, L_{n_{a_t}-\alpha_t+2}, \ldots, L_{n_{a_t}}$ one position to the right, putting $L_{n_{a_t}-\alpha_t}$ in the sequence to the left of these linear spaces and omitting $Q^{r_{k-s}}_{d_{k-s}}$ from the sequence. We have

$$\begin{aligned}
\operatorname{codim}(\Sigma_{r_{b_1}}) = \alpha_t\,(n_{a_t} - a_t) &+ \sum_{t=1}^{\beta_1} (d_{b_1} + x_{b_1} - 2(s + \beta_1) + t - 1) \\
&- (\alpha_t + 1)\,(n_{a_t} - a_t - 1) - \sum_{t=1}^{\beta_1 - 1} (d_{b_1} + x_{b_1} - 2(s + \beta_1) + t - 1) \\
&= \alpha_t + d_{b_1} + x_{b_1} - s - n_{a_t} - \beta_1
\end{aligned}$$

The only nontrivial parameterizations in the inverse image $\pi^{-1}(\Lambda)$ of a general point Λ in $\Sigma_{n_{a_t}}$ are in the row of T^t and once T^t is fixed, the rest of the row can be determined uniquely. The linear space T^t satisfies $T^{t-1} \subseteq T^t \subseteq \Lambda \cap L_{n_{a_t}}$ and hence can be parameterized by $G(\alpha_t, \alpha_t + 1)$. Thus we have

$$\begin{aligned}
\operatorname{codim}(\pi^{-1}(\Sigma_{n_{a_t}})) &= d_{b_1} + x_{b_1} - s - n_{a_t} - \beta_1 \\
&\geq \frac{d_{b_1} + r_{b_1}}{2} - n_{a_t}
\end{aligned}$$

using the property $x_i \geq k - i + 1 - \frac{d_i - r_i}{2}$ for all $1 \leq i \leq k - s$. Note that $\operatorname{codim}(\pi^{-1}(\Sigma_{n_{a_t}}))$ may be 1 in this case.

EXAMPLE 4.19. *Let* $V = \left[L_5 \subseteq Q_8^2\right]$, *then* $\widetilde{V}$ *is given by the following diagram.*

FIGURE 13. Definition of $\widetilde{V}$ for $V = \left[L_5 \subseteq Q_8^2\right]$

$$\begin{array}{ccccc} & & Q_8^{2,sing} & & \\ & & \cap\!\mathrm{I} & & \\ T^1 & \subseteq & O^{1,n_{a_1}} & \subseteq & L_5 \\ \cap\!\mathrm{I} & & \cap\!\mathrm{I} & & \cap\!\mathrm{I} \\ T^2 & \subseteq & Z^1 & \subseteq & Q_8^2 \end{array}$$

The subvariety $\Sigma_{n_{a_1}} = \left[L_4 \subseteq L_5\right]$ *has codimension 2. In the inverse image* $\pi^{-1}(\Lambda)$ *of a general point* Λ *in* $\Sigma_{r_{b_1}}$, *we have* $T^2 = \Lambda$ *and* $Z^1 = \overline{\Lambda, Q_8^{2,sing}}$. *The linear space* T^1 *satisfies* $T^1 \subseteq \Lambda \cap L_5$ *and hence can be parameterized by* $G(1,2)$. *Then* $O^{1,n_{a_1}}$ *is determined uniquely as* $O^{1,n_{a_1}} = \overline{Q_8^{2,sing}, T^1}$. *Thus* $\dim(\pi^{-1}(\Lambda)) = 1$ *and* $\operatorname{codim}(\pi^{-1}(\Sigma_{r_{b_1}})) = 2 - 1 = 1$.

EXAMPLE 4.20. *Let* $V = \left[L_4 \subseteq Q_8^1\right]$, *an orthogonal Schubert variety in* $OG(2,9)$. *The following diagram gives the definition of* $\widetilde{V}$.

FIGURE 14. Definition of $\widetilde{V}$ for $V = \left[L_4 \subseteq Q_8^1\right]$

$$\begin{array}{ccccc} & & Q_8^{1,sing} & & \\ & & \cap\!\mathrm{I} & & \\ T^1 & \subseteq & O^{1,n_{a_1}} & \subseteq & L_4 \\ \cap\!\mathrm{I} & & \cap\!\mathrm{I} & & \cap\!\mathrm{I} \\ T^2 & \subseteq & Z^1 & \subseteq & Q_8^1 \end{array}$$

The subvariety $\Sigma_{n_{a_1}} = \left[L_4 \subseteq L_5\right]$ *has codimension 3. In the inverse image* $\pi^{-1}(\Lambda)$ *of a general point* Λ *in* $\Sigma_{r_{b_1}}$, *we have* $T^2 = \Lambda$ *and* $Z^1 = \overline{\Lambda, Q_8^{1,sing}}$. *The linear space* T^1 *satisfies* $T^1 \subseteq \Lambda \cap L_4$ *and hence can be parameterized by* $G(1,2)$. *Then* $O^{1,n_{a_1}}$ *is determined uniquely as* $O^{1,n_{a_1}} = \overline{Q_8^{1,sing}, T^1}$. *Thus* $\dim(\pi^{-1}(\Lambda)) = 1$ *and* $\operatorname{codim}(\pi^{-1}(\Sigma_{r_{b_1}})) = 3 - 1 = 2$.

