On the Cohomology of the Classifying Spaces of Projective Unitary Groups and Applications

by

Xing Gu
B.A. (Nankai University) 2010
M.S. (Nankai University) 2013

Thesis submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Chicago, 2017

Chicago, Illinois

Defense Committee:
Brooke Shipley, Chair and Advisor
Benjamin Antieau, Co-advisor
Aldridge Bousfield
Henri Gillet
Ben Williams, Mathematics Department, University of British Columbia

To my beloved mother and father, Li Zhao and Zhixue Gu, to whom I owe every single success.

谨以此文献给我敬爱的母亲赵俐女士和父亲古志学先生。

ACKNOWLEDGMENTS

The author would like to express his deepest gratitude to his co-advisor Benjamin Antieau who proposed this project, pointed the author to the right references, and spent a large amount of time discussing details with the author, to his advisor Brooke Shipley for her constant encouragement, financial support and helpful advice, and to Ben Williams, who inspired the author by presenting him notes on a joint project with Benjamin Antieau, which is a invaluable inspiration. In addition, the three persons mentioned above have read several earlier versions of this paper and shared their valuable opinions with the author, which resulted in several improvements on this paper.

The author would also like to thank Masaki Kameko for his hospitality during the author's short visit to Saitama, Japan, and for a helpful conversation in which he presented many ideas related to the computation of the cohomology of $\mathbf{B}PU_n$.

Finally, the author would like to express his deepest gratitude to Aldridge Bousfield for examining several earlier versions of this thesis, and discussed its content with the author extensively, during the course of which he pointed out several problems and proposed solutions to them.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

SUMMARY

This thesis is a combination of two papers, (20) and (21), with the same names and almost same contents as Chapter 1 and Chapter 2, respectively. Non-essential changes are made for consistency.

In Chapter 1 we determine the integral cohomology ring $H^*(\mathbf{B}PU_n;\mathbb{Z})$ through degree 10, where PU_n is the projective unitary group of degree n, and $\mathbf{B}PU_n$ is its classifying space. Serre spectral sequences of various fiber sequences play an important role.

Chapter 2 concerns the topological period-index problem, an analog of the period-index problems in algebra and algebraic-geometry, which have been around since 1930's. In particular, a period-index conjecture of Colliot-Thélène was proposed in 1999 ((15)), of which most of the interesting part remains open. In this chapter we use classical tools in homotopy theory, including Postnikov tower, Serre spectral sequence, and Eilenberg-Moore spectral sequence to extend a theorem by Antieau and Williams, shedding light on the correct form of the topological version of the period-index conjecture.

CHAPTER 0

INTRODUCTION

In this introductory chapter we explain relevant concepts and the necessary background knowledge, and outline the structure of this thesis.

Chapter 1 concerns the integral cohomology ring $H^*(\mathbf{B}PU_n;\mathbb{Z})$, where PU_n is the projective unitary group of degree n and $\mathbf{B}PU_n$ is its classifying space. An easy computation shows that the rational cohomology ring of $\mathbf{B}PU_n$ is canonically isomorphic to that of $\mathbf{B}SU_n$, the classifying space of the special unitary group of degree n. However the torsion subgroup of $H^*(\mathbf{B}PU_n;\mathbb{Z})$ remains surprisingly elusive. In (1), Antieau and Williams calculated $H^*(\mathbf{B}PU_n;\mathbb{Z})$ through dimension 5. The cohomology of $\mathbf{B}PU_n$ is also considered by Kameko and Yagita ((23), (24)), as well as Kono and Mimura ((25)). In all the cases that they considered, n is either a prime p or of the form $4k+2$ for some integer k, and the cohomology theories considered reveal little about the integral cohomology. In Chapter 1 we compute the ring $H^*(\mathbf{B}PU_n;\mathbb{Z})$ through degree 10. The main theorem of this chapter is the following:

Theorem 0.0.1. *For an integer $n>1$, the graded ring $H^*(\mathbf{B}PU_n;\mathbb{Z})$, in degrees ≤ 10, is isomorphic to the following graded ring:*

$$\mathbb{Z}[e_2,\cdots,e_{j_n},x_1,y_{3,0},y_{2,1},z_1,z_2]/I_n.$$

Here e_i is of degree $2i$, $j_n = min\{5, n\}$; the degrees of $x_1, y_{3,0}, y_{2,1}$ are $3, 8, 10$, respectively; and the degrees of z_1, z_2 are $9, 10$, respectively. I_n is the ideal generated by

$$nx_1, \quad \epsilon_2(n)x_1^2, \quad \epsilon_3(n)y_{3,0}, \quad \epsilon_2(n)y_{2,1}, \quad \epsilon_3(n)z_1, \quad \epsilon_3(n)z_2,$$

$$\delta(n)e_2x_1, \quad (\delta(n)-1)(y_{2,1} - e_2x_1^2), \quad e_3x_1,$$

where

$$\delta(n) = \begin{cases} 2, & \textit{if } n = 4l+2 \textit{ for some integer } l, \\ 1, & \textit{otherwise.} \end{cases}$$

and

$$\epsilon_p(n) = \gcd(p, n).$$

Chapter 2 is a succession of (1) and (2), in which Antieau and Williams initiated the study of the topological period-index problem. Given a topological space X, let $\mathrm{Br}(X)$ be the topological Brauer group defined in (2), whose underlying set is the Azumaya algebras modulo the Brauer equivalence: $\mathcal{A}_0$ and $\mathcal{A}_1$ are called Brauer equivalent if there are vector bundles $\mathcal{E}_0$ and $\mathcal{E}_1$ such that

$$\mathcal{A}_0 \otimes \mathcal{E}_0 \cong \mathcal{A}_1 \otimes \mathcal{E}_1.$$

The multiplication is given by taking tensor product.

Azumaya algebras over X of degree r are classified by the collection of PU_r-torsors over X, i.e., the cohomology set $H^1(X; PU_r)$, where PU_r is the projective unitary group of degree r. Consider the short exact sequences of Lie groups

$$1 \to S^1 \to U_r \to PU_r \to 1 \tag{0.0.1}$$

and

$$0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{\exp} S^1 \to 1 \tag{0.0.2}$$

where the arrow $S^1 \to U_n$ is the inclusion of scalars. Then the composition of Bockstein homomorphisms

$$H^1(X; PU_r) \to H^2(X; S^1) \to H^3(X; \mathbb{Z}) \tag{0.0.3}$$

associates an Azumaya algebra $\mathcal{A}$ to a class $\alpha \in H^3(X; \mathbb{Z})$. The exactness of the sequences above implies that

1. $\alpha \in H^3(X; \mathbb{Z})_{\mathrm{tor}}$, the subgroup of torsion elements of $H^3(X; \mathbb{Z})$, and
2. the class α only depends on the Brauer equivalence class of $\mathcal{A}$.

Therefore, we established a function $H^3(X; \mathbb{Z})_{\mathrm{tor}} \to \mathrm{Br}(X)$. It is not hard to show ((19)) that this function is in fact an inclusion of subgroup. For this reason, $H^3(X; \mathbb{Z})_{\mathrm{tor}}$ is also called the cohomological Brauer group of X, and is sometimes denoted by $\mathrm{Br}'(X)$.

Serre showed ((19)) that when X is a finite CW complex, the inclusion is also surjective. Hence, for any $\alpha \in H^3(X; \mathbb{Z})_{\mathrm{tor}}$, there is some r such that a PU_r-torsor over X is associated to

α via the homomorphism (Equation 1.6.2). Let $\text{per}(\alpha)$ denote the order of α as an element of the group $H^3(X;\mathbb{Z})$, then Serre also showed ((19)) that $\text{per}(\alpha)|r$, for all r such that there is a PU_r-torsor over X associated to α in the way described above. Let $\text{ind}(\alpha)$ denote the greatest common divisor of all such r, then in particular we have

$$\text{per}(\alpha) \mid \text{ind}(\alpha). \tag{0.0.4}$$

Furthermore, Antieau and Williams showed ((2)) the following

Proposition 0.0.2. The integers $\text{per}(\alpha)$ and $\text{ind}(\alpha)$ have the same set of prime divisors when X is a finite CW complex.

Therefore, for a sufficiently large integer e we have

$$\text{ind}(\alpha) \mid \text{per}(\alpha)^e. \tag{0.0.5}$$

The topological period-index problem can be stated as follows:

For a given class $\mathcal{C}$ of finite CW complexes, find the sharp lower bound of e such that Equation 0.0.5 holds for all finite CW complex X in C and all elements in $Br(X)$.

The topological period-index problem is motivated by its analog in algebraic geometry, where the topological Brauer group of a CW complex is replaced by the usual Brauer group of a scheme, and where the period and index are defined similarly. We have the following

Conjecture 0.0.3 (Colliot-Thélène)**.** Let k be either a C_d-field or the function field of a d-dimensional variety over an algebraically closed field. Let $\alpha \in \mathrm{Br}(k)$, and suppose that $\mathrm{per}(\alpha)$ is prime to the characteristic of k. Then

$$\mathrm{ind}(\alpha) | \mathrm{per}(\alpha)^{d-1}.$$

The conjecture has been proved in a few low dimensional cases, which are summarized in (2). Very little is know in high dimensions.

There is an obvious topological analog of Conjecture 0.0.3, which is proposed by Antieau and Williams in (1) and referred to as "straw man", or the topological period-index conjecture:

Conjecture 0.0.4 (Antieau-Williams)**.** If X is a $2d$-dimensional finite CW complex, and $\alpha \in \mathrm{Br}(X)$, then

$$\mathrm{ind}(\alpha) | \mathrm{per}(\alpha)^{d-1}.$$

Notice that, if X is a complex algebraic variety of dimension d, then its underlying topological space has a $2d$-dimensional cell decomposition, hence the $2d$ in the conjecture.

Antieau and Williams disproved Conjecture 0.0.4 in (1). To state their results in consistency with this thesis, we denote by $\epsilon_p(n)$ the greatest common divisor of p and n. Typically p will be a prime number. The notations in the following theorem are altered accordingly.

Theorem 0.0.5 (Antieau-Williams,(1))**.** *Let n be a positive integer. There exists a connected finite CW complex X of dimension 6 equipped with a class $\alpha \in \mathrm{Br}(X)$ for which* $\mathrm{per}(\alpha) = n$ *and* $\mathrm{ind}(\alpha) = \epsilon_2(n)n^2$.

Furthermore, they established an upper bound of the index of an element of the Brauer group in terms of its period:

Theorem 0.0.6 (Antieau-Williams, (3))**.** *Let X be a finite $2d$-dimensional CW complex, and let $\alpha \in \mathrm{Br}'(X)$ have period $p_1^{r_1} \cdots p_k^{r_k}$. Then,*

$$\mathrm{ind}(\alpha) | m^{d-1} \prod_{i=1}^{k} p_i^{v_{p_i}((d-1)!)},$$

where v_{p_i} is the p_i-adic evaluation.

Furthermore, they made the following

Conjecture 0.0.7. The upper bound established in Theorem 0.0.6 is sharp.

In this thesis, we show that the topological period-index conjecture fails again for 8-dimensional CW complexes. The main result is the following

Theorem 0.0.8. *Let X be a topological space of homotopy type of an 8-dimensional CW-complex, and let $\alpha \in H^3(X;\mathbb{Z})_{\mathrm{tor}}$ be a topological Brauer class of period n. Then*

$$\mathrm{ind}(\alpha) | \epsilon_2(n)\epsilon_3(n)n^3. \tag{0.0.6}$$

In addition, if X is the 8-th skeleton of $K(\mathbb{Z}/n, 2)$, and α is the restriction of the fundamental class $\beta_n \in H^3(K(\mathbb{Z}/n, 2), \mathbb{Z})$, then

$$\begin{cases} \operatorname{ind}(\alpha) = \epsilon_2(n)\epsilon_3(n)n^3, & 4 \nmid n, \\ \epsilon_3(n)n^3 | \operatorname{ind}(\alpha), & 4|n. \end{cases}$$

In particular, the sharp lower bound of e such that $\operatorname{ind}(\alpha)|n^e$ for all X and α is 4.

The theorem implies that Conjecture 0.0.7 is true for 8-complexes. In particular, it shows that the topological version of the period-index conjecture fails in degree 8, as it does in degree 6.

CHAPTER 1

ON THE COHOMOLOGY OF THE CLASSIFYING SPACES OF PROJECTIVE UNITARY GROUPS

1.1 Introduction to Chapter 1

Let U_n be the unitary group of order n, and consider the unit circle group S^1 of complex numbers as the normal subgroup of scalars of U_n. The quotient group, denoted hereafter by PU_n, is called the projective unitary group of order n. Its classifying space $\mathbf{B}PU_n$ is a topological space determined by PU_n up to homotopy type, with a canonical base point, characterised by the fact that for a well behaved topological space X with a base point, the set of pointed homotopy classes of maps, $[X, \mathbf{B}PU_n]$, has a natural one-to-one correspondence with the isomorphism classes of PU_n bundles, also known as topological Azumaya algebras of degree n, over X.

The space $\mathbf{B}PU_n$ is related to a number of interesting questions. For example, in the study of the topological period-index problem we need to consider classes of Azumaya algebras of various degrees, which are in some sense related to a certain torsion class in $H^3(X;\mathbb{Z})$. For more details see (1). The cohomology of $\mathbf{B}PU_n$ is a useful tool to detect non-trivial elements in the Chow ring of $\mathbf{B}GL_n$, which has been drawing attention in recent years. More on that can be found in (32). Furthermore, the accessibility and non-triviality of the cohomology of $\mathbf{B}PU_n$ make it possible to shed light on the understanding of cohomology of classifying space of exceptional Lie groups. The author owes this idea to M. Kameko. More about it can be found in (23).

The integral cohomology $H^k(\mathbf{B}PU_n;\mathbb{Z})$, was known for $k = 0, \dots, 5$, of which section 3 of (1) is a good reference. More results on $H^k(\mathbf{B}PU_n;\mathbb{Z}/p)$ and their applications are discussed in, for example, (32). In this chapter we consider a version of the Leray-Serre spectral sequence converging to $H^*(\mathbf{B}PU_n;\mathbb{Z})$, of which many differentials can be found fairly easily. We use this spectral sequence to compute $H^k(\mathbf{B}PU_n;\mathbb{Z})$ for $k \leq 10$, the ring structure of $H^*(\mathbf{B}PU_n;\mathbb{Z})$ in this range, and some cohomological operations. The torsion free components can be easily described in terms of Chern classes, via the quotient map $P : \mathbf{B}U_n \to \mathbf{B}PU_n$, which are discussed in details in Remark 1.6.2. For readers' convenience we restate Theorem 0.0.1 as follows:

Theorem 0.0.1. *For an integer $n > 1$, the graded ring $H^*(\mathbf{B}PU_n;\mathbb{Z})$, in degrees ≤ 10, is isomorphic to the following graded ring:*

$$\mathbb{Z}[e_2, \cdots, e_{j_n}, x_1, y_{3,0}, y_{2,1}, z_1, z_2]/I_n.$$

Here e_i is of degree $2i$, $j_n = min\{5, n\}$; the degrees of $x_1, y_{3,0}, y_{2,1}$ are $3, 8, 10$, respectively; and the degrees of z_1, z_2 are $9, 10$, respectively. I_n is the ideal generated by

$$nx_1, \quad \epsilon_2(n)x_1^2, \quad \epsilon_3(n)y_{3,0}, \quad \epsilon_2(n)y_{2,1}, \quad \epsilon_3(n)z_1, \quad \epsilon_3(n)z_2,$$

$$\delta(n)e_2x_1, \quad (\delta(n)-1)(y_{2,1} - e_2x_1^2), \quad e_3x_1,$$

where

$$\delta(n) = \begin{cases} 2, & \textit{if } n = 4l+2 \textit{ for some integer } l, \\ 1, & \textit{otherwise.} \end{cases}$$

and

$$\epsilon_p(n) = \gcd(p, n).$$

For easier reference, especially in the proof, we break down the statement above into the following assertions:

1. $H^k(\mathbf{B}PU_n;\mathbb{Z}) = 0$ *for* $k = 1, 2$. $H^3(\mathbf{B}PU_n;\mathbb{Z}) \cong \mathbb{Z}/n$ *is generated by* x_1 *of order* n. x_1^2 *is of order* 2 *if* n *is even and is* 0 *otherwise.*

2. $H^4(\mathbf{B}PU_n;\mathbb{Z}) \cong \mathbb{Z}$ *is generated by* e_2*, such that* $P^*(e_2) = 2nc_2 - (n-1)c_1^2$ *when* n *is even and* $P^*(e_2) = nc_2 - \frac{n-1}{2}c_1^2$ *when* n *is odd.*

3. $H^5(\mathbf{B}PU_n;\mathbb{Z}) = 0$.

4. *Let* $n > 2$. *If* n *is even,* $H^6(\mathbf{B}PU_n;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}/2$ *is generated by* e_3 *of order infinity and* x_1^2 *of order* 2. *If* n *is odd,* $H^6(\mathbf{B}PU_n;\mathbb{Z}) \cong \mathbb{Z}$ *is generated by* e_3*, and* $x_1^2 = 0$. *In the exceptional case* $n = 2$*, the assertion holds with the absence of* e_3 *and its corresponding direct summand* $\mathbb{Z}$.

5. *If* $n = 4l+2$, $H^7(\mathbf{B}PU_n;\mathbb{Z}) \cong \mathbb{Z}/2$ *is generated by* e_2x_1. *Otherwise* $H^7(\mathbf{B}PU_n;\mathbb{Z}) = 0$ *and in particular* $e_2x_1 = 0$.

6. *Let* $n \geq 4$. *If* $3|n$, $H^8(\mathbf{B}PU_n;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/3$ *generated by* e_4 *and* e_2^2 *of order infinity and* $y_{3,0}$ *of order* 3. *Otherwise* $H^8(\mathbf{B}PU_n;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}$ *is generated by* e_2^2 *and* e_4. *In the*

exceptional cases $n = 2, 3$, this assertion holds as well, with e_4 and its corresponding direct summand $\mathbb{Z}$ absent.

7. $H^9(\mathbf{B}PU_n;\mathbb{Z}) \cong \mathbb{Z}/\epsilon_2(n)\oplus\mathbb{Z}/\epsilon_3(n)$ *is generated by* x_1^3 *of order* $\epsilon_2(n)$ *and* z_1 *of order* $\epsilon_3(n)$.

8. $e_3x_1 = 0$.

9. *If* $n \geq 5$, $H^{10}(\mathbf{B}PU_n;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/\epsilon_2(n)\oplus\mathbb{Z}/\epsilon_3(n)$ *is generated by* e_2e_3, e_5 *of order infinity,* $y_{2,1}$ *of order* $\epsilon_2(n)$ *and* z_2 *of order* $\epsilon_3(n)$. *In the exceptional cases* $n < 5$, *the assertion holds, with the absence of monomials involving* e_i *for* $i > n$ *and their corresponding direct summands* $\mathbb{Z}$.

10. $e_2x_1^2 = y_{2,1}$ *when* $n = 4l + 2$, *and* $e_2x_1^2 = 0$ *otherwise.*

Theorem 0.0.1 provides enough information for the study of the topological period-index problem, for CW-complexes of dimension less than or equal to 8, which is the subject of Chapter 2. For those of dimension 6, the problem was solved by B. Antieau and B. Williams in (1).

As can be easily read from Theorem 0.0.1, in degrees ≤ 5, where the computation is done degree-wise in (1), there is no non-trivial cup product, whereas in degree 6, non-trivial cup products begin to occur. Another way to verify this fact is through the (11), section 4.

In the case $n = 4l + 2$, The $\mathbb{Z}/2$-module structure of $H^*(\mathbf{B}PU_n;\mathbb{Z}/2)$ was studied by A. Kono and M. Mimura in their paper (25). We compare their results with Theorem 0.0.1 and show that they agree, as they ought to.

Consider the short exact sequence of Lie groups as follows

$$1 \to S^1 \to U_n \to PU_n \to 1.$$

Applying the classifying space functor $\mathbf{B}$ to it, we obtain the following fiber sequence

$$\mathbf{B}S^1 \to \mathbf{B}U_n \to \mathbf{B}PU_n.$$

Notice that $\mathbf{B}S^1$ is a model for the Eilenberg-Mac Lane space $K(\mathbb{Z}, 2)$, which is homotopy equivalent to $\mathbf{\Omega}K(\mathbb{Z}, 3)$. This fiber sequence can be shifted to the following one:

$$\mathbf{B}U_n \to \mathbf{B}PU_n \to K(\mathbb{Z}, 3). \tag{1.1.1}$$

Let ${}^UE_*^{*,*}$ be the integral cohomological Serre Spectral Sequences induced by (1.1). This is the main object of interest in this chapter.

We compare ${}^UE_*^{*,*}$ with two other Serre spectral sequences ${}^KE_*^{*,*}$ and ${}^TE_*^{*,*}$, which are induced by the well known fiber sequence $K(\mathbb{Z}, 2) \to * \to K(\mathbb{Z}, 3)$, and $\mathbf{B}T^n \to \mathbf{B}PT^n \to K(\mathbb{Z}, 3)$, respectively. Here $*$ is a contractible space; T^n and PT^n are the maximal tori of U_n and PU_n respectively.

Section 2 contains an outline and a summary of main results of exp. 2 to exp. 11 of (14), which completely determined the homology, and consequently the cohomology of Eilenberg-Mac Lane spaces of Abelian groups of integral and mod p coefficients for any prime p. This is the cornerstone of the study of the differentials of the Serre spectral sequences considered in this chapter.

In Section 3, we follow (14), or Section 2, to construct an analog of the first fiber sequence mentioned above in the category of chain complexes of Abelian groups. In particular, we obtain algebraic models for $K(\mathbb{Z},3)$, denoted by $A(3)$, and a spectral sequence $\tilde{E}_*^{*,*}$ such that $\tilde{E}_2^{s,t} \cong H^s(A(3);H^t(K(\mathbb{Z},2);\mathbb{Z}))$. In Section 4, we apply an argument on homology suspension to prove that this spectral sequence is isomorphic to ${}^KE_*^{*,*}$, the Serre spectral sequence associated to $K(\mathbb{Z},2) \to * \to K(\mathbb{Z},3)$, after E_2-page.

In Section 5 we compare ${}^UE_*^{*,*}$, ${}^TE_*^{*,*}$ with ${}^KE_*^{*,*}$ to give all the differentials of ${}^TE_*^{*,*}$ and consequently a considerable amount of the differentials of ${}^UE_*^{*,*}$, which enable one to compute $H^k(\mathbf{B}PU_n;\mathbb{Z})$, at least in the range relevant to Theorem 0.0.1.

In Section 6 we make some remarks on the Theorem 0.0.1, in particular compare it with the result of A. Kono and M. Mimura, and consider the $\mathbb{Z}/2$-algebra structure of $H^*(\mathbf{B}PU_n;\mathbb{Z}/2)$. We also consider the p-local cohomology of $\mathbf{B}PU_n$.

In the last three sections we apply the apparatus set up in Section 5 to prove Theorem 0.0.1.

1.2 Preliminary on Multiplicative Constructions

This section is a condensed and slightly verified version of exp. 2-11 of (14). The author makes no claim of originality to materials in this section. As in the introduction, all DGA's involved are graded-commutative and augmented over the base ring R, which is either $\mathbb{Z}$ or $\mathbb{Z}/p$ for some prime p. These conditions are slightly stronger then that in Cartan's original work, but do no harm to our application and make considerable simplification.

Definition 1.2.1. A *multiplicative construction* is a triple (A,N,M) of DGA's over a base ring R such that

1. As a graded A-module, $M = A \otimes_R N$.
2. The DGA structure of N is given by $N = R \otimes_A M$ where A acts on R by augmentation.

An *acyclic multiplicative construction* is one such that M is acyclic.

Remark 1.2.2. A (resp. N) is a sub-DGA of M via $a \mapsto a \otimes 1$ (resp. $n \mapsto 1 \otimes n$). We will use this fact implicitly.

Example 1.2.3. Let A be a DGA, and η be its augmentation. The *bar construction* of A is a multiplicative construction $(A, \overline{\mathcal{B}}(A), \mathcal{B}(A))$ where the DGA's $\mathcal{B}(A)$ and $\overline{\mathcal{B}}(A)$ are defined as follows:

1. $$\mathcal{B}(A) = \sum_{k \geq 0} A \otimes_R A^{\otimes k}; \overline{\mathcal{B}}(A) = \sum_{k \geq 0} A^{\otimes k}.$$

   By convention $A^{\otimes 0} = R$. For simplicity the element $a \otimes a_1 \otimes \cdots \otimes a_k$ of $\mathcal{B}(A)$ is denoted by

   $$a[a_1 | \cdots | a_k]$$

   and

   $$1 \cdot [a_1 | \cdots | a_k] = [a_1 | \cdots | a_k].$$

2. The degree is defined by

   $$deg(a[a_1 | \cdots | a_k]) = k + deg(a) + deg(a_1) + \cdots + deg(k)$$

   and the length of $a[a_1 | \cdots | a_k]$ is defined to be $deg([a_1 | \cdots | a_k])$. We define a bi-degree on $\mathcal{B}(A)$ by saying that $a[a_1 | \cdots | a_k]$ has bi-degree (s, t) if $\deg(a) = s$, $\deg([a_1 | \cdots | a_k]) = t$.

The degree of $a[a_1|\cdots|a_k]$ is obviously $s+t$. Let $\mathbf{F}_B$ be the filtration induced by the first entry s of this bi-degree.

3. A chain map $s : \mathcal{B}(A) \to \overline{\mathcal{B}}(A)$ of degree one is defined as follows:

$$s(a[a_1|\cdots|a_k]) = [a|a_1|\cdots|a_k]s(1) = 0$$

4. The differential of $\mathcal{B}(A)$ is defined by induction on k as follows:

$$\begin{cases} d([a]) = a \cdot 1 - [d(a)] - \eta(a) \cdot 1 \\ d([a_1|\cdots|a_k]) = a_1[a_2|\cdots|a_k] - sd(a_1[a_2|\cdots|a_k]), k \geq 2 \end{cases}$$

   and the differential of $\overline{\mathcal{B}}(A)$ is induced by the above.

5. $\overline{\mathcal{B}}(A)$ has a product structure induced by that of A, sometimes called the shuffle product, which makes $\overline{\mathcal{B}}(A)$ a DGA with respect to the differential defined above.

It is obvious and proved in Exp.3 of (14) that s is a chain homotopy between the identity of $\mathcal{B}(A)$ and the augmentation η. We can iterate this procedure to construct inductively $\mathcal{B}^{n+1}(A) = \mathcal{B}(\overline{\mathcal{B}}^n(A)$ and $\overline{\mathcal{B}}^{n+1}(A) = \overline{\mathcal{B}}(\overline{\mathcal{B}}^n(A))$ In the case $A = R[\Pi]$(concentrated in degree 0) where Π is an Abelian group, (22) implies $H_*(\overline{\mathcal{B}}^n(R[\Pi])) \cong H_*(K(\Pi, n))$.

Remark 1.2.4. By (22) $H_*(\overline{\mathcal{B}}^n(R[\Pi])) \cong H_*(K(\Pi, n); R)$. Later we will define a filtration on $\mathcal{B}(\overline{\mathcal{B}}^n(R[\Pi])$ such that its associated (cohomological) spectral sequence coincides with the one associated to the Serre Spectral sequence associated to the fiber sequence

$$K(\Pi, n-1) \to * \to K(\Pi, n)$$

We proceed to introduce the operations on an acyclic multiplicative construction (A, N, M) with base ring R, following section 6 to 12 of (14). Let d_A, d_N, d_M be the differentials of A, N and M respectively. Furthermore, we assume that there are augmentations $\epsilon_A, \epsilon_N, \epsilon_M$ from A, N, M respectively, to the base ring R. The subscripts will be omitted whenever there is no risk of ambiguity.

Definition 1.2.5. Assume that the homomorphism $A \to M : a \mapsto a \otimes 1$ is injective. Let $\alpha \in H_k(A)$ be represented by $a \in \mathrm{Ker} d_A$. Since M is acyclic, there is some $x \in M$ such that $d_M(x) = a$. Passing to N, we obtain an element $\bar{x} = 1 \otimes x \in A \otimes_R M \cong N$. $\bar{x}$ is easily verified to be a cycle in N. The homology class $\{\bar{x}\} \in H_{k+1}(N)$ is therefore called the *suspension* of the homology class $\alpha \in H_k(A)$, denoted by $\sigma(\alpha)$. It is easy to verify that $[\bar{x}]$ is independent of the choice of a or x and $\sigma : H_k(A) \to H_{k+1}(N)$ is a well defined homomorphism of graded Abelian groups of degree 1.

Example 1.2.6. Let $(A, N, M) = (A, \overline{\mathcal{B}}(A), \mathcal{B}(A))$, then the suspension can be realized by the homomorphism $A \to \mathcal{B}(A), a \mapsto [a]$. Notice that the presence of the bracket lifts the degree by 1.

Remark 1.2.7. This example gives an alternative description of the homology suspension which is a homomorphism $H_k(\Omega X) \to H_{k+1}(X)$ for a topological space X, which will be discussed in section 4.

Definition 1.2.8. Let A, N, M have characteristic a prime number p. Define the following R-submodule of degree $2q$ of A by

$$_pA_{2q} = \{a \in A_{2q} | d_A(a) = 0, a^p = (\epsilon(a))^p, \text{or equivalently } (a - \epsilon(a))^p = 0\}.$$

Take $x \in M_{2q+1}$ such that $d_M(x) = a - \epsilon(a)$. Then we have $d_M((a-\epsilon(a))^{p-1}x) = (a-\epsilon(a))^p = 0$. Now take $y \in M_{2pq+2}$ such that $d(y) = (a - \epsilon(a))^{p-1}x$. Passing to N we define the *transpotence* $\psi(a) = \bar{y} \in H_{2pq+2}(N)$.

Notice that $\psi : {}_pA_{2q} \to H_{2pq+2}(N)$ is not necessarily a homomorphism, even of Abelian groups, and does not necessarily pass to homology. However we have the following

Proposition 1.2.9. (H. Cartan, Proposition 5, exp. 6, (14)) All notations are as in definition 1.2.8. Suppose that

1. $a^p = 0$, for all $a \in A_{2q}$, and
2. $b \cdot d_A(b^{p-1})$ is in the image of d_A, for all $b \in A_{2q+1}$.

Then ψ passes to homology to define a map $\varphi : H_{2q}(A) \to H_{2pq+2}(N)$.

This ψ is is again not necessarily additive. But in the case that interests us, we have the following

Proposition 1.2.10. (H. Cartan, Theorem 3, exp. 6, (14)) Let A be a commutative DGA concentrated in degree 0, over a base ring R of characteristic p, a prime number. Then $\psi : {}_pA \to H_2(\overline{\mathcal{B}}(A))$ is additive when p is odd. For all $n \geq 1$ and $q \geq 1$, the transpotence $\varphi : H_{2q}(\overline{\mathcal{B}}^n(A)) \to H_{2pq+2}(\overline{\mathcal{B}}^{n+1}(A))$ induced by ψ is well defined, and if p is odd, it is additive with kernel containing all the decomposable elements of $H_{2q}(\overline{\mathcal{B}}^n(A))$.

Definition 1.2.11. For A a graded commutative, a *divided power operation* on A is a collection of maps $\gamma_k : A \to A$ for all integers $k \geq 0$, such that for any $x, y \in A$ they satisfy the following:

1. $\gamma_0(a) = 1, \gamma_1(a) = a, \deg\gamma_k(a) = k\deg a$.

2. $\gamma_k(x)\gamma_l(x) = \binom{k+l}{k}\gamma_{k+l}(x)$.

3. (Leibniz rule) $\gamma_k(x+y) = \sum_{i+j=k} \gamma_i(x)\gamma_j(y)$.

4. $$\gamma_k(xy) = \begin{cases} 0, \deg(x), \deg(y) \text{ are odd}, k \geq 2, \\ x^k\gamma_k(y), \deg(x), \deg(y) \text{ are even}, \deg(y) \geq 2. \end{cases}$$

   When the characteristic of R is 2, we have in addition, for $k \geq 2$,

   $$\gamma_k(xy) = \begin{cases} 0, \deg(x), \deg(y) > 0, \\ x^k\gamma_k(y), \deg(x) = 0. \end{cases}$$

5. $\gamma_k(\gamma_l(x)) = \binom{2l-1}{l-1}\binom{3l-1}{l-1}\cdots\binom{kl-1}{l-1}\gamma_{kl}(x)$. If in addition, A is a DGA with differential d, then we require

6. $d(\gamma_k(x)) = \gamma_{k-1}(x)d(x)$ for $k \geq 1$.

