Essential Spunnormal Surfaces via Tropical Geometry

BY

ANDREW BRASILE
B.A., Augustana College, May 2006
M.S., University of Illinois at Chicago, May 2008

THESIS

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Chicago, 2013

Chicago, Illinois

Defense Committee:

Marc Culler, Chair and Advisor; Mathematics, Statistics, and Computer Science
Peter Shalen; Mathematics, Statistics, and Computer Science
David Dumas; Mathematics, Statistics, and Computer Science
Jan Verschelde; Mathematics, Statistics, and Computer Science
Nathan Dunfield; University of Illinois at Urbana-Champaign, Mathematics

I dedicate this to my family, Robert and Candace Brasile; John, Joni and August Brasile; Michael and Heidi Brasile and Anna Carley; and Stephanie Brasile, who have been my support and inspiration throughout my life.

ACKNOWLEDGMENT

I would like to acknowledge all of the following graduate students who have given me wonderful advice throughout this program: Tuan Pham, Emily Cilli-Turner, Beth Bauer-Capocci, Nicholas Gardner, Natalie McGathey, Christopher Carlo Fazioli, Marcy Robertson, Gail Tang, Phillip Wesolek, Marc Kjerland, Kathleen Dexter, Sam Ziegler, Randall J. Stading, Holly Krieger, Paul Reschke, Christine Robinson, Ella Revzin, Luigi Lombardi, Bradley Groff, Ali Mohajer, Deniz Bilman, Dimitrios Diochnos, Cara Mullen, Karen Zaya, Michael Ramsey, Ellie Dannenberg, Jonah Gaster, and Ryan Steinbach. I would especially like to acknowledge Tuan Pham and Danko Adrovic who were an invaluable help with the development of this thesis. I would also like to thank Mehul Patel, Brett Hapeman, and Katherine McCullough for the support they have given me throughout the years. Finally, I would like to thank my advisor, Marc Culler, for his patience, understanding, and wisdom throughout this process.

AB

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

SUMMARY

Let M be a compact, orientable 3-manifold with torus boundary. An *essential surface* in M is a nonempty, properly embedded, orientable surface S such that $i_* : \pi_1(S) \to \pi_1(M)$ is injective, and no component of S is boundary parallel or a 2-sphere. If an essential surface S has nonempty boundary, then it must consists of pairwise disjoint, nonseparating closed curves on the boundary torus. Therefore, the components of ∂S are pairwise isotopic, essential closed curves on the boundary torus of M. We call the unoriented isotopy class of the components of ∂S the *boundary slope* of the essential surface S.

Let M be a compact, orientable 3-manifold with torus boundary and S be an essential surface in M. Take a basis μ, λ for $H_1(\partial M; \mathbb{Z})$. Let γ be a component of ∂S, and give γ an orientation. Then $[\gamma] = p\mu + q\lambda \in H_1(\partial M; \mathbb{Z})$ for some $p, q \in \mathbb{Z}$. Now the boundary slope of S can be represented by $p/q \in \mathbb{Q} \cup \{\infty\}$. Notice that the ratio p/q is consistent with both orientations of γ.

We denote the set of boundary slopes of essential surfaces in M by $\mathrm{bs}(M)$. The set $\mathrm{bs}(M)$ is an important object. The relationship between $\mathrm{bs}(M)$ and Dehn fillings of M has produced several fundamental results (1; 2). Furthermore, $\mathrm{bs}(M)$ gives insight into which surfaces are essential in M. Characterizing the essential surfaces in a 3-manifold is the first step in finding a hierarchy of the manifold, a decomposition which is the basis for many classification theorems.

A. Hatcher (3) showed that if M is a compact, orientable 3-manifold with torus boundary, then the set $\mathrm{bs}(M)$ is finite. There have also been many papers which classify this finite set

SUMMARY (Continued)

bs(M) for certain classes of 3-manifolds with torus boundary. A. Hatcher and W. Thurston (4) gave an algorithm for finding bs(M) when M is a 2-bridge knot complement. M. Culler, W. Jaco and J. Rubinstein (5), as well as W. Floyd and A. Hatcher (6) both gave algorithms to find bs(M) for once-punctured torus bundles. A. Hatcher and U. Oertel (7) did the same for Montesinos knot complements with N. Dunfield (8) implementing this algorithm in a computer program.

For general knot complements, W. Jaco and E. Sedgewick (9) have given an algorithm for computing bs(M) for any knot complement M using a standard triangulation of M. M. Culler has used the result that boundary slopes of the Newton polytope of the A-polynomial are boundary slopes in bs(M), which was proven by D. Cooper, M. Culler, H. Gillet, D. Long and P. Shalen (10), to find boundary slopes by computing the A-polynomial for many knot complements. However, both of these processes often involve huge calculations, and there are still relatively simple knot complements where bs(M) is unknown.

Therefore, it is still of interest to give simple criteria for a rational number to be a boundary slope of a knot complement. The main result in this thesis gives simple, sufficient conditions for a surface with nonempty boundary to be essential in a compact, orientable 3-manifold with torus boundary. These criteria are then used to list boundary slopes of essential surfaces in several knot complements. Some of these boundary slopes have not previously been calculated to the author's knowledge.

The main result of this thesis is largely based on the ideas of N. Dunfield and S. Garoufalidis (11). Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, orientable 3-manifold with torus

SUMMARY (Continued)

boundary. It was shown by N. Dunfield and S. Garoufalidis using a result from Y. Kabaya (12), that any vertex surface in M with a normal quad in each tetrahedron of $\mathcal{T}$ is essential. These are easily checked sufficient conditions for a spunnormal surface to be essential in M. In this thesis we will show any vertex surface that is contained in an isolated ray of the set of spunnormal surfaces is essential in M. This removes the previous requirement that there be a normal quad in every tetrahedron.

Chapter 1 gives some basic definitions.

Chapter 2 introduces ideal triangulations of compact, orientable 3-manifolds with torus boundary and cell decompositions of 3-manifolds in general.

We then develop the theory of spunnormal surfaces with respect to an ideal triangulation, which are surfaces that intersect the tetrahedra of the ideal triangulation in such a way that the surfaces can be represented combinatorially. The theory of spunnormal surfaces was introduced by W. Thurston (13) and is presented in Chapter 3.

In Chapter 4 the deformation variety is introduced. The deformation variety is a complex variety which parametrizes hyperbolic structures on the 3-manifold, and was also introduced by W. Thurston (13). . There is a map from the deformation variety to the character variety of the manifold. The character variety of a 3-manifold was developed in work by M. Culler and P. Shalen (14). They used Bass-Serre theory to associate ideal points of the character variety to essential surfaces in the 3-manifold. Therefore, we are interested in finding the ideal points of the deformation variety.

SUMMARY (Continued)

Chapter 5 works towards this end by introducing concepts from tropical geometry. The Fundamental Theorem of Tropical Geometry associates vectors in a set called the tropical variety to ideal points of the deformation variety. A result of B. Osserman and S. Payne (15) is then presented which gives criteria for a vector to be in the tropical variety of the deformation variety.

The main result is then presented in Chapter 6 which shows that any spunnormal surface in an isolated ray of the set of spunnormal surfaces is essential in the 3-manifold. As previously stated, this result generalizes a result of N. Dunfield and S. Groufalidis (11).

The final chapter gives examples of boundary slopes of essential surfaces found using this result.

CHAPTER 1

INTRODUCTION

Definition 1.1 (Embedded manifold) *Let M be an orientable m-manifold with boundary and N be an n-manifold with boundary where $n \leq m$. An embedding of N into M is a map $\varphi : N \to M$ which is a homeomorphism onto its image $\varphi(N)$ with the inherited subspace topology from M. We abuse notation and denote the image $\varphi(N)$ as $N \subseteq M$. We say that N is embedded in M.*

Definition 1.2 (Properly embedded manifold) *Let M be an orientable m-manifold with boundary and N be an n-manifold with boundary where $n \leq m$. Then N is properly embedded in M if N is embedded in M, $\partial N \subseteq \partial M$ and $\mathrm{Int}(N) \subseteq \mathrm{Int}(M)$.*

Definition 1.3 (Two-sided surface) *Let M be an orientable 3-manifold with boundary. A properly embedded connected surface $S \subseteq M$ is two-sided if the complement of S in a sufficiently small neighborhood of S is disconnected. If S in not connected, then S is two-sided if every component of S in two-sided.*

If M is an orientable 3-manifold with boundary, then a properly embedded surface S is two-sided if and only if every component of S is orientable.

Definition 1.4 (Incompressible surface) *Let M be a compact, orientable 3-manifold with boundary. Then a two-sided, properly embedded surface S is incompressible if the following hold:*

1

1. *The inclusion homomorphism $\pi_1(S') \to \pi_1(M)$ is injective for every component S' of S;*

2. *each component of S is distinct from a 2-sphere;*

3. *the surface S is nonempty.*

A non-orientable, properly embedded surface S is incompressible if for any regular neighborhood N of S, the two-sided, properly embedded surface ∂N is incompressible.

Let M be an orientable 3-manifold with boundary. Let S be a two-sided, properly embedded surface. Due to the Loop Theorem proven by Papakyriakopoulos, if $\pi_1(S') \to \pi_1(M)$ is not injective for every component S' of S, then there is an embedded disk $D \subseteq M$ with $\partial D \subseteq S$ where ∂D is an essential closed curve in S. Such a disk is called a *compressing disk* of S.

Definition 1.5 (Boundary-parallel surface) *Let M be an orientable 3-manifold with boundary. A properly embedded surface S is boundary-parallel if there exists a set P such that S is the frontier of P and the pair (S, P) is homeomorphic to the pair $(S \times \{0\}, S \times [0, 1])$.*

Here the *frontier* of a subset P in an orientable 3-manifold with boundary, M, is the intersection of the closure of P with the closure of $M \setminus P$.

Definition 1.6 (Essential surface) *Let M be a compact, orientable 3-manifold with torus boundary. A properly embedded surface S is essential in M if S is incompressible, and has no boundary-parallel components.*

Let M be a compact, orientable 3-manifold with torus boundary and S be an essential surface in M with nonempty boundary. Then the components of ∂S must be pairwise disjoint,

nonseparating closed curves on the boundary torus of M. Therefore, the components of ∂S must be pairwise isotopic, essential closed curves in ∂M.

Definition 1.7 (Boundary slope) *Let M be a compact, orientable 3-manifold with torus boundary. If S is an essential surface with nonempty boundary, then the unoriented isotopy class of the components of ∂S is the boundary slope of S.*

Let M be a compact, orientable 3-manifold with torus boundary and S be an essential surface in M with nonempty boundary. Take a basis μ, λ for $H_1(\partial M, \mathbb{Z})$. Let γ be a component of ∂S and give γ an orientation. Then $[\gamma] = p\mu + q\lambda \in H_1(\partial M, \mathbb{Z})$ for some integers p, q. The boundary slope of S can be represented by $p/q \in \mathbb{Q} \cup \{\infty\}$. Notice that the ratio p/q will be the same regardless of which of the two orientations we choose for γ.

Definition 1.8 (Set of boundary slopes) *Let M be a compact, orientable 3-manifold with torus boundary and choose a basis for $H_1(\partial M, \mathbb{Z})$. Then we denote the set of all boundary slopes of essential surfaces in M with respect to this basis by*

$$\mathrm{bs}(M) = \{p/q \in \mathbb{Q} \cup \{\infty\} \mid p/q \text{ is the boundary slope of an essential surface}\}$$

One fundamental result concerning boundary slopes of essential surfaces is due to A. Hatcher (3). He showed that if M is a compact, orientable 3-manifold with torus boundary, then $\mathrm{bs}(M)$ is finite. There has been extensive work done to classify this finite set of boundary slopes of essential surfaces for certain classes of 3-manifolds with torus boundary. A. Hatcher and W. Thurston (4) gave an algorithm for finding $\mathrm{bs}(M)$ when M is a 2-bridge knot complement. M.

Culler, W. Jaco and H. Rubinstein (5), and also W. Floyd and A. Hatcher (6) gave algorithms to find bs(M) for M a once-punctured torus bundle. A. Hatcher and U. Oertel (7) did the same for Montesinos knot complements. This algorithm was implemented in a program by N. Dunfield (8). Still there are relatively simple knot complements where the set of boundary slopes of essential surfaces is unknown.

CHAPTER 2

IDEAL TRIANGULATIONS AND CELL DECOMPOSITIONS

Throughout our development of ideal triangulations and spunnormal surfaces we will follow the thorough exposition of Stephan Tillmann (16).

2.1 Pseudo-Manifolds

Let $\widetilde{\Delta}$ be a (possibly countably infinite) collection of oriented, polyhedral 3-cells. An *orientation* on a polyhedral 3-cell is an ordering of its vertices. We also consider the set of subfaces of each $\widetilde{\Delta}_i \in \widetilde{\Delta}$ as being contained in $\widetilde{\Delta}$. All subfaces are taken to be closed. Now let Φ be a collection of orientation reversing linear homeomorphisms between 2-dimensional faces of $\widetilde{\Delta}$ such that each face is the domain or range of at most one map $\phi \in \Phi$.

Definition 2.1 (Pseudo-manifold) *Using the notation in the above paragraph, $\mathcal{T}$ is a pseudo-manifold if it is the space obtained by the quotient of $\widetilde{\Delta}$ by the maps in Φ denoted $\mathcal{T} = \widetilde{\Delta}/\Phi$. The quotient of a polyhedral 3-cell $\widetilde{\Delta}_i \in \widetilde{\Delta}$ will be denoted by $\Delta_i \subseteq \mathcal{T}$. The quotient of any subface $\widetilde{\sigma} \in \widetilde{\Delta}$ will be denoted by $\sigma \subseteq \mathcal{T}$.*

Notice that the only possible non-manifold points of such a quotient space are the vertices. The *d-skeleton* of a psuedo-manifold $\mathcal{T}$, denoted $\mathcal{T}^{(d)} \subseteq \mathcal{T}$, is the image of the *d*-dimensional subfaces in $\widetilde{\Delta}$ for $0 \le d \le 3$.

2.2   Ideal Triangulations and Cell Decompositions

Definition 2.2 (Ideal triangulation) *Let M be a compact, oriented 3-manifold with torus boundary. An ideal triangulation of M is a pseudo-manifold, $\mathcal{T} = \widetilde{\Delta}/\Phi$, together with a homeomorphism*

$$p : \mathcal{T} \setminus \mathcal{T}^{(0)} \to \mathrm{Int}(M).$$

We require that $\widetilde{\Delta} = \{\widetilde{\Delta}_1, ..., \widetilde{\Delta}_n\}$ is a finite set, and that each 3-cell $\widetilde{\Delta}_i \in \widetilde{\Delta}$ is a tetrahedron for $1 \leq i \leq n$. We will say the pair $(M, \mathcal{T})$ is an ideal triangulation of a compact, oriented 3-manifold with torus boundary.

Notice that tetrahedra of $\widetilde{\Delta}$ are not necessarily embedded in $\mathrm{Int}(M)$ as they may intersect themselves. This allows the tetrahedra and subsets of the tetrahedra to take on very interesting shapes after identification. For interesting examples see the introduction to W. Jaco and J. Rubinstein's paper (17).