II.B: $\Sigma_{n_{a_g}}$: $\dim(\Lambda \cap L_{n_{a_g}}) = a_g + 1$ for some $1 \leq g \leq t-1$

We have already discussed in I.C the case when there is some $Q^{r_{b_h}}_{d_{b_h}}$ in the defining sequence with $r_{b_h} = n_{a_g}$. Also, if there is $Q^{r_{b_h}}_{d_{b_h}}$ in the sequence with $r_{b_h} > n_{a_g}$ then $\Sigma_{n_{a_g}}$ will be contained in $\Sigma_{r_{b_h}}$. So it is sufficient to consider the case when $n_{a_g} > r_{b_h}$ for all $1 \leq h \leq u$, equivalently, when $n_{a_g} > r_{k-s}$.

Specializing a k-plane Λ so that it intersects $L_{n_{a_g}}$ in one more dimension is equivalent to moving $L_{n_{a_g}-\alpha_g+1}$, $L_{n_{a_g}-\alpha_g+2}$, … , $L_{n_{a_g}}$ one position to the right, putting $L_{n_{a_g}-\alpha_g}$ to the left of these isotropic linear spaces and changing $Q^{r_i}_{d_i}$ to $Q^{r_i+(n_{a_g}-r_{k-s})}_{d_i-(n_{a_g}-r_{k-s})}$ for all i with $b_1 \leq i \leq k-s$. Note that this increases x_i by $\alpha_g + 1$ for all i with $b_1 \leq i \leq k-s$. We have

$$\begin{aligned}\operatorname{codim}(\Sigma_{n_{a_g}}) = \alpha_g\left(n_{a_g} - a_g\right) &+ \alpha_{g+1}\left(n_{a_{g+1}} - a_{g+1}\right) \\ &- (\alpha_g+1)\left(n_{a_g} - a_g - 1\right) - (\alpha_{g+1}-1)\left(n_{a_{g+1}} - a_{g+1}\right) \\ &+ \beta_1(n_{a_g} - r_{k-s}) - \beta_1(\alpha_g+1)\end{aligned}$$

The only nontrivial parameterizations in the inverse image $\pi^{-1}(\Lambda)$ of a general point Λ in $\Sigma_{n_{a_g}}$ are in the g-th row of the diagram of $\widetilde{V}$ and once T^g is parameterized the remaining coordinates can be determined uniquely. The linear space T^g satisfies $T^{g-1} \subseteq T^g \subseteq L_{n_{a_g}}$ and hence can be parameterized by the Grassmannian $G(\alpha_g, \alpha_g+1)$. Thus we have

$$\operatorname{codim}(\pi^{-1}(\Sigma_{n_{a_g}})) = n_{a_{g+1}} - n_{a_g} - (a_{g+1} - a_g) + 1 + \beta_1(n_{a_g} - \alpha_g - r_{k-s} - 1).$$

Note that $n_{a_{g+1}} - n_{a_g} \geq a_{g+1} - a_g + 1$ and $n_{a_g} - \alpha_g \geq k - s + 1$ by assumption. Therefore $\operatorname{codim}(\pi^{-1}(\Sigma_{n_{a_g}})) \geq 2$ in this case.

EXAMPLE 4.21. *Let* $V = \left[L_2 \subseteq L_4 \subseteq Q^0_9\right]$, *an orthogonal Schubert variety on* $OG(3,9)$. *The following diagram gives the definition of* $\widetilde{V}$.

FIGURE 15. Definition of $\widetilde{V}$ for $V = \left[L_2 \subseteq L_4 \subseteq Q_9^0\right]$

$$\begin{array}{ccc} T^1 & \subseteq & L_2 \\ \cap\!\!| & & \cap\!\!| \\ T^2 & \subseteq & L_4 \\ \cap\!\!| & & \cap\!\!| \\ T^3 & \subseteq & Q_9^0 \end{array}$$

The subvariety $\Sigma_{n_{a_1}} = \left[L_1 \subseteq L_2 \subseteq Q_7^2\right]$ *has codimension 3. In the inverse image* $\pi^{-1}(\Lambda)$ *of a general point* Λ *in* $\Sigma_{n_{a_1}}$*, the coordinates* T^3 *and* T^2 *are determined uniquely as* $T^3 = \Lambda$ *and* $T^2 = \Lambda \cap L_4$*. The coordinate* T^1 *satisfies* $T^1 \subseteq L_2$ *and hence is parameterized by* $G(1,2)$*. Thus* $\dim(\pi^{-1}(\Lambda)) = 1$ *and* $\operatorname{codim}(\pi^{-1}(\Sigma_{r_{b_1}})) = 3 - 1 = 2$.

