On Siegel Maass Wave Forms of Weight 0

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THESIS

Submitted in partial fulfillment of the requirements

for the degree of Doctor of Philosophy in Mathematics

in the Graduate College of the

University of Illinois at Chicago, 2013

Chicago, Illinois

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ACKNOWLEDGMENTS

I am sincerely grateful to my advisor, Ramin Takloo-Bighash, for his guidance and support, without which this thesis would not have been possible. I would like to thank my committee members, Steven Hurder, Simon Marshall, David Radford, and Brooke Shipley, and the Mathematics Department at the University of Illinois at Chicago for providing a stimulating mathematical community. I owe gratitude to Professor Dr. Jr̈ug Kramer for his guidance and advice, and Humboldt Universität zu Berlin for providing excellent resources during my visit in the summer of 2011. I am also thankful to Professor Olav Richter, for his helpful insights into this project. Finally, I want to thank my husband, Matthew, for his unending patience and interest in my work, and my son, Wesley, who has been a joyful support, even when I read him research papers in place of stories. The writing of this thesis was facilitated in part by the University of Illinois at Chicago Provost's Award for graduate research and by National Security Agency Standard Grant 2010 – 2012 (Award number 081031), PI Ramin Takloo-Bighash.

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SUMMARY

There has been tremendous growth in the study of non-holomorphic automorphic forms in recent decades. Progress has been made toward a Saito-Kurokawa lift, including a non-holomorphic Shimura lift and a lift from the non-holomorphic analogue of the Kohnen plus space to Jacobi Maass forms. A large gap remains in our understanding of Siegel Maass forms, which are the non-holomorphic analogue of Siegel modular forms. Relatively few results are known with a high degree of generality, and even basic results have not been developed in some cases.

In the case of Siegel Maass wave forms of weight 0, Niwa, in 1991, utilized explicit differential operators given by Nakajima (1982) to develop the Fourier series expansion. However, Nakajima's quartic differential operator is not invariant under the action of the desired slash operator, and so we still lack a valid Fourier expansion for Siegel Maass wave forms of weight 0.

In this thesis, we introduce Siegel Maass wave forms of weight 0, which are simultaneous eigenvectors of Maass' Casimir operators, rather than the operators given by Nakajima, and follow the method of Niwa to obtain a fourth order ordinary differential equation, which must be satisfied by the Fourier coefficients of such wave forms.

In Chapter 2, we review the theory of holomorphic Siegel modular forms and the classical Saito-Kurokawa lift. In Section 3.1, we define Siegel Maass wave forms of weight 0, and in

SUMMARY (Continued)

Section 3.2, we describe non-holomorphic automorphic forms involved in a Saito-Kurokawa lift, as well as the maps between them which have previously been established.

In Section 4.1, we explicitly compute the Casimir operators which form the basis for our definition of Siegel Maass wave forms, followed by the computation of the system of differential equations satisfied by these forms, in Section 4.2. In Section 4.3, through a series of changes of variable, we reduce this system of differential equations to a single fourth order ordinary linear differential equation. The Fourier coefficient of a wave form, corresponding to the identity matrix, will satisfy this differential equation. We discuss the proof given by Niwa for the solutions to his ordinary differential equation and his method for obtaining the first solution by theta lifting, in Section 4.4, and finally we conclude in Section 4.5 by giving the Fourier coefficients corresponding to definite matrices, according to Hori.

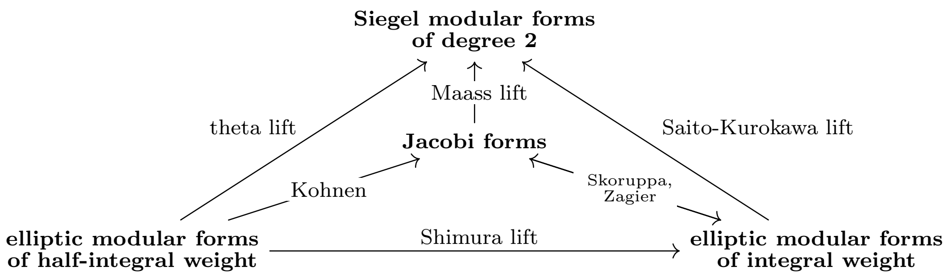
CHAPTER 1

INTRODUCTION

1.1 History and motivation

Since the mid-1900s, Siegel modular forms have become a vibrant area of inquiry. A Siegel modular form of degree n is a higher dimensional analogue of an elliptic modular form. It is a complex-valued function on the Siegel upper half space $\mathbb{H}_n$ of degree n, which is holomorphic, transforms in a certain way under the action of the symplectic group, and is bounded on Siegel's fundamental domain. For $n > 1$, this last condition is fulfilled automatically by any function satisfying the first two. Many properties of holomorphic Siegel modular forms are well known; see, for example [16]. Various "lifting" theorems have been established, which provide great insight into the relationships between Siegel modular forms and other number theoretic objects.

One such lifting theorem is the Saito-Kurokawa lift. Proved in 1979, by Maass [19–21], Andrianov [2], and Zagier [30], this established, in the degree two case, a lift from elliptic modular forms of weight $2k-2$ and level 1 to a subspace of holomorphic Siegel modular forms of weight k and level 1, called the Spezialschar, which is characterized by certain identities among the Fourier coefficients. The lifting is the composition of three maps. The first is from the Maass Spezialschar of Siegel modular forms on the full modular group, to Jacobi forms of weight k and index 1, and utilizes the Fourier-Jacobi expansion of a Siegel modular form. The second maps Jacobi forms to Kohnen's plus subspace of half-integral weight modular forms on $\Gamma_0(4)$, and the third is the Shimura isomorphism between modular forms of integral and half-integral weight. The diagram below (Skoruppa [29]) illustrates the relationships between these spaces.



In contrast, much less is known about Siegel Maass forms, which are the non-holomorphic analogue of Siegel modular forms. The holomorphicity of Siegel modular forms is replaced by the requirement that a Siegel Maass form be a simultaneous eigenfunction of certain differential operators; following Borel [3], the Casimir operators are a natural choice, since they are the images under group actions of the canonical elements of the center of the universal enveloping algebra. This makes them in general more difficult to work with (for example, even the Fourier coefficients involve much more complicated functions than holomorphic modular forms), and it is only very recent developments in areas like non-holomorphic Jacobi forms (Pitale [25], Bringmann and Richter [7], Bringmann, Raum, and Richter [6], etc.) that have made Siegel Maass forms more accessible for study.

It would be very useful to have a non-holomorphic version of the Saito-Kurokawa lift, and indeed, partial results have already been obtained. In [15], Katok and Sarnak establish a non-holomorphic Shimura correspondence, between Maass forms of weight $k - 1/2$ with respect to

$\Gamma_0(4)$ and Maass-Hecke eigenforms. Furthermore, Pitale [25] broadens previous attempts at a theory of non-holomorphic Jacobi Maass forms, and establishes a correspondence between his Jacobi Maass forms and the Kohnen plus space of Maass forms. Bringmann and Richter [7] have also improved on Pitale's definition of Maass-Jacobi forms, and Bringmann, Raum, and Richter [5] further improve it to include those with singularities. This leaves the lift from Jacobi Maass forms to Siegel Maass forms an open question.

1.2 Planned scope of thesis

It seems unlikely that the Saito-Kurokawa lift can be completed in the classical method as in Eichler and Zagier [10], whereby a lift is constructed using the Fourier-Jacobi expansion on the Siegel modular form: In the Fourier-Jacobi expansion of a Siegel Maass wave form, $F(Z) = \sum_{m\in\mathbb{Z}} \phi_m(z_1, z_2, y_3)e^{2\pi i m x_3}$, the coefficients ϕ_m generally depend on y_3 and are thus not Jacobi forms. Regardless, other approaches to obtaining a non-holomorphic Saito-Kurokawa lift will utilize a Fourier expansion for Siegel Maass forms.

Niwa [23] had established such an expansion. However, in [6], the authors remark that one of the differential operators on which Niwa's result rests, is not actually invariant under the action of the symplectic group. This differential operator was given explicitly by Nakajima [22]. As a result, a correct proof of the Fourier expansion is lacking.

In this thesis, we give a correct expansion of the differential operator given in Nakajima and follow Niwa to obtain an ordinary differential equation which is satisfied by the Fourier coefficients of a Siegel Maass wave form.

Niwa derived a Fourier expansion of generalized Whittaker functions which are simultaneous eigenfunctions of the differential operators given in Nakajima [22]. In particular, he gives the generators of the algebra of $\mathrm{Sp}(2,\mathbb{R})$-invariant differential operators on $\mathbb{H}_2$. This algebra is isomorphic to $\mathbb{C}[\Delta_1,\Delta_2]$, the commutative polynomial ring of two variables over $\mathbb{C}$. The quadratic operator, Δ_1, is the Laplacian for the invariant metric on $\mathbb{H}_2$, and Nakajima explicitly computed a choice for the quartic operator, Δ_2. It is this quartic operator which is shown in [6] to be not invariant. The quadratic operator we consider, H_1 is the same, up to a factor of -4, as Nakajima's Δ_1, and we use the quartic operator H_2 computed in Maass [18]. Then we define Siegel Maass wave forms of weight 0 to be smooth functions

$$f : \mathbb{H}_2 \to \mathbb{C},$$

which are $\mathrm{Sp}(2,\mathbb{Z})$-invariant, are common eigenfunctions of H_1 and H_2, and satisfy a growth condition.

1.3 Avenues of future research

Many questions in the study of Siegel Maass forms, including the Saito-Kurokawa lift, require a Fourier expansion. Kohnen's limit process, adapted in [6] to real-analytic Siegel Maass forms, may be further adaptable to the Siegel Maass forms above. Another approach would be to adapt the methods of Duke and Imamoḡlu [9], in using Imai's converse theorem in [14] to give another proof of the standard Saito-Kurokawa lift.

Further areas of consideration which may be more accessible with a Fourier expansion, are extending the Ikeda and Duke-Imamoḡlu liftings to Siegel Maass forms of arbitrary even degree, characterizations of the Fourier coefficients, non-holomorphic converse theorems in the vein of Imai, L-functions, considering all of the above with respect to congruence subgroups, rather than the full modular group, etc.

CHAPTER 2

THE HOLOMORPHIC THEORY

2.1 Siegel modular forms

What we now call Siegel modular forms were developed by C.L. Siegel in the 1930s, as a higher degree generalization of elliptic modular forms on $\mathrm{SL}(2,\mathbb{Z})$. This was motivated by Siegel's investigations of the Minkowski-Hasse principle for quadratic forms over $\mathbb{Q}$. Siegel modular forms represent one of the most important kinds of automorphic forms in several complex variables. Various "lifting" theorems have been proven, one of the most important of which is the Saiko-Kurokawa lift. We summarize the basic properties of holomorphic Siegel modular forms and a proof of this lift. The reader is referred to van der Geer's "Siegel Modular Forms and their Applications," in [8], Freitag [11], and Klingen [16] for a more comprehensive treatment of these topics.

2.1.1 Basic properties

The symplectic group, $\mathrm{Sp}(g,\mathbb{R})$, generalizes $\mathrm{SL}(2,\mathbb{R})$, and is defined as

$$\mathrm{Sp}(g,\mathbb{R}) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}(2g,\mathbb{R}) \,|\, j[M] = j \right\}$$

where $j = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, I denotes the $g \times g$ identity matrix, $a[b] := {}^t bab$, and ${}^t b$ denotes the transpose of b. This is equivalent to requiring that the $g \times g$ matrices A, B, C, D satisfy $A^t B = B^t A$,

$C^tD = D^tC$, and $A^tD - B^tC = 1_g$. In general, for R a Euclidean ring, $\mathrm{Sp}(g, R)$ is generated by the matrices j, $\left(\begin{smallmatrix} E & S \\ 0 & E \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} {}^tU & 0 \\ 0 & U^{-1} \end{smallmatrix}\right)$, where $S \in M_g(R)$ is symmetric, and $U \in GL_n(R)$. The subgroup $\Gamma_g := \mathrm{Sp}(g, \mathbb{Z})$ is a discrete subgroup of $\mathrm{Sp}(g, \mathbb{R})$, and is called the Siegel modular group, analogous to the modular group $\mathrm{SL}(g, \mathbb{Z}) \subset \mathrm{SL}(g, \mathbb{R})$.

The Siegel upper half space of degree g consists of all $g \times g$ complex symmetric matrices whose imaginary part is positive definite:

$$\mathbb{H}_g := \left\{ z = x + iy \in \mathrm{M}(g, \mathbb{C}) \,|\, {}^tz = z,\, y > 0 \right\}.$$

$\mathrm{Sp}(g, \mathbb{Z})$ acts on $\mathbb{H}_g$ by $M\langle z \rangle := (az + b)(cz + d)^{-1}$ where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}(g, \mathbb{Z})$ and $z \in \mathbb{H}_g$. Indeed, $\mathrm{Sp}(g, \mathbb{R})/\{\pm 1\}$ is the group of biholomorphic automorphisms of $\mathbb{H}_g$, acting faithfully and transitively, with stabilizer of iI the unitary group $U(g)$. We further define the congruence subgroups

$$\Gamma_g(n) = \{\gamma \in \mathrm{Sp}(g, \mathbb{Z}) \,|\, \gamma \equiv 1_{2g} \pmod{n}\}.$$

Finally, we come to the definition of scalar-valued (or, classical) Siegel modular forms.

Definition 1. *A scalar-valued Siegel modular form of weight k and degree (or genus) g is a holomorphic function $f : \mathbb{H}_g \to \mathbb{C}$ such that*

$$f(\gamma\langle z \rangle) = \det(cz + d)^k f(z)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_g$. When $g = 1$, we require also that f is holomorphic at ∞.

Modular forms of weight k and degree g form a finite-dimensional vector space, which we denote by $M_k(\Gamma_g)$. There is, of course, a more general notion of scalar-valued modular forms, but we shall omit it here, and in the following, "Siegel modular form" shall always mean a scalar-valued Siegel modular form. Similarly, it is also possible to define Siegel modular forms for congruence subgroups, but we shall generally only deal with the full Siegel modular group.

A Siegel modular form f has a Fourier expansion

$$f(z) = \sum_n a(n) e^{2\pi i \mathrm{tr}(nz)}$$

with $a(n) \in \mathbb{C}$ and n ranging over half-integral symmetric symmetric $g \times g$ matrices. Here, n half-integral means that the diagonal entries n_k are integers, and twice the off-diagonal entries are integers, that is, $2n_{kl} \in \mathbb{Z}$ $(k \neq l)$. We will, at times, use the notation $q^n = e^{2\pi i \mathrm{tr}(nz)}$, in analogy with the classical case. The coefficients are given by

$$a(n) = \int_{x \mod 1} f(z) e^{-2\pi i \mathrm{tr}(nz)} \mathrm{d}x$$

where $\mathrm{d}x = \prod_{k \leq l} \mathrm{d}x_{kl}$ is the Euclidean volume element in the x-space, and the integral runs over $-1/2 \leq x_{ij} \leq 1/2$. It is well-known that Siegel modular forms of odd or negative weight vanish. Let $f = \sum_n a(n) e^{2\pi i \mathrm{tr}(nz)} \in M_k(\Gamma_g)$. Then $a(n) = 0$ for n not positive semi-definite. This leads to the well-known Koecher principle:

Theorem 1. *$f \in M_k(\Gamma_g)$ is bounded on any subset of $\mathbb{H}_g$ of the form*

$$\{z \in \mathbb{H}_g \mid Im(z) = y \geq cI\} \quad (c > 0).$$

As in the case of classical modular forms, we have a notion of cusp form.