Since M is taken to have only one boundary component, $\mathcal{T}$ has only one vertex, and $\mathcal{T}$ is homeomorphic to M with ∂M collapsed to a point. If $\{\Delta_1, ..., \Delta_n\}$ are the tetrahedra in $\mathcal{T}$, then $\mathcal{T}$ must have n edges since $\chi(\mathrm{Int}(M)) = 0$. Due to work by W. Jaco and J. Rubinstein (17) every compact, orientable 3-manifold with torus boundary admits an ideal triangulation. The program SnapPy (18) developed by M. Culler, J. Weeks and N. Dunfield computes ideal triangulations for compact, orientable 3-manifolds with torus boundary.

For convenience, we will give each tetrahedron an orientation (or labeling of its vertices) using the convention of the program SnapPy (18). See Figure 1 below.

Figure 1. SnapPy orientation on a tetrahedron.
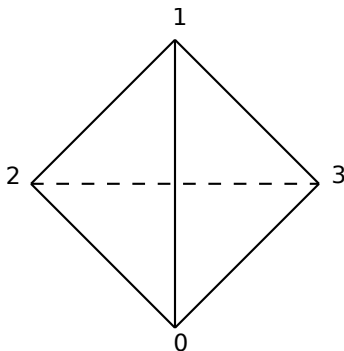
Definition 2.3 (Cell Decomposition) *Let M be an oriented 3-manifold with boundary. A cell decomposition of M is a pseudo-manifold $\mathcal{T}$ together with a homeomorphism*

$$p : \mathcal{T} \to M.$$

We say that the pair $(M, \mathcal{T})$ is a cell decomposition of an oriented 3-manifold with boundary.

Let $(M, \mathcal{T})$ be a cell decomposition of an oriented 3-manifold with boundary. Then every vertex of $\mathcal{T}$ must be a manifold point. Also, if M is noncompact, then $\mathcal{T}$ must consist of infinitely many 3-cells.

2.3   Polyhedral Complexes

Pseudo-manifolds are examples of more general spaces called polyhedral complexes. Polyhedral complexes play a key role throughout this thesis. Our development of polyhedral complexes,

which will be done as needed throughout the thesis, will follow the exceptional presentation in

D. Maclagan and B. Sturmfels' book (19). Here polyhedra can be of any dimension.

Definition 2.4 (Polyhedral complex) *Let $\widetilde{\Sigma}$ be a collection of convex polyhedra, which are not necessarily compact. We consider faces of elements in $\widetilde{\Sigma}$ as being in $\widetilde{\Sigma}$. Let Φ be a collection of linear homeomorphisms between faces of $\widetilde{\Sigma}$. Then the quotient space $\Sigma = \widetilde{\Sigma}/\Phi$ is a polyhedral complex. The image of any face $\widetilde{\sigma} \in \widetilde{\Sigma}$ will be denoted by $\sigma \subseteq \Sigma$.*

We consider polyhedral complexes in the category of piecewise linear spaces, and say that two polyhedral complexes are *isomorphic* if there is a piecewise linear homeomorphism between them.

CHAPTER 3

SPUNNORMAL SURFACES

Throughout this chapter we will continue to follow the development in S. Tillmann's paper (16).

3.1 Normal Disks

Definition 3.1 (Normal arcs and normal disks)

1. *A normal arc is a properly embedded arc in the face of a polyhedral 3-cell which has end points in the interiors of distinct edges.*

2. *A normal disk is a properly embedded disk in a polyhedral 3-cell whose boundary consists of normal arcs and intersects each edge of the polyhedral 3-cell at most once.*

Definition 3.2 (Normal isotopy of a polyhedral 3-cell) *A normal isotopy of a polyhedral 3-cell Σ is an isotopy of Σ which leaves all of the subcells of Σ fixed.*

Most of the development of normal disks in this thesis involves normal disks in the tetrahedra of an ideal triangulation. Therefore, we introduce a few more definitions for normal disks in tetrahedra.

Definition 3.3 (Normal triangles, normal quads and quad-types)

1. *A normal disk in a tetrahedron which meets three faces of the tetrahedron is a normal triangle.*

2. *A normal disk in a tetrahedron which meets four faces of the tetrahedron is a normal quad.*

3. *In any tetrahedron there are three normal isotopy classes of normal quads called the quad-types. We will denote these quad-types by Q_1, Q_2, and Q_3. See Figure 2.*

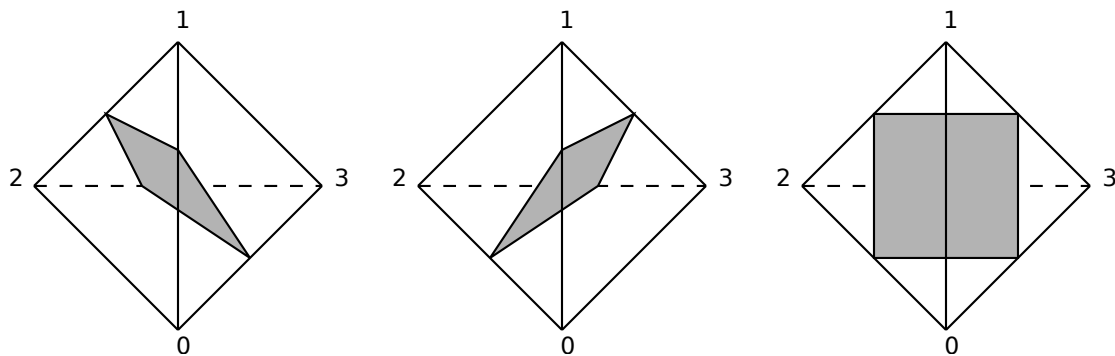Notice there are four normal isotopy classes of normal triangles.

Figure 2. The quad-types from left to right: Q_1, Q_2, Q_3.

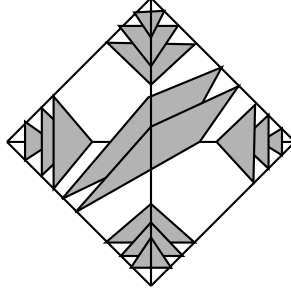## 3.2   Normal Surfaces and Spunnormal Surfaces

Definition 3.4 (Normal surface with respect to a cell decomposition) *Let $(M, \mathcal{T})$ be a cell decomposition of an oriented 3-manifold with boundary, and S be a properly embedded surface in $\mathcal{T}$. Then S is a normal surface with respect to $\mathcal{T}$ if S intersects each polyhedral 3-cell of $\mathcal{T}$ in a collection of normal disks.*

Remark 3.5 *The normalization process consisting of eight moves introduced in the third chapter of Matveev's book (20) can be used to show that if M is a compact, orientable 3-manifold with boundary, then any essential surface in M is isotopic to a normal surface with respect to any cell decomposition $(M, \mathcal{T})$. A property of this normalization process, which will be used later, is that the process never increases the number of intersections of a surface with an edge of $\mathcal{T}$.*

Definition 3.6 (Spunnormal surface with respect to an ideal triangulation) *Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Let S be a properly embedded surface in $\mathcal{T} \setminus \mathcal{T}^{(0)}$. Then S is a spunnormal surface with respect to $\mathcal{T}$ if S intersects every tetrahedron of $\mathcal{T}$ in infinitely many normal disks. See Figure 3 for the intersection of a spunnormal surface with a single tetrahedron.*

Remark 3.7 *Since normal quads of any two distinct quad-types in a single tetrahedron must intersect, the normal quads of a spunnormal surface must be of the same quad-type in any given tetrahedron.*

Figure 3. A possible intersection between a spunnormal surface and a single tetrahedron. There are infinitely many normal triangles, but only finitely many normal quads.

The following is Lemma 1.16 from Tillmann's paper (16), but we give a proof here to help conceptualize ideal triangulations and spunnormal surfaces.

Lemma 3.8 *Let* $(M, \mathcal{T})$ *be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Let* S *be a spunnormal surface with respect to* $\mathcal{T}$. *Then* S *intersects each tetrahedron in* $\mathcal{T}$ *in finitely many normal quads.*

Proof: Assume S intersects the tetrahedron $\Delta \subseteq \mathcal{T}$ in infinitely many normal quads. Let $\widetilde{\mathcal{T}}$ be the universal cover of $\mathcal{T}$ with the lifted cell decomposition. Then the lift of S, $\widetilde{S}$, is a properly embedded surface in $\widetilde{\mathcal{T}} \setminus \widetilde{\mathcal{T}}^{(0)}$. Let $\widetilde{\Delta}$ be the lift of Δ in $\mathcal{T}$.

By Remark 3.7, all of the infinitely many normal quads of $\widetilde{S}$ must have the same quad-type Q in $\widetilde{\Delta}$. Let α be the line segment connecting the midpoints of the edges of $\widetilde{\Delta}$ which normal

quads of quad-type Q do not intersect. Then α must intersect every normal quad of $\widetilde{S}$ in Δ. Therefore the normal quads must have an accumulation point on α. This contradicts that $\widetilde{S}$ is a properly embedded surface in $\widetilde{\mathcal{T}} \setminus \widetilde{\mathcal{T}}^{(0)}$. $\square$

Definition 3.9 (Vertex linking surface) *Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. A vertex linking surface is an embedded torus in $\mathcal{T} \setminus \mathcal{T}^{(0)}$ which consists of one normal triangle from each normal isotopy class of normal triangles in each tetrahedron of $\mathcal{T}$.*

Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Let S be a vertex linking surface in $\mathcal{T} \setminus \mathcal{T}^{(0)}$. Then $p(S) \subseteq M$ is a boundary-parallel torus where $p : \mathcal{T} \setminus \mathcal{T}^{(0)} \to \text{Int}(M)$ is the homeomorphism from Definition 2.2.

Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Tillmann's Lemma 1.15 (16) shows that any embedded surface in $\mathcal{T}$ that intersects the tetrahedra of $\mathcal{T}$ in only normal triangles is a collection of vertex linking components. By this and Lemma 3.8, a spunnormal surface either has a noncompact component which intersects every neighborhood of the vertex of $\mathcal{T}$; or is a closed, compact surface together with infinitely many vertex linking components. A closed surface in $\mathcal{T}$ with a finite number of components is not considered a spunnormal surface by Definition 3.6.

3.3   Associated Properly Embedded Surfaces

Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary, and S be a spunnormal surface with respect to $\mathcal{T}$. By 2.3 of N. Dunfield and S. Garoufalidis's

paper (11), we can associate to S a properly embedded surface in M, $\widehat{S}$, with every component distinct from a boundary parallel torus. A result of this association is that if $\widehat{S}$ has nonempty boundary, then it has a well-defined boundary slope whether $\widehat{S}$ is essential or not. We call the boundary slope of $\widehat{S}$ *the boundary slope of* S. We also say that S is essential if $\widehat{S}$ is essential in M.

G. Walsh (21) has proven that if M is a compact, oriented, hyperbolic 3-manifold with boundary the union of tori, then every essential surface in M which is not a *virtual fiber* is associated to a spunnormal surface with respect to any ideal triangulation $\mathcal{T}$ with essential edges. This statement was first conjectured by W. Thurston (13). An embedded surface S in a oriented, 3-manifold with boundary, M, is a *virtual fiber* if S is either a fiber of a fibration of M over the circle, or a generic fiber of an orbifold-fibration of M over an interval with mirrored endpoints.

Definition 3.10 (End of an ideal triangulation) *Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Let v be the vertex of $\mathcal{T}$. Let $N^\bullet$ be any neighborhood of v with $\partial \overline{N^\bullet}$ a vertex linking surface. Then $N = N^\bullet \setminus \{v\}$ is an end of $\mathcal{T}$. Every time we take an end of an ideal triangulation N, $N^\bullet$ will represent $N \cup \{v\}$.*

Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Then any end, N, has a product structure of the form $\mathbb{T} \times \mathbb{R}$ where $\mathbb{T}$ is the torus.

Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. If N is an end of $\mathcal{T}$, then there is a homeomorphism $p_N : (\mathcal{T} \setminus N^{\bullet}) \to M$ which defines a cell decomposition $(M, \mathcal{T} \setminus N^{\bullet})$ by truncated tetrahedra.

Remark 3.11 *Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, orientable 3-manifold with torus boundary. Let S be a spunnormal surface with respect to $\mathcal{T}$ and N be an end of $\mathcal{T}$. For any homeomorphism, $p_N : (\mathcal{T} \setminus N^{\bullet}) \to M$, S can be isotoped so that $p_N(S \cap (\mathcal{T} \setminus N^{\bullet})) = \widehat{S}$. This follows directly from the construction of $\widehat{S}$ in 2.3 of N. Dunfield and S. Garoufalidis's paper (11).*

3.4   Q-Coordinates

The next proposition is Lemma 1.19 in S. Tillmann's paper (16). We still provide a proof to help understand how spunnormal surfaces are glued together between tetrahedra.

Proposition 3.12 *Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Every spunnormal surface with respect to $\mathcal{T}$ is specified by its normal quads in each tetrahedron of $\mathcal{T}$.*

The proof of this proposition will use a few definitions involving normal arcs on a face of a tetrahedron.

Let F be a triangular face of a tetrahedron. There are three normal isotopy classes of normal arcs in F. Let $\{v_1, v_2, v_3\} = F^{(0)}$ be the vertices of F. The vertex v_i for $1 \leq i \leq 3$ is *dual* to the normal arc α if one component of $F \setminus \alpha$ contains v_i and the other component contains the

other two vertices. Every representative of a normal isotopy class of normal arcs is dual to the same vertex. See Figure 4.

Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Let S be a spunnormal surface with respect to $\mathcal{T}$. Consider the normal disks of S in the tetrahedra of $\widetilde{\Delta} = \{\widetilde{\Delta}_1, ..., \widetilde{\Delta}_n\}$ before taking the quotient. Focus on one face, F, of a tetrahedron $\widetilde{\Delta}_i$ for $1 \leq i \leq n$. The normal disks of S meet F in a countably infinite collection of normal arcs. A normal arc α of S dual to a vertex v is *innermost* if the component of $F \setminus \alpha$ not containing v does not contain any normal arcs of S from the same normal isotopy class as α. Up to normal isotopy, there is a well-ordering of the normal arcs of S in each isotopy class on F. This well-ordering starts with the innermost normal arc for that normal isotopy class, and the successor α_n of a normal arc α_{n-1} only has α_{n-1} and the preceding normal arcs in the component of $F \setminus \alpha_n$ not containing the dual vertex. See Figure 4.

Let F' be the face identified to F by a map in Φ where $\mathcal{T} = \widetilde{\Delta}/\Phi$. The normal arcs of one normal isotopy class on F must be glued to the normal arcs of a corresponding normal isotopy class on F' agreeing with the well-ordering from above.

