An Analysis of Multivariate Final-Offer Arbitration

BY

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LIST OF ABBREVIATIONS

$d\mathrm{NL}_1$	$d-$Dimensional Normal L_1 distance
$d\mathrm{NL}_2$	$d-$Dimensional Normal L_2 distance
dNNO	$d-$Dimensional Normal Net Offer
$2\mathrm{NL}_1$	Bivariate Normal L_1 distance
$2\mathrm{NL}_2$	Bivariate Normal L_2 distance
$2\mathrm{NL}_\infty$	Bivariate Normal L_∞ distance
$2\mathrm{NL}_p$	Bivariate Normal L_p distance
2NMD	Bivariate Normal Mahalanobis distance
2NNO	Bivariate Normal, Net Offer
$2\mathrm{UL}_2$	Bivariate Uniform L_2 distance
CA	Conventional Arbitration
FOA	Final-Offer Arbitration
IBIFOA	Issue-By-Issue Final-Offer Arbitration
MD	Mahalanobis distance
WPFOA	Whole Package Final-Offer Arbitration

SUMMARY

When negotiations fail, arbitration is often an effective means by which a binding resolution can be found. To address the many shortcomings of conventional arbitration, many industries have been using a variation called *Final-Offer Arbitration* since the 1970s. The mechanics are simple - rather than crafting a compromise, the judge must choose one of the two final offers proposed by the parties. Variants of the single-issue arbitration scenario, modeled as a two-player game, have been studied, but very little has been said about the game theoretic properties of the multi-issue case. In this work we define various game models for two or more issues under arbitration, study the conditions under which optimal pure strategies exist, derive these strategies, and in some cases prove that they are the unique globally optimal strategies. In particular, we look at modeling the uncertainty of arbitrator behavior with either a normal or uniform distribution, and consider a number of metrics the judge may use to make his ruling (pun intended).

CHAPTER 1

Introduction

Should negotiating parties fail to arrive at an agreeable solution, *arbitration* serves as a mechanism whereby a binding resolution may be reached. In *conventional arbitration (CA)*, the disputing parties submit their cases to an agreed upon arbiter who has full power to craft whatever fair and just settlement he sees fit. It is widely accepted, however, that CA has a number of undesirable properties, in particular what has been called the "chilling effect": since both parties know the arbiter will craft a compromise, they tend to take extreme positions. Since it is commonly held that a settlement reached through negotiation is preferable to a settlement reached through arbitration, one can view the purpose of a compulsory arbitration as motivating the parties to reach an agreement during negotiations. This is the paradox of arbitration: the best arbitration mechanism is that which is used least often.

It was Stevens (1966) who suggested a simple arbitration mechanism now known as *Final-Offer Arbitration (FOA)*. In FOA, the arbiter must select one of the *final offers* submitted by the parties and has no prerogative to craft a compromise settlement. The theory was that such uncertainty in the final outcome would combat this chilling effect driving the two parties to make final offers that are "close" to one another, or better still motivate them to reach agreement during negotiations.

Since 1975 when FOA was adopted by Major League Baseball for salary disputes, variants of FOA have been used in various states in public sectors where labor does not have the right to strike (e.g. police, firefighters). A growing body of literature has been developed by legal scholars, economists and game theorists studying both the theoretical and empirical properties of FOA.

The first theoretical model of FOA was introduced by Crawford (1979). With the assumption that both parties know with certainty the arbiter's opinion of a "fair" settlement, he showed that FOA would inevitably lead to the same outcome as conventional arbitration. Farber (1980), Chatterjee (1981), and Brams and Merrill (1983) independently developed game theoretic models of single-issue FOA for which players are uncertain of the arbiter's behavior. Farber studied the effect of risk aversion by one of the parties, and derived the strategy pair which in many cases is a Nash equilibrium. Chatterjee and Brams and Merrill model the game as zero-sum and consequently assumed both parties are risk-neutral. Brams and Merrill provide sufficient conditions for the existence of a pure equilibrium. In all three models, the arbiter is assumed by the players to choose a "fair" settlement from a probability distribution commonly known to both players and select whichever player's offer is closest in absolute value.

The basic model has been extended and analyzed in a number of ways in the literature. Samuelson (1991) developed a model of single-issue FOA where parties' knowledge of a fair settlement is asymmetric. Kilgour (1994) studied the game theoretic properties of FOA and extended the Brams-Merrill model to allow for risk-aversion on the part of the players. Dickinson (2006) further showed that optimism on the part of the players, in the form of a biased prior distribution, drives the final-offers apart. Armstrong and Hurley (2002) generalized FOA and CA into a single model and showed that optimal offers under CA will always diverge more than those under FOA. Mazalov and Tokareva (2012) considered an extension where the decision is made by multiple arbitrators.

If multiple issues are in dispute, FOA has been primarily implemented in two ways (Stern, 1975). Under *Issue-by-Issue FOA (IBIFOA)*, the arbiter may craft a compromise of sorts from the two parties' offers by choosing some components from one and some from the other. Alternatively, *Whole Package FOA (WPFOA)* requires that the arbiter select one offer in its entirety. A multi-issue model of FOA was first discussed by Crawford (1979) and further developed by Wittman (1986). Here the main concern was the existence of a Nash equilibrium under various assumptions. Wittman was also able to show in his model that increased risk-aversion leads a player to make a less extreme final-offer. Olson (1992) discussed how the single-issue model does not accurately reflect arbiter behavior when more than one issue is in dispute.

In his initial paper introducing FOA, Stevens cautions against the use of the "Whole Package" variant, stating that "such a system would run the danger of generating unworkable awards...the arbitration authority might be forced to choose between two extreme positions, each of which was unworkable"(Stevens, 1966). Tulis (2013) elaborates: "One common criticism of package final-offer arbitration is that parties may be tempted to include outrageous offers." He further claims that "issue-by-issue final offers...are more aligned with the objectives of final-offer arbitration." We argue the opposite - that both players' optimal strategy in a multiple-issue FOA is to make all final-offer components reasonable. Furthermore, the additional variance in the awards from WP, as opposed to IBI, acts as a greater motivator for the parties to reach agreement during negotiations. We show this by extending the model of Brams and Merrill to multiple-issues and proceed to explicitly construct a pure strategy pair, proving it is the unique optimal strategy pair in many cases.

The outline of this work is as follows: In Chapter 2 we review the single-issue FOA model as it appears in the work of Brams and Merrill, and in particular highlight the theorem which explicitly gives the globally optimal pure strategy solution to the game. In Chapter 3 we define the game model for a multi-issue version of the zero-sum game. Chapter 4 explores the two issue case in great detail. Where the uncertainty is modeled by a normal distribution, we look at six decision criteria which may be used by the judge to determine which offer is more reasonable, deriving the only possible optimal pure strategies in each case. In the case of the L_2 metric, the strategies are further shown to be locally optimal under certain conditions of the covariance matrix, and globally optimal under slightly more restrictive conditions. The chapter is concluded by considering uncertainty modeled by a uniform distribution on a rectangular region. In Chapter 5 we consider the game extended to arbitrary dimensional final-offer vectors, and derive the possible optimal pure strategies.

CHAPTER 2

Single-Issue Final-Offer Arbitration

For context, suppose an employer and worker's union are in negotiations for a wage increase. Negotiations stall and the parties are contractually obligated to resolve the impass via FOA. Across the industry, let us assume that the population of wages follow a normal distribution. We will also assume that the offer of the employer (Player I, the minimizer) is no greater than the demand of the workers (Player II, the maximizer).

The judge has his clerks perform a sampling of the industry to compute and average salary, which the judge takes to be a 'fair' standard for comparison to the parties' final-offers. Let ξ be the random value chosen by the judge, drawn from $\mathcal{N}(\mu, \sigma^2)$. Let x_1 be the company offer and x_2 the demand from the workers' union. The expected ruling of the judge is thus

$$x_1 P(|x_1 - \xi| < |x_2 - \xi|) + x_2 P(|x_1 - \xi| > |x_2 - \xi|)$$

or expressed another way,

$$x_1 P\left(\xi < \frac{x_1 + x_2}{2}\right) + x_2 P\left(\xi > \frac{x_1 + x_2}{2}\right).$$

Letting $F(x) = P(\xi < x)$, we may write this as

$$x_1 F\left(\frac{x_1 + x_2}{2}\right) + x_2\left(1 - F\left(\frac{x_1 + x_2}{2}\right)\right) = (x_1 - x_2) F\left(\frac{x_1 + x_2}{2}\right) + x_2.$$

Call this $K(x_1, x_2)$. Suppose pure optimal strategies x_1^*, x_2^* exist. Because the expected payoff is a continuous differentiable function of x_1 and x_2, it must be the case that $\frac{d}{dx_1}K(x_1^*, x_2^*) = 0$, that is

$$\frac{x_1^* - x_2^*}{2} f\left(\frac{x_1^* + x_2^*}{2}\right) + F\left(\frac{x_1^* + x_2^*}{2}\right) = 0. \tag{1}$$

Similarly, $\frac{d}{dx_2}K(x_1^*, x_2^*) = 0$, that is

$$\frac{x_1^* - x_2^*}{2} f\left(\frac{x_1^* + x_2^*}{2}\right) - F\left(\frac{x_1^* + x_2^*}{2}\right) + 1 = 0. \tag{2}$$

Subtracting (2) from (1), we get

$$F\left(\frac{x_1^* + x_2^*}{2}\right) = \frac{1}{2} \tag{3}$$

Thus, $\frac{x_1^* + x_2^*}{2} = \mu$. If we instead add (1) and (2) we get

$$f(\mu) = \frac{1}{\sigma\sqrt{2\pi}} = \frac{1}{x_2^* - x_1^*} \tag{4}$$

So we arrive at the solution; if pure optimal strategies x_1^*, x_2^* exist, they are given by

$$x_1^* = \mu - \frac{\sigma\sqrt{2\pi}}{2} \quad x_2^* = \mu + \frac{\sigma\sqrt{2\pi}}{2}.$$

If we consider more generally any density function f with median 0, Brams and Merrill (1983) provide the following results:

THEOREM 2.1. *Suppose the arbitrator's notion of a fair settlement has a continuous distribution with density function f and distribution function F where $F' = f$. Assume, without loss of generality that the median of F is 0.*

(1) *If $f'(0)$ exists and $f(0) > 0$ then if $|f'(0)| < 4f(0)^2$, locally optimal strategies for the minimizer and maximizer respectively are*

$$a^* = -\frac{1}{2f(0)}, b^* = \frac{1}{2f(0)}.$$

(2) If $f(0) > 0$, then the strategies given above are globally optimal if and only if the following conditions hold:

$$\int_0^x f(t)dt \leq \frac{x}{2b^* - 2x} \text{ for } 0 < x \leq \frac{1}{4f(0)},$$

$$\int_0^x f(t)dt \geq \frac{x}{2b^* + 2x} \text{ for } x > 0,$$

and the same inequalities hold for $\int_{-x}^0 f(t)dt$ in place of $\int_0^x f(t)dt$.

It is worth noting that among the distributions which satisfy the conditions for global optimality are the normal distribution and the uniform distribution. This theorem will be instrumental throughout the rest of this work.

CHAPTER 3

General Multi-Issue Final-Offer Arbitration

Our model extends the model defined by Brams and Merrill (1983). Let us consider the case where each player makes not a single valued offer, but an d-tuple $\mathbf{x}_i = (x_1^i, \ldots, x_d^i)$, $i = 1, 2$. We will assume that the issues in dispute are quantitative in nature, with both players restricted to a strategy space S which is an arbitrarily large, but *compact*, subset of $\mathbb{R}^d$. Let $v_1(\mathbf{x})$ be the valuation of a settlement vector $\mathbf{x}$ by Player I and $v_2(\mathbf{x})$ the valuation function for Player II. (consequently, Player II values this settlement as $-v(x, y)$). Suppose the arbiter has in mind a fair settlement $\xi \in \mathbb{R}^d$, and measures the "reasonable-ness" of a final-offer demand by a function $R : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. Then the payoff of the game to Player i is given by

$$(5) \qquad K_i(\mathbf{x}_1, \mathbf{x}_2) = \begin{cases} v_i(\mathbf{x}_1) & R(\mathbf{x}_1, \xi) > R(\mathbf{x}_2, \xi) \\ v_i(\mathbf{x}_2) & R(\mathbf{x}_1, \xi) < R(\mathbf{x}_2, \xi) \\ \frac{1}{2}v_i(\mathbf{x}_1) + \frac{1}{2}v_i(\mathbf{x}_2) & R(\mathbf{x}_1, \xi) = R(\mathbf{x}_2, \xi) \end{cases}$$

where the arbiter chooses either offer with equal probability if, in his opinion, they are equally reasonable. This raises three important questions: How do players value settlement bundles, how do they model the uncertainty of the arbiter's opinion of fairness, and how do we define the function R?

1. Valuation of Settlements

For the rest of this work we will assume the valuation is additive. An example of such a situation is one in which wage and workers compensation amounts are in dispute; workers' compensation may be valued at the *expected* compensation amount (in the probabilistic sense). Even issues which are not monetary, such as number of sick days, may have a straightforward monetary valuation by the parties. Furthermore, we will assume that the game is zero-sum; giving Player I the role of the Minimizer (e.g. employer) and Player II the role of the Maximizer (e.g. workers' union) we have

$$v_1(x_1, \ldots, x_d) = -\sum_{j=1}^{d} x_j = -v_2(x_1, \ldots, x_d).$$

While it has been noted assumptions of additivity in valuation "suppress multi-dimensionality and, in fact, degenerate it into a univariate case" (David Levhari, 1975), even with this simple assumption the model produces interesting results.

2. Models of Uncertainty

Both players are uncertain of the arbiter's opinion of a fair settlement $\boldsymbol{\xi}$. We will assume that the players share a prior assumption that $\boldsymbol{\xi}$ is drawn from a probability distribution with density f, and it is common knowledge (Aumann). We will specifically consider two special cases in the analysis which follows: the normal and uniform distributions.

Normal Distribution

Suppose the arbiter (or a fact-finder) is sampling from relevant industry data to form an opinion. Thus, by the Central Limit Theorem, we may suppose that their common prior distribution for $\boldsymbol{\xi}$ is a multivariate normal distribution, $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and let us assume without loss of generality that $\boldsymbol{\mu} = \mathbf{0}$.

Uniform Distribution

In this case we will suppose the players assume only that the arbiter will choose $\boldsymbol{\xi} \in \bigtimes_{j=1}^{d} [-\alpha_j, \alpha_j]$ with uniform probability density, where $\alpha_j > 0$.

3. Decision Criteria

Under WPFOA the arbiter must rule in favor of one final-offer vector in its entirety. It is in this variant that the choice of a distance criterion needs to be chosen by the arbiter. The "distance" from a final-offer vector $\mathbf{x}$ to $\boldsymbol{\xi}$ may be determined in a number of ways. The following decision metrics will be considered in the sections which follow

3.1. Net Offer Absolute Difference. In perhaps the simplest case, the arbiter may take the net offer from each player and side with whichever is closer to $\sum_j \xi_j$.

$$R_{NO}(\mathbf{x}, \boldsymbol{\xi}) = -\left|\sum_{j=1}^{d} x_j - \xi_j\right| \tag{6}$$

3.2. L2 Distance. If the arbiter uses L_2 (Euclidean) distance, we may equivalently let

$$R_{L_2}(\mathbf{x}, \boldsymbol{\xi}) = -\sum_{j=1}^{d} (x_j - \xi_j)^2 \tag{7}$$

3.3. L1 Distance. Suppose, for example, the arbiter wishes to measure deviation from fair component-wise; an L_1 distance is appropriate in this case.

$$R_{L_1}(\mathbf{x}, \boldsymbol{\xi}) = -\sum_{j=1}^{d} |x_j - \xi_j| \tag{8}$$

3.4. L Infinity Distance. If instead he finds a large component deviation particularly disagreeable, an L_∞ (Chebychev) distance may be used.

$$R_{L_\infty}(\mathbf{x}, \boldsymbol{\xi}) = -\max_j \{|x_j - \xi_j|\} \tag{9}$$

3.5. Lp Metric. We will generalize the three previous metrics with an L_p metric (i.e. $p \geq 1$)

$$R_{L_p}(\mathbf{x}, \boldsymbol{\xi}) = -\sum_{j=1}^{d} |x_j - \xi_j|^p \tag{10}$$

3.6. Mahalanobis Distance. In the case where $\boldsymbol{\xi}$ follows a normal distribution with covariance matrix Σ, a Mahalanobis distance (Mahalanobis (1936)) may be appropriate to consider

$$R_M(\mathbf{x}, \boldsymbol{\xi}) = -(\mathbf{x} - \boldsymbol{\xi})'\Sigma^{-1}(\mathbf{x} - \boldsymbol{\xi}) \tag{11}$$

CHAPTER 4

Dual-Issue Final-Offer Arbitration

Let Player I be the minimizer and Player II the maximizer in this zero-sum game. Let us consider the case where each player makes not a single valued offer, but an ordered pair (x_i, y_i), $i = 1, 2$. Let $v(x, y)$ be the valuation of a settlement vector (x, y) by Player I (consequently, Player II values this settlement as $-v(x, y)$). If the arbiter measures the "reasonable-ness" of a final-offer by a function R, then the payoff of the game is given by

$$K\big((x_1, y_1), (x_2, y_2)\big) = \begin{cases} v(x_1, y_1) & R(x_1, y_1) > R(x_2, y_2) \\ v(x_2, y_2) & R(x_1, y_1) < R(x_2, y_2) \\ \frac{1}{2}v(x_1, y_1) + \frac{1}{2}v(x_2, y_2) & R(x_1, y_1) = R(x_2, y_2) \end{cases} \tag{12}$$

where the arbiter chooses either offer with equal probability if they are equally reasonable (in his opinion). This raises three important questions: How do players value settlement bundles, how do we model the function R and how to model the uncertainty of the arbiter's opinion of fairness.

We will assume that the two issues in dispute are quantitative in nature, with both players restricted to a strategy space S which is an arbitrarily large, but *compact*, subset of $\mathbb{R}^2$[1]. Furthermore, we will assume the valuation is additive, namely $v(x, y) = x + y$.

1. Normal Distribution

Both players are uncertain of the arbiter's opinion of a fair settlement (ξ, η), but assume that the arbiter (or a fact-finder) is sampling from relevant industry data to

[1]Otherwise, the game may not even posses a value.

form an opinion. Thus, by the Central Limit Theorem, we may suppose that their common prior distribution for (ξ, η) is a bivariate normal distribution, $\mathcal{N}(\mu, \mathbf{\Sigma})$ and it is common knowledge (Aumann). Let us assume without loss of generality that $\mu = \mathbf{0}$. We will assume that these issues are positively correlated across the industry, thus

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix},$$

where $\rho > 0$.

In the multi-issue case, FOA is typically handled in one of two ways: Issue-by-Issue (IBI) or Whole-Package (WP). Under IBIFOA the arbiter rules independently on each issue presented. A compromise of sorts may be crafted in this way. If the arbiter uses the IBI mechanic, the players are engaged in two independently decided single-issue FOA games. By the Brams-Merrill Theorem (1983), we know that the unique optimal strategy pair of the players is given by

$$(13) \qquad (x_1^*, y_1^*) = \left(-\frac{\sigma_x\sqrt{2\pi}}{2}, -\frac{\sigma_y\sqrt{2\pi}}{2}\right) \quad (x_2^*, y_2^*) = \left(\frac{\sigma_x\sqrt{2\pi}}{2}, \frac{\sigma_y\sqrt{2\pi}}{2}\right).$$

Under WPFOA the arbiter must rule in favor of one final-offer vector in its entirety. It is in this variant that the choice of a distance criterion needs to be chosen by the arbiter. The "distance" from a final-offer point (x_i, y_i) to (ξ, η) may be determined in a number of ways.

1.1. Net Offer. In this model, 2NNO (Bivariate Normal, Net Offer), the judge considers only the net final-offer from each player. The payoff is

$$K(\mathbf{x}_1, \mathbf{x}_2) = \begin{cases} x_1 + y_1 & \text{if } |x_1 - \xi + y_1 - \eta| < |x_2 - \xi + y_2 - \eta| \\ x_2 + y_2 & \text{if } |x_1 - \xi + y_1 - \eta| > |x_2 - \xi + y_2 - \eta| \end{cases}$$

We notice that if we let $\Omega = \xi + \eta$, and $w_i = x_i + y_i$, this simplifies to

$$K(w_1, w_2) = \begin{cases} w_1 & \text{if } |w_1 - \Omega| < |w_2 - \Omega| \\ w_2 & \text{if } |w_1 - \Omega| > |w_2 - \Omega| \end{cases}$$

The random variable Ω follows a normal distribution $\mathcal{N}(0, \sigma_\Omega^2)$, where $\sigma_\Omega^2 = \alpha + \beta$ ($\alpha = \sigma_x^2 + \rho\sigma_x\sigma_y, \beta = \sigma_y^2 + \rho\sigma_x\sigma_y$). This solution is reducible to the univariate case, with global optimal strategies given by

$$w_1^* = -\frac{\sqrt{2\pi(\alpha+\beta)}}{2} \quad w_2^* = \frac{\sqrt{2\pi(\alpha+\beta)}}{2}.$$

This however gives each player set of solution points. Thus we have the following result:

THEOREM 4.1. *In 2NNO, any pair of strategies $\mathbf{x}_1 \in S_1$ and $\mathbf{x}_2 \in S_2$ are optimal, where*

$$S_1^* = \left\{ \left(x_1^*, -\frac{\sqrt{2\pi(\alpha+\beta)}}{2} - x_1^* \right) : x_1^* \in \mathbb{R} \right\}$$
$$S_2^* = \left\{ \left(x_2^*, \frac{\sqrt{2\pi(\alpha+\beta)}}{2} - x_2^* \right) : x_2^* \in \mathbb{R} \right\}.$$

1.2. L2 Distance. For our second model, 2NL$_2$ (Bivariate Normal L_2), we will assume that the arbiter uses L_2 (Euclidean) distance:

$$D_{L_2}\big((x,y),(\xi,\eta)\big) = \sqrt{(\xi-x)^2 + (\eta-y)^2}. \tag{14}$$

1.2.1. *Properties of 2NL2.* We now establish some properties of the game. Suppose Player I chooses pure strategy $\mathbf{a} = (x_1, x_2)$ and Player II chooses pure strategy $\mathbf{b} = (x_2, y_2)$, and the arbiter considers (ξ, η) a fair settlement. We define $C_i(\mathbf{a}, \mathbf{b})$, as the set of points in $\mathbb{R}^2$ which are strictly closer to Player i's final-offer than to the other player's, namely

$$C_1(\mathbf{a},\mathbf{b}) := \left\{ (x,y) : (x_1-x)^2 + (y_1-y)^2 < (x_2-x)^2 + (y_2-y)^2 \right\}, \tag{15}$$

$$C_2(\mathbf{a},\mathbf{b}) := \left\{(x,y) : (x_1-x)^2+(y_1-y)^2 > (x_2-x)^2+(y_2-y)^2\right\}. \tag{16}$$

It is immediately apparent that $C_1(\mathbf{a},\mathbf{b}) = C_2(\mathbf{b},\mathbf{a})$. The midset is

$$Mid(\mathbf{a},\mathbf{b}) := \left\{(x,y) : (x_1-x)^2+(y_1-y)^2 = (x_2-x)^2+(y_2-y)^2\right\}. \tag{17}$$

We observe that if $\mathbf{a} \neq \mathbf{b}$ then $Mid(\mathbf{a},\mathbf{b})$ is a line so, because (ξ,η) follows a continuous distribution, $P\big((\xi,\eta) \in Mid(\mathbf{a},\mathbf{b})\big) = 0$. We can now define the expected payoff to Player II from I

(18)

$$K(\mathbf{a},\mathbf{b}) = \begin{cases} x_1+y_1 & \mathbf{a}=\mathbf{b} \\ (x_1+y_1)P\big((\xi,\eta)\in C_1(\mathbf{a},\mathbf{b})\big) + (x_2+y_2)P\big((\xi,\eta)\in C_2(\mathbf{a},\mathbf{b})\big) & \mathbf{a}\neq\mathbf{b} \end{cases}$$

The following are some consequential properties of the game.

LEMMA 4.1. *$K(\mathbf{a},\mathbf{b}) = K(\mathbf{b},\mathbf{a})$.*

The first property concerns the **anonymity of final-offers**; the arbiter essentially does not care which player submits which final-offer.

PROOF. If $\mathbf{a}=\mathbf{b}$ the proof is trivial. Assume $\mathbf{a}\neq\mathbf{b}$.

$$\begin{aligned} K(\mathbf{a},\mathbf{b}) &= (x_1+y_1)P\big((\xi,\eta)\in C_1(\mathbf{a},\mathbf{b})\big) + (x_2+y_2)P\big((\xi,\eta)\in C_2(\mathbf{a},\mathbf{b})\big) \\ &= (x_1+y_1)P\big((\xi,\eta)\in C_2(\mathbf{b},\mathbf{a})\big) + (x_2+y_2)P\big((\xi,\eta)\in C_1(\mathbf{b},\mathbf{a})\big) \\ &= (x_2+y_2)P\big((\xi,\eta)\in C_1(\mathbf{b},\mathbf{a})\big) + (x_1+y_1)P\big((\xi,\eta)\in C_2(\mathbf{b},\mathbf{a})\big) \\ &= K(\mathbf{b},\mathbf{a}) \end{aligned}$$

$\square$

LEMMA 4.2. *Let $-\mathbf{a} = (-x_1,-y_1)$ and $-\mathbf{b} = (-x_2,-y_2)$. Then $K(-\mathbf{a},-\mathbf{b}) = -K(\mathbf{a},\mathbf{b})$.*

This is due to the symmetry of the bivariate normal distribution about $(0,0)$. If the players negate their offers then they are effectively swapping roles.

PROOF. This proof makes use of two facts: First, $(\xi,\eta) \in C_i(\mathbf{a},\mathbf{b}) \Leftrightarrow (-\xi,-\eta) \in C_i(-\mathbf{a},-\mathbf{b})$, $i = 1,2$. Secondly, (ξ,η) and $(-\xi,-\eta)$ follow the same probability distribution.

$$\begin{aligned}
K(-\mathbf{a},-\mathbf{b}) &= (-x_1 - y_1)P\big((\xi,\eta) \in C_1(-\mathbf{a},-\mathbf{b})\big) + (-x_2 + -y_2)P\big((\xi,\eta) \in C_2(-\mathbf{a},-\mathbf{b})\big) \\
&= -\Big((x_1 + y_1)P\big((\xi,\eta) \in C_1(-\mathbf{a},-\mathbf{b})\big) + (x_2 + y_2)P\big((\xi,\eta) \in C_2(-\mathbf{a},-\mathbf{b})\big)\Big) \\
&= -\Big((x_1 + y_1)P\big((-\xi,-\eta) \in C_1(\mathbf{a},\mathbf{b})\big) + (x_2 + y_2)P\big((-\xi,-\eta) \in C_2(\mathbf{a},\mathbf{b})\big)\Big) \\
&= -\Big((x_1 + y_1)P\big((\xi,\eta) \in C_1(\mathbf{a},\mathbf{b})\big) + (x_2 + y_2)P\big((\xi,\eta) \in C_2(\mathbf{a},\mathbf{b})\big)\Big) \\
&= -K(\mathbf{a},\mathbf{b})
\end{aligned}$$

$\square$

LEMMA 4.3. *Let* $-\mathbf{b} = (-x_2,-y_2)$. *Then* $K(-\mathbf{b},\mathbf{b}) = 0$.

In other words, because the probability distribution of (ξ,η) is symmetric, the arbiter is unbiased with regards to negating offers.

PROOF. This proof also relies on the fact that (ξ,η) and $(-\xi,-\eta)$ follow the same probability distribution.

$$\begin{aligned}
K(-\mathbf{b},\mathbf{b}) &= (-x_2,-y_2)P\big((\xi,\eta) \in C_1(-\mathbf{b},\mathbf{b})\big) + (x_2 + y_2)P\big((\xi,\eta) \in C_2(-\mathbf{b},\mathbf{b})\big) \\
&= (x_2 + y_2)\Big(P\big((\xi,\eta) \in C_2(-\mathbf{b},\mathbf{b})\big) - P\big((\xi,\eta) \in C_1(-\mathbf{b},\mathbf{b})\big)\Big) \\
&= (x_2 + y_2)\Big(P\big((\xi,\eta) \in C_2(-\mathbf{b},\mathbf{b})\big) - P\big((-\xi,-\eta) \in C_1(\mathbf{b},-\mathbf{b})\big)\Big) \\
&= (x_2 + y_2)\Big(P\big((\xi,\eta) \in C_2(-\mathbf{b},\mathbf{b})\big) - P\big((-\xi,-\eta) \in C_2(-\mathbf{b},\mathbf{b})\big)\Big) \\
&= (x_2 + y_2)\Big(P\big((\xi,\eta) \in C_2(-\mathbf{b},\mathbf{b})\big) - P\big((\xi,\eta) \in C_2(-\mathbf{b},\mathbf{b})\big)\Big) \\
&= 0
\end{aligned}$$

□

LEMMA 4.4. *The value of* $2NL_2$ *is zero.*

It may seem deceptively obvious that a symmetric zero-sum game must have a value of zero, but we have no guarantee that a value exists at all. For example, consider a simple symmetric game where Player I and II choose $x, y \in \mathbb{R}$, and I receives a payment from II of $x - y$. The game has no value as $\sup_y \inf_x (x - y) = -\infty$ while $\inf_x \sup_y (x - y) = \infty$.

