# Fair Resource Allocation in a Constrained Network

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THESIS

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*to my parents*

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JX

## PREFACE

This body of work is an intellectual property of Jian Xu. The works discussed here was conducted at the University of Illinois at Chicago. The works have been published in (Xu et al., 2018), (Xu et al., 2017), and one piece of work is under submission (Xu and Rao, 2019). The research work was funded by NSF Grant CNS-1149661.

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## CONTRIBUTION OF AUTHORS

The contents of Chapter 2 was published in (Xu et al., 2018), (Xu et al., 2017). Prof. Wenjing Rao was the PI for this part of the work. I was responsible for building and developing the majority of the ideas, proving the theorems and composing the manuscript. The contents of Chapter 3 was under submission (Xu and Rao, 2019). Prof. Wenjing Rao was the PI for this part. I was responsible for building and developing the majority of the ideas, proving the theorems and composing the manuscript.

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| | |
|---|---|
| IT | Improving Transfer |
| NT | Neutral Transfer |
| CNT | Cyclic Neutral Transfer |
| SNT | Straight Neutral Transfer |
| DT | Deteriorating Transfer |
| TBC | Transfer-based Comparison |
| UIC | University of Illinois at Chicago |

# SUMMARY

The dissertation discusses about the understanding of the fairest semi-matching on bipartite graphs. Let $G=(R\cup U,E)$ be a bipartite graph with two vertex sets, $R$ (representing resources), $U$ (representing users), and $E\subseteq R\times U$ as the edge set representing allocating possibilities, where an edge $e_{ij}=\{r_i,u_j\}\in E$ indicates that resource $r_i\in R$ can be assigned to user $u_j\in U$. A semi-matching $M\subseteq E$ on $G=(R\cup U,E)$ is defined as a set of edges such that each resource vertex in $R$ is incident with exactly one edge in $M$, formally $M=\{\{e_{ij}\}|\forall r_i\in R,\exists u_j,$ such that $e_{ij}\in M,$ while $\forall k\neq j, e_{ik}\notin M\}$. Intuitively, a semi-matching represents one valid allocation where each resource is allocated to one user, among all the users the resource is connected to in $G$. Consequently, a semi-matching specifies which and how many resources are assigned to each user. For $u_j\in U$, let $Q(M,u_j)$ denotes the *quota* of $u_j$ under $M$, defined as the number of edges in $M$ that include $u_j$. Then, let $Q(M)$ denotes the *quotas vector* of a semi-matching $M$, defined as $Q(M)=(Q(M,u_1),Q(M,u_2),\cdots,Q(M,u_j),\cdots,Q(M,u_n))$, where $u_j\in U$, $j\in[1,n]$. A *sorted quotas vector* $Q^{\uparrow}(M)$ is defined as $Q(M)$ sorted in non-decreasing order. The fairness of a semi-matching is usually considered based on its sorted quotas vector alone, under some fairness measures.

Our three main contributions are presented in this dissertation:

1. We prove that there always exists one (or a set of equally fair) semi-matching(s), universally agreed by all the existing fairness measures, to be the fairest among all the semi-matchings of a given bipartite graph. In other words, given that fairness measures disagree on many comparisons between semi-matchings, they nonetheless are all in agreement on the (set of) fairest semi-

matching(s) for a given bipartite graph. To prove this, we propose a preorder relationship (named Transfer-based Comparison) among the semi-matchings, showing that the greatest elements always exist in such a preordered set. We then show that such greatest elements can guarantee to be the fairest ones under the fairness measure of Majorization (Olkin and Marshall, 2016). This further indicates that such fairest semi-matchings are agreed by all the fairness measures which are compatible with Majorization. To the best knowledge of us, this is true for all existing fairness measures.

2. It has been shown that, for any given bipartite graph, there always exists a set of fairest semi-matchings which are considered equally fair by all the fairness indices. While achieving one of the fairest semi-matchings is easy, it is not obvious how all of them are related, or what they have in common. We concludes that a lot can be learned about the entire set of fairest semi-matchings (which are usually hard to enumerate) from an arbitrary one (which is easy to achieve). The main conclusions achieved are: given a bipartite graph, from one arbitrary fairest semi-matching (which is easy to achieve), we can understand some important attributes for the entire set of fairest semi-matchings: 1) the classification of the edges in the bipartite graph - whether each edge is used by all, none, or some of the fairest semi-matchings; 2) the partition of user and resource vertices in the bipartite graph - the allocating of all the fairest semi-matchings are all within the partitions, and each user vertex has a very narrow quota range (at most differ by 1, and is predictable from the knowledge gained from one fairest semi-matching) among all the fairest semi-matchings.

3. Under the scenario that the resources are divisible which indicates each resource can be split and assigned to maybe more than one users, the fairest ones with regarding to one fairness measure

are defined as *fairest fractional allocations*. We show that there always exists a (maybe infinite) set of universally agreed fairest fractional allocations, based on the similar techniques for the existence of universally agreed fairest semi-matchings. And this set of universally agreed fairest fractional allocations are denoted as $\mathcal{C}$. We study on the relationship between the set of fairest semi-matchings ($\mathcal{F}$) and the set of fairest fractional allocations ($\mathcal{C}$), and conclude that: 1) the constant vertices (resources and users) partition across the set $\mathcal{F}$ is also constant across both the set $\mathcal{F}$ and the set $\mathcal{C}$ - the allocating of all $M \in \mathcal{F}$ and all $W \in \mathcal{C}$ are all within the partitions. 2) for each user, the quota difference between one $M \in \mathcal{F}$ and one $W \in \mathcal{C}$ is either zero or bound by one, and its specific quota ranges across the set $\mathcal{F}$ and $\mathcal{C}$ can be determined from one $M \in \mathcal{F}$ or $W \in \mathcal{C}$.

# CHAPTER 1

# INTRODUCTION

## 1.1 Background and Motivation

One of the classical combinatorial optimization problem is bipartite matching problem which has wide applications such as Network Routing (Bertsekas et al., 1992), Large Computing Clusters Frameworks (Dean and Ghemawat, 2004), Google AdWords (Geddes, 2014), Data Warehousing (Thusoo et al., 2010), Shape Matching (Belongie et al., 2002), Image Recognition (Cheng et al., 1996), VLSI Design (Huang et al., 1991), Bioinformatics (Lin et al., 2004) etc. Many variants of this problem have been studied widely in literature. The bipartite graph can be weighted or unweighted, and the matching types include maximum matching, maximal matching and perfect matching. Some works focus on the algorithms to finding a matching (Kuhn, 1955) (Hopcroft and Karp, 1973) (Gabow and Tarjan, 1989) (Goel et al., 2010) (Bollobás, 2013), while some study on the enumeration of all matchings (Fukuda and Matsui, 1994) (Uno, 1997) (Uno, 2001) (Boros et al., 2004), under the offline setting (Lenstra et al., 1990) (Henzinger et al., 2011) or online setting (Karp et al., 1990) (Ghodsi et al., 2013).

This dissertation studies on a relaxation of the bipartite matching problem, which is called as the *fairest semi-matchings* problem on bipartite graph. A *semi-matching* in a bipartite graph $G = (R \cup U, E)$ where $E \subseteq R \times U$, is defined as a set of edges $M \subseteq E$ such that each vertex in $R$ is incident with exactly one edge in $M$, and a vertex in $U$ can be incident with an arbitrary number of edges in $M$. In general, valid semi-matchings can be easily obtained by matching each vertex $r \in R$ with an arbitrary vertex

$u \in U$ for which $(r,u) \in E$. The problem of finding the *fairest* semi-matching is related to the fair resource allocation problem in a system, where a set of discrete resources ($R$) need to be assigned to a set of users ($U$) in the fairest way. Meanwhile, the given bipartite graph corresponds to the allocating constraints, where the edge set $E$ indicate the possible allocating options. This is representative for a large-scale system, where resource allocation is constrained to a limited set of users (for instance, in the local area). In such cases, the optimal solution of a resource allocation problem is given by a fairest semi-matching indicating how the resources (of $R$-vertices) should be allocated to the users (of $U$-vertices), under allocating constraints (of edge set $E$).

Given a semi-matching $M$, the *quota* of user $u \in U$ is defined as the number of $R$-vertices matched with $u$ according to $M$. The corresponding *quotas vector* of $M$ denoted as $Q(M)$, is the vector of which each element $Q(M,u)$ represents the *quota* of a user $u \in U$. Based on the assumption that all vertices and edges of the bipartite graph are unweighted, the fairness of a semi-matching is usually considered based on its quotas vector, using some fairness measure to give an index.

In literature, some works focused on the bipartite graph with weighted resources which is NP-hard problem. Some approximate algorithms have been proposed (Lenstra et al., 1990) (Shmoys and Tardos, 1993) (Bezáková and Dani, 2005). And (Bezáková and Dani, 2005) showed that there is no approximation algorithm for this problem with performance better then 2 unless $P = NP$. A special case of so called “big goods/small goods” was studied by (Golovin, 2005).

Some works focused on the unweighted bipartite graph, and aim to achieve one fairest semi-matching with regard to one fairest measure by polynomial algorithms. In the job scheduling field, some algorithms have been proposed to achieve the fairest semi-matchings between jobs and computing nodes which minimize the total computation time ($L_2$-norm) or the time the last machine finishes (Max-min fairness). (Lee et al., 2011) proposed a $O(m+nlogn)$ algorithm for a special case called *nested* case[1]. (Harvey et al., 2006) reduced the problem to the min-cost problem, and proposed a $O(nm)-$ algorithm. (Fakcharoenphol et al., 2014) presented a divide and conquer algorithm with $O(\sqrt{n}mlogn)$ complexity. Recently, some approximation of optimal semi-matching have been proposed based on the distributed setting.

Many fairness measures in literature have been proposed to compare quotas vectors of semi-matchings, such as Jain's index (Jain et al., 1984), Entropy (Tse and Viswanath, 2005), $\alpha$-fairness (Mo and Walrand, 2000), Max-min fairness (Bansal and Sviridenko, 2006), Convex measure (Harvey et al., 2006), $L_p$-norm (Harvey et al., 2006), Lexicographically order (Bokal et al., 2012), etc. Lan et al. (Lan et al., 2010) constructed a family of fairness measures based on a set of axioms. Note that, for all the fairness measures, any pair of semi-matchings with the same sorted quotas vector are considered equal in fairness. Nonetheless, the existence of numerous fairness measures indicates that they do not agree on the fairness comparison between many semi-matchings (or the sorted quotas vectors of them). Few work in literature studies on the properties of the fairest semi-matchings with regard to the consistence among fairness measures. (Harvey et al., 2006) proved the existence of a fairest semi-matching with regard to both Max-min fairness and $L_2$-norm, and showed the obtained semi-matching is also the fairest with regard to any convex measures and any $L_p$-norm (where $p \in R$ and $p > 1$). (Bokal et al., 2012) showed

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[1] where for any pair of jobs, the set of connected users for job should be the subset of another job if the two sets are disjoint

that, a semi-matching is the fairest with regard to $L_2$-norm if and only if the sorted quotas vector of this semi-matching is lexicographically minimum.

## 1.2 Main Conclusions and Techniques

This dissertation focuses on the properties of all fairest semi-matchings on bipartite graphs. Our contributions include three parts:

1. We prove that for any given bipartite graph, there always exists a fairest (set of) semi-matching(s), under the special case of partially ordered Majorization measure (Olkin and Marshall, 2016). This result, combined with the previous knowledge (that all known fairness measures shown in Table I are compatible with Majorization) indicates that in fact despite their disagreement on some comparisons, those fairness measures always agree on the fairest semi-matching(s) for a given bipartite graph.

   To prove this, we define a preordered set (proset) of semi-matchings based on the way to transfer one into another (Transfer-based Comparison). This proset is shown to always have the greatest elements, as the fairest semi-matchings. Subsequently, we show that the proposed Transfer-based Comparison can strictly imply the Majorization order. In other words, the fairest semi-matchings under the proposed Transfer-based Comparison are always regarded as fairest under Majorization. To our best knowledge, all existing fairness measures in literature are compatible with Majorization. In conclusion, for any bipartite graph, there always exists a set of equally fair semi-matchings, which are universally regarded as the fairest ones, by all existing fairness measures, even though they may disagree on the comparisons among the ones that are not the fairest. As a result, a number of previously proposed algorithms which claimed to achieve the fairest semi-matchings under some specific fairness measure, in fact achieves the universally agreed fairest ones for all the known and listed fairness measures.

2. It has been shown that, for any given bipartite graph, there always exists a set of fairest semi-matchings which are considered equally fair by all the fairness indices. While achieving one of the fairest semi-matchings is easy, it is not obvious how all of them are related, or what they have in common. We conclude that a lot can be learned about the entire set of fairest semi-matchings (which are usually hard to enumerate) from an arbitrary one (which is easy to achieve). The main conclusions achieved are: from one arbitrary fairest semi-matching, one can easily identify: 1) whether an edge in the bipartite graph is used by all, or none, or some of the fairest semi-matchings; 2) a constant partition of user and resource vertices in the bipartite graph such that the allocation of all the fairest semi-matchings are all within the partitions; 3) the quota range of each user across all the fairest semi-matchings, which is very narrow and predictable.

   The conclusions are achieved mainly based on the common attributes among all fairest semi-matchings, specifically, edge usage and quota similarity for each user among all the fairest semi-matchings. a) edge usage: we prove that all fairest semi-matchings have the same set of *Neutral Transfer Covered Edges* (which constitute the paths which will not change the fairness, and will be defined formally later), and this set is actually the set of all edges used by some but not all fairest semi-matchings. b) quota similarity: we define a sequence of ranked user sets such that each set contains all users with the same quota, or contains all users covered by *Neutral Transfers* within same "quota-level". Then it is proved that each user always belongs to the same ranked user set across all the fairest semi-matchings.

3. Motivated by the observation that the fairness of the fairest semi-matchings can be improved by splitting the resources, we consider the scenario that the resources are divisible which indicates each resource can be split and allocated to maybe more than one users, and then the fairest ones with regard to one fairness measure under this scenario are called *fairest fractional allocations*. We show that under a bipartite graph there always exists a (maybe infinite) set of universally agreed fairest fractional allocations, and all of them have the same quotas vector. The set of universally agreed fairest fractional allocations are denoted as $\mathcal{C}$. We study on the relationship between the set of fairest semi-matchings $\mathcal{F}$ and the set of fairest fractional allocations $\mathcal{C}$, specifically, how the fairest semi-matchings be similar with the fairest fractional allocations.

   The main conclusion achieved are: 1) the constant vertices (users and resources) partition across all the fairest semi-matchings is also constant across all the fairest fractional allocations - the allocating of all fairest semi-matchings and all the fairest fractional allocations are always within the partitions. 2) for each user, the difference of its quotas between one fairest semi-matching and one fairest fractional allocation is limited (either 0 or bound by 1). Furthermore, the special quota range of each user across the sets $\mathcal{F}$ and $\mathcal{C}$ can be determined from one arbitrary fairest semi-matching or fairest fractional allocation.

   The proof is mainly based on the construction of a fairest fractional allocation. Under the constant vertices partition across the set $\mathcal{F}$, a bipartite sub-graph is constructed for each partition. Then a fractional allocation $W$ for the original bipartite graph $G$ is constructed, by the combining the fairest fractional allocations under the bipartite sub-graphs (one fairest fractional allocation from one bipartite sub-graph). This fractional allocation $W$ is then proved to be one fairest fractional allocation for the original bipartite graph $G$. Based on that, our main conclusions can be achieved.

## 1.3 Thesis Organization

This thesis is organized into 5 chapters. Chapter 1 present the overall introduction of this dissertation. Our three main contributions regarding the property of the fairest semi-matchings are shown in Chapter 2, Chapter 3, Chapter 4 respectively. A concluding summary of the thesis, and the discussion of a few possible extensions of our work and related open problems, are presented in Chapter 5.

**Chapter 1:** This chapter provides the reader with the background and motivations for this work and identifies the research questions that the work focuses on. A list of our contributions and main techniques are included in this chapter, as well.

**Chapter 2:** This chapter presents the work on the existence of universally agreed fairest semi-matchings. First an overall introduction of motivation, related works and main conclusions, is shown, followed by the presentation of the lack of consensus among fairness measure. Then, the key tool for the proof (the proposed Transfer-based Comparison) is introduced and analyzed. After that, the main theorems are shown and proved. Finally, some discussion on the structure of all semi-matchings and algorithms are present.

**Chapter 3:** This chapter presents the work on the understanding of all the fairest semi-matchings from one. At first, an overall introduction of the problem, motivation and main conclusions is present. Then some definitions and conclusions from Chapter 2 which serves as a preliminary in this chapter are given. Following that, two main theorems about bipartite graph edge classification among fairest semi-matchings, and the user quota stability among all the fairest semi-matchings, are present and proved.

**Chapter 4:** This chapter presents the work on the correspondence between fairest semi-matchings and fairest fractional allocations. At first, an overall introduction of the fractional allocations definition, motivation and main conclusions is present. Then, the existence of universally agreed fairest fractional allocations is proved based on the similar techniques from Chapter 2. After that, the main theorems about the correspondence between the fairest semi-matchings and the fairest fractional allocations are shown and proved.

**Chapter 5:** The first part of this chapter concludes the main contributions of this dissertation. The second part of this chapter discuss the possible extensions of this work, which include how does the bipartite graph topology impact on the fairness of the fairest semi-matchings and the edge classification profile, followed by that some hypothesis on the open question of the fairest semi-matchings for weighted bipartite graph.

# CHAPTER 2

# THE EXISTENCE OF UNIVERSALLY AGREED FAIREST SEMI-MATCHINGS FOR ANY BIPARTITE GRAPH

*Parts of this chapter have been presented in (Xu et al., 2018), (Xu et al., 2017). Copyright © 2018, Elsevier, 2017, Springer.*

## 2.1 Introduction

This paper focuses on the problem of the *Fairest Semi-matching*. A *semi-matching* in a bipartite graph $G = (R \cup U, E)$ where $E \subseteq R \times U$, is defined as a set of edges $M \subseteq E$ such that each vertex in $R$ is incident with exactly one edge in $M$, and a vertex in $U$ can be incident with an arbitrary number of edges in $M$. In general, valid semi-matchings can be easily obtained by matching each vertex $r \in R$ with an arbitrary vertex $u \in U$ for which $(r, u) \in E$.

**Definition 2.1.** ***(Semi-Matching)*** *(Harvey et al., 2006) In a bipartite graph* $G = (R \cup U, E)$ *where* $E \subseteq R \times U$*, A semi-matching* $M \subseteq E$ *on* $G$ *is defined as* $M = \{\{e_{ij}\} | \forall r_i \in R, \exists\, e_{ij} = \{r_i, u_j\} \in M, \text{while } \forall k \neq j, e_{ik} = \{r_i, u_k\} \notin M\}$.

The problem of finding the *fairest* semi-matching is related to the fair resource allocation problem in a system, where a set of discrete resources ($R$) need to be allocated to a set of users ($U$) in the fairest way. Meanwhile, the given bipartite graph corresponds to the allocating constraints, where the edge set $E$ indicate the possible allocating options. This is representative for a large-scale system, where resource allocation is constrained to a limited set of users (for instance, in the local area). In such cases, the optimal solution of a fair resource allocation problem is given by a fairest semi-matching indicating how the resource (of $R$-vertices) should be allocated to the users (of $U$-vertices), under allocating constraints (of edge set $E$).

Consequently, a semi-matching specifies which and how many resources are assigned to each user. For $u_j \in U$, let $Q(M,u_j)$ denotes the *quota* of $u_j$ under $M$, defined as the number of edges in $M$ that cover $u_j$. Then, let $Q(M)$ denotes the *quotas vector* of a semi-matching $M$, defined as $Q(M) = (Q(M,u_1),Q(M,u_2),\cdots,Q(M,u_n))$, where $u_j \in U$, $j \in [1,n]$.