EXAMPLE 4.22. *Let* $V = \left[L_5 \subseteq L_7 \subseteq Q_{20}^3\right]$*, then* $\widetilde{V}$ *is given by the following diagram.*

FIGURE 16. Definition of $\widetilde{V}$ for $V = \left[L_5 \subseteq L_7 \subseteq Q_{20}^3\right]$

$$\begin{array}{ccccc} & & Q_{20}^{3,sing} & & \\ & & \cap\!\!| & & \\ T^1 & \subseteq & O^{1,n_{a_1}} & \subseteq & L_5 \\ \cap\!\!| & & \cap\!\!| & & \cap\!\!| \\ T^2 & \subseteq & O^{1,n_{a_2}} & \subseteq & L_7 \\ \cap\!\!| & & \cap\!\!| & & \cap\!\!| \\ T^3 & \subseteq & Z^1 & \subseteq & Q_{20}^3 \end{array}$$

The subvariety $\Sigma_{n_{a_1}} = \left[L_4 \subseteq L_5 \subseteq Q_{18}^5\right]$ *has codimension 3. In the inverse image* $\pi^{-1}(\Lambda)$ *of a general point* Λ *in* $\Sigma_{n_{a_1}}$*, we have* $T^3 = \Lambda$*,* $T^2 = \Lambda \cap L_5 = \Lambda \cap L_7$*,* $Z^1 = \overline{Q_{20}^{3,sing}, \Lambda}$ *and* $O^{1,n_{a_2}} = \overline{Q_{20}^{3,sing}, \Lambda \cap L_7}$*. The linear space* T^1 *satisfies* $T^1 \subseteq L_5 \cap \Lambda$ *and hence can be parameterized by* $G(1,2)$*. Then* $O^{1,n_{a_1}}$ *is determined uniquely as* $O^{1,n_{a_1}} = \overline{Q_{20}^{3,sing}, T^1}$*. Thus* $\dim(\pi^{-1}(\Lambda)) = 1$ *and* $\operatorname{codim}(\pi^{-1}(\Sigma_{r_{b_1}})) = 3 - 1 = 2.$

III: $\Sigma_{d_{b_h}}$: $\dim(\Lambda \cap Q^{r_{b_h}}_{d_{b_h}}) = k - b_h + 2$ for some $1 \leq h \leq u$

A locus of type $\Sigma_{d_{b_h}}$ is contained in a locus of type $\Sigma_{r_{b_h}}$ when $\Sigma_{r_{b_h}}$ exists. We are interested in when this is not the case. A locus of type $\Sigma_{r_{b_h}}$ does not exist if and only if $r_{b_h} = x_{b_h}$. This is possible if either $Q^{r_{b_h}}_{d_{b_h}}$ is smooth or all sub-quadrics have the same singular locus and the k-planes Λ all contain this locus. The latter case is equivalent to the restriction variety of $(k - r_{b_h})$-planes defined by the sequence that contains only the smooth parts of the sub-quadrics. Therefore it is sufficient to consider the case when r_{b_h}, and consequently every r_i with $i \leq b_h$ is zero.

Let $\theta_1 = s + \sum_{l=1}^{h} \beta_l$ and $\theta_2 = \beta_{h+1}$. Consider the restriction variety $V_\sharp$ given by the partitions $(0), (d^{\theta_1}_{b_h}, d^{\theta_2}_{b_{h+1}}), (0,0)$. This is the transverse intersection of $OG(k,n)$ and the Type A Schubert variety Z in $G(k,n)$ defined as the closure of $Z^0 = \{W \in G(k,n) \mid \dim(W \cap F_{d_{b_h}}) = \theta_1 \text{ and } \dim(W \cap F_{d_{b_{h+1}}}) = \theta_1 + \theta_2\}$ where $F_{d_{b_h}}$ and $F_{d_{b_{h+1}}}$ are linear spaces of dimensions d_{b_h} and $d_{b_{h+1}}$ in a general full flag $F_\bullet$. Let $V^s_\sharp$ be the closure of the locus of k-planes Λ in $V_\sharp$ with the property that $\dim(\Lambda \cap Q^{r_{b_h}}_{d_{b_h}}) = \theta_1 + 1$. We have

$$\operatorname{codim}(\pi^{-1}(\Sigma_{d_{b_h}}) \subseteq \widetilde{V}) = \operatorname{codim}(\pi^{-1}(V^s_\sharp) \subseteq \widetilde{V}_\sharp).$$

Let $\widetilde{Z}$ be the Schubert variety in the flag variety $F(\theta_1, \theta_1 + \theta_2; n)$ defined by $\widetilde{Z} = \{(W_1, W_2) \in F(\theta_1, \theta_1 + \theta_2; n) \mid W_1 \subseteq F_{b_h}, W_2 \subseteq F_{b_{h+1}}\}$. We denote the projection from $\widetilde{Z}$ onto its second coordinate by $\phi : (W_1, W_2) \mapsto W_2$. This is the Bott-Samelson resolution for the ordinary Schubert variety Z. Furthermore, the resolution of singularities π for $V_\sharp$ has the transverse intersections $\widetilde{V}_\sharp = \widetilde{Z} \cap OF(\theta_1, \theta_1 + \theta_2; n)$ and $\pi^{-1}(V^s_\sharp) = \phi^{-1}(Z^{sing}) \cap OF(\theta_1, \theta_1 + \theta_2; n)$. The singular locus of Z is the closure of the locus of k-planes in Z with $\dim(Z \cap F_{d_{b_h}}) = \theta_1 + 1$. In partition notation Z is given by $(F^{\theta_1}_{d_{b_h}}, F^{\theta_2}_{d_{b_{h+1}}})$ and Z^{sing} is given by