A graded commutative algebra with a divided power operation is called a *divided power algebra*. A map of divided power algebras is a homomorphism of graded algebras compatible with the divided power operation.

Remark 1.2.12. We do not require A to have a differential, since we often wish to define a divided power operator on homology of a DGA, rather than the DGA itself.

Example 1.2.13. The prototype of a divided power algebra is $P_R(y)$, which, as a graded R-algebra, is generated by element $\gamma_k(y)$ for all $k \geq 1$ modulo the relations imposed by definition 1.2.11. Here y is of degree $2q$ for any positive integer q. By (4) of definition 1.2.11, we have $k!\gamma_k(y) = y^k$. In fact, when R is torsion free, $P_R(y)$ is isomorphic to the polynomial algebra $R[y]$ adjoining all $\frac{y^k}{k!} = \gamma_k(y)$.

On the other hand, if $R = \mathbb{Z}/p$ where p is a prime number, then (4) of definition 1.2.11 implies that $y^p = p!\gamma_p(y) = 0$. Furthermore, for $k = k_0 + k_1 p + k_2 p^2 + \cdots + k_r p^r$, where $0 \leq k_i < p-1, i = 0, 1, \cdots, r$, we have $\gamma_k(y) = \prod_{0 \leq i < r} \gamma_{k_i}(\gamma_{p^i}(y)) = \gamma_{p^i}(y)/k_i!$. In fact, as a graded $\mathbb{Z}/p$ algebra

$$P_{\mathbb{Z}/p}(y) \cong \bigotimes_{k \geq 0} \mathbb{Z}/p[\gamma_{p^k}(y)]. \tag{1.2.1}$$

A detailed discussion on divided power algebras over $\mathbb{Z}/p$, including the proofs of the statements above, can be found in section 7, exp. 7 of (14).

Example 1.2.14.

1. Let M be a free graded R-module generated by elements in odd degrees. Then we can form the free exterior $E(M)$ over a given set of generators and take the trivial divided power operations such that γ_0 is the constant map on to $1 \in R$, γ_1 is the identity, and γ_k is zero for all $k > 1$.

2. Let M be a free graded R-module generated by a set $\{e_i\}$ of elements. If R has characteristic other than 2, we require e_i to have even degree. Then we can form the the universal symmetric algebra $S(M)$ which has the underlying graded R-module $\sum_{i \geq 0} M^{\otimes i}$, with $M^{\otimes 0} = R$ in degree 0, and with a structure of DGA such that for any divided power algebra A, an R-linear map $M \to A$ can be extended uniquely to a map of divided power algebra $S(M) \to A$.

   The product on $S(M)$ is shuffle product similar to that of the reduced bar construction. In particular $e_i^k = k! e_i \otimes \cdots \otimes e_i$ with k copies of e_i's on the righthand side of the equation. There is a power operation $\{\gamma_k\}_k$ on $S(M)$ determined by $\gamma_k(e_i) = e_i \otimes \cdots \otimes e_i$ with k copies of e_i and the axioms in Definition 1.2.11.

3. Let A be a DGA and $a \in A$. Then $[a] \in \overline{\mathcal{B}}(A)$, and the shuffle product $[a]^k = k![a|\cdots|a]$ with k copies of a on the righthand side of the equation. We can define a divided power operation on $\overline{\mathcal{B}}(A)$ similar to that of $S(M)$, by requiring $\gamma_k([a]) = [a|\cdots|a]$ with k copies of a in the bracket. For more details see section 4, exp. 8 of (14).

Example 1.2.15. Let A and A' be DGA's with divided power operations. Then (3) and (4) of Definition 1.2.11 gives a unique way to extend the divided power operations to $A \otimes A'$, a DGA with differential determined by those of A and A' via the Leibniz rule. For details see Theorem 2, exp. 7 of (14).

Based on Example 1.2.14 and Example 1.2.15 we have the following

Theorem 1.2.16 (H. Cartan, Theorem 2, exp. 8, (14))**.** *Let A be a graded commutative algebra with a divided power operation as in Definition 1.2.11 and M a free graded module. Then there is a divided power algebra $U(M)$ such that any homomorphism of graded R-modules $f : M \to A$ is can be extended uniquely to a map of divided power algebras $U(M) \to A$.*

The three operations, the suspension, the transpose, and the divided power operation, as described above, are enough to describe the homology of $K(\Pi, n)$ with coefficients in $\mathbb{Z}$ or $\mathbb{Z}/p$, at least when Π is a finitely generated Abelian group. We start with the definitions of words and their heights and degrees. From now on we fix a prime number p.

Definition 1.2.17. We give parallel definitions in the cases when p is odd and $p = 2$.

1. By a *word* we mean a sequence consists of the three symbols σ, φ_p, and γ_p when p is odd, or σ, γ_2 when $p = 2$, where repetition is allowed. The *height* of a word α is the total number of σ and φ_p in α, counting repetition. We take the degree of the empty word to be 0 and inductively define the degree of the words $\sigma\alpha, \varphi_p\alpha$ and $\gamma_p\alpha$ as follows:

$$\begin{cases} \deg(\sigma\alpha) = \deg(\alpha) + 2, \\ \deg(\varphi_p\alpha) = 2p\deg(\alpha) + 2, \\ \deg(\gamma_p\alpha) = p\deg(\alpha). \end{cases} \tag{1.2.2}$$

2. When p is odd, an *admissible word*, is a word α such that (i) α is non-empty and starts and ends with σ or φ_p, and (ii) for every φ_p and γ_p appeared in α, there are even number of copies of σ on its righthand side. An admissible word is of type 1 if it ends with σ, and type 2 if it ends with φ_p.

3. When $p = 2$, an *admissible word*, is a word α that starts and ends with σ. α is of type 2 if it ends with $\gamma_2\sigma$, and of type 1 otherwise.

The words consisting of σ, φ_p and γ_p are also called p-words.

A word can be regarded as the compose of the sequence of operations σ, φ_p and γ_p, in the obvious manner. Let $\mathbb{Z}[\Pi]$ be the group ring of a finitely generated Abelian group Π, viewed as a DGA concentrated in degree 0 and with a trivial differential. Then $H_*(\mathbb{Z}[\Pi]) \cong \mathbb{Z}[\Pi]$. The alert reader will find that, an admissible word α is a well defined compose of operations on $\mathbb{Z}[\Pi]$, with image in $H_{\deg(\alpha)}(\overline{\mathcal{B}^n}(\mathbb{Z}[\Pi])) \cong H_{\deg(\alpha)}(K(\Pi, n))$, where n is the height of α. In fact all homology classes are generated this way, as we will see soon.

Definition 1.2.18. Let Π be a finitely generated Abelian group, and let ${}_p\Pi$ be the subgroup of Π of elements of order infinity or a power of p. We write $\Pi/(p\Pi) = \prod_i \Pi'_i$ and ${}_p\Pi = \prod_j \Pi''_j$ as the decomposition of $\Pi/(p\Pi)$ and ${}_p\Pi$ into direct products of cyclic groups of order infinity or a power of p.

Fix a positive integer n. Let $M^{(n)}$ be the free graded R-module generated by α_i of degree $\deg(\alpha)$ for every admissible word α of type 1 with height n and Π'_i, and α'_j of degree $\deg(\alpha')$

for every admissible word α' with height n of type 2. Let $U(M^{(n)})$ be as in Theorem 1.2.16. Notice in particular that $U(M^{(0)}) \cong \mathbb{Z}/p[\Pi]$.

In the following theorem we do not distinguish a word and the compose of operations that it represents.

Theorem 1.2.19 (H. Cartan, Théorème fondamental, exp. 9, (14)). *With the notations as above, let w_i', w_j'' be generators of Π_i', Π_j'' respectively. Take $R = \mathbb{Z}/p$, where p is an odd prime number. Let $f^{(n)} : M^{(n)} \to H_*(K(\Pi, n); \mathbb{Z}/p)$ be the homomorphism of $\mathbb{Z}/p$-modules taking α_i (resp. α_j') to $\alpha(w_i')$ (resp. $\alpha(w_j'')$). Its unique extension to $U(M^{(n)})$, $\tilde{f}^{(n)}$ given by Theorem 1.2.16, is an isomorphism of divided power algebras.*

We will give a sketch of proof of Theorem 1.2.19 since the idea is relevant to our application. To do so we need the following theorems.

Theorem 1.2.20 (H. Cartan, Theorem 2, exp.2, (14)). *Let $f : A \to A'$ be a morphism of DGA's over R. Let M (resp. M') be an acyclic chain complex over R with a graded A (resp.A')-module structure. Let I (resp. I') be the kernel of the augmentation of A (resp. A'). Then there is a morphism of chain complexes $g : M/IM \to M'/I'M'$ compatible with f in the obvious sense. The induced morphism $H_*(g)$ is independent of the choice of g. Moreover, if f is a weak equivalence, then so is g.*

Theorem 1.2.21 (H. Cartan, Theorem 5, exp.4, (14)). *Let (A, N, M) and (A', N', M') be two multiplicative constructions. Let $\tilde{N}'$ be a R-subalgebra of M containing N such that $d : \tilde{N}'_{k+1} \to Ker(d_k)$ is a degree-wise isomorphism of R-modules for all $k \geq 0$. Here $d_0 = \eta$. In particular,*

M' is acyclic. Let $f : A \to A'$ be a map of DGA's. Then there is a unique map $g : M \to M'$ of DGA's restricting to f, such that $g(N) \subset \tilde{N}'$.

Sketch of proof of Theorem 1.2.19. By Künneth formula it suffice to consider the case that Π is a cyclic group with a generator w. We proceed to show that there is a multiplicative construction $(U(M^{(n)}), U(M^{(n+1)}), L)$ with L acyclic.

One can easily show that an admissible word α of height $n+1$ is of the form

1. $\sigma\alpha'$ where α' is of height n and odd degree, or
2. $\sigma\gamma_{p^k}\alpha'$ or $\varphi_p\gamma_{p^k}\alpha'$ where $k \geq 0$ and α' is of height n and even degree.

The base case where $n = 0$ is easy. Let $L = U(M^{(n)}) \otimes_{\mathbb{Z}/p} U(M^{(n+1)})$ as a graded $\mathbb{Z}/p$-algebra, and consider $U(M^{(n)})$ and $U(M^{(n+1)})$ as its subalgebras in the obvious manner.

In the first case as above, let $x = \alpha'(w) \in U(M^{(n)})$. Then the free exterior algebra $E_{\mathbb{Z}/p}(x)$ is a subalgebra of $U(M^{(n)}$. Let $y = \alpha(w) = \sigma\alpha'(w) \in U(M^{(n+1)})$, and we have the subalgebra $P_{\mathbb{Z}/p}(y)$ of $U(M^{(n+1)})$. Define the differentials of x and y in L by $d_L(x) = 0$ and $d_L(y) = x$ together with the axioms in Definition 1.2.11. Then the $E_{\mathbb{Z}/p}(x) \otimes P_{\mathbb{Z}/p}(y)$ is acyclic.

In the second case, let $x = \alpha'(w) \in U(M^{(n)}), y_k = \sigma\gamma_{p^k}(w)$ and $z_k = \phi_p\gamma_{p^k}(w)$ for all $k \geq 0$. Define their differentials in L by $d_L(x) = 0, d_L(y_k) = x$ and $d_L(z_k) = x^{p-1}y_k$. Then one can show that

$$P_{\mathbb{Z}/p}(x) \otimes \bigotimes_{k\geq 0} E_{\mathbb{Z}/p}(y_k) \otimes \bigotimes_{k\geq 0} P_{\mathbb{Z}/p}(z_k)$$

is acyclic.

By Theorem 1.2.20 and Theorem 1.2.21 one can inductively prove the statement, the base case where $n = 1$ being standard homological algebra. $\square$

Remark 1.2.22. This argument fails for $p = 2$, in which case φ is not additive.

For a free graded R-module M, recall the graded R-algebra $S(M)$ introduced in Example 1.2.14, (2). We take $M^{(n)}$ as in Definition 1.2.18. and $f^{(n)}$ as in Theorem 1.2.19. Notice that the constructions apply to $p = 2$. The analog of Theorem 1.2.19 in the case where $p = 2$ is the following:

Theorem 1.2.23 (H. Cartan, Théorème fondamental, exp. 9, (14))**.** $f^{(n)} : M^{(n)} \to H_*(K(\Pi, n); \mathbb{Z}/2)$ *extends to* $\tilde{f}^{(n)} : S(M^{(n)}) \to H_*(K(\Pi, n); \mathbb{Z}/2)$ *which is an isomorphism of divided power algebras.*

The proof is similar to that of Theorem 1.2.19.

We proceed to consider integral homology of $K(\Pi, n)$. Recall the transpotence ψ defined in Definition 1.2.8. For an arbitrary integer l, we have a similar operation $\psi_l : {}_l\Pi \to H_2(K(\Pi, 1); \mathbb{Z}/l)$ where ${}_l\Pi$ is the subgroup of Π of l-torsion elements. ψ_l satisfies the following condition.

Proposition 1.2.24. Let $\delta_l : H_2(K(\Pi, 1); \mathbb{Z}/l) \to H_1(K(\Pi, 1); \mathbb{Z})$ be the Bockstein homomorphism and $\sigma : {}_l\Pi \to H_1(K(\Pi, 1); \mathbb{Z})$ be the suspension. Then $\sigma = \delta_l \psi_l$.

For details see section 1, exp. 11 of (14). In the case of integral cohomology, we extend the definition of an admissible p-word as follows.

Definition 1.2.25. A p-word is a sequence consists of the symbols $\sigma, \varphi_p, \gamma_p$, and ψ_{p^λ}, for some positive integer λ. An admissible p-word is a word satisfying (2) of Definition 1.2.17 except that it can end with ψ_{p^λ}. ψ_{p^λ} is of height 1 and degree 2. The degree of a p-word is therefore given as in (1) of 1.2.17. Notice we do not make the exception when $p = 2$.

In what follows we abuse notations to let words denote elements of a DGA rather then homology classes, as we did earlier. Let $\Pi = \prod_k \Pi_k$ be the decomposition of Π into cyclic groups of order infinity or a power of a prime, and let w_k be a generator of Π_k. Also we recall the the decompositions $\Pi/(p\Pi) = \prod_i \Pi'_i$ and ${}_p\Pi = \prod_j \Pi''_j$ as well as the generators $\{w'_i\}, \{w''_j\}$ as in Definition 1.2.18 and Theorem 1.2.19. We let $E(a;k)$ or $P(;k)$ denote exterior algebras or divided power algebras over $\mathbb{Z}$ generated by a single element a of degree k, suppressing the ring of coefficients, and consider them as graded $\mathbb{Z}$-algebras. We fix a positive integer n, and construct a collection of DGA's.

1. For each w_k of order infinity, take the DGA $E(\sigma^n(w_k);n)$ with trivial differential when n is odd, or $P(\sigma^n(w_k);n)$ when n is even. We denote this DGA by $A(n)_0$.
2. For each w_k of order p^λ for some prime p and positive integer λ, If n is odd take $E(\sigma^n(w_k);n) \otimes P(\sigma^{n-1}(\psi_{p^\lambda})(w_k);n+1)$, or $P(\sigma^n(w_k),n) \otimes E(\sigma^{n-1}(\psi_{p^\lambda})(w_k),n+1)$. In either case we define differential

$$d(\sigma^{n-1}\psi_{p^\lambda})(w_k)) = (-1)^{n-1}p^\lambda\sigma^n(k), d(\sigma^n(w_k)) = 0$$

   when n is even.

3. Let α' be an admissible p-word of height $n-l-1$ and degree q. Consider the pair of p-words $\sigma^l\varphi_p\alpha'$ and $\sigma^{l+1}\gamma_p\alpha'$. If n is odd, for each w_i' take $E(\sigma^{l+1}\gamma_p\alpha'(w_i'); pq+l+1) \otimes P(\sigma^l\varphi_p\alpha'(w_i'); pq+l+2)$. If n is even, for each w_i' take $P(\sigma^{l+1}\gamma_p\alpha'(w_i'); pq+l+1) \otimes E(\sigma^l\varphi_p\alpha'(w_i'), pq+l+2)$. In both cases we take differential

$$d(\sigma^l\varphi_p\alpha'(w_i')) = (-1)^{n-1}p(\sigma^{l+1}\gamma_p\alpha'(w_i')), d(\sigma^{l+1}\gamma_p\alpha'(w_i')) = 0.$$

4. Let $\alpha', \sigma^l\varphi_p\alpha'$ and $\sigma^{l+1}\gamma_p\alpha'$ be as above. If n is odd, for each w_j'' take $E(\sigma^{l+1}\gamma_p\alpha'(w_j''); pq+l+1) \otimes P(\sigma^l\varphi_p\alpha'(w_j''); pq+l+2)$. If n is even, for each w_j'' take $P(\sigma^{l+1}\gamma_p\alpha'(w_j''), pq+l+1) \otimes E(\sigma^l\varphi_p\alpha'(w_j''); pq+l+2)$. In both cases we take differential

$$d(\sigma^l\varphi_p\alpha'(w_j'')) = (-1)^{n-1}p(\sigma^{l+1}\gamma_p\alpha'(w_j'')),\ d(\sigma^{l+1}\gamma_p\alpha'(w_j'')) = 0.$$

We take all the DGA's constructed in (2), (3), (4) and denote their tensor product by $A(n)_p$. Finally we take $A(n) = A(n)_0 \otimes_{\mathbb{Z}} \bigotimes A(n)_p$. It is easily seem that there is a homomorphism of divided power algebra $f : H_*(A(n)) \to H_*(K(\Pi, n); \mathbb{Z})$ taking the class represented by the word $\alpha(k)$ (resp. $\alpha(i), \alpha(j)$ to the homology class given by the operations $\alpha(w_k)$, (resp. $\alpha(w_i', \alpha(w_j'')$. The following theorem from (14) is stated in a more modern form than the original.

Theorem 1.2.26 (H. Cartan, Theorem 1, exp. 11, (14))**.** *$f : H_*(A(n)) \to H_*(K(\Pi, n); \mathbb{Z})$ is an epimorphism. For any prime p, the restriction $f : H_*(A(n)_0 \otimes A(n)_p) \to H_*(K(\Pi, n); \mathbb{Z})$ is a p-local isomorphism.*

Remark 1.2.27. The kernel of f is not always trivial. Indeed, when $n = 2$ and $\Pi = \mathbb{Z}/2$ with a generator w_1, we take $x_k = \varphi_2\gamma_{2^k}\psi_2(w_1)$ for $k \geq 1$, $y_k = \sigma\gamma_{2^k}\psi_2(w_1)$ for $k \geq 0$, and $z = \sigma^2(w_1)$. Consider the DGA over $\mathbb{Z}$

$$A = \bigotimes_{k\geq 1} P(x_k; 2^{k+1}+2) \otimes \bigotimes_{k\geq 0} E(y_k; 2^{k+1}+1) \otimes P(z; 2)$$

with $d(x_k) = -2y_k, d(y_k) = 0$ for $k \geq 1$, $d(y_0) = -2z$ and $d(z) = 0$. In particular we have

$$d(y_0\gamma_2(z)) = -2z\gamma_2(z) = -6\gamma_3(z),$$

which implies that the homology class $\gamma_3(z)$ is of order 6, but by Theorem 1.2.19, $H_*(K(\mathbb{Z}/2, 2); \mathbb{Z})$ has only 2-primary elements. Hence $2\gamma_3 z \neq 0$ is in the kernel of f since it is a 3-torsion. However, as we see in the next section, when $\Pi = \mathbb{Z}$ we do have $A(3) \cong H_*(K(\mathbb{Z}, n); \mathbb{Z})$.

1.3 A Chain Complex Model for $K(\mathbb{Z}, 3)$

The previous section provides a method to calculate the integral homology of $K(\Pi, n)$ for any positive integer n and Π a finitely generated Abelian group, in particular $K(\mathbb{Z}, 3)$. However, for the purpose of this chapter we need information on the level of chain complex which is not necessarily revealed by homology. Therefore we proceed to construct an acyclic multiplicative construction $(A(2), A(3), M(3))$ from which there is an isomorphism to $(\overline{\mathcal{B}}^2(\mathbb{Z}[\Pi]), \overline{\mathcal{B}}^3(\mathbb{Z}[\Pi]), \mathcal{B}^3(\mathbb{Z}[\Pi]))$, where $\Pi = \mathbb{Z}$. In particular, we have

$$H_*(A(3); \mathbb{Z}) \cong H_*(K(\Pi, 3); \mathbb{Z}), H^*(A(3); \mathbb{Z}) \cong H_*(K(\Pi, 3); \mathbb{Z}).$$

Throughout the rest of this section, we write $E_R(x;k), P_R(y;l), E(x;k), P_R(y;l)$ for exterior algebras and divided power algebras with coefficients in a ring R, or $\mathbb{Z}$, and with one generator in a specified degree. **Other than this, no attempt is made to keep notations in consistency with those in Section 2.** All tensor products are with respect to the base ring $\mathbb{Z}$ unless otherwise specified.

By Theorem 1.2.26 we see that we should take $A(2) = P(u, 2)$ where u corresponds to the operation σ^2, and $A(3)$ to be the following

$$A(3) = \bigotimes_p \big[\bigotimes_{k\geq 0} P(a_{p,k}; 2p^{k+1} + 2) \otimes \bigotimes_{k\geq 1} E(b_{p,k}; 2p^k + 1)\big] \otimes E(b_1; 3)$$

where $a_{p,k} = \varphi_p \gamma_{p^k} \sigma^2(w), b_{p,k} = \sigma\gamma_{p,k}\sigma^2(w)$, and $b_1 = \sigma^3(w)$ for a generator w of $\mathbb{Z}$, and p ranges over all prime numbers. The differential is defined by

$$\bar{d}(a_{p,k}) = pb_{p,k+1}, \bar{d}(b_{p,k}) = 0, \bar{d}(b_1) = 0.$$

Notice that $A(2)$ is torsion free, and that for each p, $\bigotimes_{k\geq 0} P(a_{p,k}; 2p^{k+1}+2) \otimes \bigotimes_{k\geq 1} E(b_{p,k}; 2p^k+1)$ has no p'-torsion for any prime $p' \neq p$. Therefore the epimorphism in Theorem 1.2.26 is an isomorphism, for $n = 2, 3$.

We proceed to take $M(3) = A(2) \otimes_{\mathbb{Z}} A(3)$ as a graded algebra, and define its differential d in such a way that it makes $M(3)$ acyclic, and when pass to $A(3) = \mathbb{Z} \otimes_{A(2)} M(3)$, it induces $\bar{d}$, the differential of $A(3)$. We will use the sketch of proof of Theorem 1.2.19 as our guide. Then we define a filtration on $M(3)$ and in Section 4 we prove that the spectral sequence induced by this differential is the Serre spectral sequence associated to the fiber sequence $K(\mathbb{Z},2) \to * \to K(\mathbb{Z},3)$.

Lemma 1.3.1 (Lucas' Theorem)**.** *Let p be a prime number, $k = \sum_{r=0}^{n} k_r p^r$, and $l = \sum_{r=0}^{n} l_r p^r$ such that $0 \le k_r, l_r \le p-1$ are integers, and $k_n \neq 0$. Then*

$$\binom{k}{l} \equiv \prod_{r=0}^{n} \binom{k_r}{l_r} \mod p.$$

Here $\binom{i}{j} = 0$ for $i < j$.

Proof. For an independent variable w we have

$$(1+w)^k = \prod_{r=0}^{n} (1+w)^{k_r p^r} \equiv \prod_{r=0}^{n} (1+w^{p^r})^{k_r} \mod p.$$

The result is verified by comparing the coefficient of w^l on both sides of the equation above. $\square$

As will be made clear in Section 4, for a prime p and an integer $k \ge 0$, $\gamma_{p^{k+1}}(u)$ as an element of the E_2 page of the Serre spectral sequence mentioned above, is in the image of the transgression, i.e., differentials with domain in the bottom row and image in the leftmost column. We write $\gamma_{p^{k+1}}(u) = u^{p^{k+1}}/p^{k+1}! = u^{p^{k+1}}/[(p^{k+1}-1)!p^{k+1}]$. The following lemma says, that in the Serre spectral sequence $u^{p^{k+1}}/[(p^{k+1}-1)!p^i$ is killed by the differential of $b_{p,i}\gamma_{p^{k+1}-p^i}(u)$ successively as i ranges over $1, 2, \cdots, k+1$.

Lemma 1.3.2. *For a prime p and $1 \leq i \leq k+1$ the greatest common divisor of $\frac{(p^{k+1}-1)!}{(p^{k+1}-p^i)!}$ and $\frac{p^i!}{p^{i-1}}$ is $(p^i-1)!$. In particular there are integers $\{\lambda_i^{p,k+1}, \mu_i^{p,k+1}\}_{i=0}^k$ such that*

$$\frac{1}{(p^{k+1}-1)!p^i} = \lambda_i^{p,k+1}\frac{1}{(p^{k+1}-1)!p^{i-1}} + \mu_i^{p,k+1}\frac{1}{(p^{k+1}-p^i)!p^i!}.$$

Proof. We have

$$\frac{p^i!}{p^{i-1}} = \frac{p^i}{p^{i-1}}[(p^i-1)!] = p[(p^i-1)!]; \ \frac{(p^{k+1}-1)!}{(p^{k+1}-p^i)!} = \binom{p^{k+1}-1}{p^i-1}[(p^i-1)!]$$

So

$$(p^i-1)!|\mathrm{gcd}\{\frac{p^i!}{p^{i-1}}, \frac{(p^{k+1}-1)!}{(p^{k+1}-p^i)!}\}$$

On the other hand, $p^{k+1}-1 = (p-1)(1+p+\cdots+p^k), p^i = (p-1)(1+p+\cdots+p^{i-1})$. By Lemma 1.3.1, we have

$$\binom{p^{k+1}-1}{p^i-1} \equiv \binom{p-1}{p-1}^{i-1} = 1 \mod p,$$

in particular

$$\mathrm{gcd}\{\binom{p^{k+1}-1}{p^i-1}, p\} = 1,$$

Hence

$$\mathrm{gcd}\{\frac{p^i!}{p^{i-1}}, \frac{(p^{k+1}-1)!}{(p^{k+1}-p^i)!}\}|(p^i-1)!,$$

and the proof is completed. $\square$

Definition 1.3.3. We define $M(3)$ as follows:

1. Let $M(3)_k$ be the kth level of

$$A(2) \otimes_{\mathbb{Z}} A(3). \tag{1.3.1}$$

2. By Definition 1.2.11 it is enough to define the differential of $M(3)$ by the following:

$$d(u) = 0; \quad d(b_1) = u; \quad d(b_{p,k}) = \gamma_{p^k}(u);$$
$$d(a_{p,k}) = (pb_{p,k+1} - \Lambda_0^{p,k+1} b_1 \gamma_{p^{k+1}-1}(u)) - \sum_{i=1}^{k} \Lambda_i^{p,k+1} b_{p,i} \gamma_{p^{k+1}-p^i}(u)$$

where $\{\Lambda_i^{p,k+1}\}_{i=0}^k \subset \mathbb{Z}$ are defined as follows.

3. Fix a set of integers $\{\lambda_i^{p,k+1}, \mu_i^{p,k+1}\}_{i=0}^k$ as in 1.3.2. Define

(a)

$$\Lambda_i^{p,k+1} = \lambda_1^{p,k+1} \cdots \lambda_k^{p,k+1}, \text{ if } i = 0.$$

(b)

$$\Lambda_i^{p,k+1} = \mu_i^{p,k+1} \lambda_{i+1}^{p,k+} \cdots \lambda_k^{p,k+1}, \text{ if } i = 1, \cdots, k-1.$$

(c)

$$\Lambda_i^{p,k+1} = \Lambda_k^{p,k+1} = \mu_k^{p,k+1}, \text{ if } i = k.$$

1.3.2 ensures that d is indeed a differential.

4. We define a bi-degree on $M(3)$ as follows. Let $B\otimes C$ be a monomial in $M(3)$ such that

$$B\in A(3);\ C\in A(2).$$

Then the bi-degree of $B\otimes C$ is $(s,t)=(deg(B),deg(C))$. Clearly the total degree agree with the usual degrees, and s induces a filtration $\mathbf{F}_K$ on $M(3)$.

Remark 1.3.4. Here is a word about the unfortunate coefficients appeared in (2) and (3) as above. To obtain the correct spectral sequence with respect to the filtration $\mathbf{F}_K$, we have to take $d(a_{p,k})=pb_{p,k+1}\text{mod } \mathbf{F}_K^{2p^{k+1}+1}M(3)$. To ensure $d^2=0$, the differential of the remaining terms in $\mathbf{F}_K^{2p^{k+1}+1}M(3)$ has to be $-d(pb_{p,k+1})=u^{p^{k+1}}/[(p^{k+1}-1)!p^k]$. The coefficients $\Lambda_k^{p,k+1},\mu_k^{p,k+1}$ are designed so that $u^{p^{k+1}}/[(p^{k+1}-1)!p^i]$ is killed by successive terms as i ranges over $1,2,\cdots,k$.

The following result plays a central role in this section.

Theorem 1.3.5. *$M(3)$ is acyclic.*

The following lemma is from (14), which were used in the proof of Theorem 1.2.19.

Lemma 1.3.6. *For any integer q and a prime number p the following DGA over $\mathbb{Z}/p$ is acyclic:*

$$P_p(a;2pq+2)\otimes E_p(b;2q+1)\otimes\mathbb{Z}/p[c;2q]/c^p$$

with differential given by

$$d(c)=0;\ d(b)=c;\ d(\gamma_l(a))=c^{p-1}b\gamma_{l-1}(a),\ l\geq 1.$$

Proof. In fact, a chain homotopy can be defined on linear generators as follows:

$$c \to b;\ c^{p-1}b\gamma_{l-1}(a) \to \gamma_l(a),\ l \geq 1$$

and all the other linear generators are sent to 0. □

Lemma 1.3.6 immediately generalize to the following

Lemma 1.3.7. *Let $M(3)[p]$ be the DGA*

$$\bigotimes_{k\geq 0} P(a_{p,k}; 2p^{k+1}+2) \otimes \bigotimes_{k\geq 1} E(b_{p,k}; 2p^k+1) \otimes E(b_1; 3) \otimes P(u; 2) \otimes \mathbb{Z}/p$$

with differential $d[p]$ defined as follows:

$$d[p](u) = 0; \quad d[p](b_1) = u; \quad d[p](b_{p,k}) = \gamma_{p^k}(u);$$

$$d[p](\gamma_l(a_{p,k})) = -\Lambda_k^{p,k+1} b_{p,k}\gamma_{p^{k+1}-p^k}(u) \text{ if } k > 0,$$

$$d[p](\gamma_l(a_{p,0})) = -\Lambda_k^{p,1} b_1 \gamma_{p-1}(u).$$

Then $M(3)[p]$ is acyclic.

Proof.

$$M(3)[p] = \bigotimes_{k\geq 0} P(a_{p,k}; 2p^{k+1}+2) \otimes E(b_{p,k}; 2p^k+1) \otimes \mathbb{Z}[\gamma_{p^k}(u)]/(\gamma_{p^k}(u)^p) \otimes \mathbb{Z}/p,$$

where $b_{p,0} = b_1$. It is a tensor product of DGA's of the form in Lemma 1.3.6 indexed by k, and the result follows. $\square$

Proof of Theorem 1.3.5. We proceed to show that $M(3) \otimes \mathbb{Z}/p$ is acyclic for all primes p. Since $M(3)$ is a degree-wise finitely generated free Abelian group, this, together with the Künneth formula porves the theorem.