**Definition 2.2.** ***(Quotas Vector)*** *Given a semi-matching $M$, the quota of vertex $u \in U$ is defined as the number of edges that cover $u$ (or the number of resource vertices matched with $u$) according to $M$. The corresponding quotas vector of $M$, denoted as $Q(M)$, is the vector of which each element $Q(M,u)$ represents the quota of a vertex $u \in U$.*

A *sorted quotas vector* $Q^{\uparrow}(M)$ is defined as $Q(M)$ sorted in non-decreasing order. Assuming that both edges and vertices in the bipartite graph are unweighted, the fairness of a semi-matching is usually considered based on its sorted quotas vector alone, under some fairness measures. An example of the fairest semi-matching according to Jain's index (Jain et al., 1984) is shown in Fig. 1. The literature shows that one fairest semi-matching regarding one fairest measure, can be achieved by polynomial algorithms.

### 2.1.1 Motivation

Many fairness measures have been proposed to compare quotas vectors, such as Jain's index (Jain et al., 1984), Entropy (Tse and Viswanath, 2005), $\alpha$-fairness (Mo and Walrand, 2000), Lexicographical order (Bokal et al., 2012), Convex measure (Harvey et al., 2006), $L_p$-norm (Harvey et al., 2006), Max-min fairness (Bansal and Sviridenko, 2006), etc. (Lan et al., 2010) constructed a family of fairness measures based on a set of axioms. A list of known fairest measures are given in Table I. Note that, for all the fairness measures, any pair of semi-matchings with the same sorted quotas vector are considered equal in fairness. Nonetheless, the existence of numerous fairness measures indicates that they do not agree on the fairness comparison between many semi-matchings (or the sorted quotas vectors of them).

### 2.1.2 Related Works

*Fairness in job scheduling:* Some previous works in the area of job scheduling focus on the algorithms to achieve one fairest semi-matching with regard to some specific fairness measures. Specifically, the goal is to achieve the fairest semi-matching between jobs and computing nodes which minimize the total computation time ($L_2$-norm) or the time the last machine finishes (Max-min fairness). This problem is one type of the general job scheduling problem with $p_{ij} \in \{1, \infty\}$ where $p_{ij}$ represents the processing time of job $j$ on machine $i$. (Lee et al., 2011) proposed a $O(m + n \log n)$ algorithm for a special *nested* case, where for any pair of jobs, the set of connected machines for one job should be the subset of the set for another job if the two sets are not disjoint. (Harvey et al., 2006) reduced the problem to the min-cost problem, and proposed a $O(nm)$ algorithm. (Fakcharoenphol et al., 2014) presented a divide and conquer algorithm with time complexity of $O(\sqrt{n}m \log n)$. Recently, some approximation of optimal semi-matching were proposed based on the distributed setting (Czygrinow et al., 2012)(Konrad and Rosén, 2016).

*The properties of the fairest semi-matchings:* (Harvey et al., 2006) showed that the existence of semi-matchings which are the fairest with regard to both Max-min fairness and $L_2$-norm, and showed that the obtained semi-matching is also the fairest with regard to any convex measures and any $L_p$-norm

TABLE I: A LIST OF KNOWN FAIREST MEASURES

| Fairness measure | Expression (function of $Q = (q_1, q_2, \cdots, q_m)$) | Argument for "Compatibility with Majorization (Schur-convex)" |
|---|---|---|
| Jain's fairness (Jain et al., 1984) | $f(Q) = (\sum_{i=1}^{m} q_i)^2/(m\sum_{i=1}^{m} q_i{}^2)$ | $\beta = -1$ in (Lan et al., 2010) |
| Entropy (Tse and Viswanath, 2005) | $f(Q) = \sum_{i=1}^{m} (\frac{q_i}{\sum_j q_j})\ log(\frac{q_i}{\sum_j q_j})$ | $\beta \to 0$ in (Lan et al., 2010) |
| Max-min fairness (Bansal and Sviridenko, 2006) | $f(Q) = min_{i=1,\cdots,m}\{q_i\}$ | Similar to $\beta \to \infty$ in (Lan et al., 2010) |
| Min-max fairness (Bansal and Sviridenko, 2006) | $f(Q) = max_{i=1,\cdots,m}\{q_i\}$ | Similar to $\beta \to -\infty$ (Lan et al., 2010) |
| Lexicographical order (Bokal et al., 2012) | Sorted $Q_a^{\uparrow} = (q_1, q_2, \cdots, q_m)$, $Q_b{}^{\uparrow} = (q'_1, q'_2, \cdots, q'_m)$, $Q_a^{\uparrow} > Q_b^{\uparrow}$, if $q_i > q'_i$, for the first $i$ where $q_i$ and $q'_i$ differ. | Easy to prove from its definition |
| $L_p$-norm (Harvey et al., 2006) | $f(Q) = (\sum_i^m x_i^p)^{1/p}$ $(1 < p < \infty)$ | $L_p$-norm is convex and symmetric ,which is sufficient for schur-convex function |
| Convex measure (Harvey et al., 2006) | $f(Q) = \sum_i^m f(q_i)$ where $f$ is convex | Convex measure is convex and symmetric ,which is sufficient for schur-convex function |
| fairness family (Lan et al., 2010) | $f_\beta(Q) = sign(1-\beta) \cdot [\sum_{i=1}^{n} (\frac{q_i}{\sum_j q_j})^{1-\beta}]^{\frac{1}{\beta}}$ | Proved in (Lan et al., 2010) |
| including: | | |
| Max ratio | $\beta \to \infty : f_\beta(Q) = -max_i\{\frac{\sum_j q_j}{q_i}\}$ | |
| $\alpha$-fair utility | $\beta \in (1,\infty)$:$f_\beta(Q) = -[(1-\beta)U_{\alpha=\beta}(\frac{Q}{\sum_i q_i})]^{\frac{1}{\beta}}$ | |
| $\alpha$-fair utility | $\beta \in (0,1) : f_\beta(Q) = [(1-\beta)U_{\alpha=\beta}(\frac{Q}{\sum_i q_i})]^{\frac{1}{\beta}}$ | |
| Entropy | $\beta \to 0 : f_\beta(Q) = e^{H(\frac{Q}{\sum_i q_i})}$ | |
| Jain's fairness | $\beta = -1 : f_\beta(Q) = -(\sum_{i=1}^{m} q_i)^2/(m\sum_{i=1}^{m} q_i{}^2)$ | |
| Min ratio | $\beta \to -\infty : f_\beta(Q) = min_i\{\frac{\sum_j q_j}{q_i}\}$ | |

(where $p \in \mathbb{R}$ and $p > 1$). (Bokal et al., 2012) showed that, a semi-matching is the fairest with regard to $L_2$-norm if and only if the sorted quotas vector of this semi-matching is lexicographical minimum.

### 2.1.3 Main Results

In this chapter, we prove that for any given bipartite graph, there always exists a fairest (set of) semi-matching(s), under the special case of Majorization. This result, combined with the previous knowledge (that all known fairness measures shown in Table I are compatible with Majorization) indicates that in fact despite their disagreement on some comparisons, those fairness measures always agree on the fairest semi-matching(s) for a given bipartite graph.

### 2.1.4 Organization

This chapter is organized as follows. Section 2.2 discusses the consensus among fairness measures with regard to Majorization. Section 2.3 defines a Transfer-based Comparison between any pair of semi-matchings, which is shown to be a preorder relationship more rigorous than Majorization. Section 2.4 proves there exists a set of the greatest semi-matchings in the proposed Transfer-based Comparison. Section 2.5 provides a discussion on various issues related to the Transfer-based Comparison, and some algorithms for achieving the fairest semi-matchings. This paper is concluded in Section 2.6.

## 2.2 The Lack of Consensus Among Fairness Measures

Most of the existing and widely used fairness measures are index-based, which is a mapping of a sorted quotas vector into a real number. Using an index-based fairness measure, any arbitrary pair of quotas vectors are comparable (either one is considered fairer, or both are considered equally fair). However, different index-based fairness measures often disagree on the comparison of some pairs of

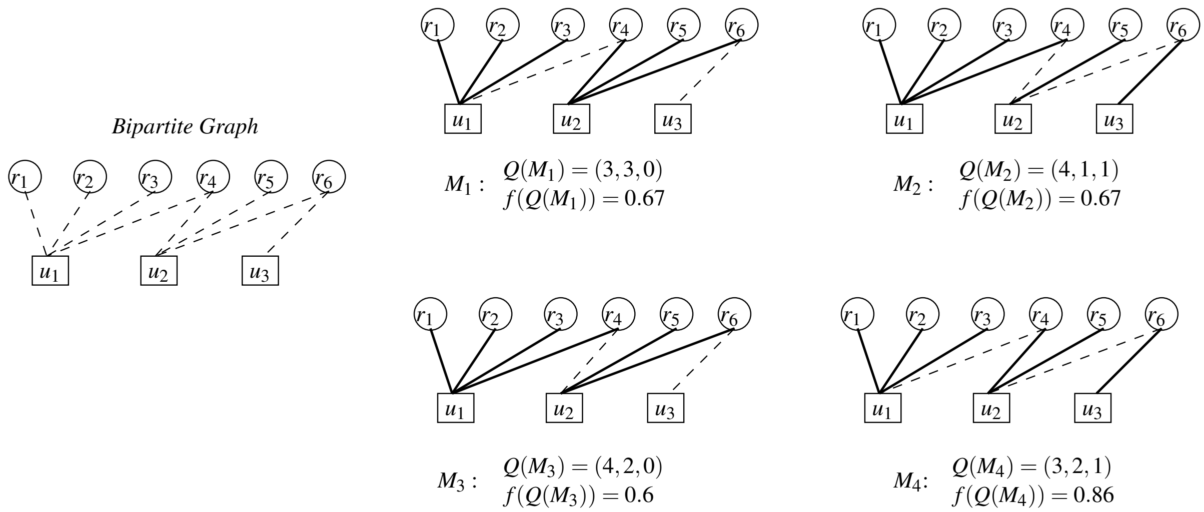



Figure 1: An example of the fairest semi-matching, where $f(Q(M)) = \frac{(\sum_{i=1}^{m} Q(M,i))^2}{m\sum_{i=1}^{m} Q(M,i)^2}$ ($m$ is the size of $Q(M)$) is based on Jain's index, and $M_4$ is the fairest one with the highest index of 0.86.

quotas vectors. For example, for the comparison of two quotas vectors $Q_1 = (0,3,3)$ and $Q_2 = (4,1,1)$, Table II illustrates the different comparison results from three fairness measures.

*Majorization* (Olkin and Marshall, 2016) is a preorder over quotas vectors, which allows some pairs to be "incomparable".

**Definition 2.3.** ***(Majorization)****(Olkin and Marshall, 2016) For $x, y \in R^n$, x is majorized by y (denoted as $x \preceq_{Maj} y$), if $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, and $\sum_{i=1}^{d} x_i^{\uparrow} \leqslant \sum_{i=1}^{d} y_i^{\uparrow}$ for $d = 1,\ldots,n$, where $x_i^{\uparrow}$ and $y_i^{\uparrow}$ are the $i^{th}$ elements of $x^{\uparrow}$ and $y^{\uparrow}$, which are sorted in ascending order.*

For example, $x = (0,3,3)$ is majorized by $y = (3,1,2)$. The sorted vectors are $x^{\uparrow} = (0,3,3)$ and $y^{\uparrow} = (1,2,3)$ respectively. Let $S_x^d = \sum_{i=1}^{d} x_i^{\uparrow}$ for $d = 1,2,3$. Then we have $S_x = (S_x^1, S_x^2, S_x^3) = (0, 0+3, 0+$

TABLE II: DISAGREEMENT AMONG VARIOUS FAIRNESS MEASURES FOR QUOTAS VECTORS

| | Jain's index | Max-min index | An index in (Lan et al., 2010) ($\beta = -2$) |
|---|---|---|---|
| Definition $f(Q)$ : $(Q = (q_1, q_2, \cdots, q_m))$ | $\frac{(\sum_{i=1}^{m} q_i)^2}{m \sum_{i=1}^{m} q_i{}^2}$ | $min_{i=1,\cdots,m}\{q_i\}$ | $\{\sum_{i=1}^{m} \big(\frac{q_i}{\sum_{j=1}^{m} q_j}\big)^{1-\beta}\}^{\frac{1}{\beta}}$ |
| $Q_1 = (0,3,3)$ | 0.67 | 0 | 2 |
| $Q_2 = (4,1,1)$ | 0.67 | 1 | 1.809 |
| Fairness Comparison | $Q_1 = Q_2$ | $Q_1 < Q_2$ | $Q_1 > Q_2$ |

$3+3) = (0,3,6)$ and $S_y = (S_y^1, S_y^2, S_y^3) = (1, 1+2, 1+2+3) = (1,3,6)$. Therefore, it meets $x \preceq_{Maj} y$. For the example pair shown in Table 1, where $x = (0,3,3)$ and $y = (4,1,1)$, they are incomparable with regard to Majorization, because $S_x = (S_x^1, S_x^2, S_x^3) = (0,3,6)$ and $S_y = (S_y^1, S_y^2, S_y^3) = (1,2,6)$, where $S_x^1 < S_y^1$ and $S_x^2 > S_y^2$.

A fairness measure $f$ is considered "compatible" with Majorization (or known as Schur-convex function), if it satisfies that $f(x) < f(y)$ when $x \preceq_{Maj} y$. (Lan et al., 2010) studied various fairness measures and constructed a family of fairness measures satisfying five axioms (continuity, homogeneity, saturation, partition, and starvation). The resulting family of fairness measures includes all the existing popularly used fairness measures such as $\alpha$-fairness, Jain's index, Entropy function, etc. All the members of this family has been proved to be compatible with Majorization. Furthermore, we have listed all the fairness measures to our best knowledge and shown in Table I that they are all compatible with Majorization.

## 2.3 Transfer-based Comparison Among Semi-matchings

### 2.3.1 Definition of Transfer-based Comparison

For a given semi-matching $M$ in a bipartite graph $G = (R \cup U, E)$, we define a *Transfer* $T$ on $M$ to be a sequence of alternating edges $(\{u_1, r_1\}, \{r_1, u_2\}, \{u_2, r_2\}, \cdots, \{u_{k-1}, r_{k-1}\}, \{r_{k-1}, u_k\})$ with $u_i \in U, r_i \in R$, $\{u_i, r_i\} \in M$ for each $i \in [1, k-1]$.

**Definition 2.4.** *(**Transfer**) Given a bipartite graph $G = (R \cup U, E)$, $T = (\{u_1, r_1\}, \{r_1, u_2\}, \{u_2, r_2\}, \cdots, \{u_{k-1}, r_{k-1}\}, \{r_{k-1}, u_k\})$ is defined as a Transfer on a semi-matching $M$, if $T$ is an alternating path with regard to $M$, that is, for each $i \in [1, k-1]$, $\{u_i, r_i\} \in M$ (thus all the $\{r_i, u_{i+1}\} \notin M$ according to the definition of semi-matching).*

**Definition 2.5.** *(**Source and destination users of a Transfer**) For a Transfer $T = (\{u_1, r_1\}, \cdots, \{r_{k-1}, u_k\})$ on bipartite graph $G = (R \cup U, E)$, $u_1$ and $u_k$ are defined as the source and destination users of Transfer $T$, denoted as $u_s(T)$ and $u_d(T)$ respectively.*

Essentially, a Transfer $T$ on $M$ is a path beginning and ending in $U$-vertices in $M$, consisting of alternating edges in and out of $M$. The *application of Transfer $T$* to semi-matching $M$ is defined as switching the matching and non-matching edges in $M$ along Transfer $T$.

**Definition 2.6.** *(**Application of a Transfer**) The application of Transfer $T$ on semi-matching $M$, denoted as $T(M) = M'$, is defined as switching the matching and non-matching edges in $M$ along Transfer $T$. The result of the application of $T$ on $M$ will change $M$ to a different semi-matching $M'$, which includes all the $\{r_i, u_{i+1}\}$ edges in $T$, but excludes all the $\{u_i, r_i\}$ edges in $T$.*

For example, in Fig. 1, semi-matching $M_3$ can be changed to $M_4$ by applying a sequence of edges $(\{u_1, r_4\}, \{r_4, u_2\}, \{u_2, r_6\}, \{r_6, u_3\})$ which constitute a valid Transfer $T$ which will change $M_3$ to $M_4$.

**Definition 2.7.** ***(Transfer Types: Improving, Deteriorating, Neutral)*** *Let $T$ represents a Transfer on a semi-matching $M$, with $Q(M, u_s(T))$ and $Q(M, u_d(T))$ representing the quotas of the source user $u_s(T)$ and destination user $u_d(T)$ in $M$ respectively. Apparently, when $u_s(T) \neq u_d(T)$, $T(M) = M'$ results in the change of quota in only two user: $Q(M', u_s(T)) = Q(M, u_s(T)) - 1$, and $Q(M', u_d(T)) = Q(M, u_d(T)) + 1$. In other words, the source user's quota decreases by 1, and the destination user's quota increases by 1 as the result of applying $T$. In the special case of $u_s(T) = u_d(T)$, $M$ and $M'$ has the same quotas vector.*

*As a result, the application of $T$ on $M$ will affect the fairness of $M$ in only three possible ways:*

- *$T$ could result in a fairer $M'$ over $M$, thus $T$ is denoted as an Improving Transfer ($IT$) if $Q(M, u_s(T)) > Q(M, u_d(T)) + 1$;*
- *$T$ could result in a more unfair $M'$ over $M$, thus $T$ is denoted as a Deteriorating Transfer ($DT$) if $Q(M, u_s(T)) < Q(M, u_d(T)) + 1$ and $u_s(T) \neq u_d(T)$;*
- *$T$ could result in $M'$ with the same fairness as $M$, thus $T$ is denoted as a Neutral Transfer ($NT$), where two cases exist:*
  - *$T$ is denoted as a Straight Neutral Transfer ($NT_S$) if $Q(M, u_s(T)) = Q(M, u_d(T)) + 1$ and obviously $u_s(T) \neq u_d(T)$;*
  - *$T$ is denoted as a Cyclic Neutral Transfer ($NT_C$) if $u_s(T) = u_d(T)$.*

*Obviously, if a transfer $T$ is $NT$, $M$ and $T(M)$ have the same sorted quotas vector.*

**Definition 2.8.** ***(Transfer Sequence)*** *A Transfer Sequence between two semi-matchings $M_1$ and $M_k$, denoted as $M_k = T_{k-1} \circ \cdots \circ T_2 \circ T_1(M_1)$, is defined as a sequence of Transfers $(T_1, T_2, \cdots, T_{k-1})$ such that $\forall i \in [1, k-1], T_i(M_i) = M_{i+1}$ holds.*

A *Transfer-based Comparison* between semi-matchings is proposed, based on properties of the Transfer Sequence between them, shown as in Definition 2.9.

**Definition 2.9.** ***(Transfer-based Comparison)*** *For any two semi-matchings $M_x$ and $M_y$, if there exists a Transfer Sequence $(T_1, T_2, \cdots, T_{k-1})$ such that $M_y = T_{k-1} \circ \cdots \circ T_2 \circ T_1(M_x)$, and for each $i \in [1, k-1]$, $T_i$ is $IT$ or $NT$, then $M_y$ is defined as not less fair than $M_x$ in terms of a Transfer-based Comparison, denoted as $M_x \preceq_T M_y$. Specifically, if all $T_i$ are $NT$, then $M_x$ is defined as equally fair with $M_y$, denoted as $M_x \approx_T M_y$; if all $T_i$ are either $IT$ or $NT$, with at least one $T_i$ being $IT$, then $M_y$ is defined as fairer than $M_x$, denoted as $M_x \prec_T M_y$.*

*If for every possible sequences $(T_1, T_2, \cdots, T_{k-1})$ such that $M_y = T_{k-1} \circ \cdots \circ T_2 \circ T_1(M_x)$, there always exist some $T_i$, $T_j$ with $i, j \in [1, k-1]$ such that $T_i$ is $IT$ and $T_j$ is $DT$, then $M_x$ and $M_y$ are defined as incomparable, denoted as $M_x \prec\succ_T M_y$.*

### 2.3.2 Attributes of Transfer-based Comparison

Such a Transfer-based Comparison aims at reserving the "strictly comparable" relationships between semi-matchings via identifiable Improving or Neutral Transfer Sequence, while leaving out the "incomparable" ones with mixtures of both Improving and Deteriorating Transfers. It is easy to prove this defined comparison meets the properties of *Reflexivity, Transitivity*. Thus it is a preorder on the set of all semi-matchings in a given bipartite graph.