PROOF. Because the strategy space S of each player is compact and the payoff K is continuous, by the general minimax theorem of Ville the game has a value v in mixed strategies (see Parthasarathy and Raghavan (1971)).

First suppose an optimal pure strategy pair $\mathbf{a}^*, \mathbf{b}^*$ exists. Suppose $v > 0$. Then for any pure strategy $\mathbf{a}$ of Player I, $K(\mathbf{a}, \mathbf{b}^*) \geq v > 0$. But by Lemma 4.3 $K(-\mathbf{b}^*, \mathbf{b}^*) = 0$, contradicting that $v > 0$. Similarly it cannot be the case that $v < 0$. Therefore $v = 0$.

Now suppose that optimal mixed strategies F_1^*, F_2^* exist. Suppose $v > 0$. Then for any mixed strategy F_1,

$$K(F_1, F_2^*) \geq v > 0. \tag{19}$$

Player II may approximate the optimal strategy F_2^* by $\hat{F}_2^*$ where probability mass is concentrated only on a finite symmetric subset $T \subset S$ such that for $\epsilon > 0$ small enough and for any mixed strategy F_1,

$$K(F_1, \hat{F}_2^*) \geq v - \epsilon > 0.^2 \tag{20}$$

[2]This claim follows from three arguments: First, by a well known theorem of Varadarajan the space of all probability measures $M(S)$ is a compact metric space in weak topology. Secondly, the set of all probability measures under weak topology concentrated on finite subsets of a compact metric space S are themselves dense in the space of all probability measures on S. Lastly, by a well known theorem of Prohorov, any compact subset T of $M(S)$ is characterized by the property that given δ positive, there exists a compact subset of C of T such that $\mu(C) > 1 - \delta$ for all μ in the set S. (See Parthasarathy (2014))

Define

$$g_1^*(x,y) = \hat{f}_2^*(-x,-y), \forall (x,y) \in T$$

and call the associated mixed strategy G_1^*.

$$\begin{aligned}
K(G_1^*, \hat{F}_2^*) &= \sum_{(\mathbf{a},\mathbf{b}) \in T \times T} g_1^*(\mathbf{a}) \hat{f}_2^*(\mathbf{b}) K(\mathbf{a},\mathbf{b}) \\
&= \sum_{(\mathbf{a},\mathbf{b}) \in T \times T} \hat{f}_2^*(-\mathbf{a}) g_1^*(-\mathbf{b}) K(\mathbf{a},\mathbf{b}) \\
&= \sum_{(\mathbf{a},\mathbf{b}) \in T \times T} \hat{f}_2^*(-\mathbf{a}) g_1^*(-\mathbf{b}) K(\mathbf{b},\mathbf{a}) \\
&= -\sum_{(\mathbf{a},\mathbf{b}) \in T \times T} g_1^*(-\mathbf{b}) \hat{f}_2^*(-\mathbf{a}) K(-\mathbf{b},-\mathbf{a})
\end{aligned}$$

With a change of variables $\mathbf{c} = -\mathbf{b}, \mathbf{d} = -\mathbf{a}$,

$$\begin{aligned}
&= -\sum_{(\mathbf{c},\mathbf{d}) \in T \times T} g_1^*(\mathbf{c}) \hat{f}_2^*(\mathbf{d}) K(\mathbf{c},\mathbf{d}) \\
&= -K(G_1^*, \hat{F}_2^*)
\end{aligned}$$

Therefore $K(G_1^*, \hat{F}_2^*) = 0$, contradicting (19), so $v \leq 0$. In a similar manner we can show that $v \geq 0$. $\square$

Having established that the game has a value, we know the players must have optimal mixed strategies. The key contribution of this paper is that in fact the players have *optimal pure strategies.*

LEMMA 4.5. (x_2, y_2) *is an optimal pure strategy for Player II if and only if* $(-x_2, -y_2)$ *is an optimal pure strategy for Player I.*

This is to say, if optimal pure strategies do exist then they must be symmetric about the origin.

PROOF. Suppose $\mathbf{b}^* = (x_2^*, y_2^*)$ is an optimal pure strategy for Player II. Because the value of the game is zero,

$$K(\mathbf{a}, \mathbf{b}^*) \geq 0, \forall \mathbf{a}. \tag{21}$$

If $-\mathbf{b}^*$ is not an optimal pure strategy for Player I then there exists $\mathbf{b}^\circ$ such that

$$K(-\mathbf{b}^*, \mathbf{b}^\circ) > 0.$$

By Lemmas 4.1 and 4.2,

$$\begin{aligned} K(-\mathbf{b}^\circ, \mathbf{b}^*) &= K(\mathbf{b}^*, -\mathbf{b}^\circ) \\ &= -K(-\mathbf{b}^*, \mathbf{b}^\circ) \\ &< 0 \end{aligned}$$

but this contradicts (21), so it must be the case that $-\mathbf{b}^*$ is an optimal pure strategy for Player I. The converse of the lemma is shown in an analogous way. $\square$

Recall from (18), that if Player I chooses $\mathbf{a} = (x_1, y_1)$ and Player II chooses $\mathbf{b} = (x_2, y_2)$, assuming $\mathbf{a} \neq \mathbf{b}$,

$$\begin{aligned} K(\mathbf{a}, \mathbf{b}) &= (x_1 + y_1)P\big((\xi, \eta) \in C_1(\mathbf{a}, \mathbf{b})\big) + (x_2 + y_2)P\big((\xi, \eta) \in C_2(\mathbf{a}, \mathbf{b})\big) \\ &= (x_1 + y_1)P\big((\xi, \eta) \in C_1(\mathbf{a}, \mathbf{b})\big) + (x_2 + y_2)[1 - \big((\xi, \eta) \in C_1(\mathbf{a}, \mathbf{b})\big)] \\ &= (x_2 + y_2) + (x_1 + y_1 - x_2 - y_2)P\big((\xi, \eta) \in C_1(\mathbf{a}, \mathbf{b})\big). \end{aligned}$$

LEMMA 4.6. *If a pure optimal strategy pair* $\mathbf{a}^* = (x_1^*, y_1^*), \mathbf{b}^* = (x_2^*, y_2^*)$ *exists, then* $x_2^* \geq 0,\ y_2^* \geq 0$ *and* $x_1^* \leq 0,\ y_1^* \leq 0.$

In other words, optimal pure strategies for Players I and II must be in quadrants III and I respectively.

PROOF. We know that if both players are playing optimally then the expected payoff is zero. Suppose only one of Player II's offers is negative[3]; WLOG let $x_2^* < 0$. By playing $(-x_2^*, -y_2^*)$, Player I is guaranteeing a zero expected payoff. Suppose Player I instead switches to $(x_2^*, -y_2^*)$. If $y_2^* = 0$ then the final offers are identical and the net award is x_2^*. Therefore let us assume $y_2^* > 0$.

$$
\begin{aligned}
K\big((x_2^*, -y_2^*), (x_2^*, y_2^*)\big) &= (x_2^* + y_2^*) + (x_2^* - y_2^* - x_2^* - y_2^*)P\big((\xi, \eta) \in C_1((x_2^*, -y_2^*), (x_2^*, y_2^*))\big) \\
&= x_2^* + y_2^*\Big(1 - 2P\big((\xi, \eta) \in C_1((x_2^*, -y_2^*), (x_2^*, y_2^*))\big)\Big)
\end{aligned}
$$

Since $C_1 = \{(x, y) : y < 0\}$, $P((\xi, \eta) \in C_1) = P(\eta < 0) = \frac{1}{2}$. Therefore,

$$K\big((x_2^*, -y_2^*), (x_2^*, y_2^*)\big) = x_2^* < 0.$$

This contradicts that (x_2^*, y_2^*) is an optimal pure strategy for Player II. Thus $x_2^* \geq 0$. Because the choice of component is arbitrary, $y_2^* \geq 0$ as well. The argument is the same to show that Player I's component offers must be non-positive in order to play optimally. □

1.2.2. *Local Optimality of Pure Strategies.* Having established some of the properties of the game in question, we now derive a pure strategy pair for the players and show that it is locally optimal. Recall that

$$K(\mathbf{a}, \mathbf{b}) = (x_2 + y_2) + (x_1 + y_1 - x_2 - y_2)P(\text{Player I wins}). \tag{22}$$

The event that "Player I wins" occurs precisely when the arbiter picks a random fair settlement (ξ, η) and

$$(x_1 - \xi)^2 + (y_1 - \eta)^2 < (x_2 - \xi)^2 + (y_2 - \eta)^2 \tag{23}$$

[3]Player II cannot possibly be playing optimally if both $x_2^* < 0$ and $y_2^* < 0$, for in this case Player I may simply agree to the Player II's final offer and happily accept a negative net settlement.

which is equivalent to

$$(x_2 - x_1)\xi + (y_2 - y_1)\eta < \frac{x_2^2 + y_2^2 - x_1^2 - y_1^2}{2} = w. \tag{24}$$

Letting $\Omega = (x_2 - x_1)\xi + (y_2 - y_1)\eta$, we have $\Omega \sim \mathcal{N}(0, \sigma_\Omega^2)$ where

$$\sigma_\Omega^2 = (x_2 - x_1)^2\sigma_x^2 + 2(x_2 - x_1)(y_2 - y_1)\rho\sigma_x\sigma_y + (y_2 - y_1)^2\sigma_y^2. \tag{25}$$

Thud Ω/σ_Ω follows a standard normal distribution. Thus we may express the expected payoff as

$$K(\mathbf{a}, \mathbf{b}) = (x_2 + y_2) + (x_1 + y_1 - x_2 - y_2)\Phi(z) \tag{26}$$

where $\Phi(z)$ is the distribution function of a standard normal random variable and

$$z = \frac{w}{\sigma_\Omega} = \frac{x_2^2 + y_2^2 - x_1^2 - y_1^2}{2\sqrt{(x_2 - x_1)^2\sigma_x^2 + 2(x_2 - x_1)(y_2 - y_1)\rho\sigma_x\sigma_y + (y_2 - y_1)^2\sigma_y^2}}. \tag{27}$$

THEOREM 4.2. *If* $\rho > \max\left\{-\frac{\sigma_x^2 + 3\sigma_y^2}{4\sigma_x\sigma_y}, -\frac{3\sigma_x^2 + \sigma_y^2}{4\sigma_x\sigma_y}\right\}$ *then the pure strategies given by*

$$(x_i^*, y_i^*) = \left((-1)^i\frac{\sqrt{2\pi(\sigma_x^2 + 2\rho\sigma_x\sigma_y + \sigma_y^2)}}{4}, (-1)^i\frac{\sqrt{2\pi(\sigma_x^2 + 2\rho\sigma_x\sigma_y + \sigma_y^2)}}{4}\right) \tag{28}$$

for players $i = 1, 2$ *constitute a locally optimal strategy pair*[4].

[4]Namely, $\exists\epsilon > 0$ such that for pure strategies $\mathbf{a}^*$ and $\mathbf{b}^*$,

$$\inf_{\mathbf{a}\in N_\epsilon(\mathbf{a}^*)} K(\mathbf{a}, \mathbf{b}^*) = \sup_{\mathbf{b}\in N_\epsilon(\mathbf{b}^*)} K(\mathbf{a}^*, \mathbf{b}) = K(\mathbf{a}^*, \mathbf{b}^*),$$

where $N_\epsilon(\cdot)$ is an $\epsilon-$neighborhood.

PROOF. If the players have locally optimal pure strategies $\mathbf{a}^*$ and $\mathbf{b}^*$ then we must have all four first derivatives zero, namely

$$\frac{\partial K}{\partial x_1} = \Phi(z) + (x_1 + y_1 - x_2 - y_2)\phi(z)\left(-\frac{x_1}{\sigma_\Omega} + \frac{(x_2 - x_1)\sigma_x^2 + (y_2 - y_1)\rho\sigma_x\sigma_y}{\sigma_\Omega^2} z\right) = 0 \tag{29}$$

$$\frac{\partial K}{\partial y_1} = \Phi(z) + (x_1 + y_1 - x_2 - y_2)\phi(z)\left(-\frac{y_1}{\sigma_\Omega} + \frac{(x_2 - x_1)\rho\sigma_x\sigma_y + (y_2 - y_1)\sigma_y^2}{\sigma_\Omega^2} z\right) = 0 \tag{30}$$

$$\frac{\partial K}{\partial x_2} = 1 - \Phi(z) + (x_1 + y_1 - x_2 - y_2)\phi(z)\left(\frac{x_2}{\sigma_\Omega} - \frac{(x_2 - x_1)\sigma_x^2 + (y_2 - y_1)\rho\sigma_x\sigma_y}{\sigma_\Omega^2} z\right) = 0 \tag{31}$$

$$\frac{\partial K}{\partial y_2} = 1 - \Phi(z) + (x_1 + y_1 - x_2 - y_2)\phi(z)\left(\frac{y_2}{\sigma_\Omega} - \frac{(x_2 - x_1)\rho\sigma_x\sigma_y + (y_2 - y_1)\sigma_y^2}{\sigma_\Omega^2} z\right) = 0 \tag{32}$$

Since $0 < \Phi(z^*) < 1$, $x_1^* + y_1^* - x_2^* - y_2^* \neq 0$. By adding (29) and (31) we have

$$0 = 1 + \frac{x_2^* - x_1^*}{\sigma_\Omega^*}(x_1^* + y_1^* - x_2^* - y_2^*)\phi(z^*) \tag{33}$$

and by adding (30) and (32) we have

$$0 = 1 + \frac{y_2^* - y_1^*}{\sigma_\Omega^*}(x_1^* + y_1^* - x_2^* - y_2^*)\phi(z^*). \tag{34}$$

From (33) and (34) we know that

$$x_2^* - x_1^* = y_2^* - y_1^* = d^* \neq 0. \tag{35}$$

Note also that

$$\sigma_\Omega^{*2} = d^{*2}(\alpha + \beta), \tag{36}$$

where $\alpha = \sigma_x^2 + \rho\sigma_x\sigma_y$ and $\beta = \rho\sigma_x\sigma_y + \sigma_y^2$. Furthermore, we now have that

$$z^* = \frac{d^*((x_2^* + x_1^*) + (y_2^* + y_1^*))}{2\sigma_\Omega^*} = \frac{x_2^* + x_1^* + y_2^* + y_1^*}{2\sqrt{\alpha + \beta}}\,\mathrm{sgn}(d^*). \tag{37}$$

We may now simplify the four equations derived from (29)-(32) as

$$0 = \Phi(z^*) + 2\phi(z^*)\left(\frac{\operatorname{sgn}(d^*)x_1^*}{\sqrt{\alpha+\beta}} - \frac{\alpha}{\alpha+\beta}z^*\right) \tag{38}$$

$$0 = \Phi(z^*) + 2\phi(z^*)\left(\frac{\operatorname{sgn}(d^*)y_1^*}{\sqrt{\alpha+\beta}} - \frac{\beta}{\alpha+\beta}z^*\right) \tag{39}$$

$$0 = 1 - \Phi(z^*) - 2\phi(z^*)\left(\frac{\operatorname{sgn}(d^*)x_2^*}{\sqrt{\alpha+\beta}} - \frac{\alpha}{\alpha+\beta}z^*\right) \tag{40}$$

$$0 = 1 - \Phi(z^*) - 2\phi(z^*)\left(\frac{\operatorname{sgn}(d^*)y_2^*}{\sqrt{\alpha+\beta}} - \frac{\beta}{\alpha+\beta}z^*\right) \tag{41}$$

By taking (38) + (39) − (40) − (41) and using (37) we get

$$\begin{aligned} 0 &= -2 + 4\Phi(z^*) + 2\phi(z^*)\left(\frac{x_1^* + y_1^* + x_2^* + y_2^*}{\sqrt{\alpha+\beta}}\operatorname{sgn}(d^*) - 2\frac{\alpha+\beta}{\alpha+\beta}z^*\right) \\ \frac{1}{2} &= \Phi(z^*) + 2\phi(z^*)(2z^* - 2z^*) \\ &= \Phi(z^*) \end{aligned}$$

Thus $z^* = 0$, and by simplifying the four equations (38)-(41) we get that $x_1^* = y_1^* = -x_2^* = -y_2^*$. We simplify $\sigma_\Omega^{*2} = 4x_1^{*2}(\alpha+\beta)$ and noting that $\phi(0) = \frac{1}{\sqrt{2\pi}}$, equation (38) becomes

$$0 = \frac{1}{2} + 2\phi(0)\left(\frac{\operatorname{sgn}(d^*)x_1^*}{\sqrt{\alpha+\beta}}\right), \tag{42}$$

or equivalently

$$x_1^* = -\frac{\sqrt{2\pi(\sigma_x^2 + 2\rho\sigma_x\sigma_y + \sigma_y^2)}}{4}\operatorname{sgn}(d^*). \tag{43}$$

We have two solutions to the first order equations, namely $d^* > 0$ and $d^* < 0$. Let us choose the solution given by $d^* > 0$. To show that $\mathbf{b}^*$ is a local maximum for Player II and $\mathbf{a}^*$ is a local minimum for Player I we look at the second partial derivatives evaluated at $(\mathbf{a}^*, \mathbf{b}^*)$. It is straightforward to verify[5] that

$$\left.\frac{\partial^2 K}{\partial x_1^2}\right|_{(\mathbf{a}^*,\mathbf{b}^*)} = \frac{\phi(0)}{\sqrt{\alpha+\beta}}\left(3 - \frac{2\alpha}{\alpha+\beta}\right).$$

This is positive if and only if $\alpha + 3\beta > 0$, or equivalently, when

$$\rho > -\frac{\sigma_x^2 + 3\sigma_y^2}{4\sigma_x\sigma_y}. \tag{44}$$

Similarly,

$$\left.\frac{\partial^2 K}{\partial y_1^2}\right|_{(\mathbf{a}^*,\mathbf{b}^*)} = \frac{\phi(0)}{\sqrt{\alpha+\beta}}\left(3 - \frac{2\beta}{\alpha+\beta}\right)$$

which is positive if and only if

$$\rho > -\frac{3\sigma_x^2 + \sigma_y^2}{4\sigma_x\sigma_y}. \tag{45}$$

Note that it is impossible for both (44) and (45) to be unsatisfied, as this would imply that $\alpha < 0$ and $\beta < 0$ and thus $\alpha + \beta < 0$. It is likewise easily shown that

$$\left.\frac{\partial^2 K}{\partial y_1 \partial x_1}\right|_{(\mathbf{a}^*,\mathbf{b}^*)} = 0,$$

thus $K_{x_1x_1}K_{y_1y_1} - K_{x_1y_1}^2 > 0$ as long as

$$\rho > \max\left\{-\frac{\sigma_x^2 + 3\sigma_y^2}{4\sigma_x\sigma_y}, -\frac{3\sigma_x^2 + \sigma_y^2}{4\sigma_x\sigma_y}\right\}.$$

It can be similarly verified that $\mathbf{b}^*$ is a local maximum for Player II when Player I plays $\mathbf{a}^*$, with the same condition on ρ. Thus we have a locally optimal pair of pure strategies. □

It is important to highlight that if the issues are too negatively correlated, pure optimal strategies need not exist. For example, suppose $\sigma_x = 1, \sigma_y = \sqrt{3}$ and $\rho = -.9$. If optimal pure strategies do exist, they must be

$$\mathbf{a}^* = (-0.588627, -0.588627), \mathbf{b}^* = (0.588627, 0.588627).$$

[5]See SageMath code in the Appendix.

However,

$$K(\mathbf{a}', \mathbf{b}^*) = -.003085 < 0 = K(\mathbf{a}^*, \mathbf{b}^*),$$

where $\mathbf{a}' = (-0.572278, -1.029105)$. Because Player I has a strict incentive to deviate to a new pure strategy, $\mathbf{a}^*$ is not optimal (and consequently, neither is $\mathbf{b}^*$).

1.2.3. *Global Optimality of Pure Strategies.* We now proceed to show that the pure strategies found in the preceding section are indeed globally optimal and thus represent the unique optimal strategy pair. We must first establish a few lemmas which will motivate the geometric interpretation of the model which follows.

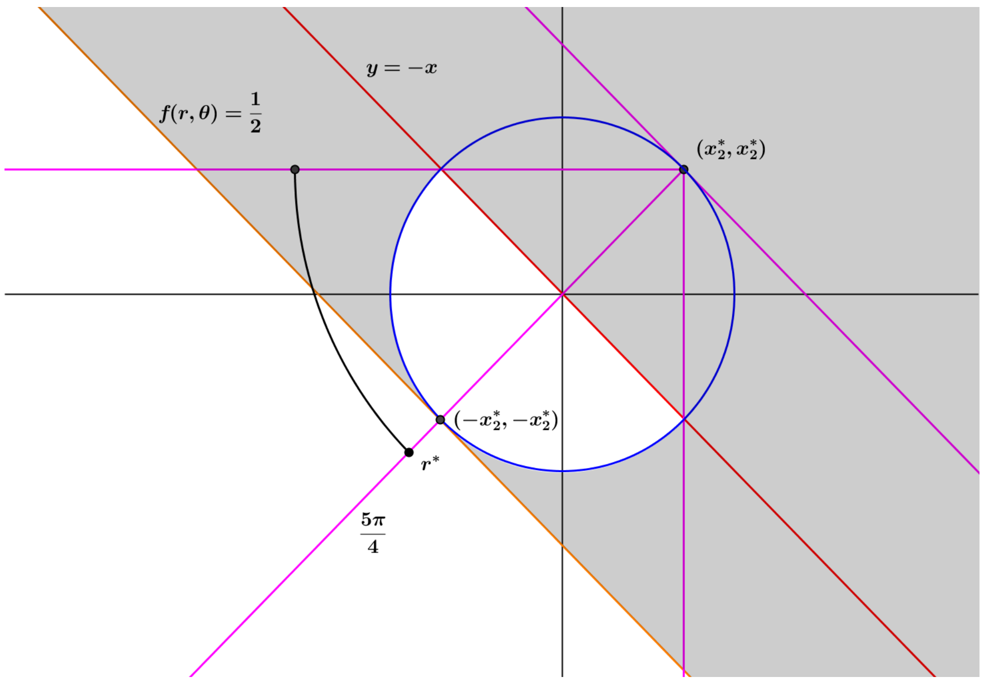

FIGURE 1. If Player II fixes his strategy $\mathbf{b}^* = (x_2^*, x_2^*)$, we show that for all pure strategies $\mathbf{a} \neq (-x_2^*, -x_2^*)$, $K(\mathbf{a}, \mathbf{b}^*) > 0$. In Lemma 4.8 we show this is true in the gray shaded region. We consider cases of z: $z = 0$, the points on the blue circle, in Proposition 4.9; $z > 0$, points outside the circle, in Proposition 4.12; $z < 0$ (points within the circle) in Proposition 4.13.

LEMMA 4.7. *Suppose Player II chooses strategy* $\mathbf{b}^* = (x_2^*, x_2^*)$*. If Player I selects pure strategy* $\mathbf{a} = (x_1, y_1)$ *then* $z(\mathbf{a}, \mathbf{b}^*) = 0$ *iff* (x_1, y_1) *lies on the circle of radius* $\sqrt{2}x_2^*$ *centered at the origin. Furthermore,* $x_1^2 + y_1^2 < 2x_2^{*2}$ *iff* $z > 0$ *and* $x_1^2 + y_1^2 > 2x_2^{*2}$ *iff* $z < 0$.

The proof is trivial.

LEMMA 4.8. *If another pure strategy* $\mathbf{a} = (x_1, y_1) \neq -\mathbf{b}^* = (-x_2^*, -x_2^*)$ *exists such that* $K(\mathbf{a}, \mathbf{b}^*) \leq 0$*, then* $x_1 + y_1 < 0$ *and either*

$$x_1^2 + y_1^2 < 2x_2^{*2} \quad \text{or} \quad x_1 + y_1 \leq -2x_2^*.$$

PROOF. Suppose $x_1 + y_1 \geq 0$. Because the net offer of Player II, $2x_2^* > 0$, and Player II has a positive probability p of being chosen by the arbiter, the expected payoff

$$K(\mathbf{a}, \mathbf{b}^*) = p(2x_2^*) + (1-p)(x_1 + y_1) > 0.$$

This contradicts our assumption.
Suppose $x_1^2 + y_1^2 \geq 2x_2^{*2}$. Then $z \leq 0$ by Lemma 4.7 and $\Phi(z) \leq \frac{1}{2}$. Suppose also that $x_1 + y_1 > -2x_2^*$. Then

$$x_1 + y_1 - 2x_2^* = -4x_2^* + \epsilon$$

for some $0 < \epsilon < 2x_2^*$. But then

$$\begin{aligned} K(\mathbf{a}, \mathbf{b}^*) &= 2x_2^* + (x_1 + y_1 - 2x_2^*)\Phi(z) \\ &= 2x_2^* - (4x_2^* - \epsilon)\Phi(z) \\ &\geq 2x_2^* - (4x_2^* - \epsilon)\frac{1}{2} \\ &= \frac{\epsilon}{2} \\ &> 0 \end{aligned}$$

and this contradicts our assumption. $\square$

We now proceed to show that if Player II chooses pure strategy $\mathbf{b}^* = (x_2^*, x_2^*)$ and I deviates from $\mathbf{a}^* = (-x_2^*, -x_2^*)$ to any other pure strategy (x_1, y_1) then it will simply result in a positive expected payoff. For the remainder of the paper we will assume $\rho \geq 0$[6] and without loss of generality $\sigma_x \leq \sigma_y$.

PROPOSITION 4.9. *For all pure strategies* $\mathbf{a} = (x_1, x_2) \neq (-x_2^*, -x_2^*)$ *such that* $x_1^2 + y_1^2 = 2x_2^{*2}$, $K(\mathbf{a}, \mathbf{b}^*) > 0$.

PROOF. If $x_1^2 + y_1^2 = 2x_2^{*2}$, $z(\mathbf{a}, \mathbf{b}^*) = 0$, so

$$\begin{aligned} K(\mathbf{a}, \mathbf{b}^*) &= 2x_2^* + (x_1 + y_1 - 2x_2^*)\Phi(0) \\ &= 2x_2^* + (x_1 + y_1 - 2x_2^*)\frac{1}{2} \\ &= x_2^* + \frac{x_1 + y_1}{2}. \end{aligned}$$

Geometrically we can see that $x_1 + y_1$ is minimized on the circle $x_1^2 + y_1^2 = 2x_2^{*2}$ at $(-x_2^*, -x_2^*)$. □

Against Player II's strategy $\mathbf{b}^* = (x_2^*, x_2^*)$, any pure strategy $\mathbf{a} = (x_1, y_1)$ may be represented in terms of r and θ as $(x_2^* + r\cos\theta, x_2^* + r\sin\theta)$. This will greatly facilitate the remaining proofs[7]. In this representation, with $t(\theta) = -(\cos\theta + \sin\theta)$, we can rewrite

$$K(\mathbf{a}, \mathbf{b}^*) = 2x_2^* + r(\cos\theta + \sin\theta)\Phi(z) = 2x_2^* - rt(\theta)\Phi(z) \tag{46}$$

and

$$z(r, \theta) = \frac{2x_2^{*2} - (x_2^* + r\cos\theta)^2 - (x_2^* + r\sin\theta)^2}{2r\sqrt{\sigma_x^2\cos^2\theta + 2\rho\sigma_x\sigma_y\cos\theta\sin\theta + \sigma_y^2\sin^2\theta}} = \frac{2x_2^* t(\theta) - r}{2\sqrt{\sigma_\theta^2}} \tag{47}$$

The following two lemmas are needed for Proposition 4.12. See the Appendix for proofs.

---

[6]It is the author's conjecture that global optimality of pure strategies will hold with a weaker condition on ρ, namely that of Theorem 4.2.

[7]For convenience we will define $t(\theta) := -(\cos\theta + \sin\theta)$ and $\sigma_\theta^2 := \sigma_x^2\cos^2\theta + 2\rho\sigma_x\sigma_y\cos\theta\sin\theta + \sigma_y^2\sin^2\theta$. Note that $t(\theta) = -\sqrt{2}\sin(\theta + \frac{\pi}{4})$.

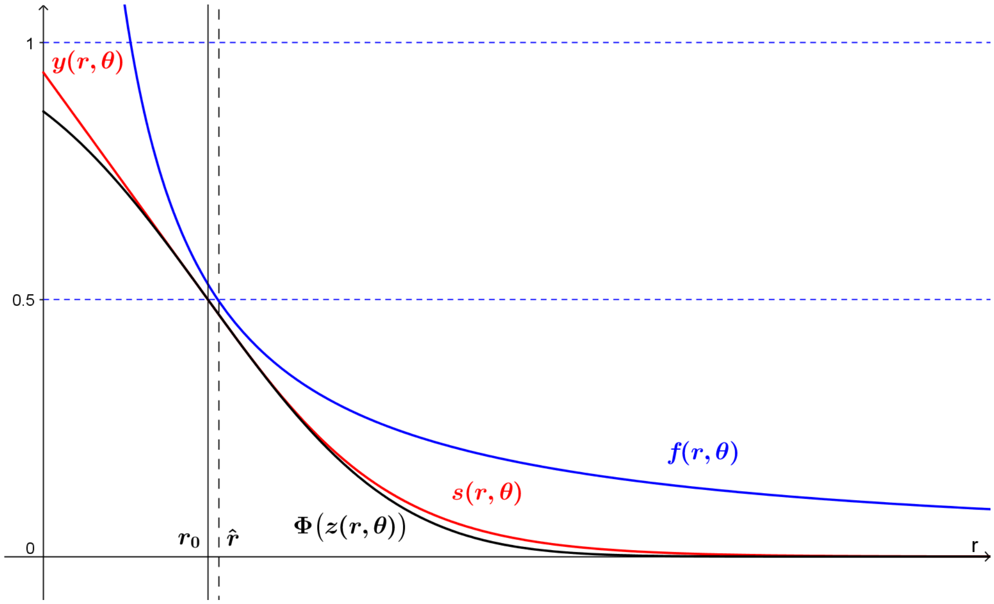

FIGURE 2. This sketch shows the curves Φ and f defined below for a fixed θ. To show that $\mathbf{a}^*$ and $\mathbf{b}^*$ are a globally optimal pure strategy pair, we show that Φ and f intersect only at r_0 when $\theta = \frac{5\pi}{4}$ (Proposition 4.9) and everywhere else $f > \Phi$. To prove this, we show that for $r > r_0$, $f > s > \Phi$, where s is a scaled logistic curve (Proposition 4.12), while for $r < r_0$, $f > y$, a line tangent to Φ (Proposition 4.13).