Proof of Proposition 3.12: Let S and S' be two spunnormal surfaces with respect to $\mathcal{T}$ such that S and S' have the same number of normal quads of the same quad-types in each tetrahedron of $\mathcal{T}$. For any face F of $\widetilde{\Delta}$ and any normal isotopy class of normal arcs on F, the normal arcs which are on the boundary of normal quads of S must come first in the well-ordering of normal arcs from this normal isotopy class discussed above. See Figure 3 and Figure 4. Then by the above discussion, the gluing of S and S' must agree on the innermost arcs in

Figure 4. A well-ordering of the normal arcs dual to v. The well-ordering continues *ad infinitum.*
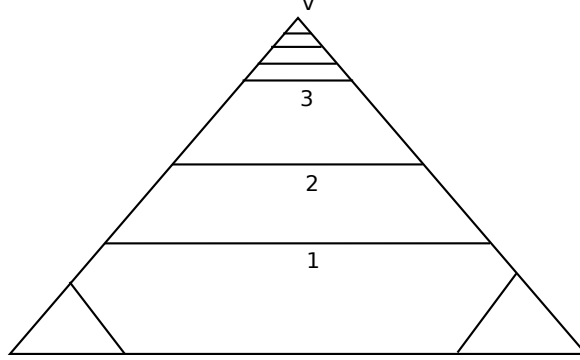
each normal isotopy class and continue through the well-ordering of each normal isotopy class of normal arcs. Therefore, the gluing of S and S' is the same on each face in $\mathcal{T}$. Therefore, S and S' are isotopic. $\square$

Definition 3.13 (Q-coordinate vector) *Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Let $\{\Delta_1, ..., \Delta_n\}$ be the tetrahedra of $\mathcal{T}$. For any spunnormal surface S the Q-coordinate vector of S is a 3n-tuple given by*

$$(q_{1,1}, q_{2,1}, q_{3,1}, q_{1,2}, ..., q_{1,n}, q_{2,n}, q_{3,n}) \in \mathbb{Z}_{\geq 0}^{3n}$$

where each $q_{i,j}$ corresponds to the number of normal quads of S of type Q_i for $1 \leq i \leq 3$ in the tetrahedron Δ_j for $1 \leq j \leq n$.

By Proposition 3.12 a spunnormal surface is completely determined by its Q-coordinate vector. Also by Remark 3.7, if $(q_{1,1}, q_{2,1}, q_{3,1}, q_{1,2}, ..., q_{3,n})$ is the Q-coordinate vector for a spunnormal surface at most one of $q_{1,j}$, $q_{2,j}$, $q_{3,j}$ can be nonzero for $1 \leq j \leq n$. All $3n$-tuples in $\mathbb{R}^{3n}_{\geq 0}$ satisfying this last condition are called *admissible*.

3.5 Q-Matching Equations

Before we get started, we will need some definitions for general polyhedral complexes.

Definition 3.14 (Relative interior of a face) *Let Σ be a polyhedral complex and σ be an n-dimensional face of Σ. The relative interior of σ, $\mathrm{rInt}(\sigma)$, is the interior of σ as an n-dimensional space, not with the subspace topology.*

**Definition 3.15 (Star neighborhood, and closed star neighborhood)**

1. *Let Σ be a polyhedral complex and P be a subset of Σ. The star neighborhood of P, $\mathrm{Str}(P)$, is the union of the relative interiors of every face τ in Σ such that $P \cap \tau \neq \emptyset$.*

2. *The closed star neighborhood of P, $\overline{\mathrm{Str}}(P)$, is the closure of $\mathrm{Str}(P)$.*

Definition 3.16 (Abstract neighborhood, equator and hemispheres) *Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Consider $\widetilde{\mathcal{T}}$ the universal cover of $\mathcal{T}$ with the lifted cell decomposition. An abstract neighborhood of an edge $e \subseteq \mathcal{T}$, $B(e) \subseteq \widetilde{\mathcal{T}}$, is the closed star neighborhood of the relative interior of a lifting $\widetilde{e} \subseteq \widetilde{\mathcal{T}}$ of e,*

$$B(e) = \overline{\mathrm{Str}}(\mathrm{rInt}(\widetilde{e})).$$

The abstract neighborhood $B(e)$ is given a cell decomposition by restricting the cell decomposition of $\widetilde{\mathcal{T}}$. The edges in each tetrahedron of $B(e)$ opposite to $\widetilde{e}$ form a closed curve in $\partial B(e)$ which we call the equator of $B(e)$. The complement of the equator in $\partial B(e)$ is separated into two components called hemispheres. See Figure 5.

Figure 5. A picture of the abstract neighborhood, $B(e)$.
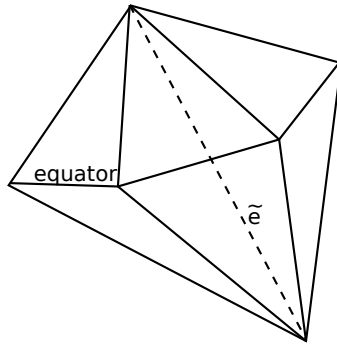
Remark 3.17 *Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Take an edge e in $\mathcal{T}$ and consider the abstract neighborhood $B(e)$ in the universal cover $\widetilde{\mathcal{T}}$ of $\mathcal{T}$. Notice that normal triangles in any tetrahedron of $B(e)$ which intersect $\widetilde{e}$ do not cross the equator. The only normal disks which intersect $\widetilde{e}$ and cross the equator are normal quads.*

Definition 3.18 (Edge pairings) *Consider a tetrahedron with its vertices labeled as in Figure 2. The edge pairings are the three sets of opposite edges, $P_1 = \{(0,1),(2,3)\}$, $P_2 = \{(0,2),(1,3)\}$, and $P_3 = \{(1,2),(0,3)\}$ where (a,b) is the edge connecting the vertex labeled a to the vertex labeled b.*

Definition 3.19 (Slope of a quad-type) *Consider a tetrahedron with its vertices labeled as in Figure 2. For each edge pairing P_i for $i \in \{1,2,3\}$ we assign a slope to each quad-type by defining the slope function $s_i : \{Q_1, Q_2, Q_3\} \to \{-, 0, +\}$ as follows:*

$$s_i(Q_j) = \begin{cases} + & \text{if } j = i \\ - & \text{if } j \equiv i+1 \ \text{mod } 3 \\ 0 & \text{if } j \equiv i+2 \ \text{mod } 3 \end{cases}$$

See Figure 6.

The output of the slope function $s_i(Q_j)$ represents the "slope" of a normal quad of quad-type Q_j at an edge from the edge pairing P_i when using the right-hand rule. Figure 2 can be used to check that the slope function is consistent for all edge pairings and all quad-types (you may need to tilt you head to check some of them).

Remark 3.20 *Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, orientable 3-manifold with torus boundary. Consider an edge $e \subseteq \mathcal{T}$ and an abstract neighborhood $B(e)$ in the universal cover $\widetilde{\mathcal{T}}$. Using the slope function for the edge pairing in which $\widetilde{e}$ is contained for each tetrahedron, we see that normal quads from quad-types with $+$ slope cross the equator from one*

Figure 6. When e is in the edge pairing P_i and q is of quad-type Q_j, $s_i(Q_j) = +$. This represents the slope of q when using the right-hand rule at e.
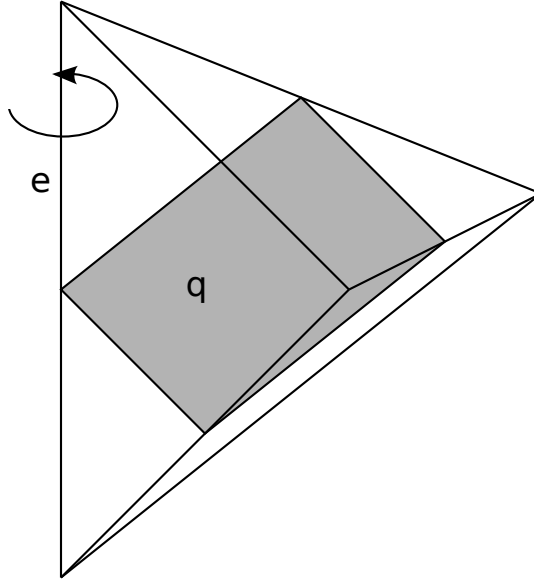
hemisphere to the other when using the right-hand rule at $\tilde{e}$. Call the former hemisphere the southern hemisphere and the latter the northern hemisphere. Then any normal quads from quad-types with $-$ slope cross the equator from the northern hemisphere to the southern hemisphere when using the right-hand rule at $\tilde{e}$.

Let D be a disk properly embedded in $B(e)$ which meets every tetrahedron of $B(e)$ in a normal disk and intersects $\tilde{e}$. Then ∂D is a closed curve in $\partial B(e)$. Let $x \in \partial D$. Then a path starting at x and following ∂D must eventually come back to x. Therefore, by Remark 3.17

and Remark 3.20, if the path intersects k normal quads from quad-types with $+$ slope, then it must intersect k normal quads from quad-types with $-$ slope, using the slope function for the edge pairing containing $\widetilde{e}$ in each tetrahedron. Since the lift of any spunnormal surface with respect to $(M, \mathcal{T})$ intersects the abstract neighborhood of some edge in such disks, we get a requirement on normal quads for them to glue up to form a spunnormal surface with respect to $\mathcal{T}$ which is reflected in the following equations.

Definition 3.21 (Q-matching equations) *Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary where $\{\Delta_1, ..., \Delta_n\}$ are the tetrahedra in $\mathcal{T}$. Let $\{e_1, ..., e_n\}$ be the edges of $\mathcal{T}$. For each edge e_k with $1 \le k \le n$, we get an equation:*

$$Q_k = \sum_{j=1}^{n} \left((a_{k,j} - c_{k,j}) q_{1,j} + (b_{k,j} - a_{k,j}) q_{2,j} + (c_{k,j} - b_{k,j}) q_{3,j} \right) = 0$$

where $a_{k,j}$ is the number of edges in the tetrahedron $\widetilde{\Delta}_j$ from the edge pairing P_1 identified to e_k, $b_{k,j}$ is the number of edges in the tetrahedron $\widetilde{\Delta}_j$ from the edge pairing P_2 identified to e_k, and $c_{k,j}$ is the number of edges in the tetrahedron $\widetilde{\Delta}_j$ from the edge pairing P_3 identified to e_k. These n equations are called the Q-matching equations.

Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Every spunnormal surface with respect to $\mathcal{T}$ has a Q-coordinate vector which is an integral admissible solution to the Q-matching equations.

Definition 3.22 (The admissible solutions to the Q-matching equations, $Q(\mathcal{T})$) *Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. The admissible solutions to the Q-matching equations form a polyhedral complex $Q(\mathcal{T}) \subseteq \mathbb{R}_{\geq 0}^{3n}$.*

Remark 3.23 *Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. For any vector $\mathbf{q} \in Q(\mathcal{T})$, the entire ray $\mathbb{R}_{\geq 0} \cdot \mathbf{q}$ is contained in $Q(\mathcal{T})$. It is sometimes convenient to consider the polyhedral complex $\mathcal{P}Q(\mathcal{T}) = Q(\mathcal{T}) \cap \{(x_1, ..., x_{3n}) \mid x_1 + \cdots + x_{3n} = 1\}$. This is a polyhedral complex consisting of rational, compact polytopes. A polytope in $\mathbb{R}^m$ is rational if the coordinates of every vertex are rational.*

Definition 3.24 (Vertex surface) *Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Let $\mathbf{q} \in \mathbb{R}_{+}^{3n}$ be a vertex of $\mathcal{P}Q(\mathcal{T})$. Then the spunnormal surface with the primitive representative of $\mathbb{R}_{+} \cdot \mathbf{q}$ as its Q-coordinate vector is a vertex surface.*

Remark 3.25 *Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Consider the Q-matching equations as linear polynomials in $\mathbb{R}[q_{1,1}, q_{2,1}, q_{3,1}, q_{2,1}, ..., q_{3,n}]$. Now choose n coordinates of the form $\{q_{i_1,1}, q_{i_2,2}, ..., q_{i_n,n}\}$ where each $i_j \in \{1, 2, 3\}$ for all $1 \leq j \leq n$. Then $Q(\mathcal{T})$ restricted to these n coordinates forms a convex polyhedral complex since any solution in the restricted coordinates is automatically admissible, and so, the solution set is only defined by linear equations.*

3.6   Geometric Sum

Definition 3.26 (Compatible spunnormal surfaces) *Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Two spunnormal surfaces are compatible if they do not have normal quads of different quad-types in any tetrahedron of $\mathcal{T}$.*

Definition 3.27 (Geometric sum) *Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Let S and S' be two compatible spunnormal surfaces with Q-coordinate vectors $\mathbf{q}$ and $\mathbf{q}' \in Q(\mathcal{T})$. The geometric sum of S and S' is the surface $S \uplus S'$ whose Q-coordinate vector is $\mathbf{q} + \mathbf{q}' \in Q(\mathcal{T})$.*

This geometric sum operation is well-defined. The geometric sum operation adds a great deal of structure to the theory of spunnormal surfaces. However, we will just need what has been stated for the main result in this thesis. For a thorough presentation see S. Tillmann's paper (16).

CHAPTER 4

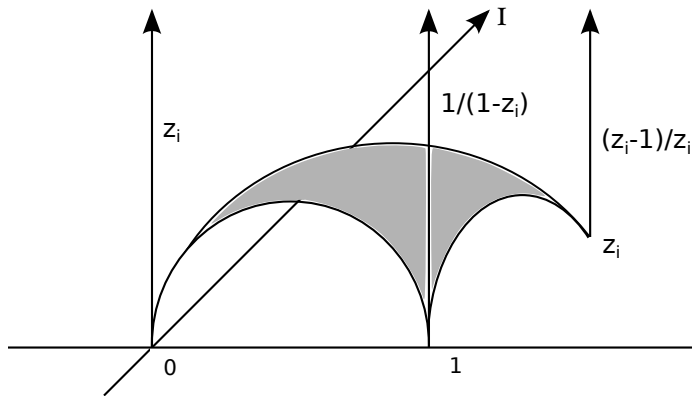# THE DEFORMATION VARIETY

4.1   <u>Ideal Tetrahedra</u>

Definition 4.1 (Ideal tetrahedron in $\widehat{\mathbb{H}}^3$) *Let $\widehat{\mathbb{H}}^3 = \mathbb{H}^3 \cup S^2_\infty$ where S^2_∞ is the sphere at infinity. An ideal tetrahedron in $\widehat{\mathbb{H}}^3$ is a tetrahedron contained in $\widehat{\mathbb{H}}^3$ with vertices on S^2_∞.*

Let $\Delta \subseteq \widehat{\mathbb{H}}^3$ be an ideal tetrahedron with the SnapPy orientation (see Figure 1) in the upper half-space model of $\mathbb{H}^3$ with S^2_∞ identified to $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Apply an orientation preserving isometry to $\mathbb{H}^3$ such that the vertex labeled 0 in the SnapPy orientation is sent to $0 \in \widehat{\mathbb{C}}$, the vertex labeled 1 in the SnapPy orientation is sent to $\infty \in \widehat{\mathbb{C}}$, and the vertex labeled 3 in the SnapPy orientation is sent to $1 \in \widehat{\mathbb{C}}$. The vertex labeled 2 in the SnapPy orientation must then be mapped to some complex number $z \in \mathbb{C} \setminus \{0, 1\}$.

Definition 4.2 (Shape parameters) *Let Δ be an ideal tetrahedron in $\widehat{\mathbb{H}}^3$ with the above identification of vertices. Shape parameters are assigned to the edges of Δ as follows. Assign the shape parameter z to the edge $(0, \infty)$ of Δ, assign the shape parameter $1/(1-z)$ to the edge $(1, \infty)$, and assign the shape parameter $(z-1)/z$ to the edge (z, ∞). Opposite edges in Δ are assigned the same shape parameter. In terms of the edge pairings of Definition 3.18, the edges of P_1 are assigned the shape parameter z, the edges of P_2 are assigned the shape parameter $1/(1-z)$, and the edges of P_3 are assigned the shape parameter $(z-1)/z$. See Figure 7.*

These shape parameters are the cross-ratios for the dihedral angles of each edge. Opposite edges of Δ are assigned the same shape parameter since the dihedral angles of opposite edges of an ideal tetrahedron in $\widehat{\mathbb{H}}^3$ are equal. Figure 7 shows an ideal tetrahedron in $\widehat{\mathbb{H}}^3$ with the vertices labeled and shape parameters given to one edge from each edge pair.