**Lemma 2.1.** ***(Preordered Set (proset))*** *The set of all semi-matchings of a bipartite graph is a preordered set with regard to the Transfer-based Comparison defined above.*

Different from the preorder of Majorization, which is defined on the quotas vectors, the Transfer-based Comparison is a preorder defined over the set of semi-matchings. We will prove that the Transfer-based Comparison implies the Majorization relationship between the corresponding quotas vectors. In other words, the Transfer-based Comparison is a "stricter" fairness measure than Majorization, by leaving out some pairs that can be compared under Majorization as incomparable. An example of a pair of semi-matchings comparable under Majorization but not comparable under Transfer-based Comparison, is shown in Figure 2.

**Lemma 2.2.** ***(Improving Transfer $\Rightarrow$ Majorization)*** *If $M_x \preceq_T M_y$, then $Q(M_x) \preceq_{Maj} Q(M_y)$.*

*Proof.* Let $Q^{\uparrow}(M_x) = (q_1^x, q_2^x, \cdots, q_i^x, \cdots, q_n^x)$ be the sorted quotas vectors of $M_x$ in ascending order and the partial sums $S_x^d = \sum_{i=1}^d q_i^x$ for $d = 1, \cdots, n$. Assume the application of an Improving Transfer (IT) $T$ to $M_x$ yields another semi-matching $M_{x'}$, which decreases the quota of source user of $T$ (suppose $q_j^x$) by 1, and increases the quota of destination vertex of $T$ (suppose $q_i^x$) by 1. Note that Transfer $T$ is $IT$, therefore the quota of its source user must be larger than the quota of its source user in $M_x$, thus $i < j$. Let $Q^{\uparrow}(M_{x'})$ represents the quotas vector of $M_{x'}$ sorted in ascending order, and $S_{x'}^d$ represents the partial sums of $Q^{\uparrow}(M_{x'})$ for $d = 1, \cdots, n$. It can be easily derived that $S_{x'}^d = S_x^d$ when $d < i$, that $S_{x'}^d > S_x^d$ when $i \leq d < j$, and that $S_{x'}^d = S_x^d$ when $j \leq d \leq n$. Likewise, if a Neutral Transfer is applied, then $S_{x'}^d = S_x^d$ when $1 \leq d \leq n$. Thus, it meets $Q(M_x) \preceq_{Maj} Q(M_{x'})$ if $M_{x'}$ can be derived by the application of an Improving or Neutral Transfer on $M_x$.

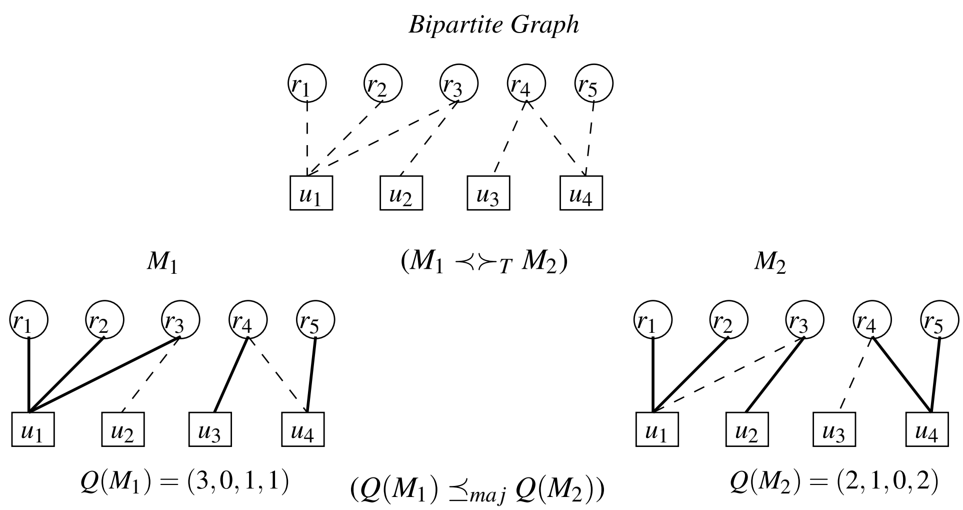



Figure 2: An example of a pair of semi-matching $M_1$ and $M_2$ which are comparable under Majorization, but are incomparable under Transfer-based Comparison, because any Transfer Sequence between $M_1$ and $M_2$ contains one $IT$ and one $DT$.

If $M_x \preceq_T M_y$, then $M_y$ can be derived by the application of a sequence of Improving or Neutral Transfers on $M_x$. Thus, it meets $Q(M_x) \preceq_{Maj} Q(M_y)$ due to the Transitivity of the preorder "$\preceq_{Maj}$". □

**Corollary 2.1.** *Any fairest semi-matching with regard to "$\preceq_T$" is the fairest with regard to "$\preceq_{Maj}$".*

**Corollary 2.2.** *Any fairest semi-matching with regard to "$\preceq_T$" is the fairest with regard to any fairness measure compatible with Majorization.*

### 2.4 The Existence of Universally Agreed Fairest Semi-matchings

In this section, we prove that the proset of semi-matchings under Transfer-based Comparison has the *greatest elements*. In other words, there exists some fairest semi-matchings which are comparable to all semi-matchings and are fairer under Transfer-based Comparison. This is done by showing there always exists a fairer semi-matching for any two incomparable ones. The proof is done in two steps. First, we show that if one semi-matching $M_x$ can be changed to another one $M_y$ via a Bitonic Transfer Sequence (essentially a Transfer Sequence where all $IT's$ are before all $DT's$), then a fairer semi-matching than both $M_x$ and $M_y$ can be (straightforwardly) found. Then, we show how to construct a Bitonic Transfer Sequence between any two incomparable semi-matchings by modifying an arbitrary non-Bitonic Transfer Sequence between them.

**Definition 2.10.** *(**Bitonic Transfer Sequence**) For a Transfer Sequence $(T_1, T_2, \cdots, T_{k-1})$ such that $M_y = T_{k-1} \circ \cdots \circ T_2 \circ T_1(M_x)$, if there exists $p \in [1, k-1]$, such that all $T_i$ (where $i \leq p$) are $IT$ (or $NT$) with at least one $T_i$ being $IT$, and all $T_j$ (where $p < j \leq k-1$) are $DT$ (or $NT$) with at least one $T_j$ being $DT$, then $(T_1, T_2, \cdots, T_{k-1})$ is called a Bitonic Transfer Sequence.*

**Lemma 2.3.** *For a pair of semi-matchings $M_x$ and $M_y$, if there exists a Bitonic Transfer Sequence $(T_1, T_2, \cdots, T_{k-1})$ such that $M_y = T_{k-1} \circ \cdots \circ T_2 \circ T_1(M_x)$, then there exists a semi-matching $M_k$ satisfying that $M_x \preceq_T M_k$ and $M_y \preceq_T M_k$.*

*Proof.* A Bitonic Transfer Sequence $(T_1, T_2, \cdots, T_{k-1})$ can be divided into two Sub-Sequences $T_{front} = (T_1, T_2, \cdots, T_j)$ and $T_{back} = (T_{j+1}, T_{j+2}, \cdots, T_{k-1})$, where each $T_i \in T_{front}$ is either $IT$ or $NT$, and each $T_j \in T_{back}$ is either $DT$ or $NT$.

Let $M_j$ be the semi-matching obtained by the application of $T_{front}$ on $M_x$. Then $M_j$ is not less fair than $M_x$ ($M_x \preceq_T M_k$). Obviously, $M_y$ can be derived by the application of $T_{back}$ on $M_j$, that is $M_y = T_{j+1} \circ T_{j+2} \cdots \circ T_{k-1}(M_j)$. Then it meets $M_y \preceq_T M_j$. $\square$

Note that, if one Transfer suppose $T_i$ is a Cyclic Neutral Transfer ($NT_C$), then if there exist at least one user covered by both $T_i$ and $T_j$, then it is regarded that $T_i$ and $T_j$ can be merged.

Note that, if one Transfer suppose $T_i$ is a Cyclic Neutral Transfer ($NT_C$), then if there exist at least one user covered by both $T_i$ and $T_j$, then it is regarded that $T_i$ and $T_j$ can be merged.

**Lemma 2.4.** *For a pair of incomparable semi-matchings $M_x$ and $M_y$ ($M_x \prec\succ_T M_y$), there always exists a Bitonic Transfer Sequence which changes $M_x$ to $M_y$.*

*Proof.* We start with an arbitrary Prime Transfer Sequence (which is defined in Definition 2.11 ) changing $M_x$ to $M_y$ ($M_x \prec\succ_T M_y$), . We will prove any such a Transfer Sequence can be re-organized to form a Bitonic Transfer Sequence which also changes $M_x$ to $M_y$.

**Definition 2.11.** ***(Prime Transfer Sequence)*** *Let $TS = (T_1, T_2, \cdots, T_{k-1})$ represent one Transfer-Sequence from one fairest semi-matching $M_1$ to another one $M_k$. $TS$ is defined as a Prime Transfer-Sequence, if $\forall T_i, T_j \in TS$, two conditions are met: 1) (Transfers can not be merged): $u_s(T_i) \neq u_d(T_j)$ and $u_s(T_j) \neq u_d(T_i)$; 2) (Transfers are non-overlapping): Let $\mathcal{E}(T_i)$ and $\mathcal{E}(T_j)$ represent the set of all edges covered by $T_i$ and $T_j$ respectively, then $\mathcal{E}(T_i) \bigcap \mathcal{E}(T_j) = \emptyset$.*

Obviously, for any two adjacent Transfers $T_a$ and $T_b$ in a Transfer Sequence (assume $T_a$ precedes $T_b$) such that $T_a(M_i) = M_a$, $T_b(M_a) = M_j$ (illustrated as $M_i \xrightarrow{T_a} M_a \xrightarrow{T_b} M_j$), swapping the order of $T_a$ and $T_b$ (such that $M_i \xrightarrow{T_b} M_b \xrightarrow{T_a} M_j$) will not change the beginning and ending semi-matchings $M_i$ and $M_j$, but will yield a different intermediate one $M_b$ rather than $M_a$. The following proves that if $T_b$ is Improving Transfer before the swapping (i.e., $T_b$ is $IT$ in $M_a \xrightarrow{T_b} M_j$), then $T_b$ remains to be Improving Transfer after the swapping (i.e., $T_b$ is $IT$ in $M_i \xrightarrow{T_b} M_b$).

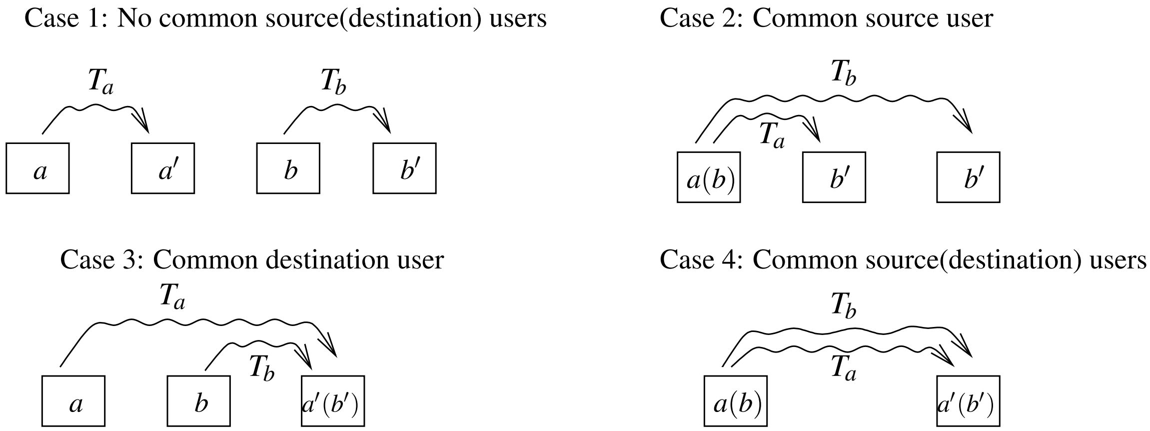



Figure 3: An illustration of four relationship types between two Transfers.

Suppose the source and destination user of $T_a$ are $a$ and $a'$ respectively, and the source and destination user of $T_b$ are $b$ and $b'$ respectively. If $T_b$ is an Improving Transfer before the swapping, then $Q(M_a,b) - Q(M_a,b') > 1$. The following shows it is also true that $Q(M_i,b) - Q(M_i,b') > 1$, by examining how the application of $T_a$ on $M_i$ changes the quotas of $b$ and $b'$. Since the Transfer Sequence guarantees $a \neq b'$ and $a' \neq b$ to begin with, then one of the following four cases (illustrated in Figure 3) should apply:

- *Case 1*: $T_a$ and $T_b$ have no common source user and no common destination user ($a \neq b$ and $a' \neq b'$). The application of $T_a$ on $M_i$ will not change the quotas of $b$ and $b'$, thus $Q(M_i,b) = Q(M_a,b)$, $Q(M_i,b') = Q(M_a,b')$. It indicates $Q(M_i,b) - Q(M_i,b') = Q(M_a,b) - Q(M_a,b') > 1$.
- *Case 2*: $T_a$ and $T_b$ have common source user but no common destination user ($a = b$ and $a' \neq b'$). The application of $T_a$ on $M_i$ will not change the quota of $b'$, but will decrease the quota of $b$

by 1, thus $Q(M_i,b) = Q(M_a,b)+1$, $Q(M_i,b') = Q(M_a,b')$. It indicates $Q(M_i,b)-Q(M_i,b') = Q(M_a,b)-Q(M_a,b')+1 > 1$.

- *Case 3*: $T_a$ and $T_b$ have no common source user but common destination user ($a \neq b$ and $a' = b'$). The application of $T_a$ on $M_i$ will not change the quota of $b$, but will increase the quota of $b'$ by 1, thus $Q(M_i,b) = Q(M_a,b)$, $Q(M_i,b') = Q(M_a,b')-1$. It indicates $Q(M_i,b)-Q(M_i,b') = Q(M_a,b)-Q(M_a,b')+1 > 1$.

- *Case 4*: $T_a$ and $T_b$ have common source user and common destination user ($a = b$ and $a' = b'$). The application of $T_a$ on $M_i$ will decrease the quota of $b$ by 1 and increase the quota of $b'$ by 1, thus $Q(M_i,b) = Q(M_a,b)+1$, $Q(M_i,b') = Q(M_a,b')-1$. It indicates $Q(M_i,b)-Q(M_i,b') = Q(M_a,b)-Q(M_a,b')+2 > 1$.

It can be concluded that $T_b$ will maintain being an $IT$ when swapped ahead. Besides, by similar deduction it can be shown that if $T_a$ is a $DT$ before the swap, then $T_a$ will maintain being a $DT$ when swapped behind; if $T_a$ is a $NT$ before the swap, then $T_a$ will maintain being a $NT$ or become to a $DT$ when swapped behind. Thus, the above-proven property of "$IT$ conservation when swapped ahead" ensures that it is possible to re-organize any Prime Transfer Sequence with a mixture of $IT's$ and $DT's$ (perhaps some $NT's$) into a Bitonic Transfer Sequence, by swapping all the $IT's$ to be ahead of all the $DT's$.

In fact, for any pair of semi-matchings, there might exist many Transfer Sequences between them. And there always exists at least one sequence containing of a Prime Transfer Sequence and a sequence of Cyclic Neutral Transfers, which changes one semi-matching to another. The construction process is shown in the Lemma 2.5. For any pair of incomparable semi-matchings $M_x$ and $M_y$, let $S_{Simp}$ and $S_{cyclic}$

represent the Prime Transfer Sequence and the sequence of Cyclic Neutral Transfers respectively, the concatenation of which changes $M_x$ to $M_y$. Obviously, $S_{Simp}$ contains both $IT's$ and $DT's$, and can be re-organized into a Bitonic Transfer Sequence. Then, adding $S_{cyclic}$ into the front or back of this Bitonic Transfer Sequence, results in yet another Bitonic Transfer Sequence changing $M_x$ to $M_y$. $\square$

**Lemma 2.5.** *For any pair of semi-matchings $M_x$ and $M_y$, there always exists a sequence containing of a Prime Transfer Sequence and a sequence of Cyclic Neutral Transfer, which changes $M_x$ to $M_y$.*

*Proof.* The notion $M_x \oplus M_y$ denotes the symmetric difference of edges set $M_x$ and $M_y$, that is, $M_x \oplus M_y = (M_x \setminus M_y) \cup (M_y \setminus M_x)$. Let $S$ represents the set of all Cyclic Neutral Transfers in $M_x \oplus M_y$. Suppose $M_{x'}$ can be derived by the application of all Cyclic Neutral Transfers in $S$ on $M_x$.

An illustration of the Transfers Construction is shown in Fig. 4. Let thin edges represent the edges of $M_{x'} \setminus M_y$, and thick edges represent the edges of $M_y \setminus M_{x'}$. An observation on $M_{x'} \oplus M_y$ is that there exist one or more users which are endpoints of only thin edges, but not thick edges. We call those users as *Starting Users*. We build a Transfer which is an alternating thin-thick sequence of edges, as follows. Starting with one Starting User and one thin edge, identity a path as long as possible, which includes alternative resources and users, and alternative thin edges and thick edges. Then delete all chosen edges from $M_{x'} \oplus M_y$, and then repeat above procedure to build more Transfers until $M_{x'} \oplus M_y$ becomes empty.

Throughout this process, we maintain that among all the obtained Transfers, the source user of one Transfer cannot be the destination user of another Transfer. Then, an arbitrarily ordered sequence of all the obtained Transfers constructs a Prime Transfer Sequence from $M_{x'}$ to $M_y$. $\square$

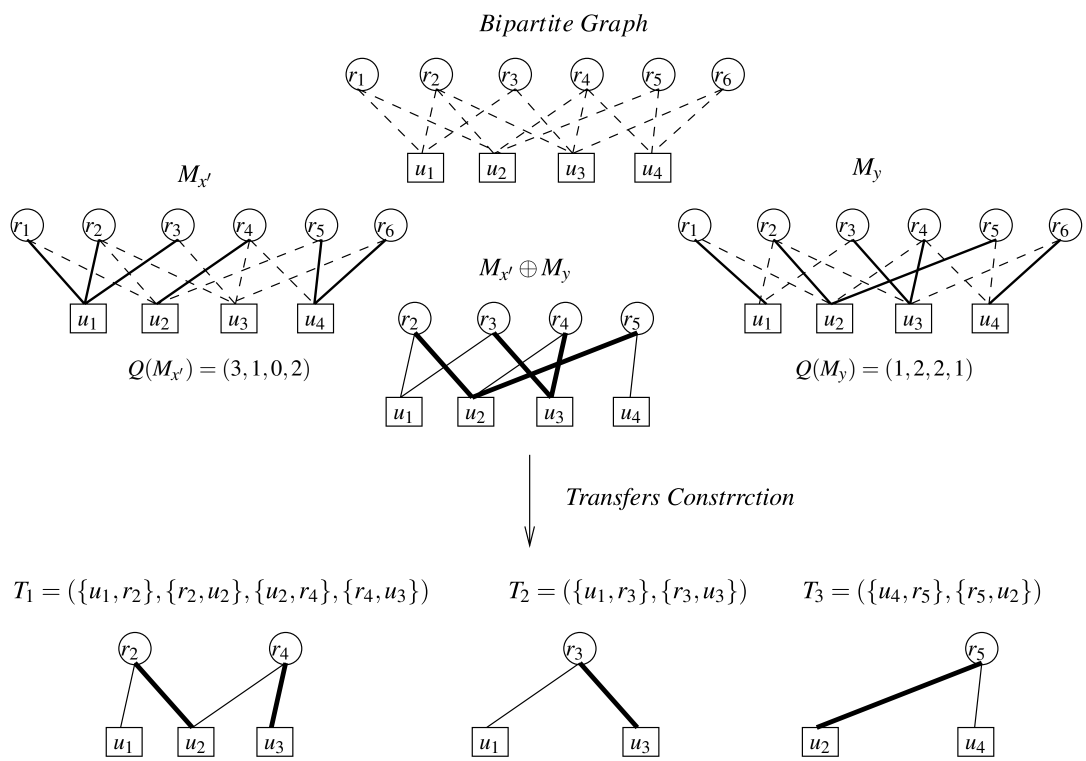



Figure 4: An illustration of Transfers Construction.