LEMMA 4.10. *For $z < 0$, the scaled logistic function*

$$s(z) := \frac{1}{1 + \exp\left(-\sqrt{\frac{8}{\pi}}z\right)} > \Phi(z).$$

PROOF. Consider $s(z) - \Phi(z)$ and find the minimum:

$$s'(z) - \phi(z) = s(z)(1 - s(z))\sqrt{\frac{8}{\pi}} - \frac{1}{\sqrt{2\pi}}e^{-z^2/2} = 0$$

$$\Leftrightarrow 4\frac{e^{-\sqrt{\frac{8}{\pi}}z}}{(1 + e^{-\sqrt{\frac{8}{\pi}}z})^2} - e^{-z^2/2} = 0$$

$$\Leftrightarrow 4\frac{e^{-\sqrt{\frac{8}{\pi}}z}}{1 + 2e^{-\sqrt{\frac{8}{\pi}}z} + e^{-2\sqrt{\frac{8}{\pi}}z}} - e^{-z^2/2} = 0$$

$$\Leftrightarrow 4\frac{e^{-\sqrt{\frac{8}{\pi}}z}}{1 + 2e^{-\sqrt{\frac{8}{\pi}}z} + e^{-2\sqrt{\frac{8}{\pi}}z}} = e^{-z^2/2}$$

$$
\begin{aligned}
&\Leftrightarrow \frac{1 + 2e^{-\sqrt{\frac{8}{\pi}}z} + e^{-2\sqrt{\frac{8}{\pi}}z}}{e^{-\sqrt{\frac{8}{\pi}}z}} = 4e^{z^2/2} \\
&\Leftrightarrow e^{\sqrt{\frac{8}{\pi}}z} + 2 + e^{-\sqrt{\frac{8}{\pi}}z} = 4e^{z^2/2} \\
&\Leftrightarrow 1 + \cosh\left(\sqrt{\frac{8}{\pi}}z\right) = 2e^{z^2/2} \\
&\Leftrightarrow 2\cosh^2\left(\sqrt{\frac{2}{\pi}}z\right) = 2e^{z^2/2} \\
&\Leftrightarrow \cosh^2\left(\sqrt{\frac{2}{\pi}}z\right) = e^{z^2/2} \\
&\Leftrightarrow \cosh\left(\sqrt{\frac{2}{\pi}}z\right) = e^{z^2/4}
\end{aligned}
$$

The three roots are at $z = 0$ and $z \approx \pm 1.7318300869742718735$. It is clear to see that $s(0) = \Phi(0)$. Since $s(-1.7318300869742718735) - \Phi(-1.7318300869742718735) \approx .017671$, this point represents a local maximum, i.e. here the two curves are furthest apart.

Next observe that

$$
\lim_{z\to-\infty} \frac{s(z)}{\Phi(z)} = \lim_{z\to-\infty} \frac{\cosh\left(\sqrt{\frac{2}{\pi}}z\right)}{e^{z^2/4}} = \lim_{z\to-\infty} \frac{e^{\sqrt{2/\pi}z} + e^{-\sqrt{2/\pi}z}}{e^{z^2/4}}
$$

$$
= \lim_{z\to-\infty} e^{\sqrt{2/\pi}z - z^2/4} + \lim_{z\to-\infty} e^{-\sqrt{2/\pi}z - z^2/4} = \infty + 0.
$$

It cannot be the case that $s(z) - \Phi(z) < 0$ at any point $z < 0$, as this would imply the existence of a local minimum of $s(z) - \Phi(z)$, of which we know there is none. Thus $s(z) > \Phi(z)$ for all $z < 0$. $\square$

LEMMA 4.11. *The general exponential curve $y = \beta e^{\alpha x}$ and line $y = mx$ have:*

$$\begin{cases} \text{1 intersection at } x^* < 0 & \text{if } m < 0 \\ \text{No intersection} & \text{if } 0 \leq m < \alpha\beta e \\ \text{1 intersection at } x^* = \frac{1}{a} & \text{if } m = \alpha\beta e \\ \text{2 intersections} & \text{if } m > \alpha\beta e \end{cases}$$

PROOF. First let $w = \alpha x$

$$y = \beta e^w \quad y = \frac{m}{\alpha} w$$

Now let let $z = \frac{y}{\beta}$

$$z = e^w \quad z = \frac{m}{\alpha\beta} w = \gamma w$$

If $\gamma = e$ then the two curves have a single intersection, and are tangent, at $w = 1$. If $0 \leq \gamma < e$ then they cannot intersect. If $\gamma > e$ they will intersect at two points. □

PROPOSITION 4.12. *For all pure strategies $\mathbf{a} = (x_1, x_2)$ such that $x_1^2 + y_1^2 > 2x_2^{*2}$, $K(\mathbf{a}, \mathbf{b}^*) > 0$.*

PROOF. The claim may be equivalently expressed as

$$K(\mathbf{a}, \mathbf{b}^*) = 2x_2^* - rt(\theta)\Phi(z) > 0 \Leftrightarrow \Phi(z) < \frac{2x_2^*}{rt(\theta)} = f(r, \theta). \tag{48}$$

From Lemma 4.10, for $z < 0$, we note that a scaled logistic function

$$s(z) := \frac{1}{1 + \exp\left(-\sqrt{\frac{8}{\pi}} z\right)} > \Phi(z).$$

We will indeed prove a stronger claim, namely that for all points $\mathbf{a}$ in this region[8],

$$s(z) < f(r, \theta)$$

[8]Strictly speaking, for $\theta \in (\pi, \frac{5\pi}{4})$ (Case 4) the proof proceeds in a different way.

$$\Leftrightarrow \quad 2x_2^*\left(\frac{1}{s(z(r+\hat{r},\theta))}-1\right) > 2x_2^*\left(\frac{1}{f(r+\hat{r},\theta)}-1\right),$$

where $\hat{r} = \frac{2x_2^*}{t(\theta)}$. Let us define the left and right side functions as $\hat{s}(r,\theta)$ and $\hat{f}(r,\theta)$ respectively. These may be explicitly written as

$$\hat{s}(r,\theta) = \sqrt{\frac{\pi(\alpha+\beta)}{2}}\exp\left(-\sqrt{\frac{8}{\pi}}\frac{2x_2^*t(\theta)-r-\frac{2x_2^*}{t(\theta)}}{2\sigma_\theta}\right) \tag{49}$$

$$= \sqrt{\frac{\pi(\alpha+\beta)}{2}}\exp\left(\left(\frac{1}{t(\theta)}-t(\theta)\right)\frac{\sqrt{\alpha+\beta}}{\sigma_\theta}\right)\exp\left(\sqrt{\frac{2}{\pi\sigma_\theta^2}}r\right), \tag{50}$$

and

$$\hat{f}(r,\theta) = 2x_2^*\left(\frac{(r+\frac{2x_2^*}{t(\theta)})t(\theta)}{2x_2^*}-1\right) = t(\theta)r. \tag{51}$$

We have simply translated and scaled s and f so that, for fixed θ, $\hat{s}(r,\theta) = be^{ar}$ and $\hat{f}(r,\theta) = mr$. By Lemma 4.11, we need show that $\forall\theta \in [\frac{3\pi}{4}, \frac{7\pi}{4}]$, $t(\theta) \leq A(\theta)e$ and attains equality only at $\theta = \frac{5\pi}{4}$, where

$$A(\theta) := a(\theta)b(\theta) = \frac{\sqrt{\alpha+\beta}}{\sigma_\theta}\exp\left(\left(\frac{1}{t(\theta)}-t(\theta)\right)\frac{\sqrt{\alpha+\beta}}{\sigma_\theta}\right). \tag{52}$$

Case 1: $\theta \in [\frac{3\pi}{4}, \pi] \cup [\frac{3\pi}{2}, \frac{7\pi}{4}]$

In this case,

$$A(\theta)e \geq e > 1 \geq t(\theta).$$

Case 2: $\theta = \frac{5\pi}{4}$

$$A\left(\frac{5\pi}{4}\right)e = \frac{\sqrt{\alpha+\beta}}{\sqrt{\frac{1}{2}(\alpha+\beta)}}\exp\left(\frac{\frac{\sqrt{2}}{2}-\sqrt{2}}{\sqrt{\frac{1}{2}(\alpha+\beta)}}\sqrt{\alpha+\beta}\right)e = \sqrt{2}e^{-1+1} = \sqrt{2} = t\left(\frac{5\pi}{4}\right).$$

Case 3: $\theta \in (\frac{5\pi}{4}, \frac{3\pi}{2})$

On this interval it is clear that $t(\theta)$ is decreasing. We will show that on this interval A

is bounded below by an increasing function $\underline{A}$ and that $A(\frac{5\pi}{4}) = \underline{A}(\frac{5\pi}{4})$. First consider the partial derivatives with respect to ρ and σ_x:

$$\frac{dA}{d\sigma_x} = \frac{1}{2}\exp\left(\left(\frac{1}{t(\theta)} - t(\theta)\right)\frac{\sqrt{\alpha+\beta}}{\sigma_\theta}\right)\left(\frac{\sigma_\theta}{\sqrt{\alpha+\beta}} + \frac{1}{t(\theta)} - t(\theta)\right)\left(\frac{d}{d\sigma_x}\frac{\alpha+\beta}{\sigma_\theta^2}\right)$$

$$\frac{d}{d\sigma_x}\frac{\alpha+\beta}{\sigma_\theta^2} = -\frac{2\sigma_y(\cos\theta - \sin\theta)\big((\rho\sigma_x^2 + \sigma_x\sigma_y)\cos\theta + (\rho\sigma_y^2 + \sigma_x\sigma_y)\sin\theta\big)}{\sigma_\theta^4}$$

$$\frac{dA}{d\rho} = \frac{1}{2}\exp\left(\left(\frac{1}{t(\theta)} - t(\theta)\right)\frac{\sqrt{\alpha+\beta}}{\sigma_\theta}\right)\left(\frac{\sigma_\theta}{\sqrt{\alpha+\beta}} + \frac{1}{t(\theta)} - t(\theta)\right)\left(\frac{d}{d\rho}\frac{\alpha+\beta}{\sigma_\theta^2}\right)$$

$$\frac{d}{d\rho}\frac{\alpha+\beta}{\sigma_\theta^2} = \frac{2\sigma_x\sigma_y(\cos\theta - \sin\theta)(\sigma_x^2\cos\theta - \sigma_y^2\sin\theta)}{\sigma_\theta^4}$$

For $\theta \in (\frac{5\pi}{4}, \frac{3\pi}{2})$:

$$\frac{d}{d\rho}\frac{\alpha+\beta}{\sigma_\theta^2} > 0, \quad \frac{d}{d\sigma_x}\frac{\alpha+\beta}{\sigma_\theta^2} > 0, \tag{53}$$

and

$$G(\theta) : \frac{\sigma_\theta}{\sqrt{\alpha+\beta}} + \frac{1}{t(\theta)} - t(\theta) > 0. \tag{54}$$

Inequalities of (53) are self-evident. To justify (54), note that (53) implies that

$$\frac{d}{d\rho}\frac{\sigma_\theta^2}{\alpha+\beta} < 0, \quad \frac{d}{d\sigma_x}\frac{\sigma_\theta^2}{\alpha+\beta} < 0. \tag{55}$$

Thus, for any fixed $\theta \in (\frac{5\pi}{4}, \frac{3\pi}{2})$,

$$\frac{\sigma_\theta^2}{\alpha+\beta} \geq \frac{\sigma_\theta^2}{\alpha+\beta}_{\rho=1,\sigma_x=\sigma_y} = \frac{t(\theta)^2}{4}.$$

Because $1 < t(\theta) < \sqrt{2}$,

$$G(\theta) = \frac{t(\theta)}{2} - \frac{1}{t(\theta)} + t(\theta) = \frac{2 - t(\theta)^2}{2t(\theta)} > 0.$$

Since A is an increasing function of ρ and σ_x,

$$A(\theta) \geq \underline{A}(\theta) := A(\theta)_{\rho=0,\sigma_x=0} = -\frac{1}{\sin\theta}\exp\left(\frac{-2\cos\theta}{\sin\theta+\cos\theta}\right).$$

The derivative

$$\frac{d\underline{A}(\theta)}{d\theta} = \exp\left(\frac{-2\cos\theta}{\sin\theta+\cos\theta}\right)\frac{\cos\theta - 2\sin^3\theta}{\sin^2\theta t(\theta)^2}$$

is positive for any $\theta \in (\frac{5\pi}{4}, \frac{3\pi}{2})$. Since $A(\frac{5\pi}{4}) = \underline{A}\left(\frac{5\pi}{4}\right)$ and $\underline{A}(\theta)$ is increasing, we are done.

Case 4: $\theta \in (\pi, \frac{5\pi}{4})$

Recall from (48) that $K > 0$ is equivalent to

$$\Phi(z) < f(r,\theta) = \frac{2x_2^*}{rt(\theta)}.$$

Let us fix $r^* > 2\sqrt{2}x_2^*$. The proof proceeds via three claims:

Claim 1: $f(r^*,\theta)$ is a decreasing function on this interval.

This claim follows immediately after noting that $t(\theta)$ is an increasing function on $(\pi, \frac{5\pi}{4})$.

Claim 2: $\Phi(z(r^*,\theta))$ is an increasing function for $\theta \in (\pi, \frac{5\pi}{4})$.

Recall that $z = \frac{2x_2^*t(\theta)-r}{2\sqrt{\sigma_\theta^2}}$. For $\theta \in (\pi, \frac{5\pi}{4})$, $r_0 = 2x_2^*t(\theta)$ is increasing and attains its maximum value of $2\sqrt{2}x_2^*$ when $\theta = \frac{5\pi}{4}$. Since $r^* > 2\sqrt{2}x_2^* \geq r_0$, $z(r^*,\theta) < 0$. Because

$$\frac{d}{d\theta}\sigma_\theta^2 = 2\rho\sigma_x\sigma_y(\cos^2\theta - \sin^2\theta) + 2(\sigma_y^2 - \sigma_x^2)\cos\theta\sin\theta > 0,$$

σ_θ^2 is increasing on $(\pi, \frac{5\pi}{4})$.

Consider

$$|z| = \frac{r^* - 2x_2^*t(\theta)}{2\sqrt{\sigma_\theta^2}}.$$

For $\theta \in (\pi, \frac{5\pi}{4})$, $t(\theta)$ increases so the numerator is decreasing. Meanwhile the denominator is increasing. Thus $\Phi(z(r^*,\theta))$ is an increasing function in $(\pi, \frac{5\pi}{4})$.

Claim 3: $\Phi(z(r^*, \frac{5\pi}{4})) < f(r^*, \frac{5\pi}{4})$.

If we fix $\theta = \frac{5\pi}{4}$, then the players are in the one-dimensional FOA game, and we already know that $\mathbf{a}^* = -\mathbf{b}^*$ (i.e. $r = 2\sqrt{2}x_2^*$) is the globally optimal strategy for

Player I to play against $\mathbf{b}^*$. Since we have fixed $r^* > 2\sqrt{2}x_2^*$, Player I is not playing optimally, so $K > 0$ which is equivalent to the claim.

From these three claims it follows that $K > 0$ for $r > 2\sqrt{2}x_2^*$ and $\theta \in (\pi, \frac{5\pi}{4})$. $\square$

Now that we have shown that against (x_2^*, x_2^*) all pure strategies (x, y) for Player I outside the circle $x^2 + y^2 = 2x_2^{*2}$ will give a positive expected payoff, we consider strategies within the circle.

PROPOSITION 4.13. *For all pure strategies* $\mathbf{a} = (x_1, x_2)$ *such that* $x_1^2 + y_1^2 < 2x_2^{*2}$, $K(\mathbf{a}, \mathbf{b}^*) > 0$.

The proof relies on the concavity of the normal distribution function for $z > 0$.

PROOF. From Lemma 4.8, we need only show that $K(\mathbf{a}, \mathbf{b}^*) > 0$ for all $\mathbf{a}$ in the semi-circle described by

$$\begin{cases} x + y < 0, \\ x^2 + y^2 < 2x_2^{*2}. \end{cases}$$

In terms of θ, we are restricting our attention to $\theta \in (\pi, \frac{3\pi}{2})$. For the angles θ in question,

$$t(\theta) > 1. \tag{56}$$

Recall from (46) that $K(\mathbf{a}, \mathbf{b}^*) > 0$ is equivalent to

$$\Phi(z) < f(r, \theta) = \frac{2x_2^*}{rt(\theta)}.$$

First we fix $\tilde{\theta} \in (\pi, \frac{3\pi}{2})$. Let $r_0 = 2x_2^* t(\tilde{\theta})$. Note by definition that $z(r_0, \tilde{\theta}) = 0$. Since $z = \frac{r_0 - r}{2\sqrt{\sigma_\theta^2}}$, it is straightforward to show that

$$\frac{d}{dr}\Phi(z)\bigg|_{z=0} = \phi(z)\frac{dz}{dr}\bigg|_{z=0} = \frac{1}{\sqrt{2\pi}}\frac{-1}{2\sqrt{\sigma_{\tilde{\theta}}^2}} = \frac{-1}{2\sqrt{2\pi\sigma_{\tilde{\theta}}^2}}.$$

Define y as the line tangent to Φ at $(r_0, \frac{1}{2})$, specifically,

$$y(r, \tilde{\theta}) = -\frac{r - r_0}{2\sqrt{2\pi\sigma^2_{\tilde{\theta}}}} + \frac{1}{2}.$$

Note Φ is a concave function for $r < r_0$. Therefore, $\Phi(z(r, \tilde{\theta})) \leq y(r, \tilde{\theta})$. To demonstrate that $f > \Phi$ for all $r < r_0$, it suffices to show that $f > y$ for all r. Since f and y are both continuous functions and $\lim_{r \to 0^+} f(r, \tilde{\theta}) = \infty \gg y(0, \tilde{\theta})$, it suffices to show that $f \neq y$ for any r. If the two curves do intersect, then there is at least one solution to the equation

$$\frac{\sqrt{2\pi(\alpha+\beta)}}{2rt(\tilde{\theta})} = -\frac{1}{2\sqrt{2\pi\sigma^2_{\tilde{\theta}}}}r + \frac{t(\tilde{\theta})\sqrt{\alpha+\beta}}{4\sqrt{\sigma^2_{\tilde{\theta}}}} + \frac{1}{2},$$

or equivalently

$$\begin{aligned} 0 &= \frac{1}{2\sqrt{2\pi\sigma^2_{\tilde{\theta}}}}r^2 - \left(\frac{t(\tilde{\theta})\sqrt{\alpha+\beta}}{4\sqrt{\sigma^2_{\tilde{\theta}}}} + \frac{1}{2}\right) r + \frac{\sqrt{2\pi(\alpha+\beta)}}{2t(\tilde{\theta})} \\ &= r^2 - \left(\frac{t(\tilde{\theta})\sqrt{2\pi(\alpha+\beta)}}{2} + \sqrt{2\pi\sigma^2_{\tilde{\theta}}}\right) r + \frac{2\pi\sqrt{(\alpha+\beta)\sigma^2_{\tilde{\theta}}}}{t(\tilde{\theta})}. \end{aligned}$$

We have a quadratic in r. Let the vertex be

$$\hat{r} = \frac{t(\tilde{\theta})\sqrt{2\pi(\alpha+\beta)}}{4} + \frac{\sqrt{2\pi\sigma^2_{\tilde{\theta}}}}{2}$$

and the discriminant, Δ, for the quadratic is

$$\Delta = \left(\frac{t(\tilde{\theta})\sqrt{2\pi(\alpha+\beta)}}{2} + \sqrt{2\pi\sigma^2_{\tilde{\theta}}}\right)^2 - \frac{8\pi\sqrt{(\alpha+\beta)\sigma^2_{\tilde{\theta}}}}{t(\tilde{\theta})}.$$

If $\Delta < 0$ then we are done. Let us assume that $\Delta \geq 0$. If $f(r^*, \tilde{\theta}) = y(r^*, \tilde{\theta})$, it must that $r^* < r_0$; for $r \geq r_0$, $f(r, \tilde{\theta}) > \Phi(z(r^*, \tilde{\theta})) > y(r^*, \tilde{\theta})$. This gives us a condition, namely $\hat{r} + \frac{\sqrt{\Delta}}{2} < r_0$.

$$\hat{r} + \frac{\sqrt{\Delta}}{2} < r_0$$

$$\frac{1}{\sqrt{2\pi}}\left(\hat{r} + \frac{\sqrt{\Delta}}{2}\right) < \frac{r_0}{\sqrt{2\pi}}$$

$$\frac{t(\tilde{\theta})\sqrt{\alpha+\beta}}{4} + \frac{\sqrt{\sigma^2_{\tilde{\theta}}}}{2} + \frac{\sqrt{\left(\frac{t(\tilde{\theta})\sqrt{\alpha+\beta}}{2} + \sqrt{\sigma^2_{\tilde{\theta}}}\right)^2 - \frac{2\sqrt{(\alpha+\beta)\sigma^2_{\tilde{\theta}}}}{t(\tilde{\theta})}}}{2} < \frac{t(\tilde{\theta})\sqrt{\alpha+\beta}}{2}$$

$$\sqrt{\sigma^2_{\tilde{\theta}}} + \sqrt{\left(\frac{t(\tilde{\theta})\sqrt{\alpha+\beta}}{2} + \sqrt{\sigma^2_{\tilde{\theta}}}\right)^2 - \frac{2\sqrt{(\alpha+\beta)\sigma^2_{\tilde{\theta}}}}{t(\tilde{\theta})}} < \frac{t(\tilde{\theta})\sqrt{\alpha+\beta}}{2}$$

$$\sqrt{\left(\frac{t(\tilde{\theta})\sqrt{\alpha+\beta}}{2} + \sqrt{\sigma^2_{\tilde{\theta}}}\right)^2 - \frac{2\sqrt{(\alpha+\beta)\sigma^2_{\tilde{\theta}}}}{t(\tilde{\theta})}} < \frac{t(\theta)\sqrt{\alpha+\beta}}{2} - \sqrt{\sigma^2_{\tilde{\theta}}}$$

$$\left(\frac{t(\tilde{\theta})\sqrt{\alpha+\beta}}{2} + \sqrt{\sigma^2_{\tilde{\theta}}}\right)^2 - \frac{2\sqrt{(\alpha+\beta)\sigma^2_{\tilde{\theta}}}}{t(\tilde{\theta})} < \left(\frac{t(\tilde{\theta})\sqrt{\alpha+\beta}}{2} - \sqrt{\sigma^2_{\tilde{\theta}}}\right)^2$$

$$\left(\frac{t(\tilde{\theta})\sqrt{\alpha+\beta}}{2} + \sqrt{\sigma^2_{\tilde{\theta}}}\right)^2 - \left(\frac{t(\tilde{\theta})\sqrt{\alpha+\beta}}{2} - \sqrt{\sigma^2_{\tilde{\theta}}}\right)^2 < \frac{2\sqrt{(\alpha+\beta)\sigma^2_{\tilde{\theta}}}}{t(\tilde{\theta})}$$

$$2t(\tilde{\theta})\sqrt{(\alpha+\beta)\sigma^2_{\tilde{\theta}}} < \frac{2\sqrt{(\alpha+\beta)\sigma^2_{\theta}}}{t(\theta)}$$

$$t(\tilde{\theta}) < \frac{1}{t(\tilde{\theta})}$$

in other words, $t(\tilde{\theta}) < 1$. But recall from (56) that $t(\tilde{\theta}) > 1$, which is a contradiction.

□

The following is the main result.

THEOREM 4.3. *If $\rho \geq 0$, then $\mathbf{a}^* = (-x_2^*, -x_2^*), \mathbf{b}^* = (x_2^*, x_2^*)$ is the unique globally optimal pure strategy pair.*

PROOF. This follows from Propositions 4.9, 4.12 and 4.13. Without any loss of generality, assume $\sigma_x \leq \sigma_y$. If Player II plays pure strategy $\mathbf{b}^*$ then for any pure strategy $\mathbf{a} = (x_1, y_1)$, $K(\mathbf{a}, \mathbf{b}^*) \geq 0$, and equality is only achieved when $\mathbf{a} = \mathbf{a}^*$.

Similarly, if Player I plays $\mathbf{a}^*$, $K(\mathbf{a}^*, \mathbf{b}) \leq 0$ and equality is only achieved when $\mathbf{b} = \mathbf{b}^*$. □

1.2.4. *Variability of Issue-by-Issue and Whole Package Outcomes.* Having shown that under an L_2 distance criterion there is a unique pure optimal strategy pair, we consider the question of whether the issue-by-issue or whole-package variant is more in line with the aims of FOA. Since FOA makes arbitration a costly alternative by its inherent uncertainty, we may compare the uncertainty (i.e. variance) between optimal strategies under the two mechanisms. It may come as no surprise that the arbitrated outcome in WPFOA has a higher variance.

PROPOSITION 4.14. *The expected payoff is zero under both Issue-by-Issue and Whole-Package variants. If both player play optimally then the variances of the awards are $\frac{\pi}{2}(\sigma_x^2 + \sigma_y^2)$ and $\frac{\pi}{2}(\sigma_x^2 + 2\rho\sigma_x\sigma_y + \sigma_y^2)$ respectively.*

PROOF. Under IBIFOA, since the components are awarded independently. Let K_x and K_y be the awards for the first and second issue in dispute respectively. The variance is

$$\begin{aligned} Var(K) &= Var(K_x + K_y) \\ &= E(K_x^2) + E(K_y^2) \\ &= \frac{1}{2}\left(2\frac{2\pi\sigma_x^2}{4}\right) + \frac{1}{2}\left(2\frac{2\pi\sigma_y^2}{4}\right) \\ &= \frac{\pi}{2}(\sigma_x^2 + \sigma_y^2). \end{aligned}$$

Under WPFOA the variance is

$$\begin{aligned}
Var(K) &= E(K^2)\\
&= \frac{1}{2}(2x_1^*)^2 + \frac{1}{2}(2x_2^*)^2\\
&= 4x_2^{*2}\\
&= \frac{\pi}{2}(\sigma_x^2 + 2\rho\sigma_x\sigma_y + \sigma_y^2).
\end{aligned}$$

□

Thus we argue that quantitative issues should be arbitrated by package rather than independently to provide a stronger motivation to the parties to reach "security in agreement" (Stevens, 1966).

1.3. L1 Distance. The L_1 decision criterion sums the component-wise absolute difference for each player. The judge rules in favor of whichever party has the lowest total absolute difference from the 'fair' settlement. When Players play strategies $\mathbf{a} = (x_1, y_1), \mathbf{b} = (x_2, y_2)$, the judge's ruling would be

$$K(\mathbf{a}, \mathbf{b}) = \begin{cases} x_1 + y_1 & \text{if } |x_1 - \xi| + |y_1 - \eta| < |x_2 - \xi| + |y_2 - \eta| \\ x_2 + y_2 & \text{if } |x_1 - \xi| + |y_1 - \eta| > |x_2 - \xi| + |y_2 - \eta| \end{cases}$$

This distance metric is known as the **Manhattan distance**, or **taxicab distance**, since to a taxi driver the distance between two points in a city is the sum of the east-west distance and the north-south distance. This model will be referred to as $2NL_1$ (Bivariate Normal L_1).

The Manhattan distance is somewhat counter-intuitive, so it is worth discussing the geometry involved. Consider two points in $\mathbb{R}^2$, $\mathbf{a} = (x_1, y_1)$ and $\mathbf{b}(x_2, y_2)$. Let

$$C_1(\mathbf{a}, \mathbf{b}) = \{(x, y) : |x_1 - x| + |y_1 - y| < |x_2 - x| + |y_2 - y|\}$$

Denote this set C_1 where there is no ambiguity. Also, let C_2 be defined similarly with the inequality in the opposite direction.

LEMMA 4.15. *Suppose* $\mathbf{a} = (x_1, y_1) \neq \mathbf{b} = (x_2, y_2)$ *with* $x_1 \leq x_2$ *and* $y_1 \leq y_2$. *Let* $d = (x_2 + y_2) - (x_1 + y_1)$. *If,* $y_2 - y_1 > x_2 - x_1$,

$$C_1(\mathbf{a}, \mathbf{b}) = \left\{ (x, y) : \begin{cases} y < y_1 + \frac{d}{2} & x < x_1 \\ y < y_1 + x_1 + \frac{d}{2} - x & x_1 \leq x < x_2 \\ y < y_2 - \frac{d}{2} & x_2 \leq x \end{cases} \right\}$$

PROOF. This proof will consider various cases to show that the two sets are equal.