Figure 7. An ideal tetrahedron in $\widehat{\mathbb{H}}^3$ with labeling.

4.2 Gluing Equations and the Deformation Variety

Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary where $\{\Delta_1, ..., \Delta_n\}$ are the tetrahedra in $\mathcal{T}$ with the SnapPy orientation, see Figure 1. Each

tetrahedron Δ_i for $1 \leq i \leq n$ can be identified with an ideal tetrahedron in $\widehat{\mathbb{H}}^3$ by an orientation preserving isometry as above, and so be assigned shape parameters for its edges:

$$z_i, \frac{1}{1-z_i}, \frac{z_i-1}{z_i}.$$

See Figure 7.

Let $\{e_1, ..., e_n\}$ be the edges of $\mathcal{T}$. Take an edge e_k for $1 \leq k \leq n$. Consider the abstract neighborhood of e, $B(e)$, with the lifted hyperbolic structure. The tetrahedra of $B(e)$ must glue up around $\widetilde{e}_k$. Therefore, the product of the shape parameters assigned to $\widetilde{e}_k$ from each tetrahedron in $B(e)$ must be 1. These equations are made explicit in the next definition.

Definition 4.3 (Gluing equations) *Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary where $\{\Delta_1, ..., \Delta_n\}$ are the tetrahedra in $\mathcal{T}$. Let $\{e_1, ..., e_n\}$ be the edges of $\mathcal{T}$. For each edge e_k with $1 \leq k \leq n$ we get an equation:*

$$\overline{R}_k = \prod_{i=1}^{n} z_i^{a_{k,j}} \left(\frac{1}{1-z_i}\right)^{b_{k,j}} \left(\frac{z_i-1}{z_i}\right)^{c_{k,j}} = 1$$

where $a_{k,j}$ is the number of edges in the tetrahedron $\widetilde{\Delta}_j$ from the edge pairing P_1 identified to e_k, $b_{k,j}$ is the number of edges in the tetrahedron $\widetilde{\Delta}_j$ from the edge pairing P_2 identified to e_k, and $c_{k,j}$ is the number of edges in the tetrahedron $\widetilde{\Delta}_j$ from the edge pairing P_3 identified to e_k. Because the product $\overline{R}_1 \cdots \overline{R}_n = 1$ for all $(z_1, ..., z_n) \in (\mathbb{C} \setminus \{0,1\})^n$, we omit the nth equation and consider $\{\overline{R}_1, ..., \overline{R}_{n-1}\}$ the gluing equations for $(M, \mathcal{T})$.

Notice that the gluing equations could be defined with any edge pair receiving the shape parameter z_i. However, the result proven in this thesis is better presented when this convention is set.

Definition 4.4 (Augmented Gluing Equations) *Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary where $\{\Delta_1, ..., \Delta_n\}$ are the tetrahedra in $\mathcal{T}$. Let $\{e_1, ..., e_n\}$ be the edges of $\mathcal{T}$. The augmented gluing equations are the gluing equations with the substitution $w_i = 1 - z_i$. Again we get one equation for each edge e_k where $1 \leq k \leq n - 1$:*

$$R_k = \prod_{i=1}^{n} (-1)^{c_{k,j}} z_i^{a_{k,j} - c_{k,j}} w_i^{c_{k,j} - b_{k,j}} = 1$$

where $a_{k,j}$ is the number of edges in the tetrahedron $\widetilde{\Delta}_j$ from the edge pairing P_1 identified to e_k, $b_{k,j}$ is the number of edges in the tetrahedron $\widetilde{\Delta}_j$ from the edge pairing P_2 identified to e_k, and $c_{k,j}$ is the number of edges in the tetrahedron $\widetilde{\Delta}_j$ from the edge pairing P_3 identified to e_k.

The following example gives the augmented gluing equations for the 5_2-knot complement. This will be a running example throughout the remainder of the thesis.

Example 4.5 (5_2-**knot complement**) *The ideal triangulation from SnapPy for the 5_2-knot complement uses three tetrahedra. The augmented gluing equations for the 5_2-knot complement are the following:*

$$R_1 = w_1 \cdot z_2^{-1} \cdot z_3 \cdot w_3^{-2} = 1,$$

$$R_2 = (-1) \cdot z_1^{-1} \cdot z_2 \cdot w_2^{-1} \cdot z_3^{-2} \cdot w_3^2 = 1.$$

$$(4.1)$$

Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary where $\{\Delta_1, ..., \Delta_n\}$ are the tetrahedra of $\mathcal{T}$. Notice that the augmented gluing equations can only have a zero in the denominator if z_i or w_i is zero for some $i \in \{1, ..., n\}$. However, any ideal tetrahedron cannot have a shape parameter with $z_i = 1$ or $z_i = 0$. Therefore, the augmented gluing equations can be expressed as polynomial equations. For $1 \leq k \leq n - 1$, let $R'_k = 0$ be the polynomial equations coming from the equation $R_k = 1$.

Example 4.6 (5_2-**knot complement**) *The polynomial augmented gluing equations for the 5_2-knot complement are:*

$$R'_1 = w_1 z_3 - z_2 w_3^2 = 0,$$

$$R'_2 = z_2 w_3^2 + z_1 w_2 z_3^2 = 0.$$

Definition 4.7 (Deformation Variety) *Let $(M, \mathcal{T})$ be an ideal triangulation of a compact,*

oriented 3-manifold with torus boundary where $\{\Delta_1, ..., \Delta_n\}$ are the tetrahedra in $\mathcal{T}$. Let

$$\mathcal{AG} = \{R_1', ...R_{n-1}', z_1 + w_1 - 1, z_2 + w_2 - 1, ..., z_n + w_n - 1\} \subseteq \mathbb{C}[z_1, ..., z_n, w_1, ..., w_n]$$

be the set of polynomial augmented gluing equations together with the linear equations $w_i = 1 - z_i$

for $1 \leq i \leq n$. Then the deformation variety, $\mathcal{D}(\mathcal{T})$ is given by

$$\mathcal{D}(\mathcal{T}) = V(\langle \mathcal{AG} \rangle) \subseteq (\mathbb{C} \setminus \{0, 1\})^{2n}$$

where $V(\langle \mathcal{AG} \rangle)$ is the zero set of the ideal generated by $\mathcal{AG}$.

Notice that the deformation variety from an ideal triangulation of a compact, oriented 3-manifold with torus boundary is not necessarily irreducible.

Remark 4.8 *Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. For the main result of this thesis we will need to assume every irreducible component of $\mathcal{D}(\mathcal{T})$ is 1-dimensional. This is true in examples, and is thought to be true in general for such an $(M, \mathcal{T})$.*

4.3   The Character Variety

Definition 4.9 (Character variety) *Let M be a compact, oriented 3-manifold with torus boundary. The character variety of M, $X(M)$, is the $SL_2(\mathbb{C})$ character variety of $\pi_1(M)$.*

The following is shown in N. Dunfield and S. Garoufalidis's paper as Lemma 3.5 (11).

Proposition 4.10 *Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, orientable 3-manifold with torus boundary. There is a regular map from $\mathcal{D}(\mathcal{T})$ to $\overline{X}(M)$ where $\overline{X}(M)$ is the $\mathrm{PSL}_2(\mathbb{C})$ character variety of $\pi_1(M)$.*

This regular map is usually called the pseudo-developing map and is not surjective.

Definition 4.11 (Ideal points of a variety) *Let V be an affine variety and $\widehat{V}$ be the projective completion of V. Then $\widehat{V} \setminus V$ is the set of ideal points of V.*

Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, orientable 3-manifold with boundary. M. Culler and P. Shalen (14) have shown that ideal points of the character variety $X(M)$ correspond to essential surfaces in M. Also, ideal points of $\overline{X}(M)$ can be associated to ideal points of $X(M)$. However, not every ideal point of $\mathcal{D}(\mathcal{T})$ maps to an ideal point of $X(M)$ using this association and the map from Proposition 4.11.

4.4 Associated Spunnormal Surfaces

Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary where $\{\Delta_1, ..., \Delta_n\}$ are the tetrahedra in $\mathcal{T}$. Let D be an irreducible curve of $\mathcal{D}(\mathcal{T})$. For an ideal point ξ of D we can associate a spunnormal surface $S(\xi)$ with Q-coordinate vector $(q_{1,1}, ..., q_{3,n})$ as follows. For each tetrahedron Δ_i for $1 \leq i \leq n$ define the entries $(q_{1,i}, q_{2,i}, q_{3,i})$ as follows. Let z_i be the shape parameter of the edge $(0, \infty)$ in Δ_i. The projective completion of D, $\widetilde{D}$, is a Riemannian surface, and z_i is a function of this surface. Let v_ξ be the discrete valuation

defined by ξ on the function field of $\widetilde{D}$. Then the coordinates $(q_{1,i}, q_{2,i}, q_{3,i})$ are assigned using the following:

$$(q_{1,i}, q_{2,i}, q_{3,i}) = \begin{cases} (v_\xi(z_i), 0, 0) & \text{if } v_\xi(z_i) > 0 \\[2mm] (0, -v_\xi(z_i), 0) & \text{if } v_\xi(z_i) < 0 \\[2mm] (0, 0, v_\xi(1 - z_i)) & \text{if } v_\xi(z_i) = 0 \text{ and } z_i(\xi) \notin \mathbb{C} \setminus \{0, 1\} \\[2mm] (0, 0, 0) & \text{if } z_i(\xi) \in \mathbb{C} \setminus \{0, 1\}. \end{cases}$$

The fact that the shape parameters around an edge in $\mathcal{T}$ must multiply to 1, means that the valuation of the shape parameters at ξ must sum to zero around the edge. This gives exactly that the Q-coordinate vector for $S(\xi)$ satisfies the Q-matching equations, and that $S(\xi)$ glues up on the faces to give a spunnormal surface. We say $S(\xi)$ is the *spunnormal surface associated to the ideal point ξ.*

Definition 4.12 (Reduction of a properly embedded surface) *Let M be a 3-manifold with boundary. A properly embedded surface S in M is said to reduce to S' if there is a sequence of compressions along compressing disks, boundary compressions, elimination of 2-spheres, and elimination of boundary parallel components which turns S into S'. We then say that S' is a reduction of S.*

This next result can be found in N. Dunfield and S. Garoufalidis's paper as Theorem 3.8 (11) and S. Tillmann's paper as Proposition 4.3 (22). This result uses the pseudo-developing map between the deformation variety and the $\text{PSL}_2(\mathbb{C})$ character variety of the fundamental group of the manifold.

Proposition 4.13 *Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Let ξ be an ideal point of an irreducible curve $D \subseteq \mathcal{D}(\mathcal{T})$. Suppose S is the spunnormal surface with respect to $\mathcal{T}$ which is associated to ξ. Also assume the associated properly embedded surface in M, $\widehat{S}$, is two-sided and has nonempty boundary with boundary slope α. Then any reduction of $\widehat{S}$ has nonempty boundary with boundary slope α. In particular, $\widehat{S}$ can be reduced to a nonempty essential surface in M with boundary slope α.*

Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, orientable 3-manifold with torus boundary. Proposition 4.14 tells us the boundary slope of a spunnormal surface associated to an ideal point of $\mathcal{D}(\mathcal{T})$ is a boundary slope in $\text{bs}(M)$.

CHAPTER 5

TROPICAL GEOMETRY

This chapter follows the presentation of concepts in tropical geometry in the paper by T. Bogart, A. Jensen, D. Speyer, B. Sturmfels and R. Thomas (23). Please see their paper for details.

5.1   Tropical Varieties

Definition 5.1 (w-weight, and initial form) *Let $f \in \mathbb{C}[x_1, ..., x_n]$ be a polynomial.*

1. *Let $\mathbf{w} \in \mathbb{R}^n$ be a vector. Then the $\mathbf{w}$-weight of a term $cx_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ in f, where $a_i \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq n$, is defined as $\mathbf{w} \cdot (a_1, ..., a_n) \in \mathbb{R}$.*

2. *Let $\mathbf{w} \in \mathbb{R}^n$ be a vector. The initial form of f with respect to $\mathbf{w}$ denoted $\mathrm{in}_{\mathbf{w}}(f)$ is the sum of all terms in f of lowest $\mathbf{w}$-weight.*

Example 5.2 (5_2-knot complement) *Take the first polynomial augmented gluing equation from the 5_2-knot complement,*

$$R'_1 = w_1 z_3 - z_2 w_3^2 = 0.$$

Consider $R'_1 \in \mathbb{C}[z_1, w_1, z_2, w_2, z_3, w_3]$. Let $\mathbf{w}_1 = (0, 1, 1, 2, 2, 1)$. Then the $\mathbf{w}_1$-weight of the term $w_1 z_3$ is 3. The $\mathbf{w}_1$-weight of the term $-z_2 w_3^2$ is also 3. Therefore, the initial form of R'_1 with respect to $\mathbf{w}_1$ is

$$\mathrm{in}_{\mathbf{w}_1}(R'_1) = w_1 z_3 - z_2 w_3^2.$$

Now let $\mathbf{w}_2 = (-2, -2, -1, -1, 1, 0)$. Then the $\mathbf{w}_2$-weight of the term $w_1 z_3$ is -1. The $\mathbf{w}_2$-weight of the term $-z_2 w_3^2$ is also -1. So

$$\mathrm{in}_{\mathbf{w}_2}(R_1') = w_1 z_3 - z_2 w_3^2.$$

Finally, let $\mathbf{w}_3 = (4, 1, -1, 0, -2, 3)$. Then the $\mathbf{w}_3$-weight of the term $w_1 z_3$ is -1. The $\mathbf{w}_3$-weight of the term $-z_2 w_3^2$ is 5. So

$$\mathrm{in}_{\mathbf{w}_3}(R_1') = w_1 z_3.$$

Definition 5.3 (Tropical hypersurface of a polynomial) *Let $f \in \mathbb{C}[x_1, ..., x_n]$. The tropical hypersurface of f is the set*

$$\mathrm{Trop}(f) = \{\mathbf{w} \in \mathbb{R}^n \mid \mathrm{in}_{\mathbf{w}}(f) \text{ is not a monomial}\}.$$

A tropical hypersurface can be thought of as the solutions to a tropical linear equation, where addition is defined by $a \oplus b = \min(\{a, b\})$, and multiplication is defined by $a \odot b = a + b$.

Example 5.4 (5_2-knot complement) *Take the first polynomial augmented gluing equation from the 5_2-knot complement,*

$$R_1' = w_1 z_3 - z_2 w_3^2 = 0.$$

Consider $R_1' \in \mathbb{C}[z_1, w_1, z_2, w_2, z_3, w_3]$. *Let* $\mathbf{w}_1 = (0, 1, 1, 0, 2, 1)$, $\mathbf{w}_2 = (1, 0, 1, 0, -1, -1)$, *and*

$\mathbf{w}_3 = (4, 1, -1, 0, -2, 3)$ *as in Example 5.2. Then, by Example 5.2,* $\mathbf{w}_1 \in \mathrm{Trop}(R_1')$, *and*

$\mathbf{w}_2 \in \mathrm{Trop}(R_1')$, *however* $\mathbf{w}_3 \notin \mathrm{Trop}(R_1')$.