$(F^{\theta_1+1}_{d_{b_h}}, F^{\theta_2-1}_{d_{b_{h+1}}})$. We have

$$\begin{aligned}\operatorname{codim}(Z^{sing}) &= \theta_1\,(d_{b_h}-\theta_1)+\theta_2\left(d_{b_{h+1}}-\theta_1-\theta_2\right)\\ &\quad-(\theta_1+1)\,(d_{b_h}-\theta_1-1)-(\theta_2-1)\left(d_{b_{h+1}}-\theta_1-\theta_2\right)\\ &= d_{b_{h+1}}-d_{b_h}+\theta_1-\theta_2+1\end{aligned}$$

In order to find $\operatorname{codim}(\phi^{-1}(Z^{sing}))$, we consider fibers of ϕ as before: For a general W in Z^{sing}, $\phi^{-1}(W)=(W_1,W_2)$ satisfies $W_1\subseteq W\cap F_{d_{b_h}}$. Such W_1 can be parameterized by the Grassmannian $G(\theta_1,\theta_1+1)$. Therefore

$$\operatorname{codim}(\phi^{-1}(Z^{sing}))=\operatorname{codim}(Z^{sing})-\dim\phi^{-1}(W)=d_{b_{h+1}}-d_{b_h}-\theta_2+1.$$

Note that this is always greater than 1.

Considering the action of $GL(n)$ on $F(\theta_1,\theta_1+\theta_2;n)$, Kleiman's Transversality Theorem shows that

$$\dim(\pi^{-1}(V^s_\sharp))=\dim(\phi^{-1}(Z^{sing}))-\operatorname{codim}(OF(\theta_1,\theta_1+\theta_2;n)\subseteq F(\theta_1,\theta_1+\theta_2;n))\ \text{ and}$$
$$\dim(V^s_\sharp)=\dim(Z^{sing})-\operatorname{codim}(OG(k,n)\subseteq G(k,n)).$$

Since we have

$$\begin{aligned}\dim OF(\theta_1,\theta_1+\theta_2;n)&=\dim OG(\theta_1+\theta_2,n)+\theta_1\theta_2\\ \dim F(\theta_1,\theta_1+\theta_2;n)&=\dim G(\theta_1+\theta_2,n)+\theta_1\theta_2\end{aligned}$$

we can conclude

$$\operatorname{codim}(\pi^{-1}(\Sigma_{d_{b_h}})\subseteq\widetilde{V})=\operatorname{codim}(\pi^{-1}(V^s_\sharp))=\operatorname{codim}(\pi^{-1}(Z^{sing}))=d_{b_{h+1}}-d_{b_h}-\theta_2+1\geq 2.$$

EXAMPLE 4.23. *Let* $V=\left[L_2\subseteq Q^0_7\subseteq Q^0_{10}\right]$. *In this case* $V_\sharp=\left[Q^0_6\subseteq Q^0_7\subseteq Q^0_{10}\right]$ *and the Type A Schubert variety* Z *is given by* $\left[L_6\subseteq L_7\subseteq L_{10}\right]$ *in* $G(3,10)$.

By the argument above, we have

$$\operatorname{codim}(\pi^{-1}(\Sigma_{d_{b_1}}) \subseteq \widetilde{V}) = \operatorname{codim}(\pi^{-1}(V^s_\sharp)) = \operatorname{codim}(\pi^{-1}(Z^{sing})) = 10 - 7 = 3.$$

The following observation summarizes our computations.

OBSERVATION 4.24. *A component of the exceptional locus of π with image of one of the types*

- $\Sigma_{r_{b_h}}$ *with* $r_{b_h} < n_s$
- $\Sigma_{n_{a_g}}$ *with* $1 \leq g \leq t-1$
- $\Sigma_{d_{b_h}}$ *for all* $1 \leq h \leq u-1$

has codimension larger than 1 (by I.B, I.C, II.B and III).

A component with image of type $\Sigma_{r_{b_h}}$ *with* $r_{b_h} \geq n_s$ *has codimension equal to 1 (by I.A and I.D).*

A component with image of type $\Sigma_{n_{a_t}}$ *has codimension given by* $\operatorname{codim}(\pi^{-1}(\Sigma_{n_{a_t}})) = d_{b_1} + x_{b_1} - s - n_{a_t} - \beta_1$ *which may be larger than or equal to 1.*

REMARK 4.25. *Observation 4.24 gives a characterization of the divisorial contractions of π.*

The following lemma allows us to give a partial description of the singular locus of a restriction variety. It is based on Lemma 1.8.

LEMMA 4.26. *The singular locus of a restriction variety $V(L_\bullet, Q_\bullet)$ is contained in the exceptional locus of π. Furthermore, a subvariety Σ satisfying* $\operatorname{codim}(\pi^{-1}(\Sigma)) > 1$ *is in the singular locus of $V(L_\bullet, Q_\bullet)$.*

PROOF. The open set $V^0(L_\bullet, Q_\bullet)$ is the locus where $\pi^{-1}(\Lambda)$ is a single point. $V^0(L_\bullet, Q_\bullet)$ is smooth since it is homogeneous under the action of $SO(n)$. Conversely, suppose $\operatorname{codim}(\pi^{-1}(\Sigma)) > 1$ and $\Lambda \in \Sigma$ is a point such that $\pi^{-1}(\Lambda)$ is positive dimensional. If Λ is smooth, then in order to check that π is a local isomorphism, it suffices to check that the Jacobian does not vanish. Since $\operatorname{codim}(\pi^{-1}(\Sigma)) > 1$ and the vanishing locus of the Jacobian is a divisor, we conclude that the Jacobian does not vanish. On the other hand, since π is not a local isomorphism around $\pi^{-1}(\Lambda)$, we conclude that Λ is a singular point. □

