Ramsey type results on hypercubes and hypergraphs

by

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SUMMARY

In this thesis, we will examine some Ramsey type problems for graphs and hypergraphs. Our starting point, and motivating question, is to determine the minimum number of colors required to color the edge set of a hypergraph G subject to the constraint that the edges of every copy of the hypergraph $H \subset G$ receive at least q colors. This problem was introduced in such generality by Erdős and Gyárfás.

We study this question in a variety of contexts for both graphs and hypergraphs. For the graph case, we focus on the situation where G is the n-dimensional hypercube and H is a hypercube, path or cycle. For the hypergraph case we consider the situation when G is a complete hypergraph and H is a complete hypergraph, path or cycle.

CHAPTER 1

INTRODUCTION

1.1 History and past results

Ramsey Theory encompasses an extensive body of work in mathematics which can be summarized by the statement "Complete chaos is not possible". The basic premise of the subject is that every large system contains a sufficiently large subsystem which is not chaotic. The subject is most often studied through the lens of combinatorial structures such as graphs and hypergraphs, though the paradigm appears in other branches of mathematics including geometry [26] and number theory [20]. We will define a graph and several basic examples of graphs before proceeding.

A graph G is a pair $(V(G), E(G))$ where $V(G)$ is the vertex set of G and $E(G) \subset \binom{V(G)}{2}$. An edge $\{u, v\} \in E(G)$ is also written as uv. The complete graph K_t (also referred to as a clique) is the graph on t vertices whose edge set is the set of all possible $\binom{t}{2}$ edges. The path on q edges P_q is the graph with vertex sex $V(P_q) = \{v_1, \dots, v_{q+1}\}$ and edge set $E(G) = \{v_1v_2, \dots, v_qv_{q+1}\}$. The cycle on q edges C_q is the graph with vertex sex $V(C_q) = \{v_1, \dots, v_q\}$ and edge set $E(G) = \{v_1v_2, \dots, v_{q-1}v_q, v_qv_1\}$. The complete bipartite graph $K_{s,t}$ is the graph with $V(K_{s,t}) = A \cup B$ where A and B are disjoint with $|A| = s$, $|B| = t$ and with $E(K_{s,t}) = \{uv \mid u \in A, v \in B\}$.

In many problems we will study, it is necessary to assign colors to the edges of a graph. A k-edge-coloring of a graph G is a function $\chi : E(G) \mapsto \{1, \ldots, k\}$. An edge-colored graph is *monochromatic* if all of its edges received the same color.

The basic problem of Ramsey Theory is to find monochromatic complete subgraphs within a larger edge-colored complete graph. The fact that such subgraphs exist in sufficiently large complete edge-colored graphs is the basic result of Ramsey Theory.

Theorem 1 (**Ramsey [46]**)**.** *For every pair of fixed positive integers $s, t > 0$, there exists an n_0 such that if $n > n_0$ and K_n is edge-colored with red and blue then there is either monochromatic K_s in red or a monochromatic K_t in blue as a subgraph of a K_n.*

Definition. *For integers $s, t > 0$, let $r(s, t)$ be the minimum value of n_0 in Theorem 1.*

Ramsey's initial motivation for Theorem 1 came from problems related to the decidability of logical systems. Determining $r(s, t)$ for fixed s and t is an extremely difficult problem. For example, despite only requiring a very elementary proof to show that $r(3, 3) = 6$, there are very few exact results known for the diagonal cases where $s = t$. In fact, the only other diagonal case for which we know the exact answer is $r(4, 4) = 18$.

Determining the values of $R(k, k)$ for $k \geq 5$ seems to be an extraordinarily difficult problem. While the bounds $43 \leq r(5, 5) \leq 49$ are known [9], it would take a vast amount of computing power to simply resolve whether or not $r(5, 5) = 43$. In a brute force calculation approach, we would need to consider all of the of the 2-edge-colorings of a K_{43} and check for monochromatic copies K_5 in each color. A K_{43} has $\binom{43}{2} = 903$ edges and each edge may receive one of two possible colors, so there are 2^{903} colorings to check. As this basic example illustrates, determining Ramsey numbers exactly often appears hopeless and therefore asymptotic bounds are desirable. The probabilistic method, pioneered by Erdős, gave the following bound on the diagonal Ramsey number $r(t,t)$.

Theorem 2 (**Erdős** [**4**])**.** *For $t \geq 2$, we have $r(t,t) > 2^{t/2}$.*

Erdős and Szekeres determined the following upper bound.

Theorem 3 (**Erdős-Szekeres** [**26**])**.** *For $t \geq 2$, we have $r(t,t) < 4^t$.*

There have been slight improvements made to the above Theorems [14, 51], however they essentially represent the current state of progress on the problem. There are many natural extensions to $r(s,t)$. Perhaps the most obvious direction to go in would be to use more than two colors. These are referred to as multicolor Ramsey numbers. We may generalize even further by asking for monochromatic copies of a given subgraph H instead of just cliques.

Definition. *Given an integer $k \geq 1$ and graphs $H_1, \ldots, H_k$, let $r_k(H_1, \ldots, H_k)$ be the minimum n_0 such that, for $n > n_0$, any k-edge-coloring of K_n contains a monochromatic copy H_i in the i-th color for some i. If $H = H_1 = H_2 = \ldots H_k$, then $r_k(H) = r_k(H_1, \ldots, H_k)$.*

As one would expect, determining $r_k(H)$ for various H is an extremely difficult problem. Greenwood and Gleason [35] showed that $r_3(K_3) = 17$ and at present this is the only non-trivial value which is known for cliques when $k \geq 3$. They also provided the following upper bound for the general problem for cliques.

Theorem 4 (Greenwood-Gleason [35]).

$$r_k(K_{i_1}, K_{i_2}, \ldots, K_{i_k}) \leq \frac{(i_1 + \ldots + i_k - k)!}{i_1! \ldots i_k!}.$$

The problem of determining $r_k(H)$ has been investigated when H is a cycle. Among the only results are $R_3(C_4) = 11$ due to Clapham [13] and $R_3(C_6) = 12$ due to Rowlison and Yang [47]. Returning to cliques, Erdős and Gyárfás [23] considered an extension of the classical Ramsey problem by requiring the subclique to have more than two colors.

Definition. *Given positive integers p and q, an edge-coloring of K_n is a called a (p, q)-coloring if every copy of $K_p \subset K_n$ contains at least q distinct colors on its edges. The minimum number of colors required for a (p, q)-coloring of K_n is denoted as $f(n, p, q)$.*

Under this definition, a $(p, 2)$-coloring avoids monochromatic cliques. The statement $f(n, p, 2) \leq k$ is equivalent to the two statements $r_k(p) > n$ and $r_{k-1}(p) \leq n$, hence determining $f(n, p, 2)$ is equivalent to determining $r_k(p)$. The problem of determining $f(n, p, q)$ is extremely difficult. Several small cases have received considerable attention, including the case of $(4, 3)$-colorings first studied by Erdős and Gyárfás.

Theorem 5 (Erdős-Gyárfás [23]).

$$f(n, 4, 3) = O(\sqrt{n})$$

Mubayi [41] proved the following result for $f(n, 4, 3)$.

Theorem 6 (Mubayi [41]).

$$f(n,4,3) \leq e^{O(\sqrt{\log n})}$$

Kostochka and Mubayi [37] later improved the trivial lower bound of $f(n,4,3) \geq f(n,4,2) = \Omega(\frac{\log n}{\log\log n})$ to $\Omega(\frac{\log n}{\log\log\log n})$ and their methods were refined by Fox and Sudakov [29] to give the current best bound $f(n,4,3) = \Omega(\log n)$. Since obtaining exact results for generalized Ramsey type problems is very hard it is often desirable to determine the asymptotic behavior of these functions, though this still is a very difficult problem. Erdős and Gyárfás [23] answered the following question: for fixed p, for which values of q is $f(n,p,q)$ linear or quadratic in n. Recent progress was made on this problem by Conlon, Fox, Lee and Sudakov [17]; they determined the largest value of q for which $f(n,p,q)$ is subpolynomial.

Theorem 7 (Conlon et. al. [17]). *For fixed positive integers p and q with $2 \leq q \leq \binom{p}{2}$, the maximum value of q for which $f(n,p,q)$ is subpolynomial in n is $p-1$.*

Ramsey's Theorem has been generalized to consider subgraphs other than cliques and to change the restriction on the number of colors that a subgraph may have. Another way is to take the large graph to be something other than K_n. The complete bipartite graph, $K_{n,n}$ has been studied at some length. Axenovich, Fúredi and Mubayi [5] studied this problem and defined the following generalized version of $f(n,p,q)$:

Definition. *Given graphs G and H, an edge-coloring of G is an (H,q)-coloring if every copy of H in G receives at least q distinct colors on its edges. The minimum number of colors required in an (H,q)-coloring is $f(G,H,q)$.*

Among other problems, they studied the function $f(G, H, q)$ for $G = K_{n,n}$ and $H = K_{p,p}$. Motivated by the work of Erdős and Gyárfás on cliques, they proved various thresholds for $f(K_{n,n}, K_{p,p}, q)$ [5].

1.2 New results on Hypercubes

In this thesis, we will obtain results for the n-dimensional hypercube and for hypergraphs. The hypercube will be studied in chapters 2-4.

Definition. *The n-dimensional hypercube Q_n is the graph with vertex set $V(Q_n) = 2^{[n]}$; if we view two vertices u and v as their corresponding subsets of $[n]$, then uv is an edge if $|v| = |u| + 1$ and $u \subset v$.*

Given a subgraph $H \subset Q_n$, we will study the function $f(Q_n, H, q)$ as defined earlier. Recall that P_q is the path with q edges. We will begin in Chapter 2 by giving some simple results on $f(Q_n, P_q, q)$. We will focus specifically on the cases where all the edge of P_q receive different colors.

Definition. *An edge-colored graph G is rainbow if no two edges receive the same color.*

We will prove the following bounds on rainbow coloring small paths in Q_n.

Theorem 8. *The following lower and upper bounds hold for $f(Q_n, P_q, q)$:*

q	*lower bound*	*upper bound*
2	n	n
3	$2n-1$	$2n$
4	n^2	$3n^2$

In the Chapter 3, we will study $f(Q_n, Q_3, q)$ and for various values of q. This appears to be a difficult problem for most of the values of k. We will give upper and lower bounds, some of which will require the use of the probabilistic techniques. The primary motivation for this is a result of Faudree, Gyárfás, Lesniak and Schlep [27].

Theorem 9 (**Faudree et. al. [27]**). *Let $n = 4$ or $n \geq 6$. Then $f(Q_n, Q_2, 4) = n$.*

Our result follows.

Theorem 10. *The following upper and lower bounds hold for $f(Q_n, Q_3, q)$:*

q	*lower bound*	*upper bound*
4	4	4
5	5	8
6	6	12
7	$\Omega(\frac{\log n}{\log\log n})$	$O(n^{1/3})$
8	$\Omega(\frac{\log n}{\log\log n})$	$O(n^{2/5})$
9	$n^{1/3}$	$O(\sqrt{n})$
10	$\sqrt{n}$	$O(n^{2/3})$
11	$n/2$	$O(n)$
12	$3n-2$	$n^{1+o(1)}$

In Chapter 4, we consider $f(Q_n, H, q)$ when $H = C_q$ and q is even (there are no odd cycles in Q_n). Since $C_4 = Q_2$, we have $f(Q_m, C_4, 4) = f(Q_n, Q_2, 4)$; it is also an easy exercise to show that $f(Q_m, C_6, 6) = f(Q_n, Q_6, 12)$. Since we will focus on rainbow colorings in Chapter 4, we will introduce the following definition for convenience.

Definition. *For q even and $n \geq q$, let $f(n, q) = f(Q_n, Q_q, q)$.*

Theorem 11. *There a positive integers c_1, c_1', c_2, c_2' such that the following hold. If $q \equiv 0\,(mod\, 4)$, then*

$$c_1 n^{k/4} < f(n,k) < c_2 n^{k/4}.$$

If $q \equiv 2\,(mod\,4)$, *then*

$$c_1' n^{\lceil k/4 \rceil} < f(n,k) < c_2' n^{\lceil k/4 \rceil}.$$

We will also prove a better result for the particular case $k = 6$.

Theorem 12. *For every* $\epsilon > 0$ *there exists* n_0 *such that for* $n > n_0$ *we have* $f(n,6) \leq n^{1+\epsilon}$.

Although this parameter has been introduced more recently, the study of $f(Q_n, C_k, q)$ has received considerable attention. Alon, Radoičić, Sudakov and Vondrák [3] studied the problem of avoiding monochromatic cycles and proved the following result.

Theorem 13 (**Alon et. al.** [**3**])**.** *Fix* $k \geq 1$ *and* $l \geq 5$. *For sufficiently large* n, *every* k*-edge-coloring of* Q_n *contains a monochromatic cycle of length* $2l$.

Alternatively, we may rephrase this as saying that $f(Q_n, C_l, 2)$ is not bounded by a function of l. Even more recently, at the end of 2013, a result similar to Theorem 11 was obtained by Balogh, Delcourt, Lidický and Palmer [7]. They asked the following closely related question: if Q_d is colored with a fixed number of colors k, then what is the largest possible number of rainbow cycles of a particular length that there can be? They proved the following result:

Theorem 14 (**Balogh et. al.** [**7**])**.** *Let* k *and* d *be fixed integers such that* $4 \leq k < d$ *and* $k \neq 5$ *and write* $d = ka + b$ *where* a *and* b *are integers such that* $a \geq 0$ *and* $0 \leq b < k$, *then the maximum number of rainbow copies of* C_4 *in a* k*-edge-coloring of* Q_d *is* $2^{d-2}\left[\binom{d}{2} - k\binom{c}{2} - ba\right]$.

They also conjectured for a 5-edge-coloring of Q_d, that $\lim_{d\to\infty} \frac{q_5(d)}{\binom{d}{2} 2^{d-2}} = 4/5$ where $q_k(d)$ is the maximum number of rainbow copies of C_4 in a k edge coloring of Q_d.

1.3 Hypergraphs: Cliques

All of the types of questions we have looked at so far can be naturally extended to hypergraphs. Our attention will be limited to l-uniform hypergraphs, which may also refer to as simply l-graphs. Let X be set; we denote collection of subsets of size l of X by $\binom{X}{l}$. An l-uniform hypergraph H (l-graph for short) is a pair $(V(H), E(H))$ with $E(H) \subset \binom{V(H)}{l}$. The complete l-graph on n vertices K_n^l has vertex set $V(K_n^l)$ and edge set $E(K_n^l) = \binom{V(K_n^l)}{l}$. Given an l-graph G and $S \in \binom{V(G)}{l-1}$, the *codegree* of (S), written $\mathrm{codeg}(S)$, is the number of edges in G which contain S.

Definition. *Fix integers $s, t \geq l \geq 2$. Let $r^l(s,t)$ be the minimum n such that every 2-edge-coloring of K_n^l yields either a monochromatic copy K_s^l in the first color or a monochromatic copy of K_t^l in the second color.*

Definition. *Fix $k, l \geq 2$ and l-graphs $H_1, \ldots H_k$. Then $r^l(H_1, \ldots, H_k)$ is the minimum n such that every k-edge-coloring of K_n^l results in a monochromatic subgraph H_i in color i for some $i \in [k]$. When $H_1 = \cdots = H_k = H$, we write $r_k^l(H)$.*

The classical Ramsey numbers seem even more difficult to determine for hypergraphs. The smallest non-trivial case is $r^3(4,4)$ and this is in fact the only classical Ramsey number for hypergraphs where the exact value is known. The upper bound $r^3(4,4) \leq 13$ was first proved by Giraud in 1969 [31]. Much later, McKay and Radziszowski [39] showed (with a computer aided proof) that Giraud's result was optimal.

Theorem 15 (**McKay-Radziszowski [39]**).

$$r^3(4,4) = 13.$$

A result of Erdős and Rado [24] provides upper and lower bounds for $r^3(t,t)$.

Theorem 16 (**Erdős-Rado [24]**). *For all $t \geq 3$, there are positive constants c and c' such that*

$$2^{ct^2} < r^3(t,t) < 2^{2^{c't}}.$$

Recently Conlon, Fox and Sudakov made significant progress on the problem of determining $r^3(s,t)$ [18]. They improved the upper bounds in Theorem 16 and also improved the previous lower bounds for this problem. They provided a new lower bound for the diagonal multicolor Ramsey number $r^3(t,t,t)$ improving the previous bound of Erdős and Hajnal.

