Computing Series Expansions of Algebraic Space Curves

by

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To Grace, Mom, and Dad

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PREFACE

This thesis is the culmination of my graduate work done under the advisement of Professor Jan Verschelde. I have hopefully presented it with the context necessary to make it understandable and anchor it within the wider field of computational algebraic geometry. Its purpose is to satisfy the UIC graduation requirements, to provide a useful guide for future students of the subject, and ultimately, to glorify God. Anything I achieve is only by His grace to me in Jesus Christ.

CONTRIBUTION OF AUTHORS

Chapter 1 introduces the context and contributions of this thesis. Chapter 2 is a moderate revision of [13], which was written with my advisor Jan Verschelde; a section to which I did not contribute significantly is excluded, but what remains is joint work. Taken together, Chapters 3 and 4 represent a minor revision of [14], which was also written jointly with my advisor; here my contributions were significant throughout.

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$\mathbb{N}$	Natural numbers, $\{0, 1, 2, \ldots\}$
$\mathbb{Q}$	Field of rational numbers
$\mathbb{R}$	Field of real numbers
$\mathbb{C}$	Field of complex numbers
k	A field
k^*	$k \setminus \{0\}$
R	A ring
$\mathbf{x}$	Tuple of variables, typically $x_1, \ldots, x_n$
$R[\mathbf{x}]$	Polynomial ring in the variables $\mathbf{x}$ over R
t	Ring variable / homotopy parameter
$\mathbf{f}(\mathbf{x})$	Polynomial system (tuple of polynomials)
I	An ideal
$R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$	Laurent polynomial ring
$\mathbb{V}(\mathbf{f})$	Zero set of $\mathbf{f}$
$\mathbb{V}(I)$	Zero set of I
$k(x)$	Field of rational functions over k
$k[[x]]$	Ring of formal power series over k

LIST OF MATHEMATICAL NOTATION (Continued)

$k((x))$	Field of formal Laurent series over k
$k\{\{x\}\}$	Field of fractional power (Puiseux) series over k
$\mathrm{val}(\cdot)$	A valuation (on a field)
$f^{(x_i)}$	The image of f under $k[\mathbf{x}] \hookrightarrow k\{\{x_i\}\}[\mathbf{x} \setminus x_i]$
$\langle \cdot, \cdot \rangle$	The dot (inner) product
$\mathrm{in}_{\mathbf{v}}(\cdot)$	The initial form with respect to $\mathbf{v}$
$\mathrm{trop}(V(f))$	Tropical hypersurface of f
$\mathrm{trop}(V(\mathbf{f}))$	Tropical prevariety of $\mathbf{f}$
$\mathrm{trop}(V(I))$	Tropical variety of I
P	A polytope; alternatively, a permutation matrix
$\mathbf{z}$, $\Delta\mathbf{z}$	Vectors of power or Laurent series
$\mathbf{x}$, $\Delta\mathbf{x}$	Series with vector coefficients corresponding to $\mathbf{z}$ and $\Delta\mathbf{z}$

SUMMARY

The primary goal of this work is to find series expansions for dimension one components of algebraic varieties by exploiting the sparse structure of the Newton polytopes of the associated polynomial system. These series may be power series, Laurent series, or Puiseux series, depending on the problem. We have three primary motivations for this. First, such expansions capture local information about the variety and can help us to better understand it. Second, this serves as a building block toward the computational algebraic geometer's dream, namely, a general solver for polynomial systems with solution sets of arbitrary dimension. Finally, and a bit more specifically, such a solution is highly applicable to homotopy continuation. In fact, using series expansions to better understand the space curve swept out by a homotopy path track served as one of the initial motivations for our investigation.

Our first contribution is a better understanding of the polyhedral behavior of series expansions for polynomial systems. Such expansions are connected to the combinatorial world through the field of tropical algebraic geometry, and more specifically through tropical varieties and prevarieties. We define what we call "hidden cone" behavior, where the prevariety fails to capture enough information to find the correct series expansion, and we give examples that exhibit such behavior in arbitrary dimensions. We show that, generically, such behavior cannot occur. Finally, we provide a numerical strategy based on polyhedral end games which enables us to complete information lacking in the tropical prevariety.