Definition 2. *The Siegel operator Φ on Siegel modular forms of degree g is defined by*

$$\Phi f = \lim_{t \to \infty} f \begin{pmatrix} z' & 0 \\ 0 & it \end{pmatrix}$$

where $z' \in \mathbb{H}_{g-1}$ and $t \in \mathbb{R}$.

This limit is defined because of the Koecher principle. The Siegel operator defines a linear map $M_k(\Gamma_g) \to M_k(\Gamma_{g-1})$, where $M_k(\Gamma_0) = \mathbb{C}$ by convention. The kernel of this linear map forms the subspace of **cusp forms**:

Definition 3. *$f \in M_k(\Gamma_g)$ is called a cusp form if $\Phi f = 0$.*

We denote the space of cusp forms by $S_k = S_k(\Gamma_g)$. Cusp forms can be characterized by their Fourier expansions.

Proposition 1. *$f \in M_k(\Gamma_g)$ $(g \geq 1)$ is a cusp form if and only if*

$$f(z) = \sum_{n>0} a(n) e^{2\pi i tr(nz)}$$

where n ranges over half-integral positive definite $g \times g$ matrices. That is, $a(n) = 0$ for all n that are semi-definite, but not definite.

When $g = 2$, we can write $Z \in \mathbb{H}_2$ as $Z = \left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right)$ where $\tau, \tau' \in \mathbb{H}$ and $z \in \mathbb{C}$ satisfy $\mathrm{Im}(z)^2 < \mathrm{Im}(\tau)\mathrm{Im}(\tau')$. We then write $f(\tau, z, \tau')$. Similarly, N positive semi-definite and half-integral can be written $\left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right)$ with $n, r, m \in \mathbb{Z}$, $n, m \geq 0$, and $r^2 \leq 4nm$, and we put $a(n, r, m)$ in place of the Fourier coefficient $a(N)$. Then the Fourier expansion of f is

$$f(Z) = \sum_{\substack{n,r,m \in \mathbb{Z} \\ n,m,4nm-r^2 \geq 0}} a(n, r, m) e^{2\pi i(n\tau + rz + m\tau')}.$$

The terms can then be rearranged to obtain the **Fourier-Jacobi expansion** of the Siegel modular form:

$$f(\tau, z, \tau') = \sum_{m=0}^{\infty} \phi_m(\tau, z) e^{2\pi i m\tau'}.$$

This expansion will play a pivotal role in the proof of the Saito-Kurokawa lift, as will be seen below.

2.1.2 Hecke operators and L-functions

In the case of elliptic modular forms, Hecke operators provide arithmetic information from the Fourier coefficients of modular forms. In particular, for $f = \sum_n a(n)q^n$ a normalized common eigenform of the Hecke operators, then the eigenvalue $\lambda(p)$ of f under the Hecke operator $T(p)$ is equal to the Fourier coefficient $a(p)$. Here, $q^n = e^{2\pi i n z}$, as is standard, and normalized means that the first Fourier coefficient, $a(1)$, is equal to 1.

Hecke operators play a similarly important role in the theory of Siegel modular forms. Let $G = \mathrm{GSp}(2g, \mathbb{Q}) := \{M \in \mathrm{GL}(2g, \mathbb{Q}) \,|\, j[M] = \nu(M)j,\ \nu(M) \in \mathbb{Q}^*\}$ be the group of rational symplectic similitudes, and let $G^+ := \{\gamma \in G \,|\, \nu(M) > 0\}$. Then G^+ is a semi-group, and $\Gamma_g \in G^+$. Let $L(\Gamma_g, G)$ be the free $\mathbb{C}$-module generated by the right cosets $\Gamma_g x$ for $x \in \Gamma_g \backslash G^+$. Let $H(\Gamma, G) = L(\Gamma_g, G)^{\Gamma_g}$ be the subspace of $L(\Gamma_g, G)$ invariant under right multiplication by Γ_g. $H(\Gamma_g, G)$ is endowed with the structure of an algebra, by defining the following product: For $T_1 = \sum_{x \in \Gamma_g \backslash G} a_x \Gamma_g x$ and $T_2 = \sum_{\in \Gamma_g \backslash G} b_y \Gamma_g y$, define

$$T_1 \cdot T_2 := \sum_{x,y \in \Gamma_g \backslash G} a_x b_y \Gamma_g xy.$$

Since $H(\Gamma_g, G)$ is generated by double cosets $\Gamma_g x \Gamma_g$, with $x \in G^+$, we have that $T_1 \cdot T_2 \in H(\Gamma_g, G)$. With this multiplication, $H(\Gamma_g, G)$ is a commutative associative algebra with unity, called the Hecke algebra. The Hecke algebra can be written as the product of local Hecke algebras: $H(\Gamma_g, G) = \otimes_p H_p = \otimes_p H(\Gamma_g, G \cap \mathrm{GL}(2g, \mathbb{Z}[p^{-1}]))$. The local Hecke algebra H_p is generated by the $g+1$ double cosets

$$T(p) = \Gamma_g \begin{pmatrix} 1_g & 0 \\ 0 & p1_g \end{pmatrix} \Gamma_g$$

and

$$T_i(p^2) = \Gamma_g \begin{pmatrix} 1_{g-i} & & & \\ & p1_i & & \\ & & p^2 1_{g-1} & \\ & & & p1_i \end{pmatrix} \Gamma_g,$$

where $0 \leq i < g$. We can also define Hecke operators $T(m)$ for $m \in \mathbb{N}$ by $T(m) := \sum x \in \Gamma_g \backslash O_m \Gamma_g x$, where $O_m := \{x \in \mathrm{GL}(2n, \mathbb{Z}) \,|\, j[x] = mj\}$. For $m = p$ prime, $T(m)$ coincides with the $T(p)$ introduced above, and for $m = p^2$, $T(m)$ is a sum $\sum_{i=0}^{g} T_i(p^2)$. It is further known that

$$\mathrm{Hom}(H_p, \mathbb{C}) \equiv (\mathbb{C}^*)^{g+1}/W_G, \tag{2.1}$$

where W_G is the Weyl group of G.

For $f \in M_k(\Gamma_p)$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G^+$, set

$$f|_{\gamma,k}(z) = r_\gamma^{gn-g(g+1)/2} \det(cz+d)^{-k} f(\gamma\langle z\rangle)$$

where $r_\gamma^{gn-g(g+1)/2}$ is a scalar factor of γ included for cohomological considerations. Now, define

$$Tf := \sum_i f|_{\gamma_i,k}.$$

This gives a linear operator on $M_k(\Gamma_g)$. Thus the Hecke algebra operates on Siegel modular forms. Moreover, the Hecke operators are Hermitian with respect to the Petersson inner product

$$\langle f, h\rangle = \int_{F_g} f(z)\overline{h(z)}(\det y)^k \frac{dydy}{(\det y)^{g+1}}$$

where F_g is a fundamental domain, $f \in M_k(\Gamma_g)$ and $h \in S_k(\Gamma_g)$. Hence, $S_k(\Gamma_g)$ has a basis of common eigenfunctions of all $T \in H(\Gamma_g, G)$. Now, if f is such an eigenfunction with eigenvalues $\lambda(T)$ for $T \in H(\Gamma_g, G)$, then the map $H_p \to \mathbb{C}$ defined by $T \to \lambda(T)$ is a homomorphism for each p, and thus, by the isomorphism Equation 2.1, is determined by $(\alpha_0, \alpha_1, \ldots, \alpha_g) \in (\mathbb{C}^*)^{g+1}/W_G$. These non-zero complex numbers, α_i, are called the **Satake parameters** of f.

Recall that, in the case of $g = 1$, for an eigenform $f = \sum_n a(n)q^n \in M_k(\Gamma_1)$ of the Hecke algebra, there is an associated Dirichlet series $\sum_{n\geq 1} a(n)n^{-s}$ for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > k/2 + 1$. When $f \in S_k(\Gamma_1)$, we have holomorphic continuation of the L-function to the whole s-plane, and it satisfies a functional equation. It has an Euler product

$$\sum_{n\geq 1} a(n)n^{-s} = \prod_p (1 - a(p)p^{-s} + p^{k-1-2s})^{-1}.$$

Similarly, for $f \in S_k(\Gamma_g)$ a common eigenform of the Hecke operators, we define the following two L-functions. The **standard zeta function** is

$$D_f(s) = \prod_p D_{f,p}(p^{-s})^{-1}$$

with local Euler factor at p

$$D_{f,p}(t) = (1-t)\prod_{i=1}^{g}(1-\alpha_i t)(1-\alpha_i^{-1}t).$$

The **spinor zeta function** is defined to be

$$Z_f(s) = \prod_p Z_{f,p}(p^{-s})^{-1}$$

with local Euler factor at p

$$Z_{f,p}(t) = (1-\alpha_0 t)\prod_{r=1}^{g}\prod_{1\le i_1<\cdots<i_r\le g}(1-\alpha_0\alpha_{i_1}\cdots\alpha_{i_r}t).$$

The standard zeta function has a meromorphic continuation to $\mathbb{C}$ and a functional equation with respect to $s \mapsto 1-s$ for all g. For $g=2$, Andrianov [1] proved that

$$\Phi_f(s) := \Gamma(s)\Gamma(s-k+2)(2\pi)^{-2s}Z_f(s)$$

is meromorphic with finitely many poles and satisfies the functional equation

$$\Phi(f, 2k-2-s) = (-1)^k\Phi_f(s).$$

2.2 The Saito-Kurokawa lift

Proved primarily by Maass [19–21], and completed by Andrianov [2] and Zagier [30], the Saito-Kurokawa lift establishes a correspondence between classical modular forms and Siegel modular forms of degree 2.

We must first define $M_k^*(\Gamma_2)$, the **Maass subspace** of Siegel modular forms. The Maass subspace consists of forms $F(Z) = \sum_{N\geq 0} a(N)e^{2\pi i \mathrm{tr}(NZ)}$ whose Fourier coefficients $a(N)$ depend only on the discriminant $d = 4mn - r^2$ and the greatest common divisor $\gcd(n, r, m)$ ($N = \left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right)$). Writing $a([n, r, m])$ for $a(N)$, the condition that $F \in M_k^*(\Gamma_2)$ can be stated as

$$a([n, r, m]) = \sum_{d>0, d|(n,r,m)} d^{k-1} a([1, \frac{r}{d}, \frac{mn}{d^2}]).$$

We denote by $S_k^*(\Gamma_2)$ the space of cusp forms in $M_k^*(\Gamma_2)$.

Theorem 2 (Saito-Kurokawa lift). *The Maass subspace $S_k^*(\Gamma_2)$ is spanned by Hecke eigenforms, which are in $1-1$ correspondence with normalized Hecke eigenforms $f \in S_{2k-2}(\Gamma_1)$, with*

$$Z_F(s) = \zeta(s-k+1)\zeta(s-k+2)L(f, s) \tag{2.2}$$

where both L-functions are the spinor L-functions.

The correspondence is the composition of three maps, which we will briefly sketch here.

The first is between Siegel modular forms in the Maass subspace, and Jacobi forms of the same weight and index 1. A Jacobi form on $\Gamma_1 = \mathrm{SL}(2, \mathbb{Z})$ is a holomorphic function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ satisfying the transformation equations

$$\phi(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}) = (c\tau + d)^k e^{\frac{2\pi i m c z}{c\tau+d}} \phi(\tau, z)$$

and

$$\phi(\tau, z + \lambda\tau + \mu) = e^{-2\pi i m(\lambda^2\tau + 2\lambda z)}\phi(\tau, z),$$

and having a Fourier expansion

$$\phi(\tau, z) = \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathbb{Z} \\ r^2 \leq 4nm}} c(n, r) e^{2\pi i(n\tau + rz)}$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1, (\lambda\ \mu) \in \mathbb{Z}^2$. The natural numbers k and m are called the weight and index of the Jacobi form, respectively. We will sometimes use the notation $q = e^{2\pi i \tau}$ and $\zeta = e^{2\pi i z}$, so that the Fourier expansion is written $\sum c(n, r) q^n \zeta^r$. We denote by $J_{k,m}$ the space of Jacobi forms of degree k and index m. A cusp form is a Jacobi form for which $c(n, r) = 0$ if $r^2 = 4nm$.

There is an operator V_l acting on Jacobi forms of weight k and index m to Jacobi forms of the same weight and index kl. It is given by

$$(\phi|_{k,m} V_l)(\tau, z) = l^{k-1} \sum_{\substack{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1 \backslash M_2(\mathbb{Z}) \\ ad - bc = l}} (c\tau + d)^{-k} e^{2\pi i m l \frac{-cz^2}{c\tau + d}} \phi(\tfrac{a\tau + b}{c\tau + d}, \tfrac{lz}{c\tau + d}).$$

The action on Fourier coefficients is

$$\phi|V_l = \sum_{n,r}\left(\sum_{a|(n,r,l)} a^{k-1} c\left(\frac{nl}{a^2}, \frac{r}{a}\right)\right) q^n \zeta^r.$$

The coefficients, $\phi_m(\tau, z)$, in the Fourier-Jacobi expansion of a Siegel modular form of weight k and degree 2 are themselves Jacobi forms of weight k and index m, as was shown by Piatetski-Shapiro [24]. This gives an injective map $\mathcal{H} : M_k(\Gamma_2) \to \prod_{m\geq 0} J_{k,m}$. We have a map in the other direction, $\mathcal{V} : J_{k,1} \to M_k(\Gamma_2)$ due to Maass [19]: For a Jacobi form of weight k and index m, the functions $\phi|V_m$ for $m \geq 0$ are the Fourier-Jacobi coefficients of a Siegel modular form $\mathcal{V}\phi$ of weight k and degree 2. The map $\mathcal{V}$ is itself injective with image the set of $F \in M_k(\Gamma_2)$ satisfying $F = \mathcal{V}(\mathcal{H}(F))$. Maass [19] shows that this is precisely the Maass subspace. Hence, $\mathcal{H}$ and $\mathcal{V}$ provide inverse isomorphisms $M_k^*(\Gamma_2) \cong J_{k,1}$.

The second map takes a subspace of half-integral weight modular forms to Jacobi forms of index 1. The Kohnen plus space of half-integral weight modular forms is defined by

$$M^+_{k-1/2}(4N) = \left\{ h \in M_{k-1/2}(\Gamma_0(4N)) \,|\, h = \sum_{\substack{m=0 \\ (-1)^{k-1}m \equiv 0,1 \pmod 4}}^{\infty} c(m)q^m \right\}$$

where $\Gamma_0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z}) \,|\, c \equiv 0 \pmod N \right\}$.