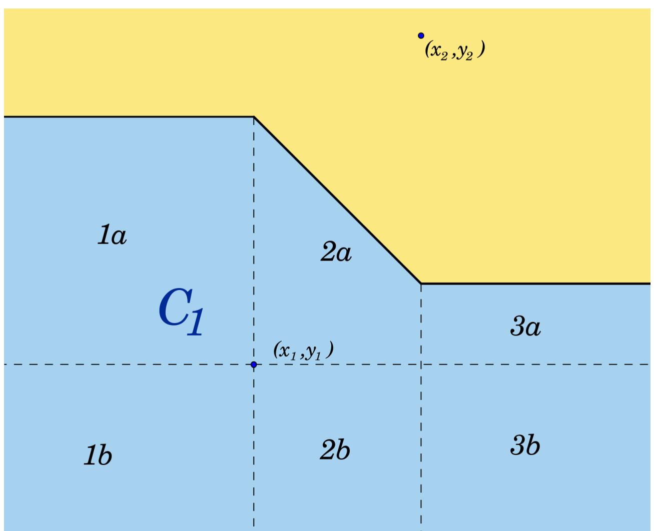

FIGURE 3. A graphical sketch of the proof of Lemma 4.15

Case 1: Let $x < x_1$ and $y < y_1 + \frac{d}{2}$.

Note that $y < y_2$. This is because

$$\begin{aligned} y &< y_1 + \frac{x_2 + y_2 - x_1 - y_1}{2} \\ &= y_2 + \frac{(x_2 - x_1) - (y_2 - y_1)}{2} \\ &< y_2 \end{aligned}$$

Let us assume that $(x, y) \notin C_1$. Thus

$$|x_1 - x| + |y_1 - y| \geq |x_2 - x| + |y_2 - y| \tag{57}$$

Case 1a: $y \geq y_1$.

In this case, Equation (57) becomes

$$\begin{aligned} x_1 - x + y - y_1 \geq x_2 - x + y_2 - y \\ 2y \geq x_2 + y_2 - x_1 + y_1 \\ y \geq y_1 + \frac{x_2 - x_1 + y_2 - y_1}{2} \\ = y_1 + \frac{d}{2} \end{aligned}$$

Which is a contradiction.

Case 1b: $y < y_1$.

In this case, Equation (57) becomes

$$x_1 - x + y_1 - y \geq x_2 - x + y_2 - y$$

$$x_1 + y_1 \geq x_2 + y_2$$

But because $x_1 \leq x_2$ and $y_1 \leq y_2$ by assumption, this implies that $x_1 + y_1 = x_2 + y_2$, and $y_2 - y_1 = x_2 - x_1$. This contradicts our assumption that $x_2 - x_1 < y_2 - y_1$.

Case 2: Let $x_1 \leq x < x_2$ and $y < y_1 + x_1 + \frac{d}{2} - x$.

Note that in this case $y < y_2$, as

$$\begin{aligned} y &< y_1 + x_1 + \frac{x_2 + y_2 - x_1 - y_1}{2} - x \\ &< y_1 + \frac{x_2 + y_2 - x_1 - y_1}{2} \\ &= y_2 + \frac{x_2 - y_2 - x_1 + y_1}{2} \\ &< y_2 \end{aligned}$$

Let us assume that $(x, y) \notin C_1$. Thus

$$|x_1 - x| + |y_1 - y| \geq |x_2 - x| + |y_2 - y| \tag{58}$$

Case 2a: $y_1 < y$.

In this case, Equation (58) becomes

$$x - x_1 + y - y_1 \geq x_2 - x + y_2 - y$$
$$2y \geq 2y_1 + 2x_1 + (x_2 + y_2 - x_2 - x_1) - 2x$$
$$y \geq y_1 + x_1 + \frac{d}{2} - x,$$

which is a contradiction.

Case 2b: $y_1 \geq y$.

In this case, Equation (58) becomes

$$x - x_1 + y_1 - y \geq x_2 - x + y_2 - y$$
$$x_2 - x_1 + 2(x - x_2) \geq y_2 - y_1$$

Because $x < x_2$, we have $x_2 - x_1 + 2(x - x_2) < x_2 - x_1$. Consequently $x_2 - x_1 > y_2 - y_1$, which is contradicts our assumption.

Case 3: $x > x_2$ and $y < y_2 - \frac{d}{2}$.

Let us assume that $(x, y) \notin C_1$. Thus

$$|x_1 - x| + |y_1 - y| \geq |x_2 - x| + |y_2 - y| \tag{59}$$

Case 3a: $y \geq y_1$.

In this case, Equation (59) becomes

$$x - x_1 + y - y_1 \geq x - x_2 + y_2 - y$$
$$2y \geq 2y_2 + x_1 - x_2 + y_1 - y_2$$
$$y \geq y_2 - \frac{d}{2}$$

Which is a contradiction of our assumption.

Case 3b: $y < y_1$.

In this case, Equation (59) becomes

$$x - x_1 + y_1 - y \geq x - x_2 + y_2 - y$$
$$x_2 - x_1 \geq y_2 - y_1,$$

which contradicts an assumption of this Lemma. Now by symmetry we have that $(x, y) \in C_2$ in any of the following cases:

$$\begin{cases} y > y_1 + \frac{d}{2} & x < x_1 \\ y > y_1 + x_1 + \frac{d}{2} - x & x_1 \leq x < x_2 \\ y > y_2 - \frac{d}{2} & x_2 \leq x \end{cases}$$

It remains to show that $(x, y) \notin C_1$ in the cases of equality, that is when

$$\begin{cases} y = y_1 + \frac{d}{2} & x < x_1 \\ y = y_1 + x_2 + \frac{d}{2} - x & x_1 \leq x < x_2 \\ y = y_2 - \frac{d}{2} & x_2 \leq x \end{cases}$$

Case 1: $x < x_1, y = y_1 + \frac{d}{2}$.

We show equality directly:

$$\begin{aligned}
|x_1 - x| + |y_1 - y| &= x_1 - x + \left|-\frac{d}{2}\right| \\
&= x_1 - x + \frac{x_2 - x_1 + y_2 - y_1}{2} \\
&= x_2 - x + y_2 + \frac{-x_2 + x_1 - y_2 - y_1}{2} \\
&= |x_2 - x| + y_2 - y_1 - \frac{x_2 - x_1 + y_2 - y_1}{2} \\
&= |x_2 - x| + |y_2 - y|
\end{aligned}$$

Case 2: $x_1 \leq x < x_2, y = y_1 + x_1 + \frac{d}{2} - x$.

In this case, note that $y_1 < y < y_2$. First we justify that $y < y_2$. As $x \geq x_1$

$$\begin{aligned}
y &= y_1 + x_1 + \frac{x_2 + y_2 - x_1 - y_1}{2} - x \\
&\leq y_1 + \frac{x_2 + y_2 - x_1 - y_1}{2} \\
&= y_2 + \frac{x_2 - y_2 - x_1 + y_1}{2} \\
&< y_2
\end{aligned}$$

And since $x < x_2$,

$$\begin{aligned}
y &= y_1 + x_1 + \frac{x_2 + y_2 - x_1 - y_1}{2} - x \\
&= y_1 + x_1 + \frac{-x_2 + y_2 - x_1 - y_1}{2} - x + x_2 \\
&> y_1 + \frac{-x_2 + y_2 + x_1 - y_1}{2} \\
&> y_1
\end{aligned}$$

Now we can show equality directly:

$$\begin{aligned}
|x_1 - x| + |y_1 - y| &= x - x_1 + y - y_1 \\
&= x - x_1 + \left(y_1 + x_1 + \frac{d}{2} - x\right) - y_1 \\
&= x - x_1 + \left(y_2 + x_2 - \frac{d}{2} - x\right) - y_1 \\
&= x_2 - x + y_2 - \left(y_1 + \frac{d}{2} - x\right) \\
&= |x_2 - x| + |y_2 - y|
\end{aligned}$$

Case 3: $x_2 \leq x, y = y_2 - \frac{d}{2}$.

$$\begin{aligned}
|x_2 - x| + |y_2 - y| &= x - x_2 + \left|y_2 - y_2 + \frac{d}{2}\right| \\
&= x - x_2 + \frac{x_2 + y_2 - x_1 - y_1}{2} \\
&= x - x_1 + y_2 - y_1 + \frac{-x_2 + x_1 - y_2 + y_1}{2} \\
&= x - x_1 + y_2 - \frac{d}{2} - y_1 \\
&= |x_1 - x| + |y_1 - y|
\end{aligned}$$

By symmetry, this set of points cannot belong to C_2 either. $\square$

LEMMA 4.16. *Suppose* $\mathbf{a} = (x_1, y_1) \neq \mathbf{b} = (x_2, y_2)$ *with* $x_1 \leq x_2$ *and* $y_1 \leq y_2$. *If* $y_2 - y_1 = x_2 - x_1$,

$$C_1(\mathbf{a}, \mathbf{b}) = \left\{ (x, y) : \begin{cases} y < y_2 & x < x_1 \\ y < y_2 - (x - x_1) & x_1 \leq x < x_2 \end{cases} \right\}$$

PROOF. **Case 1:** $x < x_1$ and $y < y_2$.

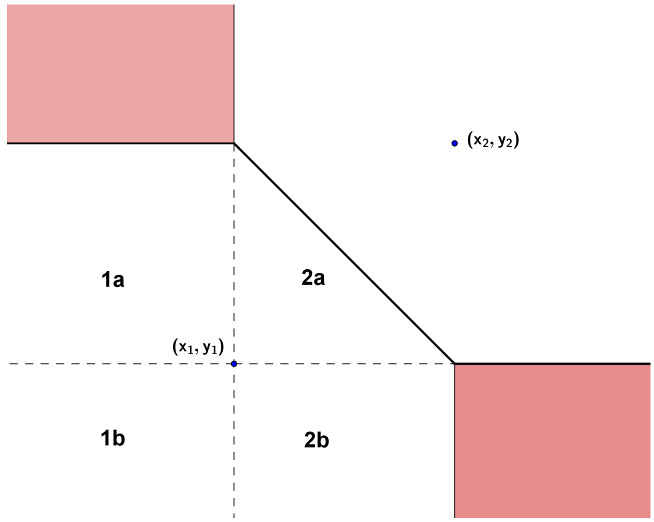

FIGURE 4. A graphical sketch of the proof of Lemma 4.16

Suppose $(x, y) \notin C_1$. Then

$$|x_1 - x| + |y_1 - y| \geq |x_2 - x| + |y_2 - y|. \tag{60}$$

Case 1a: $y > y_1$.

Equation (60) becomes

$$\begin{aligned} x_1 - x + y - y_1 &\geq x_2 - x + y_2 - y \\ 2y &\geq x_2 - x_1 + y_2 + y_1 \\ 2y &\geq y_2 - y_1 + y_2 + y_1 \\ y &\geq y_2. \end{aligned}$$

Case 1b: $y \leq y_1$.

Equation (60) becomes

$$\begin{aligned} x_1 - x + y_1 - y &\geq x_2 - x + y_2 - y \\ x_1 + y_1 &\geq x_2 + y_2 \end{aligned}$$

But since $x_1 \leq x_2$ and $y_1 \leq y_2$, this implies that $x_1 + y_1 = x_2 + y_2$. Thus $x_2 - x_1 = -(y_2 - y_1)$. With $y_2 - y_1 = x_2 - x_1$, We have that $x_2 = y_2$ and $x_1 = y_1$, another contradiction.

Case 2: $x_1 \leq x < x_2$ and $y < y_2 - (x - x_1)$.

Note that in this case $y < y_2$. Suppose $(x, y) \notin C_1$. Then

$$|x_1 - x| + |y_1 - y| \geq |x_2 - x| + |y_2 - y| \tag{61}$$

Case 2a: $y > y_1$.

Equation (61) becomes

$$\begin{aligned} x - x_1 + y - y_1 &\geq x_2 - x + y_2 - y \\ 2x + 2y &\geq x_2 + x_1 + y_2 + y_1 \\ 2(x + y) &\geq 2(x_1 + y_2) \\ y &\geq y_2 - (x - x_1) \end{aligned}$$

giving us a contradiction.

Case 2b: $y \leq y_1$.

Equation (61) becomes

$$\begin{aligned} x - x_1 + y_1 - y &\geq x_2 - x + y_2 - y \\ 2x &\geq x_2 + x_1 + y_2 - y_1 \\ 2x &\geq x_2 + x_1 + (x_2 - x_1) \\ x &\geq x_2 \end{aligned}$$

giving another contradiction.

If $x_1 \le x < x_2$ and $y = y_2 - (x - x_1)$, (x_2, y_2) and (x_1, y_1) are equidistant from (x, y). We can see this directly, noting that $y_1 \le y \le y_2$:

$$\begin{aligned}
|x_1 - x| + |y_1 - y| &= x - x_1 + y - y_1 \\
&= x - x_1 + y_2 - (x - x_1) + (x_2 - x_1 - y_2) \\
&= x_2 - x + y_2 - (y_2 - (x - x_1)) \\
&= |x_2 - x| + |y_2 - y|
\end{aligned}$$

To complete the proof, we show that if $x \le x_1$ and $y \ge y_2$, then (x_1, y_1) and (x_2, y_2) are equidistant from (x, y). By symmetry, this will imply the same for when $x \ge x_2, y \le y_1$. We show equidistance directly:

$$\begin{aligned}
|x_1 - x| + |y_1 - y| &= x_1 - x + y - y_1 \\
&= x_2 - x + y - y_2 \\
&= |x_2 - x| + |y_2 - y|
\end{aligned}$$

□

Hence we can picture the regions C_1 and C_2

LEMMA 4.17. *In $2NL_1$, if optimal pure strategies exist, they must be points along the line $y = x$.*

The idea of the proof is that if either player is playing a pure strategy not along the line $y = x$, the other player can always find a pure strategy on this line which gives a nonzero payoff (in his favor). Thus neither player has any pure optimal strategies off this line.

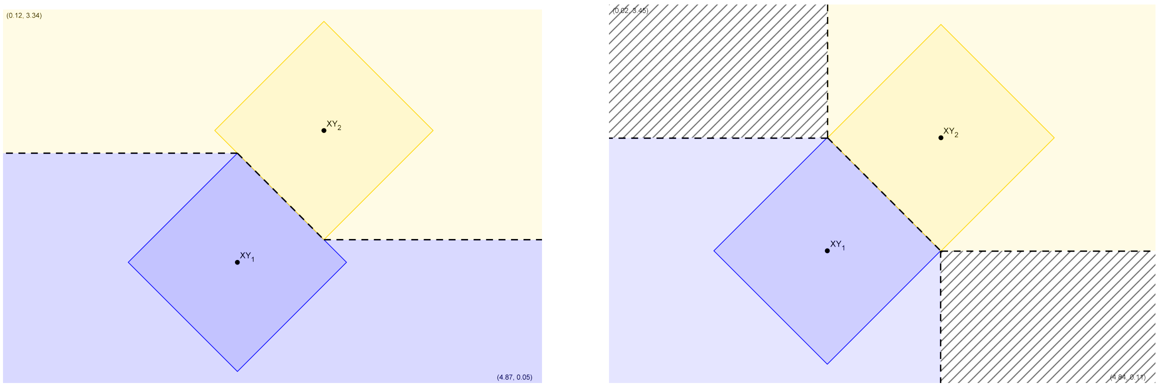

FIGURE 5. L_1 Distance: The region in blue is the set C_1, points closer to **a** than to **b** under L_1 distance. A "circle" is shown around each point

PROOF. If optimal pure strategies exist, then the game has a value and by symmetry the value must be zero. Suppose player II, maximizer, is playing $\mathbf{b}^* = (x_2^*, y_2^*)$ where $x_2^* \neq y_2^*$, and suppose this is an optimal pure strategy. WLOG, assume $y_2^* > x_2^*$. By responding by choosing strategy $-\mathbf{b}^* = (-x_2^*, -y_2^*)$, Player I guarantees a zero expected payoff. Let $\bar{\mathbf{a}} = (\bar{x}, \bar{y}) = (-\frac{x_2^*+y_2^*}{2}, -\frac{x_2^*+y_2^*}{2})$. Let $C_1^* := C_1(-\mathbf{b}^*, \mathbf{b}^*)$, the set of points closer to $(-x_2^*, -y_2^*)$ than to (x_2^*, y_2^*) under the L_1 metric. Namely,

$$C_1^* = \{(x, y) : |-x_2^* - x| + |-y_2^* - y| < |x_2^* - x| + |y_2^* - y|\}.$$

Define $\bar{C}_1 := C_1(\bar{\mathbf{a}}, \mathbf{b}^*)$, the set of points closer to $(\bar{x}, \bar{y})$ than to (x_2^*, y_2^*), namely

$$\bar{C}_1 = \{(x, y) : |\bar{x} - x| + |\bar{y} - y| < |x_2^* - x| + |y_2^* - y|\}$$

Claim: $C_1^* \subset \bar{C}_1$.

By Lemma 4.15, letting $d = x_2^* - (-x_2^*) + y_2^* - (-y_2^*) = 2x_2^* + 2y_2^*$, we have

$$C_1^* = \left\{ (x,y) : \begin{cases} y < -y_2^* + y_2^* + x_2^* = x_2^* & x < -x_2^* \\ y < -y_2^* - x_2^* + x_2^* + y_2^* - x = -x & -x^* \leq x < x_2^* \\ y <_2 y^* - x_2^* - y_2^* = -x_2^* & x_2^* \leq x \end{cases} \right\}.$$

Let $\bar{d} = x_2^* - \bar{x} + y_2^* - \bar{y} = x^* + y_2^* - 2\frac{-x_2^* - y_2^*}{2} = 2x_2^* + 2y_2^*$. Because $y_2^* > x_2^*$, $y_2^* - \bar{y} > x_2^* - \bar{x}$. Thus by Lemma 4.15,

$$\bar{C}_1 = \left\{ (x,y) : \begin{cases} y < \bar{y} + y_2^* + x_2^* = \frac{y_2^* + x_2^*}{2} & x < -x_2^* \\ y < \bar{y} + \bar{x} + x_2^* + y_2^* - x = -x & -x_2^* \leq x < x_2^* \\ y < y^* - x_2^* - y_2^* = -x_2^* & x_2^* \leq x \end{cases} \right\}.$$

Because $y_2^* > x_2^*$, $C_1^* \subset \bar{C}_1$ and consequently $P((\xi,\eta) \in \bar{C}_1) > P((\xi,\eta) \in C_1^*)$.

As $-x_2^* - y_2^* = \bar{x} + \bar{y}$, $K(\bar{\mathbf{a}}, \mathbf{b}^*) < K(-\mathbf{b}^*, \mathbf{b}^*) = v$, so (x^*, y^*) is not an optimal strategy for Player II. The argument is the same for Player I. $\square$

THEOREM 4.4. *In $2NL_1$, if pure optimal strategies exist for players $i = 1, 2$ then they are given by*

$$(x_i^*, y_i^*) = \left((-1)^i \frac{\sqrt{2\pi(\sigma_x^2 + 2\rho\sigma_x\sigma_y + \sigma_y^2)}}{4}, (-1)^i \frac{\sqrt{2\pi(\sigma_x^2 + 2\rho\sigma_x\sigma_y + \sigma_y^2)}}{4} \right).$$

PROOF. Since all optimal pure strategy points must be on the line $y = x$ for both players, we may transform the distribution by rotating $\frac{\pi}{4}$ clockwise so that solutions will be on the x-axis. The rotation matrix is

$$R = \begin{bmatrix} \cos(-\pi/4) & -\sin(-\pi/4) \\ \sin(-\pi/4) & \cos(-\pi/4) \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

And the transformed covariance matrix is

$$\begin{aligned} \bar{\Sigma} = R\Sigma R^T &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} \sigma_x^2 + \rho\sigma_x\sigma_y & \sigma_y^2 + \rho\sigma_x\sigma_y \\ -\sigma_x^2 + \rho\sigma_x\sigma_y & \sigma_y^2 - \rho\sigma_x\sigma_y \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} \sigma_x^2 + 2\rho\sigma_x\sigma_y + \sigma_y^2 & \sigma_y^2 - \sigma_x^2 \\ \sigma_y^2 - \sigma_x^2 & \sigma_x^2 - 2\rho\sigma_x\sigma_y + \sigma_y^2 \end{bmatrix}. \end{aligned}$$

The marginal distribution along the x-axis has variance $\bar{\sigma}_x^2 = \frac{1}{2}\left(\sigma_x^2 + 2\rho\sigma_y\sigma_y + \sigma_y^2\right)$.

We now look at solving a modification of the univariate case. For any strategy x of Player II (a strategy along the x-axis with the rotated distribution), the total of his component demands in the original case would be $\frac{\sqrt{2}}{2}(x+x) = \sqrt{2}x$. This, however, is true for Player I; we can think of this as simply a scaling of the payoffs so this will not change the solution. Thus, we may safely disregard this point.

The univariate solution point for Player i is

$$\left((-1)^i \frac{\sqrt{2\pi\bar{\sigma}_x^2}}{2}, 0\right).$$

Rotated back to the original axes, the solution points are

$$(-1)^i \frac{\sqrt{2\pi\bar{\sigma}_x^2}}{2}\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \left((-1)^i \frac{\sqrt{4\pi\bar{\sigma}_x^2}}{4}, (-1)^i \frac{\sqrt{4\pi\bar{\sigma}_x^2}}{4}\right).$$

□

1.4. L Infinity Distance. Now suppose the judge is offended by the single largest absolute deviation (componentwise) from what he considers fair. For example, the maximizer demands 50 units more than what the judge considers fair on every issue, while the minimizer's offer is precisely fair on all but one issue, and on that he offers 51 units less than the fair amount. In this case the judge will be so offended by the single deviation that he will side with the maximizer. This decision criterion is achieved by the L_∞ distance. We will call this game variant $2NL_\infty$ (Bivariate Normal, L_∞ distance). The L_∞ distance between two points $(x_1, y_1), (x_2, y_2)$ is

$$D_{L_\infty}\left((x_1, y_1), (x_2, y_2)\right) = \max\{|x_1 - x_2|, |y_1 - y_2|\}.$$

Under and L_∞ distance criterion, the judge makes a ruling in favor of whichever player minimizes the maximum absolute componentwise difference. This is also known as the **Chebychev distance**, or **chessboard distance**, referencing the distance that a King may travel in any number of moves. We will denote this model as $2NL_\infty$. Specifically, suppose players choose strategies $\mathbf{a} = (x_1, y_1), \mathbf{b} = (x_2, y_2)$ and the judge chooses random point $\boldsymbol{\xi} = (\xi, \eta)$. The ruling is

$$K(\mathbf{a}, \mathbf{b}) = \begin{cases} x_1 + y_1 & \text{if } \max\{|x_1 - \xi|, |y_1 - \eta|\} < \max\{|x_2 - \xi|, |y_2 - \eta|\} \\ x_2 + y_2 & \text{if } \max\{|x_1 - \xi|, |y_1 - \eta|\} > \max\{|x_2 - \xi|, |y_2 - \eta|\} \end{cases}$$

We first establish the geometry of the midset and sets C_1, C_2 as defined earlier in this work. That is, Given any two points $\mathbf{a} = (x_1, y_1), \mathbf{b} = (x_2, y_2)$,

$$C_1(\mathbf{a}, \mathbf{b}) = \{\mathbf{x} | D_{L_\infty}(\mathbf{x}, \mathbf{a}) < D_{L_\infty}(\mathbf{x}, \mathbf{b})\},$$

$$C_2(\mathbf{a}, \mathbf{b}) = \{\mathbf{x} | D_{L_\infty}(\mathbf{x}, \mathbf{a}) > D_{L_\infty}(\mathbf{x}, \mathbf{b})\},$$

$$Mid(\mathbf{a}, \mathbf{b}) = \{\mathbf{x} | D_{L_\infty}(\mathbf{x}, \mathbf{a}) = D_{L_\infty}(\mathbf{x}, \mathbf{b})\}.$$

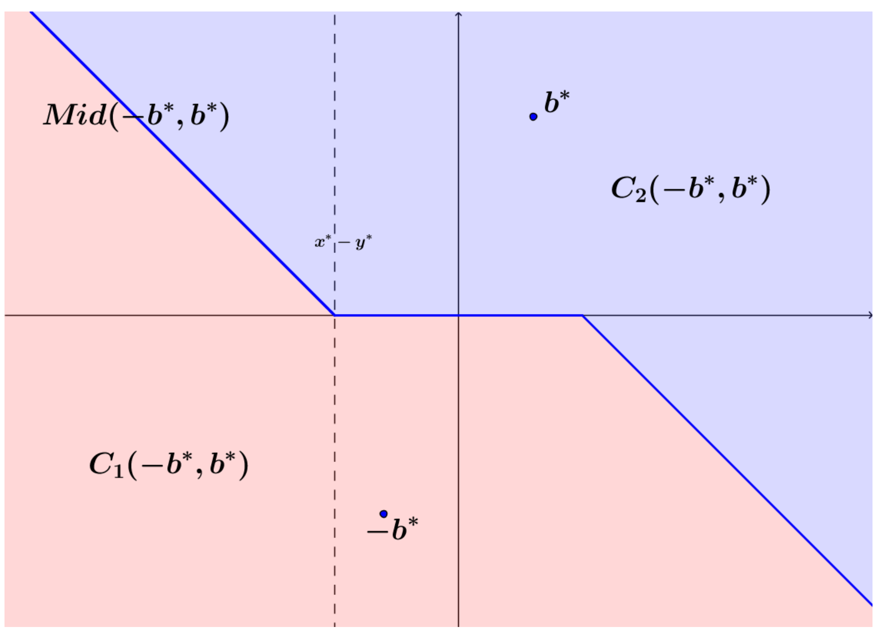

FIGURE 6. $2NL_\infty$: A graphical sketch of the Midset, C_1 and C_2 with respect to points $-\mathbf{b}^*, \mathbf{b}^*$.

LEMMA 4.18. *Suppose* $y^* > x^* > 0$. *Let* $\mathbf{b}^* = (x^*, y^*), -\mathbf{b}^* = (-x^*, -y^*)$ *and* $\mathbf{x} = (x, y)$. *If* $x \leq x^* - y^*$ *then*

$$\mathbf{x} \in \begin{cases} C_2(-\mathbf{b}^*, \mathbf{b}^*) & \textit{if } y > -x + x^* - y^* \\ Mid(-\mathbf{b}^*, \mathbf{b}^*) & \textit{if } y = -x + x^* - y^* \\ C_1(-\mathbf{b}^*, \mathbf{b}^*) & \textit{if } y < -x + x^* - y^* \end{cases} \tag{62}$$

If $x^* - y^* < x \leq 0$ *then*

$$\mathbf{x} \in \begin{cases} C_2(-\mathbf{b}^*, \mathbf{b}^*) & \textit{if } y > 0 \\ Mid(-\mathbf{b}^*, \mathbf{b}^*) & \textit{if } y = 0 \\ C_1(-\mathbf{b}^*, \mathbf{b}^*) & \textit{if } y < 0 \end{cases} \tag{63}$$

PROOF. First suppose $x \leq x^* - y^*$. Thus $x^* - x \geq y$.

Case 1: $y > -x + x^* - y^*$.

Then $y + y^* > x^* - x \geq y^*$. Therefore, $y > 0$. Obviously, $y + y^* > |y - y^*|$, so

$$|y + y^*| > \max\{|x - x^*|, |y - y^*|\},$$

that is, $\mathbf{x} \in C_2$.

Case 2: $y = -x + x^* - y^*$.

So $y + y^* = x^* - x \geq y^*$. Thus, we have

$$|y + y^*| = y + y^* = x^* - x > |x^* - (-x)|$$

and

$$|x - x^*| = x^* - x = y^* + y > |y - y^*|.$$

By these inequalities, we have $D_{L_\infty}(\mathbf{x}, \mathbf{b}^*) = x^* - x = y + y^* = D_{L_\infty}(\mathbf{x}, -\mathbf{b}^*)$, so $\mathbf{x} \in Mid(-\mathbf{b}^*, \mathbf{b}^*)$.

Case 3: $y < -x + x^* - y^*$.

Note that in this case, $|x^* - x| = x^* - x > |x^* + x|$. If $y < 0$, then clearly $|y^* - y| > |y^* + y|$ so $\mathbf{x} \in C_1$. If on the other hand $y \geq 0$, $|x^* - x| = x^* - x > y + y^* = |y - (-y^*)|$, so $\mathbf{x} \in C_1$.

Now suppose that $x^* - y^* < x \leq 0$. If $y > 0$, It is clear to see that

$$|y^* - y| > |-y^* - y| > |-x^* - x|$$

so $\mathbf{x} \in C_2$. If $y = 0$, by supposition $y^* > x^* - x$ so $D_{L_\infty}(\mathbf{x}, \mathbf{b}^*) = y^*$. This also implies that $y^* > x^* + x$ and $y^* > -x^* - x$, so $D_{L_\infty}(\mathbf{x}, -\mathbf{b}^*) = y^*$, so $\mathbf{x} \in Mid(-\mathbf{b}^*, \mathbf{b}^*)$. If $y < 0$, then by Case 3 above, $\mathbf{x} \in C_1$. $\square$

LEMMA 4.19. *Let* $\mathbf{b} = (x_2, y_2), -\mathbf{b} = (-x_2, -y_2)$ *where* $y_2 > x_2 > 0$. *If* $(x, y) \in C_1(-\mathbf{b}, \mathbf{b})$ *then* $(x + \alpha, y - \alpha) \in C_1(-\mathbf{b}, \mathbf{b})$ *for all* $\alpha > 0$.