Definition 5.5 (Newton polytope of a polynomial) *Let* $f \in \mathbb{C}[x_1, ..., x_n]$ *where*

$$f = \sum_{i=1}^m c_i x_1^{a_{1,i}} x_2^{a_{2,i}} \cdots x_n^{a_{n,i}}.$$

Let $V = \{(a_{1,i}, a_{2,i}, ..., a_{n,i}) \in \mathbb{R}^n \mid 1 \leq i \leq m\}$ *be the set of exponent vectors for each term of*

f. *Then the Newton polytope of* f, $N(f)$, *is the polyhedron formed by the convex hall of* V *in*

$\mathbb{R}^n$.

Remark 5.6 *Let* $f \in \mathbb{C}[x_1, ..., x_n]$. *Then the tropical hypersurface of* f, $\mathrm{Trop}(f)$, *is the inner*

pointing normal vectors of each nonvertex subcell of the Newton polytope $N(f)$. *A vector* $\mathbf{w} \in$

$\mathrm{Trop}(f)$ *must be normal to a nonvertex subcell of* $N(f)$ *since this guarantees that two or more*

terms of f *have the same* $\mathbf{w}$-*weight. A vector* $\mathbf{w} \in \mathrm{Trop}(f)$ *which is normal to a nonvertex*

subcell of $N(f)$ *must also be inner pointing since this guarantees that the terms corresponding*

to the vertices of the subcell to which $\mathbf{w}$ *is normal are of lowest* $\mathbf{w}$-*weight.*

Remark 5.7 *Let* $f \in \mathbb{C}[x_1, ..., x_n]$. *The tropical hypersurface* $\mathrm{Trop}(f)$ *is a polyhedral complex*

such that, if $\mathbf{w} \in \mathrm{Trop}(f)$ *then* $\mathbb{R}_+ \cdot \mathbf{w} \subseteq \mathrm{Trop}(f)$. *Therefore, it is often convenient to consider*

the projection of $\mathrm{Trop}(f)$ *onto the unit sphere* $S^{n-1} \subseteq \mathbb{R}^n$.

Example 5.8 (5_2-**knot complement**) *Consider the first polynomial augmented gluing equation for the 5_2-knot complement,*

$$R_1' = w_1 z_3 - z_2 w_3^2 = 0.$$

In order to get a picture of a Newton polytope and tropical hypersurface we make the substitution $1 - z_i = w_i$. We get the polynomial gluing equation,

$$\overline{R}_1' = (1 - z_1)z_3 - z_2(1 - z_3)^2 = 0.$$

$$\Rightarrow \overline{R}_1' = z_3 - z_1 z_3 - z_2 + 2z_2 z_3 - z_2 z_3^2 = 0.$$

Then the set V from Definition 5.5 is given by

$$V = \{(0, 0, 1), (1, 0, 1), (0, 1, 0), (0, 1, 1), (0, 1, 2)\}.$$

The Newton polytope $N(\overline{R}_1') \subseteq \mathbb{R}^3$ is pictured in Figure 8 by the bold lines. Then we can compute the inner pointing normals for each edge and face of $N(\overline{R}_1')$. This gives us $\mathrm{Trop}(\overline{R}_1')$. The tropical hypersurface $\mathrm{Trop}(\overline{R}_1')$ projected onto the unit sphere is pictured in Figure 9 by the bold arcs. The vertices are labeled by the primitive integral vectors of the rays which they represents.

By Remark 5.7, tropical hypersurfaces fall into a special category of polyhedral complexes called *fans* which we define now.

Figure 8. A picture of $N(\overline{R}'_1)$ represented by the bold lines.
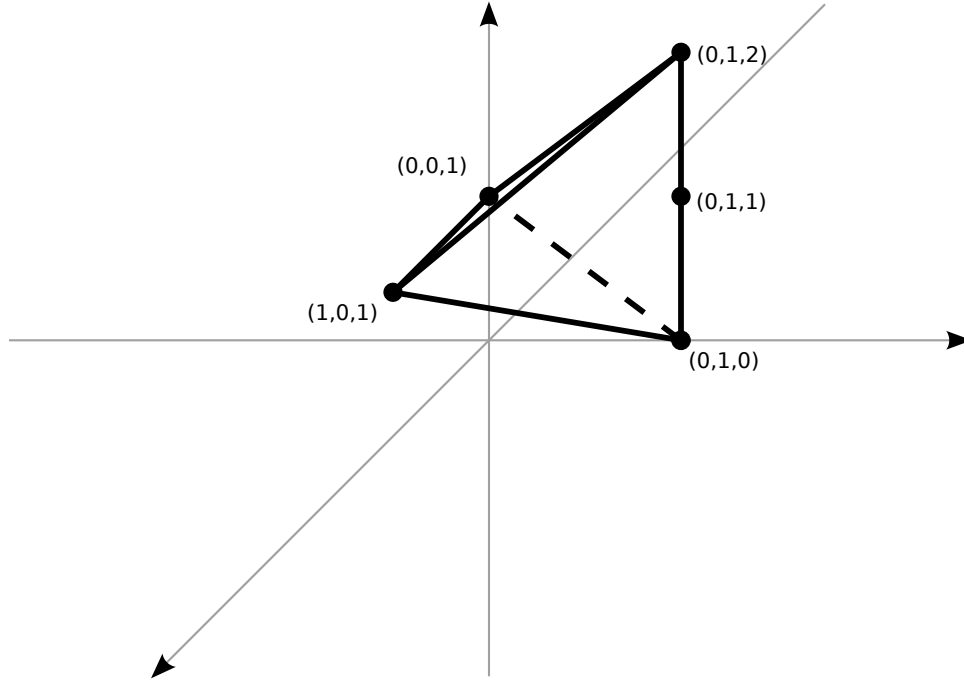
Definition 5.9 (Fan) *A fan is a polyhedral complex where every face is a cone emanating from a common vertex. We use the terms either face or cone to refer to faces of a fan. The term ray may be used to refer to a 1-dimensional cone.*

Remark 5.10 *Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary. Then, by Remark 3.23, the set of admissible solutions to the Q-matching equations, $\mathcal{Q}(\mathcal{T})$, is a fan emanating from the origin in $\mathbb{R}^{3n}$ with the origin represented by a union of infinitely many vertex linking tori.*

Figure 9. A picture of $\mathrm{Trop}(\overline{R}'_1)$ projected onto the unit sphere represented by the bold arcs with vertices labeled by primitive integral representatives of the ray.
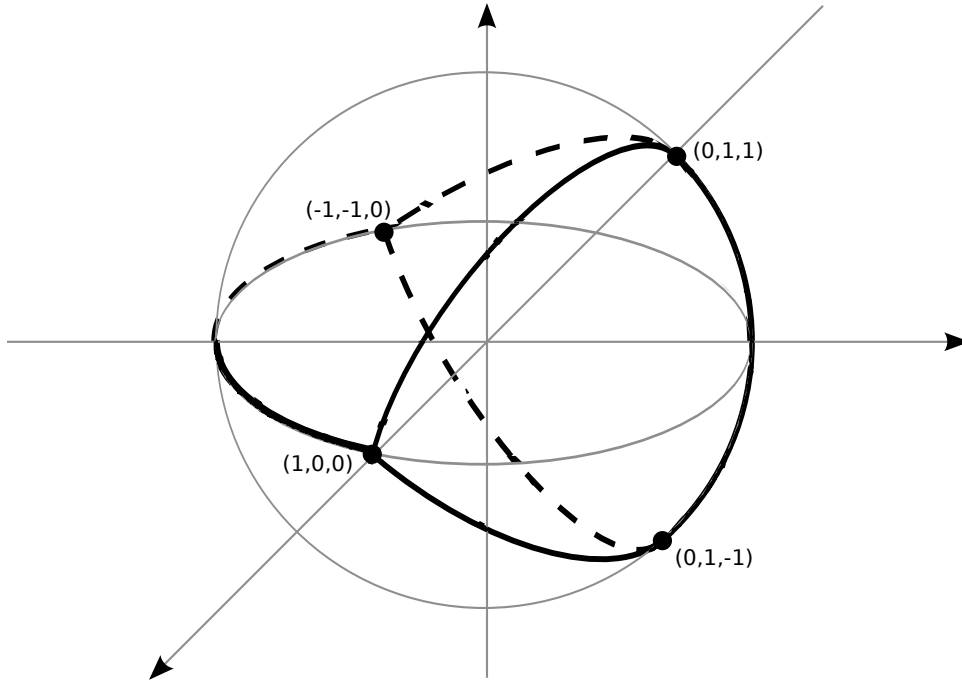
Definition 5.11 (Tropical prevariety) *Let $\{f_1, ..., f_m\} \subseteq \mathbb{C}[x_1, ..., x_n]$ be a finite collection of polynomials. Then the tropical prevariety of $\{f_1, ..., f_m\}$, $\mathrm{PreTrop}(\{f_1, ..., f_m\})$, is the inter-section of the tropical hypersurfaces of each f_i for $1 \leq i \leq m$. That is,*

$$\mathrm{PreTrop}(\{f_1, ..., f_m\}) = \mathrm{Trop}(f_1) \cap \cdots \cap \mathrm{Trop}(f_m).$$

The finite intersection of fans emanating from the same point is again a fan emanating that point. Therefore, if $\{f_1, ..., f_m\} \subseteq \mathbb{C}[x_1, ..., x_n]$, then the tropical prevariety, $\mathrm{PreTrop}(\{f_1, ..., f_m\})$, is a fan.

Example 5.12 (5_2-**knot complement**) *Let*

$$\overline{R}_2' = z_2 - 2z_2z_3 + z_2z_3^2 + z_1z_3^2 - z_1z_2z_3^2 = 0$$

be second polynomial gluing equation for the 5_2-knot complement with the terms expanded. The Newton polytope for $\overline{R}_2'$ is given in Figure 10 by the bold lines. The tropical hypersurface for $\overline{R}_2'$ projected onto the unit sphere is then given in Figure 11 by the bold arcs. If we consider the intersection of $\mathrm{Trop}(\overline{R}_1')$ with $\mathrm{Trop}(\overline{R}_2')$, we see that the tropical prevariety, $\mathrm{PreTrop}(\{\overline{R}_1', \overline{R}_2'\})$ projected onto the unit sphere, is given by the bold arc and four points in Figure 12.

Definition 5.13 (Tropical Variety) *Let $I \subseteq \mathbb{C}[x_1, ..., x_n]$ be an ideal. The tropical variety of I denoted $\mathrm{Trop}(I)$ is the intersection of the tropical hypersurfaces $\mathrm{Trop}(f)$ as f runs over all polynomials in I.*

Notice that for $f \in \mathbb{C}[x_1, .., x_n]$, the tropical hypersurface $\mathrm{Trop}(f)$ is equal to the tropical variety $\mathrm{Trop}(\langle f \rangle)$.

Example 5.14 (5_2-knot complement) *A. Jensen has written a program that utilizes the algorithm created by T. Bogart, A. Jensen, D. Speyer, B. Sturmfels and R. Thomas (23) to*

Figure 10. A picture of $N(\overline{R}'_2)$ represented by the bold lines.
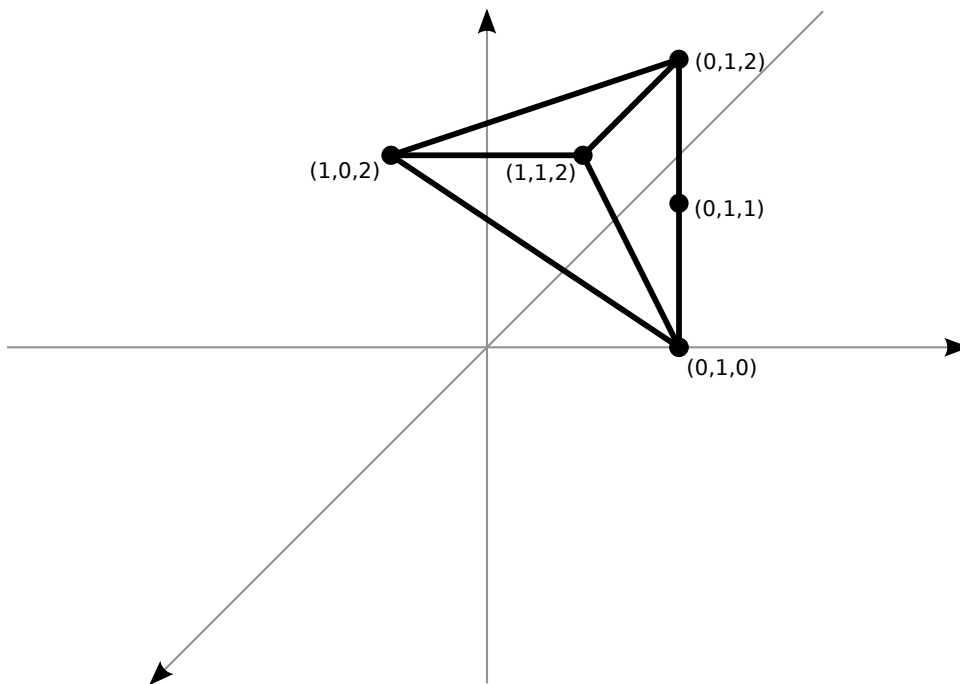
compute tropical varieties called gfan (24). Using gfan, we find that the vectors $(1, 1, -1)$, $(1, -1, 0)$, $(0, 2, 1)$, $(-1, 0, -1)$ and $(-2, -1, 1)$ are in $\mathrm{Trop}(\langle \overline{R}'_1, \overline{R}'_2 \rangle)$. See Figure 13.