COROLLARY 4.27. *Let $V(L_\bullet, Q_\bullet)$ be a restriction variety and $\pi : \widetilde{V}(L_\bullet, Q_\bullet) \to V(L_\bullet, Q_\bullet)$ the resolution of singularities in Theorem 3.6. The components of the exceptional locus whose images are of the form*

- $\Sigma_{r_{b_h}}$ *with* $r_{b_h} < n_s$
- $\Sigma_{n_{a_g}}$ *for all* $1 \leq g \leq t-1$
- $\Sigma_{n_{a_t}}$ *such that* $d_{b_1} + x_{b_1} - s - n_{a_t} - \beta_1 > 1$
- $\Sigma_{d_{b_h}}$ *for all* $1 \leq h \leq u-1$

are in the singular locus of $V(L_\bullet, Q_\bullet)$.

CHAPTER 5

More Observations On the Exceptional Locus

Note that the results of the previous chapter are inconclusive about the image of a component of the exceptional locus of π with codimension equal to 1. By Observation 4.24, the subvarieties Σ that fall under this category are the following:

- $\Sigma_{r_{b_h}}$ with $r_{b_h} \geq n_s$
- $\Sigma_{n_{a_t}}$ such that $d_{b_1} + x_{b_1} - s - n_{a_t} - \beta_1 = 1$

Under certain conditions, we can say more by studying the tangent space to the restriction variety V at a general point Λ in Σ. In this chapter, we study the type

- $\Sigma_{r_{b_h}}$ with $r_{b_h} \geq n_s$

when the sub-quadric $Q^{r_{b_h}}_{d_{b_h}}$ has even rank, that is, $d_{b_h} - r_{b_h}$ is even.

We will eventually show that this type is contained in the singular locus. Our strategy is to use the basis sequence of V in order to construct arcs through a point Λ in V by moving the basis elements of Λ. The arcs found this way give independent elements in the tangent space. Therefore, if we find more arcs through a given Λ than the dimension of V, we can conclude that Λ is a singular point. If Λ is a general point in Σ, this also implies that V is singular along Σ.

Let us illustrate this idea with examples before stating the proposition.

EXAMPLE 5.1. *Let V be the restriction variety contained in $OG(2, 10)$ given by the sequence $\left[L_1 \subseteq Q^3_7\right]$. By Remark 4.7, the only locus where π has positive dimensional fibers, hence the only locus that may be in the singular locus of V, is $\Sigma_{r_{b_1}} = \left[L_1 \subseteq L_3\right]$. By* ***I.A****, we know that* $\operatorname{codim}(\pi^{-1}(\Sigma_{r_{b_1}})) = 1$*, which is inconclusive* *about whether V is singular along $\Sigma_{r_{b_1}}$. In the following we use basis sequences to study the arcs contained in V through a general point in $\Sigma_{r_{b_1}}$.*

The basis sequence of V is given by

$$e_1 \,\big]\, e_2\; e_3\; e_4\; f_4\; e_5\; f_5 \,\big\}\, f_1\; f_2\; f_3 \; .$$

Here the sub-quadric Q^3_7 is the zero locus of the polynomial $x_4y_4 + x_5y_5$. Let us pick a general point from $\Sigma_{r_{b_1}}$ as $\Lambda = \big\langle e_1,\ e_3 \big\rangle$. Then the following arcs $\Gamma_i(t)$ are contained in V:

$$\Gamma_1(t) = \big\langle e_1,\ e_3 + te_2 \big\rangle,\ \ \Gamma_2(t) = \big\langle e_1,\ e_3 + te_4 \big\rangle,\ \ \Gamma_3(t) = \big\langle e_1,\ e_3 + tf_4 \big\rangle\ ,$$

$$\Gamma_4(t) = \big\langle e_1,\ e_3 + te_5 \big\rangle,\ \ \Gamma_5(t) = \big\langle e_1,\ e_3 + tf_5 \big\rangle\ .$$

These are 5 independent arcs contained in V passing through Λ which implies that the tangent space to V at Λ has dimension at least 5. Since the dimension of V is 4, we conclude that Λ is a singular point of V. Thus $\Sigma_{r_{b_1}}$ is in the singular locus of V. This allows us to describe the singular locus of V.

$$V^{sing} = \Sigma_{r_{b_1}} = \Big[L_1 \subseteq L_3\Big]\ .$$

EXAMPLE 5.2. *Let V be the restriction variety contained in $OG(2,12)$ given by the sequence $\Big[L_4 \subseteq Q^4_8\Big]$. By Remark 4.7, $\Sigma_{r_{b_1}} = \Big[L_3 \subseteq L_4\Big]$ is the only locus where π is positive dimensional. By **I.B**, we know that the locus $\Sigma_{r_{b_1}}$ satisfies* $\operatorname{codim}(\pi^{-1}(\Sigma_{r_{b_1}})) = 1$*, so the results of the previous chapter do not determine whether V is singular along $\Sigma_{r_{b_1}}$.*