PROOF. This is a direct consequence of Lemma 4.18 and the fact that C_1 is the symmetric reflection of C_2 about $(0, 0)$. $\square$

LEMMA 4.20. *In* $2NL_\infty$, *if pure optimal strategies exist, they must be on the line* $y = x$.

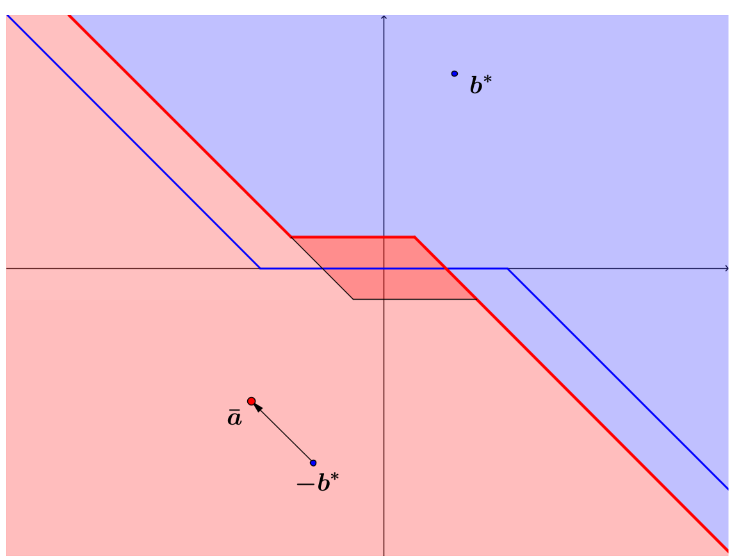

FIGURE 7. $2NL_\infty$: A graphical sketch of the proof of Lemma 4.20

PROOF. Suppose Player II, maximizer, is playing $\mathbf{b}^* = (x_2^*, y_2^*)$ where $x_2^* \neq y_2^*$, and suppose this is an optimal pure strategy. WLOG, assume $y_2^* > x_2^*$. Also $x_2^* > 0$. By responding by choosing strategy $-\mathbf{b}^* = (-x_2^*, -y_2^*)$, Player I guarantees a zero payoff. Let $\bar{\mathbf{a}} = (\bar{x}, \bar{y}) = (-\frac{x_2^*+y_2^*}{2}, -\frac{x_2^*+y_2^*}{2})$. Let $C_1^* := C_1(-\mathbf{b}^*, \mathbf{b}^*)$, the set of points closer to $(-x_2^*, -y_2^*)$ than to (x_2^*, y_2^*) under the L_∞ metric, namely

$$C_1^* = \{(x,y)|D_{L_\infty}((x,y),(-x_2^*,-y_2^*)) < D_{L_\infty}((x,y),(x_2^*,y_2^*))\}.$$

Define $\bar{C}_1 = C_1(\bar{\mathbf{a}}, \mathbf{b}^*)$, namely

$$\bar{C}_1 = \{(x,y)|D_{L_\infty}((x,y),(\bar{x},\bar{y})) < D_{L_\infty}((x,y),(x_2^*,y_2^*))\}.$$

Let C_2^* and $\bar{C}_1$ be defined similarly with the opposite inequality.

Claim 1: If $(x,y) \in \bar{C}_2$ then $(-x,-y) \in \bar{C}_1$.

Suppose $(x,y) \in \bar{C}_2$. By the symmetry of the L_∞ midset, (x,y) reflected over the midpoint of $\mathbf{b}^*$ and $\bar{\mathbf{a}}$, $\left(\frac{x_2^*-y_2^*}{4}, \frac{-x_2^*+y_2^*}{4}\right)$, will be in $\bar{C}_1$. Namely, $\left(\frac{x_2^*-y_2^*}{2} - x, \frac{y_2^*-x_2^*}{2} - y\right) \in \bar{C}_1$.

By Lemma 4.19, $\left(\frac{x_2^*-y_2^*}{2} - x + \frac{-y_2^*+x_2^*}{2}, \frac{y_2^*-x_2^*}{2} - y - \frac{-y_2^*+x_2^*}{2}\right) \in \bar{C}_1$. This point is $(-x,-y)$.

Claim 2: $(0,0) \in \bar{C}_1$. This is clear because $D_{L_\infty}(\mathbf{0}, \mathbf{b}^*) = y^*$ while $D_{L_\infty}(\mathbf{0}, \bar{\mathbf{a}}) = \frac{x_2^*+y_2^*}{2} < y^*$

By Claims 1 and 2, $P(\bar{C}_1) > \frac{1}{2}$, so

$$K(\bar{\mathbf{a}}, \mathbf{b}^*) < 0 = K(-\mathbf{b}^*, \mathbf{b}^*).$$

This contradicts that $\mathbf{b}^*$ is an optimal strategy.

$\square$

THEOREM 4.5. *In $2NL_\infty$, if pure optimal strategies exist for players $i = 1, 2$ then they are given by*

$$(x_i^*, y_i^*) = \left((-1)^i \frac{\sqrt{2\pi(\sigma_x^2 + 2\rho\sigma_x\sigma_y + \sigma_y^2)}}{4}, (-1)^i \frac{\sqrt{2\pi(\sigma_x^2 + 2\rho\sigma_x\sigma_y + \sigma_y^2)}}{4} \right).$$

PROOF. Since for either player, a pure optimal strategy must be on the line $y = x$ by Lemma 4.20, the argument is the same as in Theorem 4.4. □

1.5. $\mathbf{L_p}$ Metric.

1.5.1. *A note on the L_p Midset in R^2.* The L_p distance (or Minkowski(p) distance) between two points (x_1, y_1) and (x_2, y_2) is defined as

$$D_{L_p}\big((x_1, y_1), (x_2, y_2)\big) = \left(|x_2 - x_1|^p + |y_2 - y_1|^p\right)^{1/p}. \tag{64}$$

When $p = 1$ we have the Manhattan distance. When $p = 2$ this is Euclidean distance. The limit as $p \to \infty$ is the Chebychev distance. For any two points $(x_1, y_1), (x_2, y_2)$ we define the **midset** under Minkowski(p) distance as the set of points equidistant from both. We will denote the midset as $Mid_p[(x_1, y_1), (x_2, y_2)]$.

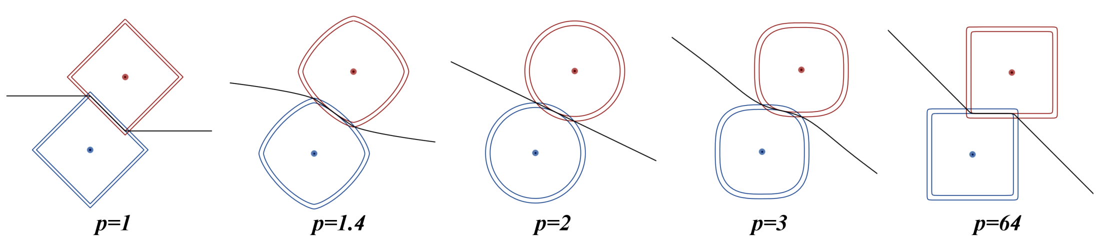

FIGURE 8. Minkowski Distance Midset: The midset between the two points is shown as a black line for $p = 1, 1.4, 2, 3,$ and 64 respectively. Two curves of constant Minkowski distance are shown around each point.

LEMMA 4.21. *For any $p \geq 1$, $Mid_p[(x_1, y_1), (x_2, y_2)]$ is symmetric about the midpoint $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$.*

PROOF. WLOG, assume the midpoint is $(0,0)$, thus $(x_1,y_1)=(-x_2,-y_2)$. Suppose point (x,y) is equidistant from (x_1,y_1) and (x_2,y_2). Thus

$$|x_1-x|^p+|y_1-y|^p=|x_2-x|^p+|y_2-y|^p.$$

But this may be written

$$|-x_2-x|^p+|-y_2-y|^p=|-x_1-x|^p+|-y_1-y|^p$$

$$\Leftrightarrow |-(x_2--x)|^p+|-(y_2--y)|^p=|-(x_1--x)|^p+|-(y_1--y)|^p$$

$$\Leftrightarrow |x_2-(-x)|^p+|y_2-(-y)|^p=|x_1-(-x)|^p+|y_1-(-y)|^p.$$

So $(-x,-y)$ is also equidistant from (x_1,y_1) and (x_2,y_2). $\square$

LEMMA 4.22. *For $p>1$, $\alpha>0$,*

$$Mid_p[(x^*,y^*),(x^*-\alpha,y^*-\alpha)]=\left\{\left(x^*+\frac{\alpha}{2}+t,y^*+\frac{\alpha}{2}+t\right),t\in\mathbb{R}\right\}.$$

PROOF. WLOG, assume $(x^*,y^*)=(\alpha/2,\alpha/2)=\mathbf{b}^*$, and $-\mathbf{b}^*=(-x^*,-y^*)$. So the set in question is simplified to the line $y=-x$. Take some point $(x,y)=(t,-t)$, for $t\in\mathbb{R}$.

$$\begin{aligned}(|x^*-x|^p+|y^*-y|^p)^{1/p}&=\left(\left|\frac{\alpha}{2}-t\right|^p+\left|\frac{\alpha}{2}-(-t)\right|^p\right)^{1/p}\\&=\left(\left|\alpha-\frac{\alpha}{2}-t\right|^p+\left|\alpha-\frac{\alpha}{2}-(-t)\right|^p\right)^{1/p}\\&=\left(\left|\frac{\alpha}{2}-\alpha-(-t)\right|^p+\left|\frac{\alpha}{2}-\alpha-t\right|^p\right)^{1/p}\\&=\left(|(y^*-\alpha)-y|^p+|(x^*-\alpha)-x|^p\right)^{1/p}.\end{aligned}$$

Now consider a point outside of this set, where $y>-x$. For convenience, let $\beta=\frac{\alpha}{2}$.

Case 1: $x=\beta+\theta, y=-\beta-\phi, 0\le\phi<\theta$.

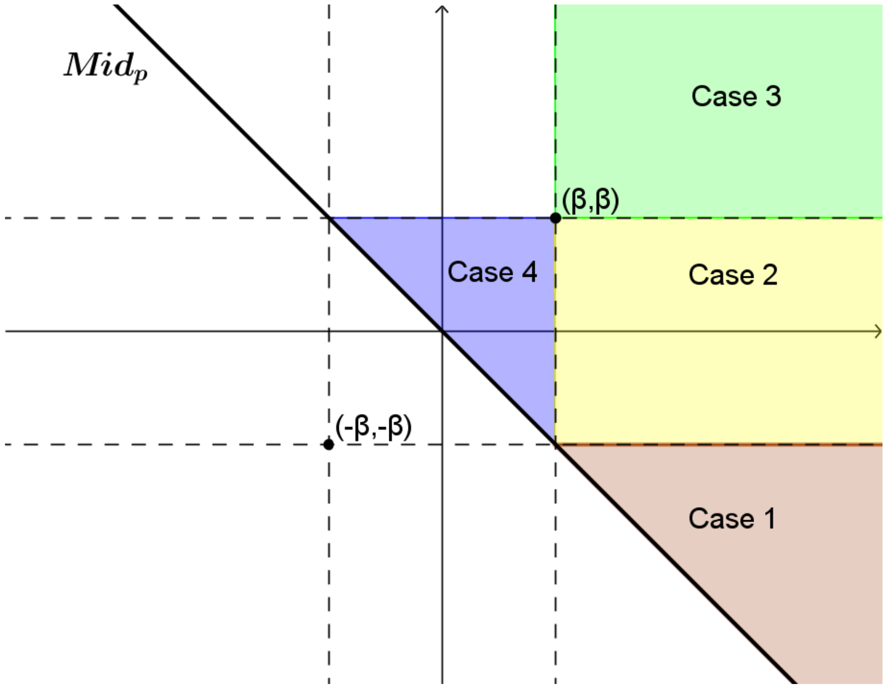

FIGURE 9. The four regions in the proof of Lemma 4.22.

Suppose $(x, y) \in Mid_p[\mathbf{b}^*, -\mathbf{b}^*]$. Then

$$|x^* - x|^p + |y^* - y|^p = |-x^* - x|^p + |-y^* - y|^p$$

$$\Leftrightarrow \theta^p + (2\beta + \phi)^p = (2\beta + \theta)^p + \phi^p.$$

For fixed $\beta > 0, \phi \geq 0$, take the derivative of each side with respect to θ.

$$p\theta^{p-1} = p(2\beta + \theta)^{p-1}$$

$$\Leftrightarrow \theta = 2\beta + \theta.$$

Since $p > 1$. But $\beta > 0$ by assumption. This is impossible.

Case 2: $x = \beta + \theta, y = \beta - \phi, 0 < \theta, 0 \leq \phi \leq 2\beta$.

Suppose $(x, y) \in Mid_p[\mathbf{b}^*, -\mathbf{b}^*]$. Then

$$|x^* - x|^p + |y - y|^p = |-x^* - x|^p + |-y^* - y|^p$$

$$\Leftrightarrow \theta^p + \phi^p = (2\beta + \theta)^p + (2\beta - \phi)^p.$$

We can state the following inequality:

$$\theta^p + \phi^p \leq \theta^p + (2\beta)^p \leq (2\beta+\theta)^p \leq (2\beta+\theta)^p + (2\beta-\phi)^p.$$

But $\theta^p + (2\beta)^p = (\theta+2\beta)^p$ only when $\theta = 0, \beta = 0$, or $p = 1$, none of which is the case by assumption.

Case 3: $x = \beta+\theta, y = \beta+\phi, \theta \geq 0, \phi \geq 0$.

Suppose $(x,y) \in Mid_p[\mathbf{b}^*, -\mathbf{b}^*]$. Then

$$|x^* - x|^p + |y - y|^p = |-x^* - x|^p + |-y^* - y|^p$$

$$\Leftrightarrow \theta^p + \phi^p = (2\beta+\theta)^p + (2\beta+\phi)^p.$$

But because $\beta > 0$ this is not possible.

Case 4: $x = \beta - \theta, y = -\beta+\phi, 0 \leq \theta \leq 2\beta, \theta < \phi \leq frm{-}e\beta$.

Suppose $(x,y) \in Mid_p[\mathbf{b}^*, -\mathbf{b}^*]$. Then

$$|x^* - x|^p + |y^* - y|^p = |-x^* - x|^p + |-y^* - y|^p$$

$$\Leftrightarrow \theta^p + (2\beta-\phi)^p = (2\beta-\theta)^p + \phi^p.$$

But since $\phi^p > \theta^p$ and $2\beta - \theta > 2\beta - \phi$, this is not possible. The remaining cases are handled by symmetry. □

LEMMA 4.23. *For $p > 1$, the curve $Mid_p\big[(x^*, y^*), (-x^*, -y^*)\big]$ is differentiable everywhere.*

PROOF. WLOG, we will consider only the case where $y^* > x^* \geq 0$, and $x \geq 0, y \leq 0$. When $0 \leq x \leq x^*, 0 \geq y \geq -x^*$,

$$\frac{dy}{dx} = -\frac{(x^*-x)^{p-1} + (x^*+x)^{p-1}}{(y^*-y)^{p-1} + (y^*+y)^{p-1}}. \tag{65}$$

When $x \geq x^*, 0 \geq y \geq -x^*$, we have

$$\frac{dy}{dx} = -\frac{(x+x^*)^{p-1} - (x-x^*)^{p-1}}{(y^*-y)^{p-1} + (y^*+y)^{p-1}}. \tag{66}$$

Evaluated at $x = x^*$, they are equal. When $-y^* \leq y \leq -x^*, x > x^*$, we have

$$\frac{dy}{dx} = -\frac{(x+x^*)^{p-1} - (x-x^*)^{p-1}}{(y^*-y)^{p-1} + (y^*+y)^{p-1}}. \tag{67}$$

When $y \geq -y^*, x > x^*$ we have

$$\frac{dy}{dx} = -\frac{(x+x^*)^{p-1} - (x-x^*)^{p-1}}{(-y+y^*)^{p-1} - (-y-y^*)^{p-1}}. \tag{68}$$

Evaluated when $y = -y^*$ both are equal. Thus the curve is differentiable everywhere.

□

LEMMA 4.24. *For $p > 1$, $x^*, y^* \geq 0, x^* \neq y^*$, $Mid_p[(x^*, y^*), (-x^*, -y^*)]$ and the line $y = -x$ intersect only at $(0,0)$.*

PROOF. Assume WLOG that $y^* > x^*$. Let us assume that there does exist a point $(\tilde{x}, -\tilde{x}) \in Mid_p\big[(x^*, y^*), (-x^*, -y^*)\big]$ with $\tilde{x} \neq 0$ (WLOG, assume $\tilde{x} > 0$).

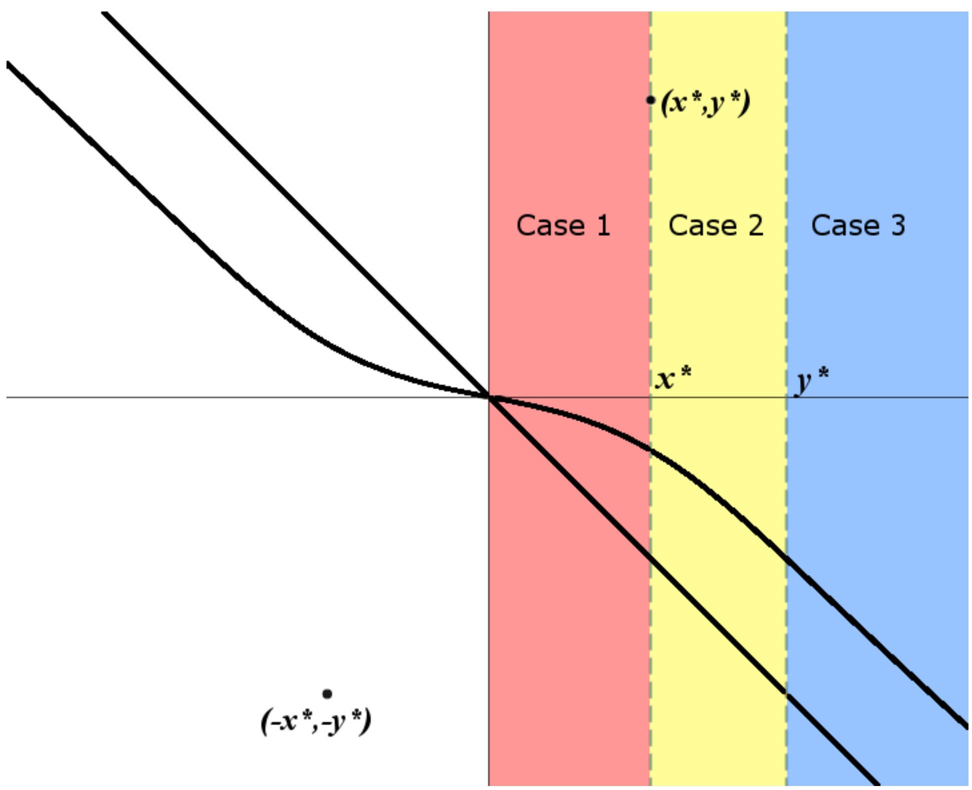

FIGURE 10. The three regions in the proof of Lemma 4.24.

Case 1: $0 < \tilde{x} \leq x^* < y^*$. The point $(\tilde{x}, -\tilde{x})$ satisfies

$$|x^* - \tilde{x}|^p + |y^* - (-\tilde{x})|^p = |-x^* - \tilde{x}|^p + |-y^* - (-\tilde{x})|^p.$$

Therefore

$$(x^* - \tilde{x})^p + (y^* + \tilde{x})^p = (x^* + \tilde{x})^p + (y^* - \tilde{x})^p. \tag{69}$$

Let $\alpha = x^* + \tilde{x}$, $\beta = x^* - \tilde{x}$, $y^* = x^* + \epsilon$, so $\alpha > \beta \geq 0$. We may write

$$(\alpha + \epsilon)^p - \alpha^p = (\beta + \epsilon)^p - \beta^p. \tag{70}$$

As $f(x) = x^p$ is convex for $p > 1$, the slope of a the secant line between (x, x^p) and $(x + \epsilon, (x + \epsilon)^p)$ increases as x increases. In other words,

$$\frac{(\alpha + \epsilon)^p - \alpha^p}{\epsilon} > \frac{(\beta + \epsilon)^p - \beta^p}{\epsilon},$$

but this contradicts the equality above.

Case 2: $0 < x^* < \tilde{x} \leq y^*$. The point $(\tilde{x}, -\tilde{x})$ satisfies

$$(\tilde{x} - x^*)^p + (y^* + \tilde{x})^p = (x^* + \tilde{x})^p + (y^* - \tilde{x})^p. \tag{71}$$

Let $\beta = y^* - \tilde{x}$, $\alpha = \tilde{x} - x^*$, so $y^* = x^* + \alpha + \beta$. We may substitute and write

$$\alpha^p + (2x^* + 2\alpha + \beta)^p = (2x^* + \alpha)^p + \beta^p. \tag{72}$$

But since $\alpha > 0$,

$$(2x^* + \alpha)^p + \beta^p \leq (2x^* + \alpha + \beta)^p < (2x^* + 2\alpha + \beta)^p + \alpha^p$$

contradicting the equality.

Case 3: $0 < x^* < y^* < \tilde{x}$. The point $(\tilde{x}, -\tilde{x})$ satisfies

$$(\tilde{x} - x^*)^p + (y^* + \tilde{x})^p = (x^* + \tilde{x})^p + (\tilde{x} - y^*)^p. \tag{73}$$

Note that since $\tilde{x} - x^* > \tilde{x} - y^*$, then $(\tilde{x} - x^*)^p > (\tilde{x} - y^*)^p$, and as $\tilde{x} + y^* > \tilde{x} + x^*$, then $(\tilde{x} + y^*)^p > (\tilde{x} + x^*)^p$. But this contradicts the equality. $\square$

As we are considering only the midset curve, subsequently in this section when referring to the derivative of the curve at point $\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y})$, we will simply say $\left.\frac{dy}{dx}\right|_{\tilde{\mathbf{x}}}$. In the following lemmas we will establish an important fact about the midset curve, that nowhere on the curve is the derivative equal to -1.

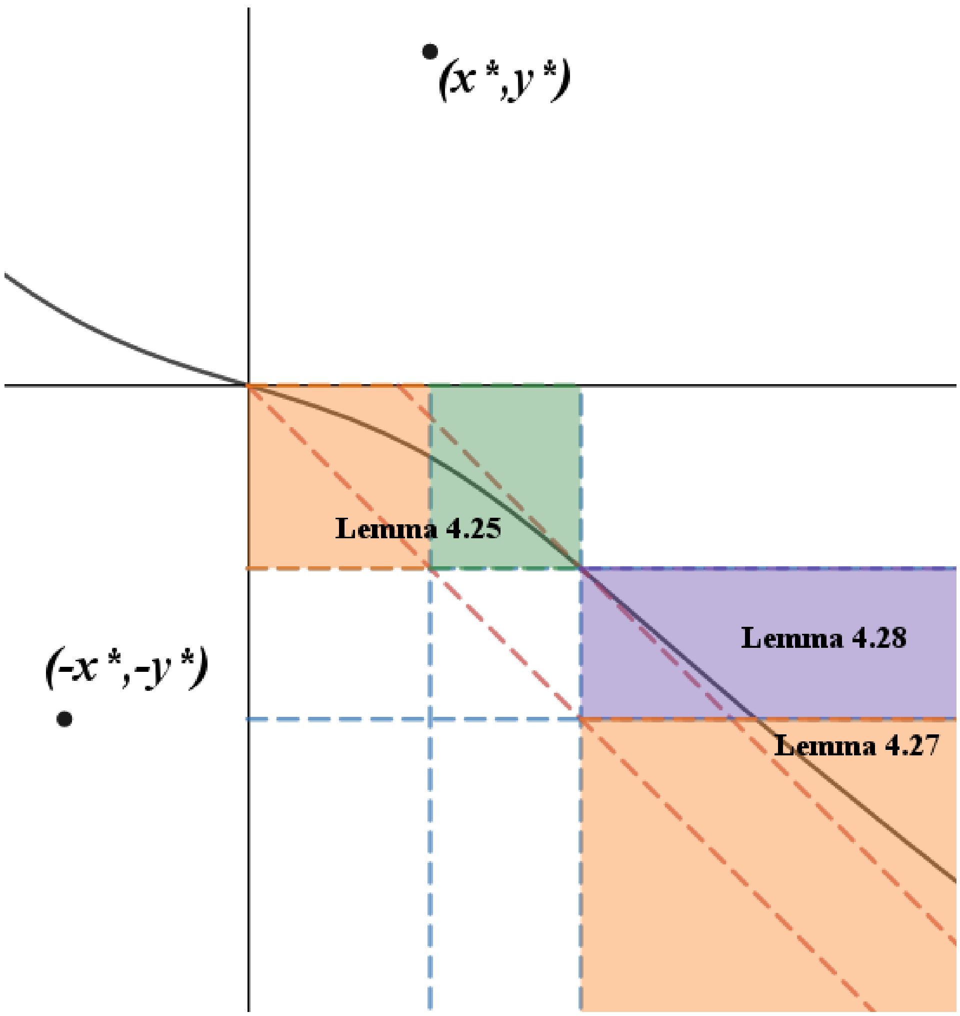

FIGURE 11. The regions in the proof of Lemmas 4.25, 4.27 and 4.28 .

LEMMA 4.25. *For $p > 1$, $y^* > x^* > 0$, let $\epsilon = y^* - x^*$. Suppose $\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y}) \in Mid_p[(x^*, y^*), (-x^*, -y^*)]$. If $0 \leq \tilde{x} \leq y^*$ then $\max(-\tilde{x}, -x^*) \leq \tilde{y} \leq \min(0, \epsilon - \tilde{x})$, and $0 > \left.\frac{dy}{dx}\right|_{\tilde{\mathbf{x}}} > -1$. Furthermore, $(0,0), (y^*, -x^*) \in Mid_p$.*

PROOF. By definition,

$$Mid_p[(x^*, y^*), (-x^*, -y^*)] = \{\mathbf{x} \,|\, |x - x^*|^p + |y - y^*|^p = |x + x^*|^p + |y + y^*|^p\}.$$

Clearly both $(0,0)$ and $(y^*, -x^*)$ satisfy this equation.

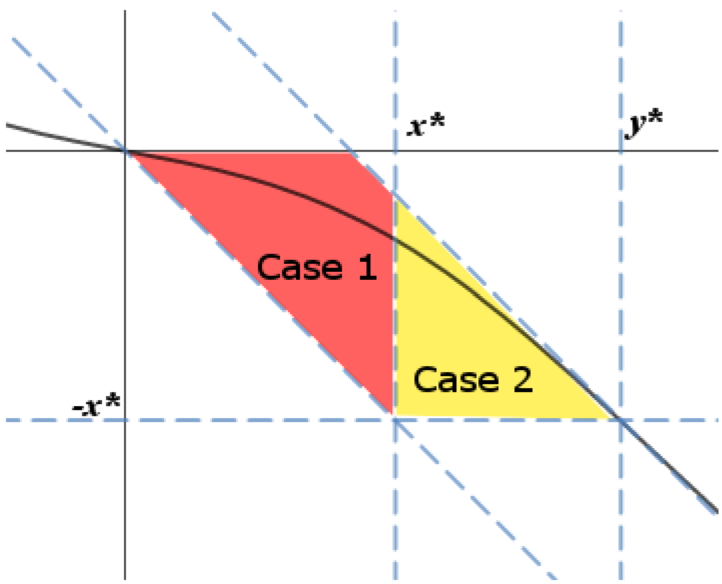

FIGURE 12. The regions in the proof of Lemma 4.25.

Case 1: First consider $\tilde{\mathbf{x}} \in [0, x^*] \times [-x^*, 0]$. The following must be true:

$$(x^* - \tilde{x})^p + (y^* - \tilde{y})^p = (x^* + \tilde{x})^p + (y^* + \tilde{y})^p.$$

By implicit differentiation, we get

$$\left.\frac{dy}{dx}\right|_{\tilde{\mathbf{x}}} = -\left[\frac{(x^* - \tilde{x})^{p-1} + (x^* + \tilde{x})^{p-1}}{(y^* - \tilde{y})^{p-1} + (y^* + \tilde{y})^{p-1}}\right]. \tag{74}$$

Within the brackets, both numerator and denominator are positive on this region of the curve, so $\left.\frac{dy}{dx}\right|_{\tilde{\mathbf{x}}} < 0$. Furthermore, evaluated at $(0,0)$ we get

$$\left.\frac{dy}{dx}\right|_{(0,0)} = -\left(\frac{x^*}{y^*}\right)^{p-1} > -1.$$

Because of this and Lemma 4.24, it must be the case that $\tilde{y} > -\tilde{x}$. Note that in this case $0 < -\tilde{y} < \tilde{x} < x^* < y^*$. Let $\alpha = x^* - \tilde{x} > 0$, $\delta = \tilde{x} - (-\tilde{y}) > 0$. Suppose $\left.\frac{dy}{dx}\right|_{\tilde{\mathbf{x}}} = -1$. Then we have

$$-1 = -\left[\frac{\alpha^{p-1} + (x^* + \tilde{x})^{p-1}}{(x^* + \tilde{x} + \epsilon - \delta)^{p-1} + (\delta + \alpha + \epsilon)^{p-1}}\right]. \tag{75}$$

If $\epsilon \geq \delta$ that would be impossible, so that means $\epsilon < \delta$. But any point on the Midset in this region must satisfy

$$(x^* - \tilde{x})^p + (y^* - \tilde{y})^p = (x^* + \tilde{x})^p + (y^* + \tilde{y})^p.$$

That may be written

$$\alpha^p + (x^* + \tilde{x} + \epsilon - \delta)^p = (x^* + \tilde{x})^p + (\delta + \alpha + \epsilon)^p.$$

But as $\alpha < \delta + \alpha + \epsilon, \alpha^p < (\delta + \alpha + \epsilon)^p$. Also since $\epsilon < \delta$, $x^* + \tilde{x} + \epsilon - \delta < x^* + \tilde{x}$ so

$$\alpha^p + (x^* + \tilde{x} + \epsilon - \delta)^p < (x^* + \tilde{x})^p + (\delta + \alpha + \epsilon)^p,$$

Presenting a contradiction. Thus everywhere in this region $\left.\frac{dy}{dx}\right|_{\tilde{\mathbf{x}}} > -1$.