Now, the coefficients $c(n,r)$ of a Jacobi form of index m depend only on the discriminant $4nm - r^2$ and on the value of r modulo $2m$, so that one can actually write $c(n,r) = c_r(4nm - r^2)$

where $c_r(N) = c_{r'}(N)$ for $r \equiv r' \pmod{2m}$. Then we have the following isomorphism between $M^+_{k-1/2}(\Gamma_0(4))$ and $J_{k,1}$:

$$\sum_{\substack{N \geq 0 \\ N \equiv 0,3 \pmod 4}} c(N) q^N \mapsto \sum_{\substack{n,r \in \mathbb{Z} \\ 4n \geq r^2}} c(4n - r^2) q^n \zeta^r.$$

This correspondence is compatible with Petersson scalar products and with the actions of the Hecke operators.

Finally, Shimura [27, 28] established an isomorphism between half-integral weight modular forms and even weight modular forms, such that the corresponding Hecke eigenvalues agree. The composition of these three maps gives a lift from $S_{2k-2}(\Gamma_1)$ to $S^*_k(\Gamma_2)$.

CHAPTER 3

THE NON-HOLOMORPHIC SETTING

3.1 Siegel Maass wave forms of weight 0

3.1.1 Basic properties

Let $\Gamma = \Gamma_2 = \mathrm{Sp}(2, \mathbb{Z})$, and recall the action in Section 2.1.1,

$$M\langle Z\rangle = (AZ + B)(CZ + D)^{-1},$$

where $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$ and $Z \in \mathbb{H}_2$. Let H_1 and H_2 be the generators of the center of the universal enveloping algebra, which is isomorphic to the algebra of all Γ-invariant differential operators. An explicit expansion of H_1 and H_2 is given in Section 4.1.

Definition 4. *A Siegel Maass wave form of weight* 0 *is a smooth function* $F : \mathbb{H}_2 \to \mathbb{C}$ *which satisfies the following conditions:*

1. F *is* Γ*-invariant:*

$$F(M\langle Z\rangle) = F(Z) \quad \forall M \in \Gamma, Z \in \mathbb{H}_2$$

2. F *is a common eigenfunction of* H_1 *and* H_2*:*

$$\begin{cases} H_1 F = d_1 F \\ H_2 F = d_2 F \end{cases}$$

for some $d_1, d_2 \in \mathbb{C}$

3. F *satisfies the growth condition*

$$|F(Z)| \leq c(\mathit{sup}\{\mathit{tr}(\mathit{Im}(Z)), \mathit{tr}(\mathit{Im}(Z)^{-1}\})^n$$

for some $0 \leq c \in \mathbb{R}$ *and some* $n \in \mathbb{N}$.

Let $\mathcal{M}(\Gamma) = \mathcal{M}_{d_1,d_2}(\Gamma)$ denote the vector space of Siegel Maass wave forms of weight 0 with eigenvalues d_1, d_2 for H_1 and H_2. By a result of Harish-Chandra [12, Theorem 1], this space is finite dimensional.

For P a parabolic subgroup of $\mathrm{Sp}(2,\mathbb{Q})$, let N be its unipotent radical. A Siegel Maass form F satisfiying the further condition

$$\int_{N(\mathbb{Z})/N(\mathbb{R})} F(n\langle Z\rangle)dn = 0$$

for any parabolic P, is called a cusp form. ($N(\mathbb{Z}) = N(\mathbb{Q}) \cap \Gamma$ and dn is Haar measure for N). Cusp forms decay rapidly along cusps of Γ, hence are bounded on $\mathbb{H}_2\backslash\Gamma$ ([4, 12]). We denote by $\mathcal{S}(\Gamma)$ the space of Siegel Maass cusp forms. This space has a Hermitian inner product,

$$(F_1, F_2) = \int_{\Gamma\backslash\mathbb{H}} F_1(Z)\overline{F_2(Z)}\frac{dZ}{\det(\mathrm{Im}(Z))^3}$$

where $dZ = dx_1 dx_2 dx_3 dy_1 dy_2 dy_3$. Finally, Hecke operators are mutually commutative and self-adjoint with respect to this inner product, so $\mathcal{S}(\Gamma)$ has a basis consisting of common eigenforms of all Hecke operators ([13]).

Applying the Γ-invariance of F for the matrix

$$M = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix} \in \Gamma,$$

where S is symmetric, we obtain

$$F(Z) = F(M\langle Z\rangle) = F((IZ + S)(0Z + I)^{-1}) = F(Z + S)$$

for $Z \in \mathbb{H}_2$. Thus, the wave form has the Fourier expansion

$$F(Z) = \sum_{N \in \mathfrak{N}_2} a(N, Y) e^{2\pi i \operatorname{tr}(NX)}$$

where $Z = X + iY$, and $\mathfrak{N}_2 = \{N \in M_2(\mathbb{Q}) \,|\, {}^tN = N,\ \text{half-integral}\}$.

Now, in analogy with holomorphic Siegel modular forms, we establish a Fourier-Jacobi expansion of Siegel Maass wave forms. Write $Z = X + iY = \left(\begin{smallmatrix} z_1 & z_2 \\ z_2 & z_3 \end{smallmatrix}\right) \in \mathbb{H}_2$ with $z_j = x_j + iy_j$. Further, write $N \in \mathfrak{N}_2$ as $N = \left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right)$. Then $\mathrm{tr}(NX) = nx_1 + rx_2 + mx_3$ and we can write

$$\begin{aligned} F(Z) &= F(z_1, z_2, z_3) \\ &= \sum_{N \in \mathfrak{N}_2} a(N, Y) e^{2\pi i \mathrm{tr}(NX)} \\ &= \sum_{n,r,m \in \mathbb{Z}} a(n, r, m, y_1, y_2, y_3) e^{2\pi i (nx_1 + rx_2 + mx_3)} \\ &= \sum_{m \in \mathbb{Z}} \phi_m(z_1, z_2, y_3) e^{2\pi i m x_3} \end{aligned}$$

by rearrangement of terms. As in the holomorphic case,

$$F(Z) = \sum_{m \in \mathbb{Z}} \phi_m(z_1, z_2, y_3) e^{2\pi i m x_3} \tag{3.1}$$

is called the **Fourier-Jacobi expansion** of the wave form.

3.1.2 Hecke operators and the Andrianov L-function

We now discuss Hecke operators for Siegel Maass wave forms, as in [13, Section 2]. For each $m \in \mathbb{Z}$, set

$$S_m = \{M \in M_4(\mathbb{Z}) \mid Mj^tM = mj\}$$

where $j = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ as in Section 2.1.1 and put

$$S = \bigcup_{m=1}^{\infty} S_m.$$

Define a function $F|M$ on $\mathbb{H}_2$, for each $F \in \mathcal{M}(\Gamma)$ and $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in S$, by

$$(F|M)(Z) = F(M\langle Z \rangle) = F((AZ+B)(CZ+D)^{-1})$$

for $Z \in \mathbb{H}_2$. Then

$$(F|M)|M' = F|MM' \quad \text{for all} \quad M, M' \in S$$

and we can define the Hecke operators $T(m)$ on $\mathcal{M}(\Gamma)$ for $m \in \mathbb{Z}$ by

$$T(m)F = m^{-3} \sum_{M \in \Gamma \backslash S_m} F|M.$$

For $M \in \Gamma$, $F|M = F$ by definition of Siegel Maass wave form, so the action of $T(m)$ is well-defined, and $T(m)F \in \mathcal{M}(\Gamma)$. For $m, m' \in \mathbb{N}$, properties of the abstract Hecke ring due to Shimura [26] imply

$$\begin{cases} T(m)T(m') &= T(m')T(m) \\ T(m)T(m') &= T(mm') \qquad \text{if } (m, m') = 1 \end{cases} \tag{3.2}$$

and that the formal power series $\sum_{\delta=0}^{\infty} T(p^{\delta})t^{\delta}$, for p prime, is given by

$$\begin{aligned}\sum_{\delta=0}^{\infty} T(p^{\delta})t^{\delta} = (1 - p^{-4}t^2) \\ \times [1 - T(p)t + \{T(p)^2 - T(p^2) - p^{-4}\}t^2 \\ - T(p)p^{-3}t^3 + p^{-6}t^4]^{-1}.\end{aligned}$$

Let $F \in \mathcal{M}(\Gamma)$ be a common eigenform of the Hecke operators $T(m)$, $m \in \mathbb{Z}$, with eigenvalues $\lambda_F(M) \in \mathbb{C}$; that is,

$$T(m)F = \lambda_F(m)F.$$

Hori [13] defines the Andrianov L-function attached to F to be

$$L_F(s) = \zeta(2s+4) \sum_{m \in \mathbb{N}} \lambda_F(m)m^{-s}.$$

Let

$$Q_{p,F}(t) = 1 - \lambda_F(p)t + \{\lambda_f(p)^2 - \lambda_F(p^2) - p^{-4}\}t^2 - \lambda_F(p)p^{-3}t^3 + p^{-6}t^4.$$

Then the Andrianov L-function has an Euler product

$$L_F(s) = \prod_p Q_{p,F}(p^{-s})$$

because of the properties Equation 3.2 of the Hecke operators. This L-function is a special case of the Langlands automorphic L-function, corresponding to the spinor representation of the dual group $\mathrm{SO}(5,\mathbb{C})$ of $\mathrm{Sp}(g,\mathbb{R})$. For $\mathrm{Re}(s)$ large enough, it is absolutely convergent for cusp forms.

3.2 Progress toward a non-holomorphic Saito-Kurokawa lift

3.2.1 Non-holomorphic Shimura lift

In [15], Katok and Sarnak develop a non-holomorphic Shimura correspondence. Let $\Gamma_1 = SL_2(\mathbb{Z})$, $\mathbb{H}$ be the upper half space, and set

$$\begin{aligned} U &= L^2_{\text{cusp}}(\Gamma_1\backslash\mathbb{H}) \\ &= \{f : \mathbb{H} \to \mathbb{C} | f(\gamma\langle z\rangle) = f(z)\, \forall\gamma \in \Gamma_1, \\ &\quad \int_{\Gamma_1\backslash\mathbb{H}} |f|^2 \frac{dxdy}{y^2} < \infty, \int_0^1 f(x,y)dx = 0 \text{ for a. e. } y\}. \end{aligned}$$

This is a Hilbert space with the usual inner product. It is invariant under the action of the Laplacian

$$\Delta_0 = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

and the Hecke operators

$$T_p(f)(z) = \sum_{n\neq 0} \{p^{1/2}b(np) + p^{-1/2}b\left(\frac{n}{p}\right)\}W_{0,ir}(4\pi|n|y)e(nx)$$

where $b(n)$ is the nth Fourier coefficient in the expansion

$$f(z) = \sum_{n\neq 0} b(n)W_{0,ir}(4\pi|n|y)e(nx),$$

$W_{\beta,\mu}(y)$ is the usual Whittaker function normalized so that $W_{\beta,\mu}(y) \sim e^{-y/2}y^{\beta}$ as $y \to \infty$, and $b(\frac{n}{p})$ is assumed to be zero if p does not divide n. These operators commute, and U has an orthogonal basis consisting of common eigenfunctions for Δ_0 and T_p. These eigenforms are called **Maass-Hecke eigenforms of weight 0**.

Now let

$$\begin{aligned} V &= L^2_{\text{cusp}}(\Gamma_0(4)\backslash\mathbb{H}, J) \\ &= \{f : \mathbb{H} \to \mathbb{C} | f(\gamma\langle z\rangle) = J(\gamma, z)f(z)\, \forall\gamma \in \Gamma_0(4), \\ &\qquad f \text{ is cuspidal and square integrable}\}, \end{aligned}$$

where $J(\gamma, z) = \frac{\theta(\gamma\langle z\rangle)}{\theta(z)}$, $\theta(z) = y^{1/4}\sum_{n=-\infty}^{\infty} e(n^2 z)$, and f cuspidal means that the zeroth Fourier coefficient is 0 at the three cusps of $\Gamma_0(4)\backslash\mathbb{H}$. Then V is a Hilbert space, invariant under the action of the weight-1/2 Laplacian

$$\Delta_{1/2} = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - \frac{1}{2}iy\frac{\partial}{\partial x}$$

and the Hecke operators T_{p^2} for $p \neq 2$. Now set

$$L = \tau_2\sigma,$$

where

$$\tau_2(f(z)) = e^{i\pi/4}\left(\frac{z}{|z|}\right)^{-1/2} f(-1/4z)$$
$$\sigma(f(z)) = \frac{\sqrt{2}}{4}\sum_{v \mod 4} f\left(\frac{a+v}{4}\right).$$

Then $\Delta_{1/2}, T_{p^2}$, and L commute and are self-adjoint, so V is spanned by common eigenforms, $f_1, f_2, \ldots$ of these operators. Let V^+ be the subspace of V on which $Lf = f$. This space will essentially take the place of Kohnen's plus space from the holomorphic setting, as we shall see. In fact, it can be characterized by Maass forms from V whose Fourier coefficients vanish for $n \equiv 2, 3 \mod 4$.

Denote by $\rho_j(n)$ the Fourier coefficients of the weight 1/2 Maass forms f_j. We will typically assume $\varphi \in U$ is normalized so that $b(1) = 1$. Associated to $\varphi \in U$ is the L-function

$$L(\varphi, s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^{s-1/2}}.$$

Let

$$\begin{aligned}\Theta(z, g) &= v^{1/2}\theta(z, g)\\ &= v^{3/4}\sum_{h\in\mathbb{Z}^3} e(u(h_2^2 - 4h_1h_2)f_3(\sqrt{v}g^{-1}(h))\end{aligned}$$

be Siegel's Θ-function coming from the Weil representation, where $z = u+iv \in \mathbb{H}_1$, $g \in \mathrm{SL}(2, \mathbb{R})$, $h = (h_1, h_2, h_3) \in \mathbb{Z}^3$, and $f_3(h) = e^{-2\pi(2h_1^2+h_2^2+2h_3^2)}$.

Now we are ready to state the non-holomorphic Shimura correspondence. If $\varphi \in U$ is a normalized Maaß-Hecke cusp form, then

$$f(z) = \int_{\Gamma_1 \backslash SL_2(\mathbb{R})} \varphi(g)\Theta(z,g)dg$$

is in V^+. Conversely,

Theorem 3. *[15, Proposition 4.1, p. 213] Let* $z = u+iv \in \mathbb{Z}, w \in \mathbb{H}$, $f(z) \in V^+$ *be a weight* $1/2$ *Maaß-Hecke form with Fourier coefficients* $\rho(n)$*. Let* $\psi(w) = \int_{\Gamma_0(4)\backslash\mathbb{H}} f(z)\overline{\Theta(z,w)}\frac{dudv}{v^2}$*. Then*

1. $\psi(w) \in U$*.*
2. ψ *is a common eignfunction of* T_p *and* Δ_0*.*
3. *If* $\rho(1) = 0$*, then* $\psi \equiv 0$*.*

   If $\rho(1) \neq 0$*, then* $\psi = 3\sqrt{(2)}\pi^{1/4}\rho(1)\phi$ *where* $\rho \in U$ *is the unique normalized Maaß form with Fourier expansion* $2\sum_{n=1}^{\infty} a(n)W_{0,2ir}(4\pi ny)\cos(s\pi nx)$ *having the same eigenvalues as* ψ*, whose Fourier coefficients are defined from the equation* $\zeta(s+1)\sum_{n=1}^{\infty}\frac{\rho(n^2)}{n^{s-1/2}} = \rho(1)\sum_{n=1}^{\infty} a(n)n^{-s}$*.*

We note here that, while Katok-Sarnak established this only for $k = 0$, one can generalize all of this to any k. We will want this more general definition in the next section, so we give it now. Let $\mathcal{G}$ be the group consisting of pairs $(\gamma, \phi(\tau))$, where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbb{R})$ and $\phi(\tau)$ is a function on $\mathbb{H}$ such that $\phi(\tau) = t\det(\gamma)^{-1/4}\left(\frac{c\tau+d}{|c\tau+d|}\right)^{1/2}$ with $t \in \mathbb{C}, |t| = 1$. The group law is given by

$$(\gamma_1, \phi_1(\tau))\dot{(}\gamma_2, \phi_2(\tau)) = (\gamma_1\gamma_2, \phi_1(\gamma_2\langle\tau\rangle)\phi_2(\tau)),$$

where $\gamma\langle\tau\rangle := \frac{a\tau+b}{c\tau+d}$.