Let $(\tilde{E}^r_{s,t}[p], d^r_{s,t})$ be the spectral sequence associated to the filtration $\mathbf{F}_K$ on $M(3) \otimes \mathbb{Z}/p$. Then obviously $E^0_{s,t}[p] = E^1_{s,t}[p]$ and $d^1_{*,*}$ is as follows:

$$d(u) = 0;\ d(b_1) = 0;\ d(b_{p,k}) = 0;$$

$$d(\gamma_l(a_{p',k})) = \begin{cases} p'\gamma_{l-1}(a_{p',k})b_{p',k+1}, & \text{if } p' \neq p \\ 0, & \text{if } p' = p \end{cases}$$

for any prime p'. Therefore we have

$$\begin{aligned} \tilde{E}^2_{*,*}[p] &\cong \bigotimes_{k\geq 0} P_p(a_{p,k}; 2p^{k+1}+2) \otimes \bigotimes_{k\geq 1} E_p(b_{p,k}; 2p^k+1) \otimes E_p(b_1; 3) \otimes P_p(u; 2) \\ &\cong \bigotimes_{k\geq 0} \left[P_p(a_{p,k}; 2p^{k+1}+2) \otimes E_p(b_{p,k}; 2p^k+1) \otimes \mathbb{Z}/p[\gamma_{p^k}(u)]/\gamma_{p^k}(u)^p\right] \end{aligned} \tag{1.3.2}$$

where $b_{p,0} = b_1$. We proceed to consider all the higher differentials on $\tilde{E}^2_{*,*}[p]$. Notice that all the non-trivial differentials on $P_p(a_{p,k}; 2p^{k+1}+2) \otimes E_p(b_{p,k}; 2p^k+1) \otimes \mathbb{Z}/p[\gamma_{p^k}(u)]/\gamma_{p^k}(u)^p$ are the following:

$$d_{2(p^{k+1}-p^k)+1}(a_{p,k}) = \Lambda_k^{p,k+1} b_{p,k}\gamma_{p^k}(u);$$

$$d_{2(p^{k+1}-p^k)+1}(b_{p,k}) = \gamma_{p^k}(u);$$

$$d_{2(p^{k+1}-p^k)+1}(\gamma_{p^k}(u)) = 0.$$

By definition $\Lambda_k^{p,k+1} = \mu_k^{p,k+1}$. By Lemma 1.3.2, $p\lambda_i^{p,k+1} + \mu_i^{p,k+1}\binom{p^{k+1}-1}{p^i-1} = 1$, which shows that $\Lambda_k^{p,k+1} = \mu_k^{p,k+1}$ is invertible mod p.

Notice that exact same statement of the definition of the filtration $\mathbf{F}_K$ in (4) of Definition 1.3.3 can be applied to define a filtration on $M(3)[p]$. A direct comparison shows that this filtration induces a spectral sequence which is identical to $\tilde{E}^2_{*,*}[p]$ after the E_2-page. The theorem then follows from Lemma 1.3.7. □

Remark 1.3.8. The construction of $M(3)$ and the proof of Theorem 1.3.5 are inspired by exp.9 and exp.11 of (14).

Several corollaries can be deduced almost immediately.

Corollary 1.3.9. Let $\Pi = \mathbb{Z}$ with a generator w. There is a morphism of DGA's $g : M(3) \to \mathcal{B}^3(\mathbb{Z}[\Pi])$ that restricts to the weak equivalence $f_2 : A(2) \to \overline{\mathcal{B}}^2(\mathbb{Z}[\Pi])$ and $f_3 : A(3) \to \overline{\mathcal{B}}^3(\mathbb{Z}[\Pi])$. Moreover, g preserves the filtrations $\mathbf{F}_K$ and $\mathbf{F}_B$, where the filtration $\mathbf{F}_B$ is defined in (2) of Example 1.2.3. In particular, when passing to homology $H_*(f_3)(b_{p,k}) = \sigma\gamma_k\sigma^2(w)$.

Proof. Notice that as a graded divided power algebra over $\mathbb{Z}$, $A(2) = P(u; 2) \cong H_*(A(2)) \cong H_*(\overline{\mathcal{B}}^2(\mathbb{Z}[\Pi])$. Let $[x] \in \overline{\mathcal{B}}^2(\mathbb{Z}[\Pi]$ represent a generator of $H_2(\overline{\mathcal{B}}^2(\mathbb{Z}[\Pi])$. Then let $f_2(\gamma_k(u)) = [x|\cdots|x]$ with k copies of x. It follows from (3) of Example 1.2.14 that $f : A(2) \to \overline{\mathcal{B}}^2(\mathbb{Z}[\Pi]$ is a weak equivalence. The rest follows immediately from 1.2.20 and 1.2.21. In particular the statement about the filtrations follows from the fact that the restriction of g on $A(i)$ is f_i for $i = 2, 3$. Therefore g preserves bi-degrees. The last statement about homology follows from Theorem 1.2.26. □

Corollary 1.3.10. Let $\tilde{E}^*_{*,*}$ be the spectral sequence associated to the filtration $\mathbf{F}_K$ on $M(3)$. Then all the higher differentials are identified by

$$d_3(b_1) = u;$$

$$d_{2p^k+1}(b_{p,k}) = \gamma_{p^k}(u), \quad k \geq 1;$$

$$d_r(\gamma_i(u)) = 0, \quad \text{for all } r, i$$

together with the Leibniz rule.

Proof. By Lemma 1.3.9, $H_*(A(3))$ is generated as an $\mathbb{Z}$-algebra by $b_{p,k}$ for all p and k. Thus $\tilde{E}^*_{*,*}$ is generated as a $\mathbb{Z}$-algebra by $b_{p,k}$ and $\gamma_k(u)$. Hence the Leibniz rule applies to give all differentials. □

Figure 1 indicates several non-trivial differentials of this spectral sequence.

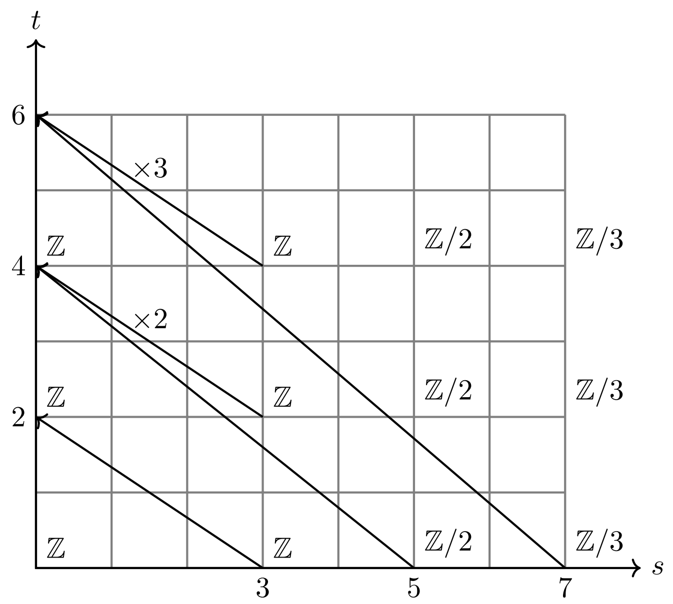

Figure 1. Some non-trivial differentials of $\tilde{E}^{*}_{*,*}$.

We proceed to study the dual complex of $M(3)$, namely $W(3)^i = Hom(M(3)_i, \mathbb{Z})$. It is a cochain complex with a $\mathbb{Z}$-module structure dual to $M(3)$. $W(3)$ has the following $\mathbb{Z}$-algebra structure:

$$W(3) = \bigotimes_p \big[\bigotimes_{k\geq 0} \mathbb{Z}[y_{p,k}; 2p^{k+1}+2] \otimes \bigotimes_{k\geq 1} E(x_{p,k}; 2p^k+1)\big] \otimes \mathbb{Z}[x_1; 3] \otimes \mathbb{Z}[v; 2]/(x_1^2 - y_{2,0}),$$

where $y_{p,k}, x_{p,k}, x_1, v$ are the dual of $a_{p,k}, b_{p,k}, b_1, u$, respectively.

Remark 1.3.11. It is well known and was brought to the author's attention by B. Antieau that $W(3)$ cannot be an anti-commutative DGA. We can verify this by checking that the differential d^* does not satisfy the Leibniz rule. We slightly abuse the notations to let d denote the differential of both $M(3)$ and $W(3)$. Consider v^2, the dual of $\gamma_2(u)$. More precisely, the pairing $\langle v^2, \xi\rangle$ is characterized by

$$\langle v^2, \xi\rangle = \begin{cases} k, & \xi = k\gamma_2(u) \\ 0, & \text{otherwise.} \end{cases}$$

To find $d(v^2)$, notice that $M(3)_5$ is generated as an Abelian group by $b_{2,1}$ and $b_1 u$, on which the differential acts as $d(b_{2,1}) = \gamma_2(u)$ and $d(b_1 u) = u^2 = 2\gamma_2(u)$. Since b_1 and u are elements in terms of the coproduct on $M(3)$, the dual of $b_1 u$ is $x_1 v$. Thus, the relation $\langle v^2, d(-)\rangle = \langle d(v^2), -\rangle$ implies $d(v^2) = 2vx_1 + x_{2,1} \neq 2vx_1$, which disproves the Leibniz formula. However the Leibniz formula is well known to hold for cohomological Serre Spectral sequences in general.

Consider the spectral sequence $\tilde{E}_*^{*,*}$ associated to the dual filtration of $\mathbf{F}_K$ on $W(3)$, denoted by $\mathbf{F}$. Dualization immediately gives $d_1^{*,*}$ and therefore $\tilde{E}_2^{*,*}$. In particular, we have

$$d_1(x_{p,k+1}) = py_{p,k}; \quad d_1(y_{p,k}) = d_1(x_1) = d_1(v) = 0, \quad k \geq 0.$$

Again, these together with the Leibniz rule determine d_1. Hence, $\tilde{E}_2^{*,*}$ is generated as a graded commutative $\mathbb{Z}$-algebra by $v, x_1, y_{p,k}, k \geq 0$ modulo the relations $py_{p,k} = 0$. Dualizing 1.3.10 immediately gives us a good understanding of the higher differentials.

Corollary 1.3.12. The higher differentials of $\tilde{E}_*^{*,*}$ satisfy

$$d_3(v) = x_1$$

$$d_r(\gamma_i(u)) = 0, \quad \text{for all } r, i$$

$$d_{2p^{k+1}-1}(p^k x_1 v^{lp^e-1}) = v^{lp^e-1-(p^{k+1}-1)} y_{p,k}, \quad k \geq e, \gcd(l,p) = 1$$

and the Leibniz rule. Here $\tilde{E}_{2p^{k+1}-1}^{3,2(lp^e-1)}$ is generated by $p^k v^{lp^e-1} x_1$.

Figure 2 shows a low dimensional picture.

Remark 1.3.13. Corollary 1.3.12 does not exhaust the higher differentials of $\tilde{E}_*^{*,*}$. For example, it can be shown that $y_{2,1}v^2$ is a 4-cocycle such that $d_5(y_{2,1}v^2)$ is non-trivial, whereas v^2 is not a 4-cocycle.

At this point the $\mathbb{Z}$-module structure of $H^*(K(\mathbb{Z},3);\mathbb{Z})$ becomes manageable. In low degrees we can even obtain the ring structure:

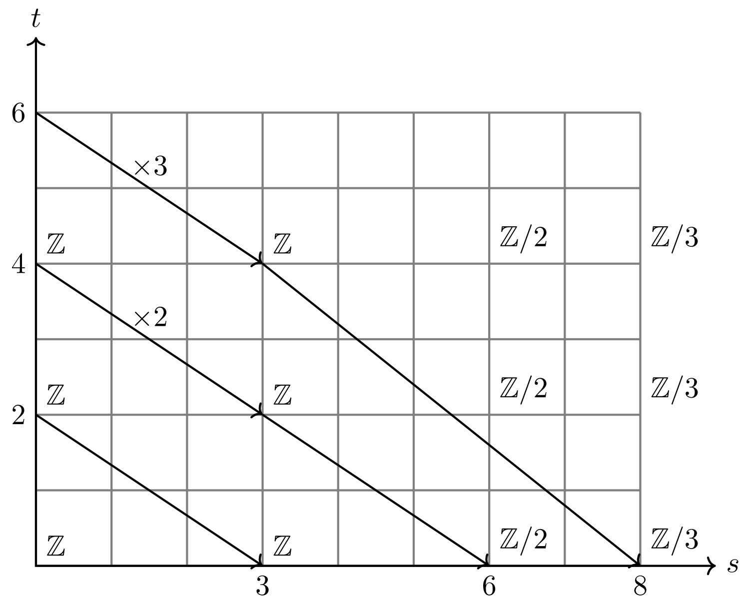

Figure 2. Some non-trivial differentials of $\tilde{E}_*^{*,*}$.

Corollary 1.3.14. In degree less then 15, $H^*(K(\mathbb{Z},3);\mathbb{Z})$ is isomorphic to the following ring:

$$H^*(K(\mathbb{Z},3)) \cong E(x_1;3) \otimes \bigotimes_{k\geq 0;p} \mathbb{Z}/p[y_{p,k};2p^{k+1}+2]/(x_1^2 - y_{2,0})$$

where p runs over all prime numbers.

The corollary fails in higher degrees. For example, in degree 15, we have the dual of $b_{2,2}b_{2,3}$, which is not a product of any $y_{p,k}$'s. The author owes the following remark to A. Bousfield.

Remark 1.3.15. The ring structure of $H^*(K(\mathbb{Z},3);\mathbb{Z})$ can be understood in the following way. It follows from the previous computation that $H^*(K(\mathbb{Z},3);\mathbb{Z})$ contains no element of order p^2, where p is a prime number, and it is a torsion group in degree greater then 3. Therefore, for each prime p the p-primary component of $H^*(K(\mathbb{Z},3);\mathbb{Z})$ above degree 3 is image of the Bockstein homomorphism from $H^*(K(\mathbb{Z},3);\mathbb{Z}/p)$, a well understood $\mathbb{Z}/p$-algebra.

The following technical result will prove useful in section 4.

Corollary 1.3.16. When $s > 3$, $\tilde{E}_2^{s,t}$ is isomorphic to the direct sum $\sum_{r\geq 2} B_r^{s,t}$, where $B_r^{s,t} = \mathrm{Im}\{d_r^{s-r,t-r+1} : \tilde{E}_2^{s-r,t-r+1} \to \tilde{E}_2^{s,t}\}$. In particular, $B_r^{s,t}$ contains no nonzero element of $\mathrm{Im} d_{r'}^{*,*}$ if $r' < r$. Notice that there are only finitely many nonzero components of the direct sum.

Proof. The first statement is as indicated in Figure 2, which also proves the second statement in the case $s = 3$. $\square$

This corollary, in plain words, asserts that in $\tilde{E}_*^{*,*}$, a subgroup generated by a monomial in v, x_1 and some $y_{p,j}$ is hit exactly once by the differential, which kills this subgroup.

It is hardly a surprise to say that the spectral sequence $\tilde{E}_*^{*,*}$ is the same as ${}^K E_*^{*,*}$, the one induced by the fiber sequence $K(\mathbb{Z},2) \to * \to K(\mathbb{Z},3)$. But we cannot make the identification just yet. This is the subject of the next section.

1.4 The Spectral Sequence and Homology Suspension

In this section we study the homology suspension to prove the following

Theorem 1.4.1. *${}^K E_*^{*,*}$ and $\tilde{E}_*^{*,*}$ are isomorphic starting at E_2-pages.*

The reader interested only in the calculation of $H^*(\mathbf{B}PU_n;\mathbb{Z})$ can safely skip the rest of this section. Let Σ and Ω be the pointed suspension and loopspace functor, respectively. For a pointed topological space X, there is a canonical map $\Sigma\Omega X \to X$ since Σ and Ω are a pair of adjoint functors. The induced map $\sigma : H_k(\Omega X;R) \cong H_{k+1}(\Sigma\Omega X;R) \to H_{k+1}(X;R)$ for any natural number k and ring R is called the homology suspension. If X is an Abelian H-space, then we follow (27) and consider the classifying space $\mathbf{B}(X)$ of X (denoted by $\mathbf{B}^\infty(X)$ in (27)), with a canonical map $X \to \Omega\mathbf{B}(X)$, and we also use the term homology suspension to refer to the following homomorphism:

$$H_k(X;R) \to H_{k+1}(\Sigma X;R) \to H_{k+1}(\Sigma\Omega\mathbf{B}(X);R) \to H_{k+1}(\mathbf{B}(X);R) \tag{1.4.1}$$

where the last homomorphism is induced by the evaluation map $\Sigma\Omega\mathbf{B}(X) \to \mathbf{B}(X), (s,\omega) \mapsto \omega(1-s)$ where ω is a loop in $\mathbf{B}(X)$. The rest of the homomorphisms above are the obvious ones from the context.

By the discussion in Section 2 of (27), if X is an Abelian H-space with a CW-complex structure, then so is $\mathbf{B}(X)$. Furthermore, By the proof of Theorem 2.3 of (27) The CW-complex of $\mathbf{B}(X)$ contains ΣX as a sub-complex, and by first and second paragraph of Section 5 of (27), the homology suspension defined by Theorem Equation 1.4.1 is identical to the following

$$H_k(X;R) \to H_{k+1}(\Sigma X;R) \to H_{k+1}(\mathbf{B}(X);R) \tag{1.4.2}$$

where the last arrow $H_{k+1}(\Sigma X;R) \to H_{k+1}(\mathbf{B}(X);R)$ is induced by the composition of the following

$$\Sigma X \to \Sigma X, \quad (s,x) \mapsto (1-s,x),$$

and the inclusion $\Sigma X \hookrightarrow \mathbf{B}(X)$.

Take $X = K(\Pi, n-1)$ for a finitely generated Abelian group Π, and take the cell-decomposition of X such that the corresponding cellular chain complex is $\overline{\mathcal{B}}^{n-1}(\mathbb{Z}[\Pi])$. By Theorem 2.3 of (27) there is a quasi-isomorphism

$$\overline{\mathcal{B}}^{n}(\mathbb{Z}[\Pi]) \to C_*(\mathbf{B}(X);\mathbb{Z}),$$

where $C_*(\mathbf{B}(X))$ is the cellular chain complex of $\mathbf{B}(X) \simeq K(\Pi, n)$, such that when passing to homology, the homology suspension Equation 1.4.2 is compatible with the suspension σ : $H_k(\overline{\mathcal{B}}^{n}(\mathbb{Z})[\Pi]) \to H_{k+1}(\overline{\mathcal{B}}^{n+1}(\mathbb{Z})[\Pi])$. Therefore we have proved the following

Proposition 1.4.2. Let $x \in H_k(\overline{\mathcal{B}}^n(\mathbb{Z})[\Pi]; R)$, then $\sigma(x)$ as in Definition 1.2.5 is the homology suspension of x in the sense of the formula Equation 1.4.1.

We turn to the study of Serre spectral sequences. Let $X \to EX \to \mathbf{B}X$ be a fiber sequence where X is a connected Abelian H-space, EX is contractible, and $\mathbf{B}X$ is the classifying space of X. Consider the corresponding homological Serre spectral sequence with coefficients in R, $E^*_{*,*}$, with $E^2_{s,t} \cong H_s(\mathbf{B}X; H_t(X; R))$. $\alpha \in H_s(\mathbf{B}X; R) \cong E_2^{s,0}$ is called*transgressive* if it is an $r-1$ cycle, or in other workds, $\alpha \in E^s_{s,0}$. Therefore $d^s(\alpha) \in E^s_{0,s-1}$. The differential $d^s_{s,0}$ is called a *transgression*. The following proposition follows from the section named *the transgression* of Chapter 8, (28). For more details see (29).

Proposition 1.4.3. In the Serre spectral sequence above, $\alpha \in E^2_{s,0}$ is transgressive if and only if it is the image of the homology suspension. Moreover, $d^s(\alpha)$ as an element in a quotient of $H_{s-1}(X; R)$, contains the pre-image of α via the homology suspension.

It is clear from Theorem 1.2.26 that $H_*(K(\mathbb{Z}, 3); \mathbb{Z}))$ is generated as a $\mathbb{Z}$-DGA by $\sigma^3(w)$ and $\sigma\gamma_{p^k}\sigma^2(w)$ for all primes p and positive integers k, where w is a generator of $\Pi = \mathbb{Z}$. By Proposition 1.4.3 and the Leibniz rule, they determine all the differentials of integral homological Serre spectral sequence associated to $K(\mathbb{Z}, 2) \to * \to K(\mathbb{Z}, 3)$. On the other hand, it is easy to check that this spectral sequence is identical to the one induced by the filtration $\mathbf{F}$ on the chain complex $M(3)$. Therefore, the filtration $\mathbf{F}$ is the one associated to the Serre spectral sequence. Hence Theorem 1.4.1 is proved.

Remark 1.4.4. The alert reader may wonder why we do not apply the cohomological version of Proposition 1.4.3 directly to prove Theorem 1.4.1 and therefore avoid all the trouble in Section

3. It is because the "dual" differentials of the transgressions in the cohomological Serre spectral sequences are not transgressions anymore, and the author, unfortunately, did not find a way to handle them in the same way as the transgressions are.

1.5 The Higher Differentials of ${}^U E_*^{*,*}$

In this section we identify some higher differentials of the spectral sequence ${}^U E_*^{*,*}$ which make it possible to prove Theorem 0.0.1. To do so, we compare it with ${}^K E_*^{*,*}$, via the following commutative diagram:

$$\begin{tikzcd}
\mathbf{B}S^1 \arrow[r, "\simeq"] & K(\mathbb{Z},2) \arrow[r] \arrow[d, "\mathbf{B}\varphi"] & * \arrow[r] \arrow[d] & K(\mathbb{Z},3) \arrow[d, equal] \\
\Phi : & \mathbf{B}T^n \arrow[r] \arrow[d, "\mathbf{B}\psi"] & \mathbf{B}PT^n \arrow[r] \arrow[d, "\mathbf{B}\psi'"] & K(\mathbb{Z},3) \arrow[d, equal] \\
\Psi : & \mathbf{B}U_n \arrow[r] & \mathbf{B}PU_n \arrow[r] & K(\mathbb{Z},3)
\end{tikzcd} \tag{1.5.1}$$

Here $\varphi : S^1 \to T^n$ is the diagonal, and ψ and ψ' are the inclusions of the maximal tori. Then for any compact Lie group G and its maximal torus T, $H^*(\mathbf{B}G;\mathbb{Q})$ is the sub-$\mathbb{Q}$-algebra of $H^*(\mathbf{B}T;\mathbb{Q})$ stable under the action of the Weyl group. In our case, we have

$$H^*(\mathbf{B}PU_n;\mathbb{Q}) \cong \mathbb{Q}[\{v_i - v_j\}]^{\mathbf{W}} \cong H^*(\mathbf{B}PT^n;\mathbb{Q})^{\mathbf{W}}, \tag{1.5.2}$$

where $\mathbf{W}$ is the Weyl group, acting on $\{v_i\}$ by permutation. (See (18).)

Let v be the multiplicative generator of $H^*(\mathbf{B}S^1;\mathbb{Z})$, v^i be the ith copy of v in $H^*(\mathbf{B}(S^1)^n;\mathbb{Z}) \cong H^*((\mathbf{B}T^n;\mathbb{Z})$, and $c_k \in H^*(\mathbf{B}U_n;\mathbb{Z})$ be the kth universal Chern class. Moreover let $\sigma_k \in H^*(\mathbf{B}T^n;\mathbb{Z})$ be the kth elementary polynomial in $v_1, \cdots, v_n$. In this case we have the splitting principal, which asserts that the above argument applies to integral cohomology ((30)). More precisely, it says that $\mathbf{B}\psi^*$ is the inclusion taking c_k to σ_k, or in other words,

$$\mathbf{B}\psi^*(\sum_{k=0}^{n} c_k w^k) = \prod_{i=1}^{n}(1 + v_i w)$$

where w is a polynomial generator.

We consider the morphisms of spectral sequences induced by Φ and Ψ. Our first result is the following

Proposition 1.5.1. The differential ${}^T d_r^{*,*}$, is partially determined as follows:

$$^T d_r^{*,2t}(v_i^t \Xi) = (\mathbf{B}\rho_i)^{*K}(d_r^{*,2t}(v^t \Xi)), \tag{1.5.3}$$

where $\Xi \in {}^T E_r^{0,*}$, a quotient group of $H^*(K(\mathbb{Z}, 3); \mathbb{Z})$, and $\rho_i : T^n \to S^1$ is the projection of the ith diagonal entry. In plain words, ${}^T d_r^{*,2t}(v_i^t \Xi)$ is simply ${}^T d_r^{*,2t}(v^t \Xi)$ with v replaced by v_i.

The proof is straightforward since the Serre spectral sequence is functorial.

We proceed to study the differentials ${}^T d_*^{0,*}$ with domain in the leftmost column.

Proposition 1.5.2. 1. The differential ${}^T d_3^{0,2t}$ is given by the "formal divergence"

$$\nabla = \sum_{i=1}^{n}(\partial/\partial v_i) : H^t(\mathbf{B}T^n) \to H^{t-2}(\mathbf{B}T^n),$$

in such a way that ${}^T d_3^{*,*} = \nabla(-)\cdot x_1$. This remains true when the ground ring $\mathbb{Z}$ is replaced by $\mathbb{Z}/m$ for any integer m.

2. The spectral sequence degenerates at ${}^TE_4^{0,*}$, or in other words, $\mathrm{Ker}(d_3^{0,*}) = \mathbb{Z}[v_1 - v_n, \cdots, v_{n-1} - v_n]$. In particular, ${}^TE_\infty^{0,*} = {}^TE_4{}^{0,*} = \mathrm{Ker} d_3^{0,*}$.

Proof. (1) is an immediate consequence of the Leibniz rule, chain rule, and Proposition 1.5.1.

Given a polynomial $\theta(v_1, \cdots, v_n)$, we change variables to rewrite it as $\tilde{\theta}(v_1 - v_n, \cdots, v_{n-1} - v_n, v_n)$. Equation 1.5.2 then tells us that $H^*(\mathbf{B}PU_n; \mathbb{Q})$ is the free commutative ring generated by $\{t_i - t_n\}_{i=1}^{n-1}$. The equations

$$\begin{aligned}
&d_3(\theta(v_1, \cdots, v_n)) \\
=& \sum_{i=1}^{n} \frac{\partial}{\partial v_i} \tilde{\theta}(v_1 - v_n, \cdots, v_{n-1} - v_n, v_n) \\
=& \frac{\partial}{\partial v_n} \tilde{\theta}(v_1 - v_n, \cdots, v_{n-1} - v_n, v_n)
\end{aligned}$$

then imply that $d_3(\theta(v_1, \cdots, v_n)) = 0$ if and only if $\tilde{\theta}(v_1 - v_n, \cdots, v_{n-1} - v_n, v_n)$ is independent of v_n, which proves (2). □

Corollary 1.5.3. ${}^Ud_3(c_k) = (n - k + 1)c_{k-1}x_1$.

Proof.

$${}^Ud_3(c_k) = {}^Td_3(\sigma_k) = \sum_{i=1}^{n} \frac{\partial \sigma_k}{\partial v_i} = (n - k + 1)c_{k-1}x_1.$$

□

Next we study ${}^Td_*^{*,*}$ in general.

Proposition 1.5.4. Let $\theta(v_1, \cdots, v_n) \in H^t(\mathbf{B}T^n)$ be an element in ${}^TE_2^{0,2t}$. As in the proof of Proposition1.5.2 we apply the change of variable $(v_1 - v_n, \ldots, v_{n-1} - v_n, v_n)$ and rewrite θ as

$$\theta = \sum_{i=0}^{t} \theta_i v_n^i$$

such that $\theta_i \in \mathrm{Ker}\nabla$, $i = 0, \cdots, t$. If $\theta(v_1, \cdots, v_n) \in H^{2t}(\mathbf{B}T^n)$ represent an element in ${}^TE_2^{0,2t}$, ξ being an element of $H^s(K(\mathbb{Z}, 3))$ for some s. If $\theta\xi \in {}^TE_r^{s,*}$ for $r > 3$, then there are $m_i \in \mathbb{Z}$ and $\theta_i' \in \mathrm{Ker}\nabla$, $i = 0, \cdots, t$ such that $m_i v_n^i \xi \in {}^TE_r^{*,*}$ for all i and that

$$\theta = \sum_{i=0}^{t} m_i v_n^i \theta_i'$$

Therefore

$${}^Td_r(\theta\xi) = {}^Td_r(\sum_{i=0}^{t} m_i v_n^i \theta_i') = \sum_{i=0}^{t} {}^Td_r(m_i v_n^i)\theta_i'$$

Proof. In this proof everything in sight is localized at a prime number p.

Case 1: s is even. Then

$${}^Td_3(\theta\xi) = \nabla(\theta)x_1\xi = \sum_{i=0}^{t} \frac{\partial}{\partial v_n}(v_n^i)\theta_i x_1 \xi = 0.$$

Comparing the exponents of v_n of each term, we have $\theta_i = 0$ unless $p|i$, in which case $v_n^i\xi$ is in the image of some ${}^Td_r$, by 1.3.12.

Case 2: s is odd. It suffice to consider the special case of $\xi = x_1$, from which the general one can be obtained from the Leibniz rule. Since everything is localized at p, $v_n^t = 0$ unless $p|t+1$. $\square$

Consider a map $\kappa : T^n \to PT^n \times S^1$ defined as follows:

$$\begin{pmatrix} \lambda_1 & 0 & \dots \\ 0 & \lambda_2 & \dots \\ \vdots & \vdots & \lambda_n \end{pmatrix} \mapsto \left(\begin{bmatrix} \lambda_1 & 0 & \dots \\ 0 & \lambda_2 & \dots \\ \vdots & \vdots & \lambda_n \end{bmatrix}, \lambda_n \right)$$

where the matrix in the square bracket denotes its class in PT^n. Then κ is a homeomorphism, its inverse being the following:

$$\left(\begin{bmatrix} \lambda_1 & 0 & \dots \\ 0 & \lambda_2 & \dots \\ \vdots & \vdots & \lambda_n \end{bmatrix}, \lambda \right) \mapsto \begin{pmatrix} \lambda\lambda_1/\lambda_n & 0 & \dots \\ 0 & \lambda\lambda_2/\lambda_n & \dots \\ \vdots & \vdots & \lambda \end{pmatrix}.$$

Passing to classifying spaces, we have $\mathbf{B}\kappa : \mathbf{B}T^n \simeq \mathbf{B}PT^n \times \mathbf{B}S^1$.

Proposition 1.5.5. Let $\theta(v_1, \cdots, v_n) \in H^t(\mathbf{B}T^n)$ be an element in ${}^TE_2^{0,2t}$. As in the proof of 1.5.2 we apply the change of variable $(v_1 - v_n, \dots, v_{n-1} - v_n, v_n)$ and rewrite θ as

$$\theta = \sum_{i=0}^{t} \theta_i v_n^i$$

such that $\theta_i \in \mathrm{Ker}\nabla$, $i = 0, \cdots, t$. Then

$$(\mathbf{B}\kappa^{-1})^*(\theta) = \sum \theta_i \otimes v_n^i$$

.

Consider the fiber sequence $\mathbf{B}PT^n \to \mathbf{B}PT^n \to *$ and let ${}^PE_*^{*,*}$ be its cohomological Serre Spectral sequence. We take the product of fiber sequences

$$(\mathbf{B}PT^n \to \mathbf{B}PT^n \to *) \times (\mathbf{B}S^1 \to * \to K(\mathbb{Z}, 3))$$

or equivalently

$$\mathbf{B}PT^n \times \mathbf{B}S^1 \to \mathbf{B}PT^n \to K(\mathbb{Z}, 3).$$

Then we have the following morphism between fiber sequences:

$$\begin{array}{ccccc} \mathbf{B}T^n & \longrightarrow & \mathbf{B}PT^n & \longrightarrow & K(\mathbb{Z},3) \\ \big\downarrow{\scriptstyle \mathbf{B}\kappa} & & \Big\| & & \Big\| \\ \mathbf{B}PT^n \times \mathbf{B}S^1 & \longrightarrow & \mathbf{B}PT^n & \longrightarrow & K(\mathbb{Z},3) \end{array}$$

with the unspecified maps being the obvious ones. This diagram is easily seen to be commutative by the discussion above.

Proposition 1.5.6. Let ${}^PE_*^{*,*}$ be the cohomological Serre Spectral sequence associated to $\mathbf{B}PT^n \to \mathbf{B}PT^n \to *$. Then the commutative diagram above induces an isomorphism of spectral sequences ${}^TE_r^{*,*} \cong {}^PE_r^{*,*} \otimes {}^KE_r^{*,*}$ for $r \geq 2$.