Case 2: Now we consider $\tilde{\mathbf{x}} \in (x^*, y^*] \times [-y^*, 0]$. $\tilde{\mathbf{x}}$ satisfies

$$(\tilde{x} - x^*)^p + (y^* - \tilde{y})^p = (x^* + \tilde{x})^p + (y^* + \tilde{y})^p.$$

By implicit differentiation, we get

$$\left.\frac{dy}{dx}\right|_{\tilde{\mathbf{x}}} = -\left[\frac{(\tilde{x} + x^*)^{p-1} - (\tilde{x} - x^*)^{p-1}}{(y^* - \tilde{y})^{p-1} + (y^* + \tilde{y})^{p-1}}\right]. \tag{76}$$

Within the brackets, both numerator and denominator are positive on this region of the curve, so $\left.\frac{dy}{dx}\right|_{\tilde{\mathbf{x}}} < 0$. And because the curve passes through $(y^*, -x^*)$, it must be the case that $\tilde{y} \geq -x^*$ in this region. Because $x^* > 0$ and $p > 1$, the numerator is never zero so $\tilde{y} = -x^*$ only when $\tilde{x} = y^*$. Suppose that $\left.\frac{dy}{dx}\right|_{\tilde{\mathbf{x}}} = -1$. Let $\beta = -\tilde{y}$, $\alpha = \tilde{x} - x^*$, and $\epsilon = y^* - x^*$. We may write:

$$-1 = -\frac{(2x^* + \alpha)^{p-1} - \alpha^{p-1}}{(x^* + \epsilon + \beta)^{p-1} + (x^* + \epsilon - \beta)^{p-1}}, \tag{77}$$

so

$$(2x^* + \alpha)^{p-1} = (x^* + \epsilon + \beta)^{p-1} + (x^* + \epsilon - \beta)^{p-1} + \alpha^{p-1}. \tag{78}$$

But we must also satisfy the equation of the curve

$$\alpha^p + (x^* + \epsilon + \beta)^p = (2x^* + \alpha)^p + (x^* + \epsilon - \beta)^p. \tag{79}$$

Note that the restrictions of this region have $\alpha \leq \epsilon$, and $\beta \leq x^*$. If

$$x^* + \alpha \geq \epsilon + \beta$$

then $x^* + \epsilon + \beta \leq 2x^* + \alpha$ and the right hand side of the equation cannot be equal to the left, so we must have that $x^* + \alpha < \epsilon + \beta$. But then $(2x^* + \alpha)^{p-1} < (x^* + \epsilon + \beta)^{p-1}$ contradicting (79). Therefore, $\left.\frac{dy}{dx}\right|_{\tilde{\mathbf{x}}} \neq -1$ in this region. When we evaluate the derivative at $(y^*, -x^*)$, i.e. $(x^* + \epsilon, -x^*)$, we get

$$\left.\frac{dy}{dx}\right|_{(y^*, -x^*)} = -\frac{(2x^* + \epsilon)^{p-1} - \epsilon^{p-1}}{(2x^* + \epsilon)^{p-1} + \epsilon^{p-1}} > -1. \tag{80}$$

Thus, in this region $0 > \left.\frac{dy}{dx}\right|_{\tilde{\mathbf{x}}} > -1$.

By the extended mean value theorem, the slope of the secant line between any (x, y) on the curve in this region and $(y^*, -x^*)$ must be greater than -1, so $y \leq y^* - x^* - x$. $\square$

LEMMA 4.26. *Suppose* $p > 1$, $0 < x^* < y^*$. *Let* $\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y}) \in Mid_p[(x^*, y^*), (-x^*, -y^*)]$. *If* $\tilde{x} > y^*$ *then* $\tilde{y} > y^* - x^* - \tilde{x}$.

PROOF. As before, let $\epsilon = y^* - x^* > 0$.

Case 1: $-y^* < \tilde{y} < 0$.

In this region, $\tilde{\mathbf{x}}$ must satisfy

$$(\tilde{x} - x^*)^p + (-\tilde{y} + y^*)^p = (\tilde{x} + x^*)^p + (\tilde{y} + y^*)^p.$$

Suppose $\tilde{y} = \epsilon - \tilde{x}$. Then

$$(\tilde{x} - x^*)^p + (x^* + \epsilon - \epsilon + \tilde{x})^p = (\tilde{x} + x^*)^p + (\epsilon - \tilde{x} + x^* + \epsilon)^p,$$

which means

$$(\tilde{x} - x^*)^p = (2\epsilon - (\tilde{x} - x^*))^p.$$

In this region $\tilde{x} > y^* = x^* + \epsilon$, so $\tilde{x} - x^* > \epsilon$. But then $2\epsilon - (\tilde{x} - x^*) < \epsilon$, clearly a contradiction. Therefore $\tilde{y} \neq \epsilon - \tilde{x}$. Because $(y^*, -x^*) \in Mid_p$ and $\left.\frac{dy}{dx}\right|_{(y^*, -x^*)} > -1$, $\tilde{y} > \epsilon - \tilde{x}$.

Case 2: $\tilde{y} \leq -y^*$

In this case, $\tilde{\mathbf{x}}$ satisfies

$$(\tilde{x} - x^*)^p + (y^* - \tilde{y})^p = (\tilde{x} + x^*)^p + (-\tilde{y} - y^*)^p.$$

Suppose $\tilde{y} = \epsilon - \tilde{x}$. Then

$$(\tilde{x} - x^*)^p + (x^* + \epsilon - \epsilon + \tilde{x})^p = (\tilde{x} + x^*)^p + (-\epsilon + \tilde{x} - x^* - \epsilon)^p,$$

which means

$$(\tilde{x} - x^*)^p = (\tilde{x} - x^* - 2\epsilon)^p.$$

This is clearly not true as $\epsilon > 0$. From Case 1, we know that at the point $(\tilde{x}(-y^*), -y^*)$ in the Midset that $-y^* > \epsilon - \tilde{x}(-y^*)$ and because the midset is continuous, it must be the case that $\tilde{y} > \epsilon - \tilde{x}$ everywhere in this region.

□

LEMMA 4.27. *Suppose* $p > 1$, $0 < x^* < y^*$. *Let* $\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y}) \in Mid_p[(x^*, y^*), (-x^*, -y^*)]$. *If* $\tilde{\mathbf{x}} \in (y^*, \infty) \times (-\infty, -y^*]$ *then* $0 > \left.\frac{dy}{dx}\right|_{\tilde{\mathbf{x}}} \neq -1$.

PROOF. The proof for $p = 2$ is trivial, so we will assume $p \neq 2$. Let $\epsilon = y^* - x^* > 0$. The derivative is

$$\left.\frac{dy}{dx}\right|_{\tilde{\mathbf{x}}} = -\left[\frac{(\tilde{x} + x^*)^{p-1} - (\tilde{x} - x^*)^{p-1}}{(-\tilde{y} + y^*)^{p-1} - (-\tilde{y} - y^*)^{p-1}}.\right] \tag{81}$$

We can verify that within the brackets both numerator and denominator are positive, so $\left.\frac{dy}{dx}\right|_{\tilde{\mathbf{x}}} < 0$.

Suppose $\left.\frac{dy}{dx}\right|_{\tilde{\mathbf{x}}} = -1$. Then $\tilde{\mathbf{x}}$ satisfies

$$(\tilde{x} + x^*)^{p-1} - (\tilde{x} - x^*)^{p-1} = (-\tilde{y} + y^*)^{p-1} - (-\tilde{y} - y^*)^{p-1}$$

and because $\tilde{\mathbf{x}} \in Mid_p$,

$$(\tilde{x} + x^*)^p - (\tilde{x} - x^*)^p = (-\tilde{y} + y^*)^p - (-\tilde{y} - y^*)^p,$$

but these are inconsistent. Thus $\left.\frac{dy}{dx}\right|_{\tilde{\mathbf{x}}} \neq -1$. □

LEMMA 4.28. *Suppose* $p > 1$, $0 < x^* < y^*$. *Let* $\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y}) \in Mid_p[(x^*, y^*), (-x^*, -y^*)]$. *If* $\tilde{\mathbf{x}} \in (y^*, \infty) \times (-y^*, -x^*)$ *then* $0 > \left.\frac{dy}{dx}\right|_{\tilde{\mathbf{x}}} \neq -1$.

PROOF. The derivative is

$$\left.\frac{dy}{dx}\right|_{\tilde{\mathbf{x}}} = -\left[\frac{(\tilde{x} + x^*)^{p-1} - (\tilde{x} - x^*)^{p-1}}{(-\tilde{y} + y^*)^{p-1} - (-\tilde{y} - y^*)^{p-1}}\right]. \tag{82}$$

We can verify that within the brackets both numerator and denominator are positive, so $\left.\frac{dy}{dx}\right|_{\tilde{\mathbf{x}}} < 0$. From Lemma 4.26 we know that $\tilde{y} > \epsilon - \tilde{x}$.

Suppose $1 < p < 2$. If $\left.\frac{dy}{dx}\right|_{\tilde{\mathbf{x}}} = -1$ then $\tilde{\mathbf{x}}$ satisfies

$$(\tilde{x} + x^*)^{p-1} - (\tilde{x} - x^*)^{p-1} = (-\tilde{y} + y^*)^{p-1} + (-\tilde{y} - y^*)^{p-1}. \tag{83}$$

Because $-\tilde{y} < \tilde{x} - \epsilon$,

$$-\tilde{y} + y^* < \tilde{x} - \epsilon + y^* = \tilde{x} + x^*,$$

and

$$-\tilde{y} - y^* < \tilde{x} - \epsilon - y^* < \tilde{x} - x^* - 2\epsilon.$$

For $\delta > 0$, the slope of the secant line on $f(t) = t^{p-1}$ between $t - \delta$ and t increases as t decreases or as δ increases. Therefore,

$$\frac{(\tilde{x} + x^*)^{p-1} - (\tilde{x} - x^*)^{p-1}}{2x^*} < \frac{(-\tilde{y} + y^*)^{p-1} - (-\tilde{y} - y^*)^{p-1}}{2x^* + 2\epsilon} < \frac{(-\tilde{y} + y^*)^{p-1} - (-\tilde{y} - y^*)^{p-1}}{2x^*}.$$

This contradicts the equality of equation 83. Suppose instead that $p > 2$. The second derivative is

$$\begin{aligned}\left.\frac{d^2y}{dx^2}\right|_{\tilde{\mathbf{x}}} &= -(p-1)\frac{((\tilde{x} + x^*)^{p-2} - (\tilde{x} - x^*)^{p-2})((y^* - \tilde{y})^{p-1} + (y^* + \tilde{y})^{p-1})}{((y^* - \tilde{y})^{p-1} + (y^* + \tilde{y})^{p-1})^2} \\ &- (p-1)\left(\left.\frac{dy}{dx}\right|_{\tilde{\mathbf{x}}}\right)\frac{((\tilde{x} + x^*)^{p-1} - (\tilde{x} - x^*)^{p-1})(-(y^* - \tilde{y})^{p-2} + (y^* + \tilde{y})^{p-2})}{((y^* - \tilde{y})^{p-1} + (y^* + \tilde{y})^{p-1})^2}\end{aligned} \tag{84}$$

If $\tilde{\mathbf{x}}$ is a point of inflection then this is equal to zero, i.e.

$$(\tilde{x} + x^*)^{p-2} - (\tilde{x} - x^*)^{p-2} = -\left(\left.\frac{dy}{dx}\right|_{\tilde{\mathbf{x}}}\right)\frac{((\tilde{x} + x^*)^{p-1} - (\tilde{x} - x^*)^{p-1})(-(y^* - \tilde{y})^{p-2} + (y^* + \tilde{y})^{p-2})}{(y^* - \tilde{y})^{p-1} + (y^* + \tilde{y})^{p-1}} \tag{85}$$

$$\Leftrightarrow (\tilde{x} + x^*)^{p-2} - (\tilde{x} - x^*)^{p-2} = \left(\left.\frac{dy}{dx}\right|_{\tilde{\mathbf{x}}}\right)^2 \left((y^* + \tilde{y})^{p-2} - (y^* - \tilde{y})^{p-2}\right) \tag{86}$$

$$\Leftrightarrow \frac{(\tilde{x} + x^*)^{p-2} - (\tilde{x} - x^*)^{p-2}}{(y^* + \tilde{y})^{p-2} - (y^* - \tilde{y})^{p-2}} = \left(\left.\frac{dy}{dx}\right|_{\tilde{\mathbf{x}}}\right)^2 \tag{87}$$

But as $\tilde{x} > x^*$ and $y^* - \tilde{y} > y^* + \tilde{y}$, the left hand side is negative so this cannot be true. From Lemma 4.25, $\frac{dy}{dx}\big|_{\tilde{\mathbf{x}}} > -1$ when $\tilde{x} < y^*$ and from Lemma 4.27 $\frac{dy}{dx}\big|_{\tilde{\mathbf{x}}} > -1$ when $\tilde{y} < -y^*$. Therefore $\frac{dy}{dx}\big|_{\tilde{\mathbf{x}}} > -1$ in this region as well, otherwise there would be a point of inflection. $\square$

THEOREM 4.6. *Suppose* $p > 1, y^* > x^* > 0$. *Let* $z^* = \frac{x^*+y^*}{2}$, $M_1 = Mid_p((x^*, y^*), (-z^*, -z^*))$, *and* $M_2 = Mid_p((-x^*, -y^*), (z^*, z^*))$. M_1 *and* M_2 *are disjoint.*

PROOF. Suppose there exists $(\tilde{x}, \tilde{y}) \in M_1 \cap M_2$. Let $\epsilon = \frac{y^*-x^*}{2}$. Note that the two midsets are identical curves with M_2 translated horizontally by ϵ and vertically by $-\epsilon$. If the two curves intersect at $(\tilde{x}, \tilde{y})$ then that means $(\tilde{x} - \epsilon, \tilde{y} + \epsilon) \in M_1$. But because the curve is differentiable everywhere, by the extended mean value theorem this implies that $\left.\dfrac{dy}{dx}\right|_{\mathbf{x}'} = -1$ on the curve at some point $\mathbf{x}'$, which is not the case by the previous lemmas. $\square$

1.5.2. *Optimal Pure Strategies under L_p Metric.* Now we shall consider what happens when the judge uses an L_p metric to measure distance. We will call this game variant $2NL_p$ (Bivariate Normal, L_p distance). Having shown that the midset curve nowhere has a derivative of -1, we now can show that in fact for any $p \geq 1$, if a solution exists in pure optimal strategies it does not matter what value p is; it will be the same solution as in the cases of L_2, L_1 and L_∞. Suppose Player II chooses $\mathbf{b}^* = (x^*, y^*)$ with $y^* > x^* > 0$. Let $\mathbf{a}' = (x', y') = \left(-\frac{x^*+y^*}{2}, -\frac{x^*+y^*}{2}\right)$. The midpoint between $\mathbf{b}^*$ and $\mathbf{a}'$ is

$$\overline{\mathbf{a}} = (\overline{x}, \overline{y}) = \left(\frac{x^* - y^*}{4}, \frac{y^* - x^*}{4}\right).$$

LEMMA 4.29. *Let* $p > 1$.

$$D_{L_p}(\mathbf{0}, \mathbf{a}') < D_{L_p}(\mathbf{0}, \mathbf{b}^*).$$

PROOF. By convexity of $f(x) = x^p$, for $p > 1$,

$$\left|\frac{x^* + y^*}{2}\right|^p = \left|\frac{1}{2}x^* + \frac{1}{2}y^*\right|^p \leq \frac{1}{2}x^{*p} + \frac{1}{2}y^{*p},$$

so

$$2\left|\frac{x^*+y^*}{2}\right|^p \leq x^{*p}+y^{*p}$$
$$\Leftrightarrow |x'-0|^p+|y'-0|^p \leq |x^*-0|^p+|y^*-0|^p.$$

The above attains equality only when $p=1$ or in the trivial case of $x^*=y^*=0$, but $y^*>x^*\geq 0$ by assumption. $\square$

Let us define

$$\bar{C}_1=\left\{\mathbf{x}|D_{L_p}\left(\mathbf{x},\mathbf{a}'\right)<D_{L_p}\left(\mathbf{x},\mathbf{b}^*\right))\right\}$$

LEMMA 4.30. $P((\xi,\eta)\in\bar{C}_1)>\dfrac{1}{2}$.

PROOF. Assume WLOG that $y^*>x^*\geq 0$. Let the strategy of Player II be $\mathbf{b}^*=(x^*,y^*)$ and Player I's strategy be $\mathbf{a}'=(x',y')=\left(-\dfrac{x^*+y^*}{2},-\dfrac{x^*,y^*}{/}2\right)$ as previously defined. Let $M_1=Mid_p(\mathbf{a}',\mathbf{b}^*)$. So $M_1=\partial\bar{C}_1$, and $(0,0)\in\bar{C}_1$ as established by Lemma 4.29. Let $M_2=Mid_p(-\mathbf{a}'-\mathbf{b}^*)$, which is M_1 reflected about the origin. Since by Theorem 4.6 the two curves are disjoint, let us refer to the region of positive area bounded by the two curves as C^+. Because $(0,0)\in C^+$, $P((\xi,\eta)\in C^+)>0$. Also note that by symmetry that $P((\xi,\eta)\notin\bar{C}_1)=P((\xi,\eta)\in\bar{C}_1\backslash C^+)$. Thus the lemma is proved. $\square$

LEMMA 4.31. *In $2NL_p$, if optimal pure strategies exist for either player, they must lie on the line $y=x$.*

PROOF. Suppose Player II is playing optimally with the pure strategy $\mathbf{b}^*=(x^*,y^*)$ with $y^*>x^*\geq 0$. Player I may respond strategy $-\mathbf{b}^*=(-x^*,-y^*)$, resulting in an expected payoff of 0. Player I may, however, switch to $\mathbf{a}'=(x',y')$ defined as above. Because his net demand remains unchanged but by Lemma 4.30

the probability that the ruling is in his favor increases from $\frac{1}{2}$, Player II was not playing optimally. $\square$

THEOREM 4.7. *In $2NL_p$, if pure optimal strategies exist for players $i = 1, 2$ then they are given by*

$$(x_i^*, y_i^*) = \left((-1)^i \frac{\sqrt{2\pi(\sigma_x^2 + 2\rho\sigma_x\sigma_y + \sigma_y^2)}}{4}, (-1)^i \frac{\sqrt{2\pi(\sigma_x^2 + 2\rho\sigma_x\sigma_y + \sigma_y^2)}}{4}\right).$$

PROOF. Since for either player, a pure optimal strategy must be on the line $y = x$ by Lemma 4.31, the argument is the same as in Theorem 4.4. $\square$

1.6. Mahalanobis Distance. The Mahalanobis distance (MD) between two points $\mathbf{x_1}, \mathbf{x_2}$ with respect to a multivariate Normal distribution with covariance matrix Σ is given by

$$D_M(\mathbf{x_1}, \mathbf{x_2}) = \sqrt{(\mathbf{x_1} - \mathbf{x_2})^T \Sigma^{-1} (\mathbf{x_1} - \mathbf{x_2})}.$$

In our case,

$$\begin{aligned}
&D_M\big((x_1, y_1), (x_2, y_2)\big) \\
=&\sqrt{\frac{1}{|\Sigma|}(x_1 - x_2, y_1 - y_2)\begin{pmatrix} \sigma_y^2 & -\rho\sigma_x\sigma_y \\ -\rho\sigma_x\sigma_y & \sigma_x^2 \end{pmatrix}\begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}} \\
=&|\Sigma|^{-1/2}\sqrt{\left(\sigma_y^2(x_1 - x_2) - \rho\sigma_x\sigma_y(y_1 - y_2), -\rho\sigma_x\sigma_y(x_1 - x_2) + \sigma_x^2(y_1 - y_2)\right)\begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}} \\
=&|\Sigma|^{-1/2}\sqrt{\sigma_y^2(x_1 - x_2)^2 - 2\rho\sigma_x\sigma_y(x_1 - x_2)(y_1 - y_2) + \sigma_x^2(y_1 - y_2)^2}.
\end{aligned}$$

We will call this game variant $2NMD$ (Bivariate Normal, Mahalanobis distance). Again we will use the following lemma

LEMMA 4.32. *In $2NMD$, (x^*, y^*) is an optimal pure strategy for Player I if and only if $(-x^*, -y^*)$ is an optimal pure strategy for Player II.*

LEMMA 4.33. *The Mahalanobis midset between* $\mathbf{x_1} = (x_1, y_1)$ *and* $\mathbf{x_2} = (x_2, y_2)$ *is the line* $Ax + By = C$ *where*

$$A = 2\sigma_y^2(x_2 - x_1) - 2\rho\sigma_x\sigma_y(y_2 - y_1),$$

$$B = 2\sigma_x^2(y_2 - y_1) - 2\rho\sigma_x\sigma_y(x_2 - x_1), \text{ and}$$

$$C = \sigma_y^2(x_2^2 - x_1^2) - 2\rho\sigma_x\sigma_y(x_2y_2 - x_1y_1) + \sigma_x^2(y_2^2 - y_1^2).$$

PROOF. By definition, the midset is

$$Mid_{MD}(\mathbf{x_1}, \mathbf{x_2}) = \{\mathbf{x} \,|\, D_M(\mathbf{x}, \mathbf{x_1}) = D_M(\mathbf{x}, \mathbf{x_2})\} \tag{88}$$

$$= \{\mathbf{x} \,|\, (\mathbf{x} - \mathbf{x_1})^T\Sigma^{-1}(\mathbf{x} - \mathbf{x_1}) = (\mathbf{x} - \mathbf{x_2})^T\Sigma^{-1}(\mathbf{x} - \mathbf{x_2})\}. \tag{89}$$

With $\mathbf{x_i} = (x_i, y_i)$, the condition becomes

$$\sigma_y^2(x - x_1)^2 - 2\rho\sigma_x\sigma_y(x - x_1)(y - y_1) + \sigma_x^2(y - y_1)^2 \tag{90}$$

$$=\sigma_y^2(x - x_2)^2 - 2\rho\sigma_x\sigma_y(x - x_2)(y - y_2) + \sigma_x^2(y - y_2)^2 \tag{91}$$

or

$$\begin{aligned}
\sigma_y^2\left[(x - x_1)^2 - (x - x_2)^2\right] - 2\rho\sigma_x\sigma_y\left[(x - x_1)(y - y_1) - (x - x_2)(y - y_2)\right]& \\
+\sigma_x^2\left[(y - y_1)^2 - (y - y_2)^2\right] &= 0 \\
\sigma_y^2\left[-2x_1x + x_1^2 + 2x_2x - x_2^2\right] - 2\rho\sigma_x\sigma_y\left[-x_1x - x_1y + x_1y_1 + x_2y + y_2x - x_2y_2\right]& \\
+\sigma_x^2\left[-2y_1y + y_1^2 + 2y_2y - y_2^2\right] &= 0 \\
Ax + By - C &= 0
\end{aligned}$$

□

THEOREM 4.8. *In* $2NMD$*, if pure optimal strategies exist for players* $i = 1, 2$ *then they are given by*

$$(x_i^*, y_i^*) = \left((-1)^i\tau\alpha, (-1)^i\tau\beta\right),$$

where $\alpha = \sigma_x^2 + \rho\sigma_x\sigma_y, \beta = \sigma_y^2 + \rho\sigma_x\sigma_y,\ \tau = \frac{\sigma_{X'}\sqrt{\pi/2}}{\sqrt{\alpha^2+\beta^2}}$ *and*

$$\sigma_{X'}^2 = \frac{\sigma_x^2\alpha^2 + 2\rho\sigma_x\sigma_y\alpha\beta + \sigma_y^2\beta^2}{\alpha^2+\beta^2}.$$

Proof. Let

$$\overline{D_M}(\mathbf{x_1}, \mathbf{x_2}) := (|\Sigma|^{1/2} D_M(\mathbf{x_1}, \mathbf{x_2})^2.$$

Suppose Player II plays pure strategy (x^*, y^*). Let $t^* = x^* + y^*$. If Player I responds with strategy $(-x^*, -y^*)$, the expected payoff is zero. If, however, Player I chooses a new point (x, y) with $x + y = -t^*$ but with a higher probability of being chosen by the judge, then he will have improved the expected payoff in his favor. Whichever player's MD to the origin is lesser will control the majority of the probability of the distribution. Let Player I attempt to minimize

$$\overline{D_M}(\mathbf{x}, \mathbf{0}) \quad \text{subject to } x + y = -t^*.$$

$$\begin{aligned}
\overline{D_M}(\mathbf{x}, \mathbf{0}) &= \sigma_y^2 x^2 - 2\rho\sigma_x\sigma_y xy + \sigma_x^2 y^2 \\
&= \sigma_y^2 x^2 - 2\rho\sigma_x\sigma_y x(-t^* - x) + \sigma_x^2(-t^* - x)^2 \\
&= \sigma_y^2 x^2 + 2\rho\sigma_x\sigma_y(x^2 + t^* x) + \sigma_x^2(x^2 + 2t^* x + t^{*2}) \\
&= (\sigma_y^2 + 2\rho\sigma_x\sigma_y + \sigma_x^2)x^2 + (2t^*\rho\sigma_x\sigma_y + 2t^*\sigma_x^2)x + \sigma_x^2 t^{*2}.
\end{aligned}$$

The minimum occurs when

$$x = -t^* \frac{\alpha}{\alpha + \beta},$$

where $\alpha = \sigma_x^2 + \rho\sigma_x\sigma_y, \beta = \sigma_y^2 + \rho\sigma_x\sigma_y$. This gives

$$y = -t^* \frac{\beta}{\alpha + \beta}.$$

Hence any optimal point must lie on the line

$$y = \frac{\rho\sigma_x\sigma_y + \sigma_y^2}{\rho\sigma_x\sigma_y + \sigma_x^2}x.$$

As before, we apply a rotation to the distribution to align this line with the x-axis. The angle of rotation is

$$\theta = -\arctan\left(\frac{\rho\sigma_x\sigma_y + \sigma_y^2}{\rho\sigma_x\sigma_y + \sigma_x^2}\right).$$

The rotation matrix is

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The new covariance matrix is

$$\begin{aligned}\Sigma' &= R\Sigma R^T \\ &= \begin{pmatrix} \sigma_x^2\cos^2\theta - 2\rho\sigma_x\sigma_y\cos\theta\sin\theta + \sigma_y^2\sin^2\theta & (\sigma_x^2 - \sigma_y^2)\sin\theta\cos\theta + \rho\sigma_x\sigma_y(\cos^2\theta - \sin^2\theta) \\ (\sigma_x^2 - \sigma_y^2)\sin\theta\cos\theta + \rho\sigma_x\sigma_y(\cos^2\theta - \sin^2\theta) & \sigma_x^2\cos^2\theta + 2\rho\sigma_x\sigma_y\cos\theta\sin\theta + \sigma_y^2\sin^2\theta \end{pmatrix}.\end{aligned}$$

The variance along the x axis is

$$\sigma_{X'}^2 = \frac{\sigma_x^2(\rho\sigma_x\sigma_y + \sigma_x^2)^2 + 2\rho\sigma_x\sigma_y(\rho\sigma_x\sigma_y + \sigma_x^2)(\rho\sigma_x\sigma_y + \sigma_y^2) + \sigma_y^2(\rho\sigma_x\sigma_y + \sigma_y^2)^2}{(\sigma_x^2 + \rho\sigma_x\sigma_y)^2 + (\rho\sigma_x\sigma_y + \sigma_y^2)^2}.$$

By Theorem 2.1, the univariate solution points are

$$\pm(\sigma_{X'}\sqrt{\pi/2}, 0).$$

Rotating the point back counter-clockwise by θ we have

$$\pm\sigma_{X'}\sqrt{\pi/2}(\cos\theta, \sin\theta) = \pm\frac{\sigma_{X'}\sqrt{\pi/2}}{\sqrt{(\sigma_x^2 + \rho\sigma_x\sigma_y)^2 + (\rho\sigma_x\sigma_y + \sigma_y^2)^2}}\left(\rho\sigma_x\sigma_y + \sigma_x^2, \rho\sigma_x\sigma_y + \sigma_y^2\right)$$

. $\square$

2. Uniform Distribution

Suppose instead that the players assume that the judge will choose some point $\boldsymbol{\xi} = (\xi, \eta)$ with uniform probability within the rectangular region $\Xi := [-\alpha, \alpha] \times [-\beta, \beta]$, where $\alpha, \beta > 0$, and this is common knowledge among players. We will refer to the interior of this set as Ξ°. Thus, we may consider $\boldsymbol{\xi} \sim$Uniform(Ξ). Without any loss of generality, assume $\beta \geq \alpha > 0$. In fact, in the analysis which follows we will assume without any loss of generality that the units have been scaled so that $\alpha = 1$. The players may present any final-offers $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$. We will call this game variant $2UL_2$ (Bivariate Uniform, L_2 distance). Again let us use the notation $C_i(\mathbf{a}, \mathbf{b})$ (or simply C_i where there is no ambiguity) to denote the set of points in $\mathbb{R}^2$ which are strictly closer to Player i's pure strategy than to the other player's. Let the **midset** $Mid(\mathbf{a}, \mathbf{b})$ be the set of points which are equidistant to $\mathbf{a}$ and $\mathbf{b}$ under the L_2 metric. In general, if players play respective strategies $\mathbf{a} = (x_1, y_1), \mathbf{b} = (x_2, y_2)$, $Mid(\mathbf{a}, \mathbf{b})$ will be defined as the line

$$y = \frac{x_1 - x_2}{y_2 - y_1}\left(x - \frac{x_1 + x_2}{2}\right) + \frac{y_1 + y_2}{2} \tag{92}$$

We let $P_1 = P(\boldsymbol{\xi} \in C_1)$ for ease of notation.