SUMMARY (Continued)

Our second contribution picks up where the tropical approach leaves off. More specifically, when sufficiently many terms of a series expansion have been obtained using the above methods, we give a modification of the Gauss-Newton algorithm for obtaining more terms. It couples an approach called linearization with structured linear algebra, and like the traditional Newton's method, is capable of doubling the number of known terms at each step. In some cases the tropical approach mentioned above is entirely unnecessary, and we provide a result that characterizes precisely when it can be avoided in favor of going straight to our (much quicker) Gauss-Newton approach.

Finally, we conclude with several examples that illustrate the above techniques. Of particular note is an investigation of the cyclic 16-roots polynomial system. The prevariety of this polynomial system has only recently been computationally feasible. The tropical version of Backelin's lemma predicts that it will contain a particular 3-dimensional polyhedral cone, but we show that this description is incomplete, and the predicted cone is in fact contained in a larger 3-dimensional cone of the prevariety.

CHAPTER 1

INTRODUCTION

1.1 Preliminaries

1.1.1 Thesis Overview

In this introductory chapter, we begin with some general definitions. We then give a summary of those results from the field which are necessary to understand our contributions, focusing on the Newton-Puiseux method, tropical geometry, and homotopy continuation. Finally, we outline our problem statement and summarize our contributions.

In Chapter 2, we give alternatives to the current symbolic methods for computing tropical information. Most of the work in this chapter is published in [13]. Chapter 3 gives an effective way of applying the method of Gauss-Newton to find more terms of power or Laurent series satisfying systems of equations. It is a natural companion to the tropical methods of Chapter 2, and mostly coincides with work published in [14]. Finally, Chapter 4 provides many interesting examples that show the power of our results. Some are taken from [13], others from [14], and still others are new.

1.1.2 General Definitions

For the most part, our base field will be the complex numbers $\mathbb{C}$. We use $R[\mathbf{x}] = R[x_1, \ldots, x_n]$ to denote the polynomial ring in the variables $\mathbf{x} = x_1, \ldots, x_n$ over the ring R. Occasionally we will give results defined over the Laurent polynomial ring $R[\mathbf{x}^{\pm 1}] = R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ where we allow variables to be raised to negative powers.

We write a polynomial f with *support set* A as

$$f(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad c_{\mathbf{a}} \in k^*, \mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}. \tag{1.1}$$

This notation is independent of whether we are working in $R[\mathbf{x}]$ or $R[\mathbf{x}^{\pm 1}]$. The degree of f is $\max\{\mathrm{sum}(\mathbf{a}) : c_{\mathbf{a}} \neq 0\}$. A *polynomial system* is a tuple of polynomials; a polynomial system is *sparse* if it has relatively few monomials given its degrees. For example, we would consider $f = x^8 y^5 - 1$ to be sparse, but not $g = x^4 - x^3 + x^2 - x + 1$. This is somewhat relative, of course, but is useful in practice.

The variable t will often be used as a polynomial ring variable which has some special meaning, for example as the parameter in a homotopy. If $\mathbf{f}$ is a polynomial system, then $\mathbb{V}(\mathbf{f})$ will denote the zero set of $\mathbf{f}$; likewise, for an ideal I, $\mathbb{V}(I)$ denotes the variety of I. For our purposes, an *algebraic space curve*, or just *space curve*, is a dimension one algebraic variety. A point $\mathbf{p}$ on a d-dimensional component of a variety $\mathbb{V}(\mathbf{f}) \subset \mathbb{C}^n$ is *regular* if the Jacobian of $\mathbf{f}$ evaluated at $\mathbf{p}$ has rank $n-d$; for a space curve cut out by $n-1$ equations in n unknown, this just says that the Jacobian is full rank. Points that are not regular are called *singular*. The geometric intuition for a singular point is a place where the tangent space does not have the expected dimension.

Finally, we will use the following notation for various modifications of a field k. We write $k\backslash\{0\}$ as k^*. For rational functions, we use the notation $k(x)$; for formal power series, $k[[x]]$. The field of formal Laurent series, or in other words series with finitely many negative exponents, will be denoted $k((x))$. And lastly, we will use $k\{\{x\}\}$ for the field of fractional power (Puiseux) series, defined as the union over $n \in \mathbb{N}$ of $k((t^{1/n}))$. Each element is a series where the exponents are fractions with bounded denominator.