The basis sequence of V is given by

$$e_1\; e_2\; e_3\; e_4 \,\big]\, e_5\; f_5\; e_6\; f_6 \,\big\}\, f_1\; f_2\; f_3\; f_4 \; .$$

The sub-quadric Q_8^4 *is given by the polynomial* $x_3y_3+x_4y_4$. *Let us pick a general point from* $\Sigma_{r_{b_1}}$ *as* $\Lambda=\langle e_3,\ e_4\rangle$. *The following* $\Gamma_i(t)$ *are independent arcs contained in* V *passing through* Λ*:*

$$\Gamma_1(t)=\langle e_3+te_1,\ e_4\rangle,\ \Gamma_2(t)=\langle e_3+te_2,\ e_4\rangle,\ \Gamma_3(t)=\langle e_3+te_5,\ e_4\rangle,$$

$$\Gamma_4(t)=\langle e_3+tf_5,\ e_4\rangle,\ \Gamma_5(t)=\langle e_3+te_6,\ e_4\rangle,\ \Gamma_6(t)=\langle e_3+tf_6,\ e_4\rangle,$$

$$\Gamma_7(t)=\langle e_3,\ e_4+te_1\rangle,\ \Gamma_8(t)=\langle e_3,\ e_4+te_2\rangle,\ \Gamma_9(t)=\langle e_3,\ e_4+te_5\rangle,$$

$$\Gamma_{10}(t)=\langle e_3,\ e_4+tf_5\rangle,\ \Gamma_{11}(t)=\langle e_3,\ e_4+te_6\rangle,\ \Gamma_{12}(t)=\langle e_3,\ e_4+tf_6\rangle\ .$$

Since the dimension of V *is 8, this shows that* V *is singular along* $\Sigma_{r_{b_1}}$. *This allows us to conclude*

$$V^{sing}=\Sigma_{r_{b_1}}=\Big[L_3\subseteq L_4\Big]\ .$$

EXAMPLE 5.3. *Let* V *be the restriction variety contained in* $OG(3,12)$ *given by the sequence* $\Big[L_2\subseteq Q_8^4\subseteq Q_{10}^2\Big]$. *In the following we show that* $\Sigma_{r_{b_1}}$*, which is defined by the sequence* $\Big[L_2\subseteq L_4\subseteq Q_{10}^2\Big]$*, and* $\Sigma_{r_{b_2}}$*, which is defined by the sequence* $\Big[L_1\subseteq L_2\subseteq Q_8^4\Big]$*, are both in the singular locus of* V. *Note that the results of the previous chapter are inconclusive for both of these loci.*

The basis sequence of V *is given by*

$$e_1\ e_2\ \big]\ e_3\ e_4\ e_5\ f_5\ e_6\ f_6\ \big\}\ f_3\ f_4\ \big\}\ f_1\ f_2.$$

The sub-quadric Q_8^4 *is the zero locus of the polynomial* $x_5y_5+x_6y_6$. *Let us pick a general point in* $\Sigma_{r_{b_1}}$ *as* $\Lambda=\langle e_2,\ e_4,\ f_3\rangle$. *The following are independent arcs contained in* $\Sigma_{r_{b_1}}$ *passing through* Λ.

$$\Gamma_1(t)=\langle e_2+te_1,\ e_4,\ f_3\rangle,\ \Gamma_2(t)=\langle e_2,\ e_4+te_1,\ f_3\rangle,\ \Gamma_3(t)=\langle e_2,\ e_4+te_5,\ f_3\rangle,$$

$$\Gamma_4(t)=\langle e_2,\ e_4+tf_5,\ f_3\rangle,\ \Gamma_5(t)=\langle e_2,\ e_4+te_6,\ f_3\rangle,\ \Gamma_6(t)=\langle e_2,\ e_4+tf_6,\ f_3\rangle,$$

$$\Gamma_7(t)=\langle e_2,\ e_4,\ f_3+te_1\rangle,\ \Gamma_8(t)=\langle e_2,\ e_4,\ f_3+te_5\rangle,\ \Gamma_9(t)=\langle e_2,\ e_4,\ f_3+tf_5\rangle,$$

$\Gamma_{10}(t) = \langle e_2,\ e_4,\ f_3{+}te_6\rangle$, $\Gamma_{11}(t) = \langle e_2,\ e_4,\ f_3{+}tf_6\rangle$, $\Gamma_{12}(t) = \langle e_2,\ e_4,\ f_3{+}t(e_3{-}f_4)\rangle$.

Since the dimension of V is 11, this shows that V is singular at Λ, and hence along $\Sigma_{r_{b_1}}$.