Proof. By Proposition 1.5.5 the commutative diagram above induces an isomorphism of E_3 pages, and the statement follows from Theorem 3.4 in (26). $\square$

Remark 1.5.7. ${}^{P}E_*^{*,*}$ is stabilized as $\mathrm{Ker}\nabla \cong H^*(\mathbf{B}PT^n)$ concentrating in the 0th column. Let $\theta(v_1, \cdots, v_n) \in H^{2t}(\mathbf{B}T^n)$ represent an element in ${}^{T}E_2^{0,2t}$, ξ be an element of $H^s(K(\mathbb{Z}, 3))$ for some s. If $\theta\xi \in {}^{T}E_r^{s,*}$, then 1.5.5 and 1.5.6 tell as that there are $m_i \in \mathbb{Z}$ and $\theta_i' \in \mathrm{Ker}\nabla$, $i = 0, \cdots, t$ such that $m_i v_n^i \xi \in {}^{T}E_r^{*,*}$ for all i and that

$$\theta = \sum_{i=0}^{t} m_i v_n^i \theta_i'$$

Comparing this with 1.5.5 we have $\theta_i = m_i\theta_i'$. The differential ${}^{T}d_r^{3,*}$ is hence determined by

$${}^{T}d_r(\theta\xi) = {}^{T}d_r(\sum_{i=0}^{t} m_i v_n^i \theta_i'\xi) = \sum_{i=0}^{t} {}^{T}d_r(m_i v_n^i \xi)\theta_i'$$

where ${}^{T}d_r(m_i v_j^i \xi)$ is determined as in 1.5.1. This is how we determine the differentials in practice. This idea is explained more concretely in 1.5.9.

Finally we are at the place to state our main theorem of this section, of which the proof is already clear.

Theorem 1.5.8. *1.5.1, 1.5.2 and 1.5.6 determine all of the differentials of* ${}^{T}E_*^{*,*}$*. Restricted to symmetric polynomials in* $v_1, \cdots, v_n$*, they determine all of the differentials* ${}^{U}d_r^{s-r,t+r-1}$ *of* ${}^{U}E_*^{*,*}$ *such that for any* $r' < r$*,* ${}^{T}d_{r'}^{s-r',t-r'+1} = 0$.

With enough patience or perhaps a computer program one can apply this apparatus to calculate $H^k(\mathbf{B}PU_n;\mathbb{Z})$ for many k and n up to group extension. The interested readers are invited to take the following example as an exercise.

Example 1.5.9. Let $n=3$. Show that ${}^{U}d_3(2c_3x_1)=0$ but $2c_3x_1$ is not a permanent cocycle.

Proof. In ${}^{T}E_*^{*,*}$ we have

$$2v_1v_2v_3 = 2(v_1-v_2)(v_2-v_3)+2[(v_1-v_3)+(v_2-v_3)]v_3^2+2v_3^3,$$

the first two terms on the right side being easily checked to be permanent cocycles. It remains to check

$${}^{T}d_3(2v_3^3x_1)=6v_3^2y_{2,1}=0$$

since $2y_{2,1}=0$. This verifies the first statement. For the second one, we have

$${}^{T}d_5(2v_3^3x_1)=0$$

since v_3^3 has order 4 in ${}^{T}E_4^{*,*}$ but $y_{3,1}$ is of order 3. However

$${}^{T}d_7(2v_3^3x_1)=y_{2,2}\neq 0,$$

hence the second statement. $\square$

We end this section by several useful corollaries. Let ∇, ξ be the same as earlier, and let $\rho = \rho(c_1, \cdots, c_n) \in H^*(\mathbf{B}U_n)$.

Corollary 1.5.10. ${}^U d_3(\rho\xi) = {}^U d_3(\rho)\xi = \nabla(\rho) x_1 \xi$.

This is immediate from the Leibniz rule and 1.5.2. One can certainly use 1.5.6 to study ${}^U d_3$, but one will find 1.5.10 much more convenient.

Corollary 1.5.11. If $\rho\xi \in {}^U E_r^{s,r-1}$, or in other words, the image of $\rho\xi$ of ${}^U d_r$ lies in bottom row, then

$$ {}^U d_r(\rho\xi) = {}^K d_r(\mathbf{B}(\psi \cdot \phi)^*(\rho)\xi), $$

where ϕ and ψ are as in Equation 1.5.1. In particular,

$$ {}^U d_r(c_k\xi) = \binom{n}{k} {}^K d_r(v^k \xi). $$

Proof. Consider the commutative diagram Equation 1.5.1 at the beginning of this section. The result is immediate upon passing to the diagram of Serre spectral sequences: $\Psi \cdot \Phi$ induces $c_k \mapsto \binom{n}{k} v^k$ on the leftmost column and the identity on the rightmost one, the latter implying that the induced morphism of spectral sequences is the identity on $H^*(K(\mathbb{Z}, 3))$ when restricted to the bottom rows of the E_2 pages. $\square$

The following two corollaries are immediate from 1.5.6.

Corollary 1.5.12. Let $\mathbb{Z}_{(p)}$ be the p-local ring for a prime number p. If $({}^K d_r^{s,t'} \otimes \mathbb{Z}_{(p)}) = 0$ for all $t' \leq t$, then so is $({}^U d_r^{s,t} \otimes \mathbb{Z}_{(p)})$. In particular, if ${}^K d_r^{s,t'} = 0$ for all $t' \leq t$, then so is ${}^U d_r^{s,t}$.

Corollary 1.5.13. For any $s > 3$ and any r, t, if ${}^T d_r^{3,t} = 0$ (resp. ${}^K d_r^{3,t} = 0$), then so is ${}^T d_r^{s,t}$ (resp. ${}^K d_r^{s,t} = 0$).

In the next section we will apply the apparatus developed in this section to compute $H^k(\mathbf{B}PU_n;\mathbb{Z})$ in interesting cases.

1.6 Some Remarks on the Integral Cohomology of $\mathbf{B}PU_n$

In this section we make some remarks on the main theorem, including a comparison of the main theorem with the computation by A. Kono and M. Mimura, in their paper (25). It turns out that one can use the apparatus developed in Section 5 almost exclusively throughout, but $H^*(\mathbf{B}PU_n;\mathbb{Z})$ has some nice general properties that simplify the calculation considerably.

Lemma 1.6.1. *Let $\bar{P} : \mathbf{B}SU_n \to \mathbf{B}PU_n$ be the map induced by the obvious quotient map $SU_n \to PU_n$.*

1. *$\bar{P}$ induces an isomorphism $H^*(\mathbf{B}PU_n;\mathbb{Q}) \xrightarrow{\cong} H^*(\mathbf{B}SU_n;\mathbb{Q})$. In particular, $\bar{P}^* : H^*(\mathbf{B}PU_n;\mathbb{Z}) \xrightarrow{\cong} H^*(\mathbf{B}SU_n;\mathbb{Z})$ is a monomorphism modulo torsion, such that Im(P^*) has the same rank as $H^*(\mathbf{B}PU_n;\mathbb{Z})$ in each dimension.*
2. *Let m be an integer such that gcd$\{m, n\} = 1$, then $H^*(\mathbf{B}PU_n;\mathbb{Z})$ has no non-trivial m-torsion.*

Proof. Let SU_n be the special unitary group of degree n. The short exact sequence of Lie groups

$$1 \to \mathbb{Z}/n \to SU_n \xrightarrow{P} PU_n \to 1$$

induces a fiber sequence

$$K(\mathbb{Z}/n, 1) \to \mathbf{B}SU_n \to \mathbf{B}PU_n.$$

We shift it to obtain another fiber sequence

$$\mathbf{B}SU_n \to \mathbf{B}PU_n \to K(\mathbb{Z}/n, 2),$$

and consider the cohomological Serre spectral sequence with coefficients in $\mathbb{Q}$ and $\mathbb{Z}/m$. The first and second statement follows respectively from the vanishing of $H^k(K(\mathbb{Z}/n;\mathbb{Q})$ and $H^k(K(\mathbb{Z}/n;\mathbb{Z}/m) = 0$, for all $k > 0$. □

Remark 1.6.2. The map $\bar{P}$ factors as $\mathbf{B}SU_n \to \mathbf{B}U_n \xrightarrow{P} \mathbf{B}PU_n$, where P is induced by the quotient map $U_n \to PU_n$. By (1) of Lemma 1.6.1, $P^* : H^*(\mathbf{B}PU_n;\mathbb{Z}) \to H^*(\mathbf{B}U_n;\mathbb{Z})$ is a monomorphism modulo torsions. Furthermore, (1) of Lemma 1.6.1 shows that the torsion free component of $H^k(\mathbf{B}PU_n;\mathbb{Z})$ is 0 if k is odd, and is a finitely generated free Abelian group of the same rank as the Abelian group of homogenous polynomials in $\mathbb{Z}[c_2, c_3, \cdots, c_n]$ of degree $k/2$, if k is even. Therefore the group structure of the torsion free component of $H^*(\mathbf{B}PU_n;\mathbb{Z})$ is determined, though it is not at all obvious what the generators of $P^*(H^*(\mathbf{B}PU_n;\mathbb{Z}))$ are, in terms of the universal Chern classes. The calculation of the torsion free components is henceforth omitted unless we need some particular generators. An alternative method to calculate the torsion free component is via representation theory. Examples can be found in (6).

We proceed to compare the main theorem with the result of Kono and Mimura. In the case that $n = 4l + 2$ for some integer l, we apply the Künneth formula to obtain $H^k(\mathbf{B}PU_n; \mathbb{Z}/2)$ from $H^k(\mathbf{B}PU_n; \mathbb{Z})$. When $l > 0$, we have the following:

$$H^1(\mathbf{B}PU_{4l+2}; \mathbb{Z}/2) = 0,$$

$$H^k(\mathbf{B}PU_{4l+2}; \mathbb{Z}/2) = \mathbb{Z}/2, \text{ for } k = 2, 3, 4, 5, 7,$$

$$H^k(\mathbf{B}PU_{4l+2}; \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2, \text{ for } k = 6, 8,$$

$$H^9(\mathbf{B}PU_{4l+2}; \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

This is in accordance with the result by A. Kono and M. Mimura:

Theorem 1.6.3 (A. Kono and M. Mimura, Theorem 4.12, (25))**.** *As a $\mathbb{Z}/2$-module*

$$H^*(\mathbf{B}PU(4l + 2); \mathbb{Z}/2) \cong \mathbb{Z}/2[a_2, a_3, x'_{8m+8}, y(I)]/R$$

where $x'_{8m+8} = \{b^2_{4m+4}\}$, (for some b_{4m+4} not relevant to our business), $1 \leq m \leq 2l$, and I runs over all sequences satisfying $1 \leq r \leq 2l$ and $1 \leq i_1 < \cdots < i_r \leq 2l$. $y(I)$ has degree $(\sum_{j=1}^{r}(4i_j+4))-2$. (The degree of a generator is equal to its subscript.) The ideal R is generated by $a_3y(I), y(I)^2 + \sum_{j=1}^{r} x'_{8i_1+8} \cdots a^2_{4i_j+2} \cdots x_{8i_r+8}$ and $y(I)y(J) + \sum_i f_i y(I_i)$, (for $f_i \in \mathbb{Z}/2$).

The notations in the theorem above are slightly adjusted from the original, for consistency. The explanations in the parentheses are added by the author. In view of this theorem, the generator(s) of each $H^k(\mathbf{B}PU_{4l+2}; \mathbb{Z}/2)$ where $1 \leq k \leq 9$ are given in the following table:

TABLE I

GENERATORS OF $H^K(\mathbf{B}PU_{4L+2};\mathbb{Z}/2)$, FOR $L > 0$.

k	2	3	4	5	6	7	8	9
generator(s)	a_2	a_3	a_2^2	a_2a_3	$a_2^3, a_3^2, y(1)$	$a_2^2a_3$	$a_2y(1), a_2^4, a_3^2a_2$	$a_2^3a_3, a_3^3$

As the reader can easily see, this agrees with the result obtain earlier by Theorem 0.0.1 and Künneth formula. In the exceptional case of $l = 0$, the element $y(1)$ does not exist, accounting for the absence of Chern classes c_3, c_4, which would contribute a direct sum component $\mathbb{Z}/2$ when $k = 6$ and 8, respectively.

Remark 1.6.4. In the statement of Theorem 1.6.3,whether "As a $\mathbb{Z}/2$-module" can be strengthened to "As a $\mathbb{Z}/2$-algebra" is an open question. However, we have a positive answer in degree ≤ 9.

Let Sq^i be ith Steenrod operation. Let $\beta : H^*(-;\mathbb{Z}/2) \to H^{*+1}(-;\mathbb{Z})$ be the Bockstein homomorphism induced by short exact sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$. Recall that Sq^1 is the following composition

$$Sq^1 : H^*(-;\mathbb{Z}/2) \to H^{*+1}(-;\mathbb{Z}) \to H^{*+1}(-;\mathbb{Z}/2), \tag{1.6.1}$$

where the second arrow is the mod 2 reduction.

Lemma 1.6.5. *In $H^*(\mathbf{B}PU(4l+2);\mathbb{Z}/2)$, we have:*

1. *$Sq^1(a_2) = a_3$.*

2. *The mod 2 reduction of reduction of e_2 (resp. e_3) is a_2^2 (resp. $y(1)$).*

Proof. The short exact sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$ induces an exact sequence

$$0 = H^2(BPU_{4l+2};\mathbb{Z}/2) \xrightarrow{\beta} H^3(BPU_{4l+2};\mathbb{Z}) \xrightarrow{\times 2} H^3(BPU_{4l+2};\mathbb{Z}). \tag{1.6.2}$$

Notice that there is a unique 2-torsion $(2l+1)x_1$ in $H^3(BPU_{4l+2};\mathbb{Z})$, and the exactness of (Equation 1.6.2) implies that $\beta(a_2) = (2l+1)x_1$. Since $H^3(BPU_{4l+2};\mathbb{Z}/2)$ is generated by a single element a_3, a_3 is the mod 2 reduction of x_1. Therefore, by (Equation 1.6.1) we have $Sq^1(a_2) = (2l+1)a_3 = a_3$. Hence we proved (1).

By Corollary 1.5.3, ${}^Ld_3(c_1) = (4l+2)x_1 = 0$ mod 2, which means that c_1 is a permanent cocycle in the mod 2 version of ${}^LE_*^{*,*}$. Hence a_2 is represented by c_1. Then (2) of Theorem 0.0.1 implies that a_2^2 is the mod 2 reduction of $(4l+1)c_1^2 - 2(4l+2)c_2 \in H^4(\mathbf{B}PU_{4l+2};\mathbb{Z})$. Hence a_2^2 is the mod 2 reduction of e_2. Notice that $y(1)$ is the only non-zero element in $H^6(BPU_{4l+2};\mathbb{Z}/2)$ that annihilates a_3. Comparing this with (8) of Theorem 0.0.1, we see that $y(1)$ is the mod 2 reduction of e_3. □

The rest of the ring structure of $H^*(BPU_{4l+2};\mathbb{Z}/2)$ in degrees ≤ 9 follows from Lemma Equation 1.6.2 and the Leibniz rule of the Bockstein homomorphism. For example, $a_2a_3 \neq 0$ since $\beta(a_2a_3) = \beta(a_2)a_3 + a_2\beta(a_3) = a_3^2 \neq 0$.

In the statement of Theorem 0.0.1, we see that for some k and n, some components of $H^k(\mathbf{B}PU_n;\mathbb{Z})$ are generated by elements in the image of the homomorphism $H^k(K(\mathbb{Z},3);\mathbb{Z}) \to H^k(\mathbf{B}PU_n;\mathbb{Z})$, which is induced by the map $\mathbf{B}PU_n \to K(\mathbb{Z},3)$ as in the fiber sequence Equation 1.1.1. In view of this we have the following

Corollary 1.6.6. 1. When n is even, the 2-torsion subgroup of $H^k(\mathbf{B}PU_n;\mathbb{Z})$ for $k = 6, 9, 10$ are generated by the cohomology operations $y_{2,0}, y_{2,0}x_1$, and $y_{2,1}$ respectively, applied to a generator of $H^3(\mathbf{B}PU_n;\mathbb{Z})$.

2. When $3|n$, and $k = 8$, the 3-torsion subgroup of $H^8(\mathbf{B}PU_n;\mathbb{Z})$ is generated by the cohomology operation $y_{3,0}$ applied to a generator of $H^3(\mathbf{B}PU_n;\mathbb{Z})$.

Proof. In the statement of Theorem 0.0.1, we see that for the k's and n's in the lemma, the corresponding torsion subgroups of $H^k(\mathbf{B}PU_n;\mathbb{Z})$ are generated by the images of elements $y_{2,0}, y_{2,0}x_1, y_{2,1}$ and $y_{3,0}$ respectively, under the homomorphism $H^k(K(\mathbb{Z},3);\mathbb{Z}) \to H^k(\mathbf{B}PU_n;\mathbb{Z})$, which is induced by the map $\mathbf{B}PU_n \to K(\mathbb{Z},3)$ as in the fiber sequence Equation 1.1.1, and the proof is completed. $\square$

We proceed to study the p-local components of $H^*(\mathbf{B}PU_n;\mathbb{Z})$, for any prime number p.

Lemma 1.6.7. *If $t+1 < p$, then*

$$^UE_\infty^{3,2t} \otimes \mathbb{Z}_{(p)} \cong {}^UE_3^{3,2t}/Im({}^Ud_3^{0,2t+2}) \otimes \mathbb{Z}_{(p)}$$

.

Proof. In view of 1.5.12, we consider the differentials out of ${}^{K}E_r^{3,2t'}$ for $r \geq 3$, which are subgroups of ${}^{K}E_3^{3,2t'}/\mathrm{Im}d_3({}^{K}E_3^{0,2t'+2}) \cong \mathbb{Z}/(t'+1)$. Since $t+1 < p$, $\gcd(t'+1,p) = 1$ for all $t' \leq t$, and the images of ${}^{K}d_r^{3,2t'}$ therefore always have order relatively prime to p, for $t' \leq t$. In other words, ${}^{K}d_r^{3,t'} \otimes \mathbb{Z}_{(p)} = 0$. Hence by 1.5.12, ${}^{U}d_r^{3,2t} \otimes \mathbb{Z}_{(p)} = 0$, and the result follows. □

Theorem 1.6.8. *If* $3 < k < 2p+1$ *then* $H^k(\mathbf{B}PU_n;\mathbb{Z}) \otimes \mathbb{Z}_{(p)}$

$$\cong \begin{cases} Ker^{U}d_3^{0,k} \otimes \mathbb{Z}_{(p)}, & k \text{ is even,} \\ H^{k-3}(\mathbf{B}U_n)/\nabla H^{k-1}(\mathbf{B}U_n) \otimes \mathbb{Z}_{(p)}, & k \text{ is odd.} \end{cases}$$

Proof. Recall that for $s < 2p+2$, $H^s(K(\mathbb{Z},3))$ contains no element of order p. Thus, for $3 < k < 2p+1$, we have

$$H^k(\mathbf{B}PU_n;\mathbb{Z}) \otimes \mathbb{Z}_{(p)} \cong {}^{U}E_\infty^{3,k-3} \otimes \mathbb{Z}_{(p)}.$$

The case that k is even follows immediately. The other case is an immediate corollary of 1.6.7. □

We break the proof of Theorem 0.0.1 into several sections. The readers could refer to Figure Figure 3 for the spectral sequence ${}^{U}E_*^{*,*}$.

1.7 $H^k(\mathbf{B}PU_n;\mathbb{Z})$ for $1 \leq k \leq 6$

For $1 \leq r \leq 5$ the results are given in (1). The interested readers can compare it to our computation.

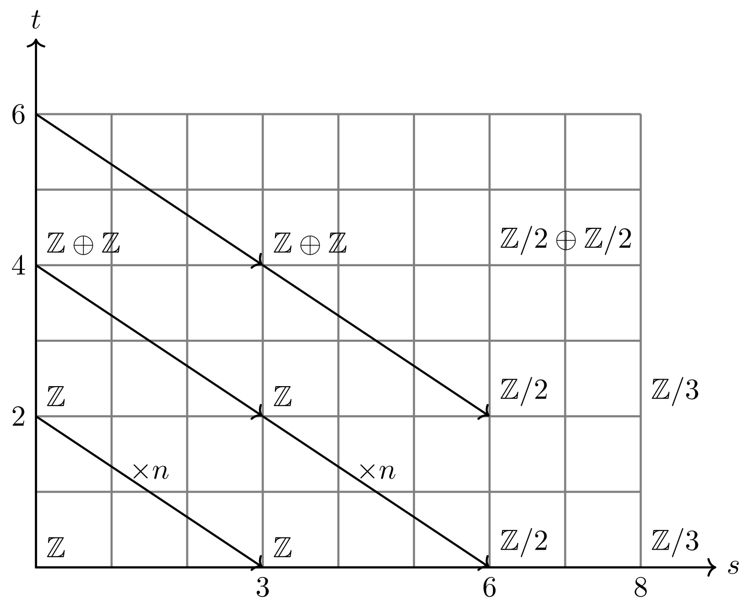

Figure 3. Some non-trivial differentials of ${}^{U}E_{*}^{*,*}$. A node with coordinate (s, t) is unmarked if ${}^{U}E_2^{s,t} = 0$.

Proof of (1) to (4) of Theorem 0.0.1. First notice ${}^UE_2^{s,t} = 0$ for $s = 1, 2, 4, 5, 7$ or t odd. By Proposition 1.5.6, ${}^Ud_3(c_1) = {}^Td_3(\sum_{i=1}^n v_i) = nx_1$. This immediately proves (1) of Theorem 0.0.1. See Figure 3 for a picture in low dimensions. By Corollary 1.5.3 we have

$$ {}^Ud_3(c_2) = (n-1)c_1x_1, {}^Ud_3(c_1^2) = 2nc_1x_1, {}^Ud_3(c_1x_1) = nx_1^2. \tag{1.7.1}$$

So ${}^UE_4^{0,4} \cong {}^UE_\infty^{0,4} \cong H^4(\mathbf{B}PU_n;\mathbb{Z})$ is easily verified as $\mathbb{Z}$. Interested readers can refer to Lemma 3.2 of (1) to identify $H^4(\mathbf{B}PU_n;\mathbb{Z})$ as a subgroup of $H^4(\mathbf{B}U_n;\mathbb{Z})$, or compute it directly. Either way, we proved (2) of Theorem 0.0.1.

Notice ${}^UE_\infty^{0,6} \cong \mathbb{Z}$, by Remark 1.6.2.

1. If n is even, by (Equation 1.7.1), ${}^Ud_3^{0,4}$ is a surjection and therefore ${}^Ud_3^{3,2} = 0$. Hence $H^5(\mathbf{B}PU_n;\mathbb{Z}) \cong {}^UE_4^{3,2} = 0$, and $H^6(\mathbf{B}PU_n;\mathbb{Z}) \cong \mathbb{Z} \oplus {}^UE_\infty^{6,0} \cong {}^UE_\infty^{0,6} \oplus {}^UE_4^{6,0} \cong {}^UE_\infty^{0,6} \oplus {}^UE_3^{6,0} \cong \mathbb{Z} \oplus \mathbb{Z}/2$, with ${}^UE_3^{6,0}$ generated by x_1^2.

2. If n is odd, again by (Equation 1.7.1), the image of ${}^Ud_3^{0,4}$ is $2c_1x_1$ which happens to be $\mathrm{Ker}{}^Ud_3^{3,2}$. By Proposition 1.5.6 ${}^Ud_3^{3,2}(c_1x_1) = nx_1^2 = ny_{2,0}$ since n is odd and $y_{2,0}$ is of order 2, which implies that ${}^UE_4^{6,0} = 0$. Hence $H^5(\mathbf{B}PU_n;\mathbb{Z}) \cong {}^UE_4^{3,2} = 0$, and $H^6(\mathbf{B}PU_n;\mathbb{Z}) \cong {}^UE_\infty^{0,6} \oplus {}^UE_4^{6,0} = \mathbb{Z}$.

In both cases we take $e_3 \in H^6(\mathbf{B}PU_n;\mathbb{Z})$ generating ${}^UE_\infty^{0,6}$. Therefore, (3) and (4) of Theorem 0.0.1 follows. Notice that e_3 is determined by this argument modulo torsion. In fact, there is a unique choice of e_3 that fits the statement of Theorem 0.0.1. This choice will be specified in Section 9, where we discuss the cup products in $H^9(\mathbf{B}PU_n;\mathbb{Z})$. □

1.8 $H^k(\mathbf{B}PU_n;\mathbb{Z})$ for $k=7,8$

Proof of (5) of Theorem 0.0.1. Notice that the only bi-degree (s,t) such that $s+t=7$ and ${}^UE_2^{s,t}$ is nontrivial is $(3,4)$. Therefore $H^7(\mathbf{B}PU_n;\mathbb{Z})\cong {}^UE_\infty^{3,4}$. We consider the p-local cohomology of $H^7(\mathbf{B}PU_n;\mathbb{Z})_{(p)}$ for each prime p separately. We consider the following relevant differentials:

1. ${}^Ud_3^{0,6}: {}^UE_3^{0,6}\to {}^UE_3^{3,4}$

$$c_3\mapsto (n-2)c_2x_1,\quad c_1c_2\mapsto [nc_2+(n-1)c_1^2]x_1,\quad c_1^3\mapsto 3nc_1^2x_1.$$

2. When localized at 2, The only nontrivial differential from ${}^UE_*^{3,4}$ is

$${}^Ud_3^{3,4}: {}^UE_3^{3,4}\to {}^UE_3^{6,2},$$

$$c_1^2x_1\mapsto 2nc_1y_{2,0}=0,\quad c_2x_1\mapsto (n-1)c_1y_{2,0}.$$

3. When localized at 3, The only nontrivial differential from ${}^UE_*^{3,4}$ is

$${}^Ud_5^{3,4}: {}^UE_5^{3,4}\to {}^UE_5^{8,0},$$

$$c_1^2x_1\mapsto n^2y_{3,0},\quad c_2x_1\mapsto \frac{n(n-1)}{2}y_{3,0}$$

In what follows we compute the localization $H^7(\mathbf{B}PU_n;\mathbb{Z})_{(p)}$ for each prime p separately. The spectral sequence is tacitly assumed to be localized at the specified prime p in each case.

Case 1. $p = 2$. In this case ${}^UE_\infty^{3,4} = \mathrm{Ker}{}^Ud_3^{3,4}/\mathrm{Im}{}^Ud_3^{0,6}$. By (2) of Lemma 1.6.1 we only need to consider the case that n is even,

in which, assuming $n > 2$ we have $\mathbb{Z}$-basis $\{c_1^3, c_1c_2, c_3\}$ and $\{c_1^2x_1, 2c_2x_1\}$ of ${}^UE_3^{0,6}$ and $\mathrm{Ker}{}^Ud_3^{3,4}$, respectively. The corresponding matrix for ${}^Ud_3^{0,6}$ is

$$\begin{bmatrix} 3n & n-1 & 0 \\ 0 & \frac{n}{2} & \frac{n-2}{2} \end{bmatrix}. \tag{1.8.1}$$

We apply invertible row and column operations to it and obtain

$$\begin{bmatrix} 1 & 0 \\ -\frac{n/2}{n-1} & 1 \end{bmatrix} \cdot \begin{bmatrix} 3n & n-1 & 0 \\ 0 & \frac{n}{2} & \frac{n-2}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3n}{n-1} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & n-1 & 0 \\ -\frac{3}{n-1}\cdot\frac{n^2}{2} & 0 & \frac{n-2}{2} \end{bmatrix}, \tag{1.8.2}$$

with the row operation corresponding to the change of basis

$$\begin{bmatrix} c_1^2x_1 & 2c_2x_1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -\frac{n/2}{n-1} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} c_1^2x_1 + \frac{n}{n-1}c_2x_1 & 2c_2x_1 \end{bmatrix}. \tag{1.8.3}$$

By (Equation 1.8.2) and (Equation 1.8.3), $\mathrm{Im}{}^Ud_3^{0,6}$ is generated by $c_1^2x_1 + \frac{n/2}{n-1}c_2x_1$ and $\gcd\{\frac{n^2}{2}, \frac{n-2}{2}\} \cdot 2c_2x_1 = \gcd\{2, \frac{n-2}{2}\} \cdot 2c_2x_1$. Where $\gcd\{2, \frac{n-2}{2}\} = 2$ if $n = 4l + 2$ for some integer l and 1 otherwise. Hence ${}^UE_4^{3,4} \cong {}^UE_\infty^{3,4}$ is isomorphic to $\mathbb{Z}$ and is generated by $2c_2x_1$ if $n = 4l + 2$, and is 0 otherwise.

In the exceptional case $n = 2$, (Equation 1.8.1) is reduced to

$$\begin{bmatrix} 6 & 1 \\ 0 & 1 \end{bmatrix},$$

and (Equation 1.8.2) is reduced to

$$\begin{bmatrix} 1 & 0 \\ -\frac{n/2}{n-1} & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -\frac{3n}{n-1} & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & 0 \end{bmatrix}, \tag{1.8.4}$$

which yields the same argument as above.

The above discussion shows that $H^7(\mathbf{B}PU_n;\mathbb{Z})_{(2)} \cong \mathbb{Z}/2$ is generated by $2c_2x_1$ if $n = 4l+2$ for some integer l, and is 0 if $4|n$.

By (2) of Theorem 0.0.1, we know that for n even, $H^4(\mathbf{B}PU_n;\mathbb{Z}) \cong \mathbb{Z}$ is generated by e_2, which is detected by $2nc_2 - (n-1)c_1^2 \in {}^UE_\infty^{0,4}$. Furthermore, when $n = 4l+2$, in $H^7(\mathbf{B}PU_n;\mathbb{Z})_{(2)} \cong \mathbb{Z}/2$ we have

$$\begin{aligned} 2c_2x_1 &= 3(4l+2)c_2x_1 = 3nc_2x_1 \\ &= [(n-1)c_1^2 + nc_2]x_1 - [(n-1)c_1^2 - 2nc_2]x_1 \\ &= [2nc_2 - (n-1)c_1^2]x_1 \in {}^UE_4^{3,4}, \end{aligned} \tag{1.8.5}$$

since by (Equation 1.8.4) and (Equation 1.8.3), $[(n-1)c_1^2 + nc_2]x_1$ is in the image of ${}^Ud_3^{0,6}$. Therefore, (Equation 1.8.5) shows that $2c_2x_1 = [2nc_2 - (n-1)c_1^2]x_1$ yields a non-trivial product in ${}^UE_\infty^{3,4}$ that detects e_2x_1.

Case 2. $p = 3$. In this case ${}^UE_\infty^{3,4} = \mathrm{Ker}{}^Ud_5^{3,4}/\mathrm{Im}{}^Ud_3^{0,6}$ and by (2) of Lemma 1.6.1 we only care about the case that $3|n$. Notice that ${}^Ud_3^{3,4} = 0$ since its target is a 2-torsion group and we are working 3-locally. A direct computation shows that $\mathrm{Ker}{}^Ud_5^{3,4} = {}^UE_3^{3,4}$ has a basis $\{c_1^2x_1, c_2x_1\}$. Again taking the basis $\{c_1^3, c_1c_2, c_3\}$ for ${}^UE_3^{0,6}$, we obtain the matrix

$$\begin{bmatrix} 3n & n-1 & 0 \\ 0 & n & n-2 \end{bmatrix} \tag{1.8.6}$$

for ${}^Ud_3^{0,6}$. We apply an invertible column operation to it and obtain

$$\begin{bmatrix} 3n & n-1 & 0 \\ 0 & n & n-2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{n}{n-2} & 1 \end{bmatrix} = \begin{bmatrix} 3n & n-1 & 0 \\ 0 & 0 & n-2 \end{bmatrix}. \tag{1.8.7}$$

Therefore ${}^Ud_3^{0,6}$ is surjective, and $H^7(\mathbf{B}PU_n;\mathbb{Z})_{(3)} = 0$ when $3|n$.

Case 3. $p > 3$. If $n > 2$, then ${}^UE_\infty^{3,4} = {}^UE_3^{3,4}/\mathrm{Im}{}^Ud_3^{0,6}$. We take the basis $\{c_1^2x_1, c_2x_1\}$ for ${}^UE_3^{3,4}$ and the matrix for ${}^Ud_3^{0,6}$ is again Equation 1.8.1. Since $p > 3$, either $3n$ or $n-2$ is invertible, so it is easy to show that ${}^Ud_3^{0,6}$ is surjective. So $H^7(\mathbf{B}PU_n;\mathbb{Z})_{(p)} = 0$ when $n > 2$. In the exceptional case $n = 2$, the same assertion follows easily from (2) of Lemma 1.6.1.