1.2 Background and Related Work

1.2.1 The Planar Newton-Puiseux Algorithm

The Newton-Puiseux algorithm dates back to Isaac Newton in the 1670's and 80's [55]. According to [22], it was "all but forgotten" in the early 19th century, but was used by Puiseux [61, 62] to prove the following:

Theorem 1.2.1 (Newton-Puiseux Theorem)**.** *If k is an algebraically closed field of characteristic 0, then the field of Puiseux series $k\{\{x\}\}$ is algebraically closed.*

Put another way, this means that a univariate polynomial with coefficients in $k\{\{x\}\}$ factors completely. For proof and an excellent exposition, see for example [19] or [79].

For a polynomial $f(x_1, x_2) = \sum_{\mathbf{a}\in A} c_{\mathbf{a}}\mathbf{x}^{\mathbf{a}} \in k[x_1, x_2]$, the Newton-Puiseux algorithm finds terms of series $s(t) \in k\{\{t\}\}$ such that $f(t, s(t)) = 0$. It does so term by term, simply by looking for conditions on the cancellation of the terms of lowest order. The first condition involves the Newton polygon of the support of f, which invites a few definitions:

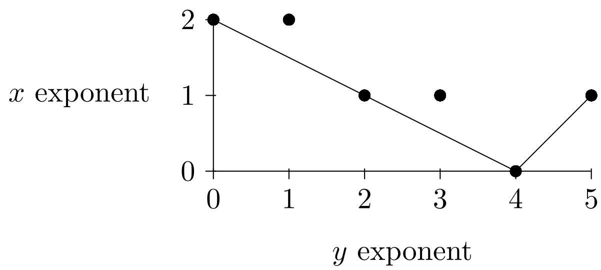


Figure 1: The lower hull of the Newton polygon of $f = -2x^2 + x^2y + xy^2 + xy^3 + y^4 + xy^5$.

Definition 1.2.2. *The* support *of a polynomial $f \in k[x_1, \ldots, x_n]$ is the set $\{\mathbf{a} \in \mathbb{N}^n\}$ such that $\mathbf{x}^a$ is a monomial of f (with nonzero coefficient).*

Definition 1.2.3. *The* Newton polytope *of a polynomial $f \in k[x_1, \ldots, x_n]$ is the convex hull of its support. When $n = 2$ we instead use the term* Newton polygon*.*

In looking for the starting term $d\,t^\gamma$ of s, it is necessary that $-\gamma$ be the slope of a segment of the Newton polygon of p, restricting to those segments that lie on the lower portion with respect to x. In addition, d must be a root of $\sum c_{\mathbf{a}} x_2^{a_2}$ where $\mathbf{a}$ runs over the points on the of the Newton polygon segment corresponding to γ; with a slight abuse of future notation we will write this as $\mathrm{in}_\gamma(f)$.

The following example illustrates these conditions.

Example 1. *Suppose we have the polynomial $f(x, y) = -2x^2 + x^2y + xy^2 + xy^3 + y^4 + xy^5 \in \mathbb{C}[x, y]$. Figure 1 shows the lower hull of the Newton polygon of f. From it we obtain two possibilities for the leading exponent, $\gamma_1 = \frac{1}{2}$ and $\gamma_2 = -1$. Using the first, we obtain* $\mathrm{in}_{\gamma_1}(f) =$

y^4+y^2-2, with roots $\pm 1, \pm i\sqrt{2}$. Using the second we obtain $\mathrm{in}_{\gamma_2}(f) = y^5 + y^4$ which has only one nonzero root -1; we discard the repeated 0 as it results in the trivial term $0 \cdot t^{-1}$. Thus the five options for the first term of a Puiseux series solution are $\pm t^{\frac{1}{2}},\ \pm i\sqrt{2}t^{\frac{1}{2}}$, and $-t^{-1}$.

$\triangle$

Finding more terms amounts to a recursive application of this idea to $f(x, x^\gamma(d+y))$ for chosen first term $d\,t^\gamma$, where we require subsequent γ to be positive. The cost is therefore linear in the number of terms desired. Pseudocode for the algorithm may be found in Appendix A. Source code documentation for our generalized Newton-Puiseux algorithm in higher dimensions can be found in Appendix B.