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$, set

$$\epsilon_d = \begin{cases} 1, & d \equiv 1 \mod 4 \\ i, & d \equiv 3 \mod 4. \end{cases}$$

Let $\theta(\tau) := y^{1/4} \sum_{n=-\infty}^{\infty} e(n^2\tau)$, and

$$j(\gamma, \tau) := \left(\frac{c}{d}\right) \epsilon_d^{-1} \left(\frac{c\tau + d}{|c\tau + d|}\right)^{1/2} = \frac{\theta(\gamma\langle\tau\rangle)}{\theta(\tau)}$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the Legendre symbol. Then there is an injective homomorphism

$$\Gamma_0(4) \longrightarrow \mathcal{G}$$

$$\gamma \longmapsto \gamma^* := (\gamma, j(\gamma, \tau)$$

Now define, for $k \in \mathbb{Z}$, a slash operator on functions on the upper half plane:

$$(f||_{k-1/2}(\gamma, \phi))(\tau) := f(\gamma\langle\tau\rangle)\phi(\tau)^{-(2k-1)}.$$

Then a smooth function $f : \mathbb{H} \to \mathbb{C}$ is called a **Maass form** of weight $k - 1/2$ with respect to $\Gamma_0(4)$ if it satisfies:

1. For every $\gamma \in \Gamma_0(4)$, we have $f||_{k-1/2}\gamma^* = f$.

2. $\Delta_{k-1/2} f = \Lambda f$ for some $\Lambda \in \mathbb{C}$ where $\Delta_{k-1/2}$ is the Laplace Beltrami operator given by

$$\Delta_{k-1/2} := y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - \left(k - \frac{1}{2} \right) iy \frac{\partial}{\partial x}.$$

3. $f(\tau) = O(y^N)$ as $y \to \infty$ for some $N > 0$.

If f vanishes at the cusps of $\Gamma_0(4)$, then f is called a Maass cusp form. We denote the space of weight $k - 1/2$ Maass (resp. cusp) forms by $\mathcal{M}_{k-1/2}(\Gamma_0(4))$ (resp. $\mathcal{S}_{k-1/2}(\Gamma_0(4))$, and now the Fourier expansion of $f \in \mathcal{M}_{k-1/2}(\Gamma_0(4))$ can be written as

$$f(\tau) = \sum_{n \in \mathbb{Z}} c(n) W_{sgn(n) \frac{k-1/2}{2}, \frac{il}{2}} (4\pi |n| y) e(nx)$$

where $\Lambda = -(1/4 + (l/2)^2)$. If f is a cusp form, then $c(0) = 0$. Define the plus space $\mathcal{M}^+_{k-1/2}(\Gamma_0(4))$ to be the subspace of those Maass forms whose Fourier coefficients $c(n)$ vanish whenever $(-1)^{k-1} n \equiv 2, 3 \mod 4$.

3.2.2 The lift to Jacobi-Maass forms

Now we can define Jacobi-Maass forms, as in Pitale [25] and look at the second isomorphism in a nonholomorphic Saito-Kurokawa lift. Let $G^J(\mathbb{R}) = \mathrm{SL}(2, \mathbb{R}) \ltimes H(\mathbb{R})$ be the Jacobi group, and denote the discrete subgroup $\mathrm{SL}(2, \mathbb{Z}) \ltimes H(\mathbb{Z})$ by Γ^J. Here, H is the Heisenberg group consisting of elements $(\lambda, \mu, \kappa) := (X, \kappa)$, where $X = (\lambda, \mu)$. There are two different coordinate systems on $G^J(\mathbb{R})$, and it will be convenient to switch back and forth between them:

1. The EZ-coordinates (due to Eichler and Zagier) $(x, y, \theta, \lambda, \mu, \kappa)$ produce the element

$$M(X, \kappa) \in G^J(\mathbb{R}),$$

   where
$$M = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$
   with $x \in \mathbb{R}$, $y \in \mathbb{R}^+$, $0 \le \theta < 2\pi$, $X = (\lambda, \mu) \in \mathbb{R}^2$, and $\kappa \in \mathbb{R}$.

2. The S-coordinates (due to Siegel) $(x, y, \theta, p, q, \kappa)$ give the element $(Y, \kappa)M \in G^J(\mathbb{R})$ where M is as above and $Y = (p, q) = XM^{-1} \in \mathbb{R}^2$.

The action of $G^J(\mathbb{R})$ on $\mathbb{H} \times \mathbb{C}$ is then given by:

1. If $g = M(X, \kappa) \in G^J(\mathbb{R})$ is given in EZ-coordinates, then for $\tau \in \mathbb{H}$, $z \in \mathbb{C}$, we have

$$g(\tau, z) := \left(M\langle\tau\rangle, \frac{z + \lambda\tau + \mu}{c\tau + d} \right)$$

   where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $M\langle\tau\rangle = \frac{a\tau+b}{c\tau+d}$ as usual.

2. In the S-coordinates, we have

$$G^J(\mathbb{R})/(SO(2) \times \mathbb{R}) \xrightarrow{\sim} \mathbb{H} \times \mathbb{C}$$

$$g = (p, q, \kappa)M \longmapsto g(i, 0) = (\tau, p\tau + q)$$

   where $\tau = M\langle i\rangle$.

There is a non-holomorphic factor of automorphy $j^{nh}_{k,m}$ $(k \in \mathbb{Z}, m \in \mathbb{N})$ for the Jacobi group, given in the EZ-coordinates by

$$j^{nh}_{k,m}(g,(\tau,z)) := e(m(\kappa - \frac{c(z+\lambda\tau+\mu)^2}{c\tau+d} + \lambda^2\tau + 2\lambda z + \lambda\mu)) \left(\frac{c\tau+d}{|c\tau+d|}\right)^{-k}.$$

In the S-coordinates, it looks much simpler:

$$j^{nh}_{k,m}(g,(i,0)) = e(m(\kappa + pz))e^{ik\theta} \text{ where } z = p(x+iy)+q.$$

The automorphy factor satisfies a cocycle condition, and we can define a slash operator on functions on $F : \mathbb{H}\times\mathbb{C} \to \mathbb{C}$ by

$$(F|_{k,m}g)(\tau,z) := j^{nh}_{k,m}(g,(\tau,z))F(g(\tau,z))$$

for $g \in G^J(\mathbb{R}), (\tau,z) \in \mathbb{H}\times\mathbb{C}$.

Now, put

$$\begin{aligned}
\mathcal{C}^{k,m}F =& \frac{5}{8}F - 2(\tau-\bar\tau)^2F_{\tau\bar\tau} - (k-1)(\tau-\bar\tau)F_{\bar\tau} - k(\tau-\bar\tau)F_\tau \\
&+ \frac{k(\tau-\bar\tau)}{8\pi im}F_{zz} + \frac{(\tau-\bar\tau)^2}{4\pi im}F_{\bar\tau zz} \\
&+ \frac{k(\tau-\bar\tau)}{4\pi im}F_{z\bar z} + \frac{(\tau-\bar\tau)(z-\bar z)}{4\pi im}F_{zz\bar z} - 2(\tau-\bar\tau)(z-\bar z)F_{\tau\bar z} + \frac{(\tau-\bar\tau)^2}{4\pi im}F_{\tau z\bar z} \\
&+ \left(\frac{(z-\bar z)^2}{2} + \frac{k(\tau-\bar\tau)}{8\pi im}\right)F_{z\bar z} + \frac{(\tau-\bar\tau)(z-\bar z)}{4\pi im}F_{z\bar z\bar z}.
\end{aligned}$$

This is the pullback of the standard Casimir operator by $j^{nh}_{k,m}$. If F is holomorphic in the z variable, then the last two lines of this will be zero; if it is also holomorphic in the τ variable, then $\mathcal{C}^{k,m}$ becomes simply $\frac{5}{8}F - k(\tau-\bar{\tau})F_\tau + \frac{k(\tau-\bar{\tau})}{8\pi i m}F_{zz}$, which bears a strong resemblance to the heat operator $8\pi i m\partial_\tau - \partial_z^2$.

Pitale ([25, Definition 3.2, p. 93]) defines a Jacobi Maass form as follows.

Definition 5. *A smooth function $F : \mathbb{H}\times\mathbb{C}\to\mathbb{C}$ is called a* ***Jacobi Maass form*** *of weight $k\in\mathbb{Z}$ and index $m\in\mathbb{N}$ with respect to Γ^J if*

1. *$(F|_{k,m}\gamma)(\tau,z) = F(\tau,z)$ for all $\gamma\in\Gamma^J$ and $(\tau,z)\in\mathbb{H}\times\mathbb{C}$,*
2. *$\mathcal{C}^{k,m}F = \lambda F$*
3. *$F(\tau,z) = O(y^N)$ as $y\to\infty$ for some $N>0$.*

Such a form will be called as **cusp form** if it further satisfies the condition

$$\int_0^1\int_0^1 F\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}(0,u,0)(\tau,z)\right)e(-(nx+ru))dxdu = 0$$

for all $n,r\in\mathbb{Z}$ with $4nm-r^2=0$. We denote by $\mathcal{J}^{nh}_{k,m}$ the vector space of all Jacobi Maass forms of weight k and index m with respect to Γ^J, and the subspace of cusp forms by $\mathcal{J}^{\text{nh, cusp}}_{k,m}$.

Now, let $\epsilon = (-1)^{k-1}$. For

$$f(\tau) = \sum_{n\in\mathbb{Z}} c(n)W_{sgn(n)\frac{k-1/2}{2},\frac{il}{2}}(4\pi|n|y)e(nx)\in\mathfrak{M}^+_{k-1/2}(\Gamma_0(4)),$$

define

$$f^{(0)}(\tau) = \sum_{n\in\mathbb{Z}} c(4n) W_{sgn(n)\frac{k-1/2}{2},\frac{il}{2}}(4\pi|n|y)e(nx)$$

and

$$f^{(1)}(\tau) = \sum_{n\in\mathbb{Z}} c(4n+\epsilon) W_{sgn(n)\frac{k-1/2}{2},\frac{il}{2}}(4\pi|n+\frac{\epsilon}{4}|y)e((n+\frac{\epsilon}{4})x).$$

Then explicit computation [25, p. 96] yields

$$f(\tau) = (f^{(0)} + f^{(1)})(4\tau),$$

and we see from the Fourier expansions of $f^{(0)}$ and $f^{(1)}$, that they are eigenfunctions of $\Delta_{k-1/2}$ with the same eigenvalue as f.

We will now restrict our attention to Jacobi Maass forms of even weight k and index $m = 1$. Define the theta series for $\tau = x + iy \in \mathbb{H}$ and $z \in \mathbb{C}, j = 0, 1$:

$$\tilde{\Theta}^{(j)}(\tau, z) := y^{\frac{1}{4}} \sum_{\substack{r\in\mathbb{Z} \\ r\equiv j \mod 2}} e(\tau\frac{r^2}{4})e(zr).$$

For $f \in \mathcal{M}^{+}_{k-1/2}(\Gamma_0(4))$ with k even, define $F_f : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ by

$$F_f(\tau, z) := f^{(0)}\tilde{\Theta}^{(0)}(\tau, z) + f^{(1)}\tilde{\Theta}^{(1)}(\tau, z).$$

It is shown in [25, Theorem 4.4], that this is precisely the desired lift:

Theorem 4. *Let $f \in \mathcal{M}^+_{k-1/2}(\Gamma_0(4))$ with k even and F_f be the smooth function defined above. Then*

1. $F_f \in \mathcal{J}^{nh}_{k,1}$,
2. $F_f \in \mathcal{J}^{nh,\ cusp}_{k,1}$ *if and only if* $f \in \mathcal{S}^+_{k-1/2}(\Gamma_0(4))$,
3. *If* $\Delta_{k-1/2} f = \Lambda f$, *then we have* $\mathcal{C}^{k,1} F_f = 2\Lambda F_f$.

Using the definitions of these theta series and the Fourier expansion of $f^{(j)}$, we obtain the Fourier expansion of F_f

$$F_f(\tau, z) = \sum_{r,n \in \mathbb{Z}} c(4n - r^2) y^{\frac{1}{4}} W_{sgn(n-\frac{r^2}{4})\frac{k-1/2}{2}, \frac{il}{2}} \left(4\pi |n - \frac{r^2}{4}| y \right) e^{-2\pi \frac{r^2}{4} y} e(nx) e(rz).$$

This implies that the map $f \mapsto F_f$ is injective, so for even k, the space $\mathcal{J}^{nh}_{k,1}$ is infinite dimensional. Finally, the image is the subspace $\hat{\mathcal{J}}^{nh}_{k,1}$ consisting of those Jacobi forms that are holomorphic in the z-variable ([25, Theorem 4.5]). So we have an isomorphism

$$\hat{\mathcal{J}}^{nh}_{k,1} \cong \mathcal{M}^+_{k-1/2}(\Gamma_0(4)).$$

This map has the added property that it is compatible with the action of the Hecke operators:

Theorem 5. *[25, Theorem 6.1] Let* $\mathcal{M}^{+}_{k-1/2}(\Gamma_0(4))$, *$k$ even, be a Hecke eigenform with eigenvalue* λ_p *for every odd prime* p. *Then the corresponding Jacobi-Maass form* F_f *is an eigenfunction of the operator*

$$T_pF := p^{k-4} \sum_{\substack{M \in SL_2(\mathbb{Z})/M_3(\mathbb{Z}) \\ \det(M)=p^2 \\ gcd(M)=1}} \sum_{(\lambda,\mu)\in(\mathbb{Z}/p\mathbb{Z})^2} F|_{k,1}(\det(M))^{-\frac{1}{2}}M(\lambda,\mu,0).$$

Moreover, if μ_p *is the eigenvalue of* F_f *under* T_p, *then*

$$\mu_p = p^{k-3/2}\lambda_p.$$

Finally, we discuss how this definition of Jacobi Maass forms in compatible with the representation theory of the Jacobi group. In particular, fix a half-integral weight Maass cusp form $f \in \mathcal{S}^{+}_{k-1/2}(\Gamma_0(4))$ (k even) such that, for every odd prime p,

$$T_pf = \lambda_p f$$

$$\Delta_{k-1/2}f = \Lambda f$$

where $\Lambda = \frac{1}{4}(s^2-1)$. Let $\tilde{\pi}_f = \otimes \pi_p$ be the irreducible cuspidal (genuine) automorphic representation of $\tilde{\mathrm{SL}}_2(\mathbb{A})$ corresponding to f. Let $F_f \in \hat{\mathcal{J}}^{nh}_{k,1}$ be the corresponding Jacobi Maass forms with

$$\mathcal{C}^{k,1} F_f = 2\lambda_p F_f = \frac{1}{2}(s^2-1)F_f,$$
$$T_p F_f = \mu_p F_f = p^{k-3/2}\lambda_p F_f.$$

Let π_F be the irreducible cuspidal automorphic representation of

$$G^J(\mathbb{A}) = G^J(\mathbb{Z})G^J(\mathbb{R})\prod_{p<\infty} G^J(\mathbb{Z}_p)$$

corresponding to F_f. To construct π_F, first lift F_f to a function ϕ_F on $G^J(\mathbb{A})$ as follows: if $g = \gamma g_\infty k_0 \in G^J(\mathbb{A})$, with $\gamma \in G^J(\mathbb{A}), g_\infty \in G^J(\mathbb{R}), k_0 \in \prod_{p<\infty} G^J(\mathbb{Z}_p)$, then set

$$\pi_F(g) := (F_f|_{k,m} g_\infty)(i,0) = j^{nh}_{k,m}(g_\infty,(i,0))F_f(g_\infty(i,0)).$$

So π_F is the space of right translates of ϕ_F, and the group $G^J(\mathbb{A})$ acts by right translation. Then [25, Theorem 7.5]

$$\pi_F = \tilde{\pi}_f \otimes \pi^1_{\mathrm{SW}}$$

where π^1_{SW} is the global Schrödinger-Weil representation of $G^J(\mathbb{A})$, which is the expected relationship.