**Corollary 2.3.** *Let $proset_G$ represents a preordered set defined by "$\preceq_T$" which contains all semi-matchings of a bipartite Graph G. For any pair of semi-matchings $M_x, M_y \in proset_G$, if $M_x \prec\succ_T M_y$, then there exists at least one semi-matching $M_z \in proset_G$ satisfying that $M_x \preceq_T M_z$ and $M_y \preceq_T M_z$.*

From Corollary 2.3 and the properties of a proset, Theorem 2.1 can be derived.

**Theorem 2.1.** ***(Existence of a set of fairest semi-matchings with regard to Transfer-based Comparison for any bipartite graph)*** *For any bipartite graph G, let $proset_G$ represents the preordered set defined by "$\preceq_T$" which contains all semi-matchings of G. There always exists a set of equally fair semi-matchings that constitute the greatest elements in $proset_G$, and all semi-matchings in this set have the same sorted quotas vector.*

**Corollary 2.4.** *All greatest elements (semi-matchings) in $proset_G$ have the same sorted quotas vector.*

*Proof.* For any pair of greatest semi-matchings $M_x, M_y \in proset_G$, there always exists a Prime Transfer Sequence containing of only Neutral Transfers. That can be derived from the following claims. If there exists one Improving Transfer $T$, then $T$ can be swapped ahead to the beginning while maintaining the improving attribute, which indicates $M_x$ is not a greatest semi-matching. Similarity, if there exists one Deteriorating Transfer, then it contradicts with that $M_y$ is a greatest semi-matching. Furthermore, the application of a Neutral Transfer will not change the sorted quotas vector of a semi-matching. Therefore, all greatest semi-matchings have the same sorted quotas vector. □

Finally, the below theorem can be achieved.

**Theorem 2.2.** *For any bipartite graph, there always exists a set of equally fair semi-matchings (with the same sorted quotas vector), which are uniformly considered the fairest by all the fairness measures which are compatible with Majorization.*

## 2.5 Discussions

### A. The proset from Transfer-based Comparison might be a semi-lattice

The proset derived from Transfer-based comparison, is not proved to be a join-semilattice (defined as the proset which has a least upper bound for any nonempty finite subset (Davey and Priestley, 2002)). In other words, we proved that the proset itself has a greatest element which is the least upper bound, but this might not be true for all of its subsets. There might exist a counter example, such as a pair of semi-matchings $M_1$, $M_2$ having three common upper bounds $M_3$, $M_4$, $M_5$ which meet $M_3 \prec\succ_T M_4$ and $M_3 \prec_T M_5$, $M_4 \prec_T M_5$. That being said, we did not find such counter examples for any given bipartite graph setting. Consequently, whether the proset is a semi-lattice remains unknown.

### B. The existence of the least elements with regard to Transfer-based Comparison is not always true

This paper has proved the existence of the greatest elements in the preordered set with regard to Transfer-based Comparison. However, the least elements in this preordered set do not always exist. This means there does not exist a uniformly agreed upon "most unfair" semi-matching set among all the fairness measures. For example, in Fig. 5, the two worst semi-matchings $M_1$ and $M_2$ are incomparable with regard to Transfer-based Comparison. Fairness measures disagree on the comparison of their

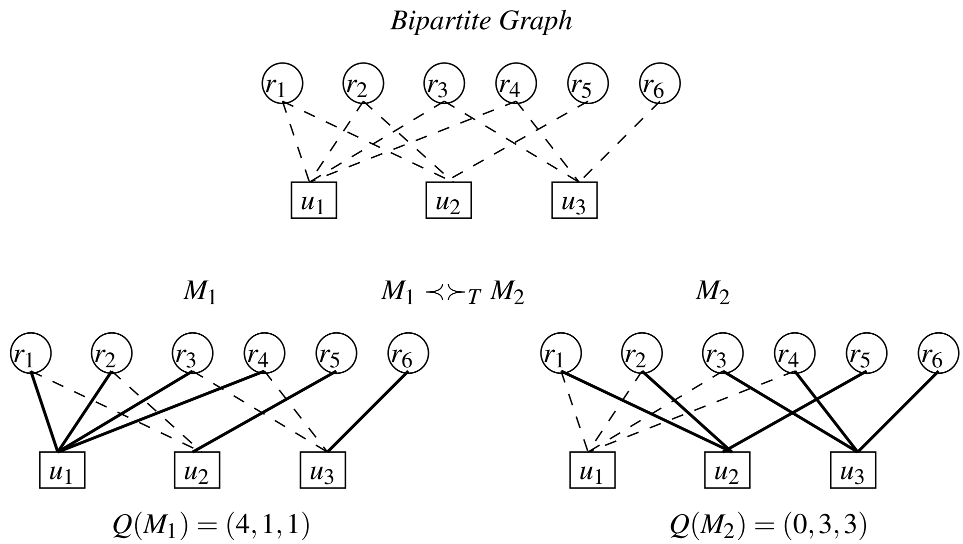



Figure 5: An example of two incomparable semi-matchings both as the most unfair ones regarding "$\preceq_T$".

quotas vectors $Q(M_1) = (4,1,1)$ and $Q(M_2) = (0,3,3)$. The details of conflicting comparisons between $Q(M_1)$ and $Q(M_2)$ have been shown in Table II.

The insight behind the existence of the greatest elements but not the least elements in this preordered set lies in the asymmetry of "swapping $IT$" ahead: we have proved that an $IT$ can be swapped ahead without losing its attribute of being an Improving Transfer, but this is not true for a $DT$. A Deteriorating Transfer, when swapped ahead, does not always maintain to be a $DT$. As a result, for any two incomparable semi-matchings $M_x$ and $M_y$, it is guaranteed that there always exists a fairer one $M_z$ that $M_x \preceq_T M_z$, $M_x \preceq_T M_z$, but not necessarily an "unfairer" one $M_{z'}$ that $M_{z'} \preceq_T M_x$, $M_{z'} \preceq_T M_y$.

**C. The algorithms for achieving one fairest semi-matching**

We shows that one fairest semi-matching under Transfer-based Comparison is also the fairest one under all known existed fairness measures shown in Table I. Furthermore, it implies that the fairest semi-matching(s) under any fairness measure, such as extensively studied lexicographically order, also dominate other semi-matchings under the preorder Transfer based Comparison and Majorization. Thus, it is true that any algorithm achieving some fairest semi-matchings under any fairness measure is in fact achieving a fairest semi-matching under all fairness measures listed in Table I.

(Kleinberg et al., 1999) proposed an algorithm with respect to lexicographically order to achieve one fairest semi-matching. Specially, they studied on the load balancing problem which is concerned with assigning uniform jobs $J$ to machine nodes $N$. For each job $J_i$, there is a set $S_i \subseteq N$ on which job $J_i$ can run. The aim is to assign each job $J_i$ to a machine in $S_i$ in a way that the assignment has the fairest loads among all machines. An iterative network flow algorithm was proposed by (Kleinberg et al., 1999) to achieve an assignment of jobs to machines, which is the fairest one under lexicographically order. This algorithm builds the assignment step by step starting with an empty assignment. In each step, it applies a max-flow algorithm to find a "partial" assignment, until eventually a legitimate assignment is reached which will turn out to be the fairest under lexicographically order. Actually, the claim that the achieved fairest assignment also dominates other assignments under all other fairness measures, can be derived from the conclusion achieved in this paper.

For the job assignment system described in (Kleinberg et al., 1999), (Harvey et al., 2006) also proposed an algorithm to achieve one fairest assignment under both minimal make-span and minimal flow time. Its essential idea is to iteratively improve an arbitrary assignment by executing one existed "cost-reducing" path until no more found. In fact, the cost-reducing path is the same with the Improving

Transfer proposed in this paper. Thus, the achieved fairest assignment in (Harvey et al., 2006) is the fairest one under Transfer-based Comparison, as well as the fairest one under all fairness measures shown in Table I. In addition, (Harvey et al., 2006) shows the runtime upper bound of the algorithm is $O(min\{|R|^{3/2},|R||U|\}\cdot|E|)$ for a bipartite graph $G=(R\cup U,E)$.

Based on our conclusion and (Harvey et al., 2006), an online algorithm can be easily derived to maintain an on-line job assignment system to be fairest. Assume initially a job assignment system is the fairest, and new jobs are coming to this system sequentially, which needs to be assigned to some machines. For each incoming job $J_i$, it will first be assigned to the least loaded machine node $N^*$ which can run job $J_i$. Subsequently, a search is conducted for an Improving Transfer $T$, starting from $N^*$ as a source vertex. We claim that, if $T$ is not found, the system is then already the fairest; if $T$ is found, then by the application of $T$ which is a chain of jobs re-assignments to the system, the system can be successfully updated to be a fairest one with the new job.

## 2.6 Conclusion

We prove that there always exist the universally agreed fairest semi-matchings in any given bipartite graph. To prove this, we define a preordered set (proset) of semi-matchings based on the way to transfer one into another. This proset (Transfer-based Comparison) is shown to always have the greatest elements, as the fairest semi-matchings. Subsequently, we show that the proposed Transfer-based Comparison can strictly imply the Majorization order. In other words, the fairest semi-matchings under the proposed Transfer-based Comparison are always regarded as fairest under Majorization. To our best knowledge, all existing fairness measures in the literature are compatible with Majorization. In conclusion, for any bipartite graph, there always exists a set of equally fair semi-matchings, which are universally regarded as the fairest ones, by all existing fairness measures, even though they may disagree on the comparisons among the ones that are not the fairest. As a result, a number of previously proposed algorithms which claimed to achieve the fairest semi-matchings under some specific fairness measure, in fact achieves the universally fairest ones for all the known and listed fairness measures.

# CHAPTER 3

# UNDERSTANDING THE ATTRIBUTES OF ALL THE FAIREST SEMI-MATCHINGS FROM AN ARBITRARY ONE

## 3.1 Introduction

In a bipartite graph $G = (R \cup U, E)$ with two sets of vertices ($R$ for resources and $U$ for users), an allocating of the resources to the users along the edges $E$ can be formally represented as a semi-matching $M \subseteq E$ (where each vertex in $R$ is incident with exactly one edge in $M$). The fairest semi-matchings on $G$, representing the fairest allocating of the resources to the users, are the main focus of this chapter. It has been shown that, for any given bipartite graph, there always exists a set of fairest semi-matchings which are considered equally fair by all the fairness measures. While achieving one of the fairest semi-matchings is easy, it is not obvious how all of them are related, or what they have in common. This chapter concludes that a lot can be learned about the entire set of fairest semi-matchings (which are usually hard to enumerate) from an arbitrary one (which is easy to achieve). We shows that, besides their common sorted quotas vector, one can draw further conclusions about all the members in this set of fairest semi-matching ($\mathcal{F}$). Specifically, by analyzing an arbitrary $M_i \in \mathcal{F}$, we can: 1) learn about the usage of edges for the bipartite graph $G$ of all the $M_j \in \mathcal{F}$; and 2) learn about the quota of each user $Q(M_j, u_k)$ where $M_j \in \mathcal{F}, u_k \in U$.

### 3.1.1 Motivational Example

An example is given with bipartite graph $G$ containing 7 users and 14 resources, with its entire set of four fairest semi-matchings $\mathcal{F} = \{M_1, M_2, M_3, M_4\}$ in Fig 6. In this example, with regard to the usage of edges among the four fairest semi-matchings, we can observe that:

- some edges, denoted by the solid lines in Fig 7.a, are used by all four fairest semi-matchings in $\mathcal{F}$: such as $\{r_1, u_1\}$, $\{r_8, u_3\}$;
- some edges, denoted by the dashed lines in Fig 7.a, are used by none of the fairest semi-matchings in $\mathcal{F}$: such as $\{r_8, u_2\}$, $\{r_{10}, u_3\}$;
- the rest of the edges, denoted by the dotted lines in Fig 7.a, are used by some (but not all) of the semi-matchings in $\mathcal{F}$: such as $\{r_2, u_1\}$, $\{r2, u_2\}$.

Moreover, among the four fairest semi-matchings in $\mathcal{F}$, one can observe that a user's final quota is largely stable: some users (such as $u_3$) always have the same quota of 2, for all $M_1, M_2, M_3$, and $M_4$, while other users have different quotas but only differ by one (such as $Q(M_1, u_7) = Q(M_2, u_7) = 1$, and $Q(M_3, u_7) = Q(M_3, u_7) = 2$). It turns out that a partition of the users and resources can be made, as is illustrated in Fig 7.b, such that all the users in the same partition will always have the same fairest quotas range (with the difference of at most 1), and the resources are always assigned to the users within same partition. Such a partition of users and resources is invariant among all the possible fairest semi-matchings in $\mathcal{F}$, for any given bipartite graph $G$.

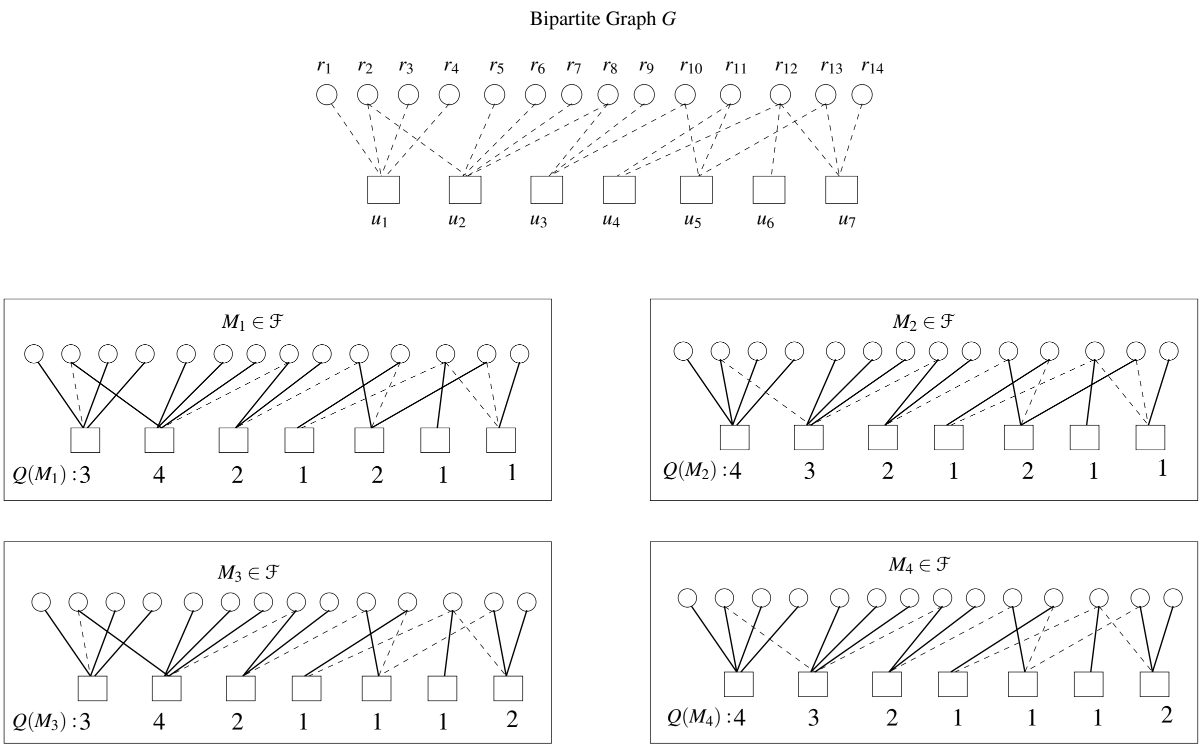
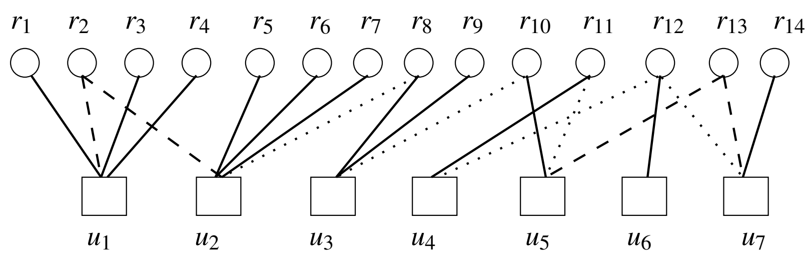



Figure 6: An example bipartite graph with $\mathcal{F} = \{M_1, M_2, M_3, M_4\}$ as illustrated, and $Q^{\uparrow}(M_i) = (1, 1, 1, 2, 2, 3, 4)$.




a) Edge classification into the edges used by all/some/none of the set $\mathcal{F}$

| | | | |
|---|---|---|---|
| $r_{11}$ $r_{12}$ | $r_{10}$ $r_{13}$ $r_{14}$ | $r_8$ $r_9$ | $r_1$ $r_2$ $r_3$ $r_4$ $r_5$ $r_6$ $r_7$ |
| $u_4$ $u_6$ | $u_5$ $u_7$ | $u_3$ | $u_1$ $u_2$ |
| $Q(M_i, u_j) = 1$ | $Q(M_i, u_j) = 1$ or 2 | $Q(M_i, u_j) = 2$ | $Q(M_i, u_j) = 3$ or 4 |

b) The stable partition of users and resources among all the set $\mathcal{F}$

Figure 7: a) Edge Classification example: solid edges are used by all $M \in \mathcal{F}$; dotted edges are never used in any $M \in \mathcal{F}$; dashed edges are used by some but not all $M \in \mathcal{F}$ b) The stable partition of users and resources among all the set $\mathcal{F}$. All users in each partition have the same fairest quota range, and each partition has one quota option or two quota options which differ by 1. The resources of each partition are always assigned to the users within same partition among all the set $\mathcal{F}$.

### 3.1.2 Main Results

The main conclusions achieved in this paper are: given a bipartite graph, from one arbitrary fairest semi-matching (which is easy to achieve), we can understand some important attributes for the entire set of fairest semi-matchings: 1) the classification of the edges in the bipartite graph - whether each edge is used by all, none, or some of the fairest semi-matchings; 2) the partition of user and resource vertices in the bipartite graph - the allocating of all the fairest semi-matchings are all within the partitions, and each user vertex has a very narrow quota range (at most differ by 1, and is predictable from the knowledge gained from one fairest semi-matching) among all the fairest semi-matchings.

### 3.1.3 Organization

Some preliminary definitions and conclusions are shown in Section 3.2. Then, section 3.3 presents the conclusions on the edge classification in the bipartite graph. Section 3.4 presents the user quota stability among all the fairest semi-matching. After that, the property of the paths between any pair of fairest semi-matchings is shown in Section 3.5, which provides the insight for the main conclusions.

## 3.2 Preliminary

This section shows some definitions and conclusions from Chapter 2, which will be used by this chapter.

### 3.2.1 Definitions

**Definition 3.1.** ***(Semi-Matching)*** *In a bipartite graph $G = (R \cup U, E)$ where $E \subseteq R \times U$, A semi-matching $M \subseteq E$ on $G$ is defined as $M = \{\{e_{ij}\} | \forall r_i \in R, \exists\, e_{ij} = \{r_i, u_j\} \in M, \text{while } \forall k \neq j, e_{ik} = \{r_i, u_k\} \notin M\}$.*

In general, valid semi-matchings can be easily obtained by matching each vertex $r_i \in R$ with an arbitrary vertex $u_j \in U$ as long as $\{r_i, u_j\} \in E$.