Theorem 17 (**Conlon et. al. [18]**). *For integers $t \geq s \geq 4$, there is a positive constant c such that*

$$r^3(s,t) \geq 2^{c\,st\,\log(t/s+1)}.$$

For fixed $s \geq 4$ and sufficiently large t

$$r^3(s,t) \leq 2^{t^{s-2}\log t}.$$

Theorem 18 (Conlon et. al. [18]). *There is a positive constant c such that*

$$r^3(t,t,t) \geq 2^{t^{c \log t}}.$$

Conlon, Fox and Sudakov used a classical technique known as the Stepping-Up Lemma which was introduced by Erdős and Hajnal [33], which they also improved in an earlier result [19]. Axenovich, Gyárfás, Liu and Mubayi investigated the problem of determining $r^3_k(H)$. One of the smallest non-trivial cases is $G = K^3_4 - e$, the unique 3-graph with four vertices and three edges.

Definition. *Let $r^l_k(H)$ be the minimum n such that every k-edge-coloring of the edges of K^l_n contains a monochromatic copy of H.*

Theorem 19 (Axenovich et. al. [6]).

$$r^3_k(K_3) \leq r^3_{4k}(K^3_4 - e) \leq r^3_{4k}(K_3) + 1$$

We will introduce another definition which is similar to one we gave earlier for graphs.

Definition. *Fix $l \geq 2$ and l-graphs G, H with $H \subset G$. Let $f_l(G,H,q)$ be the minimum number of colors in an (H,q)-coloring of G. In the special case where $G = K^l_n$ and $H = K^l_p$, we let $f_l(G,H,q) = f_l(n,p,q)$.*

A natural extension to Theorem 19 is to ask for the minimum p such that $f(n,p,3)$ is $(\log n)^{o(1)}$. Recently, Conlon and Fox observed that this minimum at most 13. We improve this below.

Theorem 20.

$$f_3(n, 7, 3) \le e^{O(\sqrt{\log\log n})}$$

We will also prove a more general theorem.

Theorem 21. *Given positive integers n, l, m with $l > 2$ and $l + 5 < m < n$. Then*

$$f_{l+1}(K_{2^n}^{l+1}, K_{m+1}^{l+1}, 3) \le 2\, f_l(K_n^l, K_m^l, 3) + 2l.$$

1.4 Hypergraphs: Paths, Cycles and other subgraphs

Hypergraph Ramsey type problems where we seek monochromatic subgraphs other than cliques have received some attention. Many of the commonly studied subgraphs in the case of standard graphs are also studied in hypergraphs. Two particular examples are paths and cycles; however, in hypergraphs, paths and cycles may be constructed in multiple ways depending on the number of vertices in the intersection of two adjacent edges.

Definition. *Given $l, q > 0$, the l-uniform loose cycle on q edges C_q^l is the l-graph with $|V(C_q^l(t))| = ql - (q-1)$ and edge set $E(C_q^l(t)) = \{e_1, \ldots, e_q\}$ where $|e_i \cap e_j| = 1$ if $j = i + 1$ or $i = 1, j = k$ and $|e_i \cap e_j| = 0$ otherwise.*

Definition. *Given $l, q, t > 0$ with $t < \lfloor l/2 \rfloor$, the t-intersecting, l-uniform path on q edges $P_q^l(t)$ is the l-graph with $|V(P_q^l(t))| = ql - (q-1)t$ and edge set $E(P_q^l(t)) = \{e_1, \ldots, e_q\}$ where $|e_i \cap e_j| = t$ if $j = i + 1$ and $|e_i \cap e_j| = 0$ otherwise. Such a path is called loose if $t = 1$.*

One may also study paths where $|e_i \cap e_{i+1}| > \lfloor l/2 \rfloor$ or cycles where $|e_i \cap e_{i+1}| > 1$, however, in this thesis we will restrict our attention to loose cycles and paths with $t < \lfloor l/2 \rfloor$. When specifically studying loose paths, the parameter $t = 1$ may be omitted for simplification of notation. An important result by Haxell et. al. [36] provided bounds the Ramsey problem for loose cycles in K_n^3 . Recall that $r^l(s,t)$ is the minimum n such that every 2-edge coloring of the K_n^l yields either a monochromatic K_s^l in the first color or a monochromatic K_t^l in the second color. They recently proved the following result on loose cycles:

Theorem 22 (**Haxell et. al. [36]**).

$$r^3(C_q^3, C_q^3) \sim 5q/4$$

Theorem [36] is the best possible asymptotic result. Even more recently, the following result on loose paths and cycles was proven [44].

Theorem 23 (**Omidi-Shahsiah [44]**). *For every $n \geq m \geq 3$, $r(P_n^3, P_m^3) = 2n + \lfloor \frac{m+1}{2} \rfloor$ holds and for $n > m \geq 3$, $r(P_n^3, C_m^3) = 2n + \lfloor \frac{m-1}{2} \rfloor$ holds.*

In Chapter 5 we will focus on rainbow coloring small subgraphs of K_n^3. Our first result is the following.

Theorem 24. *Let $n \geq 4$. Then*

$$f_3(K_n^3, P_3^3, 3) \leq \binom{n}{2} - \lfloor n^2/4 \rfloor = \begin{cases} n^2/4 - n/2, & \textit{if } n \textit{ is even} \\ n^2/4 - n/2 + 1, & \textit{if } n \textit{ is odd.} \end{cases}$$

If $n \geq 12$, then $f_3(K_n^3, P_3^3, 3) \geq n^2/4 - n/2$.

One notable aspect of studying Ramsey type problems for hypergraphs is the increased number of subgraphs that are possible due the edges being able to intersect in multiple ways. We will prove a result on a unique 3-graph known as the Pasch Configuration.

Definition. *The Pasch Configuration P is the unique 3-graph on six vertices such that any two of its edges have exactly one vertex in common.*

Our original goal was to prove that $f_3(n, P, 3) = n^{1+o(1)}$, however we were only able to prove the slightly weaker result below where we color most of the edges.

Theorem 25. *There is an edge-coloring of $H \subset K_n^3$ with $n^{1+o(1)}$ colors such that every copy of $P \subset H$ contains at least three distinct colors on its edges and $|E(H)| > \binom{n}{3} - o(n^3)$.*

Proving Ramsey results for higher uniformity or larger subgraphs is often much more difficult. We will require Turán type results to prove our remaining theorems. The focus of Turán problems is to determine the minimum number of edges in a large graph or hypergraph G required in order to force the appearance of particular subgraph H.

Definition. *Given an l-graph H, let* $\mathrm{ex}_l(n, H)$ be the maximum number of edges that an l-graph on n vertices can have without containing a copy of H.

Turán and Ramsey problems sometimes turn out to very closely related. When proving a Ramsey result, one will often require that a particular subgraph does not appear in a color class. Recall that $P_q^l(t)$ is the l-uniform, t-intersecting path on q edges. We will prove the following result.

Theorem 26. *Let l, t be fixed positive integers with $t < l$ and $l \leq 2t+1$. If there is an $S(n, 2l-t-1, l)$ on $[n]$, then*

$$f_l(K_n^l, P_2^l(t), 2) \leq (1+o(1))\tfrac{(n-l)!\,(l-t-1)!}{(n-t)!\,(2l-2t-1)!}.$$

If it is possible to partition $S(n, 2l-t-1, l)$ into copies of $S(n, 2l-t-1, t)$ and $l-t$ is a prime, then

$$f_l(K_n^l, P_2^l(t), 2) = \tfrac{(n-l)!\,(l-t-1)!}{(n-t)!\,(2l-2t-1)!}.$$

The Turán number for $P_q^l(t)$ has been studied at some length, particularly when $t = 1$. It often turns that Turán numbers are useful in graph coloring problems. Some previous results on $\mathrm{ex}_l(n, P_q^l(t))$ are given below.

Theorem 27 (Füredi-Jiang-Seiver [30]). *For all fixed $q \geq 1, l \geq 4$, and sufficiently large n we have*

$$\mathrm{ex}_l(n, P_{2t+1}^l) = \tbinom{n-1}{l-1} + \cdots + \tbinom{n-t}{l-1},$$
$$\mathrm{ex}_l(n, P_{2t+2}^l) = \tbinom{n-1}{l-1} + \cdots + \tbinom{n-t}{l-1} + \tbinom{n-t-2}{l-2}.$$

Theorem 28 (Kostochka-Mubayi-Verstraëte [38]). *Fix $l \geq 3, q \geq 4$ with $(q, l) \neq (4, 3)$. For large enough n, we have*

$$\mathrm{ex}_l(n, P_q^l) = \tbinom{n}{l} - \tbinom{n-\lfloor\frac{q-1}{2}\rfloor}{l} + \begin{cases} 0, & \text{if } q \text{ is even} \\ \tbinom{n-\lfloor\frac{q-1}{2}\rfloor-2}{l-2}, & \text{if } q \text{ is odd.} \end{cases}$$

We will study $\mathrm{ex}_l(n, P_q^l(t))$ when $t > 1$. The following is our final result of this thesis.

Theorem 29. *Fix $k, r \geq 2$, $1 < t \leq \lfloor\frac{r}{2}\rfloor$ and $l = \lfloor\frac{k-1}{2}\rfloor$. Then $ex_l(n, P_k^r(t)) \sim l\tbinom{n}{r-1}$.*

Throughout the thesis we will make use of asymptotic notation. Let $f, g : \mathbb{R}^+ \to \mathbb{R}$. Then $f = O(g)$ if there exists a constant c such that $|f| < c|g|$. We say $f = \Omega(g)$ if $g = O(f)$. If $lim_{x \to \infty} f(x)/g(x) = 0$, we say $f(x) = o(g(x))$.

CHAPTER 2

COLORING Q_N WITH RAINBOW PATHS.

In this chapter, we will prove a few basic results on coloring the hypercube with rainbow paths. In the subsequent chapters, we will prove more difficult results on coloring the hypercube with rainbow cycles, so this chapter will introduce some of the basic methods. Recall that the n-dimensional hypercube Q_n is the graph with vertex set $V(Q_n) = 2^{[n]}$; if we view two vertices u and v as their corresponding subsets of $[n]$, then uv is an edge if $|v| = |u| + 1$ and $u \subset v$. The vertex set $V(Q_n)$ may be also be regarded as the 2^n binary vectors of length n with vertices u and v being joined by an edge if and only if they differ in precisely one coordinate. We will restate an earlier definition and give a new definition which we will use throughout the sections of the thesis pertaining to Q_n.

Definition. *Given graphs G and H, an edge-coloring of G is an (H, q)-coloring if every copy of H in G receives at least q distinct colors. The minimum number of colors required in an (H, q)-coloring is $f(G, H, q)$.*

Definition. *Let $uv \in Q_n$ and let u and v represent their corresponding subsets of $[n]$ with $|u| = l$ and $|v| = l + 1$. We then say that uv is on level l of Q_n.*

We will prove the following upper and lower bounds for $f(Q_n, P_q, q)$ for $q = 2, 3, 4$:

Theorem 8. *The following lower and upper bounds hold for $f(Q_n, P_q, q)$:*

q	*lower bound*	*upper bound*
2	n	n
3	$2n-1$	$2n$
4	n^2	$3n^2$

We will prove Theorem 8 in three separate cases. Before proceeding, we will make an observation about Q_n that will be used in the proof and the subsequent chapters. The graph Q_n is obtained by joining each pair of corresponding vertices from two copies of Q_{n-1} with an edge; these edges which join the two copies of Q_n together will be referred to as *crossing edges.*

Proof. **Case 1: $q = 2$.**

An edge-coloring of a graph is called *proper* if no two incident edges receive the same color. An edge-coloring with only rainbow copies of P_2 is equivalent to a proper edge-coloring. The lower bound follows from the simple observation that each vertex in Q_n has degree n and therefore at least n colors are required.

For the upper bound, we will use induction to exhibit a proper edge-coloring of Q_n which uses n colors. For the base case, it is trivial to color Q_2 with 2 colors so that every copy P_2 two is rainbow. Now, suppose that $n > 2$ and we have shown that Q_{n-1} may be properly edge-colored with $n-1$ colors. To properly edge-color Q_n with n colors, join together two properly edge-colored copies of Q_{n-1}, each colored the with the same set of $n-1$ colors, and then color all of the crossing edges with the same new color. A P_2 is either contained in one of the properly edge-colored copies of Q_{n-1} or it contains one crossing edge and one edge from a Q_{n-1}; the crossing edges form a perfect matching, so a P_2 cannot be made up of two crossing edges. In either case, the P_2 is rainbow and we establish the upper bound of n.

Case 2: $q = 3$.

The lower bound of $2n-1$ is realized by a simple observation. Given an edge $e = uv$, then e and the set of edges which are either incident with u or incident with v must all receive distinct colors. This is a set of $1 + 2(n-1) = 2n-1$ edges.

We will now give an edge-coloring function χ which proves the upper bound. As previously stated, a vertex $v \in V(Q_n)$ is associated to a subset of $[n]$; given a vertex v, we will also use v to refer to its associated set. Let $e = uv$ be an edge with $u = \{m_1, \ldots, m_i\} \subset \{m_1, \ldots, m_{i+1}\} = v$ and let $p(e)$ denote the parity of i. Our coloring is $\chi(e) = (m_{i+1}, p(e))$. There are n possibilities for the first coordinate of χ and 2 possibilities for its second coordinate, so χ uses at most $2n$ colors.

A P_3 can span one, two or three levels of Q_n and we will consider each case separately. First, suppose that a P_3 spans three levels of Q_n. Let $P = e_1e_2e_3$ with $e_1 = v_1v_2$, $e_2 = v_2v_3$ and $e_3 = v_3v_4$ with $v_2 \subset v_3 \subset v_4$ and let $v_2 = \{m_1, \ldots, m_i\}$, $v_3 = \{m_1, \ldots, m_{i+1}\}$ and $v_4 = \{m_1, \ldots, m_{i+2}\}$. Then $\chi(e_1) = (m_i, p(e_1))$, $\chi(e_2) = (m_{i+1}, p(e_3))$ and $\chi(e_3) = (m_{i+2}, p(e_2))$ are all distinct colors.

Next, suppose that $P = e_1e_2e_3$ spans two levels of Q_n. Without loss of generality, let e_1, e_2 lie on level i of Q_n and let e_3 lie on level $i+1$. We observe that $\chi(e_3) \neq \chi(e_1), \chi(e_2)$ because $p(e_3) \neq p(e_1), p(e_2)$. Now we will show that $\chi(e_1) \neq \chi(e_2)$. Let $e_1 = vx$ with $v \subset x$ and $e_2 = vy$ with $v \subset y$ and let $v = \{m_1, \dots m_i\}$, $x = \{m_1, \dots m_i, m'\}$ and $y = \{m_1, \dots m_i, m''\}$. Because e_1 and e_2 are distinct edges, we have $m' \neq m''$ and so $\chi(e_1) = (m', p(e_1)) \neq (m'', p(e_2)) = \chi(e_2)$.

Finally, suppose that $P = e_1e_2e_3$ lies entirely on level i of Q_n for some i. Let $e_1 = ux$, $e_2 = uy$ and $e_3 = wy$ with $u \subset x, y$ and $w \subset y$. If $u = \{m_1, \dots, m_i\}$, $x = \{m_1, \dots, m_i, m'\}$ and $y = \{m_1, \dots, m_i, m''\}$, then $w = y - \{m'''\}$ for some $m''' \in y$. Since $e_1 \neq e_2$, we have $m' \neq m''$ and similarly since $e_2 \neq e_3$, we have $m'' \neq m'''$. We see that $m' \neq m'''$ by contradiction, for if $m' = m'''$ then $u = w$. Therefore, $\chi(e_1), \chi(e_2), \chi(e_3)$ are all distinct in their first coordinates.

Case 3: $q = 4$

We will first prove the lower bound. If two edges e and f lie on levels 0 or 1 of Q_n, then they must receive distinct colors. If e and f lie on level 0 then they are incident. If $e = zx$ where $z = \emptyset$ lies on level 0 and $f = vw$ with $v \subset w$, then the edges $h = zv$, e and f form a P_3. If both $e = xy$ and $f = vw$ lie on level 1 (and are not incident), then we use the edges $h = zx$ and $h' = zv$ with $z = \emptyset$ to obtain $ehh'f$ which is a P_4. Therefore, any two edges on levels 0 or 1 of Q_n must receive distinct colors and we obtain the lower bound of $n + n(n-1) = n^2$. For a vertex v, the incidence vector $\vec{v}$ is a binary vector of length n where its i-th position $\vec{v}_i = 1$ if $i \in v \subset [n]$ and $\vec{v}_i = 0$ otherwise.