LEMMA 4.34. *In $2UL_2$, if optimal pure strategies $\mathbf{a}^*, \mathbf{b}^*$ exist, then they must lie on the line $y = x$.*

PROOF. Suppose Player II plays $\mathbf{b}^* = (x_2^*, y_2^*)$ where $x_2^* \neq y_2^*$ and is playing optimally. Player I may play $-\mathbf{b}^* = (-x_2^*, -y_2^*)$ and achieve an expected payoff of zero, with $P_1 = \frac{1}{2}$. However, Player I may instead deviate to $\mathbf{a}' = \left(-\frac{x_2^* + y_2^*}{2}, -\frac{x_2^*, y_2^*}{2}\right)$. Because this point is strictly closer to $(0,0)$ than $\mathbf{b}^*$, $P\big(\boldsymbol{\xi} \in C_1(\mathbf{a}', \mathbf{b}^*)\big) > \frac{1}{2}$ while his net offer remains the same. Thus $\mathbf{b}^*$ was not an optimal pure strategy. $\square$

LEMMA 4.35. *If pure optimal strategies $\mathbf{a}^*, \mathbf{b}^*$ exist, they are given by*

$$\mathbf{a}^* = \left(-\frac{\beta}{2}, -\frac{\beta}{2}\right), \quad \mathbf{b}^* = \left(\frac{\beta}{2}, \frac{\beta}{2}\right).$$

PROOF. First note that if $\beta = 1$, then by restricting players to strategies along the line $y = x$ the players are effectively in a one-dimensional FOA game where the distribution f is a triangular distribution. In this game, Players choose strategies $z_i = 2x_i$. The distribution is symmetric with support $[-2\beta, 2\beta]$, so optimal strategies are symmetric with

$$z_1^* = -\frac{1}{2f(0)} = \frac{1}{2}2\beta = -\beta.$$

This corresponds to optimal pure strategies in the original game of $x_1^* = y_1^* = -\frac{\beta}{2}$.

Now suppose $1 < \beta$. Suppose optimal pure strategies $\mathbf{a}^* = (x_1^*, x_1^*)$ and $\mathbf{b}^* = (x_2^*, x_2^*)$ do exist. $Mid(\mathbf{a}^*, \mathbf{b}^*)$ is the line

$$y = -x + x_1^* + x_2^*.$$

We have five cases:

Case 1: $x_1^* + x_2^* < -\beta - \alpha$

In this case, because $(-\alpha, -\beta) \in C_2$, $\Xi \in C_2$ so $P_1 = 0$. Thus $K(\mathbf{a}^*, \mathbf{b}^*) = 2x_2^*$. If $x_2^* > 0$, $K(-\mathbf{b}^*, \mathbf{b}^*) = 0 < 2x_2^*$. If $x_2^* < 0$, then $K(\mathbf{a}^*, -\mathbf{a}^*) = 0 > 2x_2^*$. If $x_2^* = 0$, this means $x_1^* < -\beta - 1$. Player II may deviate to $\mathbf{b}' = (1 - \beta - x_1^*, 1 - \beta - x_1^*)$. $P\left(\boldsymbol{\xi} \in C_1(\mathbf{a}^*, \mathbf{b}')\right) = \frac{4}{8\beta} = \frac{1}{2\beta}$, so

$$\begin{aligned} K(x_1^*, x_2') &= 2 - 2\beta - 2x_1^* + \big(2x_1^* - (2 - 2\beta - 2x_1^*)\big)\frac{1}{2\beta} \\ &= 2 - 2\beta - 2x_1^* + (2x_1^* - 1 + \beta)\frac{1}{\beta} \\ &= 2x_1^*\frac{1-\beta}{\beta} + 3 - 2\beta - \frac{1}{\beta}. \end{aligned}$$

Since $x_1^* < -\beta - 1$ and $\beta > 1$,

$$2x_1^*\frac{1-\beta}{\beta} > 2\frac{(1-\beta)(-1-\beta)}{\beta} = -2\frac{1}{\beta} + 2\beta.$$

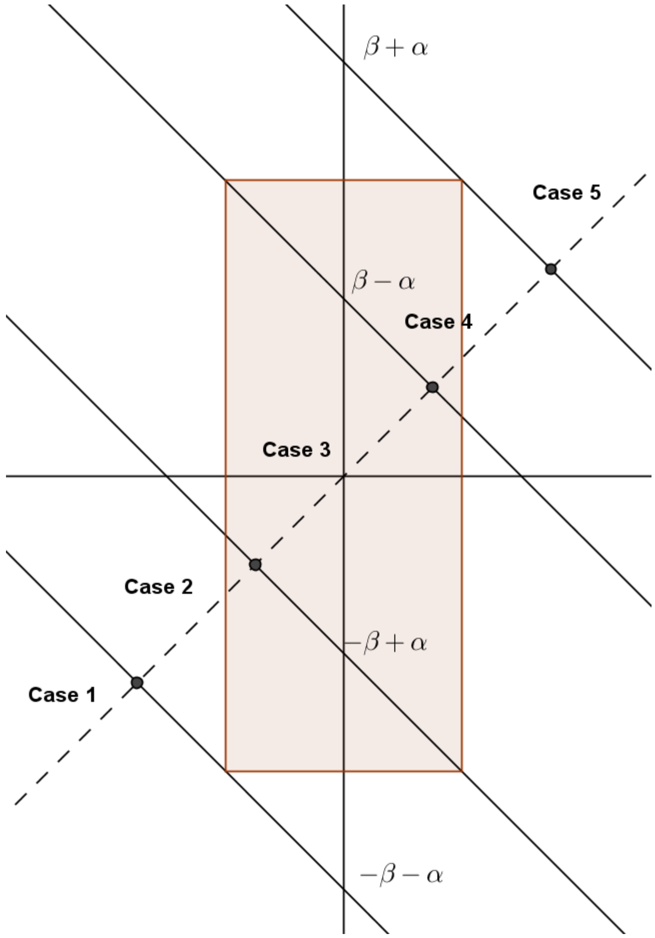

FIGURE 13. The 5 cases in Lemma 4.35

Thus

$$\begin{aligned} K(x_1^*, x_2') &> -2\frac{1}{\beta} + 2\beta + 3 - 2\beta - \frac{1}{\beta} \\ &= 3\left(1 - \frac{1}{\beta}\right) > 0. \end{aligned}$$

In any of these three sub-cases, one of the players may improve his payoff by deviating to a new strategy, so the players were not playing optimally.

Case 2: $-\beta - 1 \leq x_1^* + x_2^* < -\beta + 1$

In this case $P_1 = \dfrac{(x_1^* + x_2^* + 1 + \beta)^2}{8\beta}$, so the expected payoff is

$$K(x_1^*, x_2^*) = 2x_2^* + (2x_1^* - 2x_2^*)\frac{(x_1^* + x_2^* + 1 + \beta)^2}{8\beta}. \tag{93}$$

Thus

$$K_{x_1^*} = \frac{(x_1^* + x_2^* + 1 + \beta)^2}{4\beta} + (x_1^* - x_2^*)\frac{x_1^* + x_2^* + 1 + \beta}{2\beta}$$

Set equal to zero, and noting that $P_1 > 0$ gives us

$$0 = x_1^* + x_2^* + 1 + \beta + 2x_1^* - 2x_2^*$$

$$\Leftrightarrow \quad 2x_2^* - 2x_1^* = x_1^* + x_2^* + 1 + \beta. \tag{94}$$

Taking the derivative with respect to x_2^*, we have

$$K_{x_2^*} = 2 - \frac{(x_1^* + x_2^* + 1 + \beta)^2}{4\beta} + (x_1^* - x_2^*)\frac{x_1^* + x_2^* + 1 + \beta}{2\beta}.$$

Set equal to zero and using the substitution (94), we get

$$0 = 2 - \frac{(2x_2^* - 2x_1^*)^2}{4\beta} + (x_1^* - x_2^*)\frac{2x_2^* - 2x_1^*}{2\beta}$$

$$\Leftrightarrow \quad \frac{1}{2} = \frac{(x_2 - x_1)^2}{2\beta}.$$

But by the same substitution, $P_1 = \dfrac{(x_2^* - x_1^*)^2}{2\beta}$. However, in this case $P_1 < \dfrac{1}{2\beta} < \dfrac{1}{2}$; from this contradiction we can conclude that optimal pure strategies will not exist in this case.

Case 3: $-\beta + 1 \le x_1^* + x_2^* \le \beta - 1$

In this case, the probability that Player I is chosen by the arbitrator is

$$P_1 = \frac{x_1^* + x_2^* + \beta}{2\beta}.$$

The expected payoff then is

$$K(x_1^*, x_2^*) = 2x_2^* + (2x_1^* - 2x_2^*)\frac{x_1^* + x_2^* + \beta}{2\beta}. \tag{95}$$

If the players are playing a locally optimal strategy pair, $K_{x_1^*}(x_1^*, x_2^*) = 0$. We derive x_1^* by this first order condition: $K_{x_1^*} = \dfrac{2x_1^*}{\beta} + 1$, so $x_1^* = -\dfrac{\beta}{2}$ and similarly $x_2^* = \dfrac{\beta}{2}$.

Cases 4 and 5 are covered by cases 2 and 1 respectively by swapping the identities of the players. □

THEOREM 4.9. *In* $2UL_2$*, the strategy pair* $\mathbf{a}^* = \left(-\frac{\beta}{2}, -\frac{\beta}{2}\right), \mathbf{b}^* = \left(\frac{\beta}{2}, \frac{\beta}{2}\right)$ *is the unique globally optimal strategy pair.*

PROOF. Let Player II fix his strategy as $\mathbf{b}^*$ as given above. Consider all pure strategies $\mathbf{a} = (x_1, y_1)$. We first make two observations:

1) If $x_1 + y_1 > 0$ then $K(\mathbf{a}, \mathbf{b}^*) > 0$.

Simply put, if the net demand of both players is positive then the expected payoff of the game will certainly be positive.

2) If $x_1 + y_1 = 0$ then $K(\mathbf{a}, \mathbf{b}^*) > 0$.

This is evident because in this case, no matter the strategy of Player I, $(1, \beta)$ is closer to $\mathbf{b}^*$ than to $(x_1, -x_1)$ and thus $P_1 < 1$. To see this, note that the minimum distance to $(1, \beta)$ of all points such that $x_1 + y_1 = 0$ is attained at $\bar{\mathbf{a}} = (-\frac{\beta-1}{2}, \frac{\beta-1}{2})$.

$$D_{L_2}(\mathbf{b}^*, (1, \beta)) = \sqrt{\left(1 - \frac{\beta}{2}\right)^2 + \frac{\beta^2}{4}} = \sqrt{1 - \beta + \frac{\beta^2}{2}}$$

$$D_{L_2}(\bar{\mathbf{a}}, (1, \beta)) = \sqrt{2\left(\frac{1+\beta}{2}\right)^2} = \sqrt{\frac{1}{2} + \beta + \frac{\beta^2}{2}}$$

Since $(1, \beta) \notin C_1, P_1 < 1$.

We next make the following observations:

3) If $x_1 \geq \dfrac{\beta}{2}$, then $K\left((\beta - x_1, y_1), \mathbf{b}^*\right) \leq K(\mathbf{a}, \mathbf{b}^*)$

Suppose $x_1 \geq \dfrac{\beta}{2}$. If Player I deviates from $\mathbf{a}$ to $\mathbf{a}' = (\beta - x_1, y_1)$, his net demand is decreased. Note that $Mid(\mathbf{a}, \mathbf{b}^*)$ is

$$y = \frac{2x_1 - \beta}{\beta - 2y_1}\left(x - \frac{2x_1 + \beta}{4}\right) + \frac{2y_1 + \beta}{4}$$

and intersects the y axis at $\frac{\beta^2 - 2x_1^2 - 2y_1^2}{2(\beta - 2y_1)}$. However, $Mid(\mathbf{a}', \mathbf{b}^*)$ intersects the y-axis at $\frac{-\beta^2 + 4\beta x_1 - 2x_1^2 - 2y_1^2}{2(\beta - 2y_1)}$. Since $x_1 \geq \frac{\beta}{2} \Rightarrow \beta^2 \leq -\beta^2 + 4\beta x_1$, so P_1 is not decreased by this deviation.

4) If $y_1 \geq \frac{\beta}{2}$, then $K\left((x_1, \beta - y_1), \mathbf{b}^*\right) \leq K(\mathbf{a}, \mathbf{b}^*)$

Suppose $y_1 \geq \frac{\beta}{2}$. If Player I deviates from $\mathbf{a}$ to $\mathbf{a}' = (x_1, \beta - y_1)$, his net demand is decreased. Note that $Mid(\mathbf{a}, \mathbf{b}^*)$ is

$$x = \frac{\beta - 2y_1}{2x_1 - \beta}\left(y - \frac{2y_1 + \beta}{4}\right) + \frac{2x_1 + \beta}{4}$$

and intersects the x-axis at $\frac{\beta^2 - 2x_1^2 - 2y_1^2}{2(\beta - 2x_1)}$. However, $Mid(\mathbf{a}', \mathbf{b}^*)$ intersects the y axis at $\frac{-\beta^2 + 4\beta y_1 - 2x_1^2 - 2y_1^2}{2(\beta - 2x_1)}$. Since $y_1 \geq \frac{\beta}{2} \Rightarrow \beta^2 \leq -\beta^2 + 4\beta y_1$, so P_1 is not decreased by this deviation.

Based on the previous two points, we will assume going forward that $x_1 < \frac{\beta}{2}$ and $y_1 < \frac{\beta}{2}$.

Let 4 circles be defined with centers at $(1, -\beta), (-1, -\beta), (-1, \beta)$ and $(1, \beta)$ all intersecting the point $\mathbf{b}^*$. The regions within these circles are respectively given by:

$$(\bullet_1) \qquad x^2 - 2x + y^2 + 2\beta y + \beta - \frac{3}{2}\beta^2 \leq 0$$

$$(\bullet_2) \qquad x^2 + 2x + y^2 + 2\beta y - \beta - \frac{3}{2}\beta^2 \leq 0$$

$$(\bullet_3) \qquad x^2 + 2x + y^2 - 2\beta y - \beta + \frac{1}{2}\beta^2 \leq 0$$

$$(\bullet_4) \qquad x^2 - 2x + y^2 - 2\beta y + \beta + \frac{1}{2}\beta^2 \leq 0$$

We will refer to the boundary of $\bullet_k$ as $\bigcirc_k$, and the closure of its complements as $\blacksquare_k$. Observe first that if $\mathbf{a} \notin \text{Int} \bigcup_{k=1}^{4} \bullet_k$, then $\mathbf{a} \notin \text{Int}\bullet_k\ \forall k$ so $(\pm\alpha, \pm\beta) \notin C_1$;in other words, $\Xi^\circ \in C_2$, so $P_1 = 0$. Thus to determine the globally optimal response to $\mathbf{b}^*$ we will consider only points $\mathbf{a} \in \text{Int} \bigcup_{k=1}^{4} \bullet_k$. However, observe that if $\mathbf{a} \in \bullet_4$ then by the above, $x_1 + y_1 > 0$ and so $K > 0$. Furthermore, observe that if $\mathbf{a} \in \bullet_3 \backslash \bullet_2$, then $y_1 \geq \frac{\beta}{2}$ and if $\mathbf{a} \in \bullet_1 \backslash \bullet_2$ that $x_1 \geq \frac{\beta}{2}$. Neither of these are the case by assumption. Thus we may assume $\mathbf{a} \in \bullet_2$ and $\mathbf{a} \notin \bullet_4$. So we may restrict ourselves to the remaining cases:

(1) $\mathbf{a}$ $\in$R1:$= \bullet_2 \cap \bullet_1 \backslash (\bullet_3 \cup \bullet_4)$
(2) $\mathbf{a}$ $\in$R2:$= \bullet_2 \backslash (\bullet_3 \cup \bullet_1 \cup \bullet_4)$
(3) $\mathbf{a}$ $\in$R3:$= \bullet_2 \cap \bullet_3 \cap \bullet_1 \backslash (\bullet_4)$
(4) $\mathbf{a}$ $\in$R4:$= \bullet_2 \cap \bullet_3 \backslash (\bullet_1 \cup \bullet_4)$

Case 1: $\mathbf{a}$ $\in$R1

In this case, because $(-1, -\beta), (1, -\beta) \in C_1$, $Mid(\mathbf{a}, \mathbf{b}^*)$ intersects the left and right sides of the rectangle. So, by geometric means we have

$$P(C_1) = \frac{1}{2\beta}\left(\beta + \frac{\beta^2 - 2x_1^2 - 2y_1^2}{2\beta - 4y_1}\right) = \frac{1}{2} + \frac{\beta^2 - 2x_1^2 - 2y_1^2}{4\beta(\beta - 2y_1)}.$$

So

$$K(\mathbf{a}, \mathbf{b}^*) = \beta + (x_1 + y_1 - \beta)\left(\frac{1}{2} + \frac{\beta^2 - 2x_1^2 - 2y_1^2}{4\beta(\beta - 2y_1)}\right)$$

The first partial derivatives are

$$K_{x_1} = \frac{1}{2} + \frac{\beta^2 - 2x_1^2 - 2y_1^2}{4\beta(\beta - 2y_1)} + \frac{(\beta - x_1 - y_1)x_1}{\beta(\beta - 2y)}$$
$$K_{y_1} = \frac{1}{2} + \frac{\beta^2 - 2x_1^2 - 2y_1^2}{4\beta(\beta - 2y_1)} - \frac{\beta - x_1 - y_1}{2}\left(\frac{\beta(\beta^2 - 2x_1^2 - 2y_1^2)}{(\beta^2 - 2\beta y_1)^2} - \frac{2y_1}{\beta^2 - 2\beta y_1}\right)$$

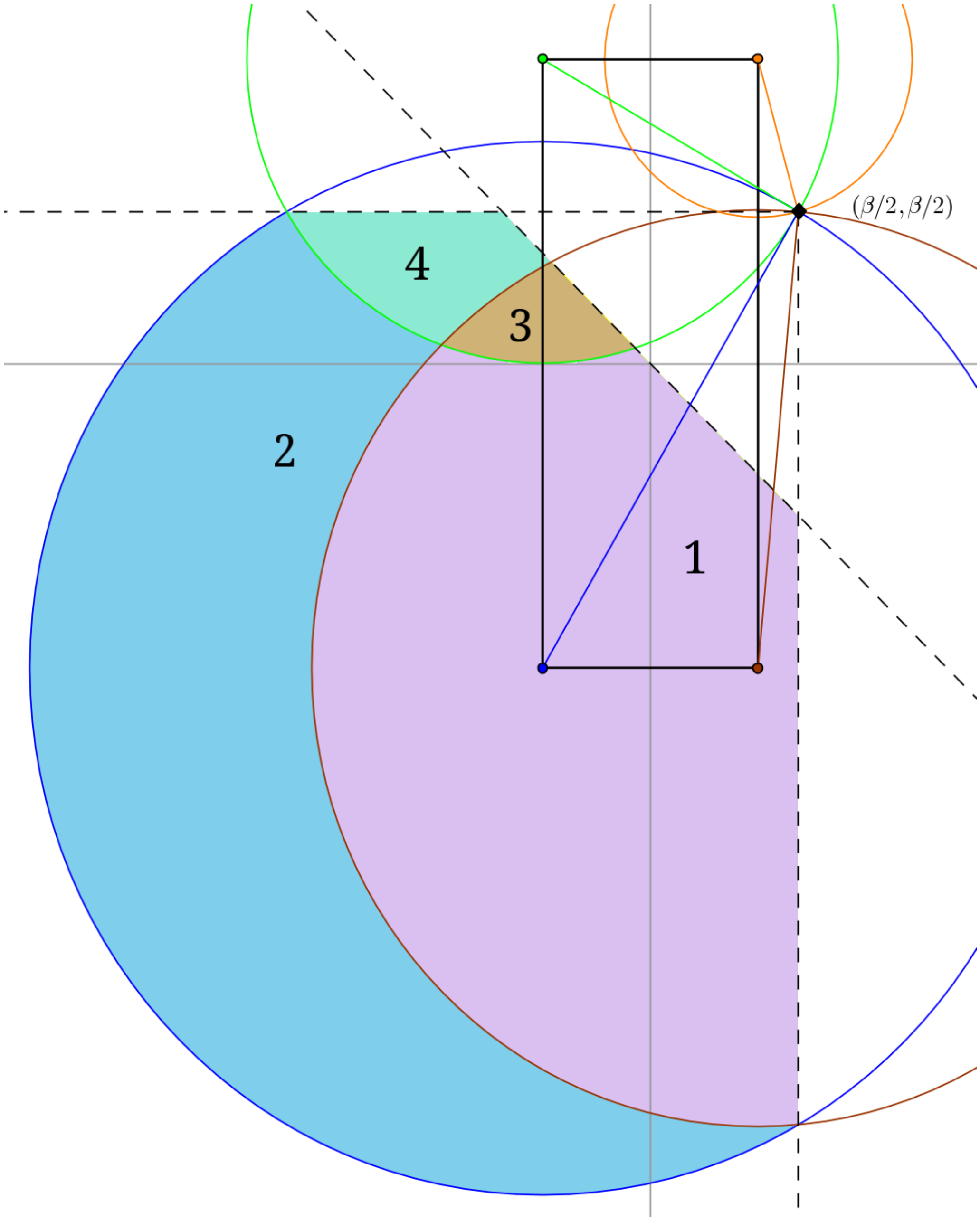

FIGURE 14. The four cases considered in Theorem 4.9

If **a** is an optimal strategy, it must be a local minimum for Player I so both first derivatives must be zero. Thus we can state:

$$x_1 = -\frac{1}{2}\left(\frac{\beta(\beta^2 - 2x_1^2 - 2y_1^2)}{(\beta^2 - 2\beta y_1)} - 2y_1\right)$$

After some algebraic manipulation we get

$$(y_1 - x_1)(2\beta - 4y_1) = \beta^2 - 2x_1^2 - 2y_1^2 \tag{96}$$

Setting $K_{x_1} = 0$ we arrive at

$$-3\beta^2 - 4\beta x + 6x_1^2 + 4\beta y_1 + 4x_1y_1 + 2y_1^2 = 0$$

$$\Leftrightarrow -3(\beta^2 - 2x^2 - 2y^2) + (y - x)(4\beta - 4y_1) = 0$$

Using the substitution from (96) we get

$$-3(y_1 - x_1)(2\beta - 4y_1) + (y_1 - x_1)(4\beta - 4y_1) = 0$$

$$\Leftrightarrow (y_1 - x_1)(-2\beta + 8y_1) = 0$$

So either $y_1 = x_1$ or $y_1 = \dfrac{\beta}{4}$. In the former case, we have already shown that the solution is $x_1 = y_1 = -\dfrac{\beta}{2}$. We can indeed check that $K_{x_1}(\mathbf{a}^*) = K_{y_1}(\mathbf{a}^*) = 0$ and $K_{x_1x_1}K_{y_1y_1} - K_{x_1y_1}^2 = \dfrac{3}{4\beta^2} > 0$, so this represents a local minimum. If instead we consider $y_1 = \dfrac{\beta}{4}$, $K_{x_1} = 0$ gives us the quadratic

$$6x_1^2 - 3\beta x_1 - \frac{15}{8}\beta^2 = 0$$

Which has solution $x_1 = \dfrac{1 \pm \sqrt{6}}{4}\beta$. We can ignore the positive solution since in that case $y_1 + x_1 > 0$. We can verify, however, that

$$K\left(\frac{1-\sqrt{6}}{4}\beta, \frac{\beta}{4}\right) = \frac{17-\sqrt{6}}{16}\beta > 0.$$

Before proceeding to Case 2, consider points between $R2$ and $R4$

Case 2/4: $\mathbf{a} \in \bigcirc_3 \cap \bullet_2 \backslash (\bullet_1 \cup \bullet_4)$

Let Player II choose point $\mathbf{b}^*$, and let Player I determine to play a strategy on this curve with probability $\dfrac{\lambda}{2}$ of being chosen by the judge, where $\lambda \in [0, 1)$. $Mid(\mathbf{a}, \mathbf{b}^*)$

will intersect points $(-1+2\lambda, -\beta)$ and $(-1, \beta)$, and thus will be defined by

$$y = -\frac{\beta}{\lambda}x + \frac{\lambda - 1}{\lambda}\beta.$$

The perpendicular line intersecting $\mathbf{b}^*$ will be

$$y = \frac{\lambda}{\beta}x + \frac{\beta - \lambda}{2}.$$

The two intersect at

$$\bar{x} = \frac{\beta\lambda(\lambda - \beta) + 2\beta^2(\lambda - 1)}{2\lambda^2 + 2\beta^2}$$

The strategy of Player I, therefore, will be

$$x(\lambda) = \frac{\beta\lambda(\lambda - \beta) + 2\beta^2(\lambda - 1)}{\lambda^2 + \beta^2} - \frac{\beta}{2}$$

with $y(\lambda) = \frac{\lambda}{\beta}x(\lambda) + \frac{\beta - \lambda}{2}$. The payoff will be given by

$$K(\lambda) = \beta + (x(\lambda) + y(\lambda) - \beta)\frac{\lambda}{2}$$

Or after some simplification

$$K(\lambda) = \frac{\beta\lambda^3 - 2\beta^2\lambda - \beta^3\lambda + 2\beta^3}{2\lambda^2 + 2\beta^2} = \frac{\beta(\beta - \lambda)\big(\beta(1 - \lambda) + (\beta - \lambda^2)\big)}{2\lambda^2 + 2\beta^2} > 0$$

since $\beta \geq 1 > \lambda \geq 0$.

Case 2: $\mathbf{a} \in$R2

In this case, because of the extreme points of Ξ only $(-1, -\beta) \in C_1$, $Mid(\mathbf{a}, \mathbf{b}^*)$ intersects the left and lower edges of Ξ. Fixing $m > 0$, consider Player I's strategy along the line

$$y = m\left(x - \frac{\beta}{2}\right) + \frac{\beta}{2}. \tag{97}$$

Case 2a: $m \geq \frac{1}{\beta}$

In this case, (97) will intersect $\bigcirc_1$ at $\mathbf{a}' = (x', y')$, where

$$x' = \frac{\beta m^2 - 6\beta m + 4 - \beta}{2(m^2+1)}$$

and $\bigcirc_2$ at $\mathbf{a}'' = (x'', y'')$, where

$$x'' = \frac{\beta m^2 - 6\beta m - 4 - \beta}{2(m^2+1)}.$$

Let $y' = y(x')$ and $y'' = y(x'')$. Note that $(1, -\beta), (-1, \frac{2}{m} - \beta) \in Mid(\mathbf{a}', \mathbf{b}^*)$ and $Mid(\mathbf{a}'', \mathbf{b}^*) \cap \Xi^\circ = \emptyset$. For any fixed $m \geq \frac{1}{\beta}$, Player I may consider any strategy $\gamma \in [0, 1)$, where $\mathbf{a}(\gamma) = (1-\gamma)(x'', y'') + \gamma(x', y')$, the payoff will be

$$K = \beta + \left(x'' + y'' + \gamma(x' + y' - x'' - y'') - \beta\right)\left(\gamma^2 \frac{1}{2m\beta}\right)$$

$$K_\gamma = (x' + y' - x'' - y'')\gamma^2 \frac{1}{2m\beta} + (x'' + y'' + \gamma(x' + y' - x'' - y'') - \beta)\left(\gamma \frac{1}{2m\beta}\right)$$

Which has critical points at $\gamma_1 = 0$ and

$$\gamma_2 = \frac{2(\beta - x'' - y'')}{3(x' + y' - x'' - y'')} = \frac{3\beta m + 2 + \beta}{6}.$$

Since $m \geq \dfrac{1}{\beta}$, $\gamma_2 \geq \dfrac{5+\beta}{6} \geq 1$, and from Case 1, we know that $K(1) > 0$. Thus for for all points $\mathbf{a}$ in this case, $K > 0$.