Summarizing the three cases above, (5) of Theorem 0.0.1 follows. $\square$

Proof of (6) Theorem 0.0.1. It suffice to consider the torsion component. Therefore the relevant entries in ${}^UE_2^{*,*}$ are ${}^UE_2^{8,0}$ and ${}^UE_2^{6,2}$. The relevant differentials are ${}^Ud_3^{3,4}$ and ${}^Ud_5^{3,4}$, as given in (2) and (3) in the proof for $k = 7$, together with the following one:

$$ {}^Ud_3^{6,2} : {}^UE_3^{6,2} \to {}^UE_3^{9,0}, \quad c_1y_{2,0} \mapsto nx_1y_{2,0} = nx_1^3. \tag{1.8.8} $$

Again we consider $H^8(\mathbf{B}PU_n;\mathbb{Z})_{(p)}$ for each prime p separately. Remember that in each of the following cases we assume that ${}^UE_*^{*,*}$ is localized at the specified prime p.

Case 1. $p = 2$. In this case the torsion component of $H^8(\mathbf{B}PU_n;\mathbb{Z})_{(2)}$ is isomorphic to ${}^UE_\infty^{6,2} = \mathrm{Ker}^Ud_3^{6,2}/\mathrm{Im}^Ud_3^{3,4}$. And by (2) of Lemma 1.6.1 we consider only the case that n is even. By (Equation 1.8.8), $\mathrm{Ker}^Ud_3^{6,2} = \mathbb{Z}\{c_1y_{2,0}\}$. By the differentials given in (2) in the proof for (5) of Theorem 0.0.1, $\mathrm{Im}^Ud_3^{3,4} = \mathbb{Z}\{c_1y_{2,0}\}$. Therefore the torsion of $H^8(\mathbf{B}PU_n;\mathbb{Z})_{(2)}$ is 0 when n is even.

Case 2. $p = 3$. In this case the torsion component of $H^8(\mathbf{B}PU_n;\mathbb{Z})_{(3)}$ is isomorphic to ${}^UE_2^{8,0}/\mathrm{Im}^Ud_5^{3,4}$, where ${}^UE_2^{8,0} \cong \mathbb{Z}/3$ is generated by a single element $y_{3,0}$. By the differentials given in (3) of the proof for $k = 7$, $\mathrm{Im}^Ud_5^{3,4} = 0$ when $3|n$. Therefore the torsion component of $H^8(\mathbf{B}PU_n;\mathbb{Z})_{(3)}$ is isomorphic to $\mathbb{Z}/3$ and generated by $y_{3,0}$.

For $p > 3$ there is no p-torsion in the relevant range of ${}^UE_*^{*,*}$. Hence (6) of Theorem 0.0.1 follows. $\square$

1.9 $H^k(\mathbf{B}PU_n;\mathbb{Z})$ for $k = 9, 10$

The study of $H^9(\mathbf{B}PU_n;\mathbb{Z})$ requires extra work when n is even, since the differential ${}^Ud_9^{0,8}$ cannot be determined by Theorem 1.5.8. Indeed, the differential ${}^Td_3^{6,2} : {}^TE_3^{6,2} \to {}^TE_3^{9,0}$ is not trivial when n is even. We consider the exceptional isomorphism of Lie groups $PU_2 \cong SO_3$. The following result is due to E. Brown. For more general cases see (13).

Proposition 1.9.1. $H^*(\mathbf{B}PU_2;\mathbb{Z}) \cong H^*(\mathbf{B}SO_3;\mathbb{Z}) \cong \mathbb{Z}[e_2] \otimes_{\mathbb{Z}} \mathbb{Z}/2[x_1]$, where e_2 is of degree 4, and we abuse the notation x_1 to let it denote the image of $x_1 \in H^3(K(\mathbb{Z},3);\mathbb{Z})$ under the homomorphism induced by the second arrow of the fiber sequence $\mathbf{B}U_2 \to \mathbf{B}PU_2 \to K(\mathbb{Z},3)$.

Recall that in Theorem 0.0.1 we let e_2 denote the generator of $H^4(\mathbf{B}PU_n;\mathbb{Z}) \cong \mathbb{Z}$ for all $n > 1$. Part (1) of the following lemma is due to A. Bousfield.

Lemma 1.9.2. *For $n > 0$ even, let $\Delta : \mathbf{B}U_2 \to \mathbf{B}U_n$ denote the inclusion of block diagonal matrices, and $\Delta' : \mathbf{B}PU_2 \to \mathbf{B}PU_n$ be its induced homomorphism on quotients. Then we have the following assertions.*

1. $(\Delta')^*(x_1) = x_1$. *In particular, $x_1^3 \in H^9(\mathbf{B}PU_n;\mathbb{Z})$ is non-zero.*
2. $(\Delta')^*(e_2) = (\frac{n}{2})^2 e_2$. *In particular, when $n = 4l+2$ for some integer l, $ex_1^2 \in H^{10}(\mathbf{B}PU_n;\mathbb{Z})$ is non-zero.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathbf{B}S^1 & \longrightarrow & \mathbf{B}U_2 & \longrightarrow & \mathbf{B}PU_2 \\
\big\downarrow{=} & & \big\downarrow{\Delta} & & \big\downarrow{\Delta'} \\
\mathbf{B}S^1 & \longrightarrow & \mathbf{B}U_n & \longrightarrow & \mathbf{B}PU_n
\end{array}$$

which induces another commutative diagram as follows:

$$\begin{tikzcd} \mathbf{B}U_2 \arrow[r] \arrow[d, "\Delta"] & \mathbf{B}PU_2 \arrow[r] \arrow[d, "\Delta'"] & K(\mathbb{Z},3) \arrow[d, "="] \\ \mathbf{B}U_n \arrow[r] & \mathbf{B}PU_n \arrow[r] & K(\mathbb{Z},3) \end{tikzcd} \tag{1.9.1}$$

This diagram induces a homomorphism of Serre spectral sequences, such that its restriction on the bottom row of the E_2 pages is the identity. In particular it takes x_1 to itself. Moreover, it follows from Proposition 1.9.1 that $(\Delta')^*(x_1^3) = x_1^3 \in H^9(\mathbf{B}PU_2;\mathbb{Z})$ is non-zero, which completes the proof of (1).

It is well known that the U_n-bundle over $\mathbf{B}U_2$ induced by Δ is the Whitney sum of $\frac{n}{2}$ copies of the universal U_2-bundle, of which the total Chern class is

$$(1+c_1+c_2)^{\frac{n}{2}} = 1 + \frac{n}{2}c_1 + (\frac{n}{2}c_2 + \binom{n/2}{2}c_1^2) + (\text{terms of higher degrees}).$$

Therefore we have

$$\begin{aligned} &\Delta^* : H^2(\mathbf{B}U_n;\mathbb{Z}) \to H^2(\mathbf{B}U_2;\mathbb{Z}), \\ &c_1 \mapsto \frac{n}{2}c_1, \quad c_2 \mapsto \frac{n}{2}c_2 + \binom{n/2}{2}c_1^2 = \frac{n}{2}c_2 + \frac{n(n-2)}{8}c_1^2. \end{aligned} \tag{1.9.2}$$

In particular,

$$\Delta^*((n-1)c_1^2 - 2nc_2) = (n-1)(\frac{n}{2})^2c_1^2 - 2n[\frac{n}{2}c_2 + \frac{n(n-2)}{8}c_1^2] = (\frac{n}{2})^2(c_1^2 - 4c_2).$$

Notice that when $n = 2$, $P^*(e_2) = c_1^2 - 4c_2$, and the equation above implies $(\Delta')^*(e_2) = (\frac{n}{2})^2 e$. When $n = 4l+2$, we have $(\Delta')^*(e_2x_1^2) = (2l+1)^2 e_2x_1^2 = e_2x_1^2 \in H^{10}(\mathbf{B}PU_2;\mathbb{Z})$ which is non-zero by Proposition 1.9.1, and (2) follows. □

Proof of (7), (8) of Theorem 0.0.1. The relevant entries in ${}^UE_2^{*,*}$ are ${}^UE_2^{3,6}$ and ${}^UE_2^{9,0}$. We study the localization of ${}^UE_*^{*,*}$ at each prime p separately.

Case 1. $p = 2$. Again we only consider the case that n is even. When $n > 2$, with respect to the basis $\{c_1^3x_1, c_1c_2x_1, c_3x_1\}$ for ${}^UE_2^{3,6}$ and $\{c_1^2y_{2,0}, c_2y_{2,0}\}$ for ${}^UE_2^{6,4}$, the differential ${}^Ud_3^{3,6} : {}^UE_2^{3,6} \to {}^UE_2^{6,4}$ is represented by the following matrix:

$$\begin{bmatrix} 3n & n-1 & 0 \\ 0 & n & n-2 \end{bmatrix}. \tag{1.9.3}$$

Since $y_{2,0}$ is of order 2, and that n is even, (Equation 1.9.3) implies that $\mathrm{Ker}^Ud_3^{3,6}$ has a basis $\{c_1^3x_1, 2c_1c_2x_1, c_3x_1\}$. Taking $\{y_{2,1}\}$ as a basis for ${}^UE_2^{10,0}$, ${}^Ud_7^{3,6}$ is represented by the matrix

$$\begin{bmatrix} \frac{n^3}{2} & \frac{n^2(n-1)}{2} & \frac{n(n-1)(n-2)}{12} \end{bmatrix}. \tag{1.9.4}$$

A closer inspect shows that all three entries are even. Therefore $\mathrm{Ker}^Ud_3^{3,6} = \mathrm{Ker}^Ud_7^{3,6} = \{c_1^3x_1, 2c_1c_2x_1, c_3x_1\}$. With this basis and $\{c_1^4, c_1^2c_2, c_1c_3, c_2^2, c_4\}$ as a basis for ${}^UE_2^{0,8}$, ${}^Ud_3^{0,8} : {}^UE_2^{0,8} \to \mathrm{Ker}^Ud_7^{3,6} \subset {}^UE_7^{3,6}$ is represented by the following matrix:

$$\begin{bmatrix} 4n & n-1 & 0 & 0 & 0 \\ 0 & n & \frac{n-2}{2} & n-1 & 0 \\ 0 & 0 & n & 0 & n-3 \end{bmatrix} \tag{1.9.5}$$

We apply an invertible column operation on it as follows:

$$\begin{bmatrix} 4n & n-1 & 0 & 0 & 0 \\ 0 & n & \frac{n-2}{2} & n-1 & 0 \\ 0 & 0 & n & 0 & n-3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{n}{n-1} & -\frac{n-2}{2(n-1)} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.9.6}$$
$$= \begin{bmatrix} 4n & n-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & n-1 & 0 \\ 0 & 0 & n & 0 & n-3 \end{bmatrix}.$$

Therefore ${}^U d_3^{0,8} : {}^U E_2^{0,8} \to \mathrm{Ker}{}^U d_7^{3,6}$ is onto. So ${}^U E_\infty^{3,6} = 0$.

In the exceptional case $n = 2$, we have $c_3, c_4 = 0$, and (Equation 1.9.4), (Equation 1.9.5), are respectively reduced to

$$\begin{bmatrix} 4 & 2 \end{bmatrix}, \begin{bmatrix} 8 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

Consequently, ${}^U d_3^{0,8}: {}^U E_2^{0,8} \to \mathrm{Ker}{}^U d_7^{3,6}$ is onto as well. It remains to consider

$$ {}^U d_9^{0,8}: {}^U E_9^{0,8} \to {}^U E_9^{9,0} \cong {}^U E_2^{9,0} = \mathbb{Z}/2\{x_1^3\}, $$

Since ${}^U E_9^{9,0} \cong \mathbb{Z}/2$ is generated by x_1^3, ${}^U d_9^{0,8}$ is either surjective or 0, depending on whether x_1^3 is 0 or not. By Lemma 1.9.2, $x_1^3 \neq 0$ when n is even. Therefore ${}^U d_9^{0,8} = 0$.

Summarising all above, when n is even, $H^9(\mathbf{B}PU_n;\mathbb{Z})_{(2)} = \mathbb{Z}/2$ is generated by x_1^3. We proceed to study the cup products in $H^9(\mathbf{B}PU_n;\mathbb{Z})_{(2)}$. Since ${}^U E_\infty^{3,6} = 0$, $x_1e_3 = x_1^3$ or $x_1e_3 = 0$. This merely depends on a choice of e_3: if the former case is true, then we simply replace e_3 by $e_3 + x_1^2$ to obtain the latter case. Therefore the 2-local case of (8) of Theorem 0.0.1 follows.

Case 2. $p = 3$. The only non-trivial target of differentials with domain ${}^U E_2^{3,6}$ is ${}^U E_2^{8,2}$. By Theorem 1.5.8, ${}^U d_5^{3,6}: {}^U E_2^{3,6} \to {}^U E_2^{8,2}$ is trivial, since ${}^T d_5^{3,6}$ is so. Hence ${}^U E_\infty^{3,6} \cong {}^U E_2^{3,6}/\mathrm{Im}{}^U d_3^{0,8}$. When $n > 3$, We take basis $\{c_1^4, c_1^2c_2, c_1c_3, c_2^2, c_4\}$ for ${}^U E_3^{0,8}$ and $\{c_1^3x_1, c_1c_2x_1, c_3x_1\}$ for ${}^U E_2^{3,6}$. Then the matrix representing ${}^U d_3^{0,8}$ is the following:

$$ \begin{bmatrix} 4n & n-1 & 0 & 0 & 0 \\ 0 & 2n & n-2 & 2(n-1) & 0 \\ 0 & 0 & n & 0 & n-3 \end{bmatrix}. \tag{1.9.7} $$

We only consider the case $3|n$, in which we apply an invertible column operation to Equation 1.9.7 and obtain

$$\begin{bmatrix} 4n & n-1 & 0 & 0 & 0 \\ 0 & 2n & n-2 & 2(n-1) & 0 \\ 0 & 0 & n & 0 & n-3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{n}{n-1} & -\frac{n-2}{2(n-1)} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 4n & n-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(n-1) & 0 \\ 0 & 0 & n & 0 & n-3 \end{bmatrix}, \tag{1.9.8}$$

which shows that for $n > 3$ we have $H^9(\mathbf{B}PU_n;\mathbb{Z})_{(3)} \cong {}^UE_\infty^{3,6} \cong {}^UE_2^{3,6}/\mathrm{Im}^U d_3^{0,8} \cong \mathbb{Z}/3$, and that it is generated by c_3x_1. We denote this cohomology by z_1.

In the exceptional case $n = 3$, the vanishing of c_4 makes the matrices (Equation 1.9.7) and (Equation 1.9.8) reduce to

$$\begin{bmatrix} 4n & n-1 & 0 & 0 \\ 0 & 2n & n-2 & 2(n-1) \\ 0 & 0 & n & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4n & n-1 & 0 & 0 \\ 0 & 0 & 0 & 2(n-1) \\ 0 & 0 & n & 0 \end{bmatrix},$$

respectively, from which the same assertion follows easily. Next, we consider the product e_3x_1. The matrix (Equation 1.9.3) shows that for $n > 2$, ${}^UE_\infty^{0,6}$ is generated by $c_1^3 - \frac{3n}{n-1}c_1c_2 + \frac{3n^2}{(n-1)(n-2)}c_3$. On the other hand ${}^UE_\infty^{3,6} \cong \mathbb{Z}/3$ is generated by c_3x_1, as we just proved. Therefore,

$$[c_1^3 - \frac{3n}{n-1}c_1c_2 + \frac{3n^2}{(n-1)(n-2)}c_3]x_1 = \frac{3n^2}{(n-1)(n-2)}c_3]x_1 = 0 \in {}^UE_\infty^{3,6},$$

which proves the 3-local case of (8) of Theorem 0.0.1.

Case 3. $p > 3$. (Equation 1.9.5) clearly represents a surjection, even when the 3rd and 5th columns, corresponding to the image of c_1c_3 and c_4, are removed. This completes the proof of (7) and (8) of Theorem 0.0.1. □

Proof of (9), (10) of Theorem 0.0.1. Case 1. $p = 2$. In this case the relevant entries are ${}^UE_2^{6,4}$ and ${}^UE_2^{10,0}$. First suppose that $n > 2$. Fix a basis $\{c_1^2y_{2,0}, c_2y_{2,0}\}$ for ${}^UE_2^{6,4}$ and $\{c_1x_1y_{2,0}\} = \{c_1x_1^3\}$ for ${}^UE_2^{9,2}$, and ${}^Ud_3^{6,4} : {}^UE_2^{6,4} \to {}^UE_2^{9,2}$ is represented by the matrix

$$\begin{bmatrix} 2n & n-1 \end{bmatrix}, \tag{1.9.9}$$

which shows that $\mathrm{Ker}{}^Ud_3^{6,4}$ is generated by $\{c_1^2y_{2,0}\}$. On the other hand, by (Equation 1.9.3), $\mathrm{Im}{}^Ud_3^{3,6}$ is also generated by $\{c_1^2y_{2,0}\}$. Therefore

$${}^UE_\infty^{6,4} \cong {}^UE_5^{6,4} \cong \mathrm{Ker}{}^Ud_3^{6,4}/\mathrm{Im}{}^Ud_3^{3,6} = 0. \tag{1.9.10}$$

By (Equation 1.9.4) ${}^U d_7^{3,6} = 0$. Hence

$$ {}^U E_\infty^{10,0} \cong {}^U E_2^{10,0}/\mathrm{Im}{}^U d_7^{3,6} \cong {}^U E_2^{10,0} \cong \mathbb{Z}/2. \tag{1.9.11}$$

is generated by $x_1 y_{2,0}$. In the exceptional case $n = 2$, $c_3 = 0$ and all the arguments above hold as well, since all the differentials of $c_3 x_1$ are 0 anyway.

Therefore, the torsion component of $H^{10}(\mathbf{B}PU_n; \mathbb{Z})_{(2)}$ is isomorphic to $\mathbb{Z}/2$, generated by $y_{2,1}$ when n is even.

Case 2. $p = 3$. We take $3|n$. In this case the only relevant entry is ${}^U E_2^{8,2}$. By Theorem 1.5.8 no non-trivial differential lands in this entry, since it is the case in ${}^T E_*^{*,*}$. On the other hand ${}^U E_2^{8,2}$ is generated by $c_1 y_{3,0}$, such that ${}^U d_3(c_1 y_{3,0}) = n x_1 y_{3,0} = 0 \in {}^U E_2^{11,0} \cong \mathbb{Z}/3$. Therefore the torsion component of $H^{10}(\mathbf{B}PU_n; \mathbb{Z})_{(3)} \cong \mathbb{Z}/3$, generated by $\{x_1 y_{3,0}\}$ when $3|n$. We denote this class by z_2.

Case 3. $p > 3$. There is no (s, t) such that $s + t = 10$ and that ${}^U E_2^{s,t}$ is p-locally non-trivial, except for ${}^U E_2^{0,10}$ which accounts for the well understood torsion free component.

(9) of Theorem 0.0.1 follows from the discussion above.

We proceed to study cup products in $H^{10}(\mathbf{B}PU_n; \mathbb{Z})$. Recall that when n is even, $H^4(\mathbf{B}PU_n; \mathbb{Z}) \cong \mathbb{Z}$ is generated by an element e_2 such that $P^*(e) = 2nc_2 - (n-1)c_1^2 \in H^4(\mathbf{B}U_n; \mathbb{Z})$. By (Equation 1.9.10), $e_2 x_1^2 \in {}^U E_\infty^{6,4}$ is a coboundary. Therefore, either $ex_1^2 = 0$, or $ex_1^2 = y_{2,0}$. By Lemma 1.9.2, the former is not true. Therefore we have the latter, and we proved (10) of Theorem 0.0.1. $\square$

CHAPTER 2

THE TOPOLOGICAL PERIOD-INDEX PROBLEM OVER 8-COMPLEXES

2.1 Introduction to Chapter 2

The goal of this chapter is to prove Theorem 0.0.8 given in the introduction. Recall the following theorem from the introduction:

Theorem 0.0.8. *Let X be a topological space of homotopy type of an 8-dimensional CW-complex, and let $\alpha \in H^3(X;\mathbb{Z})_{\mathrm{tor}}$ be a topological Brauer class of period n. Then*

$$\mathrm{ind}(\alpha)|\epsilon_2(n)\epsilon_3(n)n^3. \tag{2.1.1}$$

In addition, if X is the 8-th skeleton of $K(\mathbb{Z}/n,2)$, and α is the restriction of the fundamental class $\beta_n \in H^3(K(\mathbb{Z}/n,2),\mathbb{Z})$, then

$$\begin{cases} \mathrm{ind}(\alpha) = \epsilon_3(n)n^3, & 4 \nmid n, \\ \epsilon_3(n)n^3 | \mathrm{ind}(\alpha), & 4|n. \end{cases}$$

In particular, the sharp lower bound of e such that $\mathrm{ind}(\alpha)|n^e$ for all X and α is 4.

We decompose Theorem 0.0.8 into two sub-theorems as follows, since the proof of the second sub-theorem requires special attention.

Theorem 2.1.0.1. Let X be a topological space of homotopy type of an 8-dimensional CW-complex, and let $\alpha \in H^3(X;\mathbb{Z})_{\text{tor}}$ be a topological Brauer class of period n. Then

$$\text{ind}(\alpha) | \epsilon_2(n)\epsilon_3(n)n^3. \tag{2.1.2}$$

In addition, if X is the 8-th skeleton of $K(\mathbb{Z}/n, 2)$, and α is the restriction of the fundamental class $\beta_n \in H^3(K(\mathbb{Z}/n, 2), \mathbb{Z})$, then

$$\begin{cases} \text{ind}(\alpha) = \epsilon_3(n)n^3, & n \text{ odd}, \\ \epsilon_3(n)n^3 | \text{ind}(\alpha), & n \text{ even}. \end{cases}$$

Theorem 2.1.0.2. Let $n = 2l$ for some odd integer l and let X and α be as above. Then $\text{ind}(\alpha) \nmid \epsilon_3(n)n^3$.

Part of the proof relies on twisted complex K-theory, which shows (Equation 2.1.1) for any 8-dimensional CW complex X and any $\alpha \in \text{Br}(X)$. Details are discussed in Section 2. We prove the second paragraph of the theorem with classical obstruction theory, which is outlined as follows.

Let m, n be integers. Then $\mathbb{Z}/n$ is a closed normal subgroup of SU_{mn} in the sense of the following monomorphism of Lie groups:

$$\mathbb{Z}/n \hookrightarrow SU_{mn} : t \mapsto e^{2\pi\sqrt{-1}t/n}\mathbf{I}_{mn},$$

where $\mathbf{I}_r$ is the identity matrix of degree r. We define the quotient group to be $P(n, mn)$. In particular, $P(n, n)$ is the projective unitary group PU_n, and we have the following short exact sequence of Lie groups:

$$1 \to \mathbb{Z}/n \to P(n, mn) \xrightarrow{\varphi} PU_{mn} \to 1.$$

The homotopy groups of $P(n, mn)$ in low degrees relative to mn are well known:

$$\pi_i(P(n, mn)) \cong \begin{cases} \mathbb{Z}/n, & \text{if } i = 1, \\ \mathbb{Z}, & \text{if } 1 < n < 2mn, \text{ and } n \text{ is odd}, \\ 0, & \text{if } 1 < n < 2mn, \text{ and } n \text{ is even}, \\ \mathbb{Z}/(mn)!, & \text{if } i = 2mn. \end{cases} \tag{2.1.3}$$

This follows since $P(n, mn)$ has SU_{mn} as a simply connected n-cover, whose homotopy groups in low dimensions follows from Bott periodicity ((8)). Consider its classifying space $\mathbf{B}P(n, mn)$, and we have a map $\mathbf{B}P(n, mn) \to K(\mathbb{Z}/n, 2)$ which is the projection of $\mathbf{B}P(n, mn)$ onto the first non-trivial stage of its Postnikov tower. This map also classifies the generator of $H^2(\mathbf{B}P(n, mn); \mathbb{Z}/n)$.

Given a connected CW-complex X such that $H^2(X; \mathbb{Z}) = 0$, and $\alpha \in \mathrm{Br}'(X)$ of period n, there is a unique class $\alpha' \in H^2(X; \mathbb{Z}/n)$ such that $B(\alpha') = \alpha$, where B is the Bockstein homomorphism. Then α' is classified by a map $X \to K(\mathbb{Z}/n, 2)$. Therefore we have a lifting problem as shown by the following diagram:

$$\begin{tikzcd} & \mathbf{B}P(n, mn) \arrow[d] \\ X \arrow[ur, dashed] \arrow[r, "\alpha'"] & K(\mathbb{Z}/2, n) \end{tikzcd} \tag{2.1.4}$$

It can be shown, as we do in later sections, that α is classified by a PU_{mn}-torsor over X if and only if the lifting problem above has a solution. If X is a finite CW complex, then it suffices to study maps from X into successive stages of the Posnikov tower of $\mathbf{B}P(n, mn)$, which occupies most of this paper.

Theorem 1.5.8, in addition to Theorem 0.0.5 of Antieau and Williams, provides evidence for the following

Conjecture 2.1.1 (Antieau-Williams, (6))**.** Let X be a finite $2d$-dimensional CW-complex, and let $\alpha \in \mathrm{Br}(X)$ have period $m = p_1^{r_1} \cdots p_k^{r_k}$. Then,

$$\mathrm{ind}(\alpha) = m^{d-1} \prod_{i=1}^{k} p_i^{v_{p_i}((d-1)!)},$$

where v_{p_i} is the p_i-adic evaluation.

In (6), Antieau and Williams proved $\mathrm{ind}(\alpha) = m^{d-1} \prod_{i=1}^{k} p_i^{v_{p_i}((d-1)!)}$.

In Section 2 we recapture the cohomology of Eilenberg-Mac Lane spaces necessary for our purpose; in Section 3 we introduce the twisted K-theory and the associated Atiyah-Hirzebruch spectral sequence; Sections 4, 5, and 6 are devoted to the study of the classifying spaces

$\mathbf{B}P(n, mn)$, in particular their Postnikov towers, which is the technical core of this paper. In section 7 an Eilenberg-Moore spectral sequence is introduced to prove Theorem 2.1.0.2

2.2 Preliminary on the Cohomology of Eilenberg-Mac Lane Spaces

As mentioned in the introduction, the objects of interest are various stages of the Postnikov tower of the space $\mathbf{B}P(n, mn)$. It follows from (Equation 2.1.3) that the relevant Eilenberg-Mac Lane spaces are of the forms $K(\mathbb{Z}/n, 2)$ and $K(\mathbb{Z}, n)$. All the assertions made in this section are essentially consequences of (14).

Consider the Eilenberg-Mac Lane space $K(\mathbb{Z}, n)$ for $n \geq 3$. By (14), the integral cohomology ring $H^*(K(\mathbb{Z}, n); \mathbb{Z})$ in degree $\leq n + 3$ is isomorphic to the following graded ring:

$$\mathbb{Z}[\iota_n, \Gamma_n]/(2\Gamma_n), \tag{2.2.1}$$

where ι_n, of degree n, is the so-called fundamental class, and Γ_n, of degree $n+3$, is a class of order 2. We denote by $\bar{\iota}_n, \bar{\Gamma}_n$ the mod 2 reduction of ι_n and Γ_n in $H^*(K(\mathbb{Z}, n); \mathbb{Z}/2)$, respectively. Either by (14) or by the Künneth Theorem, there is a class $\Gamma'_n \in H^{n+2}(K(\mathbb{Z}, n); \mathbb{Z}/2)$ such that $B(\Gamma'_n) = \Gamma_n$, where B denotes the Bockstein homomorphism. Moreover, we consider the Steenrod square Sq^r and write $\overline{\mathrm{Sq}^r}$ for the following composition:

$$H^*(-; \mathbb{Z}) \xrightarrow{\text{mod } 2} H^*(-; \mathbb{Z}/2) \xrightarrow{\mathrm{Sq}^r} H^{*+r}(-; \mathbb{Z}/2),$$

where the first arrow denotes the mod 2 reduction.

Lemma 2.2.1. *If $n > 3$, then $\Gamma'_n = \mathrm{Sq}^2(\bar{\iota}_n)$. In terms of cohomology operations, this means*

$$\Gamma_n = B \circ \overline{\mathrm{Sq}^2} : H^n(-;\mathbb{Z}) \to H^{n+3}(-;\mathbb{Z}).$$

Then the Adem relation $\mathrm{Sq}^3 = \mathrm{Sq}^1 \mathrm{Sq}^2$ implies $\overline{\Gamma}_n = \overline{\mathrm{Sq}^3}(\iota_n)$.

Proof. First we consider the case $n = 3$. The path fibration $K(\mathbb{Z},2) \to * \to K(\mathbb{Z},3)$ induces a cohomological Serre spectral sequence ${}^3E_*^{*,*}$ with coefficients in $\mathbb{Z}/2$, such that ${}^3E_2^{3,0} \cong H^3(K(\mathbb{Z},3);\mathbb{Z}/2) \cong \mathbb{Z}/2$ is generated by $\bar{\iota}_3$; ${}^3E_2^{0,2} \cong H^2(K(\mathbb{Z},2);\mathbb{Z}/2) \cong \mathbb{Z}/2$ is generated by $\bar{\iota}_2$; ${}^3E_2^{5,0} \cong H^5(K(\mathbb{Z},3);\mathbb{Z}/2) \cong \mathbb{Z}/2$ is generated by Γ'_3; and ${}^3E_2^{0,4} \cong H^4(K(\mathbb{Z},2);\mathbb{Z}/2) \cong \mathbb{Z}/2$ is generated by $\bar{\iota}_2^2$. The vanishing of the E_∞ page in positive total degrees implies

$$d_2(\bar{\iota}_2) = \bar{\iota}_3, \tag{2.2.2}$$

and

$$d_4(\bar{\iota}_2^2) = \Gamma'_3. \tag{2.2.3}$$

Notice that $\bar{\iota}_2^2 = \mathrm{Sq}^2(\bar{\iota}_2)$. Moreover, by Corollary 6.9 of (26), Steenrod squares commute with transgressions in Serre spectral sequences. The following equation then follows from (Equation 2.2.2) and (Equation 2.2.3):

$$\Gamma'_3 = d_4(\bar{\iota}_2^2) = d_4(\mathrm{Sq}^2(\bar{\iota}_2)) = \mathrm{Sq}^2 d_2((\bar{\iota}_2)) = \mathrm{Sq}^2(\bar{\iota}_3). \tag{2.2.4}$$

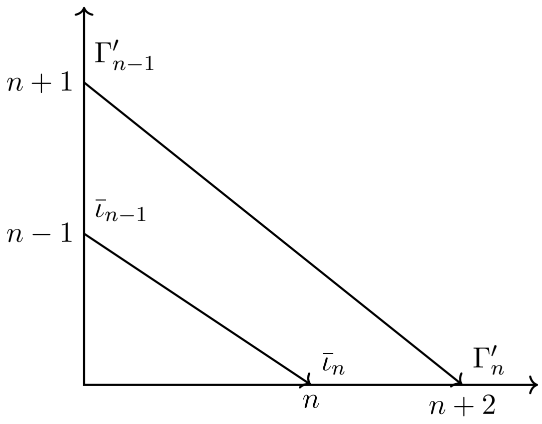

Figure 4. Low dimensional trasgressions in the mod 2 cohomological Serre spectral sequence induced by $K(\mathbb{Z}, n-1) \to * \to K(\mathbb{Z}, n)$.

This proves lemma in the case $n = 3$. We verify the general case by induction on n. Consider the path fibration $K(\mathbb{Z}, n-1) \to * \to K(\mathbb{Z}, n)$. Again, by the vanishing of the E_∞ page in positive total degrees, we have $d_n(\bar{\iota}_{n-1}) = \bar{\iota}_n$ and $d_{n+2}(\Gamma'_{n-1}) = \Gamma'_n$. See Figure Figure 4 for an indication of the relevant differentials. Since all the differentials in sight are transgressions, the induction is complete. $\square$

We proceed to consider $K(\mathbb{Z}/n, 2)$, for any positive integer n. By (14), the integral cohomology of $K(\mathbb{Z}/n, 2)$ in degree ≤ 8 is isomorphic to the following graded commutative ring:

$$\mathbb{Z}[\beta_n, Q_n, R_n, \rho_n]/(n\beta_n, \epsilon_2(n)\beta_n^2, \epsilon_2(n)nQ_n, \epsilon_3(n)nR_n, \epsilon_3(n)\rho_n), \tag{2.2.5}$$

where $\deg(\beta_n) = 3, \deg(Q_n) = 5, \deg(R_n) = 7$, and $\deg(\rho_n) = 8$. In other words, there is exactly one generator in each of the degrees $3, 5, 6, 7$, which are, respectively, $\beta_n, Q_n, \beta_n^2, R_n$, of order

$$n, \epsilon_2(n)n, \epsilon_2(n), \epsilon_3(n)n,$$

and 2 generators in degree 8, $\beta_n Q_n$ and ρ_n, of order $\epsilon_2(n)$ and $\epsilon_3(n)$, respectively. Notice that when n is odd, the generators $\epsilon_2(n)\beta_n^2$ and $\beta_n Q_n$ are trivial. When n is coprime to 3, $\rho_n = 0$.