For an edge $e = uv$ with $u \subset v$, recall that l_e is the level of e. Let

$$a(e) = (\sum_{1 \leq i \leq n} \vec{v}_i \, i) \, mod \, n.$$

Let $p(e)$ be the position i where $\vec{u}$ and $\vec{v}$ differ and let $b(e) = l_e \, mod\, 3$. We color the edge e with

$$\chi(e) = (a(e), p(e), b(e)).$$

Since there are n possibilities for $a(e)$, n possibilities for the $p(e)$ and three possibilities for $b(e)$, χ uses at most $3n^2$ colors as desired. We will now show that χ edge colors Q_n is a $(P_4, 4)$-coloring of Q_n.

First, suppose that $P = e_1e_2e_3e_4$ spans four levels of Q_n and $e_1 = v_1v_2$, $e_2 = v_2v_3$, $e_3 = v_3v_4$ and $e_4 = v_4v_5$ with $v_2 \subset v_3 \subset v_4 \subset v_5$. Then $p(e_1), p(e_2), p(e_3), p(e_4)$ are all distinct and therefore $\chi(e_1), \chi(e_2), \chi(e_3), \chi(e_4)$ are all distinct.

Second, suppose that $P_4 = e_1e_2e_3e_4$ spans three levels of Q_n. Without loss of generality, suppose that e_1,e_2 are on level i, e_3 is on level $i+1$ and e_4 is on level $i+2$. We observe that $b(e_3) \neq b(e_4)$ and $b(e_1) = b(e_2) \neq b(e_3), b(e_4)$ and we also see that $\chi(e_1) \neq \chi(e_2)$ because e_1 and e_2 are incident.

Next, suppose that $P = e_1e_2e_3e_4$ spans two levels of Q_n. There are two ways that this can occur. The first possibility is that e_1, e_2 are on level i and e_3, e_4 are on level $i+1$. Then $b(e_1) = b(e_2) \neq b(e_3) = b(e_4)$. Furthermore, since e_1 and e_2 are incident, $p(e_1) \neq p(e_2)$ and likewise $p(e_3) \neq p(e_4)$. So, $\chi(e_1), \chi(e_2), \chi(e_3), \chi(e_4)$ are all distinct. The second possibility is that, without loss of generality, e_1, e_2, e_3 lie on level i and e_4 lies on level $i+1$. In this case, $b(e_4) \neq b(e_1) = b(e_2) = b(e_3)$ so we just need to show that $\chi(e_1), \chi(e_2), \chi(e_3)$ are all distinct. This is showing that a P_3 contained on a single level of Q_n is rainbow under our coloring χ. For such a path $P = e_1e_2e_3$, the edges differ in their second coordinate. It's trivial to see that this is the case for incident edges, so we will show that $p(e_1) \neq p(e_3)$. Let $e_1 = xy$ with $x \subset y$

and $e_3 = uv$ with $u \subset v$; consequently, $e_2 = xv$. If $p(e_1) = p(e_3)$, then for some $w \in V(Q_n)$, we must have $y = x \cup \{w\}$ and $v = u \cup \{w\}$. But we must also have that $v = x \cup \{w'\}$ were $w' \neq w$ which is a contradiction.

Finally, suppose that $P = e_1e_2e_3e_4$ is entirely on level i of Q_n for some i. By the previous case, we know that $P' = e_1e_2e_3$ and $P'' = e_2e_3e_4$ are rainbow paths. Therefore, it suffices to show that $\chi(e_1) \neq \chi(e_4)$. Let $e_1 = xy$ with $x \subset y$ and $e_4 = uv$ with $u \subset v$. We will look at the symmetric difference $|y \triangle w|$. Since $y \neq v$ we know that $|y \triangle w| \neq 0$ and since $|y| = |w|$ we know that $|y \triangle w|$ must be even. If $|y \triangle w| = 6$, then it is not possible for e_1 and e_4 to lie on the same P_4, so $|y \triangle w| = 2$ or 4. If $|y \triangle w| = 2$, then $x = u = \{m_1, \ldots, m_i\}$, $y = \{m_1, \ldots, m_i, m'\}$ and $w = \{m_1, \ldots, m_i, m''\}$ with $m' \neq m''$. In this case we have $m' = a(e_1) \neq a(e_4) = m''$ and we have $\chi(e_1) \neq \chi(e_4)$. If $|y \triangle w| = 4$, then $x = \{m_1, \ldots, m_{i-1}, m'\}$, $y = \{m_1, \ldots, m_{i-1}, m', m''\}$, $u = \{m_1, \ldots, m_{i-1}, m'''\}$ and $w = \{m_1, \ldots, m_{i-1}, m''', m''''\}$ with m', m'', m''', m'''' all distinct. In this case $m'' = p(e_1) \neq p(e_4) = m''''$ and so $\chi(e_1) \neq \chi(e_4)$.

$\square$

CHAPTER 3

COLORING SUBCUBES OF Q_N

Our goal in this chapter is to study $f(Q_n, Q_3, q)$, however we will first determine the upper and lower bounds for $f(Q_n, Q_2, 3)$. Let us recall the definition provided earlier:

Definition. *Fix graphs G and H and and integer $1 \leq q \leq |E(H)|$. Then $f(G, H, q)$ is the minimum number of colors required to edge color G so that the edges in every copy of H in G receive at least q distinct colors.*

One of the motivations for studying $f(Q_n, Q_3, q)$ was the following result of Faudree, Gyárfás, Lesniak and Schelp [27].

Theorem 9. *Let $n = 4$ or $n \geq 6$. Then $f(Q_n, Q_2, 4) = n$.*

Any two incident edges of Q_n lie in a copy of $C_4 = Q_2$ and this implies that $f(Q_n, Q_2, 4) \geq n$ since any $(Q_2, 4)$-coloring is a proper coloring of Q_n. The content of Theorem 9, therefore, is an explicit coloring which demonstrates that the upper bound is also n. For $f(Q_n, Q_3, q)$ we will not be able to provide explicit colorings for the upper bounds for several values of q. We will rely on a probabilistic technique know as the Local Lemma.

Theorem 30 (**Local Lemma, Lovás [4]**). *Let $A_1, A_2, \ldots A_n$ be events in an arbitrary probability space. Suppose that each A_i is mutually independent of all but at most d of the other events and $Pr[A_i] \leq p$ for all i. If $ep(d+1) \leq 1$ then $Pr[\bigwedge_{i=1}^{n} \bar{A}_i] > 0$.*

Specifically, we will use Theorem 30 to obtain upper bounds for $f(Q_n, Q_3, q)$ for $7 \leq q \leq 11$. In the following section, we will study $f(Q_n, Q_2, q)$ and in the section after we will study $f(Q_n, Q_3, q)$.

3.1 Many colors on the 2-dimensional subcubes

The goal of this section is to obtain bounds for $f(Q_n, Q_2, 3)$. It trivial to show that $f(Q_n, Q_2, 1) = 1$ and $f(Q_n, Q_2, 2) = 2$. It was also shown in [27] that $f(Q_n, Q_2, 4) = n$. We will prove the following:

Theorem 31. *There exists an n_0 such that for $n > n_0$ we have $f(Q_n, Q_2, 3) = 4$.*

We will require a result of Offner [43] for the lower bound:

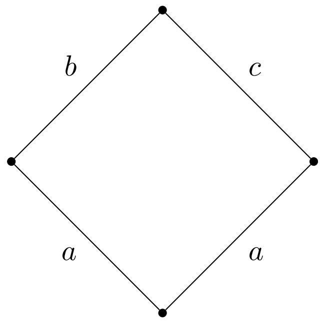
Definition. *A d-polychromatic coloring of Q_n with p colors is an edge coloring of Q_n such that all copies of Q_d have all p colors on their edges. The maximum number of colors with which it is possible to d-polychromatically color any Q_n is p_d.*

Theorem 32 (**Offner [43]**)**.**

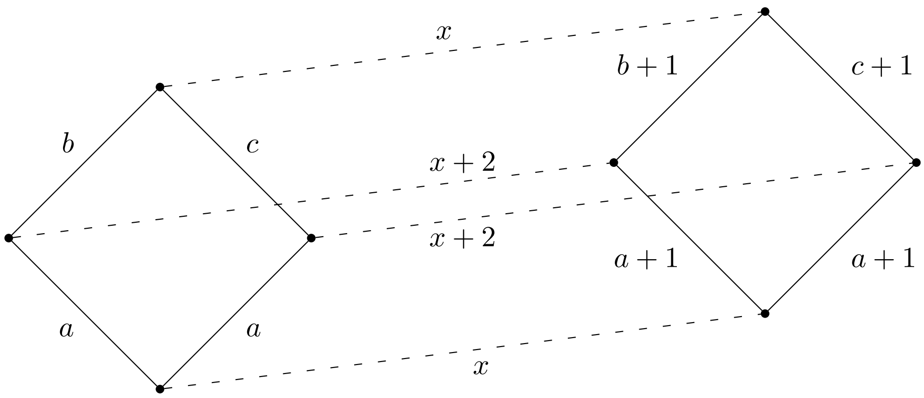
$$p_d = \begin{cases} \frac{(d+1)^2}{4} & \text{if } d \text{ is odd} \\ \frac{d(d+2)}{4} & \text{if } d \text{ is even.} \end{cases}$$

We will now to prove Theorem 31.

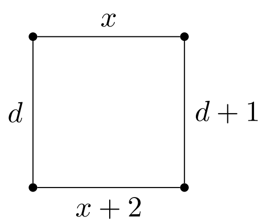
Proof. First we will prove the upper bound with an explicit coloring where our palette of colors is the elements of $\mathbb{Z}_4$. Let $a, b, c \in \mathbb{Z}_4$ be distinct. We will proceed by induction on n. For our base case, Q_2 is colored as shown in the following figure:



Now, in order to color Q_n, we consider two copies of Q_{n-1} with identical colorings such that every Q_2 receives three distinct colors. In one of the copies, if i is the color which an edge e initially received, then we recolor e with the color $i + 1$. Now we will color the crossing edges going between the two copies. We use the fact the Q_{n-1} is a bipartite graph and partition the vertices of one of the copies into two independent sets A and B. Fix any $x \in \mathbb{Z}_4$. If a crossing edge e is incident to a vertex in A we color it with x and if e incident to a vertex in B we color it with $x + 2$. Explicitly, Q_3 would be colored as shown in the following figure:



We know that any Q_2 entirely contained in either copy of Q_{n-1} receives three colors on its edges, so we need only worry about those copies of Q_2 with edges contained in each copy of Q_{n-1}. Such a Q_2 is colored as follows where $d, x \in \mathbb{Z}_4$ are not necessarily distinct.



There must be at least three distinct elements among $\{d, d+1, x, x+2\} \subset \mathbb{Z}_4$ and therefore $f(Q_n, Q_2, 3) \leq 4$.

The trivial lower bound is $f(Q_n, Q_2, 3) \geq 3$. However, by Theorem 32 we know that $f(Q_n, Q_2, 3) \neq 3$. If it were possible to color Q_n with 3 colors so that every copy of Q_2 contains 3 colors on its edges, then such a coloring of Q_n would be a 2-polychromatic coloring of with 3 colors, but Theorem 32 tells us that a 2-polychromatic coloring of Q_n can use at most 2 colors. Consequently, $f(Q_n, Q_2, 3) \geq 4$.

$\square$

3.2 many colors on the 3-dimensional subcubes

Now we consider $f(Q_n, Q_3, q)$. Let us state our main result here.

Theorem 10. *The following upper and lower bounds hold for $f(Q_n, Q_3, q)$:*

q	*lower bound*	*upper bound*
4	4	4
5	5	8
6	6	12
7	$\Omega(\frac{\log n}{\log\log n})$	$O(n^{1/3})$
8	$\Omega(\frac{\log n}{\log\log n})$	$O(n^{2/5})$
9	$n^{1/3}$	$O(\sqrt{n})$
10	$\sqrt{n}$	$O(n^{2/3})$
11	$n/2$	$O(n)$
12	$3n-2$	$n^{1+o(1)}$

Determining $f(Q_n, Q_3, q)$ is trivial when $q \in \{1, 2, 3\}$. Indeed, it is easy to see that $f(Q_n, Q_3, q) = q$ for these cases by coloring alternating levels of Q_n with colors 1 through q.

We will first prove a lemma which gives a general upper bound for $f(Q_n, Q_3, q)$.

Lemma 33. *Fix $1 \le q \le 12$. Then $f(Q_n, Q_3, q) = O(n^{\frac{2}{13-q}})$.*

Proof. Color Q_n randomly with c colors where each color appears with probability $1/c$ on a given edge, independently of all other colors. Each copy of Q_3 has an associated bad event A_i which represents that copy of Q_3 receiving less than q colors on its edges. For some constant α not depending on n, there is $\alpha\binom{c}{q-1}$ ways to color a Q_3 with less than q colors and there are c^{12} edge colorings of Q_3 in total from a palette of c colors. Therefore, $Pr[A_i] = O(\frac{c^{q-1}}{c^{12}})$. A copy of Q_3 shares an edge with $O(n^2)$ other copies of Q_3; we observe that a C_6 is determined by 3 edges and that every copy of Q_3 contains a fixed number of copies of C_6. So, if we pick an edge e in a Q_3, there are at most $(n-1)^2$ copies of C_6 which contain e. Therefore, there are $O(n^2)$ copies of C_6 and similarly $O(n^2)$ copies of Q_3 which share an edge with a Q_3. Consequently, we may apply the Local Lemma with parameters $p = O(c^{q-13})$ and $d = O(n^2)$. Then $ep(d+1) < 1$ as long as $c = O(n^{\frac{2}{13-q}})$.

□

We will use the upper bound from Lemma 33 for $f(Q_n, Q_3, q)$ when $7 \leq q \leq 11$. In the remaining cases we will use other techniques to provide better upper bounds. We will now prove Theorem 10. The case $q = 12$ will be addressed in the next chapter, so we proceed with all other values of q below.

Case 1: $q = 4$

Proof. The lower bound is trivial. We obtain the upper bound by using the same coloring we used to prove Theorem 31. A Q_3 is two copies of Q_2 joined by crossing edges. Each copy of Q_2 contains at least 3 colors on its edges. Using our coloring, one copy of Q_2 will have the colors $\{a, b, c\}$ on its edges and the other copy will have $\{a+1, b+1, c+1\}$ on its edges where

$a, b, c \in \mathbb{Z}_4$ are distinct. The subset $\{a, b, c, a+1, b+1, c+1\} \in \mathbb{Z}_4$ contains 4 distinct elements and therefore $f(Q_n, Q_3, 4) \leq 4$ □

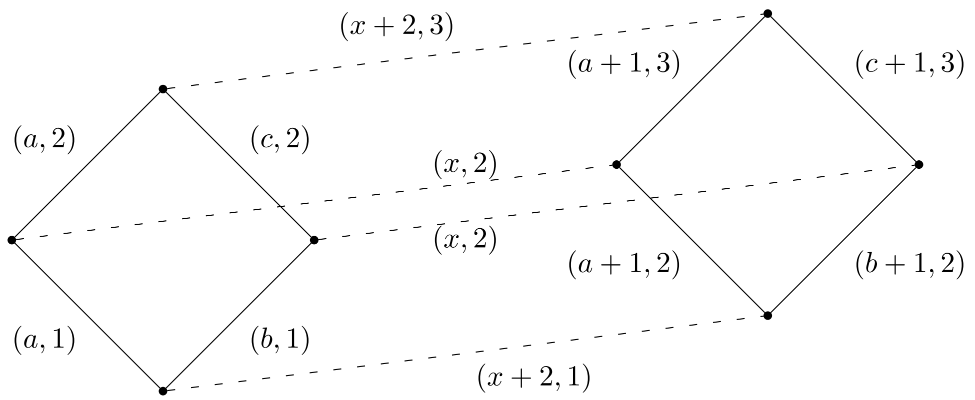
Case 2: $q = 5$

Proof. The lower bound is trivial. For the upper bound, we will use the edge coloring we used in Theorem 31 with a few modifications. Our palette of colors will now be $\mathbb{Z}_8$ instead of $\mathbb{Z}_4$ and when we adjoin two copies of Q_{n-1} we will color the crossing edges with x and $x+4$ (instead of x and $x+2$). These modifications still guarantee that a copy of Q_2 receives three distinct colors on its edges and so a copy of Q_3 will contain the set $S = \{a, b, c, a+1, b+1, c+1, x, x+4\}$ of colors on its edges where a, b, c are distinct. Now, we consider the subset $S' = \{a, b, c, a+1, b+1, c+1\}$. If $|S'| > 5$ then we are done. If $|S'| < 5$, then because $a, b, c \in \mathbb{Z}_8$ are distinct, it must be the case, without loss of generality, that $b = a+1$ and $c = a+2$. Then S' is a set of four consecutive elements and so either $x \notin S'$ or $x+4 \notin S'$. Therefore, $|S| \geq 5$ and $f(Q_n, Q_3, 5) \leq 8$.