Case 2b: $0 < m < \frac{1}{\beta}$

In this case (97) will intersect $\bigcirc_3$ at $\mathbf{a}''' = (x''', y''')$, where

$$x''' = \frac{\beta m^2 + 2\beta m - 4 - \beta}{2(m^2+1)}.$$

Let $y''' = y(x''')$. Note that $(-1, \beta), (-1 + 2m\beta, -\beta) \in Mid(\mathbf{a}''', \mathbf{b}^*)$. Player I may consider any strategy $\gamma \in [0, 1)$, where $\mathbf{a}(\gamma) = (1-\gamma)(x'', y'') + \gamma(x''', y''')$, the payoff will be

$$K(\gamma) = \beta + \left(x'' + y'' + \gamma(x''' + y''' - x'' - y'') - \beta\right)\left(\gamma^2 \frac{m\beta}{2}\right)$$

$$K_\gamma = (x''' + y''' - x'' - y'')\gamma^2 \frac{m\beta}{2\alpha} + (x'' + y'' + \gamma(x''' + y''' - x'' - y'') - \beta)\left(\gamma \frac{m\beta}{2}\right)$$

Which has critical points at $\gamma_1 = 0$ and

$$\gamma_2 = \frac{2(\beta - x'' - y'')}{3(x''' + y''' - x'' - y'')} = \frac{3\beta m + 2 + \beta}{6\beta m}$$

Since $m < \dfrac{1}{\beta}$ and $1 \leq \beta$,

$$\gamma_2 = \frac{1}{2} + \frac{2+\beta}{6\beta m} > \frac{1}{2} + \frac{2+\beta}{6} = \frac{5+\beta}{6} \geq 1.$$

From Case 2/4 we know that $K(1) > 0$.

Before proceeding to Case 3, we look at strategies between R3 and R4

Case 3/4: $\mathbf{a} \in \bigcirc_3 \cap \bullet_2 \cap \bullet_1 \backslash \bullet_4$

Let Player II choose point $\mathbf{b}$, and let Player I determine to play a strategy on this curve with probability $\dfrac{2-\lambda}{2}$ of being chosen by the judge. Assuming $(-1, -\beta) \in C_1$, $Mid(\mathbf{a}, \mathbf{b}^*)$ will intersect points $(-1 + 2(1-\lambda), \beta)$ and $(1, -\beta)$, namely:

$$y = -\frac{\beta}{\lambda}x + \frac{(1-\lambda)\beta}{\lambda}.$$

The perpendicular line intersecting $\mathbf{b}$ will be

$$y = \frac{\lambda}{\beta}x + \frac{\beta - \lambda}{2}.$$

The two intersect at

$$\bar{x} = \frac{\beta\lambda(\lambda - \beta) + 2\beta^2(1-\lambda)}{2\lambda^2 + 2\beta^2}$$

The strategy of Player I, therefore, will be

$$x(\lambda) = \frac{\beta\lambda(\lambda-\beta)+2\beta^2(1-\lambda)}{\lambda^2+\beta^2} - \frac{\beta}{2}$$

with $y(\lambda) = \frac{\lambda}{\beta}x(\lambda) + \frac{\beta-\lambda}{2}$. $K(x, y)$ will thus be

$$K(\lambda) = \beta + (x(\lambda) + y(\lambda) - \beta)\frac{2-\lambda}{2}$$

Which can be expressed as

$$K(\lambda) = \beta\left(\frac{4(\beta+\lambda)(\lambda-1)^2 + (\beta-\lambda)^2 + \lambda^2(1-\lambda) + (3+\lambda)\beta^2}{2\lambda^2+2\beta^2}\right).$$

From this, it is clear that $K(\lambda) > 0$ for all $\lambda \in [0, 1]$.

Case 3: $\mathbf{a} \in$ R3

In this case, since $Mid(\mathbf{a}, \mathbf{b}^*)$ intersects the top and right sides of the rectangle, $P_1 \geq \frac{1}{2}$. Suppose Player I determines to be chosen with probability $1-p$, where $p \in [0, \frac{1}{2}]$, and wishes $Mid(\mathbf{a}, \mathbf{b}^*)$ to intersect the top-side of the rectangle at $(1-\bar{x}, \beta)$, where $\bar{x} \in [0, 2]$. In this case, $Mid(\mathbf{a}, \mathbf{b}^*)$ will intersect the right side of the rectangle at

$$\bar{y} = \beta - \frac{8\beta p}{\bar{x}^2}$$

and will have a slope of

$$\bar{m} = -\frac{8\beta p}{\bar{x}^2}.$$

The strategy which will attain this for Player I (should it be possible) will be at

$$x(\bar{x}) = \frac{\frac{\beta}{\bar{m}} + 2\bar{m}\bar{x} - 2\beta}{\bar{m} + \frac{1}{\bar{m}}} - \frac{\beta}{2}, y(\bar{x}) = -\frac{1}{\bar{m}}\left(\frac{\frac{\beta}{\bar{m}} + 2\bar{m}\bar{x} - 2\beta}{\bar{m} + \frac{1}{\bar{m}}} - \beta\right) + \frac{\beta}{2}$$

For a fixed p, this will determine a contour:

The payoff in this case will be

$$K(p, \bar{x}|\mathbf{b}^*) = \frac{(4\beta p + \beta)\bar{x}^4 - 32\beta p^2\bar{x}^3 - 16(\beta^2 - 2\beta)p^2\bar{x}^2 + 64\beta^3 p^2}{64\beta^2 p^2 + \bar{x}^4}$$

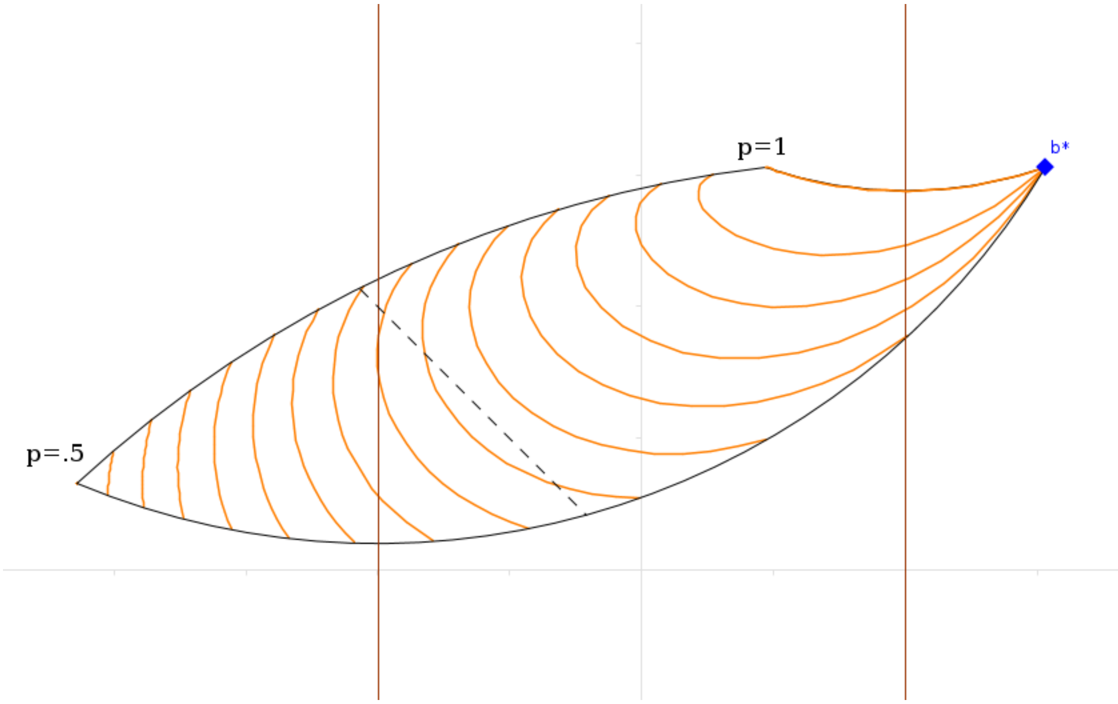

FIGURE 15. Parameterization of strategies in Case 3 of Theorem 4.9

If we let $\bar{p} = 4p$, and ignore the factor β, the numerator is

$$(\bar{p}+1)\bar{x}^4 + \bar{p}^2(-\bar{x}^2(2\bar{x}+\beta-2)+4\beta^2) \tag{98}$$

Since we know that on the border of the subregion restricted by $x+y<0$, $K(\mathbf{a},\mathbf{b}^*) > 0$, if $K(\mathbf{a},\mathbf{b}^*) \le 0$ in this sub-region then there must be a local minimum of this expression on the interior of the region (i.e. $\bar{x},\bar{p} \in (0,2), \beta \ge 1$); at this point, since the partial derivatives are all equal to zero the partial with respect to β,

$$\bar{p}^2(-\bar{x}^2 + 8\beta) = 0.$$

But clearly this is impossible. Thus $\forall \mathbf{a} \in$R3, $K(\mathbf{a},\mathbf{b}^*) > 0$.

Case 4: $\mathbf{a} \in$ R4

In this case, the midset will intersect the top and bottom sides of the rectangle; $(-1,\beta),(-1,-\beta) \in C_1$. So

$$P_1 = \frac{1}{2} + \frac{\beta^2 - 2x^2 - 2y^2}{4(\beta - 2x)}$$

So

$$K(\mathbf{a}, \mathbf{b}^*) = \beta + (x + y - \beta)\left(\frac{1}{2} + \frac{\beta^2 - 2x^2 - 2y^2}{4(\beta - 2x)}\right)$$

First derivative

$$K_x = \frac{1}{2} + \frac{\beta^2 - 2x^2 - 2y^2}{4(\beta - 2x)} - \frac{\beta - x - y}{2}\left(\frac{\beta^2 - 2x^2 - 2y^2}{(\beta - 2x)^2} - \frac{2x}{\beta - 2x}\right)$$

$$K_y = \frac{1}{2} + \frac{\beta^2 - 2x^2 - 2y^2}{4(\beta - 2x)} + \frac{(\beta - x - y)y}{\beta - 2x}$$

At an optimal strategy, a local minimum for Player I, both first derivatives must be zero, thus we can state:

$$y = \frac{1}{2}\left(2x - \frac{\beta^2 - 2x^2 - 2y^2}{\beta - 2x}\right)$$

After some algebraic manipulation

$$y = \frac{\beta}{2} - x \pm \frac{1}{2}\sqrt{3\beta^2 - 8\beta x + 8x^2} \tag{99}$$

If $y = \frac{\beta}{2} - x + \frac{1}{2}\sqrt{3\beta^2 - 8\beta x + 8x^2}$, then $x + y > 0$, which is not the case. But we shall show that the curve $y = \frac{\beta}{2} - x - \frac{1}{2}\sqrt{3\beta^2 - 8\beta x + 8x^2}$ only intersects $\bullet_3$ at $(-\frac{\beta}{2}, -\frac{\beta}{2})$ when $\beta = 1$.

Consider the line $y = \frac{1}{3}x - \frac{1}{3}$. The hyperbolic curve in question only intersects this line at $(-\frac{1}{2}, -\frac{1}{2})$ when $\beta = 1$. To see this, set equal

$$y = \frac{1}{3}x - \frac{1}{3} = \frac{\beta}{2} - x - \frac{1}{2}\sqrt{3\beta^2 - 8\beta x + 8x^2}$$

$$\Leftrightarrow -4x^2 + (12\beta - 16)x - 9\beta^2 + 6\beta + 2 = 0$$

The discriminant is $384(1 - \beta)$.

The circle, $\bigcirc_3$ may be expressed

$$x^2 + 2x + (y - \beta)^2 - \beta - \frac{1}{2}\beta^2 = 0$$

Substituting $y = \frac{1}{3}x - \frac{1}{3}$ we get

$$x^2 + 2x + (\frac{1}{3}x - \frac{1}{3} - \beta)^2 - \beta - \frac{1}{2}\beta^2 = 0$$

$$\Leftrightarrow 20x^2 + (32 - 12\beta)x + 9\beta^2 - 6\beta + 2 = 0$$

The discriminant is $-\frac{8}{9}(\beta - 1)(2\beta + 3)$, so we only have real solutions when $\beta \in [-\frac{3}{2}, 1]$, but since $\beta \geq 1$, we only have an intersection when $\beta = 1$.

We have thus shown that for all strategies $\mathbf{a} \in \mathbb{R}^2$, $K(\mathbf{a}, \mathbf{b}^*) \geq 0$, with equality only when $\mathbf{a} = \mathbf{a}^* = -\mathbf{b}^*$. By symmetry of the game, $\mathbf{a}^*, \mathbf{b}^*$ are the unique globally optimal pure strategies of Players I and II respectively. $\square$

CHAPTER 5

d-Issue Final-Offer Arbitration

We will now generalize some of the previous results to higher dimension. Now let us suppose Players I and II are to present final offers to the judge $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, respectively. The judge chooses a fair settlement vector $\boldsymbol{\xi}$ from an d-variate Normal distribution with mean μ and covariance matrix $\Sigma = (\sigma_{ij})_{d\times d}$. Also let us assume that players' valuation of a settlement vector is additive, as before. Thus the payoff is

$$K(\mathbf{x}, \mathbf{y}) = \begin{cases} \sum_{i=1}^{d} x_i & \text{if } R(\mathbf{x}, \boldsymbol{\xi}) > R(\mathbf{y}, \boldsymbol{\xi}) \\ \sum_{i=1}^{d} y_i & \text{if } R(\mathbf{x}, \boldsymbol{\xi}) < R(\mathbf{y}, \boldsymbol{\xi}) \\ \frac{1}{2}\sum_{i=1}^{d} x_i + y_i & \text{if } R(\mathbf{x}, \boldsymbol{\xi}) = R(\mathbf{y}, \boldsymbol{\xi}) \end{cases}$$

with the judge choosing either offer with equal likelihood in the case that both offers are equally reasonable. We shall consider only three reasonableness metrics in this chapter: Net Offer, L_1 Distance and L_2 Distance.

1. Net Offer

We will call this game variant $dNNO$ ($d-$Dimensional Normal, Net Offer). As with the bivariate case, we shall show that optimal solution points, should they exist, must lie on a particular $d-1$ dimensional hyperplane for each player.

Ignoring the case when both players are equally reasonable (since it occurs with probability zero) the payoff is

$$K(\mathbf{x}, \mathbf{y}) = \begin{cases} \sum_i x_i & \text{if } |\sum_i (x_i - \xi_i)| < |\sum_i (y_i - \xi_i)| \\ \sum_i y_i & \text{if } |\sum_i (x_i - \xi_i)| > |\sum_i (y_i - \xi_i)| \end{cases}.$$

Again if we let $\zeta = \sum_i \xi_i$, and $w_x = \sum_i x_i$, and $w_y = \sum_i y_i$ this simplifies to

$$K(w_x, w_y) = \begin{cases} w_x & \text{if } |w_x - \zeta| < |w_x - \zeta| \\ w_y & \text{if } |w_x - \zeta| > |w_y - \zeta| \end{cases}$$

with the reasonable assumption that $x_i < 0, y_i > 0$. The random variable ζ follows a normal distribution with mean 0 and variance $\sigma_\zeta^2 = \sum_i \sum_j \sigma_{ij}$. The optimal pure strategies in this univariate case are

$$w_x^* = -\sigma_\zeta \sqrt{\pi/2}, \quad w_y^* = \sigma_\zeta \sqrt{\pi/2}.$$

This however gives each player a set of optimal strategies which can be chosen independently. Namely, the set of optimal strategies for Player I is

$$X^* = \left\{ -\sigma_\zeta \sqrt{\pi/2}\alpha : \alpha \in \Delta_d \right\}$$

And the solution set for Player II is

$$Y^* = \left\{ \sigma_\zeta \sqrt{\pi/2}\beta : \beta \in \Delta_d \right\}$$

Where Δ_d is the d-dimensional simplex.

2. L1 Distance

If the judge instead uses the L_1 distance to measure reasonableness, the payoff will be

$$K(\mathbf{x}, \mathbf{y}) = \begin{cases} \sum_i x_i & \text{if } \sum_i |x_i - \xi_i| < \sum_i |y_i - \xi_i| \\ \sum_i y_i & \text{if } \sum_i |x_i - \xi_i| > \sum_i |y_i - \xi_i| \end{cases}.$$

We will call this game variant dNL_1 ($d-$Dimensional Normal, L_1 distance). The results from the 2-dimensional case generalize well to d dimensions. First a lemma:

LEMMA 5.1. *In dNL_1, both players' optimal pure strategies, if they exist, must lie on the line $x_1 = x_2 = \cdots = x_d$.*

PROOF. First observe that if the players have optimal pure strategies, then by the symmetry of the game, its value must be zero. Suppose Player II is playing optimally with strategy $\mathbf{y}^* = (y_1^*, \ldots, y_d^*)$ not on the line $y_1 = \cdots = y_d$. WLOG, assume $y_1^* \neq y_2^*$. Player I may of course respond by choosing strategy $\mathbf{x}^* = -\mathbf{y}^*$, giving an expected payoff of zero. By switching to strategy $\mathbf{x}'$ where $x_1' = x_2' = \frac{x_1^*+x_2^*}{2}$, Player I increases the probability that he receives the judge's ruling (by lemma 4.17), yet net demand remains unchanged, thus the new expected payoff is negative. Therefore, Player II could not have been playing optimally. The same is true for Player I by symmetry. $\square$

Given that any optimal pure strategies for either player must exist on the line $x_1 = \cdots = x_d$, we may apply a rotation to the distribution to align the vector $\mathbf{1}_d$ with e_1.

LEMMA 5.2. *The rotation matrix to align $\mathbf{1}_d$ to $\mathbf{e}_1$, where $\mathbf{e}_1 = (1, 0, \cdots, 0)$ is*

$$R_d = \begin{bmatrix} \sqrt{\frac{1}{d}} & \sqrt{\frac{1}{d}} & \sqrt{\frac{1}{d}} & \cdots & \sqrt{\frac{1}{d}} \\ -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 & \cdots & 0 \\ -\sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{6}} & \sqrt{\frac{2}{3}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\sqrt{\frac{1}{d(d-1)}} & -\sqrt{\frac{1}{d(d-1)}} & -\sqrt{\frac{1}{d(d-1)}} & \cdots & \sqrt{\frac{d-1}{d}} \end{bmatrix}$$

Specifically, in the first row each entry $(R_d)_{1j} = \sqrt{1/d}$. For $i = 2, \ldots, d$,

$$(R_d)_{ij} = \begin{cases} 0 & i > j \\ \sqrt{\frac{i-1}{i}} & i = j \\ -\sqrt{\frac{1}{i(i-1)}} & i < j \end{cases}$$

PROOF. That the matrix R_d is a rotation matrix is apparent from two facts, that $\det(R_d) = 1$ and $R_d R_d^T = I$.

Claim 1: $\det(R_d) = 1$

First note that $\det(R_1) = 1$. Let us assume that $\det(R_{d-1}) = 1$. Let $R_d(ij)$ be the matrix resulting by removing row i and column j.

$$\begin{aligned}\det(R_d) &= \sqrt{\frac{d-1}{d}}\det(R_d(dd)) + (-1)^{d-1}\sqrt{\frac{1}{d}}\det(R_d(1d)) \\ &= \sqrt{\frac{d-1}{d}}\sqrt{\frac{d-1}{d}}\det(R_{d-1}) + (-1)^{d-1}\sqrt{\frac{1}{d}}\det(R_d(1d)) \\ &= \frac{d-1}{d} + (-1)^{d-1}\sqrt{\frac{1}{d}}\det(R_d(1d)).\end{aligned}$$

Claim 1a: $\det(R_d(1d)) = (-1)^{d-1}\sqrt{\frac{1}{d}}$

$$\det(R_d(1d)) = \det\begin{bmatrix} -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 & \cdots & 0 \\ -\sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{6}} & \sqrt{\frac{2}{3}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\sqrt{\frac{1}{(d-1)(d-2)}} & -\sqrt{\frac{1}{(d-1)(d-2)}} & -\sqrt{\frac{1}{(d-1)(d-2)}} & \cdots & \sqrt{\frac{d-2}{d-1}} \\ -\sqrt{\frac{1}{d(d-1)}} & -\sqrt{\frac{1}{d(d-1)}} & -\sqrt{\frac{1}{d(d-1)}} & \cdots & -\sqrt{\frac{1}{d(d-1)}} \end{bmatrix}$$

$$= \sqrt{\frac{1}{2}}\sqrt{\frac{2}{3}}\sqrt{\frac{3}{4}}\cdots\sqrt{\frac{d-2}{d-1}}\sqrt{\frac{d-1}{d}}\det\begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{d-2} & -\frac{1}{d-2} & \frac{1}{d-2} & \cdots & 1 \\ -\frac{1}{d-1} & -\frac{1}{d-1} & -\frac{1}{d-1} & \cdots & -\frac{1}{d-1} \end{bmatrix}$$

$$= \sqrt{\frac{1}{d}}\det(R'_{d-1}).$$

Claim1b: $\det(R'_d) = (-1)^d$, where R'_d is defined as follows:

$$(R'_d)_{ij} = \begin{cases} 0 & j > i+1 \\ 1 & j = i+1 \\ -\frac{1}{i} & j \le i \end{cases}$$

Clearly $\det(R'_1) = (-1)^1$. Suppose $\det(R'_k) = (-1)^k$ for $k = 1, \ldots, d-1$. Note that

$$R'_d = \begin{bmatrix} R'_{d-1} & \mathbf{e}_{d-1} \\ -\frac{1}{d}\mathbf{1}^T & -\frac{1}{d} \end{bmatrix},$$

where $\mathbf{e}_{d-1} = (0, 0, \cdots, 0, 1)^T$.

$$\begin{aligned}
\det(R'_d) &= \frac{-1}{d}\det(R'_{d-1}) - \det\begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{d-2} & -\frac{1}{d-2} & \frac{1}{d-2} & \cdots & 1 \\ -\frac{1}{d} & -\frac{1}{d} & -\frac{1}{d} & \cdots & -\frac{1}{d} \end{bmatrix} \\
&= -\frac{1}{d}(-1)^{d-1} - \frac{d-1}{d}\det(R'_{d-1}) \\
&= -\frac{1}{d}(-1)^{d-1} - \frac{d-1}{d}(-1)^{d-1} \\
&= \left(-\frac{1}{d} - \frac{d-1}{d}\right)(-1)^{d-1} \\
&= (-1)(-1)^{d-1} \\
&= (-1)^d.
\end{aligned}$$

By Claim 1b and Claim 1a, Claim 1 is proven.

Claim 2: $R_d R_d^T = I$

$(R_d R_d^T)_{11} = \sum_{i=1}^{d} \frac{1}{d} = 1$.

For $i > 1$, $(R_d R_d^T)_{ii} = (i-1)\frac{1}{i(i-1)} + \frac{i-1}{i} = 1$.

For $j > 1$, $(R_d R_d^T)_{1j} = \sqrt{\frac{1}{d}}\left(\sum_{k=1}^{j-1} -\sqrt{\frac{1}{j(j-1)}} + \sqrt{\frac{j-1}{j}}\right) = \sqrt{\frac{1}{d}}\left(-\sqrt{\frac{j-1}{j}} + \sqrt{\frac{j-1}{j}}\right) = 0$.

For $i > 1$, $R_d R_d^T)_{i1} = 0$, as above.

Claim: the dot product of any two rows i and j,$i > 1, j > 1, i \neq j$, is zero.

WLOG assume let $i > j$. All entries in row i where nonzero corresponding entries exist in row j are identical, say α. The sum of the entries in row j is $(j-1)\sqrt{\frac{1}{j(j-1)}} + \sqrt{\frac{j-1}{j}} =$

0.

It suffices to verify that $R_d \mathbf{1}_d = \sqrt{d}\mathbf{e}_1$, and this is easily seen to be true, since the sum of each row $j > 1$ of R is zero. $\square$

After rotation, the variance along the x_1 axis is $\sigma'_{11} = R\Sigma R^T = \frac{1}{d}\sum_i \sum_j \sigma_{ij}$. The optimal strategies for Players I and II in the univariate case correspond, respectively, to

$$\tilde{\mathbf{x}}^* = -\sqrt{\frac{\pi \sum_i \sum_j \sigma_{ij}}{2d}}\mathbf{e}_1, \tilde{\mathbf{y}}^* = \sqrt{\frac{\pi \sum_i \sum_j \sigma_{ij}}{2d}}\mathbf{e}_1$$

Rotated back to the original axis, we have the following result:

THEOREM 5.1. *In dNL_1, if optimal pure strategies $\mathbf{x}^*, \mathbf{y}^*$ exist for Players I and II, they are given by*

$$\mathbf{x}^* = -\frac{\sqrt{\sum_i \sum_j \sigma_{ij}\pi/2}}{d}\mathbf{1}_d, \mathbf{y}^* = \frac{\sqrt{\sum_i \sum_j \sigma_{ij}\pi/2}}{d}\mathbf{1}_d.$$

3. L2 Distance

Now suppose the judge awards to whichever player is closer to his fair settlement opinion under Euclidean distance. They payoff is

$$K(\mathbf{x}, \mathbf{y}) = \begin{cases} \sum_i x_i & \text{if } \sum_i (x_i - \xi_i)^2 < \sum_i (y_i - \xi_i)^2 \\ \sum_i y_i & \text{if } \sum_i (x_i - \xi_i)^2 > \sum_i (y_i - \xi_i)^2 \end{cases}.$$

We will call this game variant dNL_2 ($d-$Dimensional Normal, L_2 distance).

LEMMA 5.3. *In dNL_2, if optimal pure strategies $\mathbf{x}^*, \mathbf{y}^*$ exist for Players I and II respectively, they both must lie on the line $x_1 = x_2 = \cdots = x_d$.*

PROOF. First observe that if the players have optimal pure strategies, then by the symmetry of the game, its value must be zero. Suppose Player II is playing optimally with strategy $\mathbf{y}^* = (y_1^*, \ldots, y_d^*)$ not on the line $y_1 = \cdots = y_d$. WLOG, assume $y_1^* \neq y_2^*$. Player I may of course respond by choosing strategy $\mathbf{x}^* = -\mathbf{y}^*$, giving an expected payoff of zero. Let $w^* = \sum_i x_i$. By deviating to any other strategy $\mathbf{x}'$ such that $\sum_i x_i' = -w^*$, Player I's net reward, should he be chosen by the judge, does not change. The point on this plane that is closest to the origin is $\mathbf{x}' = -\frac{w^*}{d}\mathbf{1}_d$, where $\mathbf{1}_d$ is the d-dimensional vector of all components 1. As this strategy minimizes the distance to the origin for Player I, the half-space of points closer to $\mathbf{x}'$ contains the origin, so the expected value of the game is negative. Thus Player II was not playing optimally. $\square$

By Lemma 5.2 we have the same conclusion as in dNL_1, namely

THEOREM 5.2. *In dNL_2, if optimal pure strategies $\mathbf{x}^*, \mathbf{y}^*$ exist for Players I and II, they are given by*

$$\mathbf{x}^* = -\frac{\sqrt{\sum_i \sum_j \sigma_{ij}\pi/2}}{d}\mathbf{1}_d, \mathbf{y}^* = \frac{\sqrt{\sum_i \sum_j \sigma_{ij}\pi/2}}{d}\mathbf{1}_d.$$

CHAPTER 6

Conclusion

We have developed a model of multi-issue final-offer arbitration as a zero-sum game where both players are risk-neutral, issues under dispute are quantitative and the values are additive. In the bivariate case where the judge's opinion is drawn from a normal distribution, the players' optimal pure strategies (should they exist) for any L_p metric with $p \in [1, \infty]$ are identical irrespective of p. For $p = 2$, if the two components are not too negatively correlated, these pure strategies are locally optimal. If we further assume that the issues are positively correlated, these represent the unique optimal strategy pair. Furthermore, it was observed that in this case whole-package FOA leads to an outcome with greater variance than IBI, and would act as a greater motivator to reach agreement in negotiations. The unique possible pure optimal strategies for a variant where the judge uses Mahalanobis distance to measure reasonableness of offers were derived. A game variant was studied where the judge draws an opinion from a bivariate uniform distribution, and the only possible pure optimal strategies were derived and proven to be globally optimal. Lastly, a game variant was studied where the final offers are arbitrarily large vectors.

Even among the game variants considered in this work, there is work that may be done to strengthen the results. It is our conjecture that the results of local and global optimality in the L_2 case apply also to the L_1, L_∞ and L_p cases, and a similar result could be shown for the Mahalanobis distance. It is also our conjecture that the restriction on ρ in Theorem 4.3 may be weakened to math that of Theorem 4.2. It is also of interest how the proof of global optimality in the L_2 case extend to the $d-$dimensional game.

This represents only an initial model of the multi-issue FOA game. As this is the first attempt (to the authors' knowledge) to formulate a detailed model of the higher-dimensional FOA game, the intention here is not to create an exhaustively general model. Instead we wish to delineate a tractable model from which we can glean some insights, highlight the many interesting ways in which it can be extended and generalized, as well as discuss the challenges in doing so. Many variants are worthy of consideration. It may be the case that the final-offer vectors must be standardized before making the ruling so that components of differing units may be compared. Certainly, the players' valuation of a settlement may be more complicated than the sum of the components. By weighting the components differently, the game immediately becomes non-zero sum. Under what conditions will pure optimal strategies exist? The level of analysis required for this variant is beyond that of this work, but it may be that some of the techniques used could be modified to tackle this case.

Finally, it is worth considering an extension of final-offer arbitration to n-player games, where based on the evidence provided by the parties, the judge draws a fair settlement from a Dirichlet distribution. This would have applications for inheritance splitting, for example, where the heirs cannot agree on a fair split and need to bring in an arbiter. To our knowledge, final-offer arbitration has not been used in an $n-$player scenario but we feel it would be an effective means to encourage agreement among the participants.

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