**Definition 3.2.** ***(Quotas Vector)*** *Given a semi-matching $M$, the quota of vertex $u \in U$ is defined as the number of edges that cover $u$ (or the number of resource vertices matched with $u$) according to $M$. The corresponding quotas vector of $M$, denoted as $Q(M)$, is the vector of which each element $Q(M, u)$ represents the quota of a vertex $u \in U$.*

**Definition 3.3.** *(**Transfer**) Given a bipartite graph $G = (R \cup U, E)$, $T = (\{u_1, r_1\}, \{r_1, u_2\}, \{u_2, r_2\}, \cdots, \{u_{k-1}, r_{k-1}\}, \{r_{k-1}, u_k\})$ is defined as a Transfer on a semi-matching $M$, if $T$ is an alternating path with regard to $M$, that is, for each $i \in [1, k-1]$, $\{u_i, r_i\} \in M$ (thus all the $\{r_i, u_{i+1}\} \notin M$ according to the definition of semi-matching).*

**Definition 3.4.** *(**Source and destination users of a Transfer**) For a Transfer $T = (\{u_1, r_1\}, \cdots, \{r_{k-1}, u_k\})$ on bipartite graph $G = (R \cup U, E)$, $u_1$ and $u_k$ are defined as the source and destination users of Transfer $T$, denoted as $u_s(T)$ and $u_d(T)$ respectively.*

**Definition 3.5.** *(**Application of a Transfer**) The application of Transfer $T$ on semi-matching $M$, denoted as $T(M) = M'$, is defined as switching the matching and non-matching edges in $M$ along Transfer $T$. The result of the application of $T$ on $M$ will change $M$ to a different semi-matching $M'$, which includes all the $\{r_i, u_{i+1}\}$ edges in $T$, but excludes all the $\{u_i, r_i\}$ edges in $T$.*

**Definition 3.6.** *(**Transfer Types: Improving, Deteriorating, Neutral**) Let $T$ represents a Transfer on a semi-matching $M$, with $Q(M, u_s(T))$ and $Q(M, u_d(T))$ representing the quotas of the source user $u_s(T)$ and destination user $u_d(T)$ in $M$ respectively. Apparently, when $u_s(T) \neq u_d(T)$, $T(M) = M'$*

*results in the change of quota in only two user:* $Q(M', u_s(T)) = Q(M, u_s(T)) - 1$, *and* $Q(M', u_d(T)) = Q(M, u_d(T)) + 1$. *In other words, the source user's quota decreases by 1, and the destination user's quota increases by 1 as the result of applying* $T$. *In the special case of* $u_s(T) = u_d(T)$, $M$ *and* $M'$ *has the same quotas vector.*

*As a result, the application of* $T$ *on* $M$ *will affect the fairness of* $M$ *in only three possible ways:*

- $T$ *could result in a fairer* $M'$ *over* $M$, *thus* $T$ *is denoted as an Improving Transfer (IT) if* $Q(M, u_s(T)) > Q(M, u_d(T)) + 1$;
- $T$ *could result in a more unfair* $M'$ *over* $M$, *thus* $T$ *is denoted as a Deteriorating Transfer (DT) if* $Q(M, u_s(T)) < Q(M, u_d(T)) + 1$ *and* $u_s(T) \neq u_d(T)$;
- $T$ *could result in* $M'$ *with the same fairness as* $M$, *thus* $T$ *is denoted as a Neutral Transfer (NT), where two cases exist:*
  - $T$ *is denoted as a Straight Neutral Transfer* ($NT_S$) *if* $Q(M, u_s(T)) = Q(M, u_d(T)) + 1$ *and obviously* $u_s(T) \neq u_d(T)$;
  - $T$ *is denoted as a Cyclic Neutral Transfer* ($NT_C$) *if* $u_s(T) = u_d(T)$.

*Obviously, if a transfer* $T$ *is* $NT$, $M$ *and* $T(M)$ *have the same sorted quotas vector.*

**Definition 3.7.** ***(Transfer Sequence)*** *A Transfer Sequence between two semi-matchings* $M_1$ *and* $M_k$, *denoted as* $M_k = T_{k-1} \circ \cdots \circ T_2 \circ T_1(M_1)$, *is defined as a sequence of Transfers* $(T_1, T_2, \cdots, T_{k-1})$ *such that* $\forall i \in [1, k-1], T_i(M_i) = M_{i+1}$ *holds.*

**Definition 3.8.** ***(Prime Transfer Sequence)*** *Let* $TS = (T_1, T_2, \cdots, T_{k-1})$ *represent one Transfer-Sequence from one fairest semi-matching* $M_1$ *to another one* $M_k$. $TS$ *is defined as a Prime Transfer-Sequence, if*

*$\forall T_i, T_j \in TS$, two conditions are met: 1) (Transfers can not be merged): $u_s(T_i) \neq u_d(T_j)$ and $u_s(T_j) \neq u_d(T_i)$; 2) (Transfers are non-overlapping): any pair of Transfers $T_i, T_j \in TS$ do not have any common edge.*

Note that, if one Transfer suppose $T_i$ is a Cyclic Neutral Transfer ($NT_C$), then if there exist at least one user covered by both $T_i$ and $T_j$, then it is regarded that $T_i$ and $T_j$ can be merged.

### 3.2.2 Existence and Attributes of The Fairest Semi-matching Set $\mathcal{F}$

Though the disagreement among numerous fairness measures on semi-matchings comparison, (Xu et al., 2017) proved the existence of the set of *universally agreed* fairest semi-matchings, and all these fairest semi-matchings have the same sorted quotas vector.

**Theorem 3.1.** ***(The existence of a fairest set of semi-matchings for any bipartite graph)*** *(Xu et al., 2017) For any bipartite graph, there always exist a set of the fairest semi-matchings denoted as $\mathcal{F}$ which are universally agreed to be equal in fairness by all fairness measures. Furthermore, a semi-matching belongs to the fairest set iff there exists no Improving Transfer to be applied.*

**Corollary 3.1.** ***(The set of all the fairest semi-matchings have the same sorted quotas vector)*** *(Xu et al., 2017) For any bipartite graph, the semi-matchings in the fairest set $\mathcal{F}$ might have different quotas vector, but they all have the same sorted quotas vector, that is, $\forall M_i, M_j \in \mathcal{F}, Q^{\uparrow}(M_i) = Q^{\uparrow}(M_j)$.*

## 3.3 Bipartite Graph Edge Classification Among Fairest Semi-matchings

### 3.3.1 Definitions

**Definition 3.9.** ***(Edge Classification Sets)*** *Let $\mathcal{F}$ represent the set of all the fairest semi-matchings on a bipartite graph $G = (R \cup U, E)$. $\forall e \in E$, it belongs to one of the three mutually exclusive sets:*

- $\mathcal{E}_{all} = \bigcap_{M_i \in \mathcal{F}} M_i$ *(used by all);*
- $\mathcal{E}_{none} = \overline{\bigcup_{M_i \in \mathcal{F}} M_i}$ *(used by none);*
- $\mathcal{E}_{some} = \bigcup_{M_i \in \mathcal{F}} M_i - \bigcap_{M_i \in \mathcal{F}} M_i$ *(used by some).*

**Definition 3.10.** ***(Covered Edge Set of a Transfer)*** *The Covered Edge Set of a Transfer* $T = (\{u_1, r_1\}, \{r_1, u_2\}, \{u_2, r_2\}, \cdots, \{u_{k-1}, r_{k-1}\}, \{r_{k-1}, u_k\})$ *is defined as* $\mathcal{E}(T) = \{\{u_1, r_1\}, \{r_1, u_2\}, \{u_2, r_2\}, \cdots, \{u_{k-1}, r_{k-1}\}, \{r_{k-1}, u_k\}\}$.

**Definition 3.11.** ***(Neutral Transfers Covered Edge Set)*** *A Neutral Transfers Covered Edge Set on a semi-matching $M$, denoted as $\mathcal{E}_{NT}(M)$, is defined as the union of Covered Edges Sets of all Neutral Transfer under $M$. That is, $\mathcal{E}_{NT}(M) = \cup_{all\ NT_i\ under\ M} \mathcal{E}(NT_i)$, where $NT_i$ represents one Neutral Transfer under $M$.*

### 3.3.2 Edge Classification Can Be Derived From One Fairest Semi-matchings

**Theorem 3.2.** ***(The Neutral Transfers Covered Edge Set $\mathcal{E}_{NT}(M)$ is constant among all the fairest semi-matchings $M \in \mathcal{F}$, and is furthermore equal to $\mathcal{E}_{some}$)*** *Under any given bipartite graph $G = (R \cup U, E)$, it always has $\mathcal{E}_{some} = \mathcal{E}_{NT}(M_i) = \mathcal{E}_{NT}(M_j)$ where $\forall M_i, M_j \in \mathcal{F}$.*

*Proof.* Let $M_k \in \mathcal{F}$ represents one arbitrarily given fairest semi-matching under a bipartite graph $G = (R \cup U, E)$, then from Lemma 3.1 it has $\mathcal{E}_{some} = \bigcup_{all\ M_i \in \mathcal{F}} M_k \oplus M_i$, where $M_k \oplus M_i = (M_k \setminus M_i) \cup (M_i \setminus M_k)$. Then $\bigcup_{all\ M_i \in \mathcal{F}} M_k \oplus M_i = \mathcal{E}_{NT}(M_k)$ can be proved if the below two arguments are true.

1) $\bigcup_{all\ M_i \in \mathcal{F}} M_k \oplus M_i \subseteq \mathcal{E}_{NT}(M_k)$:

We prove it by showing $\forall M_i \in \mathcal{F}$ it has $M_k \oplus M_i \subseteq \mathcal{E}_{NT}(M_k)$. Let $TS$ represents an arbitrary Prime

Transfer-Sequence from $M_k$ to $M_i$. Then $\forall e \in M_k \oplus M_i$, there exists one Transfer $T \in TS$, s.t. $e \in \mathcal{E}(T)$. From Lemma 3.5, it has that Transfer $T$ must be a Neutral Transfer on $M_k$ which indicates $e \in \mathcal{E}_{NT}(M_k)$. Therefore, $\forall e \in M_k \oplus M_i$, it has $e \in \mathcal{E}_{NT}(M_k)$, in other words, $M_k \oplus M_i \subseteq \mathcal{E}_{NT}(M_k)$.

2) $\mathcal{E}_{NT}(M_k) \subseteq \bigcup_{all\ M_i \in \mathcal{F}} M_k \oplus M_i$:

$\forall e \in \mathcal{E}_{NT}(M_k)$, there exists a Neutral Transfer $T$ on $M_k$, such that $e \in \mathcal{E}(T)$. By the application of $T$ on $M_k$, another fairest semi-matching, to say $M_{k'}$ will be obtained. Then, it is obvious that $e \in M_k \oplus M_{k'} \subseteq \bigcup_{all\ M_i \in \mathcal{F}} M_k \oplus M_i$. That indicates $\mathcal{E}_{NT}(M_k) \subseteq \bigcup_{all\ M_i \in \mathcal{F}} M_k \oplus M_i$.

Therefore, it is true that $\mathcal{E}_{some} = \mathcal{E}_{NT}(M_k)$. Moreover, because $M_k$ is an arbitrary given fairest semi-matching, then straight-forwardly it can be deduced that $\mathcal{E}_{some} = \mathcal{E}_{NT}(M_i) = \mathcal{E}_{NT}(M_j)$ where $M_i, M_j \in \mathcal{F}$. $\square$

**Lemma 3.1.** *($\mathcal{E}_{some} = \bigcup_{all\ M_i \in \mathcal{F}} M_k \oplus M_i$, $\forall M_k \in \mathcal{F}$) Under a bipartite graph $G = (R \cup U, E)$, let $M_k \in \mathcal{F}$ represents one arbitrarily given fairest semi-matching. Then it has $\mathcal{E}_{some} = \bigcup_{all\ M_i \in \mathcal{F}} M_k \oplus M_i$ where $M_k \oplus M_i = (M_k \setminus M_i) \cup (M_i \setminus M_k)$.*

*Proof.* From the definition it has $\mathcal{E}_{some} = \bigcup_{\text{all } M_i, M_j \in \mathcal{F}} M_j \oplus M_i$, where $M_j \oplus M_i = (M_j \setminus M_i) \cup (M_i \setminus M_j)$. Then for an arbitrarily given fairest semi-matching $M_k \in \mathcal{F}$, let $\mathcal{E}'_{some} = \bigcup_{\text{all } M_i \in \mathcal{F}} M_k \oplus M_i$. Obviously, $\mathcal{E}'_{some} \subseteq \mathcal{E}_{some}$, the following will prove $\mathcal{E}_{some} \subseteq \mathcal{E}'_{some}$, then it has $\mathcal{E}_{some} = \mathcal{E}'_{some}$.

Assume there exists one edge $e \in M_i \oplus M_{i'} \subseteq \mathcal{E}_{some}$ and $e \notin \mathcal{E}'_{some}$. That indicates either $M_i$ or $M_{i'}$ covers edge $e$. Suppose $e \in M_i$ and $e \notin M_{i'}$, then it can be obtained that if $M_k$ does not cover $e$, then $e \in M_k \oplus M_i$; if $M_k$ covers $e$, then $e \in M_k \oplus M_{i'}$, thus $e \in \mathcal{E}'_{some}$, which is a contradictory.

$\square$

**Corollary 3.2.** ***(Edge classification among $\mathcal{E}_{all}$, $\mathcal{E}_{none}$, and $\mathcal{E}_{some}$ can be performed efficiently from an arbitrary $M \in \mathcal{F}$ )*** *Let $M \in \mathcal{F}$ represents one arbitrary fairest semi-matching on a bipartite graph $G = (R \cup U, E)$. The set $\mathcal{E}_{some}$ can be identified by the finding of $\mathcal{E}_{NT}(M)$. Then, for any remaining edge $e \notin \mathcal{E}_{some}$, if $e \in M$, then $e \in \mathcal{E}_{all}$, otherwise, $e \in \mathcal{E}_{none}$.*

## 3.4 User Quota Stability Among All The Fairest Semi-matchings

### 3.4.1 Definitions

**Definition 3.12.** ***(Users Partition)*** *Given a fairest semi-matching $M \in \mathcal{F}$, a sequence of ranked user sets $\mathscr{P}_U(M) = (\mathcal{U}^0(M), \mathcal{U}^1(M), \ldots, \mathcal{U}^i(M), \ldots, \mathcal{U}^{2m}(M))$, where each $\mathcal{U}^i(M)$ is a set of user and $m$ is the number of resources, is defined as: if a user $u$ is covered by a Straight Neutral Transfer $T$, then $u \in \mathcal{U}^i(M)$ where $i = Q(M, u_s(T)) + Q(M, u_d(T))$; otherwise, if $u$ is not covered by any Straight Neutral Transfer, then $u \in \mathcal{U}^i(M)$ where $i = 2 \times Q(M, u)$.*

Obviously, for any user $u$, there exists a set which covers $u$. And it can be shown that for any two sets $\mathcal{U}^i(M)$ and $\mathcal{U}^j(M)$ where $i \neq j$ it has $\mathcal{U}^i(M) \cap \mathcal{U}^j(M) = \emptyset$. If at least one of $i$ and $j$ is even, then from the definition it has $\mathcal{U}^i(M) \cap \mathcal{U}^j(M) = \emptyset$. If both $i$ and $j$ are odd, $\mathcal{U}^i(M) \cap \mathcal{U}^j(M) = \emptyset$ can be proved based on the following claim. One user can not be simultaneously covered by two Straight Neutral Transfers $NT_S(q, q-1)$ and $NT_S(q', q'-1)$ where $q \neq q'$, otherwise, one Improving Transfer can be formed on the fairest semi-matching $M \in \mathcal{F}$ contradicts with Theorem 3.1.

### 3.4.2 A User's Quota Range Is Narrow in The Set $\mathcal{F}$ and Predictable from Any $M \in \mathcal{F}$

**Theorem 3.3.** ***(The users partition is constant among all the fairest semi-matchings)*** *Let $M_x, M_y \in \mathcal{F}$ represent two arbitrary fairest semi-matchings in a bipartite graph $G = (R \cup U, E)$, then for any $i \in [0, 2m], \mathcal{U}^i(M_x) = \mathcal{U}^i(M_y)$. In other words, we can say $\mathscr{P}_U(M_x) = \mathscr{P}_U(M_y)$.*

*Proof.* The theorem can be derived from Lemma 3.2 and Lemma 3.3. Suppose there exists a user $u$ such that $u \in \mathcal{U}^i(M_x)$ and $u \in \mathcal{U}^j(M_y)$ where $i \neq j$. Then both $i$ and $j$ must be odd otherwise it should has $i = j$, which can be derived from Lemma 3.3 and the fact that the sets of a partition are pairwise disjoint. Furthermore, from Lemma 3.2 it should has $i = j$. A contradictory is then obtained. □

**Lemma 3.2.** *$\forall u \in U$, suppose both $\mathcal{U}^i(M_x)$ and $\mathcal{U}^j(M_y)$ cover $u$ where $i \neq j$, then $i$ and $j$ can not be both odd.*

*Proof.* Suppose both $i$ and $j$ are odd, and $u$ is covered in Straight Neutral Transfer $NT_S(q_s, q_d)$ and $NT_S(q'_s, q'_d)$ under $M_x$ and $M_y$ respectively. Then there exist two fairest semi-matchings $M_{x'}, M_{y'} \in \mathcal{F}$, such that $|Q(M_{x'}, u) - Q(M_{y'}, u)| \geq 2$. This contradicts with Lemma 3.4 that $|Q(M_{x'}, u) - Q(M_{y'}, u)| \leq 1$. The finding of $M_{x'}$ and $M_{y'}$ is based on the below two claims: 1) There exists another fairest semi-matching $M_{x'}$ such that the quotas of $u$ under $M_x$ and $M_{x'}$ are $q_s$ and $q_d$ (or "$q_d$ and $q_s$") respectively. That is because $Q(M_x, u)$ is either $q_s$ or $q_d$ (otherwise one Improving Transfer can be formed), then if $Q(M_x, u) = q_s$, another fairest semi-matching $M_{x'}$ with $Q(M_{x'}, u) = q_d$ can be obtained by application of one Straight Neutral Transfer $NT'_S(q_s, q_d)$ on $M_x$, where $NT'_S(q_s, q_d)$ is a sub-sequence of $NT_S(q_s, q_d)$ which starts with $u$ as the the source user. Similarity, if $Q(M_x, u) = q_d$, another fairest semi-matching

$M_{x'}$ with $Q(M_{x'}, u) = q_s$ can be obtained. And this claim can also be applied on $M_y$; 2) If $q'_s > q_s$, then it has $|q'_s - q_d| \geq 2$; else if $q_s > q'_s$, then it has $|q_s - q'_d| \geq 2$. $\square$

**Lemma 3.3.** *$\forall u \in U$, if $\mathcal{U}^i(M_x)$ covers $u$ where $i$ is even, then $\mathcal{U}^i(M_y)$ also covers $u$.*

*Proof.* $i$ is even, indicates that under $M_x$ either $u$ is covered by a Cyclic Neutral Transfer and not covered by all Straight Neutral Transfer, or $u$ is not covered by any Neutral Transfer. Then this lemma can be derived from the below two claims.