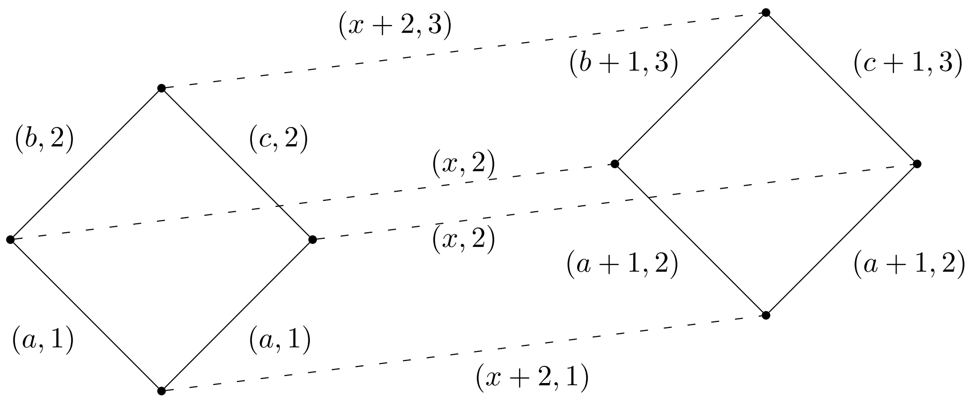
Before proceeding to the next case, notice that this would not work if we used $\mathbb{Z}_p$ where $p \leq 7$ as our palette of colors. For if in the set S we had $a = 1, b = 2, c = 3$ and $x = 4$ then $|S| \leq 4$. Thus, if a better upper bound is possible, we would need to use a different strategy for coloring the edges. □

Case 3: $q = 6$

Proof. The lower bound is trivial. For the upper bound we will use a modified version of the coloring used in Theorem 31. We now color the edge e with the tuple (i, r) where i is the color which e originally received in the proof of Theorem 31 and r is the congruence class *modulo* 3 of the level of Q_n that e lies. We are now using 12 colors in total and we have two cases to examine. Since we are using the coloring from Theorem 31 for our first coordinate, we know that a Q_3 is made up of two joined copies of Q_2 which each have at least three distinct colors in their first coordinates and the first coordinates of the second copy of Q_2 match those of the first copy of Q_2 but shifted up by one. We will assume that the two copies of Q_2 each receive only three distinct colors in their first coordinates. Furthermore, the coloring in Theorem 31 guarantees that if two edges f and f' in a Q_2 receive the same color, then f and f' are incident. Without loss of generality, there are two cases we must consider. The first case is shown below:



Since a, b and c are distinct, we immediately see that $(a,1), (b,1), (a,2), (b,2), (a+1,3)$ and $(c+1,3)$ are all distinct. In the second case, a Q_3 would appear as follows:



We are done if a and $x+2$ are distinct, as $(a,1),(x+2,1),(b,2),(c,2),(b+1,3)$ and $(c+1,3)$ are all distinct. However, suppose that $a = x+2$. If level one of the above Q_3 contains three distinct colors then we are done. Suppose this does not happen. Without loss of generality, assume that $b = a+1$ and $c = x$ (note that $x \neq a+1$ since $a = x+2$). Then $b+1$ and $c+1$ are two consecutive numbers and neither is equal to $x+2$, so we have at least one color from level zero, two colors from level 1 and three colors from level 2 and so $f(Q_n, Q_3, 6) \leq 12$. $\square$

Case 4: $q = 7, 8$

Proof. The upper bounds for $f(Q_n, Q_3, 7)$ and $f(Q_n, Q_3, 8)$ are obtained by applying Lemma 33. We will prove the lower bound for $f(Q_n, Q_3, 7)$ and also use it as the lower bound for $f(Q_n, Q_3, 8)$. We will prove the lower bound by contradiction when we assume that it is finite. Our proof will use $r_k(3)$; recall that $r_k(3)$ is the smallest n such that every edge coloring of K_n with k colors contains a monochromatic triangle. We proceed with our proof.

Suppose that χ is a $(Q_3, 7)$-coloring with s colors and let n be sufficiently large. On level 0 of Q_n, we can, by the Pigeonhole Principle, find a set of E of edges of size n/s which all receive the same color x. Any two edges e, e' on level 0 of Q_n are incident and therefore are contained in a C_4. There are $\binom{s}{2} + s$ possible pairs colors other two edges f, f' of the C_4 may receive. Given such a copy of a C_4 with edges e, e', f, f', let $g : \binom{E}{2} \to \{1, \cdots, \binom{s}{2} + s\}$ where $g(\{e, e'\})$ is mapped to corresponding pair of colors which f and f' received. If there are $e_1, e_2, e_3 \in E$ such that $g(\{e_1, e_2\}) = g(\{e_2, e_3\}) = g(\{e_1, e_3\})$ then we may extend e_1, e_2, e_3 to a Q_3 with at most six colors; the edges on level 0 will all receive the same color, the edges on level 1 will receive at most two distinct colors and the edges on level 2 will receive at most three distinct colors. Avoiding this is equivalent to avoiding a monochromatic triangle on $K_{\frac{n}{s}}$. Since $r_k(3)$ is finite for all $k \in \mathbb{N}$, for large enough n we will find a monochromatic triangle in $K_{\frac{n}{s}}$. So we must have $|E| < r_k(3)$, where $k = \binom{s}{2} + s < s^2$. We use the upper bound $r_k(3) < 3k!$ [10] to obtain

$$|E| = n/s < 3\, s^2! < 3\, s^{2s}.$$

Solving for s in terms of n gives $s = \Omega(\frac{\log n}{\log \log n})$. $\square$

Case 5: $q = 9$

Proof. The upper bound is from Lemma 33. The lower bound is from the following counting argument. Suppose that we have a $(Q_3, 9)$-coloring of Q_n with $n^{1-\epsilon}$ colors. On the level 0 of Q_n, we can find a set of edges of size n^ϵ which all receive the same color. Since any two edges on level 0 of Q_n can be extended to a Q_2, we have $n^\epsilon - 1$ copies of Q_2 which all intersect in an edge e and whose bottoms edges all receive the same color. The top two edges in each such Q_2 receives one of the available $\binom{n^{1-\epsilon}}{2}$ available pairs of colors. If any two of these copies of Q_2

which intersect in edge e are assigned the same pair of colors to their top edges, then we can extend them to a Q_3 with at most eight colors on its edges. The bottom edges of the Q_3 all receive the same color, the middle edges receive at most four distinct colors and the top edges receive at most three colors. Therefore $\binom{n^{1-\epsilon}}{2} \geq n^{\epsilon} - 1$ must hold which implies $\epsilon \leq 2/3$. So we require at least $n^{1-2/3}$ colors. □

Case 6: $q = 10$

Proof. The upper bound is from Lemma 33. For the lower bound, we use a similar argument to the one we used in the previous case when $q = 9$. Suppose that we have a $(Q_3, 10)$-coloring of Q_n with $n^{1-\epsilon}$ colors. On the level 0 of Q_n, we can find a set of edges of size n^{ϵ} which all receive the same color. Since any two edges on level 0 of Q_n can be extended to a Q_2, we have $n^{\epsilon}-1$ copies of Q_2 which all intersect in an edge e and whose bottoms edges all receive the same color. The top two edges in each such Q_2 receives one of the available $\binom{n^{1-\epsilon}}{2}$ available pairs of colors. If any two of these copies of Q_2 which intersect in edge e have a common color on their top edges, then we can extend them to a Q_3 with at most nine colors on its edges; the bottom edges of the Q_3 all receive the same color, the middle edges receive at most four distinct colors and the top edges receive at most three colors. Therefore the inequality is $n^{1-\epsilon} \geq 2(n^{\epsilon} - 1)$ which implies $\epsilon \leq 1/2$. So we require at least $n^{1-1/2}$ colors. □

Case 7: $q = 11$

Proof. The upper bound is from Lemma 33. For the lower bound, notice that if we use fewer than $n/2$ colors, we will be able to find three edges on level 0 of Q_n which all receive the same color; these three edges can be extended to a Q_3 which contains at most ten distinct colors on its edges. $\square$

CHAPTER 4

COLORING Q_N WITH RAINBOW CYCLES

In the previous section, we determined upper and lower bounds for $f(Q_n, Q_3, q)$. Now we study rainbow colorings of cycles where we are able to obtain more precise results. Rainbow cycles have also been well studied as subgraphs of K_n. Erdős, Simonovits and Sós [25] introduced $AR(n, H)$, the maximum number of colors in an edge coloring of K_n such that it contains no rainbow copy of H, and provided a conjecture when H is a cycle and showed that their conjecture was true when $H = C_3$. Alon [1] proved their conjecture for cycles of length four and Montellano-Ballesteros and Neumann-Lara [40] proved the conjecture for all cycles. More recently, Choi [12] gave a shorter proof of the conjecture. We will now present our results which first appeared in [42].

4.1 Our results

Since we will consider only rainbow colorings in this chapter, we will use the following notations as a matter of convenience. A C_q-rainbow coloring of Q_n is an edge coloring of Q_n such that every copy of C_q is rainbow.

Definition. *For $4 \leq q \leq 2^n$, let $f(n, q)$ be the minimum number of colors in a C_q-rainbow coloring of Q_n.*

Definition. *For $4 \leq q \leq n$, let $g(n, q)$ be the minimum number of colors in a Q_q-rainbow coloring of Q_n.*

As stated in the previous sections, the smallest case $f(n,4)$ was studied by Faudree, Gyárfás, Lesniak and Schelp [27] who proved that the trivial lower bound of n is tight by providing, for all $n \geq 6$, a C_4-rainbow coloring with n colors. We consider larger q. Our first result determines the order of magnitude of $f(n,q)$ for $q \equiv 0 \pmod 4$.

Theorem 11 . *Fix a positive $q \equiv 0$ (mod 4). There are constants $c_1, c_2 > 0$ depending only on q such that*

$$c_1 n^{q/4} < f(n,q) < c_2 n^{q/4}.$$

The case $q \equiv 2 \pmod 4$ seems more complicated. Our results imply that for such fixed q there are positive constants c_1', c_2' with

$$c_1' n^{\lfloor q/4 \rfloor} < f(n,q) < c_2' n^{\lceil q/4 \rceil}.$$

We believe that the lower bound is closer to the truth. As evidence for this, we tackle the smallest case in this range, $q = 6$. As we will observe later, the lower bound $f(n,6) \geq n$ is trivial for $n \geq 3$, and we obtain the following upper bound.

Theorem 12. *For every $\epsilon > 0$ there exists n_0 such that for $n > n_0$ we have $f(n,6) \leq n^{1+\epsilon}$.*

Theorem 12 will also prove the case $q = 12$ of Theorem 10.

Observation. In Q_n, any C_6 can be extended to a Q_3. Therefore, if Q_n is colored so that every Q_3 is rainbow then every C_6 is also rainbow so $f(Q_n, Q_3, 12) \geq f(Q_n, C_6, 6)$. Furthermore, any two edges of a Q_3 are contained in a C_6 (which is also contained in that Q_3). If Q_n is edge colored so that every C_6 is rainbow then every Q_3 must also be rainbow. So $f(Q_n, Q_3, 12) \leq f(Q_n, C_6, 6)$ and therefore $f(Q_n, Q_3, 12) = f(Q_n, C_6, 6)$.

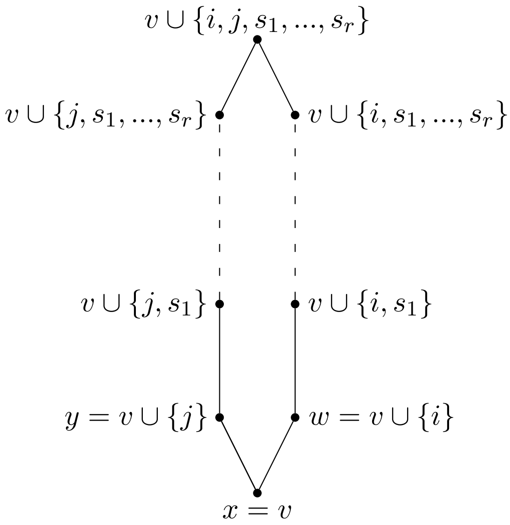
Recall the observation that $g(n, 3) = f(n, 6)$. Since $C_4 = Q_2$, the following corollary can also be considered an analogue of the result in [27] to subcubes.

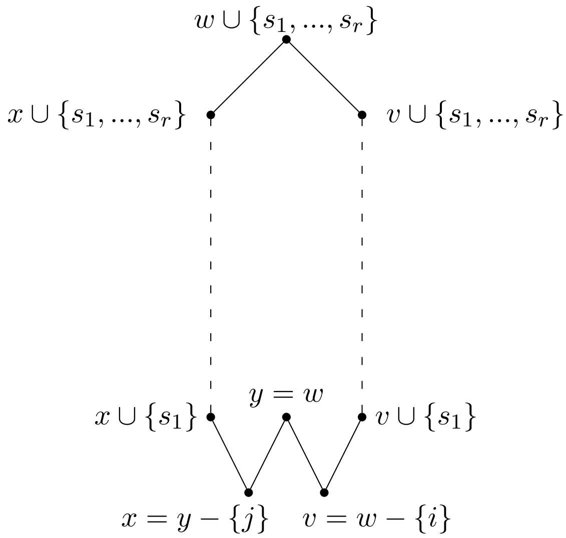
Corollary. *As $n \to \infty$, we have $g(n, 3) = n^{1+o(1)}$.*

We will consider the vertices of Q_n as binary vectors of length n or as subsets of $[n] = \{1, \ldots, n\}$, depending on the context (with the natural bijection $\vec{v} \leftrightarrow v$ where $\vec{v}$ is the incidence vector for $v \subset [n]$, i.e. $\vec{v}_i = 1$ iff $i \in v$). In particular, whenever we write $v - w$ we mean set theoretic difference, $v \cup w$ or $v \cap w$ we mean set union/intersection and when we write $\vec{v} \pm \vec{w}$ we mean vector addition/subtraction modulo 2. We write e_i for the standard basis vector, so e_i is one in the ith coordinate and zero in all other coordinates. Given an edge $f = uv$ of Q_n where $\vec{v} = \vec{u} + e_s$ for some s, we say that v is the top vertex of f and u is the bottom vertex. We will say the an edge is on level i of Q_n if its bottom vertex corresponds to a vector with $i - 1$ ones and the top vertex to a vector with i ones.

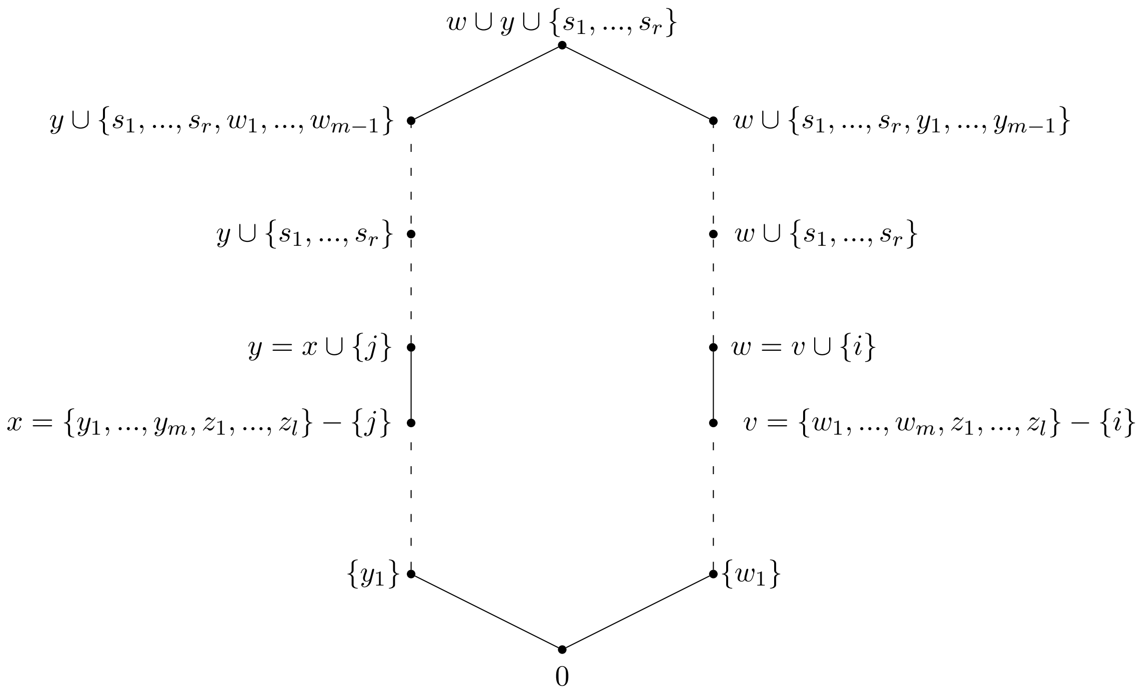
4.2 Proof of Theorem 11

The lower bound in Theorem 11 follows from the easy observation that in a C_q-rainbow coloring all edges at level $q/4$ must receive distinct colors. Indeed, given any two such edges $f_1 = vw$ and $f_2 = xy$, where $\vec{w} = \vec{v} + e_i$ and $\vec{y} = \vec{x} + e_j$, it suffices to find a copy of C_q containing f_1 and f_2. If f_1 and f_2 are incident then it is clear that we can find a C_q containing them as long as $n > q$ which we may clearly assume. The two cases are illustrated below where $r = q/2 - 2$ and $s_i \notin w \cup y$ for all $i \in \{1, ..., r\}$.