Remark. *The functions* $\phi_m(z_1, z_2, y_3)$ *in Equation 3.1 are dependent on* y_3*, so the Fourier-Jacobi coefficients of a Siegel Maass wave form are not, in general, themselves Jacobi Maass forms. Therefore, one does not expect to be able to lift Jacobi Maass forms to Siegel Maass wave forms in a manner analogous to the holomorphic case, by means of Fourier-Jacobi coefficients (see Section 2.2). So a non-holomorphic Saito-Kurokawa lift is likely not obtainable by mimicking the classical construction.*

CHAPTER 4

FOURIER EXPANSION OF WAVE FORMS

4.1 Generators of the central enveloping algebra

For functions $f : \mathbb{H}_2 \to \mathbb{C}$, and for fixed $\alpha, \beta \in \mathbb{C}$ with $\alpha - \beta \in \mathbb{Z}$, define the slash operator

$$(f|_{\alpha,\beta}M)(Z) := \det(CZ+D)^{-\alpha}\det(C\bar{Z}+D)^{-\beta} f(M\langle Z\rangle)$$

for all $M \in \Gamma_2$.

The center of the universal enveloping algebra of $\mathrm{Sp}(2,\mathbb{R})$ is generated by the Casimir elements. The images of these elements under the slash operator are a quadratic and quartic operator, called the Casimir operators, and they generate the $\mathbb{C}$-algebra of differential operators invariant under the slash operator above. Maass [18] computed these operators H_1, H_2, and we use the notation of [6], with $\alpha = \beta = 0$ to write them explicitly. As in [6], let

$$\partial_Z := \begin{pmatrix} \partial_1 & \frac{1}{2}\partial_2 \\ \frac{1}{2}\partial_2 & \partial_3 \end{pmatrix} \quad \text{and} \quad \bar{\partial}_Z := \begin{pmatrix} \partial_1 & \frac{1}{2}\partial_2 \\ \frac{1}{2}\bar{\partial}_2 & \bar{\partial}_3 \end{pmatrix},$$

where $\partial_i = \frac{\partial}{\partial z_i} = \frac{1}{2}(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i})$ and $\bar{\partial}_i = \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i})$. Set $K_\alpha = \alpha I + (Z - \bar{Z})\partial_{\bar{Z}}$, $\Lambda_\beta = -\beta I_2 + (Z - \bar{Z})\partial_{\bar{Z}}$, and

$$\begin{aligned}\Omega_{\alpha,\beta} :&= \Lambda_{\beta-\frac{3}{2}} K_\alpha + \alpha(\beta - \frac{3}{2})I \\ &= -4Y^t(Y\partial_{\bar{Z}})\partial_Z - 2i\beta Y\partial_Z + 2i\alpha Y\partial_{\bar{Z}}.\end{aligned}$$

Defining $A^{(1)}_{\alpha,\beta} := \Omega_{\alpha,\beta} - \alpha(\beta - \frac{3}{2})I$, then [6] gives that the Casimir operators generating the $\mathbb{C}$-algebra of differential operators invariant with respect to the slash operator on the Siegel upper half space $\mathbb{H}_2$ are

$$H_1^{(\alpha,\beta)} := \mathrm{tr}(A^{(1)}_{\alpha,\beta}) \quad \text{and} \tag{4.1}$$

$$H_2^{(\alpha,\beta)} := \mathrm{tr}(A^{(1)}_{\alpha,\beta}A^{(1)}_{\alpha,\beta}) - \mathrm{tr}(\Lambda_\beta A^{(1)}_{\alpha,\beta}) + \frac{1}{2}\mathrm{tr}(\Lambda_\beta)\mathrm{tr}(A^{(1)}_{\alpha,\beta}), \tag{4.2}$$

where $\mathrm{tr}(A)$ denotes the trace of $A \in M_2(\mathbb{C})$.

Setting $\alpha = \beta = 0$, $H_1 := H_1^{(0,0)}$, and $H_2 := H_2^{(0,0)}$, and $d = y_1y_3 - y_2^2$ we obtain by explicit computation

$$\begin{aligned}H_1 &= -4\sum_{i,j=1}^{3} y_i\, y_j\, \partial_i\bar{\partial}_j + 4d(\partial_1\bar{\partial}_3 + \bar{\partial}_1\partial_3 - \frac{1}{2}\partial_2\bar{\partial}_2) \\ &= -4\big(y_1^2\partial_1\bar{\partial}_1 + y_1y_2\partial_1\bar{\partial}_2 + y_2^2\partial_1\bar{\partial}_3 + y_1y_2\bar{\partial}_1\partial_2 + \frac{1}{2}y_1y_3\partial_2\bar{\partial}_2 + \frac{1}{2}y_2^2\partial_2\bar{\partial}_2 \\ &\quad + y_2y_3\partial_2\bar{\partial}_3 + y_2^2\bar{\partial}_1\partial_3 + y_2y_3\bar{\partial}_2\partial_3 + y_2^2\partial_3\bar{\partial}_3\big)\end{aligned} \tag{4.3}$$

and

$$
\begin{aligned}
H_2 &= 16\sum_{i,j,k,l=1}^{3} y_i\, y_j\, y_k\, y_l\, \partial_i \partial_j \bar{\partial}_k \bar{\partial}_l - 32d\big(\sum_{i,j=1}^{3} y_i\, y_j \partial_i \bar{\partial}_j\big)\big(\partial_1 \bar{\partial}_3 + \bar{\partial}_1 \partial_3 - \frac{1}{2}\partial_2 \bar{\partial}_2\big) \\
&+ 16d^2\big(\partial_1 \bar{\partial}_3 + \bar{\partial}_1 \partial_3 - \frac{1}{2}\partial_2 \bar{\partial}_2\big)^2 - 32d^2(\partial_1 \partial_3 - \frac{1}{4}\partial_2^2)(\bar{\partial}_1 \bar{\partial}_3 - \frac{1}{4}\bar{\partial}_2^2) \\
&+ 4i\sum_{i,j,k=1}^{3} y_i\, y_j\, y_k\, \partial_i \bar{\partial}_j \bar{\partial}_k - 4id(\sum_{i=1}^{3} y_i \bar{\partial}_i)(\partial_1 \bar{\partial}_3 + \bar{\partial}_1 \partial_3 - \frac{1}{2}\partial_2 \bar{\partial}_2) \\
&- 8id(\sum_{i=1}^{3} y_i \bar{\partial}_i)(\bar{\partial}_1 \bar{\partial}_3 - \frac{1}{4}\bar{\partial}_2^2) \\
&= 2\Big(8y_1^4 \partial_1^2 \bar{\partial}_1^2 + 16y_1^3 y_2 \partial_1^2 \bar{\partial}_1 \bar{\partial}_2 + 16y_1^3 y_2 \partial_1 \bar{\partial}_1^2 \partial_2 + 8y_1^2 y_2^2 \partial_1^2 \bar{\partial}_2^2 + 16y_1^2 y_2^2 \partial_1^2 \bar{\partial}_1 \bar{\partial}_3 \\
&+ 24y_1^2 y_2^2 \partial_1 \bar{\partial}_1 \partial_2 \bar{\partial}_2 + 8y_1^2 y_2^2 \bar{\partial}_1^2 \partial_2^2 + 16y_1^2 y_2^2 \partial_1 \bar{\partial}_1^2 \partial_3 + 16y_1 y_2^3 \partial_1^2 \bar{\partial}_2 \bar{\partial}_3 + 8y_1 y_2^3 \partial_1 \partial_2 \bar{\partial}_2^2 \\
&+ 16y_1 y_2^3 \partial_1 \bar{\partial}_1 \partial_2 \bar{\partial}_3 + 8y_1 y_2^3 \bar{\partial}_1 \partial_2^2 \bar{\partial}_2 + 16y_1 y_2^3 \partial_1 \bar{\partial}_1 \bar{\partial}_2 \partial_3 + 16y_1 y_2^3 \bar{\partial}_1^2 \partial_2 \partial_3 \\
&+ 8y_2^4 \partial_1^2 \bar{\partial}_3^2 + 8y_2^4 \partial_1 \partial_2 \bar{\partial}_2 \bar{\partial}_3 + y_2^4 \partial_2^2 \bar{\partial}_2^2 + 4y_2^2 \bar{\partial}_1 \partial_2^2 \bar{\partial}_3 + 4y_2^4 \partial_1 \bar{\partial}_2^2 \partial_3 + 8y_2^4 \bar{\partial}_1 \partial_2 \bar{\partial}_2 \partial_3 \\
&+ 8y_2^4 \bar{\partial}_1^2 \partial_3^2 + 8y_1^3 y_3 \partial_1 \bar{\partial}_1 \partial_2 \bar{\partial}_2 + 8y_1^2 y_2 y_3 \partial_1 \partial_2 \bar{\partial}_2^2 + 16y_1^2 y_2 y_3 \partial_1 \bar{\partial}_1 \partial_2 \bar{\partial}_3 \\
&+ 8y_1^2 y_2 y_3 \bar{\partial}_1 \partial_2^2 \bar{\partial}_3 + 16y_1^2 y_2 y_3 \partial_1 \bar{\partial}_1 \bar{\partial}_2 \partial_3 + 24y_1 y_2^2 y_3 \partial_1 \partial_2 \bar{\partial}_2 \bar{\partial}_3 + 6y_1 y_2^2 y_3 \partial_2^2 \bar{\partial}_2^2 \\
&+ 8y_1 y_2^2 y_3 \bar{\partial}_1 \partial_2^2 \bar{\partial}_3 + 8y_1 y_2^2 y_3 \partial_1 \bar{\partial}_2^2 \partial_3 + 32y_1 y_2^2 y_3 \partial_1 \bar{\partial}_1 \partial_3 \bar{\partial}_3 + 24y_1 y_2^2 y_3 \bar{\partial}_1 \partial_2 \bar{\partial}_2 \partial_3 \\
&+ 16y_2^3 y_3 \partial_1 \partial_2 \bar{\partial}_3^2 + 8y_2^3 y_3 \partial_2^2 \bar{\partial}_2 \bar{\partial}_3 + 16y_2^3 y_3 \partial \bar{\partial}_2 \partial_3 \bar{\partial}_3 + 8y_2^3 y_3 \partial_2 \bar{\partial}_2^2 \partial_3 \\
&+ 16y_2^3 y_3 \bar{\partial}_1 \partial_2 \partial_3 \bar{\partial}_3 + 16y_2^3 y_3 \bar{\partial}_1 \bar{\partial}_2 \partial_3^2 + y_1^2 y_3^2 \partial_2^2 \bar{\partial}_2^2 + 4y_1^2 y_3^2 \bar{\partial}_1 \partial_2^2 \bar{\partial}_3 + 4y_1^2 y_3^2 \partial_1 \bar{\partial}_2^2 \partial_3 \\
&+ 8y_1 y_2 y_3^2 \partial_2^2 \bar{\partial}_2 \bar{\partial}_3 + 16y_1 y_2 y_3^2 \bar{\partial}_1 \partial_2 \partial_3 \bar{\partial}_3 + 8y_1 y_2 y_3^2 \partial_2^2 \bar{\partial}_2 \bar{\partial}_3 + 16y_1 y_2 y_3^2 \partial_1 \bar{\partial}_2 \partial_3 \bar{\partial}_3 \\
&+ 8y_2^2 y_3^2 \partial_2^2 \bar{\partial}_3^2 + 16y_2^2 y_3^2 \partial_1 \partial_3 \bar{\partial}_3^2 + 24y_2^2 y_3^2 \partial_2 \bar{\partial}_2 \partial_3 \bar{\partial}_3 + 8y_2^2 y_3^2 \bar{\partial}_2^2 \partial_3^2 + 16y_2^2 y_3^2 \bar{\partial}_1 \partial_3^2 \bar{\partial}_3 \\
&+ 8y_1 y_3^3 \partial_2 \bar{\partial}_2 \partial_3 \bar{\partial}_3 + 16y_2 y_3^3 \partial_2 \partial_3 \bar{\partial}_3^2 + 16y_2 y_3^3 \bar{\partial}_2 \partial_3^2 \bar{\partial}_2 + 8y_3^4 \partial_3^2 \bar{\partial}_3^2
\end{aligned}
$$

$$+ i\big(2y_1^3\partial_1\bar\partial_1^2 + 4y_1^2y_2\partial_1\bar\partial_1\bar\partial_2 + 2y_1^2y_2\bar\partial_1^2\partial_2 + y_1y_2^2\partial_1\bar\partial_2^2 + 6y_1y_2^2\partial_1\bar\partial_1\bar\partial_3 + 3y_1y_2^2\bar\partial_1\partial_2\bar\partial_2$$
$$+ 2y_1y_2^2\bar\partial_1^2\partial_3 + 2y_2^3\partial_1\bar\partial_2\bar\partial_3 + 4y_2^3\bar\partial_1\partial_2\bar\partial_3 + 2y_2^3\bar\partial_1\bar\partial_2\partial_3 + y_1^2y_3\partial_1\bar\partial_2^2 - 2y_1^2y_3\partial_1\bar\partial_1\bar\partial_3$$
$$+ y_1^2y_3\bar\partial_1\partial_2\bar\partial_2 + 2y_1y_2y_3\partial_1\bar\partial_2\bar\partial_3 + 2y_1y_2y_3\partial_2\bar\partial_2^2 + 2y_1y_2y_3\bar\partial_1\bar\partial_2\partial_3 + 2y_2^2y_3\partial_1\bar\partial_3^2$$
$$+ 3y_2^2y_3\partial_2\bar\partial_2\bar\partial_3 + y_2^2y_3\bar\partial_2^2\partial_3 + 6y_2^2y_3\bar\partial_1\partial_3\bar\partial_3 + y_1y_3^2\partial_2\bar\partial_2\bar\partial_3 + y_1y_3^2\bar\partial_2^2\partial_3 - 2y_1y_3^2\bar\partial_1\partial_3\bar\partial_3$$
$$+ 2y_2y_3^2\partial_2\bar\partial_3^2 + 4y_2y_3^2\bar\partial_2\partial_3\bar\partial_3 + 2y_3^3\partial_3\bar\partial_3^2)\Big). \tag{4.4}$$

4.2 Siegel Maass wave forms of weight 0

Now, for $d_1, d_2 \in \mathbb{C}$, we consider the space $\mathcal{W}$ of generalized Whittaker functions, that is, functions

$$f : \mathbb{H}_2 \to \mathbb{C}, \quad f(Z) = g(Y)e^{2\pi i \operatorname{tr}(X)},$$

which satisfy

$$H_1 f = d_1 f,$$
$$H_2 f = d_2 f$$

where $Z = X + iY \in \mathbb{H}_2$, g some functions of Y, and the trace of X is $\operatorname{tr}(X) = x_1 + x_3$.