5.2   Tropical Bases

Theorem 5.15 *For every ideal $I \subseteq \mathbb{C}[x_1, ..., x_n]$, the tropical variety of I is the tropical preva-riety of some finite set $\{f_1, ..., f_m\} \subseteq \mathbb{C}[x_1, ..., x_n]$.*

    T. Bogart, A. Jensen, D. Speyer, B. Sturmfels and R. Thomas give an algorithm for com-puting such a set for any ideal (23).
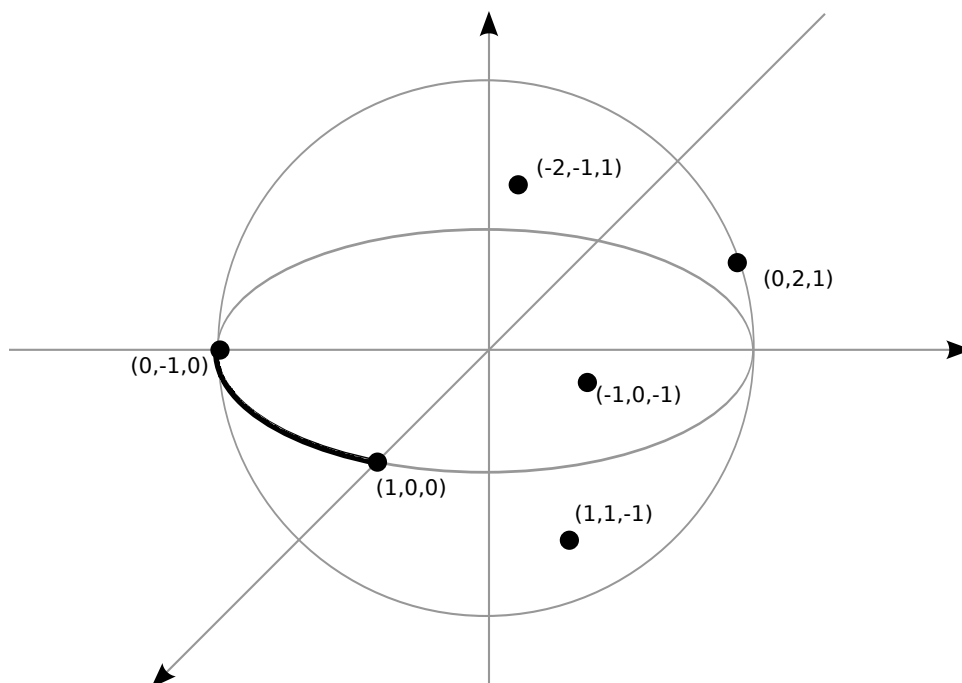
Figure 11. A picture of $\mathrm{Trop}(\overline{R}_2')$ projected onto the unit sphere represented by the bold arcs with vertices labeled by primitive integral representatives of the ray.
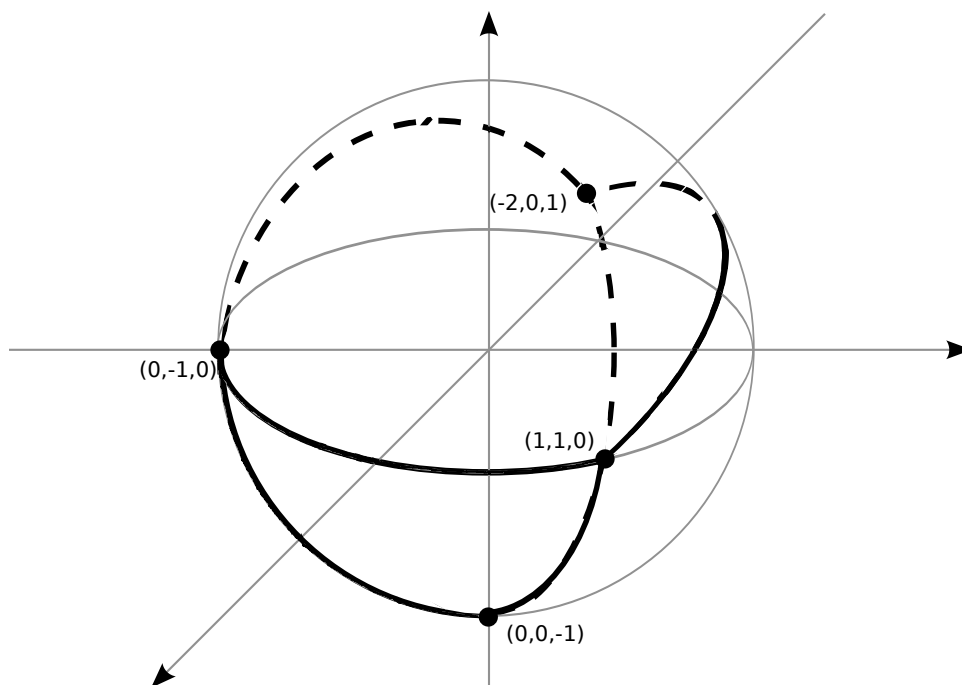
Definition 5.16 (Tropical basis) *Let $I \subseteq \mathbb{C}[x_1, ..., x_n]$. If $\{f_1, ..., f_m\} \subseteq \mathbb{C}[x_1, ..., x_n]$ is a set such that $\mathrm{Trop}(I) = \mathrm{PreTrop}(\{f_1, ..., f_m\})$ and $I = \langle f_1, ..., f_m \rangle$, then $\{f_1, ..., f_m\}$ is a tropical basis for I.*

Remark 5.17 *By Theorem 5.15, every ideal has a tropical basis. Every tropical variety is then the intersection of finitely many fans and is therefore a fan itself.*

.

Figure 12. A picture of $\text{PreTrop}(\{\overline{R}'_1, \overline{R}'_2\})$ projected onto the unit sphere represented by the bold arcs and dots with vertices labeled by primitive integral representatives of the ray.

Example 5.18 (5_2-knot complement) *The tropical basis found by gfan for $\langle \overline{R}'_1, \overline{R}'_2 \rangle$ is*

$$\{-z_2 z_3^2 - z_1 z_3 + 2 z_2 z_3 - z_2 + z_3,$$

$$-z_1 z_2 z_3^2 + z_1 z_3^2 + z_2 z_3^2 - 2 z_2 z_3 + z_2,$$

$$z_1 z_2 z_3^2 - z_1 z_3^2 + z_1 z_3 - z_3\}$$

Figure 13. A picture of $\text{Trop}(\langle \overline{R}'_1, \overline{R}'_2 \rangle)$ projected onto the unit sphere represented by dots labeled by primitive integral representatives of the ray.
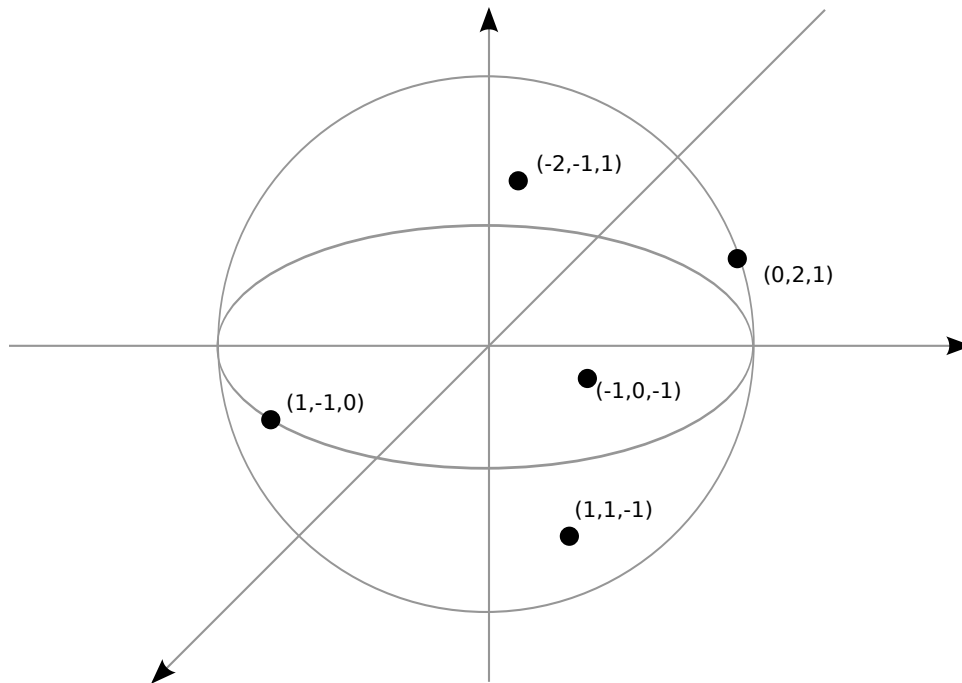
5.3   The Fundamental Theorem of Tropical Geometry

Definition 5.19 (Puiseux series) *A Puiseux series is a series of the form*

$$p(t) = c_1 t^{a_1} + c_2 t^{a_2} + \ldots$$

where $c_i \in \mathbb{C}$, $c_1 \neq 0$ unless p is identically 0, $a_i \in \mathbb{Q}$, $a_i \leq a_{i+1}$, and the a_i have a common denominator. All conditions are for all $i \geq 1$. The set of Puiseux series creates an algebraically closed field $\mathbb{C}\{\{t\}\}$. This field comes with a valuation

$$v : \mathbb{C}\{\{t\}\}^* \to \mathbb{Q}$$

given by

$$c_1 t^{a_1} + c_2 t^{a_2} + ... \mapsto a_1.$$

Theorem 5.20 (Fundamental Theorem of Tropical Geometry) *Let $I \subseteq \mathbb{C}[x_1, ..., x_n]$ be an ideal. Then*

$$\mathrm{Trop}(I) = \overline{\{(v(p_1), v(p_2), ..., v(p_n)) \in \mathbb{Q}^n \mid (p_1, ..., p_n) \in P(I)\}}$$

where P(I) is the set of all n-tuples of Puiseux series, $(p_1, ..., p_n) \in (\mathbb{C}\{\{t\}\}^*)^n$ such that $(p_1(t), p_2(t), ..., p_n(t)) \in V(I)$ for every t where the $p_i(t)$ are defined.

5.4 Proper Intersections

Definition 5.21 (Dimension of a fan) *Let Σ be a fan. The dimension of Σ, $\dim(\Sigma)$, is the maximum of the dimensions of the cones in Σ.*

Definition 5.22 (Proper intersection) *Let I and I' be two ideals in $\mathbb{C}[x_1, ..., x_n]$. We say $\mathrm{Trop}(I)$ and $\mathrm{Trop}(I')$ intersect properly at a point $\mathbf{w} \in \mathbb{R}^n$ if $\mathrm{Trop}(I) \cap \mathrm{Trop}(I')$ has codimension equal to $\mathrm{codim}(V(I)) + \mathrm{codim}(V(I'))$ in a neighborhood of $\mathbf{w}$.*

The following is shown by B. Osserman and S. Payne as Theorem 1.1 (15).

Theorem 5.23 (Osserman and Payne) *Let I and I' be two ideals in $\mathbb{C}[x_1, ..., x_n]$. Suppose* $\mathrm{Trop}(I)$ *and* $\mathrm{Trop}(I')$ *intersect properly at* $\mathbf{w}$. *Then* $\mathbf{w}$ *is in* $\mathrm{Trop}(I + I')$.

Definition 5.24 (Isolated ray) *Let Σ be a fan and R be a 1-dimensional cone of Σ. Then R is an isolated ray if R is not a subcone of any higher dimensional cones.*

The following is a consequence of Theorem 5.23.

Proposition 5.25 *Let $\{P_1, ..., P_{m-1}\} \subseteq \mathbb{C}[x_1, ..., x_m]$ where every irreducible component of $V(\langle P_1, ..., P_{m-1}\rangle) \subseteq \mathbb{C}^m$ is 1-dimensional. Then if $\mathbf{w} \in \mathbb{R}^m$ is contained in an isolated ray of $\mathrm{PreTrop}(\{P_1, ..., P_{m-1}\})$, then $\mathbf{w} \in \mathrm{Trop}(\langle P_1, ..., P_{m-1}\rangle)$.*

Proof: If $\mathbf{w}$ is the origin, then $\mathbf{w}$ is clearly in $\mathrm{Trop}(\langle P_1, ..., P_{m-1}\rangle)$. So assume $\mathbf{w}$ is not the origin. We will show for every $1 \leq j \leq m - 1$, the open star neighborhood of $\mathbf{w}$, $\mathrm{Str}_j(\mathbf{w}) \subseteq \mathrm{PreTrop}(\{P_i, ..., P_j\})$ is contained in $\mathrm{Trop}(\langle P_1, ..., P_j\rangle)$ by induction on j. The base case trivial. Now assume $\mathrm{Str}_{j-1}(\mathbf{w}) \subseteq \mathrm{PreTrop}(\{P_1, ..., P_{j-1}\})$ is also contained in $\mathrm{Trop}(\langle P_1, ..., P_{j-1}\rangle)$. Now,

$$\mathrm{Str}_j(\mathbf{w}) \subseteq \mathrm{Str}_{j-1}(\mathbf{w}) \subseteq \mathrm{Trop}(\langle P_1, ..., P_{j-1}\rangle)$$

and

$$\mathrm{Str}_j(\mathbf{w}) \subseteq \mathrm{Trop}(P_j).$$

Therefore, $\mathrm{Str}_j(\mathbf{w})$ is in the intersection of these two tropical varieties. Also, $\mathrm{Str}_j(\mathbf{w})$ has codimension $m - j$ since we must cut down the dimension of $\mathrm{Str}_{i-1}(\mathbf{w})$ by one each time

we intersect with $\mathrm{Trop}(P_i)$ for $1 < i \leq m - 1$ in order to end with $\mathrm{Str}_{m-1}(\mathbf{w})$ equal to an isolated ray. Therefore, every point of $\mathrm{Str}_j(\mathbf{w})$ is a proper intersection of $\mathrm{Trop}(\langle P_1, ..., P_{j-1}\rangle)$ and $\mathrm{Trop}(P_j)$ since we must cut down the dimension of $V(\langle P_1, ..., P_{i-1}\rangle)$ by one each time we intersect with $V(\langle P_i\rangle)$ for $1 < i \leq m - 1$ in order to end with every irreducible component of $V(\langle P_1, ..., P_{m-1}\rangle)$ being 1-dimendsional. Then $\mathrm{Str}_j(\mathbf{w}) \subseteq \mathrm{Trop}(\langle P_1, ..., P_{j-1}\rangle + \langle P_j\rangle)$ by the theorem of Osserman and Payne (15), Theorem 5.23 in this thesis. Finally $\mathrm{Trop}(\langle P_1, ..., P_{j-1}\rangle + \langle P_j\rangle) = \mathrm{Trop}(\langle P_1, ..., P_j\rangle)$. $\square$

Example 5.26 (5_2-**knot complement**) *Just by looking at the tropical prevariety* $\mathrm{PreTrop}(\{\overline{R}'_1, \overline{R}'_2\})$ *(see Figure 12), Proposition 5.25 shows that*

$$\{(-2, -1, 1), (0, 2, 1), (1, 1, -1), (-1, 0, -1)\} \subseteq \mathrm{Trop}(\langle \overline{R}'_1, \overline{R}'_2\rangle)$$

Therefore, by the Fundamental Theorem of Tropical Geometry (Theorem 5.20), these vectors correspond to ideal points of $\mathcal{D}(\mathcal{T})$. However, there are some difficulties with using the regular gluing equations and not the augmented gluing equations.

For example, Let ξ be the ideal point of $\mathcal{D}(\mathcal{T})$ corresponding to the the vector $(0, 2, 1)$, and v_ξ be the associated valuation. We know that $v_\xi(z_2) = 2$ and $v_\xi(z_3) = 1$. Therefore, the associated surface should have 2 normal quads of quad-type Q_1 in the tetrahedron Δ_2, and 1 normal quad of quad-type Q_1 in the tetrahedron Δ_3. However, the zero in the first coordinate only tells $v_\xi(z_1) = 0$. This could represent $z_1 \to 1$ as we approach the ideal point, and we should be

assigning some number of normal quads of type Q_3 to Δ_1; or $z_1 \to x \in \mathbb{C} \setminus \{0, 1\}$ in which case no quads should be assigned to Δ_1.

Even if a spunnormal surface can be assigned to an ideal point, then it still is unknown whether that spunnormal surface is actually essential. Proposition 4.14 does tell us that its boundary slope is the boundary slope of an essential surface. The main theorem of this thesis will give sufficient conditions for a spunnormal surface corresponding to an ideal point to be essential.

CHAPTER 6

MAIN RESULT

In this chapter we prove the main theorem of the thesis. However, before the main theorem, we prove a result which is in S. Tillmann's paper (22) as Proposition 3.1.