Consider the canonical inclusion $g_{m,n} : \mathbb{Z}/n \to \mathbb{Z}/mn$, which induces a map $g_{m,n}^{(i)} : K(\mathbb{Z}/n, i) \to K(\mathbb{Z}/mn, i)$ for any integer $i > 0$.

For any prime number p such that $p|r$, a straight forward computation of homology of groups shows that we have the isomorphism

$$(g_{m,n}^{(1)})_* : H_2(K(\mathbb{Z}/n, 1); \mathbb{Z}/p) \cong H_2(K(\mathbb{Z}/mn, r); \mathbb{Z}/p) \cong \mathbb{Z}/p. \tag{2.2.6}$$

See, for example, Section 6.2 of (33). On the other hand, recall “la transpotence”

$$\psi_p : H_{2q}(K(\mathbb{Z}/r, i); \mathbb{Z}/p) \to H_{2pq+2}(K(\mathbb{Z}/r, i+1); \mathbb{Z}/p)$$

defined in Section 6 of (14), which, by the example on page 6-08, (14), is $\mathbb{Z}/p$-linear when p is odd. We adopt the notations in Section 11 of (14). Let A_r be the group ring of $\mathbb{Z}/r$ generated by a single element u_r. Then, it is an easy consequence of Section 9 of (14) that $H_2(K(\mathbb{Z}/r,1);\mathbb{Z}/p)$ is generated by $\psi_p(u_r)$. It then follows from (Equation 2.2.6) that we have

$$(g^{(1)}_{m,n})_*(\psi_p(u_n)) = \mu\psi_p(u_{nm}).$$

For some $\mu \in (\mathbb{Z}/p)^*$. When p is odd, since ψ_p is functorial and $\mathbb{Z}/p$-linear, we have

$$(g^{(2)}_{m,n})_*((\psi_p)^2(u_n)) = \mu(\psi_p)^2(u_{nm}), \tag{2.2.7}$$

an equation of elements of $H_8(K(\mathbb{Z}/mn,2);\mathbb{Z}/p)$. When $p=3$, the discussion above leads to the following

Lemma 2.2.2. *The induced homomorphism $H^8(g_{m,n})$ is a 3-local isomorphism if $3|n$ and 0 otherwise.*

Proof. When $3 \nmid n$, we have $H^8(K(\mathbb{Z}/n,2);\mathbb{Z}) = 0$, and there is nothing to prove. When $3|n$, by the universal coefficient theorem, it suffices to show that

$$(g^{(2)}_{m,n})_* : H_7(K(\mathbb{Z}/n,2);\mathbb{Z}) \to H_7(K(\mathbb{Z}/mn,2);\mathbb{Z})$$

is an isomorphism. It follows from Section 11 of (14) that

$$H_7(K(\mathbb{Z}/r,2);\mathbb{Z}) \cong \mathbb{Z}/3$$

is generated by the Bockstein of $(\psi_3)^2(u_r)$, when $3|r$. The lemma then follows from (Equation 2.2.7). $\square$

Consider the short exact sequence

$$0 \to \mathbb{Z}/n \to \mathbb{Z}/mn \to \mathbb{Z}/m \to 0$$

which induces the following fiber sequence of spaces:

$$K(\mathbb{Z}/m, 1) \to K(\mathbb{Z}/n, 2) \to K(\mathbb{Z}/mn, 2).$$

We denote the induced cohomological Serre spectral sequence in integral coefficients by ${}^H E_*^{*,*}$.

Lemma 2.2.3. *Let ${}^H E_*^{*,*}$ be as above. If $\epsilon_2(n)n|m$, then*

$$H^5(K(\mathbb{Z}/n, 2); \mathbb{Z}) \cong {}^H E_\infty^{3,2} = \frac{\epsilon_2(m)}{\epsilon_2(n)} {}^H E_3^{3,2} / \operatorname{Im} {}^H d_3^{0,4},$$

and if $\epsilon_3(n)n|m$, then

$$H^7(K(\mathbb{Z}/n, 2); \mathbb{Z}) \cong {}^H E_\infty^{3,4} = \frac{\epsilon_3(m)}{\epsilon_3(n)} {}^H E_3^{3,4} / \operatorname{Im} {}^H d_3^{0,6}.$$

Remark 2.2.4. In particular, we have

$$H^5(K(\mathbb{Z}/n, 2); \mathbb{Z}) \cong \mathbb{Z}/\epsilon_2(n)n,$$

and

$$H^7(K(\mathbb{Z}/n,2);\mathbb{Z}) \cong \mathbb{Z}/\epsilon_3(n)n.$$

Proof. Consider the E_2-page

$$
{}^HE_2^{s,t} \cong H^s(K(\mathbb{Z}/mn,2);H^t(K(\mathbb{Z}/m,1);\mathbb{Z})) \cong
\begin{cases}
H^s(K(\mathbb{Z}/mn,2);\mathbb{Z}), \text{ if } t=0. \\
H^s(K(\mathbb{Z}/mn,2);\mathbb{Z}/m), \text{ if } t>0 \text{ and } t \text{ is even,} \\
0, \text{ if } t \text{ is odd.}
\end{cases}
$$

This follows from the fact that, as a ring,

$$H^*(K(\mathbb{Z}/m,2);\mathbb{Z}) \cong \mathbb{Z}[v]/(mv), \tag{2.2.8}$$

where v is of degree 2. For obvious degree reasons, we have

$$ {}^Hd_3^{0,2} : {}^HE_3^{0,2} \cong \mathbb{Z}/m \to {}^HE_3^{3,0} \cong \mathbb{Z}/mn, v \mapsto n\beta_{mn}, \tag{2.2.9}$$

i.e., the canonical inclusion $\mathbb{Z}/m \to \mathbb{Z}/mn$. Since the spectral sequence is multiplicative, it follows from (Equation 2.2.9) that

$$ {}^Hd_3^{3,2} : {}^HE_3^{3,2} \cong \mathbb{Z}/m \to {}^HE_3^{6,0} \cong \mathbb{Z}/2, v\beta_{mn} \mapsto n\beta_{mn}^2, \tag{2.2.10}$$

which is surjective if n is odd, and 0 if n is even. By Leibniz rule, we have

$$ {}^{H}d_3^{0,4} : {}^{H}E_3^{0,4} \cong \mathbb{Z}/m \to {}^{H}E_3^{3,2} \cong \mathbb{Z}/m, v^2 \mapsto 2nv\beta_{mn}. \tag{2.2.11} $$

It follows from obvious degree reasons that ${}^{H}E_\infty^{3,2} = \operatorname{Ker} {}^{H}d_3^{3,2} / \operatorname{Im} {}^{H}d_3^{0,4}$. Therefore, by (Equation 2.2.10), (Equation 2.2.11) and $\epsilon_2(n)n|m$, we have

$$ {}^{H}E_\infty^{3,2} = \langle \frac{2}{\epsilon_2(n)} v\beta_{mn}\rangle / \langle 2nv\beta_{mn}\rangle = \frac{\epsilon_2(m)}{\epsilon_2(n)} {}^{H}E_3^{3,2} / \operatorname{Im} {}^{H}d_3^{0,4} \cong \mathbb{Z}/\epsilon_2(n)n, \tag{2.2.12} $$

which is isomorphic to $H^5(K(\mathbb{Z}/n, 2); \mathbb{Z})$, and the first equation follows.

We proceed to prove the second equation in the lemma. By Leibniz rule and (Equation 2.2.9), we have

$$ d_3^{0,6}(v^3) = 3nv^2\beta_{mn}, \text{ and } d_3^{3,4}(v^2\beta_{mn}) = 2v\beta_{mn}^2 = 0 $$

since $2\beta_{mn} = 0$, from which it follows that

$$ {}^{H}E_4^{3,4} = {}^{H}E_3^{3,4} / \operatorname{Im} d_3^{0,6} = {}^{H}E_3^{3,4} / 3n {}^{H}E_3^{3,4} \cong \mathbb{Z}/\epsilon_3(m)n. \tag{2.2.13} $$

For degree reasons the only potentially nontrivial differential into or out of ${}^{H}E_4^{3,4}$ is

$$ {}^{H}d_5^{3,4} : {}^{H}E_4^{3,4} \to {}^{H}E_4^{8,0}, $$

where the codomain ${}^{H}E_4^{8,0}$ is a quotient group of $H^8(K(\mathbb{Z}/mn, 2); \mathbb{Z})$ in which ρ_{mn} is nontrivial. It follows from Lemma 2.2.2 that

$$ {}^{H}d_5^{3,4} \begin{cases} = 0, \text{if } 3|n, \\ \\ \text{onto } \rho_{mn}, \text{ otherwise.} \end{cases} \tag{2.2.14} $$

Hence, when $\epsilon_3(n)n|m$, we have

$$ {}^{H}E_\infty^{3,4} = \operatorname{Ker} {}^{H}d_5{}^{3,4} = \frac{\epsilon_3(m)}{\epsilon_3(n)} {}^{H}E_3^{3,4} / \operatorname{Im} d_3^{0,6} \cong \mathbb{Z}/\epsilon_3(n)n, \tag{2.2.15} $$

which is isomorphic to $H^7(K(\mathbb{Z}/n, 2); \mathbb{Z})$, and the desired equation follows. □

The cohomology of $K(\mathbb{Z}/n, 2)$ with coefficients in $\mathbb{Z}/2$ is of particular interest to us. In fact, we have a beautiful description of the cohomology ring $H^*(K(\mathbb{Z}/2, q); \mathbb{Z}/2)$ for any $q > 0$. We denote the fundamental class of $H^q(K(\mathbb{Z}/2, q); \mathbb{Z}/2)$ by b. Recall that a finite sequence of positive integers $I = (i_1, i_2, \cdots, i_r)$ is called admissible if $i_k \geq 2i_{k+1}$, for $k = 1, \cdots, r-1$. The excess of I is defined as

$$ e(I) = i_1 - i_2 - \cdots - i_r. $$

The following well-known theorem can be found, for example, in (28), in a slightly different form.

Theorem 2.2.5 (Theorem 4, Chapter 9, (28)). *When n is even, the ring $H^*(K(\mathbb{Z}/n, q); \mathbb{Z}/2)$ is the polynomial ring with generators*

$$\operatorname{Sq}^I(b) = \operatorname{Sq}^{i_1} \operatorname{Sq}^{i_2} \cdots \operatorname{Sq}^{i_r}(b)$$

where I runs through admissible sequences of excess $e(I) < q$, with the exception, in the case $4|n$, and $i_r = 1$, $\operatorname{Sq}^I(b)$ is replaced by

$$\operatorname{Sq}^{i_1} \operatorname{Sq}^{i_2} \cdots \operatorname{Sq}^{i_{r-1}}(b'),$$

where b' is the mod 2 reduction of the generator of $H^{q+1}(K(\mathbb{Z}/n, q); \mathbb{Z})$, and $\operatorname{Sq}^1(b) = 0$.

In the special case of $q = 2$, we have

Corollary 2.2.6. When n is even, we have the isomorphism

$$H^*(K(\mathbb{Z}/n, 2); \mathbb{Z}/2) = \mathbb{Z}/2[b_2, b_3, b_5]$$

where $b_2 = b$ is the fundamental class, $\operatorname{Sq}^1 b_2 = 0$ when $4|n$ and $\operatorname{Sq}^1 b_2 = b_3$ otherwise, and $b_5 = \operatorname{Sq}^2 b_3$.

We conclude this section with the following.

Proposition 2.2.7. The mod 2 reduction of $R_2 \in H^7(K(\mathbb{Z}/2, 2); \mathbb{Z})$ is $b_2^2 b_3 \in H^7(K(\mathbb{Z}/2, 2); \mathbb{Z}/2)$. In particular, it is nontrivial.

Proof. It is well known ((28), for example) that Sq^1 is the composition of the Bockstein homomorphism followed by the mod 2 reduction. It follows from (Equation 2.2.5) that $H^8(K(\mathbb{Z}/2,2);\mathbb{Z})$ is a 2-torsion group, from which it follows that the mod 2 reduction

$$H^8(K(\mathbb{Z}/2,2);\mathbb{Z}) \to H^8(K(\mathbb{Z}/2,2);\mathbb{Z}/2)$$

is injective. Therefore, it suffices to show $\mathrm{Sq}^1(b_2^2 b_3) = 0$, which implies the Bockstein homomorphism sends $b_2^2 b_3$ to 0. Indeed, by Cartan's formula, we have

$$\begin{aligned} & \mathrm{Sq}^1(b_2^2 b_3) \\ = & \mathrm{Sq}^1(b_2)b_2b_3 + b_2\,\mathrm{Sq}^1(b_2)b_3 + b_2^2\,\mathrm{Sq}^1(b_3) \\ = & 2\,\mathrm{Sq}^1(b_2)b_2b_3 + b_2^2\,\mathrm{Sq}^1\,\mathrm{Sq}^1(b_2) \\ = & 0, \end{aligned}$$

where the last equation follows from the Adem relation $\mathrm{Sq}^1\,\mathrm{Sq}^1 = 0$. □

2.3 Twisted K-Theory and Atiyah-Hirzebruch Spectral Sequence

For a connected topological space X and a class $\alpha \in \mathrm{Br}'(X) = H^3(X;\mathbb{Z})_{\mathrm{Tor}}$, Donovan-Karoubi ((17)) and Atiyah-Segal ((9))defined the twisted complex K-theory of X with respect to α, which we denote by $KU(X)_\alpha$, following the convention in (2).

Similar to the usual, untwisted complex K-Theory, there is a twisted version of the Atiyah-Hirzebruch spectral sequence, $\tilde{E}_*^{*,*}$, such that

$$\tilde{E}_2^{s,t} \cong \begin{cases} H^s(X;\mathbb{Z}), & \text{if } t \text{ is even,} \\ 0, & \text{if } t \text{ is odd,} \end{cases}$$

and converges to $KU(X)_\alpha$ when X is a finite CW complex. For more details, see (9) and (10). The spectral sequence is closely related to the index of α, as shown in the following

Theorem 2.3.1. *Let X be a finite CW complex and let $\alpha \in Br(X)$. Consider $\tilde{E}_*^{*,*}$, the twisted Atiyah-Hirzebruch spectral sequence with respect to α with differentials $\tilde{d}_r^{s,t}$ with bi-degree $(r, -r+1)$. In particular, $\tilde{E}_2^{0,0} \cong \mathbb{Z}$, and any $\tilde{E}_r^{0,0}$ with $r > 2$ is a subgroup of $\mathbb{Z}$ and therefore generated by a positive integer. The subgroup $\tilde{E}_3^{0,0}$ (resp. $\tilde{E}_\infty^{0,0}$) is generated by* $\operatorname{per}(\alpha)$ *(resp.* $\operatorname{ind}(\alpha)$*).*

Moreover, we have a rank map $KU^0(X)_\alpha \to \mathbb{Z}$ (See Section 2.5 of (2)) of which the image is generated by $\operatorname{ind}(\alpha)$. Theorem 2.3.1 is an immediate consequence of Proposition 2.21 and Lemma 2.23 of (2). It has the following consequence:

Corollary 2.3.2. Let X be a connected 8-dimensional CW-complex, and let $\alpha \in \mathrm{Br}'(X) = H^3(X;\mathbb{Z})_{\mathrm{tor}}$ be such that $\operatorname{per}(\alpha) = n$. Then $\operatorname{ind}(\alpha) | \epsilon_2(n)\epsilon_3(n)n^3$.

Proof. First we fix a CW-complex structure on the Eilenberg-Mac Lane space $K(\mathbb{Z}/n, 2)$ and take X to be $\mathrm{sk}_8(K(\mathbb{Z}/n, 2))$, the 8th skeleton of $K(\mathbb{Z}/n, 2)$. Then the corresponding twisted Atiyah-Hirzebruch spectral sequence is shown in Figure Figure 5, where one readily sees that the only differentials out of $\tilde{E}_*^{0,0}$ with non-trivial codomains are $\tilde{d}_3^{0,0}$, $\tilde{d}_5^{0,0}$ and $\tilde{d}_7^{0,0}$, whose codomains are, respectively, subquotients of $\tilde{E}_2^{3,-2} \cong H^3(K(\mathbb{Z}/n);\mathbb{Z})$, $\tilde{E}_2^{5,-4} \cong H^5(K(\mathbb{Z}/n);\mathbb{Z})$ and $\tilde{E}_2^{7,-6} \cong H^7(K(\mathbb{Z}/n);\mathbb{Z})$. As discussed in Section 1, the three groups above are all cyclic, of order n, $\epsilon_2(n)n$, and $\epsilon_3(n)n$ respectively, from which the desired result follows for $X = \mathrm{sk}_8(K(\mathbb{Z}/n, 2))$.

For a general X and α, choose $\alpha' \in H^2(X;\mathbb{Z}/n)$ such that $B(\alpha') = \alpha$, where B is the Bockstein homomorphism. then α' is classified by a cell map $f : X \to K(\mathbb{Z}/n, 2)$ such that $f^*(\beta_n) = \alpha$, where β_n is the canonical generator of $H^2(K(\mathbb{Z}/n, 2);\mathbb{Z})$ as discussed in Section 2. The corollary then follows from the functoriality of the twisted Atiyah-Hirzebruch spectral sequence. The idea of the proof is indicated in Figure Figure 5. $\square$

2.4 The Space $\mathbf{B}P(n, mn)$ and Its Low Dimensional Postnikov Decomposition

Let m, n be integers. Recall that in Section 1 we defined a Lie group $P(n, mn)$ which fits in the following exact sequence:

$$1 \to \mathbb{Z}/m \to P(n, mn) \xrightarrow{\varphi} PU_{mn} \to 1.$$

Applying the classifying space functor, we obtain a fiber sequence

$$\mathbf{B}\mathbb{Z}/m \to \mathbf{B}P(n, mn) \xrightarrow{\mathbf{B}\varphi} \mathbf{B}PU_{mn}. \tag{2.4.1}$$

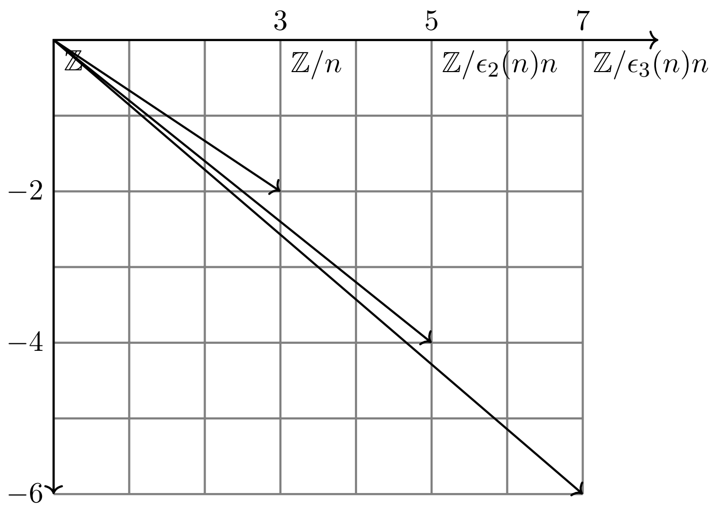

Figure 5. The twisted Atiyah-Hirzebruch spectral sequence associated to the 8th skeleton of $K(\mathbb{Z}/n, 2)$.

The space $\mathbf{B}P(n, mn)$ plays an important role in the study of topological period-index problem. As we mentioned in the introduction, $\pi_1(P(n, mn)) \cong \mathbb{Z}/n$, and consequently $\mathbf{B}P(n, mn)$ is a simply connected space with $\pi_2(\mathbf{B}P(n, mn)) \cong \mathbb{Z}/n$. Therefore we have a projection onto the 2nd stage of Postnikov Tower $\mathbf{B}P(n, mn) \to K(\mathbb{Z}/n, 2)$. In particular, $P(n, n) = PU_n$, and we have the following commutative diagram

$$\begin{tikzcd} \mathbf{B}P(n, mn) \arrow[r] \arrow[d] & \mathbf{B}PU_{mn} \arrow[d] \\ K(\mathbb{Z}/n, 2) \arrow[r] & K(\mathbb{Z}/mn, 2) \end{tikzcd} \tag{2.4.2}$$

where the vertical arrows are the projections to the 2nd stages of the respective Postnikov towers.

Proposition 2.4.1. The bottom arrow in the diagram Equation 2.4.2 is induced by the canonical inclusion $\mathbb{Z}/n \hookrightarrow \mathbb{Z}/mn$.

In fact, this follows from the fact that $\mathbf{B}\psi$ induces a morphism on the 2nd homotopy groups which is the inclusion described above.

Delooping the first term of the fiber sequence Equation 2.4.1, we obtain another fiber sequence

$$\mathbf{B}P(n, mn) \xrightarrow{\mathbf{B}\varphi} \mathbf{B}PU_{mn} \xrightarrow{\eta} K(\mathbb{Z}/m, 2), \tag{2.4.3}$$

which leads to the following

Proposition 2.4.2. The second arrow η in the fiber sequence (Equation 2.4.3) fits in the following commutative diagram:

$$\begin{tikzcd}
\mathbf{B}P(n, mn) \arrow[r] \arrow[d] & \mathbf{B}PU_{mn} \arrow[d] \arrow[dr, "\eta"] & \\
K(\mathbb{Z}/n, 2) \arrow[r] & K(\mathbb{Z}/mn, 2) \arrow[r] & K(\mathbb{Z}/m, 2)
\end{tikzcd}$$

in which the square is diagram (Equation 2.4.2) and the bottom row is a fiber sequence induced by delooping twice the canonical short exact sequence

$$0 \to \mathbb{Z}/n \to \mathbb{Z}/mn \to \mathbb{Z}/m \to 0.$$

Let X be a finite CW complex and let $\alpha \in \mathrm{Br}(X) = H^3(X;\mathbb{Z})_{\mathrm{tor}}$ be of order n. Recall the lifting problem (Equation 2.1.4) discussed in the introduction, as shown by the following diagram.

$$
\begin{tikzcd}
 & \mathbf{B}P(n,mn) \arrow[d] \\
X \arrow[ur, dashed] \arrow[r, "\alpha'"] & K(\mathbb{Z}/2,n)
\end{tikzcd}
\tag{2.4.4}
$$

We have the following

Proposition 2.4.3. Let X, α be as above. Furthermore, suppose that $H^2(X;\mathbb{Z}) = 0$. Then α is classified by an Azumaya algebra of degree mn if and only if the lifting in diagram (Equation 2.4.4) exists.

Proof. The "if" part follows easily by post-composing a lift $X \to \mathbf{B}P(n,mn)$ with $\mathbf{B}\psi$. To prove the "only if" part, suppose that α is classified by a map $f : X \to \mathbf{B}PU_{mn}$. Since $H^2(X;\mathbb{Z}) = 0$, there is a unique $\alpha' \in H^2(X;\mathbb{Z})$ such that $B(\alpha') = \alpha$, where B is the Bockstein homomorphism. Moreover, let α'' be the image of α' under the canonical map $K(\mathbb{Z}/n,2) \to K(\mathbb{Z}/mn,2)$, then α'' is the unique class in $H^2(X;\mathbb{Z}/mn)$ such that $B(\alpha'') = \alpha$. The uniqueness of α'' indicates that the map f above fits in the following commutative diagram

$$
\begin{tikzcd}
 & \mathbf{B}P(n,mn) \arrow[r] \arrow[d] & \mathbf{B}PU_{mn} \arrow[r, "\eta"] \arrow[d] & K(\mathbb{Z}/m,2) \arrow[d, "="] \\
X \arrow[ur, dashed] \arrow[urr, "f"] \arrow[r, "\alpha'"] & K(\mathbb{Z}/n,2) \arrow[r] & K(\mathbb{Z}/mn,2) \arrow[r] & K(\mathbb{Z}/m,2)
\end{tikzcd}
\tag{2.4.5}
$$

where the square in the middle is the one in Proposition Equation 2.4.2, and both the top and bottom rows of the 3 by 2 rectangular diagram are fiber sequences. The bottom row being a contractible map, a simple diagram chasing shows that the lifting indicated by the dashed arrow exists. □

We denote integral cohomological Serre spectral sequence associated to (Equation 2.4.1) by $(E_*^{*,*}, d_*^{*,*})$, of which the E_2 page is

$$E_2^{s,t} \cong H^s(\mathbf{B}PU_{mn}; H^t(\mathbf{B}\mathbb{Z}/m)) \cong \begin{cases} H^s(\mathbf{B}PU_{mn}; \mathbb{Z}), & \text{if } t = 0; \\ H^s(\mathbf{B}PU_{mn}; \mathbb{Z}/m), & \text{if } t > 0 \text{ is even}; \\ 0, & \text{if } t \text{ is odd}. \end{cases} \tag{2.4.6}$$

This follows from the fact that, as a ring,

$$H^*(\mathbf{B}\mathbb{Z}/m) \cong \mathbb{Z}[v]/(mv), \tag{2.4.7}$$

where v is of degree 2. As for the cohomology of $\mathbf{B}PU_{mn}$, we have the following

Theorem 2.4.4 ((20), Theorem 1.1)**.** *For an integer $n > 1$, the graded ring $H^*(\mathbf{B}PU_n; \mathbb{Z})$, in degrees ≤ 10, is isomorphic to the following graded ring:*

$$\mathbb{Z}[e_2, \cdots, e_{j_n}, x_1, y_{3,0}, y_{2,1}, z_1, z_2]/I_n.$$

Here e_i is of degree $2i$, $j_n = min\{5, n\}$; the degrees of $x_1, y_{3,0}, y_{2,1}$ are $3, 8, 10$, respectively; and the degrees of z_1, z_2 are $9, 10$, respectively. I_n is the ideal generated by

$$nx_1, \quad \epsilon_2(n)x_1^2, \quad \epsilon_3(n)y_{3,0}, \quad \epsilon_2(n)y_{2,1}, \quad \epsilon_3(n)z_1, \quad \epsilon_3(n)z_2,$$

$$\delta(n)e_2x_1, \quad (\delta(n)-1)(y_{2,1} - e_2x_1^2), \quad e_3x_1,$$

where

$$\delta(n) = \begin{cases} 2, & \textit{if } n = 4l+2 \textit{ for some integer } l, \\ 1, & \textit{otherwise.} \end{cases}$$

The degreewise cohomology groups with coefficients in an arbitrary ring follow immediately from the theorem above, together with the Künneth theorem. We will simply refer to Theorem 0.0.1 for them.

Consider the quotient map $SU_{mn} \to P(n, mn)$, which is a simply connected cover with Deck transformation group $\mathbb{Z}/n$. Therefore, we have

$$\begin{cases} H^1(\mathbf{B}P(n, mn; \mathbb{Z})) \cong H^2(\mathbf{B}P(n, mn); \mathbb{Z}) = 0, \\ H^3(\mathbf{B}P(n, mn); \mathbb{Z}) \cong H_2(\mathbf{B}P(n, mn); \mathbb{Z}) \cong \pi_2(\mathbf{B}P(n, mn)) \cong \pi_1(P(n, mn)) \cong \mathbb{Z}/n, \end{cases} \tag{2.4.8}$$

which leads to the following

Lemma 2.4.5. *In the spectral sequence $(E_*^{*,*}, d_*^{*,*})$, the differential $d_3^{0,2}$ is a monomorphism. By choosing the generator of $E_2^{0,2} \cong \mathbb{Z}/m$ correctly, $d_3^{0,2}$ can be taken as the canonical inclusion*

$\mathbb{Z}/m \hookrightarrow \mathbb{Z}/mn$. In particular, $H^3(\mathbf{B}P(n,mn);\mathbb{Z})$ is generated by x_1', the image of x_1 under the homomorphism $H^3(\mathbf{B}PU_{mn};\mathbb{Z}) \to H^3(\mathbf{B}P(n,mn);\mathbb{Z})$ induced by the quotient map.

Lemma 2.4.5 has the following consequence:

Proposition 2.4.6. Let m, n be positive integers. Then $\epsilon_2(n)n|m$ if and only if

$$H^5(\mathbf{B}P(n,mn);\mathbb{Z}) \cong \mathbb{Z}/\epsilon_2(n)n.$$

Proof. See Figure Figure 6 for the spectral sequence discussed here. Notice $E_2^{5,0} = 0$ from Theorem 0.0.1. Then for obvious degree reasons the only nontrivial entry of the E_2-page of total degree 5 is $E_2^{3,2} \cong \mathbb{Z}/m$, from which it follows that

$$H^5(\mathbf{B}P(n,mn);\mathbb{Z}) \cong E_\infty^{3,2}.$$

The proposition then follows from the same computation as in the proof of the first statement of Lemma 2.2.3, only with β_{mn} replaced by x_1. □

Remark 2.4.7. Indeed, we have the commutative diagram

$$\begin{array}{ccccc} K(\mathbb{Z}/m,1) & \longrightarrow & \mathbf{B}P(n,mn) & \longrightarrow & \mathbf{B}PU_{mn} \\ \downarrow{=} & & \downarrow & & \downarrow \\ K(\mathbb{Z}/m,1) & \longrightarrow & K(\mathbb{Z}/n,2) & \longrightarrow & K(\mathbb{Z}/mn,2) \end{array}$$

which induces a morphism of Serre spectral sequences taking the generator Q_n of $H^5(K(\mathbb{Z}/n,2);\mathbb{Z})$ to the generator of $H^5(\mathbf{B}P(n,mn);\mathbb{Z})$ when $\epsilon_2(n)n|m$.

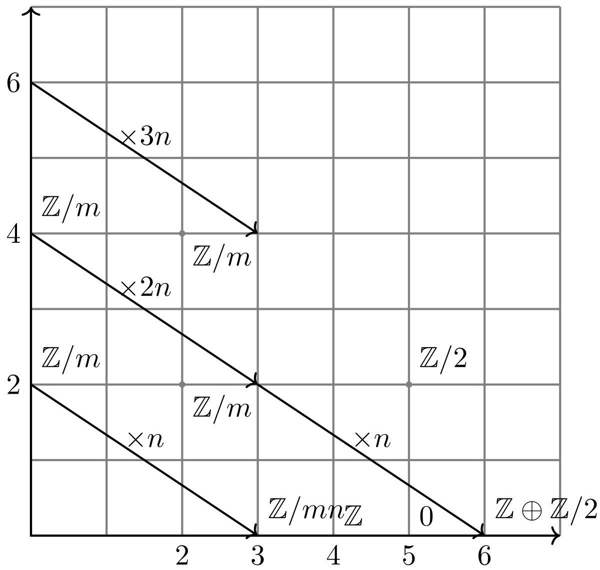

Figure 6. The E_3-page of the spectral sequence $E_*^{*,*}$.

In the proof of Proposition 2.4.6, we observe that when n is odd and m is even, the differentials $d_3^{0,2}$ and $d_3^{3,2}$ annihilate the 2-torsion elements in $E_3^{*,0} \cong H^*(\mathbf{B}P(n,mn);\mathbb{Z})$. This is a special case of a more general argument. By the definition of $P(n,mn)$, we have the following fiber sequence

$$\mathbf{B}SU_{mn} \to \mathbf{B}P(n,mn) \to K(\mathbb{Z}/n,2).$$

We consider the associated cohomological Serre spectral sequence with integral coefficients, of which the E_2-page has no p-torsion for any prime p not dividing n, from which we deduce

Lemma 2.4.8. *For a prime p, $H^*(\mathbf{B}P(n,mn);\mathbb{Z})$ has no nontrivial p-torsion if $p \nmid n$.*

Lemma 2.4.8 has the following immediate

Corollary 2.4.9. Let p be a prime such that $p|m$ and $p \nmid n$. Then all p-torsion element of $E_2^{*,0}$ vanish in the E_∞-page.