The sub-quadric Q^2_{10} is the zero locus of the polynomial $x_3y_3 + x_4y_4 + x_5y_5 + x_6y_6$. Let us pick a general point in $\Sigma_{r_{b_2}}$ as $\Lambda = \langle e_1,\ e_2,\ f_4\rangle$. The following are independent arcs contained in $\Sigma_{r_{b_1}}$ passing through Λ.

$$\Gamma_1(t) = \langle e_1 + te_3,\ e_2,\ f_4\rangle,\ \Gamma_2(t) = \langle e_1 + te_5,\ e_2,\ f_4\rangle,\ \Gamma_3(t) = \langle e_1 + tf_5,\ e_2,\ f_4\rangle,$$

$$\Gamma_4(t) = \langle e_1 + te_6,\ e_2,\ f_4\rangle,\ \Gamma_5(t) = \langle e_1 + tf_6,\ e_2,\ f_4\rangle,\ \Gamma_6(t) = \langle e_1,\ e_2 + te_3,\ f_4\rangle,$$

$$\Gamma_7(t) = \langle e_1,\ e_2 + te_5,\ f_4\rangle,\ \Gamma_8(t) = \langle e_1,\ e_2 + tf_5,\ f_4\rangle,\ \Gamma_9(t) = \langle e_1,\ e_2 + te_6,\ f_4\rangle,$$

$$\Gamma_{10}(t) = \langle e_1,\ e_2 + tf_6,\ f_4\rangle,\ \Gamma_{11}(t) = \langle e_1,\ e_2,\ f_4 + te_3\rangle,\ \Gamma_{12}(t) = \langle e_1,\ e_2,\ f_4 + te_5\rangle,$$

$$\Gamma_{13}(t) = \langle e_1,\ e_2,\ f_4 + te_5\rangle,\ \Gamma_{14}(t) = \langle e_1,\ e_2,\ f_4 + te_6\rangle,\ \Gamma_{15}(t) = \langle e_1,\ e_2,\ f_4 + tf_6\rangle.$$

Since this is larger than the dimension of V, we conclude V is singular along $\Sigma_{r_{b_2}}$.

Therefore, the singular locus of V is given by

$$\begin{aligned} V^{sing} &= \Sigma_{r_{b_1}} \ \cup\ \Sigma_{r_{b_2}} \\ &= \left[L_2 \subseteq L_4 \subseteq Q^2_{10}\right] \ \cup\ \left[L_1 \subseteq L_2 \subseteq Q^4_8\right]. \end{aligned}$$

The pattern seen in these examples generalizes in a straight-forward way. In the proof of the following proposition, we observe that a restriction variety V is singular at a point by establishing more tangent vectors at that point than the dimension of V. We use basis sequences to study the arcs through a point similar to the examples above.

PROPOSITION 5.4. *If* $Q^{r_{b_h}}_{d_{b_h}}$ *has even rank, then* $\Sigma_{r_{b_h}}$ *is in the singular locus of the restriction variety* V.

PROOF. Given a basis of a point V, we can obtain arcs through the point by moving each basis element in a way that will still obey the rank conditions defining V. Since the sub-quadric $Q^{r_{b_h}}_{d_{b_h}}$ has even rank, the basis sequence of V has some f_i to the left of the bracket of $Q^{r_{b_h}}_{d_{b_h}}$.

$$\cdots\ f_i\ \}\ \cdots$$

Thus the basis of a general element in V contains f_i, or some other $f_\bullet$ in case e_i is in the chosen basis for the point. The arcs that can be obtained by moving this basis element can only be chosen among the basis elements v of $Q^{r_{b_h}}_{d_{b_h}}$ such that $f_i + tv$ is in the zero locus of the polynomial giving $Q^{r_{b_h}}_{d_{b_h}}$. This excludes e_i, which is a basis element in the span of $Q^{r_{b_h}}_{d_{b_h}}$, from the possible choices of v.

In contrast, the basis of a general element Λ in $\Sigma_{r_{b_h}}$ contains some e_j chosen from the span of $Q^{r_{b_h},sing}_{d_{b_h}}$. The arcs obtained $e_j + tv$ have at least one more possibility compared to $f_i + tv$. This is because f_j is not in the span of $Q^{r_{b_h}}_{d_{b_h}}$ and its exclusion is not effective; the basis elements outside the span of $Q^{r_{b_h}}_{d_{b_h}}$ are already excluded to respect the rank conditions defining V.

Therefore, we can obtain at least one more arc through a general point of $\Sigma_{r_{b_h}}$ than through a general point in V. This shows that the dimension of the tangent space to V at a general point of $\Sigma_{r_{b_h}}$ is larger than the dimension of V. We conclude V is singular along $\Sigma_{r_{b_h}}$. □

REMARK 5.5. *In particular, all loci of type* $\Sigma_{r_{b_h}}$ *are in the singular locus of* V *if* V *is a Schubert variety of Type D in the orthogonal Grassmannian.*

We summarize our knowledge of the singular locus of a general restriction variety V in the following corollary.

COROLLARY 5.6. *The components of the exceptional locus of* π *whose images are of the form*

- $\Sigma_{r_{b_h}}$ *with* $r_{b_h} < n_s$
- $\Sigma_{r_{b_h}}$ *with* $d_{b_h} - r_{b_h}$ *an even number*
- $\Sigma_{n_{a_g}}$ *for all* $1 \leq g \leq t-1$
- $\Sigma_{n_{a_t}}$ *such that* $d_{b_1} + x_{b_1} - s - n_{a_t} - \beta_1 > 1$
- $\Sigma_{d_{b_h}}$ *for all* $1 \leq h \leq u-1$

are in the singular locus of $V(L_\bullet, Q_\bullet)$.