Proposition 6.1 *Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, orientable 3-manifold with torus boundary where $\{\Delta_1, ..., \Delta_n\}$ are the tetrahedra of $\mathcal{T}$. Let*

$$\mathcal{AG} = \{R_1', ..., R_{n-1}', z_1 + w_1 - 1, ..., z_n + w_n - 1\} \subseteq \mathbb{C}[z_1, w_1, \ldots, z_n, w_n]$$

where $\{R_1', ..., R_{n-1}'\}$ are from the polynomial augmented gluing equations for $(M, \mathcal{T})$. Then $\mathrm{PreTrop}(\mathcal{AG})$ and $Q(\mathcal{T})$ are isomorphic as piecewise linear spaces.

Proof: We define a map $\varphi : Q(\mathcal{T}) \to \mathrm{PreTrop}(\mathcal{AG})$ given by

$$\varphi((q_{1,1}, q_{2,1}, q_{3,1}, ..., q_{1,n}, q_{2,n}, q_{3,n})) = (x_1, y_1, ..., x_n, y_n)$$

where

$$(x_i, y_i) = \begin{cases} (q_{1,i}, 0) & \text{if } q_{1,i} > 0 \\ (-q_{2,i}, -q_{2,i}) & \text{if } q_{2,i} > 0 \\ (0, q_{3,i}) & \text{if } q_{3,i} > 0 \\ (0, 0) & \text{if } q_{1,i} = q_{2,i} = q_{3,i} = 0 \end{cases}$$

49

Then φ is well-defined since any $\mathbf{q} \in Q(\mathcal{T})$ must be admissible.

Also, for each (x_i, y_i) from a vector in the range of φ for $1 \leq i \leq n$, either

1. $x_i = y_i$, and $x_i, y_i \leq 0$,

2. $x_i > 0$ and $y_i = 0$, or

3. $x_i = 0$ and $y_i > 0$.

In any case, $\varphi(\mathbf{q}) \in \text{PreTrop}(\{z_i + w_i - 1\}_{i=1}^n)$ for every $\mathbf{q} \in Q(\mathcal{T})$.

Recall from Definition 4.4 and Definition 3.21, the augmented gluing equations are of the form

$$R_k = \prod_{i=1}^n (-1)^{c_{k,j}} z_i^{a_{k,j}-c_{k,j}} w_i^{c_{k,j}-b_{k,j}} = 1 \tag{6.1}$$

and the Q-matching equations are of the form

$$Q_k = \sum_{j=1}^n \left((a_{k,j} - c_{k,j})q_{1,j} + (b_{k,j} - a_{k,j})q_{2,j} + (c_{k,j} - b_{k,j})q_{3,j} \right) = 0$$

for edges e_k in $\mathcal{T}$ for $1 \leq k \leq n-1$. The form of R_k in equation 6.1 makes it clear that for a vector $(x_1, y_1, ..., x_n, y_n) \in \mathbb{R}^{2n}$ to be in $\text{PreTrop}(R_k')$ for $1 \leq k \leq n-1$ it must satisfy:

$$\left(x_1, y_1, ..., x_n, y_n \right) \cdot \left((a_{k,1} - c_{k,1}), (c_{k,1} - b_{k,1}), ..., (a_{k,n} - c_{k,n}), (c_{k,n} - b_{k,n}) \right) = 0. \tag{6.2}$$

When each case is considered in the definition of φ, we see that if $\mathbf{q} \in Q(\mathcal{T})$, then $\varphi(\mathbf{q}) \in$ PreTrop($\{R'_1, ..., R'_{n-1}\}$).

For example, consider $\mathbf{q} = (q_{1,1}, q_{2,1}, q_{3,1}, ..., q_{1,n}, q_{2,n}, q_{3,n}) \in Q(\mathcal{T})$. Let $i \in \{1, ..., n\}$. Assume $q_{2,i} > 0$. Then $(x_i, y_i) = (-q_{2,i}, -q_{2,i})$ in $\varphi(\mathbf{q})$. Then the contribution of x_i and y_i to the inner product of equation 6.2 is

$$-q_{2,i}(a_{k,i} - c_{k,i}) - q_{2,i}(c_{k,i} - b_{k,i}) = (b_{k,i} - a_{k,i})q_{2,i}.$$

This is exactly the contribution from $q_{1,i}$, $q_{2,i}$, $q_{3,i}$ in the gluing equation $Q_k = 0$. Therefore, if $\mathbf{q} \in Q(\mathcal{T})$, then $\varphi(\mathbf{q}) \in$ PreTrop($\mathcal{AG}$).

This argument works backwards and so an inverse function can be defined.

The restriction that only one of $q_{i,1}, q_{i,2}, q_{i,3}$ are nonzero for every $\mathbf{q} \in Q(\mathcal{T})$ implies that φ is linear on each face of $Q(\mathcal{T})$. Therefore φ is a piecewise linear homeomorphism from $Q(\mathcal{T})$ to the tropical prevariety PreTrop($\mathcal{AG}$). $\square$

We now prove the main result of the thesis. Our proof follows the structure of the proof of Theorem 1.1 in N. Dunfield and S. Garoufalidis's paper (11).

Theorem 6.2 *Let $(M, \mathcal{T})$ be an ideal triangulation of a compact, oriented 3-manifold with torus boundary where $\{\Delta_1, ..., \Delta_n\}$ are the tetrahedra of $\mathcal{T}$. Assume every irreducible component of $\mathcal{D}(\mathcal{T})$ is 1-dimensional. Assume further $\mathbf{q} \in \mathbb{Z}_{\geq 0}^{3n}$ is the Q-coordinate vector of the surface S, and is contained in the relative interior of an isolated ray of $Q(\mathcal{T})$. If the associated properly embedded surface, $\widehat{S}$, has nonempty boundary, then $\widehat{S}$ is an essential surface in M.*

Proof of main theorem: Since $\varphi : Q(\mathcal{T}) \to \mathrm{PreTrop}(\mathcal{AG})$ in the proof of Proposition 6.1 is a piecewise linear isomorphism, $\varphi(\mathbf{q})$ is contained in an isolated ray of the tropical prevariety $\mathrm{PreTrop}(\mathcal{AG})$. Now every irreducible component of $\mathcal{D}(\mathcal{T})$ is assumed to be 1-dimensional, and so, by Proposition 5.25, $\varphi(\mathbf{q}) \in \mathrm{Trop}(\langle \mathcal{AG} \rangle)$. Then by the Fundamental Theorem of Tropical Geometry (Theorem 5.20), there exist Puiseux series $p_1, ..., p_n$ such that $\varphi(\mathbf{q}) = (v(p_1), ..., v(p_n))$. Notice, since $\mathbf{q}$ is not the origin, $v(p_i) \neq 0$ for some $i \in \{1, ..., n\}$.

Therefore, S is associated to some ideal point ξ of $\mathcal{D}(\mathcal{T})$. By assumption, $\widehat{S} \subseteq M$ has nonempty boundary. Let α be the boundary slope of $\widehat{S}$. By Proposition 4.14, if $\widehat{S}$ is two-sided, then any reduction of $\widehat{S}$ has nonempty boundary with boundary slope α. Therefore, if $\widehat{S}$ is two-sided, then $\widehat{S}$ can be reduced to an essential surface in M which also has boundary slope α.

Still, we want to show that $\widehat{S}$ itself is essential. First, we reduce to the case that S is a vertex surface, and $\widehat{S}$ is two-sided and connected. If S is not a vertex surface, then it is some number of copies of the vertex surface for the isolated ray in which it is contained. A spunnormal surface is essential if and only if any number of copies of that spunnormal surface is essential. So we can assume S is a vertex surface. The definition of an essential non-orientable surface, Definition 1.6, allows us to reduce to the case where $\widehat{S}$ is two-sided.

As for connectivity, assume $\widehat{S}$ is not connected, then S would be the geometric sum of disjoint spunnormal surfaces associated to the components of $\widehat{S}$. Let S_C be one of these spunnormal surfaces associated to a component C of $\widehat{S}$. Then S_C is compatible with S. Now $Q(\mathcal{T})$ is convex when restricted to the n coordinates representing compatible quad-types of S and S_C, and $\mathbf{q}$ is

contained in an isolated ray of $Q(\mathcal{T})$. Therefore, the Q-coordinate vector $\mathbf{q}_C \in \mathbb{Z}_{\geq 0}^{3n}$ of S_C must be in this isolated ray, $\mathbb{R}_+ \cdot \mathbf{q}$. However, $\mathbf{q}_C + \mathbf{w} = \mathbf{q}$ for some vector $\mathbf{w} \in \mathbb{Z}_{\geq 0}^{3n}$. This contradicts that S is the primitive representative of $\mathbb{R}_+ \cdot \mathbf{q}$. So $\widehat{S}$ is connected.

If $\widehat{S}$ is not essential as claimed, there are two possibilities:

1. There is a compressing disk for $\widehat{S}$.

2. $\widehat{S}$ is boundary parallel.

The second case is ruled out since $\widehat{S}$ can be reduced to an essential surface.

So we focus on the first case, and proceed by contradiction.

Before starting, let N be an end of $\mathcal{T}$ such that $S \cap (\mathcal{T} \setminus N^\bullet)$ maps to $\widehat{S}$ by a homeomorphism $p_N : (\mathcal{T} \setminus N^\bullet) \to M$ where $N^\bullet$ is the end N with the vertex of $\mathcal{T}$. To simplify notation, let $M' = \mathcal{T} \setminus N^\bullet$ and $\widehat{S}' = S \cap M'$. Then M' is homeomorphic to M and $\widehat{S}'$ maps to $\widehat{S}$ by this homeomorphism.

Now let D be a compressing disk for $\widehat{S}$. Then there is a compressing disk D' for $\widehat{S}'$ which maps to D by p_N. Compress S along D' in M' to obtain a surface S_1. Now isotope S_1 off of S. Which can be done, because S was assumed to be two-sided.

Cut $\mathcal{T} \setminus \mathcal{T}^{(0)}$ open along S to obtain a noncompact, orientable 3-manifold with boundary P_S. Here, P_S is a homeomorphic to M' cut along $\widehat{S}'$ with annuli corresponding to components of $\partial M' \setminus \partial \widehat{S}'$ removed. Consider the cell decomposition $(P_S, \mathcal{T}_S)$ where $\mathcal{T}_S$ consists of infinitely many cells formed by cutting the tetrahedra of $\mathcal{T}$ by the normal disks of S. Notice that S_1 is

disjoint from the boundary of P_S which consists of two copies of S. Let $N_S \subseteq P_S$ be N cut along S. Then each component of N_S is homeomorphic to $A \times I$ where A is an open annulus, and cells of $\mathcal{T}_S$ completely contained in N_S are homeomorphic to $\delta \times I$ where δ is a triangle. Therefore, S_1 intersects each of these cells in normal triangles parallel to the two triangular faces of each cell. See Figure 14.

To simplify notation, let $M'_S = \mathcal{T}_S \setminus N_S$. Then M'_S is M' cut along S. Let $S'_1 = S_1 \cap M'_S$.

Now reduce S'_1 inside of M'_S by compressing along compressing disks and removing boundary parallel annuli to obtain an essential surface S'_2. Any reduction in M'_S is a reduction in M', and so, by Proposition 4.14, S'_2 is a nonempty, essential surface in M'_S with nonempty boundary. If S'_2 is not connected, replace it by a connected component with nonempty boundary.

Then we can construct an embedded surface S_2 in P_S by attaching the closed boundary curves of half-open annuli to the components of $\partial S'_2$. We require that these attached annuli do not intersect the boundary of N_S. Notice that S_2 meets each cell of $\mathcal{T}_S$ contained in N_S in triangles parallel to the two triangular faces of the cell. Then S_2 is disjoint from the boundary P_S. See Figure 14.

Now take every cell of $\mathcal{T}_S$ which is not completely contained in N_S. These form a cell decomposition of a compact, oriented 3-manifold with boundary which is isotopic to M'_S. Now by Remark 3.5, we can normalize S_2 in this compact cell decomposition. Notice, that since S'_2 does not contain any boundary parallel annuli, the normalization process does not push any annuli outside of M'_S. The normalization then produces a surface S_3 which is normal with

respect to $(P_S, \mathcal{T}_S)$. It could be that S_3 is not isotopic to S_2 if M is not irreducilbe. Still, S_3 is topologically the same as S_2, and is normal with respect to $(P_S, \mathcal{T}_S)$. Also, S_3 and the boundary of P_S are still disjoint since the normalization process does not increase the intersection with edges of $\mathcal{T}_S$. The possibilities for normal disks of S_3 in $\mathcal{T}_S$, shown in Figure 14, forces S_3 to be spunnormal with respect to $(M, \mathcal{T})$ and disjoint from S when the two copies of S in the boundary of P_S are glued back together.

Figure 14. The possible cells of $\mathcal{T}_S$. Dark faces represent normal disks of S. Light normal disks are the possible normal disks for S_3 which cannot intersect dark faces.
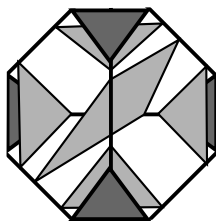
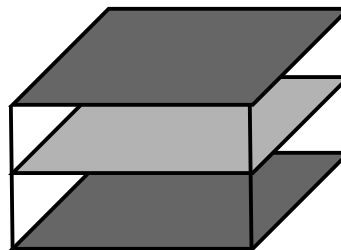Normal disk in cell cut by two triangles
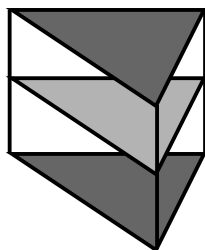
Normal disks in cell cut by quad and two triangles
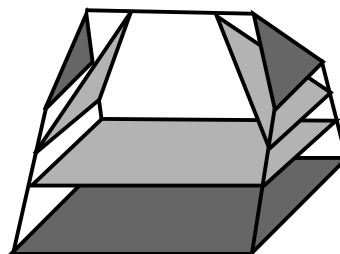
Normal disks in cell cut by triangles with no quads
(Quad inside can be of any type)

Normal disk in cell cut by two quads

Then S_3 is a spunnormal surface with respect to $(M, \mathcal{T})$ and is compatible with S. Now $Q(\mathcal{T})$ is convex when restricted to n coordinates representing compatible quad-types of S and S_3, and the Q-coordinate vector of S is contained in an isolated ray of $Q(\mathcal{T})$. Therefore, the Q-coordinate vector of S_3 must be in the same isolated ray of $Q(\mathcal{T})$. Now S_3 was constructed to be two-sided, and therefore must be some number of copies of the two-sided connected surface S. However, S_3 was also constructed to be connected, and therefore, must be isotopic to S. This contradicts that we compressed along D and further reduced S to obtain S_3. Therefore, the first case is ruled out and S is essential. $\square$

CHAPTER 7

EXAMPLES

In this chapter, we give sample Python code which defines a function, *the A function*, that finds examples of spunnormal surfaces that satisfy the hypotheses of the main theorem, Theorem 6.2. The function takes in a compact, orientable 3-manifold with torus boundary, M, and outputs vertex surfaces which are in an isolated ray of $Q(T)$ where T is the ideal triangulation used by SnapPy for M. Theorem 6.2 then shows that these surfaces are essential and their boundary slopes are in bs(M).