**Claim 1.** *If under $M_x$, $u$ is covered by a Cyclic Neutral Transfer and not covered by any Straight Neutral Transfer, then under $M_y$, $u$ can not be covered by any Straight Neutral Transfer.*

Suppose under $M_y$, $u$ is covered by one Straight Neutral Transfer $NT_S(q'_s, q'_d)$. Based on the proof of Lemma 3.2, there exists another fairest semi-matching $M_{y'}$ such that the quotas of $u$ under $M_y$ and $M_{y'}$ are $q'_s$ and $q'_d$ (or "$q'_d$ and $q'_s$") respectively. Thus, under at least one of $M_y$ and $M_{y'}$, the quota of $u$ is not equal to $Q(M_y, u)$. In a word, there exists $M_x$ and $M_y$ (or $M_{y'}$) such that $Q(M_x, u) \neq Q(M_y, u)$ (or $Q(M_x, u) \neq Q(M_{y'}, u)$). Then $u$ must be covered in at least one Straight Neutral Transfer under $M_x$, otherwise, $u$ should have the same quota among all fairest semi-matchings (which can be derived from Lemma 3.5). Thus a contradictory is obtained.

**Claim 2.** *If under $M_x$, $u$ is not covered by any Neutral Transfer, then under $M_y$, $u$ is also not covered by any Neutral Transfer.*

From Theorem 3.2, this claim can be easily derived.

$\square$

**Lemma 3.4.** ***(A user's quota is either constant or differs by at most 1 across all the fairest semi-matchings)*** *Let $M_x, M_y \in \mathcal{F}$ represent two arbitrary fairest semi-matchings on a bipartite graph $G = (R \cup U, E)$. $\forall u \in U$, it has $|Q(M_x,u) - Q(M_y,u)| \leq 1$.*

*Proof.* From Lemma 3.5, in any Prime Transfer-Sequence $TS$ from $M_x$ to $M_y$, any pair of Transfers are disjoint in both source and destination users. In addition, the application of a Transfer $T$ only changes the quotas of source and destination users by one (the quota of source user decreases by one, and the quota of destination user increases by one) while all other users covered by $T$ do not change. Therefore, $\forall u$, if $u$ is covered by one Transfer $T \in TS$, and $u$ is the source user of $T$, then $Q(M_x,u) - Q(M_y,u) = 1$; if $u$ is covered by one Transfer $T \in TS$, and $u$ is the destination user of $T$, then $Q(M_x,u) - Q(M_y,u) = -1$; else $u$ is not covered by another Transfers in $TS$, then $Q(M_x,u) - Q(M_y,u) = 0$. $\square$

**Corollary 3.3.** ***(The quota variation of any user among all the fairest semi-matchings can be predicted from one arbitrary fairest semi-matching)*** *Let $M$ represents one arbitrary fairest semi-matching in a bipartite graph $G = (R \cup U, E)$. $\forall u \in U$, assume $u \in \mathcal{U}^i(M)$, if $i$ is odd, its quota from any fairest semi-matchings is either $(i+1)/2$ or $(i-1)/2$; if $i$ is even, then its quota from any fairest semi-matchings is exactly $i/2$.*

Furthermore, it can be derived that each set of users always be assigned with the same set of resources among all the possible fairest semi-matchings, which is formally shown in the below Theorem.

**Theorem 3.4.** ***(A constant partition of users and resources across the set $\mathcal{F}$)*** *Given a fairest semi-matching $M \in \mathcal{F}$, a sequence of ranked resource sets $(\mathcal{R}^0(M), \mathcal{R}^1(M), \cdots \mathcal{R}^i(M), \cdots, \mathcal{R}^{2m}(M))$ where $m$ is the number of resources, is defined as: if under $M$ a resource $r$ is assigned to the user $u \in \mathcal{U}^i(M)$, then*

*$r \in \mathcal{R}^i(M)$. We claim that for any $i \in [0,2m]$, it has $\mathcal{R}^i(M_x) = \mathcal{R}^i(M_y)$ where $M_x, M_y \in \mathcal{F}$ represent two arbitrary fairest semi-matchings in a bipartite graph. In other words, the vertices (users and resources) partition $(\mathcal{V}^0(M), \mathcal{V}^1(M), \cdots \mathcal{V}^i(M), \cdots, \mathcal{V}^{2m}(M))$ where $\mathcal{V}^i(M) = \{\mathcal{U}^i(M), \mathcal{R}^i(M)\}$, is constant across the set $\mathcal{F}$ such that $\forall i \in [0,2m]$, $\mathcal{V}^i(M_x) = \mathcal{V}^i(M_y)$ where $M_x, M_y \in \mathcal{F}$.*

*Proof.* Suppose there exists a resource $r$ such that $r \in \mathcal{R}^i(M_x)$ and $r \in \mathcal{R}^j(M_y)$ where $i \neq j$, and specifically $r$ is assigned to $u_i$ and $u_j$ under $M_x$ and $M_y$ respectively (that is, $\{r, u_i\} \in M_x$, $\{r, u_j\} \in M_y$). Then it should has $u_i \in \mathcal{U}^i$, $u_j \in \mathcal{U}^j$ where $i \neq j$, which will be proved as a contradictory.

Let $TS$ represents a Prime Transfer Sequence from $M_x$ to $M_y$. Then the sequence $(\{u_i, r\}, \{r, u_j\})$ should be a sub-sequence of a Transfer $T \in TS$, and $T$ should be a Neutral Transfer based on Lemma 3.5. Thus, $u_i$ and $u_j$ should be from the same user set based on the definition of the sequence of ranked user set. $\square$

## 3.5 Property of The Transfer Sequences Between Any Two Fairest Semi-matching

**Lemma 3.5.** ***(In any Prime Transfer Sequence from one fairest semi-matching to another, all Transfers are Neutral Transfers, and no two Transfers share source or destination users)*** *Let $M_x, M_y \in \mathcal{F}$ represent two arbitrary fairest semi-matchings on $G = (R \cup U, E)$. Let $TS = (T_1, T_2, \cdots, T_{k-1})$ represents an arbitrary Prime Transfer Sequence from $M_x$ to $M_y$ ($T_{k-1} \circ \cdots \circ T_2 \circ T_1(M_x) = M_y$). Then, 1) $\forall T_i \in TS$, it is a Neutral Transfer on $M_x$, that is, $Q(M_x, u_s(T_i)) - Q(M_x, u_d(T_i)) = 1$ or $u_s(T_i) = u_d(T_i)$. 2) any pair of Transfers in $TS$ are disjoint in both source and destination users. That is, $\forall T_i, T_j \in TS$, it has $u_s(T_i) \neq u_s(T_j)$ and $u_d(T_i) \neq u_d(T_j)$.*

*Proof.* The two claims in the lemma are proved as below.

1) *$\forall T_i \in TS$, it is a Neutral Transfer, that is, $Q(M_x, u_s(T_i)) - Q(M_x, u_d(T_i)) = 1$ or $u_s(T_i) = u_d(T_i)$*

*Proof.* We prove by contradictory that $T_i$ can not be an Improving or Deteriorating Transfer.

a) Any $T_i \in TS$ is also a Transfer on $M_x$, because $TS$ is a Prime Transfer Sequence between $M_x$ and $M_y$ which indicates $\mathcal{E}(T_i) \cap \mathcal{E}(T_j) = \emptyset$ where $T_i, T_j \in TS$.

b) $T_i$ can not be an Improving Transfer, otherwise, the fairest semi-matching $M_x$ can be improved by the application of $T_i$.

c) Suppose $T_i$ is a Deteriorating Transfer, then it has $Q(M_x, u_s(T_i)) \leq Q(M_x, u_d(T_i))$ and $u_s(T_i) \neq u_d(T_i)$. We define a "reverse" order of $T_i = (e_1, e_2, \cdots, e_{p-1}, e_p)$, that is $T_i' = (e_p, e_{p-1}, \cdots, e_2, e_1)$. Then it has $u_d(T_i) = u_s(T_i')$, $u_s(T_i) = u_d(T_i')$. We prove that actually $T_i'$ is an Improving Transfer on $M_y$. At first, it is easy to achieve that $T_i'$ is a Transfer on $M_y$ because of the edges switching process from the application of a Transfer. Then, the inequality can be achieved that $Q(M_y, u_s(T_i') - Q(M_y, u_d(T_i')) > 1$ (or $Q(M_y, u_d(T_i) - Q(M_y, u_s(T_i)) > 1$). That is because any pair of Transfers $T_i, T_j \in TS$ can not be merged ($u_s(T_i) \neq u_d(T_j)$ and $u_s(T_j) \neq u_d(T_i)$), thus $u_s(T_i)$ can not increase its quota and $u_d(T_i)$ can not decrease its quota, by the application of any Transfer of $TS$. Then it can derived that $Q(M_x, u_s(T_i)) > Q(M_y, u_s(T_i))$ and $Q(M_x, u_d(T_i)) < Q(M_y, u_d(T_i))$, which indicates $Q(M_y, u_s(T_i') - Q(M_y, u_d(T_i')) = Q(M_y, u_d(T_i)) - Q(M_y, u_s(T_i)) > Q(M_x, u_d(T_i)) - Q(M_x, u_s(T_i)) + 1 > 1$.

□

2) *Any pair of Transfers in $TS$ are disjoint in both source and destination users. That is, $\forall T_i, T_j \in TS$, it has $u_s(T_i) \neq u_s(T_j)$ and $u_d(T_i) \neq u_d(T_j)$.*

*Proof.* Suppose there exists a pair of adjacent Transfers in $T_i, T_{i+1} \in TS$ which are not disjoint. That means $T_i$ and $T_{i+1}$ have the same source user or destination user or both source and destination users. We will prove that the case of the same source user will induce a contradictory. Then all other cases can be proved with the same idea.

a) Suppose $T_i$ and $T_{i+1}$ have the same source user and different destination users (both of them are not Cyclic Neutral Transfers, otherwise they can be merged, which conflicts with the definition of Prime Transfer Sequence), that is, $u_s(T_i) = u_s(T_{i+1})$ and $u_d(T_i) \neq u_d(T_{i+1})$.

   Let $M_{i+1} = T_i(M_i)$ where $T_i \in TS$, $i \in [1, k-1]$ (note that, $M_1 = M_x$, $M_k = M_y$). Based on the first claim (any $T_i \in TS$ is a Neutral Transfer on $M_x$), it can be concluded that $\forall i \in [1, k-1]$, $M_i$ is a fairest semi-matching, and each $T_i$ is a Neutral Transfer under $M_i$. Then we prove that $Q(M_i, u_s(T_{i+1})) - Q(M_i, u_d(T_{i+1})) > 1$ and it indicates there exists an Improving Transfer $T_{i+1}$ on the fairest semi-matching $M_i$ which is a contradictory. The application of $T_i$ on $M_i$ ($T_i(M_i) = M_{i+1}$) decreases the quota of the common source user $u_s(T_i)$ (or $u_s(T_{i+1})$) and does not change the quota of $u_d(T_{i+1})$, then it has that $Q(M_i, u_s(T_{i+1})) - Q(M_i, u_d(T_{i+1})) = Q(M_{i+1}, u_s(T_{i+1})) - Q(M_{i+1}, u_d(T_{i+1})) + 1 > 1$, where $Q(M_{i+1}, u_s(T_{i+1})) - Q(M_{i+1}, u_d(T_{i+1})) = 1$ ($T_{i+1}$ is a Neutral Transfer on $M_{i+1}$),

b) The other cases can be proved with similar idea. After that, any pair of Transfers $T_i$ and $T_j$ (might not be adjacent) can be proved to be disjoint based on swapping of adjacent Transfers.

$\square$

$\square$

### 3.6 Conclusion

This work focuses on the properties of the set of all the fairest semi-matching $\mathcal{F}$ under a given bipartite graph. It is known that all the fairest semi-matchings have the same sorted quotas vector, but it is not obvious how all of them are related, or what they have in common. This paper concludes that two important properties can be derived from one arbitrary fairest semi-matching: 1) the edge classification of the bipartite graph into $\mathcal{E}_{all}$, $\mathcal{E}_{some}$, $\mathcal{E}_{none}$, and 2) precisely how much (either constant, or differ 1) a user's quota might vary, among all the fairest semi-matchings. Furthermore, given one arbitrary fairest semi-matching, a partition of all the vertices (resources and users) in the bipartite graph can be achieved, such that it is guaranteed among all the possible fairest semi-matchings that the allocating of resources to users are strictly within the partitions.

To derive the above conclusions and prove their validity, we show that: a) all the fairest semi-matchings have the same set of Neutral Transfers Covered Edge Set, which is equal to $\mathcal{E}_{some}$; b) a sequence of ranked user sets can be constructed, such that a user will always belong to the same ranked set under all fairest semi-matchings. To achieve the proofs, we rely on the insight about the property of paths between any two fairest semi-matchings, included in Section 3.5: in any Prime Transfer Sequence between any pair of fairest semi-matchings, all the Transfers have to be Neutral and disjoint.

# CHAPTER 4

# UNDERSTANDING THE CORRESPONDENCE BETWEEN THE FAIREST SEMI-MATCHINGS AND THE FAIREST FRACTIONAL ALLOCATIONS (BINARY VS. FRACTIONAL WEIGHT)

## 4.1 Introduction

Obviously, under the relaxed constraint that a resource is no longer indivisible but can be split in any fractional ways to be assigned to multiple users, one can achieve better fairness in even the fairest semi-matchings. However, it is not straightforward whether similar conclusions (the existence of a fairest set of semi-matchings) can be drawn as the case of indivisible resources. Furthermore, it is unclear whether the two scenarios (indivisible vs. divisible resources) share any commonality. In this chapter, we consider the generalized scenario where the resources are divisible which indicates each resource can be split and assigned to maybe more than one users. Under this scenario, the fairest ones with regard to one fairness measure is called *fairest fractional allocations*, which can be achieved by linear programming techniques. However, it is not obvious that the fairest fractional allocation regarding one fairness measure is always the fairest regarding all fairness measures. We show that, based on the similar techniques used in Chapter 2, it can be proved that there always exists a set of fairest fractional allocations, universally agreed by all fairness measures, denoted as $\mathcal{C}$. We study on the relationship between the fairest semi-matchings and the fairest fractional allocations, specifically, how the fairest semi-matching be similar with the fairest fractional allocations. We prove that the constant vertices partition across all the fairest semi-matchings is also constant across all the fairest fractional allocations - the allocating of all fairest semi-matchings and all the fairest fractional allocations are always within the partitions. Moreover, for each user, the difference of its quotas between one fairest semi-matching and one fairest fractional allocation is limited (either 0 or bound by 1) and predictable.

### 4.1.1 Definition of a Fractional Allocation

In a bipartite graph $G = (R \cup U, E)$ where $R$ represents a set of resources, $U$ represents the set of users, $E \subseteq R \times U$ represents the set of edges, which are the possible allocations options from resources to users. A fractional allocation under a bipartite graph, is defined as the assigning of weights for each edge in $E$ such that the weigh sum of all edges incident to the same resource should be equal to one.

**Definition 4.1.** ***(Fractional Allocation)*** *In a bipartite graph $G = (R \cup U, E)$, a fractional allocation is defined as a vector of weights for all edges in E, denoted as $W = (w(e_{ij}))_{|E|}$ where $e_{ij} = \{r_i, u_j\}$, such that $w(e_{ij}) \in \mathbb{R}$, $w(e_{ij}) \in [0,1]$, and $\forall i \in [1,m]$, $\sum_{j\in[1,n]} w(e_{ij}) = 1$ where m and n represent the number of resources and users respectively.*

Given a fractional allocation $W$, for $u_j \in U$, the *quota* of $u_j$ denoted as $Q(W, u_j)$, is defined as the sum of weights for all edges incident to $u_j$. Formally, $Q(W, u_j) = \sum_{i\in[1,m]} w(e_{ij})$ where $e_{ij} = \{r_i, u_j\}$, $m$ represents the number of all resources. Let $Q(W)$ denotes the *quotas vector* of a fractional allocation $W$, which is defined as $Q(W) = (Q(W, u_1), Q(W, u_2), \cdots, Q(W, u_n))$, where $u_j \in U, j \in [1,n]$.

**Definition 4.2.** ***(Quotas Vector)*** *Given a fractional allocation W under a bipartite graph $G = (R \cup U, E)$, the quota of user $u_j \in U$ is defined as $Q(W, u_j) = \sum_{i\in[1,m]} w(e_{ij})$ where $e_{ij} = \{r_i, u_j\}$, m represents the number of all resources. The corresponding quotas vector of W, denoted as $Q(W)$, is the vector of which each element $Q(W, u_j)$ represents the quota of a user $u_j \in U$.*

Assuming that both edges and vertices in the bipartite graph are unweighted, the fairness of a fractional allocation is usually considered based on its quotas vector alone, under some fairness measures.

### 4.1.2 Main Results

The main conclusion achieved are:

- The property of the fairest fractional allocations.

  1. For any bipartite graph, there always exist a set of fractional allocations which are universally agreed to be the fairest by all fairness measures. Furthermore, those fairest fractional allocations have the same quotas vector.

- The correspondence between the fairest semi-matchings and fairest fractional allocation.

  1. There exists a non-trivial constant vertices (users and resources) partition such that the allocating of all the fairest semi-matchings and fairest fractional allocations are all within the partitions. And the vertices partition can be achieved from one arbitrary fairest semi-matching or one arbitrary fairest fractional allocation.

  2. For each user $u \in U$, its quota difference between one fairest semi-matching $M$ and one fairest fractional allocation $W$ is either 0 or bound by 1. Furthermore, the specific quota range for a user $u$ across all fairest semi-matchings and all fairest fractional allocations respectively can be determined from one arbitrary fairest semi-matching or fairest fractional allocation.

### 4.1.3 Organization

This chapter is organized as below. Section 4.2 shows the existence of universally agreed fairest fractional allocations, which includes two subsections: one for the definitions, another one for the main proof. Section 4.3 presents the correspondence between the fairest semi-matchings and the fairest fractional allocations, which also includes two subsections: one for the preliminary work which shows the related techniques/conclusions from Chapter 3, another for the proofs of the main conclusions. Then, this chapter is concluded in Section 4.4.

## 4.2 The Existence of Universally Agreed Fairest Fractional Allocations

This section proves that, given a bipartite graph, there always exist a set of fractional allocations, universally agreed by all fairness measures, to be the fairest among all fractional allocations. Furthermore, this set of fairest fractional allocations have the same quotas vector. The proof is mainly based on the similar techniques used to prove the existence of universally agreed fairest semi-matchings in Chapter 2, and the definitions of Transfer and Transfer-based Comparison are generalized to be applied on fractional allocations.