Now, suppose f_1 and f_2 are not incident. We know that $|x \triangle v| \leq q/2 - 2$ since x and v are each sets of size $q/4 - 1$. By successively deleting elements of v and x in the appropriate order, we can obtain a v, x-path of length $q/2 - 2$. Then, since w and y are sets of size $q/4$, we may find a w, y-path of length $q/2$ between them by successively adding the elements of y to w and vice versa along with extra elements as needed. The two paths along with the edges vw and xy form a cycle of length q. This is shown in the following diagram. Let $y - w = \{y_1, ..., y_m\}$, $w - y = \{w_1, ..., w_m\}$ and $w \cap y = \{z_1, ..., z_l\}$ where $m + l = q/4$. Let $\{s_1, ..., s_r\}$ again be a set such that $s_i \notin y \cup w$ with $r = q/4 - m$.



For the upper bound we need a classical construction of generalized Sidon sets by Bose and Chowla. A B_t-set $S = \{s_1, \ldots, s_n\}$ is a set of integers such that if $1 \leq i_1 \leq i_2 \leq \cdots \leq i_t \leq n$ and $1 \leq j_1 \leq j_2 \leq \cdots \leq j_t \leq n$, then

$$s_{i_1} + \cdots + s_{i_t} \neq s_{j_1} + \cdots + s_{j_t}$$

unless $(i_1, \ldots, i_t) = (j_1, \ldots, j_t)$. A consequence of this is that if P, Q are non empty disjoint subsets of $[n]$ with $|P| = |Q| \leq t$, then

$$\sum_{i \in P} s_i \neq \sum_{j \in Q} s_j. \tag{4.1}$$

The result below is phrased in a form that is suitable for our use later.

Theorem 34 (**Bose-Chowla [11]**). *For each fixed $t \geq 2$, there is a constant $A > 1$ such that for all n, there is a B_t-set $S = \{s_1, \ldots, s_n\} \subset \{1, 2, \ldots, \lfloor An^t \rfloor\}$.*

Now we provide the upper bound construction for Theorem 11.

Construction 1. Let $t = q/4 - 1$ and $S = \{s_1, \ldots, s_n\} \subset \{1, 2, \ldots, \lfloor An^t \rfloor\}$ be a B_t-set as above. For each $v \in V(Q_n)$, let

$$a(v) = \sum_{i=1}^{n} \vec{v}_i s_i = \sum_{i:\vec{v}_i=1} s_i.$$

Given $vw \in E(Q_n)$ with $\vec{w} = \vec{v} + e_j$, let $M = \lceil qAn^t \rceil$, and let

$$d(vw) = a(v) + Mj.$$

Suppose further that vw is at level p and p' is the congruence class of p modulo $q/2$. Then the color of the edge vw is

$$\chi(vw) = (d(vw), p').$$

Let us now argue that this construction yields the upper bound in Theorem 11.

Proof. First, the number of colors is at most

$$\max_{vw} d(vw) \times \frac{q}{2} \leq (n \cdot \max s_i + Mn)\frac{q}{2} \leq \frac{nq}{2}An^t + \frac{nq}{2}M < q^2An^{t+1} = q^2An^{q/4}$$

as desired. Now we show that this is a C_q-rainbow coloring. Suppose for contradiction that H is a copy of C_q in Q_n and $f_1 = vw, f_2 = xy$ are distinct edges of H with $\chi(f_1) = \chi(f_2)$. Since H spans at most $q/2$ levels, f_1 and f_2 cannot lie in levels that differ by more than $q/2$, so $\chi(f_1) \neq \chi(f_2)$ unless f_1 and f_2 are in the same level which we may henceforth assume. Let v, x be the bottom vertices of f_1, f_2, and $\vec{w} = \vec{v} + e_i, \vec{y} = \vec{x} + e_j$. Assume without loss of generality that $i \leq j$. If $v = x$, then

$$a(v) + Mi = d(vw) = d(xy) = a(x) + Mj = a(v) + Mj.$$

This implies that $i = j$ and contradicts the fact that $f_1 \neq f_2$. We may therefore assume that $v \neq x$. Similarly, if $w = y$, then $i < j$ and

$$a(w) - s_i + Mi = a(v) + Mi = d(vw) = d(xy) = a(x) + Mj = a(y) - s_j + Mj = a(w) - s_j + Mj.$$

This implies the contradiction $s_j - s_i = M(j - i) \geq M > An^t > s_j - s_i$. Consequently, we may assume that vw and xy share no vertex. If $|v \triangle x| > q/2$, then any v, x-path in Q_n has length more than $q/2$ so there can be no cycle of length q containing both v and x, contradiction. So we may assume that $|v \triangle x| \leq q/2$. Now $\chi(vw) = \chi(xy)$ implies that

$$a(v) + Mi = d(vw) = d(xy) = a(x) + Mj$$

and this yields

$$M(j-i) = Mj - Mi = a(v) - a(x) = a(v-x) - a(x-v) \leq \frac{|v \triangle x|}{2} An^t \leq \frac{q}{4} An^t < M.$$

Consequently, we may assume that $i = j$, $a(v) = a(x)$, $a(v-x) = a(x-v)$ and $|v \triangle x| = |w \triangle y|$. If $|v - x| = |x - v| \leq q/4 - 1$, then

$$a(v-x) = \sum_{i \in v-x} s_i \neq \sum_{j \in x-v} s_j = a(x-v)$$

due to (4.1), the definition of S and $t = q/4 - 1$. So we may assume that $|v-x| = |x-v| = q/4$ and $|w \triangle y| = |v \triangle x| = q/2$. This implies that $\mathrm{dist}_{Q_n}(w, y) = \mathrm{dist}_{Q_n}(v, x) = q/2$. Together with edges f_1, f_2, we conclude that C must have at least $q + 2$ edges, contradiction. $\square$

4.3 Proof of Theorem 12

We will first show the lower bound $f(n, 6) \geq 3n - 2$ for $n \geq 3$ by providing a set F of edges of size $3n - 2$ which must all receive distinct colors. Fix an edge f on the first level of Q_n. Our set F contains the edges of all the C_4 which contain f. There are $n - 1$ such C_4 and any two distinct C_4 may not share more than one edge, so $|F| = 4(n-1) - (n-2) = 3n - 2$.

Now, suppose $f_1, f_2 \in F$ receive the same color. If f_1 and f_2 lie on a single C_4, then let h be one of the two remaining edges and take another C_4 which contains h. Delete h and the remaining edges of the two C_4 form a C_6. If f_1, f_2 lie on different C_4, then they lie on C_4 which share the edge f. As before, delete f and the remaining six edges form a C_6. So all the edges in F must receive distinct colors.

To obtain the upper bound, we will give an explicit coloring that makes use of a classical construction of Behrend on sets of integers with no arithmetic progression of size three. Let $r_3(N)$ denote the maximum size of a subset of $\{1,\ldots,N\}$ that contains no 3-term arithmetic progression.

Theorem 35 (**Behrend** [**8**])**.** *There is a $c>0$ such that if N is sufficiently large, then*

$$r_3(N) > N^{1-\frac{c}{\sqrt{\log N}}}.$$

Behrend's result clearly implies that for $\epsilon > 0$ and sufficiently large N we have $r_3(N) > N^{1-\epsilon}$. The error term ϵ was improved recently by Elkin [21] (see [34] for a simpler proof) and using Elkin's result would give corresponding improvements in our result.

Construction 2. Let $\epsilon > 0$ and n be sufficiently large. Put $N = \lceil n^{1+\epsilon} \rceil$ and let $S = \{s_1,\ldots,s_n\} \subset \{1,\ldots,N\}$ contain no 3-term arithmetic progression. Such a set exists by Behrend's Theorem since

$$n > n^{1-\epsilon^2} = n^{(1-\epsilon)(1+\epsilon)} > N^{1-2\epsilon}.$$

Let

$$a(v) = \sum_{i=1}^{n} \vec{v}_i\, s_i.$$

Consider the edge vw, where $\vec{w} = \vec{v} + e_k$. Let

$$d(vw) = a(v) + 2s_k \in Z_{2N}.$$

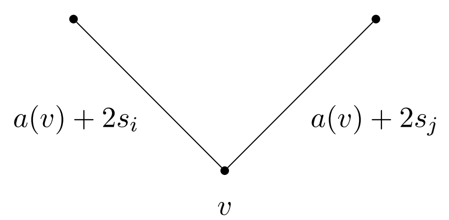
We emphasize here that we are computing $d(vw)$ modulo $2N$. Suppose further that vw is at level p and p' is the congruence class of p modulo 3. Then the color of the edge vw is

$$\chi(vw) = (d(vw), p').$$

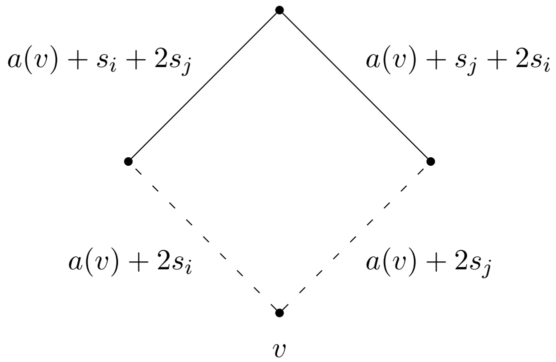
The number of colors used is at most $6N < n^{1+2\epsilon}$ as required.

Proof of the upperbound in Theorem 12

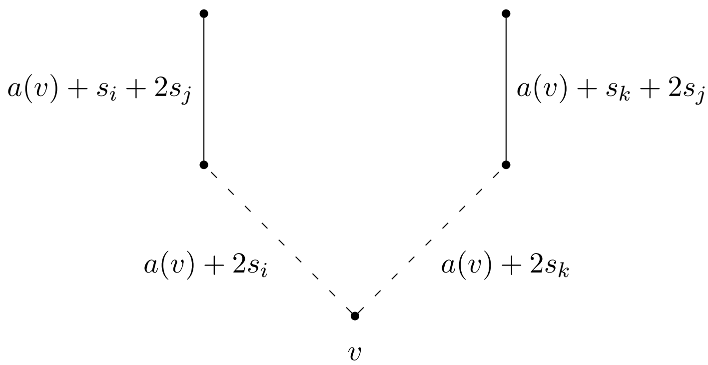
Proof. We will now show that this is a C_6-rainbow coloring. Due to the second coordinate, it suffices to show that any two edges f_1, f_2 of a C_6 which are on the same of level of Q_n receive different colors. If f_1 and f_2 are incident, then they meet either at their top vertices or bottom vertices. If incident at their bottom vertices, the edges are colored as follows and thus are distinctly colored:

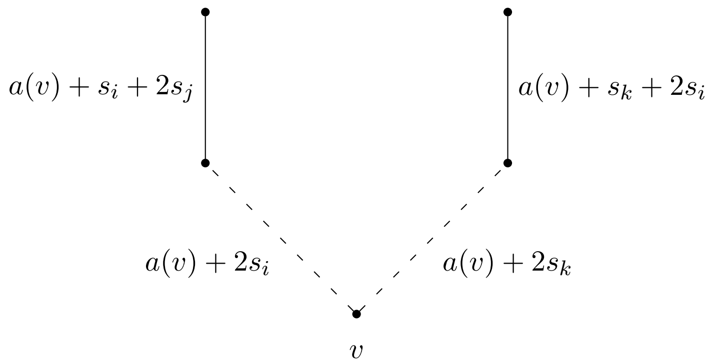


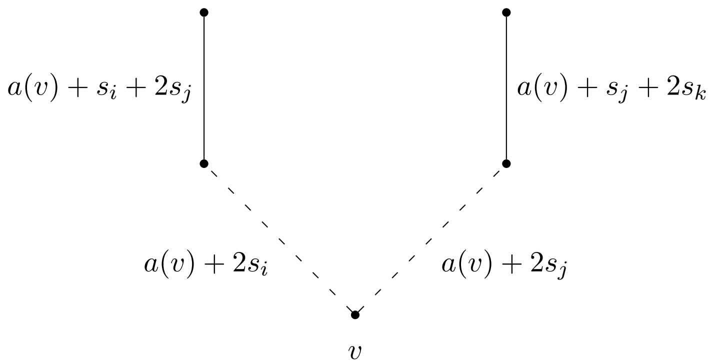
If incident at their top vertices, the edges lie on a C_4 and are therefore distinctly colored



If f_1 and f_2 are not incident, then there must be a path of length two between their bottom vertices. For if not, then they could not lie on a C_6 as the shortest path between their top vertices has length at least two. Moreover, the top vertices of f_1 and f_2 have symmetric difference precisely two since there is a path of length two between them. With these conditions, there are three ways the edges may be colored.







In the first coloring, $s_i + 2s_j \neq s_k + 2s_j$ holds due to i and k being distinct. In the second and third colorings, $s_i + 2s_j \neq s_k + 2s_i$ and $s_i + 2s_j \neq s_j + 2s_k$ hold due to our set S being free of three term arithmetic progressions. $\square$

Our results imply a tight connection between C_q-rainbow colorings in the cube and constructions of large generalized Sidon sets. When $q \equiv 0 \pmod 4$ Construction 1 gives the correct order of magnitude, however for $q \equiv 2 \pmod 4$ the same method does not work. In this case an approach similar to Construction 2 would work provided we can construct large sets that do not contains solutions to certain equations.

Conjecture. *Fix* $4 \leq q \equiv 2\,(mod\,4)$*. Then* $f(n,q) = n^{\lfloor q/4 \rfloor + o(1)}$.

For the first open case $q = 10$, we can show that $f(n, 10) = n^{2+o(1)}$ provided one can construct a set $S \subset [N]$ with $|S| > N^{1/2-o(1)}$ that contains no nontrivial solution to any of the following equations:

$$x_1 + x_2 = x_3 + x_4$$

$$x_1 + x_2 + x_3 = x_4 + 2x_5$$

$$x_1 + 2x_2 = x_3 + 2x_4.$$

Ruzsa [48, 49] defined the genus $g(E)$ of an equation

$$E : \quad a_1x_1 + \cdots + a_kx_k = 0$$

as the largest m such that there is a partition $S_1 \cup \ldots \cup S_m$ of $[k]$ where the S_i are disjoint, non-empty and for all j,

$$\sum_{i \in S_j} a_i = 0. \tag{4.2}$$

A solution $(x_1, \ldots, x_k)$ of E is trivial if there are l distinct numbers among $\{x_1, \ldots, x_k\}$ and (Equation 4.2) holds for a partition $S_1 \cup \ldots \cup S_l$ of $[k]$ into disjoint, non-empty parts such that $x_i = x_j$ if and only if $i, j \in S_v$ for some v. Ruzsa showed that if $S \subset [n]$ has no nontrivial solutions to E then $|S| \le O(n^{1/g(E)})$. The question of whether there exists S with $|S| = n^{1/g(E)-o(1)}$ remains open for most equations E. The set of equations above has genus two so it is plausible that one can prove the conjecture for $k = 10$ using this approach. For the general case, we can provide a rainbow coloring if our set S contains no nontrivial solutions to any of the three equations below with $m = \lfloor k/4 \rfloor$.

$$x_1 + \cdots + x_m = x_{m+1} + \cdots + x_{2m}$$

$$x_1 + \cdots + x_m + x_{m+1} = x_{m+2} + \cdots + x_{2m} + 2x_{2m+1}$$

$$x_1 + \cdots + x_{m-1} + 2x_m = x_{m+1} + \cdots + x_{2m-1} + 2x_{2m}.$$

The set of equations above has genus $m = \lfloor k/4 \rfloor$, so if Ruzsa's question has a positive answer, then we would be able to construct a set of the desired size.