Recall from Theorem 0.0.1 that the torsion subgroup of $E_2^{8,0} \cong H^8(\mathbf{B}PU_r;\mathbb{Z})$ is $\mathbb{Z}/3$ if $3|r$ and 0 otherwise.

Corollary 2.4.10. If $3|m$ and $3 \nmid n$, then the differential

$$d_5^{3,4} : E_5^{3,4} \to E_5^{8,0} = E_2^{8,0}$$

is a surjection onto its subgroup $\mathbb{Z}/3$.

Proof. It follows for degree reasons that $d_5^{3,4}$ is the only possibly nontrivial differential towards $E_*^{8,0}$, and in particular, it follows that $E_5^{8,0} = E_2^{8,0}$. The fact that $d_5^{3,4}$ is onto $\mathbb{Z}/3$ follows from Corollary 2.4.9. □

We proceed to consider the Postnikov tower of $\mathbf{B}P(n, mn)$. Recall the low-dimensional homotopy groups of $\mathbf{B}P(n, mn)$:

$$\pi_i(\mathbf{B}P(n, mn) \cong \begin{cases} \mathbb{Z}/n, & i = 2, \\ \mathbb{Z}, & 2 < i < 2mn, \quad i \text{ even}, \\ 0, & 0 < i < 2mn, \quad i \text{ odd}. \end{cases} \tag{2.4.9}$$

We denote the ith stage of the Postnikov tower of a simply connected topological space X by $X[i]$, and the ith k-invariant by κ_i. Then we have part of the Postnikov system of $\mathbf{B}P(n,mn)$ as follows:

$$\begin{tikzcd} K(\mathbb{Z},4) \arrow[r] & \mathbf{B}P(n,mn)[4] = \mathbf{B}P(n,mn)[5] \arrow[d] & \\ & \mathbf{B}P(n,mn)[3] = K(\mathbb{Z}/n,2) \arrow[r, "\kappa_3"] & K(\mathbb{Z},5) \end{tikzcd} \tag{2.4.10}$$

In general we have $\mathbf{B}P(n,mn)[2i] = \mathbf{B}P(n,mn)[2i+1]$ for all $n>0$ even and $i<n$, since in such cases we have $\pi_{2i+1}(\mathbf{B}P(n,mn)) = 0$. By (Equation 2.2.5) and Proposition 2.4.6, we have

$$H^5(\mathbf{B}P(n,mn);\mathbb{Z}) \cong H^5(K(\mathbb{Z}/n,2);\mathbb{Z}) \cong \mathbb{Z}/\epsilon_2(n)n \cong H^5(K(\mathbb{Z}/n,2)\times K(\mathbb{Z},4);\mathbb{Z})$$

if and only if $\epsilon_2(n)n|m$, which implies the following

Proposition 2.4.11. Let m,n be positive integers. Then $\epsilon_2(n)n|m$ if and only if in the Postnikov tower of $\mathbf{B}P(n,mn)$, we have $\kappa_3 = 0$. or equivalently, we have

$$\mathbf{B}P(n,mn)[5] = \mathbf{B}P(n,mn)[4] \simeq K(\mathbb{Z}/n,2)\times K(\mathbb{Z},4).$$

Remark 2.4.12. This is essentially the main result of (1).

The integral cohomology groups of $\mathbf{B}P(n,mn)$ in degree ≤ 5 are immediate from the proposition above. In particular, we have

Corollary 2.4.13. As in Proposition 2.4.11, we assume that $\epsilon_2(n)n|m$.

1. $H^4(\mathbf{B}P(n,mn);\mathbb{Z}) \cong \mathbb{Z}$. We denote its generator by e_2'.

2. Recall the map $\mathbf{B}\varphi : \mathbf{B}P(n, mn) \to \mathbf{B}PU_{mn}$ induced by the quotient map φ. The induced homomorphism

$$(\mathbf{B}\varphi)^* : H^4(\mathbf{B}PU_{mn}; \mathbb{Z}) \cong \mathbb{Z} \to H^4(\mathbf{B}P(n, mn); \mathbb{Z}) \cong \mathbb{Z}$$

is the multiplication by $\epsilon_2(n)mn$.

Proof. The statement (1) follows immediately from Proposition 2.4.11. To prove (2), consider the spectral sequence $E_*^{*,*}$ as in (Equation 2.4.6). Notice $E_2^{5,0} \cong H^5(\mathbf{B}PU_{mn}; \mathbb{Z}) = 0$, from which it follows that

$$E_\infty^{2,2} = E_2^{2,2} \cong H^2(\mathbf{B}PU_{mn}; \mathbb{Z}/m) \cong \mathbb{Z}/m. \tag{2.4.11}$$

For the same reason we have $E_\infty^{0,4} \cong \mathrm{Ker} d_3^{0,4}$. By the Leibniz rule we have $d_3(v^2) = 2vd_3(v) = 2nvx_1$, which implies that $E_\infty^{0,4}$ is the subgroup of $E_2^{0,4}$ of $\epsilon_2(n)n$-torsion elements, i.e.,

$$E_\infty^{0,4} \cong \mathrm{Ker} d_3^{0,4} = E_4^{0,4} \cong \mathbb{Z}/\epsilon_2(n)n. \tag{2.4.12}$$

The equations (Equation 2.4.11) and (Equation 2.4.12), together with (1) of the corollary imply (2). □

We proceed to make a similar assertion on $H^6(\mathbf{B}P(n, mn); \mathbb{Z})$. To do so we need the following

Lemma 2.4.14. *When $\epsilon_2(n)n|m$, the Abelian group $H^7(\mathbf{B}P(n,mn);\mathbb{Z})$ is isomorphic to $\mathbb{Z}/\epsilon_3(n)n\times \mathbb{Z}/n\times\mathbb{Z}/2$ modulo a cyclic subgroup. In particular, $H^7(\mathbf{B}P(n,mn);\mathbb{Z})$ is **not** a cyclic group when n is even.*

Proof. It follows from 2.4.11 that

$$H^7(\mathbf{B}P(n,mn)[5];\mathbb{Z})\cong H^7(K(\mathbb{Z}/n,2)\times K(\mathbb{Z},4);\mathbb{Z})\cong \mathbb{Z}/\epsilon_3(n)n\times\mathbb{Z}/n\times\mathbb{Z}/2.$$

Therefore

$$H^7(\mathbf{B}P(n,mn);\mathbb{Z})\cong H^7(\mathbf{B}P(n,mn)[6];\mathbb{Z})\cong \mathbb{Z}/\epsilon_3(n)n\times\mathbb{Z}/n\times\mathbb{Z}/2/(\kappa_5),$$

and the result follows. □

Corollary 2.4.15. Suppose $\epsilon_2(n)n|m$.

1. We have
$$H^6(\mathbf{B}P(n,mn);\mathbb{Z})\cong\begin{cases}\mathbb{Z}\oplus\mathbb{Z}/2, & \text{if } n \text{ is even,}\\ \mathbb{Z}, & \text{if } n \text{ is odd.}\end{cases}$$

2. When n is even, the subgroup $\mathbb{Z}/2$ of $H^6(\mathbf{B}P(n,mn);\mathbb{Z})$ is generated by $(x_1')^2$, where $x_1'=\mathbf{B}\varphi^*(x_1)$. Furthermore, $\mathbf{B}\varphi$ induces a homomorphism
$$H^6(\mathbf{B}PU_{mn};\mathbb{Z})/(x_1^2)\cong\mathbb{Z}\to H^6(\mathbf{B}P(n,mn);\mathbb{Z})/(\mathbf{B}\varphi(x_1^2))\cong\mathbb{Z}$$

which is the multiplication by

$$\begin{cases} \epsilon_3(\frac{m}{n})nm^2, \text{if } n \text{ is even, or } m, n \text{ are both odd,} \\ \\ \epsilon_3(\frac{m}{n})nm^2/2, \text{if } n \text{ is odd, and } 4|m, \\ \\ \epsilon_3(\frac{m}{n})nm^2/4, \text{if } n \text{ is odd, and } m = 2(2l+1) \text{ for some integer } l. \end{cases}$$

Proof. Consider the 6th stage of the Postnikov tower of $\mathbf{B}P(n, mn)$ as described by the following diagram:

$$\begin{array}{ccccc} K(\mathbb{Z}, 6) & \longrightarrow & \mathbf{B}P(n, mn)[6] & & \\ & & \downarrow & & \\ & & \mathbf{B}P(n, mn)[5] = K(\mathbb{Z}/n, 2) \times K(\mathbb{Z}, 4) & \xrightarrow{\kappa_5} & K(\mathbb{Z}, 7) \end{array}$$

from which it follows that

$$H^6(\mathbf{B}P(n, mn); \mathbb{Z}) \cong \mathbb{Z} \oplus H^6(K(\mathbb{Z}/n, 2) \times K(\mathbb{Z}, 4); \mathbb{Z}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2, \text{ if } n \text{ is even,} \\ \\ \mathbb{Z}, \text{ if } n \text{ is odd.} \end{cases}$$

from which (1) follows.

Consider the spectral sequence $E_*^{*,*}$. When n is even, we have

$$E_2^{6,0} \cong H^6(\mathbf{B}PU_{mn}; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2,$$

of which the 2-torsion subgroup is generated by x_1^2. By Lemma 2.4.5, the image of $d_3^{3,2}$ is generated by $nx_1^2 = 0$, since n is even. Therefore we have

$$d_3^{3,2} = 0. \tag{2.4.13}$$

For obvious degree reasons there is no other nontrivial differential hitting the entry $(6,0)$. Hence the first half of (2) follows.

To prove the second half of (2), it suffice to show that $E_\infty^{s,t}$ such that $s+t=6, t>0$ are all finite, of which the product of the cardinality is equal to the number given in each case. On the E_2-page, the nontrivial entries $E_2^{s,t}$ such that $s+t=6, t>1$ are $E_2^{4,2} \cong \mathbb{Z}/m$, $E_2^{2,4} \cong \mathbb{Z}/m$ and $E_2^{0,6} \cong \mathbb{Z}/m$.

For $E_2^{0,6}$, we have

$$d_3^{0,6} : E_2^{0,6} \cong \mathbb{Z}/m \to E_2^{3,4} \cong \mathbb{Z}/m$$

the multiplication by $3n$, by the Leibniz rule. Hence we have

$$E_4^{0,6} = \operatorname{Ker} d_3^{0,6} \cong \mathbb{Z}/\epsilon_3(\frac{m}{n})n. \tag{2.4.14}$$

We argue case by case, as follows.

Case 1: n is even, or m, n are both odd. In this case, either $4|mn$, or mn is odd. Then it follows from Theorem 0.0.1 that

$$E_2^{7,0} \cong H^7(\mathbf{B}PU_{mn};\mathbb{Z}) = 0.$$

Then for degree reasons there is no nontrivial differential into or out of $E_2^{4,2}$. Hence we have

$$E_\infty^{4,2} \cong E_2^{4,2} \cong \mathbb{Z}/m. \tag{2.4.15}$$

Next we consider the entry $E_*^{2,4}$. For degree reasons the only possibly nontrivial differential into or out of it is $d_3^{2,4}$, of which the codomain, according to the Künneth's theorem and Theorem 0.0.1, is

$$E_3^{5,2} \cong H^5(\mathbf{B}PU_{mn};\mathbb{Z}/m) \cong \begin{cases} 0, & \text{if } n \text{ is odd, and consequentely so is } m, \\ \mathbb{Z}/2, & \text{if } n \text{ is even.} \end{cases}$$

We proceed to show $d_3^{2,4} = 0$ in both cases. When n is odd this is obvious. When n is even, it follows from Lemma 2.4.14 that $E_\infty^{5,2} \neq 0$, for otherwise $H^5(\mathbf{B}P(m,n);\mathbb{Z})$ would be cyclic, a contradiction. Therefore, $E_\infty^{5,2} = E_2^{5,2} \cong \mathbb{Z}/2$, from which it follows that $d_3^{2,4} = 0$. Hence, we have

$$E_\infty^{2,4} \cong E_2^{2,4} \cong \mathbb{Z}/m. \tag{2.4.16}$$

Again for degree reasons, the only potentially nontrivial differential out of $E_4^{0,6}$ is into $E_4^{5,2} \cong \mathbb{Z}/2$. It follows from Lemma 2.4.14 that this differential is 0. Therefore we have

$$E_\infty^{0,6} \cong E_4^{0,6} \cong \mathbb{Z}/\epsilon_3(\frac{m}{n})n. \tag{2.4.17}$$

Case 1 now follow from (Equation 2.4.15), (Equation 2.4.16) and (Equation 2.4.17).

Case 2: n is odd, and $4|m$. In this case it again follows from Theorem 0.0.1 that $H^7(\mathbf{B}P_{mn}; \mathbb{Z}) = 0$. Then for the same reason as in Case 1 we have

$$E_\infty^{4,2} \cong E_2^{4,2} \cong \mathbb{Z}/m. \tag{2.4.18}$$

Consider $E_*^{2,4}$. The only potentially nontrivial differential into or out of $E_*^{2,4}$ is

$$d_3^{2,4} : E_3^{2,4} \cong \mathbb{Z}/m \to E_3^{5,2} \cong \mathbb{Z}/2,$$

where $E_3^{5,2} \cong H^5(\mathbf{B}P_{mn}; \mathbb{Z}/m) \cong \mathbb{Z}/2$ follows from Künneth's theorem. Since n is odd, it follows from Corollary 2.4.9 that $E_\infty^{5,2} = 0$. However, for degree reasons there is no nontrivial differential into or out of $E_*^{5,2}$ except for possibly $d_3^{2,4}$. (Notice that $E_2^{8,0}$ has no 2-torsion, by Theorem 0.0.1.) Therefore, $d_3^{2,4}$ is surjective and it follows that

$$E_\infty^{2,4} \cong \mathbb{Z}/\frac{m}{2}. \tag{2.4.19}$$

For degree reasons there is no nontrivial entry of total degree 7 on the E_4-page. Hence it follows that

$$E_\infty^{0,6} \cong E_4^{0,6} \cong \mathbb{Z}/\epsilon_3(\frac{m}{n})n. \tag{2.4.20}$$

Case 2 then follows from (Equation 2.4.18), (Equation 2.4.19) and (Equation 2.4.20).

Case 3: n is odd, and $m = 2(2l+1)$ for some integer l. In this case it follows from Theorem 0.0.1 that

$$E_2^{7,0} \cong H^7(\mathbf{B}PU_{mn};\mathbb{Z}) \cong \mathbb{Z}/2,$$

and moreover, the differential

$$d_3^{4,2} : E_3^{4,2} \cong \mathbb{Z}/m \to E_2^{8,0} \cong \mathbb{Z}/2$$

is onto, since, due to Theorem 0.0.1, $H^7(\mathbf{B}PU_{mn};\mathbb{Z})$ is generated by the cup product e_2x_1. For degree reasons there is no other nontrivial differentials into or out of $E_3^{4,2}$. Hence we have

$$E_\infty^{4,2} \cong \mathbb{Z}/\frac{m}{2}. \tag{2.4.21}$$

For $E_\infty^{2,4}$ and $E_\infty^{0,6}$ the same arguments as in Case 2 applies and we have

$$E_\infty^{4,2} \cong \mathbb{Z}/\frac{m}{2} \tag{2.4.22}$$

and

$$E_{\infty}^{0,6} \cong \mathbb{Z}/\epsilon_3(\frac{m}{n})n. \tag{2.4.23}$$

Therefore, Case 3 follows. □

The study of the next non-trivial stage of the Postnikov tower requires some auxiliary results on the cohomology of the classifying spaces of some Lie groups, which is the topic of the next section.

2.5 The Cohomology of Classifying Spaces of Some Lie Groups

Recall the integral cohomological Serre spectral sequence associated to the fiber sequence $\mathbf{B}U_r \to \mathbf{B}PU_r \to K(\mathbb{Z},3)$, which we denote by ${}^UE_*^{*,*}$, and the formula for the differential ${}^Ud_3$ described in Corollary 1.5.3

$${}^UE_3^{s,t} \cong {}^UE_2^{s,t} \cong H^s(K(\mathbb{Z},3);H^t(\mathbf{B}U_r;\mathbb{Z})).$$

Let $c_k \in H^{2k}(\mathbf{B}U_r;\mathbb{Z})$ be the kth Chern class, and x_1 be the generator of $H^3(K(\mathbb{Z},3);\mathbb{Z})$.

In low dimensions, for example, ${}^UE_3^{0,4}$ and ${}^UE_3^{0,6}$, ${}^Ud_3$ is the only non-trivial differential out of them. Therefore, the kernel of ${}^Ud_3^{0,*}$ gives the image of the homomorphism $H^*(\mathbf{B}PU_r;\mathbb{Z}) \to H^*(\mathbf{B}U_r;\mathbb{Z})$ induced by the quotient map $\mathbf{B}U_r \to \mathbf{B}PU_r$. A straightforward calculation gives the following

Lemma 2.5.1. *The image of the homomorphism* $H^*(\mathbf{B}PU_r;\mathbb{Z}) \to H^*(\mathbf{B}U_r;\mathbb{Z})$ *in degree* 4 *and* 6 *are generated respectively by*

$$\epsilon_2(r)(rc_2 - \frac{r-1}{2}c_1^2)$$

and

$$\frac{\epsilon_3(r)}{\epsilon_2(r)\epsilon_2(\frac{r-2}{\epsilon_2(r-2)})}\big[r^2c_3 - r(r-2)c_1c_2 + \frac{(r-1)(r-2)}{3}c_1^3\big].$$

By pre-composing the quotient map with the inclusion $SU_r \hookrightarrow U_r$, we obtain another quotient map $SU_r \to PU_r$. Applying the classifying space functor and taking integral cohomology, we obtain the homomorphism

$$H^*(\mathbf{B}PU_r;\mathbb{Z}) \to H^*(\mathbf{B}SU_r;\mathbb{Z}).$$

Recall that the inclusion $SU_r \hookrightarrow U_r$ induces a homomorphism

$$H^*(\mathbf{B}U_r;\mathbb{Z}) \cong \mathbb{Z}[c_1,\cdots,c_n] \to H^*(\mathbf{B}SU_r;\mathbb{Z}) \cong \mathbb{Z}[c_2,\cdots,c_n]$$

which annihilates c_1 and takes c_i to itself, for $i > 1$. Therefore, Lemma 2.5.1 immediately implies the following

Lemma 2.5.2. *The image of the homomorphism*

$$H^*(\mathbf{B}PU_r;\mathbb{Z}) \to H^*(\mathbf{B}SU_r;\mathbb{Z})$$

in degree 4 *and* 6 *are generated respectively by* $\epsilon_2(r)rc_2$ *and*

$$\frac{\epsilon_3(r)r^2}{\epsilon_2(r)\epsilon_2(\frac{r-2}{\epsilon_2(r-2)})}c_3.$$

We conclude this section with the following

Proposition 2.5.3. Let n and m be such that $\epsilon_2(n)n|m$. Consider the quotient map $SU_{mn} \to P(n, mn)$. The induced homomorphism

$$H^6(\mathbf{B}P(n,mn);\mathbb{Z}) \to H^6(\mathbf{B}SU_{mn};\mathbb{Z}) = \mathbb{Z}[c_3]$$

has image generated by

$$\frac{\epsilon_3(mn)}{\epsilon_3(m/n)\epsilon_2(n)}nc_3.$$

Proof. Notice that the quotient map $SU_{mn} \to PU_{mn}$ can be factorized as $SU_{mn} \to P(n, mn) \to PU_{mn}$, and the proposition follows from Lemma 2.5.2 and Corollary 2.4.15, once we notice the following equation:

$$\frac{\epsilon_3(mn)m^2n^2}{\epsilon_2(mn)\epsilon_2(\frac{mn-2}{\epsilon_2(mn-2)})} = \begin{cases} \frac{\epsilon_3(mn)m^2n^2}{\epsilon_2(n)}, & n \text{ is even(hence so is } m\text{), or } m, n \text{ are odd,} \\ \frac{\epsilon_3(mn)m^2n^2}{2\epsilon_2(n)}, & n \text{ is odd, and } 4|m, \\ \frac{\epsilon_3(mn)m^2n^2}{4\epsilon_2(n)}, & n \text{ is odd, and } m = 2(2l+1) \text{ for some } l. \end{cases}$$

□

2.6 Proof of Theorem 2.1.0.1

We proceed to consider $H^7(\mathbf{B}P(n, mn); \mathbb{Z})$. Considered the fiber sequence

$$\mathbf{B}SU_{mn} \to \mathbf{B}P(n, mn) \to K(\mathbb{Z}/n, 2)$$

and the associated integral cohomological Serre spectral sequence ${}^SE_*^{*,*}$ with

$${}^SE_2^{s,t} \cong H^s(K(\mathbb{Z}/n, 2); H^t(\mathbf{B}SU_{mn}; \mathbb{Z})).$$

We have the following

Lemma 2.6.1. *Suppose that $\epsilon_2(n)n|m$. Recall that $H^3(K(\mathbb{Z}/n, 2); \mathbb{Z})) \cong \mathbb{Z}/n$ is generated by an element β_n, and that $H^7(K(\mathbb{Z}/n, 2); \mathbb{Z})) \cong \mathbb{Z}/\epsilon_3(n)n$ is generated by R_n. In the spectral sequence ${}^SE_*^{*,*}$, we have ${}^Sd_3^{0,6}(c_3) = 2c_2\beta_n$ with kernel generated by*

$$\frac{n}{\epsilon_2(n)}c_3,$$

and

$${}^Sd_7^{0,6}(\frac{n}{\epsilon_2(n)}c_3) = \frac{\epsilon_3(n)\epsilon_3(m/n)}{\epsilon_3(mn)}nR_n.$$

All the other differentials out of ${}^SE_*^{0,6}$ are trivial.

In particular ${}^Sd_3^{0,6}$ is the only non-trivial differential out of ${}^SE_*^{0,6}$ when $\epsilon_3(n)n|m$.

Proof. See Figure Figure 7 for the differentials of the spectral sequence ${}^{S}E_{*}^{*,*}$ discussed here. For degree reason the only potentially non-trivial differential out of ${}^{S}E_{2}^{0,4}$ is ${}^{S}d_{5}^{0,4}$. By Proposition 2.4.6 we have

$${}^{S}E_{\infty}^{5,0} \cong \mathbb{Z}/\epsilon_2(n)n \cong {}^{S}E_{2}^{5,0},$$

from which it follows that ${}^{S}d_{5}^{0,4} = 0$. The statement about ${}^{S}d_{3}^{0,6}$ follows from an easy comparison of the cohomological Serre spectral sequences between the fiber sequences

$$\mathbf{B}SU_{mn} \to \mathbf{B}P(n, mn) \to K(\mathbb{Z}/n, 2)$$

and

$$\mathbf{B}U_{mn} \to \mathbf{B}PU_{mn} \to K(\mathbb{Z}, 3),$$

the E_3-page of the latter of which is well understood in Chapter 1. Therefore, the statement about ${}^{S}d_{7}^{0,6}$ follows from Proposition 2.5.3. $\square$

For future convenience we introduce the following notation:

$$I(m, n) = \frac{\epsilon_3(m/n)\epsilon_3(n)}{\epsilon_3(mn)} n. \tag{2.6.1}$$

Lemma 2.6.1 has the following immediate consequence:

Corollary 2.6.2. Assume that n is odd, and $n|m$. Then we have

$$H^7(\mathbf{B}P(n, mn); \mathbb{Z}) \cong \mathbb{Z}/I(m, n),$$

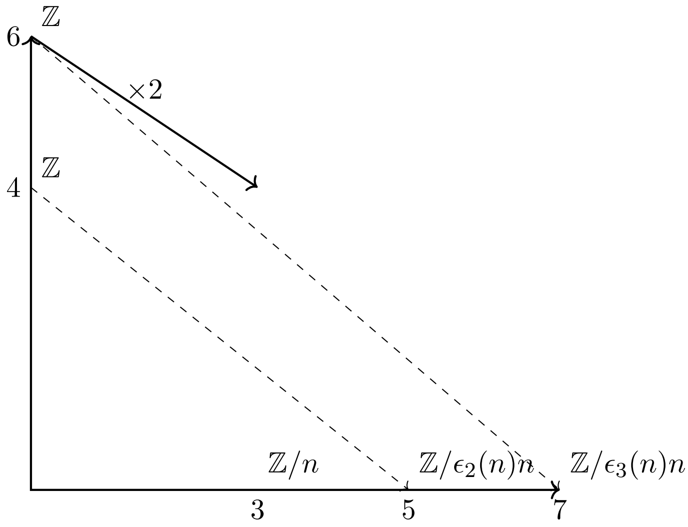

Figure 7. Low dimensional differentials of the spectral sequence ${}^S E_*^{*,*}$, when $\epsilon_3(n)n|m$. The dashed arrows represent trivial differentials.

which is generated by $R_n(x_1'')$. Here x_1'' generates $H^2(\mathbf{B}P(n,mn);\mathbb{Z}/n)$. Moreover, $B(x_1'') = x_1'$ where B is the Bockstein homomorphism.

The general case is more complicated. Recall Lemma 2.4.14, which says that $H^7(\mathbf{B}P(n,mn);\mathbb{Z})$ is not a cyclic group when n is even. With a little more work we can impose a strong restriction on the k-invariant κ_5.

Remark 2.6.3. Since we have the homotopy equivalence

$$\mathbf{B}P(n,mn)[4] = \mathbf{B}P(n,mn)[5] \simeq K(\mathbb{Z}/n,2) \times K(\mathbb{Z},4),$$

the cohomology rings $H^*(\mathbf{B}P(n,mn);\mathbb{Z})$ and $H^*(K(\mathbb{Z}/n,2)\times K(\mathbb{Z},4);\mathbb{Z})$ are isomorphic in degree ≤ 6. In view of this, in what follows we do not explicitly distinguish the elements $x_1', R_n(x_1''), e_2'$ with β_n, R_n, ι_4.

Lemma 2.6.4. *Assume $\epsilon_2(n)n|m$. Then we have*

$$\kappa_5 = \lambda_1 R_n \times 1 + \lambda_2\beta_n \times \iota_4 + 1 \times \Gamma_4$$

where $\lambda_1 \in \mathbb{Z}/\epsilon_3(n)n$, $\lambda_2 \in \mathbb{Z}/n$. Furthermore, the subgroups of $\mathbb{Z}/n$ generated by $2\lambda_2$ contains 2.

In particular, if n is odd, then, up to a scalar multiple, we have

$$\kappa_5 = \lambda_1 R_n \times 1 + \beta_n \times \iota_4 + 1 \times \Gamma_4$$

Proof. Suppose

$$\kappa_5 = \lambda_1 R_n \times 1 + \lambda_2\beta_n \times \iota_4 + \lambda_3 \times \Gamma_4,$$

where $\lambda_1 \in \mathbb{Z}/\epsilon_3(n)n, \lambda_2 \in \mathbb{Z}/n, \lambda_3 \in \mathbb{Z}/2$. Lemma 2.6.1 implies that $H^7(\mathbf{B}P(n,mn);\mathbb{Z})$ is generated by $R_n \times 1$ and $\beta_n \times \iota_4$, since they generate ${}^SE_2^{3,4}$ and ${}^SE_2^{7,0}$, the only non-trivial entries on the E_2-page with total degree 7. In particular, the class $1\times\Gamma_4$ is a linear combination of them, from which it follows that $\lambda_3 = 1$.

Let $\langle\kappa_5\rangle$ be the subgroup of $H^7(\mathbf{B}P(n,mn);\mathbb{Z})$ generated by κ_5. Lemma 2.6.1 implies that $2\beta_n \times \iota_4$ is in the subgroup generated by $R_n \times 1$, since ${}^S E_\infty^{3,4}$, generated by $\beta_n \times \iota_4$, has order 2. Hence, there is some scalar $\Lambda \in \mathbb{Z}/\epsilon_3(n)n$ such that

$$2\beta_n \times \iota_4 + \Lambda R_n \times 1 \in \langle\kappa_5\rangle. \tag{2.6.2}$$

Let $\langle\beta_n \times \iota_4, R_n \times 1\rangle$ be the subgroup of $H^7(\mathbf{B}P(n,mn);\mathbb{Z})$ generated by $\beta_n \times \iota_4$ and $R_n \times 1$. Then (Equation 2.6.2) implies

$$2\beta_n \times \iota_4 + \Lambda R_n \times 1 \in \langle\kappa_5\rangle \cap \langle\beta_n \times \iota_4, R_n \times 1\rangle = \langle 2\kappa_5\rangle,$$

where the identification of subgroups follows from the fact

$$2\kappa_5 \in \langle\kappa_5\rangle \cap \langle\beta_n \times \iota_4, R_n \times 1\rangle$$

and that 2 is a prime number.

From the above, it follows that $2\beta_n \times \iota_4 + \Lambda R_n \times 1$ is a multiple of

$$2\kappa_5 = 2(\lambda_1 R_n \times 1 + \lambda_2 \beta_n \times \iota_4 + \lambda_3 \times \Gamma_4) = 2(\lambda_1 R_n \times 1 + \lambda_2 \beta_n \times \iota_4),$$

which implies that the ideal of $\mathbb{Z}/n$ generated by $2\lambda_2$ contains 2, and the lemma follows. $\square$

We make the following important observation:

Lemma 2.6.5. *Let X be the 8-skeleton of $K(\mathbb{Z}/n,2)$, with a Brauer class*

$$\alpha \in H^3(X;\mathbb{Z})_{\mathrm{tor}} = \mathrm{Br}(X)$$

the restriction of the fundamental class $\beta_n \in H^3(K(\mathbb{Z}/n,2);\mathbb{Z})$. If α is classified by a PU_{mn}-torsor, then $\epsilon_2(n)\epsilon_3(n)n|m$.

Proof. For the obvious reason we do not distinguish cohomology classes of X and $K(\mathbb{Z}/n,2)$ in degree ≤ 7. It follows from Proposition 2.4.11 that $\epsilon_2(n)n|m$. It remains to prove that $\epsilon_3(n)n|m$. Assume $\epsilon_3(n)n \nmid m$ for a contradiction. Since we already have $\epsilon_2(n)n|m$, it follows that $3|n$ and $3 \nmid \frac{m}{n}$. Hence we have

$$I(m,n) = n. \tag{2.6.3}$$

Since $H^2(X;\mathbb{Z}) = 0$, there is a unique element $\alpha' \in H^2(X;\mathbb{Z}/n)$ such that $B(\alpha') = \alpha$, where B is the Bockstein homomorphism. Therefore, the lifting problem shown by the following diagram

$$\begin{tikzcd} & \mathbf{B}P(n,mn)[5] = K(\mathbb{Z},4)\times K(\mathbb{Z}/n,2) \arrow[r, "\kappa_5"] \arrow[d] & K(\mathbb{Z},7) \\ X \arrow[ur, dashed, "f_5"] \arrow[r, "\alpha'"] & \mathbf{B}P(n,mn)[3] = K(\mathbb{Z}/n,2) & \end{tikzcd}$$

has a unique solution f_5 since $H^4(X;\mathbb{Z}) = 0$, and for the same reason, the composition $\kappa_5 \cdot f_5$ is $\lambda_1 R_n \in H^7(X;\mathbb{Z})$, for some $\lambda_1 \in \mathbb{Z}/\epsilon_3(n)n$, according to Lemma 2.6.4. On the other hand, it follows from Lemma 2.6.1 and (Equation 2.6.3) that in $H^7(\mathbf{B}P(n,mn);\mathbb{Z})$ we have $nR_n = 0$, whereas R_n is of degree $3n$ in $H^7(\mathbf{B}P(n,mn)[5];\mathbb{Z})$. Therefore, we have

$$0 \neq nR_n \in \langle \kappa_5 \rangle \in H^7(\mathbf{B}P(n,mn)[5];\mathbb{Z}).$$

This implies that nR_n is a multiple of $\lambda_1 R_n$, and in particular, $\kappa_5 \cdot f_5 = \lambda_1 R_n \neq 0$. Hence f_5 does not lift to $\mathbf{B}P(n,mn)[6]$, a contradiction. □

We proceed to study the next non-trivial stage of the Postnikov tower of $\mathbf{B}P(n,mn)$, namely $\mathbf{B}P(n,mn)[6]$. Recall from Lemma 2.4.14 that when n is even, we have

$$\begin{aligned} &H^7(\mathbf{B}P(n,mn);\mathbb{Z}) \cong H^7(\mathbf{B}P(n,mn)[6];\mathbb{Z}) \\ \cong &H^7(K(\mathbb{Z},4)\times K(\mathbb{Z}/n,2);\mathbb{Z})/(\kappa_5) \cong \mathbb{Z}/\epsilon_3(n)n \oplus \mathbb{Z}/n \oplus \mathbb{Z}/2/(\kappa_5), \end{aligned}$$

where the components $\mathbb{Z}/\epsilon_3(n)n$, $\mathbb{Z}/n$ and $\mathbb{Z}/2$ are generated by $R_n \times 1$, $\beta_n \times \iota_4$ and $1 \times \Gamma_4$.