Observe that

$$\partial_i(g(Y)e^{2\pi i\mathrm{tr}(X)}) := \frac{1}{2}(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i})(g(Y)e^{2\pi i\mathrm{tr}(X)}) = \begin{cases} (\pi i - \frac{1}{2}\frac{\partial}{\partial y_i})g(Y)e^{2\pi i(x_1+x_3)} & \text{if } i = 1,3 \\ -\frac{1}{2}i\frac{\partial}{\partial y_i}g(Y)e^{2\pi i(x_1+x_3)} & \text{if } i = 2 \end{cases}$$

and

$$\bar{\partial}_i(g(Y)e^{2\pi i\mathrm{tr}(X)}) := \frac{1}{2}(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i})(g(Y)e^{2\pi i\mathrm{tr}(X)}) = \begin{cases} (\pi i + \frac{1}{2}\frac{\partial}{\partial y_i})g(Y)e^{2\pi i(x_1+x_3)} & \text{if } i = 1,3 \\ \frac{1}{2}i\frac{\partial}{\partial y_i}g(Y)e^{2\pi i(x_1+x_3)} & \text{if } i = 2. \end{cases}$$

By applying H_1 and H_2 to the product $g(Y)e^{2\pi i\mathrm{tr}(X)}$, we obtain that g satisfies

$$\begin{aligned}(-4)\frac{1}{8}\big(4y_2y_3\frac{\partial^2}{\partial y_2\partial y_3} + 4y_2^3\frac{\partial^2}{\partial y_3\partial y_1} + 2y_3^2\frac{\partial^2}{\partial y_3^2} + 4y_2y_1\frac{\partial^2}{\partial y_2\partial y_1} \\ + y_3y_1\frac{\partial^2}{\partial y_2^2} + y_2^2\frac{\partial^2}{\partial y_2^2} + 2t_1^2\frac{\partial^2}{\partial y_1^2} - 8\pi^2y_3^2 - 16\pi^2y_2^2 - 8\pi^2y_1^2\big)g = d_1g \qquad (4.5)\end{aligned}$$

and

$$\begin{aligned}\Big(2(y_1+y_3)\big(2\pi^3(y_1^2+2y_2^2+y_3) - 4\pi^3d - \pi^2(d-(y_1^2+y_2^2))\frac{\partial}{\partial y_1} - \pi^2(d-(y_2^2+y_3^2))\frac{\partial}{\partial y_3} \\ + \pi^2y_2(y_1+y_3)\frac{\partial}{\partial y_2}\big) + \pi y_1(d-(y_1^2+y_2^2))\frac{\partial^2}{\partial y_1^2} + \pi y_3(d-(y_2^2+y_3^2))\frac{\partial^2}{\partial y_3^2} \\ - \pi y_2^2(y_1+y_3)\big(\frac{\partial^2}{\partial y_2^2} + 2\frac{\partial^2}{\partial y_1\partial y_3}\big) - 2\pi y_2\big((y_1^2+y_2^2)\frac{\partial^2}{\partial y_1\partial y_2} + (y_2^2+y_3^2)\frac{\partial^2}{\partial y_2\partial y_3}\big) \\ + 16\pi^4\big((y_1^2+y_2^2)^2 + (y_2^2+y_3^2)^2 + 2y_2^2(y_1^2+y_3^2)^2\big) \\ - 4\pi^2\big(4y_2(y_1^2+2y_2^2+y_3^2)(y_1\frac{\partial^2}{\partial y_1\partial y_2} + y_2\frac{\partial^2}{\partial y_1\partial y_3} + y_3\frac{\partial^2}{\partial y_2\partial y_3})\end{aligned}$$

$$
\begin{aligned}
&+2(2y_2^2d+(y_1^2+y_2^2)^2)\frac{\partial^2}{\partial y_1^2}+2(2y_2^2d+(y_2^2+y_3^2)^2)\frac{\partial^2}{\partial y_3^2}\\
&+((y_1y_3+y_2^2)(y_1^2+2y_2^2+y_3^2)-d^2)\frac{\partial^2}{\partial y_2^2})-\frac{1}{2}(y_1^3\frac{\partial^3}{\partial y_1^3}+y_1y_2y_3\frac{\partial^3}{\partial y_2^3}+y_3^3\frac{\partial^3}{\partial y_3^3})\\
&-\frac{3}{2}y_2(y_1^2\frac{\partial^3}{\partial y_1^2\partial y_2}+y_3^2\frac{\partial^3}{\partial y_2\partial y_3^2})-(y_2^2+\frac{1}{2}y_1y_3)(y_1\frac{\partial^3}{\partial y_1\partial y_2^2}+y_3\frac{\partial^3}{\partial y_2^2\partial y_3})\\
&+(\frac{1}{2}y_1y_3-2y_2^2)(y_1\frac{\partial^3}{\partial y_1^2\partial y_3}+y_3\frac{\partial^3}{\partial y_1\partial y_3^2})-y_2(y_1y_3+2y_2^2)\frac{\partial^3}{\partial y_1\partial y_2\partial y_3}\\
&+(y_1^4\frac{\partial^4}{\partial y_1^4}+y_3^4\frac{\partial^4}{\partial y_3^4})+\frac{1}{8}(y_2^4+6y_1y_2^2y_3+y_1^2y_3^2)\frac{\partial^4}{\partial y_2^4}+4y_2(y_1^3\frac{\partial^4}{\partial y_1^3\partial y_2}+y_3^3\frac{\partial^4}{\partial y_2\partial y_3^3})\\
&+(y_1y_3+5y_2^2)(y_1^2\frac{\partial^4}{\partial y_1^2\partial y_2^2}+y_3^2\frac{\partial^4}{\partial y_2^2\partial y_3^2})+2y_2(y_1y_3+y_2^2)(y_1\frac{\partial^4}{\partial y_1\partial y_2^3}+y_3\frac{\partial^4}{\partial y_2^3\partial y_3})\\
&+4y_2^2(y_1^2\frac{\partial^4}{\partial y_1^3\partial y_3}+y_3^2\frac{\partial^4}{\partial y_1\partial y_3^3})+4y_2(y_1y_3+2y_2^2)(y_1\frac{\partial^4}{\partial y_1^2\partial y_2\partial y_3}+y_3\frac{\partial^4}{\partial y_1\partial y_2\partial y_3^2})\\
&+(y_1y_3(y_1y_3+5y_2^2)+3y_2^2(y_1y_3+y_2^2))\frac{\partial^4}{\partial y_1\partial y_2^2\partial y_3}\Big)g=d_2g. \qquad (4.6)
\end{aligned}
$$

4.3 Reduction of the differential equations

Since Y is symmetric and positive definite, we can make the change of variables

$$
\begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix} = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}
$$

where $a[b] := {}^t bab$, and we put $g(Y) = h(\theta, t_1, t_2)$. In particular,

$$y_1 = t_1 \cos^2\theta + t_2 \sin^2\theta,$$
$$y_2 = -\frac{1}{2}(t_1 - t_2)\sin 2\theta,$$
$$y_3 = t_1 \sin^2\theta + t_2 \cos^2\theta,$$

and, via the Jacobian,

$$\begin{aligned}
\frac{\partial}{\partial y_1} &= -\frac{1}{2}(t_1 - t_2)^{-1}\sin 2\theta\frac{\partial}{\partial\theta} + \cos^2\theta\frac{\partial}{\partial t_1} + \sin^2\theta\frac{\partial}{\partial t_2}\\
\frac{\partial}{\partial y_2} &= -\ (t_1 - t_2)^{-1}\cos 2\theta\frac{\partial}{\partial\theta} - \sin 2\theta\frac{\partial}{\partial t_1} + \sin 2\theta\frac{\partial}{\partial t_2}\\
\frac{\partial}{\partial y_3} &= \frac{1}{2}(t_1 - t_2)^{-1}\sin 2\theta\frac{\partial}{\partial\theta} + \sin^2\theta\frac{\partial}{\partial t_1} + \cos^2\theta\frac{\partial}{\partial t_2}.
\end{aligned}$$

From Equation 4.5 and Equation 4.6, we then obtain the differential equations satisfied by $h(\theta, t_1, t_2)$:

$$\begin{aligned}
&-(-4)\frac{1}{8}(t_1 - t_2)^{-2}\Big(8\pi^2(t_1 - t_2)^2(t_1^2 + t_2^2) - 2t_1t_2(t_1 - t_2)\big(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2}\big)\\
&- 2(t_1 - t_2)^2\big(t_1^2\frac{\partial^2}{\partial t_1^2} + t_2^2\frac{\partial^2}{\partial t_2^2}\big) - t_1t_2\frac{\partial^2}{\partial\theta^2}\Big)h(\theta, t_1, t_2) = d_1 h(\theta, t_1, t_2)
\end{aligned} \tag{4.7}$$

and

$$\frac{1}{8}(t_1 - t_2)^{-4}\Big(8(t_1 - t_2)^4(t_1^4 + t_2^4)\pi^4 - 32(t_1 - t_2)^6(t_1 + t_2)\pi^3$$

$$
\begin{aligned}
&+ 16\pi^2 t_1 (t_1 - t_2)^3 (t_1^3 - 5t_1^2 t_2 + 3t_1 t_2^2 - t_2^3) \frac{\partial}{\partial t_1} \\
&+ 16\pi^2 t_2 (t_1 - t_2)^3 (3t_1^3 - 3t_1^2 t_2 + 5t_1 t_2^2 - t_2^3) \frac{\partial}{\partial t_2} \\
&- 32\pi^2 t_1 t_2 (t_1 - t_2)(t_1^2 - t_1 t_2 + t_2^2) \frac{\partial^2}{\partial \theta^2} \\
&- 64\pi^2 (t_1 - t_2)^4 \big(t_1^4 \frac{\partial^2}{\partial t_1^2} + t_2^4 \frac{\partial^2}{\partial t_2^2} \big) \\
&- 8\pi (t_1 - t_2)^5 \big(t_1^2 \frac{\partial^2}{\partial t_1^2} - t_2^2 \frac{\partial^2}{\partial t_2^2} \big) \\
&+ 8 t_1 t_2 (t_1 - t_2)(5t_1^2 - 9t_1 t_2 + 5t_2^2) \big(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} \big) \\
&- 4 t_1 t_2 (t_1 - t_2)^2 \big(t_1 (10t_1 - 9t_2) \frac{\partial^2}{\partial t_1^2} + t_2 (9t_1 - 10t_2) \frac{\partial^2}{\partial t_2^2} \big) \\
&- 4 (t_1 - t_2)^3 \big(t_1^3 (t_1 - 5t_2) \frac{\partial^2}{\partial t_1^3} + t_2^3 (5t_1 - t_2) \frac{\partial^3}{\partial t_2^3} \big) \\
&+ 8 (t_1 - t_2)^4 \big(t_1^4 \frac{\partial^4}{\partial t_1^4} + t_2^4 \frac{\partial^4}{\partial t_2^4} \big) \\
&+ 8 t_1 t_2 (t_1 - t_2)^2 (5t_1^2 - 9t_1 t_2 + 5t_2^2) \frac{\partial^2}{\partial t_1 \partial t_2} \\
&- 12 t_1 t_2 (t_1 - t_2)^4 \big(t_1 \frac{\partial^3}{\partial t_1^2 \partial t_2} + t_2 \frac{\partial^3}{\partial t_1 \partial t_2^2} \big) \\
&- 4 t_1 t_2 (t_1 - t_2) \big(t_1 (9t_1 - 8t_2) \frac{\partial^3}{\partial \theta^2 \partial t_1} + t_2 (8t_1 - 9t_2) \frac{\partial^3}{\partial \theta^2 \partial t_2} \big) \\
&+ 8 t_1 t_2 (t_1 - t_2)^2 \big(t_1^2 \frac{\partial^4}{\partial \theta^2 \partial t_1^2} + t_1 t_2 \frac{\partial^4}{\partial \theta^2 \partial t_1 \partial t_2} + t_2^2 \frac{\partial^4}{\partial \theta^2 \partial t_2^2} \big) \\
&+ t_1^2 t_2^2 \frac{\partial^4}{\partial \theta^4} \Big) h(\theta, t_1, t_2) = d_2 h(\theta, t_1, t_2).
\end{aligned}
\tag{4.8}
$$