4.3.1 The necessary equations

In the previous section, we gave a set of equations such that if we could find a set of integers which did not contain any solutions to any of the equations then we could use it to obtain a rainbow coloring. This section will show that such a set is indeed sufficient.

Let $K = \{q \in \mathbb{N} : q \geq 6 \text{ and } q \equiv 2 \, (mod\ 4)\}$. For $q \in K$, let E_q be the set of the following three equations where m $= \lfloor q/4 \rfloor$ and $x_i \neq y_j$:

$$x_1 + \cdots + x_m = y_1 + \cdots + y_m$$

$$x_1 + \cdots + x_m + x_{m+1} = y_1 + \cdots + y_{m-1} + 2y_m$$

$$x_1 + \cdots + x_{m-1} + 2x_m = y_1 + \cdots + y_{m-1} + 2y_m$$

Suppose we have colored the edges of Q_n using the set of colors $C \subset \mathbb{N}$ so that the edge vw, where w is the top vertex (i.e. $\vec{w} = \vec{v} + e_j$), receives the color $\chi(vw) = (d(vw), p)$ where $d(vw) = a(v) + 2j$ and p is the level of Q_n that vw is on *modulo* $q/2$. We will show that if

C does not satisfy any of the equations in $E_q \cup E_{q-4} \cup \cdots \cup E_6$ then all cycles of length q are rainbow.

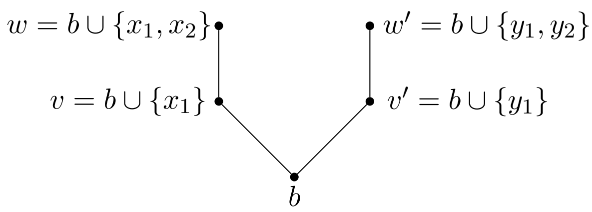
We will proceed by induction. For our base case $q = 6$, we want to show that if our set of colors C contains no solutions to $x_1 = y_1$ or $x_1 + x_2 = 2y_1$ then all cycles of length 6 are rainbow.

Sine a C_q spans at most $q/2$ levels of Q_n, two edges in a C_6 which are on different levels of Q_n cannot receive the same color. So, let vw and $v'w'$ (where v and v' are the bottom vertices) be two edges of a C_6 which are on the same level of Q_n.

First, we see that $|v \triangle v'| = 0$ or 2. For if $|v \triangle v'| \geq 4$, then the shortest possible path connecting v and v' has length 4. Since $|v \triangle v'| \geq 4$, then $|w \triangle w'| \geq 2$ so the shortest path connecting v and v' has length 2 which means that the shortest cycle that vw and $v'w'$ could lie on has length 8 which is a contradiction.

If $|v \triangle v'| = 0$, then $v = v'$. Then we must have $w \neq w'$ since vw and $v'w'$ are distinct edges, so $\chi(vw) = a(v) + 2e_i$ and $\chi(v'w') = a(v) + 2e_j$ where $i \neq j$. Since $x_1 \neq y_1$, we have $\chi(vw) \neq \chi(v'w')$

If $|v \triangle v'| = 2$, then $|w \triangle w'| = 0$ or 2. If $|w \triangle w'| = 0$, then vw and $u'w'$ lie on a C_4 and clearly receive different colors. If $|w \triangle w'| = 2$, then let the path of v,b,v' be a path of length 2 connecting v and v' where $v = b \cup \{x_1\}$, $w = b \cup \{x_1, x_2\}$, $v' = b \cup \{y_1\}$ and $w' = b \cup \{y_1, y_2\}$ as shown below:

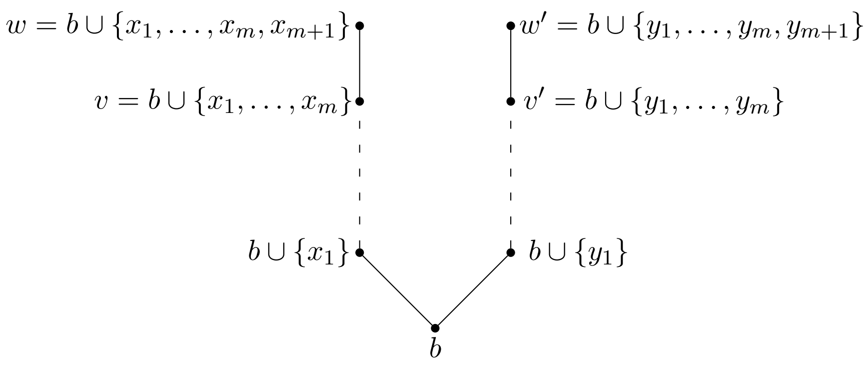


Since $|w \triangle w'| = 2$, we know that $x_i = y_j$ for some pair (i, j). If $x_1 \neq y_1$, but if $x_2 = y_2$ then $\chi(vw) \neq \chi(v'w')$ since C does not satisfy $x_1 = y_1$. If $x_1 = y_2$ or $x_2 = y_1$, then $\chi(vw) \neq \chi(v'w')$ since c does not satisfy $x_1 + x_2 = 2y_1$. Therefore, if our set of colors C does not satisfy any of the equations in E_6 then all C_6 are rainbow.

Now, we will show by induction that if Q_n is colored with a set C which does not satisfy any equations in $E_q \cup E_{q-4} \cup \ldots \cup E_6$ then then all C_q are rainbow. As in our base case, two edges of C_q which are on different levels of Q_n will receive different colors so we will look at edges which are on the same level.

Again, let vw and $v'w'$ be two such edges of a C_q. Then $|v \triangle v'| \leq 2\lfloor q/4 \rfloor$. Suppose not and that $|v \triangle v'| \geq 2\lfloor q/4 \rfloor + 2$. Then $|w \triangle w'| \geq 2\lfloor q/4 \rfloor$. Then the smallest possible path P_1 from v to v' contains at least $2\lfloor q/4 \rfloor + 2$ edges and the smallest possible path P_2 from w to w' contains at least $2\lfloor q/4 \rfloor$ edges. So the smallest possible cycle containing vw and $v'w'$ has at least $|P_1| + |P_2| + 2 \geq (|2\lfloor q/4 \rfloor + 2) + (2\lfloor q/4 \rfloor) + 2 = 2(q-2)/4 + 2 + 2(q-2)/4 + 2 = q + 2$ edges which is a contradiction.

Now, suppose that $|v \triangle v'| = 2m$ where $m = \lfloor q/4 \rfloor$. Then there is a path of length $2m$ from v to v' where $x_i \neq y_j$ for $i, j \leq m$ as shown below:

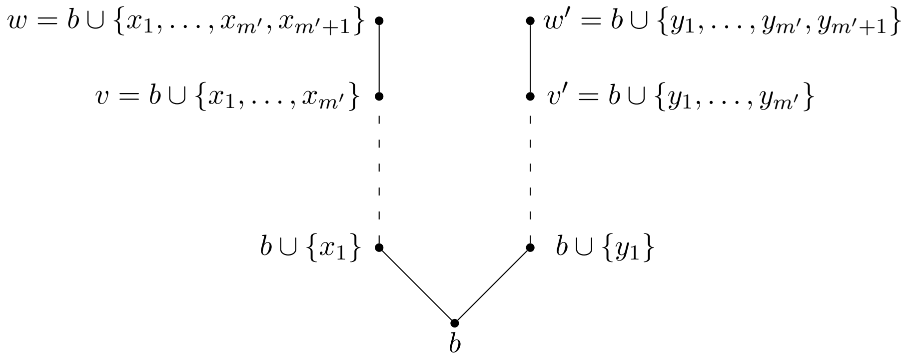


Since $|v \triangle v'| = 2m$, either $|w \triangle w'| = 2m$ or $|w \triangle w'| = 2m - 2$. If $|w \triangle w'| = 2m + 2$, then uv and $u'v'$ do not lie on a cycle of length q. Therefore $|w \cap w'| = 1$ or 2 and this gives us three cases to check:

- $x_{m+1} = y_i$ for some $i \in \{1, \ldots, m\}$
- $x_{m+1} = y_i$ for some $i \in \{1, \ldots, m\}$ and $y_{m+1} = x_i$ for some $i \in \{1, ..., m\}$
- $x_{m+1} = y_{m+1}$

In the first case, $\chi(vw) \neq \chi(v'w')$ provided that $x_1 + \cdots + x_m + x_{m+1} \neq y_1 + \cdots + y_{i-1} + y_{i+1} + cdots + y_m + 2y_{m+1}$. In the second case, $\chi(vw) \neq \chi(v'w')$ provided that $x_1 + cdots + x_{i-1} + x_{i+1} + ldots + x_m + 2x_{m+1} \neq y_1 + \cdots + y_{i-1} + y_{i+1} + + y_m + 2y_{m+1}$. Finally, in the third case, $\chi(vw) \neq \chi(v'w')$ provided that $x_1 + \cdots + x_m + 2x_{m+1} \neq y_1 + \cdots + y_m + 2y_{m+1}$. The equation for each case is an element in our set E_q of forbidden equations.

Now, suppose that $|v \triangle v'| \leq 2m$. Then there is path from v to v' of length at most $2m - 2$ where $m' < m$ and $x_i \neq y_j$ for $i, j \leq m'$ as follows:



The three cases to check are identical to the ones for $|v \triangle v'| = 2m$ but with m replaced with m'. So now the three forbidden equations lie in $E_{q-4} \cup \ldots \cup E_6$. And so by our inductive hypothesis $\chi(vw) \neq \chi(v'w')$ since vw and $v'w'$ lie on a smaller cycle and our C does not contain any solutions to $E_{q-4} \cup \ldots \cup E_6$.

CHAPTER 5

HYPERGRAPH PROBLEMS

We will now turn our attention to hypergraph problems. As previously stated, the classic Ramsey problems on hypergraphs have received considerable attention, particularly the case of 3-uniform hypergraphs [18, 24, 31, 39]. Just as in the case of 2-graphs, Ramsey type problems on hypergraphs have been considered for hypergraphs other than cliques. Various small 3-graphs including $K_4^3 - e$ were studied in [6]. Extensions of paths and cycles were studied in [36, 44]. A typical technique is to color a l-graph G which satisfies certain conditions and use it to color an $(l+1)$-graph H which also satisfies certain conditions. We will use this technique in several of the following sections.

5.1 Rainbow coloring paths of length three

A loose path of length three, denoted P_3^3, is the 3-graph with edges e_1, e_2, e_3 such that $|e_1 \cap e_2| = |e_2 \cap e_3| = 1$ and $e_1 \cap e_3 = \emptyset$. In this section, we will determine the number of colors required to color the edges of K_n^3 so that all copies of P_3^3 have three colors on their edges.

Definition. *Let $k_3(G)$ denote the number of triangles in the graph G.*

We will use the following theorem and corollary to prove Theorem 24.

Theorem 36 (McKay [28]). *Let G be a graph with n vertices. If $|E(G)| = m \geq n^2/4$, then*

$$k_3(G) \geq \tfrac{(4m-n^2)(m)}{3n}.$$

Consequently, if $|E(G)| \geq n^2/4 + c$ *for some positive integer* c *then* $k_3(G) \geq cn/3$.

Our main result in this section is stated below.

Theorem 24. *Let* $n \geq 4$. *Then*

$$f_3(K_n^3, P_3^3, 3) \leq \binom{n}{2} - \lfloor n^2/4 \rfloor = \begin{cases} n^2/4 - n/2, & \text{if } n \text{ is even} \\ n^2/4 - n/2 + 1, & \text{if } n \text{ is odd.} \end{cases}$$

If $n \geq 12$, *then* $f_3(K_n^3, P_3^3, 3) \geq n^2/4 - n/2$.

Proof. We will first prove the upper bound. Partition $V(K_n^3)$ into $X \cup Y$ where $|X| = \lfloor n/2 \rfloor$ and $|Y| = \lceil n/2 \rceil$. Color all the pairs $\{u, v\}$ where $\{u, v\} \subset X$ or $\{u, v\} \subset Y$ with distinct colors. This requires $\binom{n}{2} - \lfloor n^2/4 \rfloor$ colors. Now, color the triple $e = uvw$ with the color that one of the pairs in e received. Since $|e \cap X| \geq 2$ or $|e \cap Y| \geq 2$, at least one of the pairs in e was colored and therefore all triples are colored. We see that a P_3^3 is rainbow under this coloring; if two distinct triples e, e' receive the same color then $|e \cap e'| = 2$, but if e and e' lie in a copy of P_3^3 then $|e \cap e'| = 0$ or $|e \cap e'| = 1$ and so they received their colors from different pairs. This establishes the upper bound.

We will now prove the lower bound. Suppose that K_n^3 has been edge-colored so that all copies of $P_3^3 \subset K_n^3$ are rainbow. There are two configurations within a color class which are forbidden. The first is two edges having no vertices in common and the second is two edges intersecting in a single vertex. If one of these configurations appear within a color class, it may be extended to a P_3^3 with at most two colors on its edges. We will separate the color classes into two types:

small color classes which contain at most four edges and *large* color classes which contain five or more edges. If a color class is large, then all of its edges must intersect in a common pair of vertices. If a color class is small, then it may have the same structure as a large color class or it may be a subgraph of K_4^3. Let $C_1, \dots, C_t, C_{t+1}, \dots, C_k$ be the color classes where $C_1, \dots, C_t$ are the small color classes and $C_{t+1}, \dots, C_k$ are the large color classes. Let $H \subset K_n^3$ be the set of edges which were colored by one of the large color classes. Let us also define a graph $G \subset K_n$. As we previously stated, in a large color class all of the edges contain a fixed pair of vertices. For a large color class C_j, let e_j denote this pair of vertices. Let $E(G) = \{e_{t+1}, \dots, e_k\}$. Furthermore, since k is the total number of colors used we may assume that $k < n^2/4 - n/2$, otherwise we are done. We see that $|E(\bar{G})| = \binom{n}{2} - k + t = n^2/4 + (n^2/4 - n/2 - k + t)$ and that $|H| \geq \binom{n}{3} - 4t$.

Now, we will show that $k_3(\bar{G}) + |H| \leq \binom{n}{3}$. Suppose this were not true and that $k_3(\bar{G}) + |H| > \binom{n}{3}$. Then some triangle T in $\bar{G}$ is a triple of H. Because $T \in H$, it was not colored by one of the small color classes. However, since T is a triangle in $\bar{G}$, it was not colored by one of the large color classes either. This gives the contradiction that T is uncolored. By Theorem 36, since $|E(\bar{G})| = n^2/4 + (n^2/4 - n/2 - k + t)$ we have that $k_3(\bar{G}) \geq (n/3)(n^2/4 - n/2 - k + t)$ and so

$$(n/3)(n^2/4 - n/2 - k + t) + \binom{n}{3} - 4t \leq k_3(\bar{G}) + |H| \leq \binom{n}{3}$$

This simplifies to

$$n^3/12 - n^2/6 - 4t + tn/3 \leq kn/3$$

which yields

$$k \geq n^2/4 - n/2 - 12t/n + t.$$

Therefore $f(K_n^3, P_3^3, 3) \geq n^2/4 - n/2$ for $n \geq 12$. $\square$

5.2 Rainbow coloring $P_2^l(t)$ in K_n^l

In this section we attempt to the generalize the results on $f(K_3^3, P_3^3, m)$ to higher uniformity. We will determine upper and lower bounds for $f(K_n^l, P_2^l(t), 2)$ where $P_2^l(t)$ is the l-graph consisting of two edges e_1, e_2 with $|e_1 \cap e_2| = t$. Also, recall that given an l-graph H, $\mathrm{ex}_l(n, H)$ is the maximum number of edges that an l-graph on n vertices can have without containing a copy of H. We will require the following definitions:

Definition. *Let $S(n, l, t)$ denote a collection of l-element subsets of $[n]$ such that every t-element subset is contained in exactly one l-element subset. An element $S(n, l, t)$ is referred to as a block.*

Definition. *Let $m(n, l, \bar{t})$ denote the maximum size of a collection of l-element subsets of $[n]$ such that no two intersect in exactly t elements.*

Our result on $f_l(K_n^l, P_2^l(t), 2)$ is the following.