Lemma 2.6.6. *Suppose $\epsilon_3(n)n|m$. Then we have $E_\infty^{3,4} \cong \mathbb{Z}/\epsilon_3(n)n$. Moreover, as a direct sum component of $H^7(\mathbf{B}P(n,mn);\mathbb{Z})$, $E_\infty^{3,4}$ is generated by R_n.*

Proof. Consider the following commutative diagram

$$\begin{tikzcd} K(\mathbb{Z}/m,1) \arrow[r] \arrow[d, "="] & \mathbf{B}P(n,mn) \arrow[r] \arrow[d] & \mathbf{B}PU_{mn} \arrow[d] \\ K(\mathbb{Z}/m,1) \arrow[r] & K(\mathbb{Z}/n,2) \arrow[r] & K(\mathbb{Z}/mn,2) \end{tikzcd} \tag{2.6.4}$$

It follows from the proof of Lemma 2.2.3 that the morphism of spectral sequences induced by (Equation 2.6.4) restricts to an isomorphism

$$ {}^{H}E_4^{3,4} \cong E_4^{3,4}. $$

We proceed to show that it restricts to an isomorphism

$$ {}^{H}E_\infty^{3,4} \cong E_\infty^{3,4}. \tag{2.6.5} $$

For degree reasons, $d_3^{0,6}$ is the only non-trivial differential reaching the entry $E_*^{3,4}$. By the Leibniz rule, we have

$$ E_3^{3,4}/\operatorname{Im} d_3^{0,6} \cong (\mathbb{Z}/m)/3n \cong \mathbb{Z}/\epsilon_3(m)n, $$

where the last equation follows from the fact that $\epsilon_3(n)n|m$. On the other hand, it follows from Leibniz rule that $d_3^{3,4}(v^2x_1) = 2nvx_1 = 0$. Therefore we have $d_3^{3,4} = 0$ and

$$ E_4^{3,4} = E_4^{3,4} \cong \mathbb{Z}/\epsilon_3(m)n. $$

For degree reasons, the only potentially nontrivial differential into or out of $E_4^{3,4}$ is

$$ d_5^{3,4} : E_5^{3,4} \to E_5^{8,0}. $$

It then follows from (Equation 2.2.14) in the proof of Lemma 2.2.3 that $d_5^{3,4} = 0$ when $3|n$. On the other hand, it follows from Corollary 2.4.10 that

$$d_5^{3,4} : E_5^{3,4} \cong \mathbb{Z}/\epsilon_3(m)n \to E_5^{8,0} \cong H^8(\mathbf{B}PU_{mn}; \mathbb{Z})$$

is onto the subgroup of $E_5^{8,0}$ of order 3 when $3 \nmid n$ and $3|m$. Therefore, (Equation 2.6.5) holds, and the desired assertion follows from Lemma 2.2.3. $\square$

The next lemma concerns $\beta_n \times \iota_4$, which, as an element of $H^7(\mathbf{B}P(n, mn); \mathbb{Z})$, is identified with $e_2' x_1'$. Recall that $E_2^{0,4} \cong \mathbb{Z}/m$ is generated by v^2.

Lemma 2.6.7. *The group $E_\infty^{0,4} \cong \mathbb{Z}/\epsilon_2(m)n$ is the subgroup of $E_2^{0,4}$ generated by $\frac{m}{\epsilon_2(m)n} v^2$.As a quotient of $H^4(\mathbf{B}P(n, mn); \mathbb{Z})$, $E_\infty^{0,4}$ is generated by the element represented by e_2', the generator of $H^4(\mathbf{B}P(n, mn); \mathbb{Z})$. Moreover, suppose that $\epsilon_2(n)\epsilon_3(n)n|m$. Then we can choose v, up to an invertible scalar coefficient, so that in $E_\infty^{0,4}$ there is a relation $e_2' = \frac{m}{\epsilon_2(m)n} v^2$.*

Proof. The fact that $E_\infty^{0,4} \cong \mathbb{Z}/\epsilon_2(m)n$ and that it is generated by $\frac{m}{\epsilon_2(n)n} v^2$ follows from Lemma 2.4.5 and the Leibniz rule. The rest follows from Corollary 2.4.13. $\square$

Theorem 2.6.8. *Suppose that $\epsilon_2(n)\epsilon_3(n)n|m$. Then*

$$H^7(\mathbf{B}P(n, mn); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/\epsilon_3(n)n, & n \text{ odd}, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/\epsilon_3(n)n, & n \text{ even}. \end{cases} \tag{2.6.6}$$

Furthermore,

1. if n is odd, then, up to an invertible scalar,

$$\kappa_5 = \frac{\epsilon_3(n)m}{\epsilon_3(m)n}\lambda R_n \times 1 + \beta_n \times \iota_4 + 1 \times \Gamma_4$$

2. if n is even, then, up to an invertible scalar,

$$\kappa_5 = \frac{\epsilon_3(n)m}{\epsilon_2(m)\epsilon_3(m)n}\lambda R_n \times 1 + \lambda_2\beta_n \times \iota_4 + 1 \times \Gamma_4 \quad \mathrm{mod} \quad 2 - \mathrm{torsion}.$$

where $\lambda \in \mathbb{Z}/\epsilon_3(n)n$ is invertible and λ_2 is as in Lemma 2.6.4.

Proof. The first case of (Equation 2.6.6) is just Corollary 2.6.2. For the other case, it follows from Lemma 2.6.6 and Corollary 2.6.2 that we have a short exact sequence

$$0 \to \mathbb{Z}/\epsilon_3(n)n \to H^7(\mathbf{B}P(n,mn);\mathbb{Z}) \to \mathbb{Z}/2 \to 0.$$

By Lemma 2.4.14, the group $H^7(\mathbf{B}P(n,mn);\mathbb{Z})$ is not cyclic. Therefore we have

$$H^7(\mathbf{B}P(n,mn);\mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/\epsilon_3(n)n,$$

for n even.

To prove (1) and (2) we need to find the coefficient λ_1 as in Lemma 2.6.4. This is accomplished by studying the element $e_2'x_1' \in H^7(\mathbf{B}P(n,mn);\mathbb{Z})$. In particular, we locate it in the spectral sequence $E_*^{*,*}$.

It follows from Lemma 2.6.7 that in $E_3^{3,4}$ we have the relation $e_2'x_1' = \frac{m}{\epsilon_2(m)n}v^2x_1$. On the other hand, it follows from Lemma 2.6.6 that $E_\infty^{3,4}$ is generated by $\frac{\epsilon_3(m)}{\epsilon_3(n)}v^2x_1$ which is identified with λR_n for some invertible element $\lambda \in \mathbb{Z}/\epsilon_3(n)n$. Hence, in $E_\infty^{3,4}$ we have the relation

$$e_2'x_1' = \frac{\epsilon_3(n)m}{\epsilon_2(m)\epsilon_3(m)n}\lambda R_n \tag{2.6.7}$$

Since we have

$$H^7(\mathbf{B}P(n,mn);\mathbb{Z}) \cong \begin{cases} E_\infty^{3,4}, & n \text{ odd}, \\ E_\infty^{3,4} \oplus E_\infty^{5,2} \cong E_\infty^{3,4} \oplus \mathbb{Z}/2, & n \text{ even}, \end{cases}$$

The desired statement (1),(2) then follows immediately from (Equation 2.6.7). □

Proof of Theorem 2.1.0.1. Let X be an 8-complex and $\alpha \in \mathrm{Br}(X)$. The first paragraph of the theorem, that $\mathrm{ind}(\alpha)|\epsilon_2(n)\epsilon_3(n)n^3$ follows immediately from Corollary 2.3.2.

We proceed to prove the second paragraph. Let X, α and α' be as in Lemma 2.6.5. Consider α' as a map $X \to K(\mathbb{Z}/n, 2)$. Then it follows from Proposition 2.4.11, that α' has a lift to $\mathbf{B}P(n,mn)[4]$ if and only if $\epsilon_2(n)n|m$, in which case

$$\mathbf{B}P(n,mn)[4] \simeq K(\mathbb{Z}/n,2) \times K(\mathbb{Z},4)$$

and we have a unique lift f_5 as in the following diagram

$$\begin{tikzcd} & \mathbf{B}P(n,mn)[5] = K(\mathbb{Z},4)\times K(\mathbb{Z}/n,2) \arrow[r, "\kappa_5"] \arrow[d] & K(\mathbb{Z},7) \\ X \arrow[ur, dashed, "f_5"] \arrow[r, "\alpha'"] & \mathbf{B}P(n,mn)[3] = K(\mathbb{Z}/n,2) & \end{tikzcd} \tag{2.6.8}$$

as discussed in Lemma 2.6.5. Therefore, a lift of α'_n to $\mathbf{B}P(n,mn)[5]$ exists if and only if

$$\kappa_5 \cdot f_5 = 0 \in H^7(X;\mathbb{Z}) \cong H^7(K(\mathbb{Z}/n,2);\mathbb{Z}).$$

On the other hand, the projection

$$\mathbf{B}P(n,mn)[4] \simeq K(\mathbb{Z}/n,2)\times K(\mathbb{Z},4) \to K(\mathbb{Z}/n,2)$$

splits, from which it follows that the homomorphism $H^*(f_5)$ is a quotient homomorphism sending exactly all classes in $H^*(K(\mathbb{Z},4);\mathbb{Z})$ to 0. Recall from Proposition 2.4.11 and Lemma 2.6.5 that we only need to consider m and n such that $\epsilon_2(n)\epsilon_3(n)n|m$. Therefore, it follows from Theorem 2.6.8 that

$$\kappa_5 \cdot f_5 = f_5^*(\kappa_5) = \begin{cases} \frac{\epsilon_3(n)m}{\epsilon_3(m)n}\lambda R_n, \quad n \text{ odd}, \\ \\ \frac{\epsilon_3(n)m}{\epsilon_2(m)\epsilon_3(m)n}\lambda R_n \quad \bmod \quad 2\text{−torsion}, \quad n \text{ even}. \end{cases} \tag{2.6.9}$$

When n is odd, $\kappa_5 \cdot f_5 = 0$ if and only if $\epsilon_3(n)n|\frac{\epsilon_3(n)m}{\epsilon_3(m)n}$, i.e., $\epsilon_3(m)n^3|mn$. Since $n|m$, we have $\epsilon_3(n)|\epsilon_3(m)$, hence $\epsilon_3(n)n^3|mn$, for all m, n such that the lift of α' to $\mathbf{B}P(n,mn)[6]$ exists, which, according to Proposition 2.4.3, is true if and only if α can be classified by a PU_{mn}-torsor. Therefore, $\epsilon_3(n)n^3 | \operatorname{ind}(\alpha)$. Then it follows that $\epsilon_3(n)n^3 = \operatorname{ind}(\alpha)$, as desired.

When n is even, the same argument can be made with the indeterminacy of 2-torsions:

$$\kappa_5 \cdot f_5 = \frac{\epsilon_3(n)m}{\epsilon_2(m)\epsilon_3(m)n}\lambda R_n = 0 \mod 2-\text{torsions},$$

where $\lambda \in \mathbb{Z}/\epsilon_3(n)n$ is invertible. In other words, we have $\frac{\epsilon_3(n)n}{2} | \frac{\epsilon_3(n)m}{\epsilon_2(m)\epsilon_3(m)n}$, i.e., $\epsilon_3(m)n^3 | mn$. Since $n|m$, we have $\epsilon_3(n)n^3 | mn$ for all m, n such that α is classified by a PU_{mn}-torsor. Hence $\epsilon_3(n)n^3 | \operatorname{ind}(\alpha)$. □

2.7 Proof of Theorem 2.1.0.2

It follows from Proposition 2.4.11 that $\mathbf{B}P(n, mn)[5] \simeq K(\mathbb{Z}/n, 2) \times K((\mathbb{Z}, 4)$ when $\epsilon_2(n)n|m$. As will be shown later, the essential case of this section is $n = 2$, which we treat first. Consider the diagonal inclusion

$$\Delta_0 : SU_2 \hookrightarrow SU_{2m}.$$

Passing to the quotient spaces of the respective $\mathbb{Z}/2$ actions given by the scaler multiplication of $e^{\pi\sqrt{-1}}$, we have another inclusion

$$\Delta_1 : PU_2 \hookrightarrow P(2, 2m).$$

Passing to classifying spaces, we have

$$\mathbf{B}\Delta_1 : \mathbf{B}PU_2 \hookrightarrow \mathbf{B}P(2, 2m)$$

Lemma 2.7.1. $\mathbf{B}\Delta_1$ *induces an isomorphism on* $H^3(-;\mathbb{Z})$.

Proof. By the Hurewicz theorem, it suffices to show that $\Delta_1 : PU_2 \hookrightarrow P(2, 2m)$ induces an isomorphism of fundamental groups, which follows from the fact that the $\mathbb{Z}/2$ actions on their respective simply connected covers SU_2 and SU_{2m} commute with the diagonal inclusion. $\square$

Recall the well-known exceptional isomorphism $PU_2 \cong SO_3$, from which it follows

$$\mathbf{B}PU_2 \cong \mathbf{B}SO_3,$$

and in particular,

$$H^*(\mathbf{B}PU_2; \mathbb{Z}/2) \cong H^*(\mathbf{B}SO_3; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3],$$

where w_2, w_3 are the Stiefel-Whitney classes of the universal SO_3-bundle over $\mathbf{B}SO_3$.

Lemma 2.7.2. *In the setting above, we have*

$$\mathrm{Sq}^1(w_2) = w_3.$$

Proof. The first nontrivial stage of the Postnikov tower of $\mathbf{B}SO_3$ is $K(\mathbb{Z}/2, 2) \simeq \mathbf{B}SO_3[2]$, and w_2 is the Postnikov map. It follows from $\pi_3(\mathbf{B}SO_3) = 0$ that we have $K(\mathbb{Z}/2, 2) \simeq$

$\mathbf{B}SO_3[3]$, from which it follows that w_2 induces an isomorphism over mod 2 cohomology groups in dimensions less than 4. The lemma then follows from Corollary 2.2.6. $\square$

Recall that we denote the generator of $H^3(\mathbf{B}P(n,mn);\mathbb{Z})$ by x_1'. Also recall the element $R_2(x_1') \in H^7(\mathbf{B}P(2,2m);\mathbb{Z})$, or R_2 for short. Let overhead bars indicate the mod 2 reduction of integral cohomology classes. We have the following

Corollary 2.7.3. In mod 2 cohomology we have $(\mathbf{B}\Delta_1)^*(\bar{R}_2) = w_2^2 w_3$. In particular $(\mathbf{B}\Delta_1)^*(\bar{R}_2)$ is nontrivial.

Proof. It follows from Lemma 2.7.1 and Lemma 2.7.2 that $(\mathbf{B}\Delta_1)^*(\bar{x}_1') = w_3$ and $(\mathbf{B}\Delta_1)^*(x_1'') = w_2$, where x_1'' is the generator of $H^2(\mathbf{B}P(2,2m);\mathbb{Z}/2)$. The rest follows from Proposition 2.2.7. $\square$

Recall from Proposition 2.4.13 that $H^4(\mathbf{B}P(2,2m);\mathbb{Z})$ is generated by e_2'.

Lemma 2.7.4. *Let $4|m$, and $(\mathbf{B}\Delta_1)^*$ be the homomorphism induced by $\mathbf{B}\Delta_1$ between integral cohomology groups. Then*

$$(\mathbf{B}\Delta_1)^*(e_2') = \frac{m}{4} e_2,$$

where e_2 and e_2' are the generators of $H^4(\mathbf{B}PU_2;\mathbb{Z})$ and $H^4(\mathbf{B}P(2,2m);\mathbb{Z})$, respectively.

Proof. Consider the following composition

$$\mathbf{B}PU_2 \xrightarrow{\mathbf{B}\Delta_1} \mathbf{B}P(2,2m) \to \mathbf{B}PU_{2m},$$

where the second arrow is induced by the quotient homomorphism. By Theorem 0.0.1 and Proposition 2.4.13, $H^4(-;\mathbb{Z})$ of all 3 spaces involved are isomorphic to $\mathbb{Z}$. It follows from Lemma 1.9.2 that $H^4(-;\mathbb{Z})$ of the composition is multiplication by m^2, and from Proposition 2.4.13 that $H^4(-;\mathbb{Z})$ of the second arrow is multiplication by $4m$. Therefore $H^4(\mathbf{B}\Delta_1;\mathbb{Z})$ is multiplication by

$$m^2/4m = \frac{m}{4}.$$

□

Corollary 2.7.5. Let $m = 4$ and $\bar{e}'_2$ be the mod 2 reduction of e'_2. Then in the mod 2 cohomology, we have

$$(\mathbf{B}\Delta_1)^*(\bar{e}'_2) = w_2^2.$$

Proof. Notice that

$$H^4(\mathbf{B}PU_2;\mathbb{Z}/2) \cong H^4(\mathbf{B}SO_3;\mathbb{Z}/2)$$

is generated by w_2^2. Therefore w_2^2 is the mod 2 reduction of e_2. The rest follows from Lemma 2.7.4. □

The author owes the following lemma to A. Bousfield.

Lemma 2.7.6. *Let $4|m$, and recall the definition of $\bar{e}'_2$ from Corollary 2.7.5. We have*

$$\mathrm{Sq}^3(\bar{e}'_2) \neq 0.$$

Proof. Consider the following fiber sequence:

$$\mathbf{B}P(2,2m) \to \mathbf{B}PU_{2m} \to K(\mathbb{Z}/m,2),$$

and denote its associate Serre spectral sequence in $\mathbb{Z}/2$ coefficients by ${}^2E_*^{*,*}$. Recall that $H^2(\mathbf{B}P(2,2m);\mathbb{Z}/2)$ is generated by x_1'', and $H^3(\mathbf{B}P(2,2m);\mathbb{Z}/2)$ by $\bar{x}_1'$. Also recall from Corollary 2.2.6 which asserts

$$H^*(K(\mathbb{Z}/n,2);\mathbb{Z}/2) = \mathbb{Z}/2[b_2,b_3,b_5]$$

where $b_5 = \mathrm{Sq}^2(b_3)$ and $\mathrm{Sq}^1(b_2) = 0$. For obvious degree reasons, the differential

$$ {}^2d_3^{0,2} : {}^2E_3^{0,2} \cong H^2(\mathbf{B}P(2,2m);\mathbb{Z}/2) \to {}^2E_3^{3,0} \cong H^3(\mathbf{B}P(2,2m);\mathbb{Z}/2), x_1'' \mapsto b_3 \tag{2.7.1}$$

is an isomorphism. Therefore, the element x_1'' is transgressive and we have

$$ {}^2d_4^{0,3} : {}^2E_4^{0,3} \cong H^3(\mathbf{B}P(2,2m);\mathbb{Z}/2) \to {}^2E_4^{4,0} \cong H^3(\mathbf{B}P(2,2m);\mathbb{Z}/2),$$
$$\bar{x}_1' = \mathrm{Sq}^1(x_1'') \mapsto \mathrm{Sq}^1(b_3) = 0,$$

from which it follows that

$$ {}^2d_4^{2,3}(b_2 \otimes \bar{x}_1') = 0. \tag{2.7.2}$$

It follows from Proposition 2.4.11 that

$$H^4(\mathbf{B}P(2,2m);\mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$

and moreover, it is generated by $(x_1'')^2$ and $\bar{e}_2'$. Since $x_1'' \in {}^2E_3^{0,2}$ is transgressive, so is $(x_1'')^2 = \mathrm{Sq}^2(x_1'')$. Furthermore, we have

$${}^2d_5^{0,4}((x_1'')^2) = \mathrm{Sq}^2({}^2d_5^{0,4}(x_1'')) = \mathrm{Sq}^2(b_3) = b_5. \tag{2.7.3}$$

For obvious degree reasons, this is the only nontrivial differential reaching ${}^2E_*^{5,0}$. Hence, we have

$${}^2E_\infty^{5,0} \cong \mathbb{Z}/2. \tag{2.7.4}$$

On the other hand, it follows from (Equation 2.7.2) that $b_2 \otimes \bar{x}_1' \in^2 E_3^{2,3}$ is a permanent cocycle, whereas

$$H^5(\mathbf{B}PU_2m;\mathbb{Z}/2) \cong \mathbb{Z}/2,$$

a consequence of Theorem 0.0.1. Therefore, it follows from (Equation 2.7.4) that $b_2 \otimes \bar{x}_1'$ is a coboundary. For degree reasons and (Equation 2.7.3), we have

$${}^2d_2^{0,4}(\bar{e}_2') = b_2 \otimes \bar{x}_1'. \tag{2.7.5}$$

We recall a theorem regarding Steenrod operations in spectral sequences, proved independently by Araki ((7)) and Vázquez ((31)). We quote this theorem from (26) as follows:

Theorem 2.7.7 (Theorem 6.15, (26))**.** *On the mod p cohomology spectral sequence associated to a fibration $F \to E \to B$, there are operations*

$$\textit{for } p \textit{ odd} \begin{cases} {}_F\,\mathrm{P}^s : E_r^{a,b} \to E_r^{a,b+2s(p-1)}, 1 \leq r \leq \infty, \\ \\ {}_B\,\mathrm{P}^s : E_r^{a,b} \to E_r^{a+(2s-b)(p-1),pb}, 2 \leq r \leq \infty, \end{cases}$$

$$\textit{for } p = 2 \begin{cases} {}_F\,\mathrm{Sq}^i : E_r^{a,b} \to E_r^{a,b+i}, 1 \leq r \leq \infty, \\ \\ {}_B\,\mathrm{Sq}^i : E_r^{a,b} \to E_r^{a+i-b,2b}, 2 \leq r \leq \infty, \end{cases}$$

that converge to the action of $\mathcal{A}_p$ on $H^*(E;\mathbb{Z}/p)$, commute with the differentials in the spectral sequence, satisfy analogues of Cartan's formula and the Adem relations and reduce to the $\mathcal{A}_p$-action on $H^*(F;\mathbb{Z}/p)$ and $H^*(B;\mathbb{Z}/p)$. when $r = 2$ and $a = 0$ or $b = 0$ (that is, on $E_2^{*,0}$ and $E_2^{0,*}$). Here $\mathcal{A}_p$ denotes the mod p Steenrod algebra.

Furthermore, these operations satisfy a list of axioms similar to those characterizing Steenrod operations. In particular, we have

$$^F\,\mathrm{Sq}^i = 0 : E_r^{a,b} \to E_r^{a,b+i}, i < 0 \text{ or } i > b. \tag{2.7.6}$$

For the complete list of the axioms, see, for example, (7).

It follows from (Equation 2.7.5) that

$$^2d_2^{0,7}(\mathrm{Sq}^3(\bar{e}_2')) =_F \mathrm{Sq}^3(b_2 \otimes \bar{x}_1') = b_2 \otimes (\bar{x}_1')^2 \neq 0.$$

In particular, $\mathrm{Sq}^3(\bar{e}_2') \neq 0$. $\square$

In order to reduce the proof of Theorem 2.1.0.2 to the case that $n = 2$, we need the following

Theorem 2.7.8 (Theorem 1.3, (3)). *Let $(X, \mathcal{O}_X)$ be a connected locally ringed topos, and let $\alpha = \alpha_1 + \cdots + \alpha_t$ be the prime decomposition of a Brauer class $\alpha \in \mathrm{Br}_{top}(X)$ so that each $\mathrm{per}(\alpha_i) = p_i^{a_i}$ for distinct primes $p_1, \cdots, p_t$. Then*

$$\mathrm{ind}(\alpha) = \mathrm{ind}(\alpha_1) \cdots \mathrm{ind}(\alpha_t).$$

Proof of Theorem 2.1.0.2. Suppose $n = 2l$ where l is an odd number. Write $\alpha = \alpha_1 + \alpha_2$, where α_1 and α_2 are of order 2 and l respectively. It follows from Theorem 2.1.0.1 that $\mathrm{ind}\,\alpha_2 | \epsilon_3(l)l^3$. By Theorem 2.7.8, it suffices to show that $\mathrm{ind}\,\alpha_1 \nmid 2^3$. Hence, it suffices to prove the theorem for $n = 2$.

Recall that any n

$$\kappa_5 = \lambda_1 R_n \times 1 + \lambda_2 \beta_n \times \iota_4 + 1 \times \Gamma_4.$$

In the case $n = 2$, this means that λ_1 and λ_2 are either 0 or 1. Therefore, it suffices to determine λ_1 and λ_2 in the mod 2 cohomology group. It follows from Corollary 2.7.3 and Corollary 2.7.5 that in $H^7(\mathbf{B}PU_2; \mathbb{Z}/2)$, we have

$$\begin{aligned}0 =&(\mathbf{B}\Delta)^*(\lambda_1 R_2 + \bar{x}_1'\bar{e}_2' + \mathrm{Sq}^3(\bar{e}_2'))\\ =&(\mathbf{B}\Delta)^*(\lambda_1 R_2) + w_2^2 w_3 + \mathrm{Sq}^3(w_2^2)\\ =&\lambda_1 w_2^2 w_3 + \lambda w_2^2 w_3 + [\mathrm{Sq}^2(w_2)w_2 + w_2\,\mathrm{Sq}(w_2)]\\ =&(\lambda_1 + \lambda_2)w_2^2 w_3\end{aligned}$$

which implies $\lambda_1 + \lambda_2 = 0$. On the other hand, it follows from Lemma 2.7.6 that λ_1 and λ_2 cannot be both 0. So we have

$$\lambda_1 = \lambda_2 = 1.$$

Therefore, when $m = 4$, we have

$$\kappa_5 = R_2 \times 1 + \beta_2 \times \iota_4 + 1 \times \Gamma_4.$$

Hence, the obstruction class for lifting β_2 to $\mathbf{B}P(2, 8)[5]$ is $R_2 \neq 0$, and the desired result follows. $\square$

We conclude this paper with the following immediate corollary of Theorem 2.1.0.2:

Corollary 2.7.9. Let $X = \mathrm{sk}(K(\mathbb{Z}/n, 2))$. Then in the twisted Atiyah-Hirzebruch spectral sequence $(\tilde{E}_*^{*,*}, \tilde{d}_*^{*,*})$, the differential $\tilde{d}_7^{0,0}$ is an epimorphism if $4 \nmid n$.

Proof. Theorem 0.0.5 implies that $\tilde{d}_5^{0,0}$ is an epimorphism for all n, thus has kernel generated by $\epsilon_2(n)n$. The corollary then follows from Theorem 1.5.8 and Theorem 2.3.1. $\square$

Cited Literature

1. B. Antieau and B. Williams, *The Topological Period-Index Problem over 6-Complexes*, J. Top. 7 (2014), 617-640.

2. B. Antieau and B. Williams, *The Topological Period-Index Problem for Twisted topological K-theory*, Geometry & Topology 18 (2014), no. 2, 1115-1148.

3. B. Antieau and B. Williams, *The Topological Period-Index Conjecture*, unpublished.

4. B. Antieau and B. Williams, *The Prime Divisors of the Period and Index of a Brauer Class*, J.

5. B. Antieau and B. Williams, *The Prime Divisors of the Period and Index of a Brauer Class*, J. Pure Apl. Alg. 219 (2015), no. 6, 2218-2224.

6. B. Antieau, *On the Integral Tate Conjecture for Finite Fields and Representation Theory*, Algebraic Geometry 3 (2016), no. 2, 138-149.

7. Araki, S., *Steenrod reduced powers in the spectral sequence associated to a fibering I, II*, Mem. Fac. Sci. Kyusyu Univ. Series (A) Math. 11(1957), 15-64, 81-97. (194)

8. M. Atiyah, *K-Theory*, Westview Press, 1994.

9. M. Atiyah and G. Segal, *Twisted K-Theory*, Ukr. Mat. Visn. 1 (2004), no. 3, 287-330.

10. M. Atiyah and G. Segal, *Twisted K-Theory and Cohomology*, Inspired by S. S. Chern, Nankai Tracts Math., vol. 11, World Sci. Publ., Hackensack, NJ, 2006, pp. 5-43.

11. P. F. Baum and W. Browder, *The Cohomology of Quotients of Classical Groups*, Topology Vol. 3, Iss. 4, (June 1965), 305-336

12. A. Borel, *Topology of Lie Groups and Characteristic Classes*, Bull. Amer. Math. Soc. Vol. 61, no. 5 (1955), 397-432.

13. E. H. Brown, Jr, *The Cohomology of BSO_n and BO_n with Integer Coefficients*, Proceedings of the American Mathematical Society, Vol. 85, No. 2 (Jun., 1982), pp. 283-288

14. H. Cartan and J-P. Serre, *Séminaire Henri Cartan*, Vol. 7, no. 1, 1954-1955.

15. J.-L. Colliot-Thélène, *Conjectures de type local-global sur image des groupes de Chow dans la cohomologie étale*, Algebraic K -theory (Seattle, WA, 1997), 1-12, Proc. Sympos. Pure Math.,67, Amer. Math. Soc., Providence, RI, 1999.

16. J.-L. Colliot-Thélène, *Exposant et Indice d'Algébres Simples Centrales Non Ramifiées*, Enseign. Math. (2) 48 (2002), no. 1-2, 127-146. With an appendix by Ofer Gabber.

17. P. Donovan and M. Karoubi, *Graded Brauer Groups and K-Theory with Local Coefficients*, Inst. Hautes Études Sci. Publ. Math. (1970), no. 38, 5-25.

18. W. G. Dwyer and C. W. Wilkerson, *The Elementary Geometric Structure of Compact Lie Groups*, unpublished.

19. A. Grothendieck, *Le groupe de Brauer. I. Algébres d'Azumaya et Interprétations Diverses,* Séminaire Bourbaki, Vol. 9, Soc. Math. France, Paris, 1995, pp. Exp. No. 290, 199-219.

20. X. Gu, *On the Cohomology of Classifying Spaces of Projective Unitary Groups*, arXiv:1612.00506 (2016).

21. X. Gu, *The topological period-index problem over* 8*-complexes*, draft.

22. S. Eilenberg and S. Mac Lane, *On the Groups* $H(\Pi, n)$*, I*, Ann. of Math. Second Series, Vol. 58, No. 1 (Jul., 1953), pp. 55-106.

23. M. Kameko and N. Yagita, *The Brown-Peterson Cohomology of the Classifying Spaces of the Projective Unitary Groups* PU_p *and exceptional Lie groups*, Trans. Amer. Math. Soc. Vol. 360, No. 5, (May 2008), pp. 2265-2284.

24. M. Kameko, *Representation Theory and the Cycle Map of a Classifying Space*, arXiv:1601.06199

25. A. Kono and M. Mimura, *On the Cohomology of the Classifying Spaces of* $PSU(4n+2)$ *and* $PO(4n+2)$, Publ. RIMS, Kyoto Univ. 10 (1975), 691-720.

26. J. McCleary, *A User's Guide to Spectral Sequences, Second Edition*, Cambrige University Press, 2001.

27. R. J. Milgram, *The Bar Construction and Abelian H-spaces*, Illinois J. Math. Volume 11, Issue 2 (1967), 242-250.

28. R. E. Mosher, M. C. Tangora, *Cohomology Operations and Applications in Homotopy Theory*, Dover Publication Inc., Mineola, New York, 2008. (Originally published: New York: Harper & Row Publishers, 1968.)

29. J-P. Serre, *Homologie singulière des espaces fibrés*, Ann. Math. (2) 54 (1951), 425-505.

30. R. M. Switzer, *Algebraic Topology–Homotopy and Homology*, Springer, 1975.

31. R. Vázquez, *Nota sobre los cuadrados de Steenrod en la sucession espectral de un espacio fibrado*, Bol. Soc. Mat. Mexicana 2(1957), 1-8. (194)

32. A. Vistoli *On the Cohomology and Chow Ring of the Classifying Space of PGL_p*, J. Reine Angew.Math. 610 (2007), 181-227.

33. C. Weibel, *An Introduction to Homological Algebra*, Cambrige University Press, 1994.