### 4.2.1 Definitions

**Definition 4.3.** ***(Transfer on a fractional allocation)*** *Under a fractional allocation $W$ of a bipartite graph $G = (U \cup R, E)$, let an edge sequence $S = (\{u_1, r_1\}, \{r_1, u_2\}, \{u_2, r_2\}, \ldots, \{r_{k-1}, u_k\})$ represents a path starting and ending with users, and containing alternative users and resources. A Transfer of the amount $\delta w$ along with the edges sequence $S$, denoted as $T^{\delta w}$ where $\delta w \in R$ and $\forall i \in [1,k]$, $\delta w \leq w(\{u_i, r_i\})$, is defined as a change of weights, such that $\forall i \in [1, k-1]$, $w(\{u_i, r_i\})$ has a decrease of the amount $\delta w$, and $w(\{r_i, u_{i+1}\})$ has an increase of the amount $\delta w$. The application of Transfer*

$T^{\delta w}$ *on* $W$ *is the execution of the weight changes defined from* $T^{\delta w}$*, which results in another fractional allocation* $W'$*, denoted as* $T^{\delta w}(W) = W'$*.*

**Remark.** *If the condition* $\forall i \in [1,k]$*,* $\delta w \leq w(\{u_i, r_i\})$ *doesn't meet, then the change of weights along with the edge sequence S is not regarded as a Transfer.*

**Definition 4.4.** ***(Source and Destination users of a Transfer on a fractional allocation)*** *Let* $T^{\delta w}$ *represents the Transfer along the edges sequence* $(\{u_1, r_1\}, \cdots, \{r_{k-1}, u_k\})$*.* $u_1$ *and* $u_k$ *are defined as the source and destination users of Transfer* $T^{\delta w}$*, denoted as* $u_s(T^{\delta w})$ *and* $u_d(T^{\delta w})$ *respectively.*

**Definition 4.5.** ***(Transfer Types: Improving, Deteriorating, Neutral)*** *Let* $T^{\delta w}$ *represents a Transfer on a fractional allocation W with* $Q(W, u_s(T^{\delta w}))$ *and* $Q(W, u_d(T^{\delta w}))$ *representing the quotas of the source user* $u_s(T^{\delta w})$ *and destination user* $u_d(T^{\delta w})$ *in W respectively. Apparently, when* $u_s(T^{\delta w}) \neq u_d(T^{\delta w})$*,* $T^{\delta w}(W) = W'$ *results in the change of quota in only two user:* $Q(W', u_s(T^{\delta w})) = Q(W, u_s(T^{\delta w})) - \delta w$*, and* $Q(W', u_d(T^{\delta w})) = Q(W, u_d(T^{\delta w})) + \delta w$*. In other words, the source user's quota decreases by* $\delta w$*, and the destination user's quota increases by* $\delta w$ *as the result of the application of* $T^{\delta w}$*. In the special case of* $u_s(T^{\delta w}) = u_d(T^{\delta w})$*, W and* $W'$ *has the same quotas vector.*

*As a result, the application of* $T^{\delta w}$ *on W will affect the fairness of W in only three possible ways:*

- $T^{\delta w}$ *could result in a fairer* $W'$ *over W, thus* $T^{\delta w}$ *is denoted as an Improving Transfer (*$IT^{\delta w}$*) if* $Q(W, u_s(T^{\delta w})) > Q(W, u_d(T^{\delta w})) + \delta w$*;*
- $T^{\delta w}$ *could result in a more unfair* $W'$ *over W, thus* $T^{\delta w}$ *is denoted as a Deteriorating Transfer (*$DT^{\delta w}$*) if* $Q(W, u_s(T^{\delta w})) < Q(W, u_d(T^{\delta w})) + \delta w$ *and* $u_s(T^{\delta w}) \neq u_d(T^{\delta w})$*;*

- *$T^{\delta w}$ could result in $W'$ with the same fairness as $W$, thus $T^{\delta w}$ is denoted as a Neutral Transfer ($NT^{\delta w}$), where two cases exist:*
  - *$T^{\delta w}$ is denoted as a Straight Neutral Transfer ($NT_S^{\delta w}$) if $Q(A, u_s(T^{\delta w})) = Q(A, u_d(T^{\delta w})) + \delta w$ and $u_s(T^{\delta w}) \neq u_d(T^{\delta w})$;*
  - *$T^{\delta w}$ is denoted as a Cyclic Neutral Transfer ($NT_C^{\delta w}$) if $u_s(T^{\delta w}) = u_d(T^{\delta w})$.*

**Definition 4.6.** ***(Transfer-based Comparison between fractional allocations)*** *For any pair of fractional allocations $W_x$ and $W_y$, if there exists a Transfer Sequence $(T_1^{\delta w_1}, T_2^{\delta w_2}, \cdots, T_k^{\delta w_k})$ such that $T_k^{\delta w_k} \circ \cdots \circ T_2^{\delta w_2} \circ\ T_1^{\delta w_1}(W_x) = W_y$. $\forall i \in [1,k]$, if $T_i^{\delta w_i}$ is $IT^{\delta w}$ or $NT^{\delta w}$, then $W_y$ is defined as not less fair than $W_x$ in terms of a Transfer-based Comparison, denoted as $W_x \preceq_{T^{\delta w}} W_y$. Specifically, if all $T_i^{\delta w}$ are either $NT^{\delta w}$, then $W_x$ is defined as equally fair with $W_y$, denoted as $W_x \approx_{T^{\delta w}} W_y$; if at least one $T_i^{\delta w}$ is $IT^{\delta w}$, then $W_y$ is defined as fairer than $W_x$, denoted as $W_x \prec_{T^{\delta w}} W_y$.*

*If for every Transfer Sequence $(T_1^{\delta w_1}, T_2^{\delta w_2}, \cdots, T_k^{\delta w_k})$ such that $T_k^{\delta w_k} \circ \cdots \circ T_2^{\delta w_2} \circ\ T_1^{\delta w_1}(W_x) = W_y$, there always exist some $T_i^{\delta w_i}$, $T_j^{\delta w_j}$ where $i, j \in [1,k]$ such that $T_i^{\delta w}$ is $IT^{\delta w}$, $T_j^{\delta w}$ is $DT^{\delta w}$, then $W_x$ and $W_y$ are incomparable, denoted as $W_x \prec\succ_{T^{\delta w}} W_y$.*

**Remark.** *With the restriction that $\delta w = 1$, the above definitions: Transfer, Transfer Type, Transfer-based Comparison, becomes to the corresponding definitions on semi-matchings, which are defined in Chapter 2.*

### 4.2.2 Existence and Attributes of The Fairest Fractional Allocations Set

In this subsection, the key lemmas/theorems are shown for the proof of the existence of the universally agreed fairest fractional allocations, which corresponds to the key lemmas/theorems for the existence of universally agreed fairest semi-matchings from Chapter 2. The proofs for these lemmas/theorems will be omitted here, which are similar with the corresponding proofs from Chapter 2, except the generalization of $\delta w = 1$ (for semi-matchings) to $\delta w \in \mathbb{R}$, $\delta w \in [0,1]$ (for fractional allocations).

**Lemma 4.1.** ***(Existence of a greater element than both of incomparable fractional allocations with regard to Transfer-based Comparison)*** *For any pair of fractional allocations $W_x, W_y$, if $W_x \prec\succ_{T^{\delta w}} W_y$, then there exists at least one fractional allocation $F_z$ satisfying that $W_x \preceq_{T^{\delta w}} F_z$ and $W_y \preceq_{T^{\delta w}} F_z$.*

**Theorem 4.1.** ***(Existence of the fairest fractional allocations with regard to Transfer-based Comparison)*** *For any bipartite graph, there always exist a (maybe infinite) set of the fairest fractional allocations with regarding to the Transfer based Comparison.*

**Theorem 4.2.** ***(Existence of universally agreed fairest fractional allocations, which has no Improving and Straight Neutral Transfers to be applied)*** *The set of all fairest fractional allocations with regarding to the Transfer based Comparison is the set of fractional allocations which are universally considered the fairest by all fairness measures. Furthermore, a fractional allocation belongs to the fairest set iff there exists no Improving Transfer $IT^{\delta w}$ and Straight Neutral Transfer $NT_S^{\delta w}$ to be applied.*

**Remark.** ***(No Improving Transfer $IT^{\delta w=1}$ existed on a fairest semi-matching)*** *One different attribute of the fairest semi-matchings is that: a semi-matching belong to the fairest set iff there exists no Improving Transfer $IT^{\delta w=1}$ to be applied.*

**Remark.** ***(In a fractional allocation, the existence of a Straight Neutral Transfer $NT_S^{\delta w}$, results in the existence of an Improving Transfer $IT^{\delta w}$)*** *In a fractional allocation $W$, if there exists one Straight*

*Neutral Transfer $NT_S^{\delta w}$ along an edge sequence s, then an Improving Transfer $IT^{\delta w'}$ where $\delta w' < \delta w$ along the same edge sequence s should exist on $W$.*

**Corollary 4.1.** ***(A local search algorithm to achieve one fairest fractional allocation)*** *Given a bipartite graph, starting with one arbitrary fractional allocation, by an iterative application of existed Improving Transfers $IT^{\delta w}$ until no more found, then one fairest fractional allocation can be achieved.*

**Corollary 4.2.** ***(The set of all the fairest fractional allocations have the same quotas vector)*** *In any bipartite graph G, all the fractional allocations have the same quotas vector, that is, $Q(W_i) = Q(W_j)$ where $W_i$, $W_j$ represents two arbitrary fairest fractional allocations.*

**Remark.** ***(The set of all the fairest allocation have the same sorted quotas vector)*** *One different attribute of the fairest semi-matchings is that: all the fairest semi-matchings have the same sorted quotas vector but possibly different quotas vector.*

Under a fairest semi-matching, a Straight Neutral Transfer might exists, and the application of a Straight Neutral Transfer will change its quota vector but keep the same sorted quota vector. However, under a fairest fractional allocation, there do not exist any Straight Neutral Transfer, and only might exist some Cyclic Neutral Transfers which do not change the quota vector.

## 4.3 The Correspondence Between Fairest Semi-matchings and Fairest Fractional Allocations

This subsection studies on the correspondence between the set of fairest semi-matchings ($\mathcal{F}$) and the fairest fractional allocations denoted as $\mathcal{C}$. In Chapter 3, we show that the quota variation of any user among all the fairest semi-matchings is very narrow (at most differ by one) and can be predicted from one arbitrary fairest semi-matching. This is proved by defining a partition of users (a sequence of ranked user sets), and showing this partition is constant among all fairest semi-matchings. Furthermore, each partition of users are always be allocated with same set of resources. In other words, there exists a partition of all users and resources such that the allocating of all the fairest semi-matchings are all within the partitions. In this chapter, based on above conclusions and techniques, we achieve that: 1) the quotas difference of each user between one semi-matching $M \in \mathcal{F}$ and one fractional allocation $W \in \mathcal{C}$ is either one or bound by one. Moreover, the specific quota range of each user across the set $\mathcal{F}$ and $\mathcal{C}$ can be determined from one semi-matching $M \in \mathcal{F}$ or one fractional allocation $W \in \mathcal{C}$. 2) the constant partition of users and resources across the set $\mathcal{F}$ (allocating of all semi-matchings in the set $\mathcal{F}$ are all within the partitions) is also constant across the set $\mathcal{F}$ and the set $\mathcal{C}$ (allocating of all semi-matchings in the set $\mathcal{F}$ and all fractional allocations in the set $\mathcal{C}$ are all within the partitions).

### 4.3.1 Preliminary: Related Techniques/Conclusions From Chapter 3

**Definition 4.7.** *(**Users Partition**) Given a semi-matching $M \in \mathcal{F}$, a sequence of ranked user sets $\mathscr{P}_U(M) = (\mathcal{U}^0(M), \mathcal{U}^1(M), \cdots \mathcal{U}^i(M), \cdots, \mathcal{U}^{2m}(M))$ etc, where each $\mathcal{U}^i$ is a set of user and $m$ is the number of resources, is defined as: if a user $u$ is covered by a Straight Neutral Transfer $NT_S$, then $u \in \mathcal{U}^i(M)$ where $i = Q(M, u_s(NT_S)) + Q(M, u_d(NT_S))$; otherwise, if $u$ is not covered by any Straight Neutral Transfer $NT_S$, then $u \in \mathcal{U}^i$ where $i = 2 \times Q(M, u)$.*

**Theorem 4.3.** ***(The users partition is constant among all the fairest semi-matchings)*** *Let $M_x, M_y \in \mathscr{F}$ represent two arbitrary fairest semi-matchings in a bipartite graph $G = (R \cup U, E)$, then for any $i \in [0, 2m], \mathcal{U}^i(M_x) = \mathcal{U}^i(M_y)$. In other words, we can say $\mathscr{P}_U(M_x) = \mathscr{P}_U(M_y)$.*

**Theorem 4.4.** ***(A constant partition of users and resources across the set*** $\mathcal{F}$***)*** *Given a fairest semi-matching $M \in \mathcal{F}$, a sequence of ranked resource sets $(\mathcal{R}^0(M), \mathcal{R}^1(M), \cdots \mathcal{R}^i(M), \cdots, \mathcal{R}^{2m}(M))$ where $m$*

*is the number of resources, is defined as: if under $M$ a resource $r$ is assigned to the user $u \in \mathcal{U}^i(M)$, then $r \in \mathcal{R}^i(M)$. We claim that for any $i \in [0,2m]$, it has $\mathcal{R}^i(M_x) = \mathcal{R}^i(M_y)$ where $M_x, M_y \in \mathcal{F}$ represent two arbitrary fairest semi-matchings in a bipartite graph. In other words, the vertices (users and resources) partition $\mathscr{P}(M) = (\mathcal{V}^0(M), \mathcal{V}^1(M), \cdots \mathcal{V}^i(M), \cdots, \mathcal{V}^{2m}(M))$ where $\mathcal{V}^i(M) = \{\mathcal{U}^i(M), \mathcal{R}^i(M)\}$, is constant across the set $\mathcal{F}$ such that $\forall i \in [0,2m]$, $\mathcal{V}^i(M_x) = \mathcal{V}^i(M_y)$ where $M_x, M_y \in \mathcal{F}$.*

**Corollary 4.3.** ***(The quota variation of any user among all the fairest semi-matchings can be predicted from one arbitrary fairest semi-matching)*** *Let $M$ represents one arbitrary fairest semi-matching in a bipartite graph $G = (R \cup U, E)$. $\forall u \in U$, assume $u \in \mathcal{U}^i(M)$, if $i$ is odd, its quota from any fairest semi-matchings is either $(i+1)/2$ or $(i-1)/2$; if $i$ is even, then its quota from any fairest semi-matchings is exactly $i/2$.*

For the sake of clarification, we will omit the notation of a specific semi-matching, and let $\mathscr{P} = (\mathcal{V}^0, \mathcal{V}^1, \cdots \mathcal{V}^i, \cdots, \mathcal{V}^{2m})$ where $\mathcal{V}^i = \{\mathcal{U}^i, \mathcal{R}^i\}$ represent the constant vertices (users and resources) partition among all the fairest semi-matchings.

### 4.3.2 Main Conclusions

**Theorem 4.5.** ***(For each user, its quota difference among one arbitrary $M \in \mathcal{F}$ and one arbitrary $W \in \mathcal{C}$ is either 0 or bound by 1, and its specific quotas range across $\mathcal{F}$ and $\mathcal{C}$ can be determined from the vertices partition $\mathscr{P}$)*** *In a bipartite graph $G = (R \cup U, E)$, $\forall u \in U$, assume $u \in \mathcal{U}^i$, then $\forall M \in \mathcal{F}$, $W \in \mathcal{C}$, if $i$ is odd, $Q(M,u) = (i+1)/2$ or $(i-1)/2$, $Q(W,u) \in ((i-1)/2, (i+1)/2)$; if $i$ is even, $Q(M,u) = Q(W,u) = i/2$.*

*Proof.* Let $\mathscr{P} = (\mathcal{V}^0, \mathcal{V}^1, \cdots \mathcal{V}^i, \cdots, \mathcal{V}^{2m})$ where $\mathcal{V}^i = \{\mathcal{U}^i, \mathcal{R}^i\}$ represent the constant vertices (users and resources) partition among all the fairest semi-matchings.

*Construction of a bipartite sub-graph for each vertices set* $\mathcal{V}^i$: for each $\mathcal{V}^i = \{\mathcal{U}^i, \mathcal{R}^i\}$, a bipartite sub-graph can be constructed as $G^i = (\mathcal{R}^i \cup \mathcal{U}^i, E^i)$ where $E^i = \mathcal{R}^i \times \mathcal{U}^i \subset E$, such that $\forall e = \{r, u\}$ where $r \in \mathcal{R}^i(M)$, $u \in \mathcal{U}^i$, $e \in E^i$ if and only if $e \in E$.

Let $W^i$ represents one fairest fractional allocation for $G^i$. We show in Lemma 4.2 that $\forall u \in \mathcal{U}^i$, if $i$ is even, then $Q(W^i, u) = i/2$; if $i$ is odd, then $Q(W^i, u) \in ((i-1)/2, (i+1)/2)$. Furthermore, it can be proved that the allocation $W_{comb} = \{W^1, W^2, \cdots, W^i, \cdots, W^{2m}\}$ is a fairest fractional allocation for the original bipartite graph $G$ (that is, $W_{comb} \in \mathcal{C}$), which is shown in Lemma 4.3.

Therefore, $\forall u \in \mathcal{U}^i$, if $i$ is odd then it has $\forall M \in \mathcal{F}$, $Q(M, u) = (i+1)/2$ or $(i-1)/2$ (from Corollary 4.3), and $Q(W_{comb}, u) \in ((i-1)/2, (i+1)/2)$; if $i$ is even, then $\forall M \in \mathcal{F}$, $Q(M, u) = i/2$ (from Corollary 4.3) and $Q(W_{comb}, u) = i/2$. Furthermore, because all the fairest fractional allocations $W \in \mathcal{C}$ have the same quotas vector, thus under any $W \in \mathcal{C}$ it has, if $i$ is odd, $Q(W, u) \in ((i-1)/2, (i+1)/2)$; if $i$ is even, $Q(W, u) = i/2$. □

**Lemma 4.2.** *For each $W^i$ and $u \in \mathcal{U}^i$, if $i$ is even, $Q(W^i, u) = i/2$; if $i$ is odd, $Q(W^i, u) \in ((i-1)/2, (i+1)/2)$.*

*Proof.* We prove it from the two cases: $i$ is even and $i$ is odd.

- ($i$ is even) It indicates for any fairest semi-matching, all users in $\mathcal{U}^i$ get exactly the same quota $i/2$ and the set of all their allocated resources is $\mathcal{R}^i$. Thus, in the bipartite sub-graph $G^i = (\mathcal{R}^i \cup \mathcal{U}^i, E^i)$, a fairest fractional allocation $W^i$ can not achieve a fairer allocation, then $\forall u \in \mathcal{U}^i$, $Q(W^i, u) = i/2$.
- ($i$ is odd) It indicates for any fairest semi-matching each user in $\mathcal{U}^i$ gets a quota of $(i-1)/2$ or $(i+1)/2$, and the set of all their allocated resources is $\mathcal{R}^i$, then in the bipartite sub-graph $G^i =$

$(\mathcal{R}^i \cup \mathcal{U}^i, E^i)$, a fairest fractional allocation $W^i$ should meets $\forall u \in \mathcal{U}^i$, $Q(W^i, u) \in [(i-1)/2, (i+1)/2]$.

Moreover, we can prove that $\forall u \in \mathcal{U}^i$, $Q(W^i, u)$ can not be $(i-1)/2$ or $(i+1)/2$. In the bipartite sub-graph $G^i = (\mathcal{R}^i \cup \mathcal{U}^i, E^i)$, an initial fractional allocation $W^i_{init}$ can be constructed as follows: each user in $\mathcal{U}^i$ has the same resources allocation between $W^i_{init}$ and one arbitrary fairest semi-matching under $G$. Then, starting with $W^i_{init}$, one fairest fractional allocation $W^i$ under $G^i$ can be achieved by the iterative applications of Improving Transfers $IT^{\delta w}$. We show that $\forall u \in \mathcal{U}^i$, $Q(W^i, u)$ can not be $(i+1)/2$, based on the following two claims: 1) if $Q(W^i_{init}, u) \neq (i+1)/2$, then $Q(W^i, u) \neq (i+1)/2$. The reason is that under $W^i_{init}$ each user in $\mathcal{U}^i$ has a quota of either $(i-1)/2$ or $(i+1)/2$, thus the iterative applications of Improving Transfers can not increase the quota of one user up to $(i+1)/2$. 2) if $Q(W^i_{init}, u) = (i+1)/2$, then $Q(W^i, u) < (i+1)/2$. Suppose $Q(W^i_{init}, u) = Q(W^i, u) = (i+1)/2$, then a contradictory will be shown. By the definition of $\mathcal{U}^i$ where $i$ is odd, under $W^i_{init}$ $u$ should be covered by one Neutral Transfer $NT^{\delta w=1}$ (assume along the edge sequence $S = (\{u_1, r_1\}, \ldots, \{r_{k-1}, u_k\})$ with $u_1 = u$ and $u_k$ as the source and destination users). We show that if $Q(W^i, u) = (i+1)/2$, then under $W^i$ there should still exist an Improving Transfer $IT^{\delta w}$ along the same edge sequence $S$. The reasons are: a) because under $W^i_{init} = (w(e))_{|E^i|}$, $\forall j \in [1, k-1]$, $w(\{u_j, r_j\}) = 1$, then under $W^i = (w(e))_{|E^i|}$, it should have $\forall j \in [1, k-1]$, $w(\{u_j, r_j\}) > 0$. In other words, $w(\{u_j, r_j\})$ can not be decreased from one to zero by the applications of Improving Transfers on $W^i_{init}$. Therefore, under $W^i$ the weight change along the same edge sequence $S$ can still form a Transfer. b) Assume $Q(W^i, u) = (i+1)/2$. then it should have $Q(W^i, u_1) > Q(W^i, u_k)$ where $u_1 = u$, which indicates an Improving Transfer $IT^{\delta w}$

along the edge sequence $S$ can be found on $W^i$. In conclusion, $\forall u \in \mathcal{U}^i$, $Q(W^i,u) \neq (i+1)/2$. Furthermore, by similar deduction, it has $\forall u \in \mathcal{U}^i$, $Q(W^i,u) \neq (i-1)/2$.