Theorem 26. *Let l, t be fixed positive integers with $t < l \leq 2t+1$. If there is an $S(n, 2l-t-1, l)$ on $[n]$, then*

$$f_l(K_n^l, P_2^l(t), 2) \leq (1 + o(1)) \tfrac{(n-l)!\,(l-t-1)!}{(n-t)!\,(2l-2t-1)!}.$$

If it is possible to partition $S(n, 2l-t-1, l)$ into copies of $S(n, 2l-t-1, t)$ and $l-t$ is a prime, then

$$f_l(K_n^l, P_2^l(t), 2) = \tfrac{(n-l)!\,(l-t-1)!}{(n-t)!\,(2l-2t-1)!}.$$

The proof of Theorem 26 will require the following results.

Theorem 37 (**Frankl [32]**)**.** *If $l \le 2t+1$ and $l-t$ is a prime power then*

$$m(n, l, \bar{t}) \le \tbinom{2l-t-1}{l}\tbinom{n}{t}/\tbinom{2l-t-1}{t}.$$

If $l-t$ is a prime, then equality is achieved only for an $S(n, 2l-t-1, t)$.

It is conjectured that Theorem 37 holds even when $l-t$ is not a prime power.

Theorem 38 (**Pippenger-Spencer [45]**)**.** *Let $r \ge 2$ be fixed and $D \to \infty$. Let H be an r-graph with $d(v) = (1+o(1))D$ for every $v \in V(H)$ and codeg$(u, v) = o(D)$ for any two $u, v \in (V(H)$. Then $E(H)$ can be partitioned into $(1+o(1))D$ matchings.*

Proof of Theorem 26

We start by proving the first part of the theorem. Let $G = K_n^l$ and S be an $S(n, 2l-t-1, l)$ on $V(G)$. Every t-element subset is contained in at most $D = \binom{n-t}{l-t}/\binom{2l-2t-1}{l-t}$ blocks of S. Let H be the $\binom{2l-t-1}{t}$-graph with $|V(H)| = \binom{n}{t}$ where each $v \in V(H)$ corresponds to some $A \subset V(G)$ with $|A| = t$. In H, $\binom{2l-t-1}{t}$ vertices form an edge of H if and only if their corresponding t-element subsets make up a block of S. The degree of a vertex in H is at most D. Given $u, v \in V(H)$, their codegree is maximal when then their corresponding t-element subsets in $V(G)$ have intersection size $t-1$ and thus codeg$(u, v) \le \binom{n}{l-t-1}/\binom{2l-t-1}{l-t-1} = O(n^{l-t-1}) = o(D)$. We apply Theorem

38 and decompose the edges of H into $m = (1+o(1))D$ matchings $M_1, \ldots, M_m$. Every M_i corresponds to a collection of blocks $M_i' = \{A_1, \ldots, A_w\} \subset S$ in G with $A_i \cap A_j \leq |t-1|$ for $1 \leq i, j \leq m$. An l-element subset X is colored with $\chi(X) = i$ where $X \subset s \in M_i$. Therefore,

$$f(K_n^l, P_2^l(t), 2) \leq (1+o(1))D = (1+o(1))\frac{(n-t)!/((n-l)!(l-t)!)}{(2l-2t-1)!/((l-t-1!)(l-t)!)} = (1+o(1))\frac{(n-l)!\,(l-t-1)!}{(n-t)!\,(2l-2t-1)!}.$$

We now prove the second part of Theorem 26. The lower bound is clearly $\binom{n}{l}/\mathrm{ex}_l(n, P_2^l(t)) = \binom{n}{l}/m(n,l,\bar{t})$. If the conditions of Theorem 37 are satisfied and equality holds, we obtain:

$$f(K_n^l, P_2^l(t), 2) \geq \frac{\binom{n}{l}\binom{2l-t-1}{t}}{\binom{n}{t}\binom{2l-t-1}{l}} = \frac{(n!)/((n-l)!\,l!)\,(2l-2t-1)!/((l-t-1)!\,t!)}{(n!)/((n-t)!\,t!)\,(l-t-1)!/((2l-2t-1)!\,l!)} = \frac{(n-l)!\,(l-t-1)!}{(n-t)!\,(2l-2t-1)!}.$$

We now show the upper bound. If it is possible to partition $S(n, 2l-t-1, l)$ into $A_1, \ldots, A_m$ where each A_i is a copy of $S(n, 2l-t-1, t)$, then any two l-element subsets which are contained in the blocks of some A_j may receive the same color. This gives us an upper bound of $|S(n, 2l-t-1, l)|/|S(n, 2l-t-1, t)|$. It is easy to see that $|S(n,m,t)| = \binom{n}{t}/\binom{m}{t}$. Indeed, an $S(n,m,t)$ requires one m-element subset for every t-element subset, but every m-element subset contains $\binom{m}{t}$ t-element subsets and so we have overcounted each m-element subset that many times. We obtain the upper bound

$$f(K_n^l, B, 2) \leq \frac{\binom{n}{l}\binom{2l-t-1}{t}}{\binom{n}{t}\binom{2l-t-1}{l}} = \frac{(n-l)!\,(l-t-1)!}{(n-t)!\,(2l-2t-1)!}.$$

□

5.3 Turán number for r-uniform t-linear paths.

In the previous section, we saw that $f(n, P_2^r(t), 2) \geq \frac{\binom{n}{r}}{\mathrm{ex}_r(n, P_2^r(t))}$. It is often the case that results on Turán numbers are useful in graph coloring problems.

Definition. *Let $P_k^r(t)$ be the r-uniform path $e_1, e_2, \ldots, e_k$ such that:*

i) $|e_i \cap e_{i+1}| = t$ *for* $i = 1, \ldots, k-1$

ii) $|e_i \cap e_j| = \emptyset$ *if* $|i-j| \neq 1$

In this section, we will consider the problem of determining $\mathrm{ex}_r(n, P_k^r(t))$. We will require the following definitions. When $t = 1$, this problem has been solved [38] so we will consider $t > 1$.

Definition. *The complete r-partite r-graph with parts of size t is the r-graph with vertex set $X_1 \cup \cdots \cup X_r$, $|X_i| = t$ and edge set $X_1 \times \cdots \times X_r$. We denote this graph by $K^r_{t,\cdots,t}$.*

Definition. *Given an r-graph H on n vertices, let*

$$\partial H = \{f : |f| = r-1 \text{ and } \exists\, e \in E(H) \text{ with } f \subset e\}.$$

The following theorem is the main result of this section.

Theorem 29. *Fix $k, r \geq 2$, $t \leq \lfloor \frac{r}{2} \rfloor$ and $l = \lfloor \frac{k-1}{2} \rfloor$. Then $\mathrm{ex}_r(n, P_k^r(t)) \sim l \binom{n}{r-1}$.*

We will first prove the lower the bound.

Proof of lower bound of Theorem 29. Fix $S \subset V(K_n^r)$ with $|S| = l$. Let $H \subset K_n^r$ where $e \in E(H)$ iff $e \cap S \neq \emptyset$. Suppose H were to contain a copy P of $P_k^r(t)$. Since every edge of P must contain at least one vertex in S, then there must be at least one vertex of S which is contained in $\frac{k}{l} > 3$ edges of P which is a contradiction. Therefore, $\mathrm{ex}_r(n, P_k^r(t)) \geq |H| > l\binom{n-l}{r-1}$.

Before proceeding with the proof of the upper bound and its prerequisites, we will observe that we can easily find a $P_k^r(t)$ if the $(r-1)$-element subsets of vertices of an r-graph H have sufficiently large codegree. When such is the case, we are able to start with an edge e and, given any subset T of its vertices, find a second edge e' which intersects e in precisely the set T. We can use the following simple lemma to chain edges together to build the desired path.

Lemma 39. *Let $r \geq 2, a \geq 1$ and H be an r-graph such that every set of $r-1$ vertices has codegree at least $r+a$. Given an edge e along with sets $T \subset e$ and $A \cap e = \emptyset$ with $|A| = a$, there is an edge e' such that the $e' \cap e = T$ and $e' \cap A = \emptyset$.*

Proof. Let $B = A \cup e$ and $e' \in E(H)$ such that $T \subset e'$. Suppose that $|e \cap e'| > |T|$ and that there does not exist an f such that $|e \cap f| < |e \cap e'|$. Since $|e \cap e'| > T$, there exists $x \in B$ such that $x \notin T$ and $x \in e'$. Let $f' = e' - x$. Since every set of $r-1$ vertices has codegree at least $r+a$, there exists a $y \notin B$ such that $f' \cup y = e''$ is an edge with $|e \cap e''| < |e \cap e'|$ which is a contradiction. $\square$

Definition. *Let H be a k-graph on $[n]$ and $S \subset [k]$. An edge-coloring $\chi : E(H) \to \mathbb{N}$ is S-canonical if for any two edges $a = a_1 a_2 \cdots a_k$, $b = b_1 b_2 \cdots b_k$ with $a_1 \leq a_2 \leq \cdots \leq a_k$ and $b_1 \leq b_2 \leq \cdots \leq b_k$, we have $\chi(a) = \chi(b)$ if and only if for all i, $a_i = b_i$ when $i \in S$ and $a_i \neq b_i$ when $i \notin S$.*

Theorem 40 (**Erdős-Rado [24]**)**.** *For every $p \geq k \geq 2$, there exists an n such that if $\psi : \binom{[n]}{k} \to \mathbb{N}$, then there are subsets $S \subset [k]$ and $Y \subset [n]$ with $|Y| = p$ such that ψ restricted to $\binom{Y}{k}$ is an S-canonical coloring.*

We also define an S-canonical coloring for $K^r_{t,\ldots,t}$ and an l-multicoloring.

Definition. *Let χ be an edge-coloring of $K^r_{t,\cdots,t}$ with parts $X_1, \ldots, X_r$ and $S \subset [r]$. The edge-coloring χ is S-canonical if for any two edges $a = a_1 a_2 \cdots a_r$ and $b = b_1 b_2 \cdots b_r$ where $a_i, b_i \in X_i$, we have $\chi(a) = \chi(b)$ if and only if for all i, $a_i = b_i$ when $i \in S$ and $a_i \neq b_i$ when $i \notin S$.*

Definition. *Fix $r \geq 2$, $l > 1$ and an r-graph H. An l-multicoloring is a function χ which assigns to each $e \in E(H)$ a set of colors of size l.*

Finally, we will require the following lemmas.

Lemma 41. *For all p, there is an n such that if $X_i \cap X_j = \emptyset$, $|X_i| = n$ for $1 \leq i \leq k$ and $\chi : X_1 \times \cdots \times X_k \to \mathbb{N}$, then there exist $X'_1, \ldots, X'_k$ with $|X'_i| = p$ for $1 \leq i \leq k$ and $S \subset [k]$ such that χ restricted to $X'_1, \ldots, X'_k$ is S-canonical.*

Proof. We apply Theorem 40 with inputs k and kp and obtain the necessary value n. Let $\chi : X_1 \times \cdots \times X_k \to \mathbb{N}$ with $|X_i| = n$ and $X_i = \{x_{ia_1}, \ldots, x_{ia_n}\}$ where $x_{ia_\alpha} < x_{ia_\beta}$ when $\alpha < \beta$. Let $Y = \{y_1, \cdots, y_n\}$ with $y_\alpha < y_\beta$ when $\alpha < \beta$. We define $\psi : \binom{Y}{k} \to \mathbb{N}$ below:

$$\psi(y_{j_1}, \ldots, y_{j_k}) = \chi(x_{1a_{j_1}}, \ldots, x_{ka_{j_k}}).$$

By Theorem 40, there is a $Y' \subset Y$ with $|Y'| = kp$ such that ψ restricted to Y' is S-canonical. We relabel the elements as $Y' = \{b_1, \cdots, b_{kp}\}$. Now, let $X'_i = \{x_{ib_1}, \ldots, x_{ib_{kp}}\}$ and $X''_i = \{x_{ib_{i*}}, \cdots, x_{ib_{ip}}\}$ where $i* = (i-1)p+1$. We will show that χ restricted to $X''_1 \times \cdots \times X''_k$

is S-canonical. Let $e = \{e_1, \ldots, e_k\}$ and $f = \{f_1, \ldots, f_k\}$ be edges of Y such that $e_i, f_i \in \{y_{i^*}, \ldots, y_{ip}\}$. If $\psi(e) = \psi(f)$, then $e_i = f_i$ for all $i \in S$ and we have

$$\psi(e_1 \ldots e_k) = \chi(x_{1e_1} \ldots x_{ke_k}),\ \psi(f_1 \ldots f_k) = \chi(x_{1f_1} \ldots x_{kf_k}).$$

The edges $e' = \{x_{1e_1}, \ldots, x_{ke_k}\}$ and $f' = \{x_{1f_1}, \ldots, x_{kf_k}\}$ also have $x_{ie_i} = x_{if_i}$ for $i \in S$. Similarly, if $\psi(e) \neq \psi(f)$ then $e_i \neq f_i$ for some $i \in S$ and $\chi(e') = \psi(e) \neq \psi(f) = \chi(f')$. Therefore, χ restricted to $X_1'' \times \cdots \times X_k''$ is S-canonical. □

Lemma 42. *For every $p > 1$, there exists a t such that if $X_1, \cdots X_r$ are pairwise disjoint of size t and the edges of $K^r_{t,\ldots,t}$ are l-multicolored, then there are $X_i' \subset X_i$ with $|X_i'| = p$ for all i and there are colorings $\chi_1, \cdots, \chi_l$ such that some χ_j restricted to $H' = H(X_1', \cdots, X_r')$ is either rainbow, or χ_i restricted to H' is S-canonical where $S \subsetneq [r]$ for $1 \leq i \leq r$. Furthermore, if all χ_i are S-canonical with $S \subsetneq [r]$ then a color which appears in a monochromatic coloring χ_α will not be present in χ_β for $\alpha \neq \beta$.*

Proof. Pick a color from each edge and call this coloring χ_1. Applying Lemma 41 allows us to restrict to subsets of each X_i so that χ_1 is canonical. Now, pick among the remaining colors for each edge to obtain χ_2 and repeat so that we obtain subsets $X_i'' \subset X_i$ and $H'' = H(X_1'', \cdots, X_r'')$. If any coloring χ_i is rainbow, then we are done so assume they are all S-canonical. We see that the condition of the lemma is satisfied; we removed a color from the list on each edge to obtain a coloring, so if χ_i is monochromatic for some i, then its color cannot used by χ_j for $j \neq i$. □

We will require one more result of Erdős on the Turán number for $K^r_{t,\ldots,t}$.

Theorem 43 (Erdős [22]). *Fix $r \geq 2, t \geq 1$. Then, as $n \to \infty$, $ex_r(n, K^r_{t,\ldots,t}) \leq O(n^{r-\frac{1}{t^{r-1}}})$.*

Proof of upper bound of Theorem 29. For $\epsilon > 0$, let H be an r-graph with $|E(H)| = l\binom{n}{r-1} + \epsilon n^{r-1}$. Let H' be the graph obtained by deleting every edge which contains an $(r-1)$-element subset with codegree less than $l+1$. We deleted at most $l\binom{n}{r-1}$ edges and so $|E(H')| > \epsilon n^{r-1}$.

If $|\partial H'| < \epsilon' n^{r-1}$ where $\epsilon' = \frac{\epsilon}{rk}$, then successively delete edges that contain an $(r-1)$-element subset with codegree at most rk. The number of edges deleted is less than $|\partial H'|rk < \epsilon n^{r-1}$. We denote the r-graph that remains as H''. We may build a $P^r_k(t)$ by using Lemma 39 a total of $k-1$ times with the parameter $a = kr$ to increase the path by one edge with each application of the lemma until we obtain a $P^r_k(t)$.

If $|\partial H'| > \epsilon' n^{r-1}$, partition $V(H') = X \cup Y$ where $v \in V(H')$ is in X with probability 1/2. We say that $e \in \partial H'$ is good if $e \subset X$ and if for $e_1, \ldots e_{l+1} \in E(H')$ where $e \subset e_i$, we have $e_i - e \in Y$. The probability that an $e \in \partial H'$ is good is $(1/2)^{r+l}$. Let B be the number of good $(r-1)$-element subsets in $\partial H'$. Then $E(B) = (1/2)^{r+l}\epsilon' n^{r-1}$ and by Theorem 43, when n is sufficiently large we can find an $H'' \cong K^{r-1}_{t,\ldots,t}$. Any $e \in E(H''')$ may be extended to $l+1$ different edges in H'. Thus, we consider H''' to be $(l+1)$-multicolored. We apply Lemma 42 to H''' to obtain an $H'''' \cong K^{r-1}_{k,\ldots,k}$ with colorings $\chi_1, \cdots, \chi_m$ with $m = l+1$ as specified in the lemma. There are several cases to consider. We will proceed by constructing a pseudo paths whose edges are sets of size $r-1$ and then use the colorings $\chi_1, \ldots \chi_m$ to add a vertex to each $(r-1)$-element subset and extend the pseudo path to a proper $P^r_k(t)$ in H. Let $X_1, \ldots X_{r-1}$ be the parts of H''''.