Being periodic with respect to θ, h has a Fourier expansion

$$h(\theta, t_1, t_2) = \sum_{n \in \mathbb{Z}} B_n(t_1, t_2) e^{2ni\theta}.$$

Applying Equation 4.7 and Equation 4.8 to each term of the sum above, we obtain that $B_n(t_1, t_2)$ satisfies

$$\begin{aligned} -\frac{1}{4}(t_1 - t_2)^{-2}\Big(&4(t_1 - t_2)^2(t_1^2 + t_2^2)\pi^2 + 2n^2 t_1 t_2 - (t_1 - t_2)^2\big(t_1^2 \frac{\partial^2}{\partial t_1^2} + t_2^2 \frac{\partial^2}{\partial t_2^2}\big) \\ &- t_1 t_2 (t_1 - t_2)\big(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2}\big)\Big) B_n = d_1 B_n \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} \frac{1}{8}(t_1 - t_2)^{-4}\Big(&8(t_1 - t_2)^4(t_1^4 + t_2^4)\pi^4 - 32(t_1 - t_2)^6(t_1 + t_2)\pi^3 + 16n^4 t_1^2 t_2^2 \\ &+ 4t_1 t_2\big(32\pi^2(t_1 - t_2)^2(t_1^2 - t_1 t_2 + t_2^2) - (56t_1^2 - 96t_1 t_2 + 56t_2^2)\big)n^2 \\ &+ 8t_1(t_1 - t_2)\big(2\pi^2(t_1 - t_2)^2(t_1^3 - 5t_1^2 t_2 + 3t_1 t_2^2 - t_2^3) + t_2(5t_1^2 - 9t_1 t_2 + 5t_2^2) \\ &+ 2n^2 t_1 t_2 (9t_1 - 8t_2)\big)\frac{\partial}{\partial t_1} \\ &+ 8t_2(t_1 - t_2)\big(2\pi^2(t_1 - t_2)^2(3t_1^3 - 3t_1^2 t_2 + 5t_1 t_2^2 - t_2^3) - t_1(5t_1^2 - 9t_1 t_2 + 5t_2^2) \\ &+ 2n^2 t_1 t_2 (8t_1 - 9t_2)\big)\frac{\partial}{\partial t_2} \\ &- 4t_1^2(t_1 - t_2)^2\big(8n^2 t_1 t_2 + 16\pi^2 t_1^2 (t_1 - t_2)^2 + 2\pi(t_1 - t_2)^3 + t_2(10t_1 - 9t_2)\big)\frac{\partial^2}{\partial t_1^2} \\ &- 4t_2^2(t_1 - t_2)^2\big(8n^2 t_1 t_2 + 16\pi^2 t_2^2 (t_1 - t_2)^2 - 2\pi(t_1 - t_2)^3 + t_1(9t_1 - 10t_2)\big)\frac{\partial^2}{\partial t_2^2} \end{aligned}$$

$$
\begin{aligned}
&- 4(t_1 - t_2)^3 \big(t_1^3(t_1 - 5t_2)\frac{\partial^2}{\partial t_1^3} + t_2^3(5t_1 - t_2)\frac{\partial^3}{\partial t_2^3}\big) \\
&+ 8(t_1 - t_2)^4 \big(t_1^4 \frac{\partial^4}{\partial t_1^4} + t_2^4 \frac{\partial^4}{\partial t_2^4}\big) \\
&- 8t_1t_2(t_1 - t_2)^2 (4n^2 t_1 t_2 - 5t_1^2 + 9t_1t_2 - 5t_2^2)\frac{\partial^2}{\partial t_1 \partial t_2} \\
&- 12t_1t_2(t_1 - t_2)^4 \big(t_1 \frac{\partial^3}{\partial t_1^2 \partial t_2} + t_2 \frac{\partial^3}{\partial t_1 \partial t_2^2}\big)\Big) B_n = d_2 B_n. \qquad (4.10)
\end{aligned}
$$

Now, since Equation 4.9 and Equation 4.10 are invariant under $n \to -n$, we can assume $n \geq 0$ in the following. Set $x = t_1 - t_2$ and $y = t_1 + t_2$ with $y > 0$ and $|x| < y$, and put

$$B_n(t_1, t_2) = C_n(x, y).$$

Then $t_1 = \frac{1}{2}(x + y)$, $t_2 = -\frac{1}{2}(x - y)$, and using the chain rule to write the partial derivatives of B_n with respect to t_1 and t_2 in terms of x and y, we obtain the following differential equations satisfied by $C_n(x, y)$:

$$
\begin{aligned}
(-4)\frac{1}{8x^2}\Big(&4\pi x^2(x^2 + y^2) + (y^2 - x^2)n^2 - x^2(x^2 + y^2)\frac{\partial^2}{\partial y^2} \\
&- x^2(x^2 + y^2)\frac{\partial^2}{\partial x^2} - x(y^2 - x^2)\frac{\partial}{\partial x} - 4x^3 y \frac{\partial^2}{\partial x \partial y}\Big) C_n = d_1 C_n, \qquad (4.11)
\end{aligned}
$$

$$\frac{1}{8x^4}\Big(52n^2x^4 + n^4x^4 - 24n^2\pi^2x^6 + \pi^4x^8 + 32\pi^3x^6y - 48n^2x^2y^2 - 2n^4x^2y^2$$
$$+ 16n^2\pi^4x^4y^2 + 6\pi^4x^6y^2 - 4n^2y^4 + n^4y^4 + 8n^2\pi^2x^2y^2 + \pi^4x^4y^4$$
$$+ (-36n^2x^4y + 16\pi^2x^6y + 36n^2x^2y^3)\frac{\partial}{\partial y}$$
$$+ (2n^2x^6 - 8\pi^2x^8 - 8\pi x^6y + 4n^2x^4y^2 - 48\pi^2x^6y^2 - 6n^2x^2y^4 - 8\pi^2x^4y^4)\frac{\partial^2}{\partial y^2}$$
$$- 4x^6y\frac{\partial^3}{\partial y^3} + (x^8 + 6x^2y^2 + x^4y^4)\big(\frac{\partial^4}{\partial y^4} + \frac{\partial^4}{\partial x^4}\big)$$
$$+ (-19x^5 - 34n^2x^5 + 24\pi^2x^7 + 18x^3y^2 + 32n^2x^3y^2 + xy^4 + 2n^2xy^4 - 8\pi^2x^3y^4)\frac{\partial}{\partial x}$$
$$+ (-8\pi x^7 + 20x^5y + 16n^2x^5y - 64\pi^2x^7y - 8\pi x^5y^2 - 20x^3y^3 - 16n^2x^3y^3 - 64\pi^2x^5y^3)\frac{\partial^2}{\partial x\partial y}$$
$$+ (-6x^7 - 12x^5y^2 + 6x^3y^4)\frac{\partial^3}{\partial x\partial y^2} + (16x^7y + 16x^5y^3)\big(\frac{\partial^4}{\partial x\partial y^3} + \frac{\partial^4}{\partial x^3\partial y}\big)$$
$$+ (19x^6 + 6n^2x^6 - 8\pi^2x^8 - 8\pi x^6y - 18x^4y^2 - 4n^2x^4y^2 - 48\pi^2x^6y^2 - x^2y^4$$
$$- 2n^2x^2y^4 - 8\pi^2x^4y^4)\frac{\partial^2}{\partial x^2} + (-24x^6y + 12x^4y^3)\frac{\partial^3}{\partial x^2\partial y}$$
$$+ (6x^8 + 36x^6y^2 + 6x^4y^4)\frac{\partial^4}{\partial x^2\partial y^2} + (-6x^7 + 2x^3y^4)\frac{\partial^3}{\partial x^3}\Big)C_n = d_2C_n. \tag{4.12}$$

Now, set $C_n(x,y) = (x^2 - y^2)\sum_{k=0}^{\infty} a_k(y)x^k$ where $a_k(y) = 0$ for any $k < m$, where m is a positive integer such that $a_m(y) \neq 0$. Then, Equation 4.11 implies

$$(-4)\frac{1}{8}\big((k^2 + 2k + n^2 - 4\pi^2y^2 - 2)a_k + ((k+2)^2 - n^2)y^2a_{k+2} - 4\pi^2a_{k-2}$$
$$+ 4(k+1)ya_k' + a_{k-2}'' + y^2a_k''\big) = d_1a_k. \tag{4.13}$$

When $k = m-2$, by virtue of the fact that m is a smallest positive integer such that $a_m(y) \neq 0$, we have that $a_k = a_{k-2} = a'_k = a'_k = a''_k = a''_{k-2} = 0$, and thus

$$((k+2)^2 - n^2)y^2 a_{k+2} = (m^2 - n^2)y^2 a_m = 0.$$

Since $a_m \neq 0$ and $y > 0$, we must have that $m = \pm n$, and since $n, m > 0$, $m = n$.

Lemma 1. *With m as above, if $k \not\equiv m \pmod 2$, then $a_k(y) = 0$.*

Proof. We use strong induction. For $k = m-1$, as above, we have that $a_k = a_{k-2} = a'_k = a'_k = a''_k = a''_{k-2} = 0$, and hence

$$((k+2)^2 - n^2)y^2 a_{k+2} = ((m+1)^2 - n^2)y^2 a_{m+1} = 0.$$

Since $y > 0$, if $a_{m+1} \neq 0$, we must have

$$(m+1)^2 - n^2 = m^2 + 2m + 1 - n^2 = 2m + 1 = 0.$$

But this is a contradiction since m is a positive integer. Thus, $a_{m+1} = 0$.

Now, suppose $k = m + j$, where j is a positive odd integer so that $k \not\equiv m \pmod 2$, and $a_k = a_{m+j} = 0$ and $a_i = 0$ for odd $i < k = m + j$. We show that $a_{m+j+2} = 0$. By assumption, $a_{m+j} = a_{m+j-2} = 0$ hence also their derivatives. So, setting $k = m + j$, Equation 4.13 becomes

$$((m + j + 2)^2 - n^2)y^2 a_{m+j+2} = 0.$$

Since $y > 0$, this implies that either $a_{m+j+2} = 0$ or $(m+j+2)^2 - n^2 = 0$. Suppose $a_{m+j+2} \neq 0$.Then $0 = m^2+2mj+4(m+j)+j^2+4-n^2 = 2mj+4(m+j)+j^2+4$, but m and j are positive, so this is a contradiction. Hence $a_{m+j+2} = 0$, and, by induction, we have the result. $\square$

Furthermore, Equation 4.12 yields the following differential equation for a_k:

$$
\begin{aligned}
&\frac{1}{8}\Big((48k - 12k^2 - 4k^3 + k^4 - 8n^2 - 16kn^2 + 6k^2n^2 + n^4 - 8k\pi y - 8k^2\pi y \\
&\quad + 16\pi^2y^2 - 48k\pi^2y^2 - 48k^2\pi^2y^2 + 16n^2\pi^2y^2 + \pi^4y^4)a_k \\
&\quad - 2\pi^2(4k^2 - 16k + 12n^2 - 8 - 16\pi y - 3\pi^2y^2)a_{k-2} + \pi^4 a_{k-4} \\
&\quad - 2(24 - 30k^2 - 18k^3 - 3k^4 - 10n^2 + 2kn^2 + 2k^2n^2 + n^4 - 4n^2\pi^2y^2 \\
&\quad + 16\pi^2y^2 + 16k\pi^2y^2 + 4k^2\pi^2y^2)y^2a_{k+2} \\
&\quad + ((k+2)^2 - n^2)((k+4)^2 - n^2)y^4a_{k+4} \\
&\quad + 4(4 - 17k + 4k^3 - 3n^2 + 4kn^2 - 2k\pi y - 8\pi^2y^2 - 16k\pi^2y^2)ya_k' \\
&\quad - 8\pi(k + 8k\pi y - 6\pi y)a_{k-2}' \\
&\quad + 4((k+2)^2 - n^2)(5 + 4k)y^3a_{k+2}' \\
&\quad + 4(-3 + 6k + 9k^2 + n^2 - 2\pi^2y^2)y^2a_k'' \\
&\quad + 2(-6 - 6k + 3k^2 + n^2 - 4\pi y - 24\pi^2y^2)a_{k-2}'' \\
&\quad - 8\pi^2a_{k-4}'' + 6((k+2)^2 - n^2)y^4a_{k+2}'' \\
&\quad + 4(4k - 3)ya_{k-2}^{(3)} + 8(2k+1)y^3a_k^{(3)} \\
&\quad + y^4a_k^{(4)} + 6y^2a_{k-2}^{(4)} + a_{k-4}^{(4)}\Big) = d_2a_k
\end{aligned}
\tag{4.14}
$$

for all $k \in \mathbb{Z}$.

Next, we derive a single ordinary differential equation for a_n. We set $k = m = n$ in Equation 4.13 and solve for a_{n+2}, obtaining,

$$a_{n+2} = -\frac{1}{4(n+1)y^2}\big(2(d_1 + n^2 + n - 2\pi^2 y^2 - 1)a_n + 4(n+1)ya_n' + y^2 a_n''\big).$$

Similarly, setting $k = m + 2 = n + 2$ in Equation 4.13, we find

$$\begin{aligned} a_{n+4} = -\frac{1}{8(n+2)y^2}\big(&2(d_1 + n^2 + 3n - 2\pi^2 y^2 + 2)a_{n+2} + 4(n+3)a_{n+2}' \\ &+ y^2 a_{n+2}'' + +a_n'' - 4\pi^2 a_n\big) \end{aligned}$$

are easily computed, where $a_k^{(j)}$ denotes the jth derivative of a_k. Then substituting these values, the Differential Equation 4.14 yields the following fourth order linear ordinary differential equation for a_n:

$$\begin{aligned} -((8n^2 &- 40n - 4)d_1 - 4d_1^2 + 8d_2 + 8 - 20n - 36n^2 - 4n^3 + 4n^4 \\ &+ 8n\pi y + 8n^2\pi y + 48\pi^2 y^2 + 15\pi^4 y^4)a_n(y) \\ &- 4(4(n-2)d_1 - 14 - 17n + 3n^2 + 4n^3 + 2n\pi y - 4\pi^2 y^2)ya_n'(y) \\ &- 4(2d_1 - 5 + 9n + 6n^2 - 4\pi^2 y^2)y^2 a_n''(y) \\ &- 4(4n+5)y^3 a_n^{(3)}(y) - 4y^4 a_n^{(4)}(y) = 0. \end{aligned} \tag{4.15}$$

4.4 Solutions of the ordinary differential equation

In this section, we discuss the method of Niwa to obtain solutions to his ordinary differential equation [23, Equation (1.12)]. As was explained earlier, in [23], Niwa used the differential operators Δ_1 and Δ_2 given by Nakajima [22] and obtained an ordinary differential equation, analogous to our Equation 4.15, satisfied by a_n. Since the quartic operator of Nakajima is not invariant under the necessary slash operator, Niwa's ordinary differential equation is inaccurate. However, his method for obtaining the first of the four solutions utilizes basic properties of the Whittaker functions and the differential operators, rather than the ordinary differential equation itself, we conjecture that this first solution also satisfies our ordinary differential equation.

Conjecture 1. *The solution*

$$C_n(y) = \int_1^\infty \int_1^\infty P_{v_1}^n(z_1) P_{v_2}^n(z_2)(z_1^2-1)^{n/2}(z_2^2-1)^{n/2} e^{-2\pi z_1 z_2 y} dz_1 dz_2$$

given by Niwa [23, Proposition 1] is a solution to Equation 4.15, after a suitable change of parameters. Furthermore, it is the only solution of rapid decay to this differential equation.