□

**Lemma 4.3.** *$W_{comb} = \{W^1, W^2, \cdots, W^{2m}\}$ is a fairest fractional allocation for the bipartite graph $G$.*

*Proof.* Obviously, $W$ is fractional allocation under $G$. Then we prove that $W$ is a fairest fractional allocation by showing there exists no Improving Transfer $IT^{\delta w}$ and no Straight Neutral Transfer $NT_S^{\delta w}$ to be applied on $W$.

Assume there exists one Transfer $T^{\delta w}$ on the allocation $W$ which is an Improving Transfer or Straight Neutral Transfer. Obviously, all the edges included in $T^{\delta w}$ must be from at least two different bipartite sub-graph, otherwise, it contradicts with that $W^i$ is a fairest fractional allocation under $G^i = (\mathcal{R}^i \cup \mathcal{U}^i, E^i)$. Suppose there exist two adjacent edges from $T^{\delta w}$: $e_a = \{u_a, r_a\}$ and $e_b = \{r_a, u_b\}$ such that $u_a \in \mathcal{U}^i$ and $u_b \in \mathcal{U}^j$. Then from Lemma 4.4 it should has $i < j$. Thus, it can be deduced that $Q(W,u_a) \leq Q(W,u_b)$. Then based on an iterative deduction, it can be achieved that $Q(W,u_s) \leq Q(W,u_d)$ where $u_s$ and $u_d$ represent the source user and destination user of Transfer $T^{\delta w}$. That indicates $T^{\delta w}$ can not be either an Improving Transfer $IT^{\delta w}$ or a Straight Neutral Transfer $NT_S^{\delta w}$. Therefore, $W$ is one fairest fractional allocation under $G$. □

**Lemma 4.4.** *If there exists an edge $e = \{r,u\}$ such that $r \in \mathcal{V}^i$, $u \in \mathcal{V}^j$ where $i \neq j$, then $i < j$.*

*Proof.* Suppose there exists an edge $e = \{r,u\}$ such that $r \in \mathcal{V}^i$, $u \in \mathcal{V}^j$ where $i > j$. Let $M \in \mathcal{F}$ represents one fairest semi-matching under bipartite graph $G$, and let $u' \in \mathcal{V}^i$ represents the user such that $\{r,u'\} \in$

$M$. Then, the edge sequence $T = (\{u', r\}, \{r, u\})$ is a Transfer on $M$. And it has $Q(M, u') \geq Q(M, u)$ because of $i > j$.

- If $Q(M, u') > Q(M, u) + 1$, then $T$ is an Improving Transfer on the fairest semi-matching $M$.
- If $Q(M, u') = Q(M, u) + 1$, then $T$ is a Neutral Transfer, which means both $u'$ and $u$ should be from the same user set $U^k$ where $k = Q(M, u') + Q(M, u)$.
- If $Q(M, u') = Q(M, u)$, then based on the definition of the sequence of user sets (Definition 4.7), because users $u'$ and $u$ are from difference user sets but have the same quota, then it is easy to derive that either $u'$ is covered by a Straight Neutral Transfer with the destination user of $u'$, or $u$ is covered by a Straight Neutral Transfer with the source user of $u$. Any one of the two Straight Neutral Transfers, combined with the Transfer $T$, can form a Straight Neutral Transfer which covers both $u'$ and $u$. That indicates both $u'$ and $u$ are from the same user set $U^k$ where $k = 2 \times Q(M, u') + 1$ or $k = 2 \times Q(M, u) - 1$.

All the three cases lead to a contradictory. Thus it should meets $i < j$. □

**Theorem 4.6.** ***(The vertices partition is also constant among all fairest fractional allocations)*** *In a bipartite graph $G = (R \cup U, E)$, let $\mathscr{P} = (\mathcal{V}^0, \mathcal{V}^1, \cdots \mathcal{V}^i, \cdots, \mathcal{V}^{2m})$ where $\mathcal{V}^i = \{\mathcal{U}^i, \mathcal{R}^i\}$ represent the constant vertices (users and resources) partition among all the fairest semi-matchings. For any edge $e_{cross} = \{r, u\}$ where $r \in \mathcal{V}^i$, $u \in \mathcal{V}^j$ and $i \neq j$, then $\forall W \in \mathcal{C}$ where $W = (w(e))_{|E|}$, it has $w(e_{cross}) = 0$.*

*Proof.* It has been proved that $W_{comb} = (w(e))_{|E|}$ is a fairest fractional allocation which meets $w(e_{cross}) = 0$. The following shows that can be met for any fairest fractional allocation $W \in \mathcal{C}$.

Let $\mathcal{V}^h = \{\mathcal{R}^h, \mathcal{U}^h\}$ represents the "highest" vertices set such that $\forall \mathcal{V}^i \in \mathscr{P}$, it has $h \geq i$. From Lemma 4.4, it can be derived that under $\mathcal{V}^h$ there do not exist any "outgoing edges" (edges $e = \{r, u\}$ such that $r \in \mathcal{V}^h$, $u \in \mathcal{V}^i$ where $i \neq h$), and there might exist only some "incoming edges" (edges $e = \{r, u\}$ such that $r \in \mathcal{V}^i$, $u \in \mathcal{V}^h$ where $i \neq h$). Then, those possible "incoming edges" will not be used (the weight is 0) by any fairest fractional allocation $W \in \mathcal{C}$. Otherwise, the total quotas of all users in $\mathcal{U}^h$ under $W$ will be more than that under $W_{comb}$, and that indicates $Q(W) \neq Q(W_{comb})$ where $W, W_{comb} \in \mathcal{C}$ which is a contradictory. By the iterative deduction (on the remaining vertices sets), we can achieve that all the $e_{cross} = \{r, u\}$ where $r \in \mathcal{V}^i$, $u \in \mathcal{V}^j$ and $i \neq j$, will not be used by any fairest fractional allocation, that is $\forall W = (w(e))_{|E|} \in \mathcal{C}$, it has $w(e_{cross}) = 0$. □

**Corollary 4.4.** ***(The vertices partition $\mathscr{P}$ can be obtained from one arbitrary fairest fractional allocation)*** *For one arbitrary fairest fractional allocation $W \in \mathcal{C}$ where $W = (w(e))_{|E|}$, $\forall u \in U$, if $Q(W, u)$ is an integer, then $u \in \mathcal{U}^i$ where $i = 2 \times Q(W, u)$; else, $u \in \mathcal{U}^i$ where $i = \lfloor Q(W, u) \rfloor + \lceil Q(W, u) \rceil$. After that, $\forall r \in R$, if the edge $e = \{r, u\}$ meets $w(e) \neq 0$ and $u \in \mathcal{U}^i$, then $r \in \mathcal{R}^i$.*

**Corollary 4.5.** ***The specific quotas range for each user among all fairest semi-matchings and all fairest fractional allocations, can be predicted from one arbitrary fairest semi-matching or one arbitrary fairest fractional allocation.***

*Under a bipartite graph, the vertices partition $\mathscr{P}$ can be obtained from one arbitrary fairest semi-matching or one arbitrary fairest fractional allocation (achieved from Theorem 4.4 and Corollary 4.4). Then the specific quotas range for each user among all fairest semi-matchings and all fairest fractional allocations, can be predicted from the vertices partition $\mathscr{P}$ (achieved from Theorem 4.5).*

## 4.4 Conclusion

This chapter studies on the corresponding between the set of fairest semi-matchings (denoted as $\mathcal{F}$) and the set of fairest fractional allocations (denoted as $\mathcal{C}$). The fairest fractional allocations are defined as the fairest ones under the scenario of divisible resources. The main contributions of this chapter include: 1) It is proved that there always exist a set of universally agreed fairest fractional allocations under a bipartite graph, and they have the same quotas vector. 2) The constant vertices (resources and users) partition across the set $\mathcal{F}$ is also constant across both the set $\mathcal{F}$ and the set $\mathcal{C}$ - the allocating of all $M \in \mathcal{F}$ and all $W \in \mathcal{C}$ are all within the partitions. 3) for each user, the quota difference between one $M \in \mathcal{F}$ and one $W \in \mathcal{C}$ is either zero or bound by one, and its specific quota ranges across the set $\mathcal{F}$ and $\mathcal{C}$ can be determined from one $M \in \mathcal{F}$ or $W \in \mathcal{C}$.

# CHAPTER 5

# CONCLUSION AND OPEN PROBLEMS

## 5.1 Conclusions

This dissertation focuses on the properties of the fairest semi-matchings in bipartite graphs. The previous works related to the semi-matching problem, mainly focus on the algorithms design to achieve one fairest semi-matching with respect to one fairness measure. One big concern about the fairest semi-matchings is that whether they are always the fairest no matter which fairness measure is picked? This dissertation addresses this concern and proves the existence of a set of fairest semi-matching(s) which are universally agreed by all fairness measures, and moreover, all of them have the same sorted quotas vector. This work is our first main contribution.

Then we explore deeper about this set of fairest semi-matchings $\mathcal{F}$ on how all of them are related or what they have in common. The main contribution achieved are: given a bipartite graph, from one arbitrary fairest semi-matching (which is easy to achieve), we can understand some important attributes for the entire set of fairest semi-matchings: 1) the classification of the edges in the bipartite graph - whether each edge is used by all, none, or some of the fairest semi-matchings; 2) the partition of user and resource vertices in the bipartite graph - the allocating of all the fairest semi-matchings are all within the partitions, and each user vertex has a very narrow quota range (at most differ by 1, and is predictable from the knowledge gained from one fairest semi-matching) among all the fairest semi-matchings. This work is our second main contribution.

Furthermore, we consider the scenario that the resources are divisible which indicates each resource can be split and allocated to maybe more than one users. In this work, we explore that the similarity between the set of fairest semi-matching ($\mathcal{F}$) and the set of fairest fractional allocations ($\mathcal{C}$). And we conclude that the constant vertices partition across all the fairest semi-matchings is also constant across all the fairest fractional allocations - the allocating of all fairest semi-matchings and all the fairest fractional allocations are always within the partitions. Moreover, for each user, the difference of its quotas between one fairest semi-matching and one fairest fractional allocation is limited (either 0 or bound by 1) and predictable. This work is our third main contribution.

All our main contributions are formally concluded as below.

1. Though the disagreement among numerous fairness measures on semi-matchings comparison, we prove that there always exist a set of the fairest semi-matchings $\mathcal{F}$ which are universally agreed to be equal in fairness by all fairness measures. And a semi-matching belongs to the fairest set iff there exists no Improving Transfer to be applied. Then it can be deduced that the semi-matchings $M$ in the fairest set $\mathcal{F}$ might have different quotas vector, but they all have the same sorted quotas vector, that is, $\forall M_i, M_j \in \mathcal{F}, Q^{\uparrow}(M_i) = Q^{\uparrow}(M_j)$.

   To prove that, a Transfer based Comparison among semi-matchings is defined, which induces a preordered set of all semi-matchings. Then we prove that there always exist a greater element than a pair of incomparable elements in this preordered set. Based on that, the existence of the greatest elements can be proved, and on the greatest elements no Improving Transfer can be applied. Subsequently, we show that the proposed Transfer-based Comparison can strictly imply the Majorization order. In other words, the fairest semi-matchings under the proposed Transfer-based Comparison are always regarded as fairest under Majorization. To our best knowledge, all existing fairness measures in literature are compatible with Majorization. In conclusion, for any bipartite graph, there always exists a set of equally fair semi-matchings, which are universally regarded as the fairest ones, by all existing fairness measures. Moreover, it can be derived that all fairest semi-matchings have the same sorted quotas vector.

2. This piece of work focuses on the understanding of all fairest semi-matching from one. We prove that from one arbitrary fairest semi-matching $M$, one can determine

   - The classification of all edges into $\mathcal{E}_{all}$, $\mathcal{E}_{some}$, $\mathcal{E}_{none}$. We prove that all fairest semi-matchings have the same set of Neutral Transfer Covered Edges which is equal to $\mathcal{E}_{some}$. Then an approach to identify the edge classification based on one fairest semi-matching can be derived.
   - Each user has a very narrow quota range (at most differ by 1, and is predictable from the knowledge gained from one fairest semi-matching) among all the fairest semi-matchings. In order to achieve that, a sequence of *ranked user sets* under a fairest semi-matching $M$ is defined as $\mathscr{P}(M) = (\mathcal{U}^0(M), \mathcal{U}^1(M), \cdots \mathcal{U}^i(M), \cdots, \mathcal{U}^{2m}(M))$ where $m$ is the number of resources, which can be shown as a partition of all users. Then we prove that all fairest semi-matching has the same user partition. Thus the following conclusion can be achieved: $\forall u \in U$, assume $u \in \mathcal{U}^i(M)$, if $i$ is odd, its quota from any fairest semi-matchings is either $(i+1)/2$ or $(i-1)/2$; if $i$ is even, then its quota from any fairest semi-matchings is exactly $i/2$.
   - A constant vertices (users and resources) partition across the set $\mathcal{F}$ - the allocating of all the fairest semi-matchings are all within the partitions. In order to achieve that, a sequence

of ranked resource sets, $\mathcal{R}^0(M), \mathcal{R}^1(M), \cdots \mathcal{R}^i(M), \cdots, \mathcal{R}^{2m}(M)$ where $m$ is the number of resources is defined as: if under $M$ a resource $r$ is assigned to the user $u \in \mathcal{U}^i(M)$, then $r \in \mathcal{R}^i(M)$. We prove that for any $i \in [0, 2m]$, it has $\mathcal{R}^i(M_x) = \mathcal{R}^i(M_y)$ where $M_x, M_y \in \mathcal{F}$.

3. This piece of work focuses on the similarity between the fairest semi-matchings and the fairest fractional allocations. The main theorems are shown as below. In a bipartite graph $G = (R \cup U, E)$, let $\mathscr{P} = (\mathcal{V}^0, \mathcal{V}^1, \cdots \mathcal{V}^i, \cdots, \mathcal{V}^{2m})$ where $\mathcal{V}^i = \{\mathcal{U}^i, \mathcal{R}^i\}$ represent the constant vertices (users and resources) partition across the set $\mathcal{F}$.

   - The constant vertices partition across the set $\mathcal{F}$ is also constant across the set $\mathcal{C}$. For any edge $e_{cross} = \{r, u\}$ where $r \in \mathcal{V}^i$, $u \in \mathcal{V}^j$ and $i \neq j$, then $\forall W = (w(e))_{|E|} \in \mathcal{C}$, it has $w(e_{cross}) = 0$.
   - $\forall u \in U$, assume $u \in \mathcal{U}^i$, then $\forall M \in \mathcal{F}$, $W \in \mathcal{C}$, if $i$ is odd, $Q(M, u) = (i+1)/2$ or $(i-1)/2$, $Q(W, u) \in ((i-1)/2, (i+1)/2)$; if $i$ is even, $Q(M, u) = Q(W, u) = i/2$.

## 5.2 Open Problems

This section discusses a few possible extensions of our work and also reviews related open problems.

1. **How does the graph topology impact on the fairness of the fairest semi-matchings and the number of all fairest semi-matchings?**

   Obviously, the bipartite graph topology such as dense/sparse, balanced/unbalance, etc. will determine some properties of the set of the fairest semi-matchings, such as the fairness of the fairest ones, the size of the set, etc. One extension of our work could be the research on how they are determined by the graph topology. Some possible benefits of this research includes but not limited to: the achievable *optimal* fairness (the fairness of the fairest semi-matchings) might be estimated, before the achieving of one fairest semi-matching which could be expensive especially for large-scale network; the estimated number of the optimal solutions could be one important factor for some theoretical or practical applications.

Some insights of this work are shows as below. 1) With regarding to the optimal fairness, one impact factor might be the skewness of the *vertices degree* where the degree of each user/resource is defined as the number of connected resources/users in a bipartite graph. Another impact factor might be *connectivity* of the bipartite graph, which can describe the "freedom" of the resource re-allocating. For instance, under the *full access* where every resource can be accessed by every user, then the evenly distributed allocations can be achieved. 2) With regarding to the number of all fairest semi-matchings, the Neutral Transfers play a key role. As shown in this dissertation, the application of one Neutral Transfer will generate one new fairest semi-matching. Thus, the number of Neutral Transfers and how they overlapped with each other should be important factors for the total numbers of the fairest semi-matchings. Furthermore, the similarity between the fairest fractional allocations and the fairest semi-matching might be helpful for this number counting. One fairest fractional allocation can be achieved by solving a linear programming problem, and the rank of coefficient matrix related to the linear programming might be correspondent to the number count of all fairest semi-matchings.

2. **How does the graph topology impact on the edge classification profile?**

As shown in this dissertation, from one arbitrary fairest semi-matching, all edges can be classified into the edges used by all/some/none ($\mathcal{E}_{all}$, $\mathcal{E}_{some}$, $\mathcal{E}_{none}$) of the fairest semi-matchings. The edge classification profile can be described as the size ratios of $\mathcal{E}_{all}$, $\mathcal{E}_{some}$, $\mathcal{E}_{none}$ over $\mathcal{E}$. One hypothesis is that a bipartite graph with dense edges will show a high ratio of $\mathcal{E}_{some}$, because of the possible large number of alternative allocating options. Moreover, whether a graph is balanced or unbalanced, might also affect the ratios, but it is not obvious about the relationship between them. The exploration of how the edge classification profile affected by the graph topology, is value-able. The edges utilization for achieving the optimal solution thus can be estimated, which provides a guide on the construction of an "efficient" or "cheap" graph for achieving a fairest allocation.

3. **The fairest semi-matching problem on weighted bipartite graph**

   One related open question in literature is the fairest semi-matching problem on weighted bipartite graph. The settings of that problem are the same with that of the problem studied in this dissertation except the resources are weighted, and the quota of a user is defined as the sum of all weights from its allocated resources. Then the goal is for achieving a fairest semi-matching and the fairness is considered based on the sorted quotas vector. This problem has been proved to be a NP-hard problem, and many approximate algorithms have been proposed to improve the solution.

   Some insights of this problem can be derived from the conclusions achieved in this paper. We have concluded some on the relationship between the fairest semi-matchings (with unit indivisible resources) and the fairest fractional allocations (with unit divisible resources), and shown the fairest semi-matchings are similar with the fairest fractional allocations. Now for the weighted bipartite graph, if we relax the resources as the ones which can be divided into unit resources, then it actually becomes to the same problem (the fairest semi-matching on unweighted bipartite graph) studied in this dissertation. We hypothesize that, the relationship between the optimal solutions under weighted resources and unweighted resources can be deduced from the same idea for the relationship between the optimal solutions under unit indivisible resources and unit divisible resources. If this hypothesis can be testified, it might be able to provide the foundation for approximate algorithms, or the analysis of approximate ratio for some algorithms.

# APPENDIX

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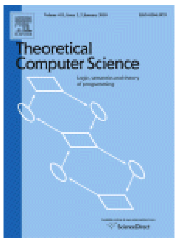


**Title:** The existence of universally agreed fairest semi-matchings in any given bipartite graph

**Author:** Jian Xu,Soumya Banerjee,Wenjing Rao

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**Jian Xu** and Soumya Banerjee, Wenjing Rao, "The Existence of University Agreed Fairest Semi-matchings in Any Given Bipartite Graph", Theoretical Computer Science, 2018

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