The following is the Python code for *the A function* discussed above. It requires the boundary_slopes module from the t3m package, and the cypari module. Everything can be downloaded, or easily found, on the website, http://www.math.uic.edu/t3m/.

```
from boundary_slopes import *

from cypari.gen import pari

def A(x):

        M=OneCuspedManifold(x)

        M.find_normal_surfaces()

        count=0

        for idx in range(len(M.NormalSurfaces)):

                print 'Spun␣surface␣', idx
```

```
S=M.NormalSurfaces[idx]

print 'Coefficients:␣',S.Coefficients


isBad=0

if 0 in S.Coefficients:

        z=list(S.Coefficients).count(0)

        if z>=9:

                isBad=isBad+1

                print '␣␣␣␣␣Not␣Computed'

        else:

                L=list(S.Coefficients)

                p=[ ]

                for i in range(z):

                        p=p+[L.index(0)+i]

                        L.remove(0)


                for i in range(len(f(z))):

                        for j in range(z):

                                S.Quadtypes[p[j

                                ]]=f(z)[i][j]
```

```
                                        if isGood(S,M)==0:

                                                isBad=isBad+1

                                                break

                else:

                        if isGood(S,M)==0:

                                isBad=isBad+1

                if isBad==0:

                        print S.info(M)

                        count=count+1

        print 'Count␣=␣', count

        print 'Manifold:␣', x
```

The following function is used in *the A function* to check the dimension of $Q(\mathcal{T})$ for some choice of n coordinates.

```
def isGood(S, M):

        cols=[3*m+n for m,n in enumerate(S.Quadtypes)]

        T=M.QuadMatrix.rows

        mymatrix=[M.QuadMatrix[i,j] for i in range(T) for j in

            cols]

        Z=pari.matrix(T,T,mymatrix)

        Z.matsnf()
```

```
if list(Z.matsnf()).count(0)==1:

        return 1

else:

        return 0
```

This final function is used in *the A function* when generating all of the 3^m coordinates that must be checked when m is the number of tetrahedra in which the surface does not have a normal quad.

```
def f(m):

    if m<=1:

            return [[0],[1],[2]]

    previous=f(m-1)

    result=[ ]

    for vector in previous:

            result = result+[vector + [0], vector + [1],

                vector +[2]]

    return result
```

The boundary slopes produced from *the A function* have been checked for consistency with N. Dunfield's list of boundary slopes for Montesinos knots up to 10 crossings (8).

Example 7.1 (5_2-**knot complement**) *The following is the output from the A function for the 5_2-knot complement. The quad-types match up with those of Definition 3.2 as follows: Q13 is Q_1, Q03 is Q_2, and Q23 is Q_3.*

```
>>> A('5_2')

slice   0 :      3 positive     3 negative     3 zero

slice   1 :      6 positive     3 negative     0 zero

slice   2 :      0 positive     0 negative     6 zero

DONE.  15 vertices were filtered;   0 were interior.


Spun surface   0

Coefficients:   [2 2 1]

SpunSurface.

 Slope: (-10, -1); Boundary components: 1; Euler characteristic:

    -2

 Incompressible: None

  Tet 0: Quad Type  Q13, weight 2

  Tet 1: Quad Type  Q03, weight 2

  Tet 2: Quad Type  Q23, weight 1

None
```

```
Spun surface  1

Coefficients:  [1 1 1]

SpunSurface.

 Slope: (-4, -1); Boundary components: 1; Euler characteristic:

    -1

 Incompressible: None

  Tet 0: Quad Type  Q13, weight 1

  Tet 1: Quad Type  Q13, weight 1

  Tet 2: Quad Type  Q03, weight 1

None


Spun surface  2

Coefficients:  [2 2 1]

SpunSurface.

 Slope: (10, 1); Boundary components: 1; Euler characteristic: -2

 Incompressible: None

  Tet 0: Quad Type  Q23, weight 2

  Tet 1: Quad Type  Q23, weight 2

  Tet 2: Quad Type  Q23, weight 1

None
```

```
Spun surface  3

Coefficients:  [1 1 1]

SpunSurface.

 Slope: (4, 1); Boundary components: 1; Euler characteristic: -1

 Incompressible: None

  Tet 0: Quad Type  Q03, weight 1

  Tet 1: Quad Type  Q23, weight 1

  Tet 2: Quad Type  Q03, weight 1

None


Spun surface  4

Coefficients:  [1 2 1]

SpunSurface.

 Slope: (0, -1); Boundary components: 1; Euler characteristic: -1

 Incompressible: None

  Tet 0: Quad Type  Q23, weight 1

  Tet 1: Quad Type  Q13, weight 2

  Tet 2: Quad Type  Q13, weight 1

None


Spun surface  5
```

```
Coefficients:  [2 1 1]

SpunSurface.

 Slope: (0, 1); Boundary components: 1; Euler characteristic: -1

 Incompressible: None

  Tet 0: Quad Type  Q03, weight 2

  Tet 1: Quad Type  Q03, weight 1

  Tet 2: Quad Type  Q13, weight 1

None



Count =  6

Manifold:  5_2
```

Notice that the vectors $(1, 1, -1)$ and $(-2, -1, 1)$ found to be in $\mathrm{Trop}(\langle \overline{R}'_1, \overline{R}'_2 \rangle)$, see Figure 8 and Proposition 5.22, correspond to 'Spun surface 1' and 'Spun surface 5' respectively from the output of A('5_2').

The main theorem of this thesis, Theorem 6.2, shows that the six spunnormal surfaces in the output of A('5_2') are essential, and therefore their boundary slopes are in $\mathrm{bs}(M)$ for M the 5_2-knot complement. Then we see from the output that $\{0, 4, 10\} \subseteq \mathrm{bs}(M)$. From N. Dunfield's table of boundary slopes for Montesinos knots (8), $\{0, 4, 10\} = \mathrm{bs}(M)$. In this case the A function actually found all of $\mathrm{bs}(M)$ which is rather rare for Montesinos knots up to 10 crossings.

All of these spunnormal surfaces have a normal quad in every tetrahedron, and so these spunnormal surfaces are already know to be essential by N. Dunfield and S. Garoufalidis (11).

Table I gives examples of non-Montesinos knots for which *the A function* finds boundary slopes of essential spunnormal surfaces with no normal quads in some tetrahedron. As far as the author knows, these boundary slopes are not previously known. These slopes are marked with an asterisks (*) and do not satisfy the criteria of N. Dunfield and S. Garoufalidis (11), but do satisfy the criteria of the main theorem in this thesis, Theorem 6.2. That is, the essential spunnormal surface or spunnormal surfaces with that boundary slope do not have a normal quad in some tetrahedron of the ideal triangulation used by SnapPy.

TABLE I

A LIST OF BOUNDARY SLOPES OF NON-MONTESINOS KNOTS.

9_{47}	4*	16					
10_{85}	-20	-15	-14	-6*	-2		
10_{87}	-14*	-10	-9	0	4		
10_{90}	-7	-6*	12	18			
10_{93}	-16*	-8*	-6*	2			
10_{94}	6	18*	28				
10_{102}	-14*	-10*	-1	12	18		
10_{103}	-20	-14	-12	-6	8*		
10_{104}	-12*	6	10	14	16		
10_{106}	-14	-8	-6	-4/3	6	14*	18*
10_{108}	-2	0	11	16*			
10_{110}	-18	-4*	3	14			
10_{112}	-28	-18*	-13	6	8	14	
10_{114}	-30	-18	-12	6*	7	14	
10_{116}	-18	-14*	-13	-12	-9	8	14
10_{117}	-14	7	8*	12	18		
10_{118}	-26	-16	-12*	0	12*	16	26
10_{119}	-30	-7	-6*	-4	12	18	24
10_{123}	-12*	12*					

* boundary slopes from essential spunnormal surfaces with no normal quads in a tetrahedron

CITED LITERATURE

1. Boyer, S., Culler, M., Shalen, P., and Zhang, X.: Characteristic subsurfaces, character varieties and Dehn fillings. Geometry and Topology, 12:233–297, 2008.

2. Boyer, S. and Zhang, X.: On Culler-Shalen seminorms and Dehn fillings. Ann. of Math., 148(2):737–801, 1998.

3. Hatcher, A.: On the boundary curves of incompressible surfaces. Pac. Jour. of Math., 99(2):373–377, 1982.

4. Hatcher, A. and Thurston, W.: Incompressible surfaces in 2-bridge knot complements. Invent. Math., 79:225–246, 1985.

5. Culler, M., Jaco, W., and Rubinstein, H.: Incompressible surfaces in once-punctured torus bundles. Proc. London Math. Soc., 45(3):385–419, 1982.

6. Floyd, W. and Hatcher, A.: Incompressible surfaces in punctured-torus bundles. Topology Appl., 13:263–282, 1982.

7. Hatcher, A. and Oertel, U.: Boundary slopes for Montesinos knots. Topology, 42:453–480, 1989.

8. Dunfield, N.: A table of boundary slopes of Montesinos knots. Topology, 40:309–315, 2001.

9. Jaco, W. and Sedgewick, E.: Decision problems in the space of Dehn fillings. Topology, 42:845–906, 2003.

10. Cooper, D., Culler, M., Gillet, H., Long, D., and Shalen, P.: Plane curves associated to character varieties of 3-manifolds. Invent. Math., 118:47–84, 1994.

11. Dunfield, N. and Garoufalidis, S.: Incompressibility criteria for spun-normal surfaces. Trans. Amer. Math. Soc., 64:6109–6137, 2012.

12. Kabaya, Y.: A method to find ideal points from ideal triangulations. J. Knot Theory Ramifications, 19:509–524, 2010.

13. Thurston, W.: The geometry and topology of three-manifolds. http://www.msri.org/publications/books/gt3m, 1978.

14. Culler, M. and Shalen, P.: Varieties of group representations and splittings of 3-manifolds. Ann. of Math., 117(2):109–146, 1983.

15. Osserman, B. and Payne, S.: Lifting tropical intersections. http://arxiv.org/abs/1109.5733, 2010.

16. Tillmann, S.: Normal surfaces in topologically finite 3-manifolds. Enseign. Math., 54:329–380, 2008.

17. Jaco, W. and Rubinstein, H.: 0-efficient triangulations of 3-manifolds. Differential Geom., 53(1):61–168, 2003.

18. Culler, M., Dunfield, N., and Weeks, J.: SnapPy, a computer program for studying the geometry and topology of 3-manifolds, May 2013. http://snappy.computop.org.

19. Maclagan, D. and Sturmfels, B.: Introduction to tropical geometry. http://homepages.warwick.ac.uk/staff/D.Maclagan/papers/TropicalBook27.1.13.pdf, 2013.

20. Matveev, S.: Algorithmic Topology and Classification of 3-Manifolds. Springer, Berlin, second edition, 2007.

21. Walsh, G.: Incompressible surfaces and spunnormal form. Geometricae Dedicata, 151:221–231, 2011.

22. Tillmann, S.: Degenerations of ideal hyperbolic triangulations. arXiv:math/0508295, 2011.

23. Bogart, T., Jensen, A., Speyer, D., Sturmfels, B., and Thomas, R.: Computing tropical varieties. J. Symbolic Comput., 42(1-2):54–73, 2007.

24. Jensen, A.: Gfan, January 2011. http://www.math.tu-berlin.de/ jensen/software/gfan/g-fan.html.

Andrew Brasile

Curriculum Vitae

851 S. Morgan (M/C 249)
Chicago, IL 60607
mobile (815) 404-0467
☎ (815) 338-5241
FAX (312) 996-1491
✉ abrasi2@math.uic.edu
math.uic.edu/~abrasi2

Education

2008–2013 **PhD in Pure Mathematics**, *University of Illinois at Chicago*, Chicago, IL.
Advisor: Marc Culler. Thesis title: *Essential spunnormal surfaces via tropical geometry*.

2006–2008 **MS in Pure Mathematics**, *University of Illinois at Chicago*, Chicago, IL.

2002–2006 **BS in Mathematics and Secondary Education**, *Augustana College*, Rock Island, IL.
Graduated magna cum laude.

Teaching Experience

2006–2013 **Course Discussions**, *Pre-Calculus, Business Calculus, Calculus II*, University of Illinois at Chicago, Chicago, IL.

2007–2012 **Emerging Scholars Program (ESP)**, *Pre-Calculus, Calculus II, Multivariable Calculus, Introduction to Proof*, University of Illinois at Chicago, Chicago, IL.
ESP workshops employ the Teisman Method of facilitating problem solving and peer communication by presenting challenging mathematical problems and implementing innovative techniques of cooperative learning.

Summer 2011 **Summer Enrichment Workshop (SEW) Instructor**, *Intermediate Algebra*, University of Illinois at Chicago, Chicago, IL.
SEW is an intensive five-week program that was developed for students who need more preparation in mathematics before enrolling in mathematics courses required by their major.

2008–2011 **Assuring STEM Credential Expansion through Nurturing Diversity (ASCEND) Program**, *Program Coordinator*, University of Illinois at Chicago, Chicago, IL.
ASCEND is a summer program funded by an NSF grant dedicated to increasing minority participation in Science, Technology, Engineering and Mathematics (STEM) subjects.

2007–2011 **ASCEND Program**, *AP Calculus BC Group Leader*, University of Illinois at Chicago, Chicago, IL.

Spring 2006 **Student Teacher**, *AP Calculus AB, Geometry*, Pleasant Valley High School, Bettendorf, IA.

University and Community Involvement

2012–2013 **GED Tutor**, *Erie Neighborhood House*, Chicago, IL.

2011–2013 **Adult Basic Education (ABE) Tutor**, *Renaissance Apartments & Fitness for Life Center, The Renaissance Collaborative*, Chicago, IL.

Spring 2012 **Completed ABE Tutor Training**, *Literacy Works*, Chicago, IL.

Spring 2012 **High School Tutor**, *AP Calculus AB*, UIC College Prep High School, Chicago, IL.

Spring 2011	**Mathematics Tutor**, *Community Youth Development Institute (CYDI) Alternative High School*, Chicago, IL.
Fall 2010	**Mathematics Tutor**, *National Society of Black Engineers (NSBE), University of Illinois at Chicago*, Chicago, IL.
2008-2009	**President of Math Graduate Students Association**, *University of Illinois at Chicago*, Chicago, IL.
Spring 2009	**Mathematics Department Mentoring Awards Nomination Committee**, *University of Illinois at Chicago*, Chicago, IL.
2008-2009	**Department of Mathematics Representative to Graduate Student Council**, *University of Illinois at Chicago*, Chicago, IL.
2004-2005	**Beling Scholar**, *Augustana College*, Rock Island, IL.

Presentations

April 9, 2012	***Boundary slopes of knot complements***, *Graduate Student Colloquium*, University of Illinois at Chicago, Chicago, IL.
March 23, 2012	***Cooperative learning workshops: Discussion on organization and methods***, *2012 Chicago Symposium Series*, Northwestern University, Chicago, IL.
September 28, 2011	***Paradoxical decompositions and hyperbolic 3-manifolds***, *Graduate Geometry, Topology and Dynamics Seminar*, University of Illinois at Chicago, Chicago, IL.
April 20, 2011	***The Thurston compactification***, *Graduate Geometry, Topology and Dynamics Seminar*, University of Illinois at Chicago, Chicago, IL.
May 5, 2010	***Reducing Heegaard splittings***, *Graduate Geometry, Topology and Dynamics Seminar*, University of Illinois at Chicago, Chicago, IL.
August 10, 2006	***Weights for simple graphs***, *Pi Mu Epsilon Student Lectures*, Math Fest 2006, Knoxville, TN.
April 7, 2005	***Weights for simple graphs***, *Illinois Section of the Mathematical Association of America (ISMAA) Conference*, Knox College, Knox, IL.