First, we sketch Niwa's proof of the solutions to the ordinary differential equation [23, Equation (1.12)]. Niwa introduces parameters λ_1 and λ_2 to describe these solutions, yielding another ordinary differential equation [23, Equation (1.14)]. His choice of parameters is

$$d_1 = \frac{\lambda_1 + \lambda_2 - 2}{8}, \qquad d_2 = \frac{(\lambda_1 - \lambda_2)^2}{256} - \frac{\lambda_1 + \lambda_2}{32} + \frac{3}{64}. \tag{4.16}$$

For $v \in \mathbb{C}$ and $m \in \mathbb{Z}$, $P_v^m(z)$ and $Q_v^m(z)$ denote the associated Legendre functions of the first and second kind, respectively. That is, they are independent solutions of Legendre's differential equation

$$\frac{d}{dz}((1-z^2)\frac{d}{dz})u(z) + v(v+1)u(z) - \frac{m^2}{1-z^2}u(z) = 0. \tag{4.17}$$

Set

$$\begin{aligned} c_{11} &= -Q_{v_2}^0(0) \qquad c_{12} = P_{v_2}^0(0) \\ c_{21} &= -\tfrac{d}{dz}Q_{v_2}^0(0) \quad c_{22} = \tfrac{d}{dz}Q_{v_2}^0(0). \end{aligned}$$

Then $c_{12} \neq 0$, $c_{22} \neq 0$ for $-1 < Re(v_2) < 0$. Set ([23, Equation (1.19)])

$$\begin{aligned} R_{v_2}^0(z) &= c_{11}P_{v_2}^0(z) + c_{12}Q_{v_2}^0(z) \\ S_{v_2}^0(z) &= c_{21}P_{v_2}^0(z) + c_{22}Q_{v_2}^0(z). \end{aligned} \tag{4.18}$$

Then [23, Proposition 1] describes the solutions to Niwa's ordinary differential equation:

Proposition 2. *Put* $\nu_i = \frac{-1+\sqrt{1+4\lambda_i}}{2}$ *for* $i = 1, 2$, *and assume that* $-1 < Re(\nu_i) < 0$ *and that* λ_1, λ_2 *are not integers. Then there exist polynomials* h_1, h_2 *in* y^{-1} *of degree* $n-1$ *such that*

$$\begin{aligned} A_n(y) &= \int_0^\infty \left(\int_1^\infty R_{\nu_1}^0(z_1)P_{\nu_2}^n(z_2)(-2\pi y z_2)^{-n}(z_2^2-1)^{n/2}e^{-2\pi z_1 z_2 y}dz_2 \right) dz_1 - h_1, \\ B_n(y) &= \int_0^\infty \left(\int_1^\infty S_{\nu_1}^0(z_1)P_{\nu_2}^n(z_2)(-2\pi y z_2)^{-n}(z_2^2-1)^{n/2}e^{-2\pi z_1 z_2 y}dz_2 \right) dz_1 - h_2, \\ C_n(y) &= \int_1^\infty \int_1^\infty P_{\nu_1}^n(z_1)P_{\nu_2}^n(z_2)(z_1^2-1)^{n/2}(z_2^2-1)^{n/2}e^{-2\pi z_1 z_2 y}dz_1 dz_2 \end{aligned}$$

are linearly independent solutions of the ordinary differential equation. If λ_1 and λ_2 are furthermore real, then there exists a polynomial h_3 in y^{-1} of degree n such that the three functions above, together with

$$D_n(y) = \int_0^1 \left(\int_{-1}^1 P_{\nu_1}^n(z_1) P_{\nu_2}^n(-z_2)(z_1^2-1)^{n/2}(z_2^2-1)^{n/2} e^{-2\pi z_1 z_2 y} dz_2 \right) dz_1 \tag{4.19}$$

$$+ i^{-n} \frac{\Gamma(\nu_1+n+1)}{\Gamma(\nu_1-n+1)} \int_0^1 \left(\int_1^\infty P_{\nu_1}^0(z_1) P_{\nu_2}^n(-z_2)(-2\pi y z_2)^{-n} \right. \tag{4.20}$$

$$\left. \times (z_2^2-1)^{n/2} e^{-2\pi z_1 z_2 y} dz_2 \right) dz_1 + h_3 \tag{4.21}$$

generate all the solutions of the ordinary differential equation

By direct computation, Niwa shows the existence of h_1, h_2, and h_3 such that these are solutions. We can see by inspection that $yA_n(y)$ and $yB_n(y)$ are bounded, that $C_n(y)$ decays rapidly, and that $D_n(y)$ grows rapidly as $y \to \infty$. Furthermore, we see A_n and B_n have different asymptotic expansions by considering

$$\int_0^\epsilon \left(\int_1^\infty R_{\nu_1}^0(z_1) P_{\nu_2}^n(z_2)(-2\pi y z_2)^{-n}(z_2^2-1)^{n/2} e^{-2\pi z_1 z_2 y} dz_2 \right) dz_1,$$

$$\int_0^\epsilon \left(\int_1^\infty S_{\nu_1}^0(z_1) P_{\nu_2}^n(z_2)(-2\pi y z_2)^{-n}(z_2^2-1)^{n/2} e^{-2\pi z_1 z_2 y} dz_2 \right) dz_1$$

for small ϵ, so the functions A_n, B_n, C_n, D_n are linearly independent.

Using the recurrence relation [23, Equation (1.9.1)] and partial integration with Legendre's differential equation, Niwa obtains (Proposition 2) that the functions

$$\begin{aligned} C_n(x,y) =&(x^2-y^2)\int_1^\infty \int_1^\infty P_{\nu_1}^n(z_1)P_{\nu_2}^n(z_2) \\ &\times J_n(2\pi i(z_1^2-1)^{1/2}(z_2^2-1)^{1/2}x)e^{-2\pi z_1 z_2 y}dz_1dz_2 \end{aligned} \tag{4.22}$$

are solutions to his Equations (1.8.1) and (1.8.2), where J_n denotes the Bessel function of the first kind. Note that our Equation 4.11 and Equation 4.12 correspond to Niwa's equations (1.8.1) and (1.8.2).

Finally, Niwa ([23, Theorem 1]) proves

Theorem 6. *Put* $\nu_i = \frac{-1+\sqrt{1+4\lambda_i}}{2}$ *for* $i = 1,2$ *and assume* $-1 < Re(\nu_i) < 0$ *and that* $\lambda_1, \lambda_2 \in \mathbb{R}\setminus\mathbb{Z}$.

Let $f(X+iY) = g(Y)e^{2\pi i tr(X)}$ *be a generalized Whittaker function (i.e. it satisfies* $\Delta_i F = d_i F$ *for* $i = 1,2$ *with* d_1, d_2 *as in Equation 4.16). Assume that* $g(Y)$ *is a real analytic function of* $t_1 - t_2$ *with*

$$Y = \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix} = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

and that for all positive integers m, a, b, c, *the convergence*

$$(tr(Y))^m \frac{\partial^{a+b+c}}{\partial y_1^a \partial y_2^b \partial y_3^c} g(Y) \to 0$$

holds uniformly on compact sets of $y_1 - y_3$ *and* y_2 *when* $tr(Y) \to 0$. *Then* $g(Y)$ *is expanded as*

$$g(Y) = \sum_{n \in \mathbb{Z}} b_n C_n(t_1 + t_2, t_1 - t_2) e^{2ni\theta}$$

where $b_n \in \mathbb{C}$ *and* $C_n(x, y)$ *is the function defined above.*

Now, we describe Niwa's derivation of the solution $C_n(y)$. He considers a Siegel modular form lifted from a Maass wave cusp form by theta correspondence ([23, Section 2]). He then constructs the other solutions by considering integral representations similar to that for the first solution.

Niwa [23, Section 2] defines a theta function, involving a Dirichlet character χ modulo N an odd squarefree integer,

$$\theta(Z, z_1, z_2)$$

where $Z \in \mathbb{H}_2, z_1, z_2 \in \mathbb{H}_1$. In Theorem 2, p. 181, he derives a system of differential equations satisfied by $\theta(Z, z_1, z_2)$, when acted on by the differential operators Δ_1 and Δ_2. Now, let φ_1 and φ_2 be Maass wave cusp forms with character χ; so φ_i satisfies

$$y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \varphi_i(z) = \lambda_i \varphi_i(z)$$

for $i = 1, 2$. Setting $d_0 z = y^{-2} dx dy$ for $z = x + iy$, define ([23, Equation (2.6)])

$$F_{\varphi_1, \varphi_2}(Z) = \int_{\Gamma \backslash \mathbb{H}_1} \int_{\Gamma \backslash \mathbb{H}_1} \theta(Z, z_1, z_2) \varphi_1(z_1) \varphi_2(z_2) d_0 z_1 d_0 z_2 \tag{4.23}$$

where $\Gamma = \Gamma_0(N)$. Then Niwa asserts that

$$\Delta_1 F_{\varphi_1,\varphi_2}(Z) = d_1 F_{\varphi_1,\varphi_2}(Z), \qquad \Delta_1 F_{\varphi_1,\varphi_2}(Z) = d_1 F_{\varphi_1,\varphi_2}(Z), \tag{4.24}$$

with d_1, d_2 defined by Equation 4.16 for λ_1, λ_2, and further that

$$F_{\varphi_1,\varphi_2}(\sigma Z) = \chi(d) F_{\varphi_1,\varphi_2}(Z)$$

holds for σ in the set

$$\sigma \in \left\{ \sigma = \left(\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \right) \middle| \begin{array}{l} \sigma \in \mathrm{Sp}(2,\mathbb{Q}), \\ a_{21}, a_{31}, a_{32}, a_{41}, a_{42} \in N\mathbb{Z}, \\ N a_{13} \in \mathbb{Z},\ \text{other } a_{ij} \in \mathbb{Z} \end{array} \right\}$$

and with lower right block $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then there is a lattice $\mathcal{T} \subset \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) | a, b, c \in \mathbb{Z} \right\}$ containing $\left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right)$ such that we can expand

$$F_{\varphi_1,\varphi_2}(Z) = \sum_{T \in \mathcal{T}} A(T, Y) e^{2\pi i \operatorname{tr}(TX)}. \tag{4.25}$$

Then by Equation 4.24, the function

$$W(Y) = A\left(\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \sqrt{N}^{-1} & 0 \\ 0 & 1 \end{pmatrix} Y \begin{pmatrix} \sqrt{N}^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)$$

is a generalized Whittaker function. Niwa then claims that direct calculation shows

$$W\left(\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\right) = \sum_{n\in\mathbb{Z}} b_n C_n(t_1 - t_2, t_1 + t_2) e^{ni\theta}$$

where C_n is as in Equation 4.22.

4.5 Conclusion

From this analysis, we expect that there is a unique solution to Equation 4.15 that decays rapidly, and that, in fact, Niwa's solution $C_n(Y)$ will satisfy our ordinary differential equation. Then Niwa's Theorem 1 gives the expansion for our $g(Y)$. Observe that the first term in the Fourier expansion of a Siegel Maass wave form is a generalized Whittaker function. Hence, for $F(Z) = \sum_{N\in\mathfrak{N}_2} A(N,Y)e^{2\pi i \mathrm{tr}(NX)}$, we expect

$$A(I,Y) = \sum_{n\in\mathbb{Z}} a_{I,n}(F) C_n(x,y) e^{2ni\theta}.$$

Then by Hori [13, p. 202] and uniqueness of Fourier expansion, we can write, for N definite,

$$a(N,Y) = \sum_{n\in\mathbb{Z}} a_{N,n}(F) W_{N,n}(Y), \quad a_{N,n}(F) \in \mathbb{C}$$

where

$$W_N(n, Y) = C_n(N^{1/2} Y N^{1/2}) \qquad \text{if } N > 0$$

$$W_N(n, Y) = C_n((-N)^{1/2} Y (-N)^{1/2}) \qquad \text{if } N < 0. \tag{4.26}$$

CITED LITERATURE

1. A.N. Andrianov, Euler products that correspond to Siegel's modular forms of genus 2, Uspehi Mat. Nauk **29**, no. 3 (177) (1974) 43-110.

2. A.N. Andrianov, Modular descent and the Saito-Kurokawa conjecture. Inv. Math **53** (1979) 267-280.

3. A. Borel, Introduction to automorphic forms, in: Algebraic groups and discontinuous subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965). Amer. Math. Soc. (1966) 199-210.

4. A. Borel, H. Jacquet, Automorphic forms and automorphic representations. Proc. of symp. in pure math.. **33-1** (1979) 189-202.

5. K. Bringmann, M. Raum, O. Richter, Harmonic Maass-Jacobi forms with singularities and a theta-like decomposition. (2012), *preprint*.

6. K. Bringmann, M. Raum, O. Richter, Kohnen's limit process for real-analytic Siegel modular forms. Adv. Math. **231** 2 (2012) 1100-1118.

7. K. Bringmann, O. Richter, Zagier-type dualities and lifting maps for harmonic Maass-Jacobi forms. Adv. Math **225** (2010) 2298-2315.

8. J. Brunier, G. van der Geer, G. Harder, D. Zagier, The 1-2-3 of Modular Forms. ed. K. Ranestad. In: Universitext, Springer-Verlag, Berlin, 2008.

9. W. Duke, Ö. Imamoḡlu, A converse theorem and the Saito-Kurokawa lift. International Math Research Notices **7** (1996) 347-355.

10. M. Eichler, D. Zagier, The Theory of Jacobi Forms. Birkhäuser, Boston, 1985.

11. E. Freitag, Siegelsche Modulfunktionen, in Grundlehren der mathematischen Wissenchaften **254**. Springer-Verlag, Berlin, 1983.

12. Harish-Chandra, Automorphic Forms on Semisimple Lie Groups, in Lecture Notes in Mathematics **62**. Springer-Verlag, Berlin/Heidelberg/New York (1968).

13. A.Hori, Andrianov's L-functions associated to Siegel wave forms of degree two. Math. Ann. **303** (1995) 195-226.

14. K. Imai, Generalization of Hecke's correspondence to Siegel modular forms. Am. J. Math. **102** 5 (1980) 903-936.

15. S. Katok, P. Sarnak, Heegner points, cycles and Maass forms. Israel J. Math **84** (1993) 193-227.

16. H. Klingen, Introductory lectures on Siegel modular forms. Cambridge University Press, Cambridge, 1990.

17. Langlands, Robert. P. Euler Products. Yale University Press, New Have, Conn. (1971).

18. H. Maass, Siegel's modular forms and Dirichlet series, in Lecture Notes in Mathematics **216**. Springer-Verlag, Berlin/Heidelberg/New York (1971).

19. H. Maass, Über eine Spezialschar von Modulformen zweiten Grades. Inv. Math. **53** (1979) 95-104.

20. H. Maass, Über eine Spezialschar von Modulformen zweiten Grades, II. Inv. Math. **53** (1979) 249-253.

21. H. Maass, Über eine Spezialschar von Modulformen zweiten Grades, III. Inv. Math. **53** (1979) 255-265.

22. S. Nakajima, On invariant differential operators on bounded symmetric domains of type IV. Proc. Japan Acad. **58**, Ser. A. (1982) 235-238.

23. S. Niwa, On generalized Whittaker functions of Siegel's upper half space of degree 2. Nagoya Math J. **121** (1991) 171-184.

24. I.I. Piatetski-Shapiro, Automorphic functions and the geometry of classical domains. Gordon and Breach, New York/ London/ Paris (1969).

25. A. Pitale, Jacobi Maass forms. Abh. Math. Semin. Univ. Hambg. **79** (2009) 87-111.

26. G. Shimura. On modular correspondences for $Sp(n, \mathbb{Z})$ and their congruence relations. Proc. Nat. Acad. Sci. U.S.A. **49** (1963) 824-828.

27. G. Shimura. Modular forms of half integral weight, in Modular Functions of One Variable, I. Lecture Notes No. 320, Springer-Verlag, Berlin/Heidelberg/New York (1973) 57-74.

28. G. Shimura. On modular forms of half integral weight. Ann. Math. **97** (1973) 440-481.

29. N.-P. Skoruppa, Developments in the theory of Jacobi forms. Acad. Sci. USSR, Inst. Appl. Math., Khabarovsk (1990) 167-185.

30. D. Zagier, Sur la conjecture de Saito-Kurokawa (d'après H. Maass), Séminaire Delange-Pisot-Poitou 1979-1980, in Prog. Math **12**. Birkhäuser-Verlag, Boston/Basel/Stuttgart (1980) 371-394.

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