Case 1. If some coloring χ_i is rainbow, then take a pseudo path $P = e_1 \cdots e_k$ such that $|e_i \cap e_{i+1}| = t$ for $1 \leq i < k-1$. We extend each edge with the coloring χ_i. Since χ_i is rainbow, when $i \neq j$ e_i and e_j will be extended with different colors and we obtain a $P_k^r(t)$.

Case 2. Suppose that none of the colorings are rainbow. Let $\chi_1, \cdots \chi_p$ be monochromatic and let $\chi_{p+1}, \cdots, \chi_m$ be canonical colorings. We will extend the first $2p$ edges of the pseudo path using $\chi_1, \cdots, \chi_p$. For $e_1, \ldots, e_{2p}$, if i is odd then $|e_i \cap e_{i+1}| = t-1$ and if i is even then $|e_i \cap e_{i+1}| = t-1$. For $1 \leq j \leq p$, use χ_j to extend e_{2p-1} and e_{2p}. Coloring $e_{2p+1}, \ldots, e_k$ reduces to the final remaining case where all χ_i are canonical.

Case 3. Suppose $\chi_1, \ldots, \chi_m$ are canonical and let $P = e_1 e_2 \ldots e_k$ be our pseudo path. We will require one of the available S-canonical colorings which we will refer to as χ and we let $|S| = s$. Our pseudo path is constructed as follows: if $s > 1$, then $e_i \cap e_{i+1} = t$ for all i and if i is even then e_i and e_{i+1} intersect in parts $X_1, \cdots, X_t$ otherwise e_i and e_{i+1} intersect in parts $X_{r-1-t}, \cdots, X_{r-1}$. We may reorder the parts to ensure that there are $i, i' \in S$ such that $i \in \{1, \ldots, t\}$ and $i' \in \{r-1-t, \ldots, r-1\}$. No two edges of P intersect in precisely the parts corresponding to S, so every e_j receives its own color and we obtain $P_k^r(t)$. If $s = 1$, then, since $t > 1$, no two edges e_i and e_{i+1} will intersect precisely one part so each edge is extended with a different color.

5.4 Pasch Configurations in K_n^3

In this section, we will prove a Ramsey-type result on one of the simplest examples of a 3-graph with four edges.

Definition. *The Pasch configuration P is the 3-graph on six vertices with four edges such that any two edges of P contain exactly one vertex in their intersection.*

Our goal was to prove that $f(K_n^3, P, 3) = n^{1+o(1)}$, however we were only able to prove the following weaker result:

Theorem 25. *There is an edge-coloring of $H \subset K_n^3$ with $n^{1+o(1)}$ colors such that every copy of $P \subset H$ receives at least three distinct colors and $|E(H)| > \binom{n}{3} - o(n^3)$.*

Our proof of Theorem 25 will require the following result on decomposing a graph G into induced matchings. An induced matching is a set $E' \subset E(G)$ such that no two $e, f \in E'$ are incident or joined by a third edge $h \in E(G)$.

Theorem 44 (**Alon-Moitra-Sudakov [2]**)**.** *There is a graph G on N vertices with $|E(G)| = \binom{N}{2} - o(N^2)$ that can be decomposed into induced matchings, each of size $N^{1-o(1)}$.*

We now prove Theorem 25.

Proof of Theorem 25

Let $V(K_n) = [n]$. By Theorem 44, there is graph $G \subset K_n$ with $|E(G)| = \binom{n}{2} - o(n^2)$ such that G can be decomposed into induced matchings $M_1, \ldots, M_l$ each of size $n^{1-o(1)}$ where $l \leq n^{1+o(1)}$. Now we color the edges of K_n^3. Let $e = xyz \in E(K_n^3)$ and assume that $x < y < z$. If $xy \in M_i$ for some i, then assign color i to e, otherwise e remains uncolored. Because each edge of K_n lies in $n-2$ triples, this leaves at most $o(n^3)$ edges of K_n^3 uncolored. We will now show that every $P \subset G$ contains at least three colors on its edges. Let $V(P) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ with the ordering of the vertices to be specified in each case. We will proceed by contradiction with two cases to consider.

Case 1. Suppose $P \subset G$ only receives two colors on its edges and three of its edges e, f, h receive the same color. Assume that $e = v_1v_2v_3$, $f = v_3v_4v_5$ and $h = v_5v_6v_1$ with $v_1 < v_2 < v_3$. It must be the case the $v_5, v_6 < v_1$ otherwise e and h would receive distinct colors. Now, we must also have the $v_3, v_4 < v_5$ otherwise f and h would receive distinct colors. But this is not possible since $v_5 < v_1 < v_3$. So, it is not possible for three edges in P to receive the same color.

Case 2. Suppose $P \subset G$ only receives two colors on its edges and two edges receive one color and the other two edges receive a second color. Let $E(P) = \{e, e', f, f'\}$ and assume that e, e' receive color i and f, f' receive color j. Let $e = v_1v_2v_3$ with $v_1 < v_2 < v_3$. Let $e' = v_3v_4v_5$, $f = v_5v_6v_1$ and $f' = v_2v_4v_6$. There are two subcases to consider. First, suppose that $v_4, v_5 < v_3$. Then, because G has been partitioned into induced matchings, f' must receive its color from v_6v_2 or v_6v_4. Since f and f' receive the same color, f must receive its color from v_1v_5, but this contradicts the matchings being induced. In the second subcase we may assume, without loss of generality, that $v_3, v_4 < v_5$. If $v_6, v_4 < v_2$ and $v_5, v_1 < v_6$ then we obtain the contradiction $v_5 < v_6 < v_2 < v_3 < v_5$. If $v_6, v_2 < v_4$, then we obtain $v_6 < v_4 < v_5 < v_6$. Therefore, P must receive at lease three colors on its edges.

5.5 Stepping up type problems

At he beginning of this thesis, we introduced Ramsey Theory with the classic problem of monochromatic cliques in a larger two colored clique. We will now conclude this chapter and this thesis with a hypergraph problem on cliques. We will prove two results where we color a clique on n vertices so that certain conditions are satisfied and then use it to color a clique on 2^n vertices which also satisfies certain conditions without using too many more colors. Recall the following definition and the following theorem. The ideas of the proofs, in particular the edge-coloring functions, in this section will closely follow those given in [6].

Definition. *Fix integers $l \geq 2$ and $p, q \geq l$. An edge-coloring of K_n^l is a (p,q)-coloring if every subgraph K_p contains at least q distinct colors its edges. The minimum number of colors required for a (p,q)-coloring of K_n^l is denoted as $f_l(n,p,q)$.*

We will also use the following previously stated theorem.

Theorem 6 (Mubayi [41]).

$$f(n,4,3) \leq e^{O(\sqrt{\log n})}$$

In analogy with the of Erdős and Gyárfás for graphs, we consider the problem of determining the smallest p such that $f_3(n,p,3) = (\log n)^{o(1)}$. As mentioned in the introduction, $r^3(4,4) = 13$ and a result of Axenovich et. al. [6] implies that $f_3(n,4,2) = O(\log \log n)$. The (4,2)-coloring given by this bound is also a (13,3)-coloring, for if a K_{13}^3 receives at most two colors, then, since $r^3(4,4) = 13$, it contains a monochromatic K_4^3 contradicting the fact that we have a (4,2)-coloring. Consequently, $f_3(n,13,3) = O(\log \log n)$. Hence, the minimum p such that $f_3(n,p,3) = (\log n)^{o(1)}$ is at most 13. We obtain the following result for $p = 7$.

Theorem 20.

$$f_3(n,7,3) \leq e^{O(\sqrt{\log \log n})}$$

Proof. Let $H = K_{2^n}^3$ with $V(H) = \{0,1\}^n$ and let $G = K_n$ with $V(G) = [n]$. Given $x, y \in V(K_{2^n}^3)$, we say that $x < y$ if the integer given by the binary representation of x is less than the integer given by the binary representation of y. In other words, $x < y$ if at the first position i that x and y differ, we have $x_i = 0$ and $y_i = 1$. For $x, y \in V(H)$ with $x < y$, let $p(x, y)$ be the first position in the binary sequences where x and y differ. Given x, y, z with $x < y < z$, it is easy to see that $p(x, y) \neq p(y, z)$. Furthermore, if $p(x, y) < p(y, z)$ then $p(x, z) = p(x, y)$ and if $p(x, y) > p(y, z)$, then $p(x, z) = p(y, z)$. For integers m, n let

$$s(m,n) = \begin{cases} 1, & \text{if } n \geq m \\ 2, & \text{if } n < m \end{cases}$$

Let g be an edge-coloring function of K_n which produces no monochromatic copies of K_3 or two-colored copies of K_4. Let $xyz \in E(H)$ with $x < y < z$; then $p(x, y)p(y, z) \in E(G)$. Let

$$h(xyz) = (g(p(x, y)p(y, z)), s(p(x, y), p(y, z))).$$

Now, suppose we have colored the edges of H with h and let K be a copy of K_7^3 with $V(K) = \{v_1, \ldots, v_7\}$ where $v_1 < v_2 < \cdots < v_7$. Let $v' = v_i$, $v'' = v_{i+1}$ where $p(v_i, v_{i+1})$ is minimized for $1 \leq i \leq 6$. We have a set of five vertices $X = \{x_1, x_2, x_3, x_4, x_5\}$ with $x_1 < x_2 < x_3 < x_4 < x_5$ such that $p(x_i, xi + 1)$ is minimized for either $i = 1$ or $i = 4$. Let $E = E(X) \subset E(H)$.

Case 1. Suppose that all edges in E receive the same value for the second coordinates of their colors. If $p(x_i, xi + 1)$ is minimized at $i = 1$ then $p(x_1, x_2) < p(x_2, x_3) < p(x_3, x_4) < p(x_4, x_5)$

and if $p(x_i, xi+1)$ is minimized at $i = 4$ then $p(x_1, x_2) > p(x_2, x_3) > p(x_3, x_4) > p(x_4, x_5)$. In either case, the first coordinates of their colors received by E contain all the colors spanned by the edges of a copy of K_4 in G. Therefore, there are at least three distinct colors on the edges in E.

Case 2. Suppose the edges in E do not all have identical second coordinates in their colors. The first subcase is when $p(v_i, v_{i+1})$ is minimized at $i = 1$. We know that $p(x_1, x_2) < p(x_2, x_3)$. If $p(x_2, x_3) < p(x_3, x_4)$, then consider $X' = \{x_1, x_2, x_3, x_4\}$. All the edges in $E(X')$ receive 2 for the second coordinates of their colors and at least two distinct values among the first coordinates of the colors since G contains no monochromatic triangles. Some edge in E receives 1 for the second coordinate of its color by our initial assumption and therefore we have at least three colors on $E(X')$. If $p(x_2, x_3) > p(x_3, x_4) > p(x_4, x_5)$ then the same argument applies for $X'' = \{x_2, x_3, x_4, x_5\}$. If $p(x_3, x_4) < p(x_4, x_5)$, then we apply the argument to $X''' = \{x_1, x_3, x_4, x_5\}$. The second subcase is when $p(v_i, v_{i+1})$ is minimized at $i = 4$ and the argument identical to the case $i = 1$.

We started with a K_n which contained no monochromatic copies of K_3 or 2-edge-colored copies of K_4 and obtained, using only twice as many colors, a $K^3_{2^n}$ containing no 2-edge-colored copies of K^3_7. We used $e^{O(\sqrt{\log n})}$ colors on $E(K^3_{2^n})$, yielding

$$f_3(n, 7, 3) \leq e^{O(\sqrt{\log \log n})}.$$

$\square$

We will now show that, given a k-edge-coloring of $G = K_n^l$ which contains no 2-edge-colored copies of K_m^l where $l+5 < m$, it is possible to edge-color $H = K_{2^n}^{l+1}$ with $2k+2l$ colors so that it contains no 2-edge-colored copies K_{m+1}^{l+1}.

Theorem 21. *Let n, l, m be positive integers with $l > 2$ and $l+5 < m < n$. Then*

$$f_{l+1}(K_{2^n}^{l+1}, K_{m+1}^{l+1}, 3) \leq 2\, f_l(K_n^l, K_m^l, 3) + 2l.$$

Proof. Let $g : \binom{[n]}{l} \mapsto [k]$ be an edge-coloring of K_n^l which contains no 2-edge-colored K_m^l. We use g to construct an edge-coloring $h : \binom{[2^n]}{l+1} \mapsto [2k+2l]$ on $H = K_{2^n}^{l+1}$ which contains no 2-edge-colored copies of K_{m+1}^{l+1}. Let $V(H) = \{0,1\}^n$. Let $v_1 v_2 \cdots v_{l+1} \in E(H)$ and

$v_1 < v_2 < \cdots < v_{l+1}$. Let f_i denote the index of the coordinate where v_i and v_{i+1} first differ. Let

$$h(v_1 v_2 \cdots v_{l+1}) = \begin{cases} (g(f_1 f_2 \cdots f_l), 1) & \text{if } (f_1, f_2, \cdots, f_l) \text{ is an increasing sequence} \\ (g(f_1 f_2 \cdots f_l), 2) & \text{if } (f_1, f_2, \cdots, f_l) \text{ is a decreasing sequence} \\ (i, 3) & \text{if } f_1 < f_2 < \cdots f_i > f_{i+1} \\ (i, 4) & \text{if } f_1 > f_2 > \cdots f_i < f_{i+1} \end{cases}$$

Now, let $K' = K_{m+2}^{l+1}$ with $V(K') = \{v_1, v_2, \ldots v_{m+2}\}$ and $v_1 < v_2 < \cdots v_{m+1}$. There are several cases to consider:

Case 1. Suppose that $f_1 < f_2 < \cdots < f_m$ or $f_1 > f_2 > \cdots > f_m$. Then all of the colors of used to color some copy of K_m^l in G are present in the colors received by $E(K')$. Every copy of K_m^l in G receives at least three distinct colors on its edges and therefore there are at least three distinct colors on $E(K')$ as well.

Case 2. Suppose that $f_1 < f_2 < f_3 > f_4$ or that $f_1 > f_2 > f_3 < f_4$. In the first case $h(v_1v_2\cdots v_{l+1}) = (3,3)$, $h(v_2v_3\cdots v_{l+2}) = (2,3)$ and $h(v_3v_4\cdots v_{l+3}) = (t,4)$ for some t. In the second case, the three distinct colors are $(3,4),(2,4)$ and $(t,3)$ for some t.

Case 3. Now, assume that $f_1 < f_2 > f_3$. Suppose that $f_3 > \cdots > f_i < f_{i+1}$. If $4 \leq i \leq m$, then, for some $j > 2$ where $i \in \{j,\ldots,j+l\}$, the edges $v_1\ldots v_{l+1}$, $v_2\ldots v_{l+2}$ and $v_j\cdots v_{j+l}$ receive three distinct colors. If $i = m+1$, then we have $f_2 > \cdots > f_{m+1}$ which yields three distinct colors by Case 1. So, assume that $f_1 < f_2 > f_3 < f_4$. If $f_3 < f_4 < f_5$, then the edges $v_1\cdots v_{l+1}$, $v_2\cdots v_{l+2}$ and $v_3\cdots v_{l+3}$ all receive distinct colors. So, we will assume that $f_1 < f_2 > f_3 < f_4 > f_5$. If $f_1 < f_3$ and f' is the position where v_1 and v_3 first differ, then we have $f' < f_3 < f_4$ and we have three distinct colors by an argument similar to Case 2. Suppose that $f_1 > f_3$. We will examine the cases $f_3 < f_5$ and $f_3 > f_5$ separately. First, suppose that $f_3 < f_5$. We see that if $f_5 > f_6$, then we have $f_4 > f_5 > f_6$ and we are in Case 1 or Case 2. But, if $f_5 < f_6$ and f'' is the position where v_3 and v_5 differ, then we have $f'' < f_5 < f_6$ and we again follow the argument of one of the previous cases. Our final case is when $f_3 > f_5$. We then have $f_1 > f_3 > f_5$. If f''' is the position where v_1 and v_3 differ and f'''' is the position where v_3 and v_5 differ, then we have $f''' > f'''' > f_5$ and once again use the arguments from either Case 1 or Case 2 to find three distinct colors. If $f_1 > f_2 < f_3$, then the argument is very similar. $\square$

APPENDICES

Appendix A

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