CHAPTER 6

Examples

In this chapter we present examples illustrating how our results can be used to determine the singular locus of a restriction variety when π has no divisorial contractions.

EXAMPLE 6.1. *Let* $V = \left[L_2 \subseteq Q_4^0\right]$*, the Fano variety of lines contained on a smooth quadric surface. By Remark 4.7, the exceptional locus of* π *is empty from which we can conclude that* V *is smooth. The restriction variety* V *is actually isomorphic to* $\mathbb{P}^1$ *in this example.*

EXAMPLE 6.2. *Let* $V = \left[Q_5^0 \subseteq Q_8^0\right]$*. This is the variety of projective lines contained in a 6-dimensional smooth quadric that intersect a 3-dimensional smooth sub-quadric. By Remark 4.7, the only locus in the image of the exceptional locus of* π *is* $\Sigma_{d_{b_1}} = \left[Q_4^0 \subseteq Q_5^0\right]$*. By Corollary 4.27, this is in the singular locus of* V*, therefore*

$$V^{sing} = \left[Q_4^0 \subseteq Q_5^0\right] .$$

EXAMPLE 6.3. *Let* $V = \left[L_3 \subseteq Q_7^1\right]$*. By Remark 4.7 there are two types of subvarieties to consider:*

$$\Sigma_{n_{a_1}} = \left[L_2 \subseteq L_3\right] \text{ and } \Sigma_{r_{b_1}} = \left[L_1 \subseteq L_4\right] \cup \left[L_1 \subseteq L_4'\right]$$

since when the corank of Q_7^1 *is increased by 2, it breaks down into* L_4 *and* L_4'*.*

By Corollary 4.27, we can conclude that $\Sigma_{r_{b_1}}$ *is in the singular locus of* V*.*

On the other hand, by Observation 4.24, we have

$$\operatorname{codim}(\pi^{-1}(\Sigma_{n_{a_1}})) = d_1 + x_1 - s - n_1 - 1 = 2\ ,$$

thus $\Sigma_{n_{a_1}}$ *is also in the singular locus of* V.

Therefore we have

$$V^{sing} = \Big[L_2 \subseteq L_3\Big] \ \cup\ \Big[L_1 \subseteq L_4\Big] \ \cup\ \Big[L_1 \subseteq L'_4\Big]$$

Note that V *is an orthogonal Schubert variety in* $OG(2,8)$. *In permutation notation, its singular locus is given by*

$$(73845162)^{sing} = (32854176)\ \cup\ (51736284)\ \cup\ (41763285).$$

EXAMPLE 6.4. *Let* $V = \Big[L_4 \subseteq Q_8^1\Big]$. *By Remark 4.27, the loci we need to consider are*

$$\Sigma_{n_{a_1}} = \Big[L_3 \subseteq L_4\Big] \ \text{ and }\ \Sigma_{r_{b_1}} = \Big[L_1 \subseteq L_4\Big]$$

since increasing the corank of Q_8^1 *by 3 results in a double copy of* L_4. *Since the latter locus is contained in the former, we only need to consider* $\Sigma_{r_{b_1}}$. *By Corollary 4.24, this locus is in the singular locus of* V. *Therefore we have*

$$V^{sing} = \Big[L_3 \subseteq L_4\Big]\ .$$

This is another orthogonal Schubert variety in $OG(2,9)$. *In permutation notation we have*

$$(849753162)^{sing} = (439852176).$$

EXAMPLE 6.5. *Let* $V = \Big[L_1 \subseteq Q_7^2 \subseteq Q_8^1\Big]$. *By Remark 4.7, the resolution of singularities* π *has no exceptional locus. Therefore* V *is smooth. Note that* V *is an orthogonal Schubert variety in* $OG(2,9)$ *given by* (871654932) *in permutations.*

EXAMPLE 6.6. *Let* $V = \left[Q_7^2 \subseteq Q_9^0\right]$. *By Remark 4.7, the loci that may be in the singular locus of* V *are*

$$\Sigma_{r_{b_1}} = \left[L_2 \subseteq Q_9^0\right] \text{ and } \Sigma_{d_{b_1}} = \left[Q_6^3 \subseteq Q_7^2\right].$$

By Corollary 4.24, V *is singular along both of these loci, thus*

$$V^{sing} = \Sigma_{r_{b_1}} \cup \Sigma_{d_{b_1}}.$$

Note that V *is a Schubert variety in* $OG(2,9)$. *In permutations, its singular locus is given by*

$$(978654231)^{sing} = (927654381) \cup (769852143).$$

EXAMPLE 6.7. *Let* $V = \left[L_2 \subseteq Q_6^2 \subseteq Q_8^0\right]$. *By Remark 4.7, the only locus where* π *has positive dimensional fibers is* $\Sigma_{r_{b_1}}$ *and by Observation 4.24,* $\operatorname{codim}(\pi^{-1}(\Sigma_{r_{b_1}})) = 1$. *By Corollary 5.6,* V *is singular along* $\Sigma_{r_{b_1}}$, *therefore we have*

$$V^{sing} = \Sigma_{r_{b_1}} = \left[L_1 \subseteq L_2 \subseteq Q_6^2\right].$$

Note that V *is a Schubert variety in* $OG(3,8)$. *In permutation notation, its singular locus is given by*

$$(82645371)^{sing} = (62154873).$$

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