1.2.1.1 Extensions and Improvements

In [25] Duval presents a modification for computing what she calls rational Puiseux expansions, defined for a polynomial $f(x,y) \in k[x,y]$ as a system $\{(\overline{x}_1, \overline{y}_1), ..., (\overline{x}_m, \overline{y}_m)\} \subset k[[t]]^2$ of non-equivalent irreducible parameterizations with the $\overline{x}_i$ monomials, such that the set is invariant under the Galois action of the algebraic closure of k on k itself. Here non-equivalence means no parameterization can be obtained from another by means of a linear substitution $(\overline{x}(z(t)), \overline{y}(z(t)))$, and irreducibility means there is no $k \in \mathbb{Z}$ positive such that $(\overline{x}, \overline{y}) \in k[[t^k]]^2$. Aside from its mathematical implications, this has the algorithmic advantage that no fractional exponents are necessary, which simplifies symbolic computations; traditional Puiseux series can be obtained easily by means of a change of variables.

Poteaux and Rybowicz present a symbolic-numeric extension of Duval's algorithm in [60] for k a finite extension of $\mathbb{Q}$. Their algorithm first computes the rational Puiseux system modulo some prime p in order to obtain exact information, where p is chosen to leave the Newton polygons unchanged. It then uses this information to guide floating-point computations where numerical instability might under normal conditions result in drastic errors. For a polynomial p and Puiseux series solution s of f, they define the *regularity index* as the number of terms necessary to distinguish s from the other series solutions of f. For a given series, the algorithm terminates when as many terms have been computed as its regularity index. Beyond that point one may quickly compute more terms via quadratic Newton iteration per [47]. This approach provides an effective way of overcoming the numerical instability inherent in the classical algorithm.

1.2.2 Tropical Geometry

Several investigations into generalizing the Newton-Puiseux algorithm to higher dimensions exist; see for example [53], or the succinct and perhaps earliest work [52]. However, the proper framework for such an investigation lies in the field of *tropical geometry*. In essence, this field studies the polyhedral skeleton of algebraic geometry; the Newton polygon in Figure 1 is a glimpse into the tropical world. We continue with a few definitions. Much of this section parallels [51], although that work is given in greater generality than our study, so we make simplifications where appropriate.

For a field k, a *valuation* on k is a map $\mathrm{val} : k \to \mathbb{R} \cup \{\infty\}$ that, for all $a, b \in k$, satisfies

- $\mathrm{val}(a) = \infty$ if and only if $a = 0$,
- $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$, and
- $\mathrm{val}(a + b) \geq \min\{\mathrm{val}(a), \mathrm{val}(b)\}$.

We will only make use of two sorts of valuation: the *trivial valuation* on k, which takes every nonzero element to 0, and the valuation on Puiseux series taking a series $s(t)$ to its lowest exponent. Both valuations necessarily send 0 to ∞.

If $f(\mathbf{x}) \in k[\mathbf{x}]$ or $k[\mathbf{x}^{\pm 1}]$ is a polynomial, we will write $f^{(x_i)}$ for the image of f under the natural map

$$k[\mathbf{x}] \hookrightarrow k\{\{x_i\}\}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]. \tag{1.2}$$

In other words, $f^{(x_i)}$ is f considered as a polynomial in the variables $\mathbf{x} \setminus x_i$ with coefficients in $k\{\{x_i\}\}$. We will use this notation for polynomial systems $\mathbf{f} = (f_1, \ldots, f_m)$ as well, setting

$$\mathbf{f}^{(x_i)} = (f_1^{(x_i)}, \ldots, f_m^{(x_i)}). \tag{1.3}$$

The *initial form* of a polynomial over a field k depends on the valuation used. The general case requires first defining value groups, residue fields, and the splitting of a valuation, but for our purposes, the following suffices. We will state the definitions over $\mathbb{C}[\mathbf{x}]$, but they extend naturally to $\mathbb{C}[\mathbf{x}^{\pm 1}]$. Let $f = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$ be a polynomial in $\mathbb{C}[\mathbf{x}]$, and let $\langle \cdot, \cdot \rangle$ denote the inner product. If we set $M = \min\{\langle \mathbf{v}, \mathbf{a}\rangle : \mathbf{a} \in A, c_{\mathbf{a}} \neq 0\}$, then the initial form of f with respect to $\mathbf{v} \in \mathbb{R}^n$ is

$$\mathrm{in}_{\mathbf{v}}(f) = \sum_{\langle \mathbf{a},\mathbf{v}\rangle = M} c_{\mathbf{a}}\mathbf{x}^{\mathbf{a}} \in \mathbb{C}[\mathbf{x}]. \tag{1.4}$$

Alternatively, we can define the initial form for a polynomial f in $\mathbb{C}\{\{t\}\}[\mathbf{x}]$, where we use the nontrivial valuation on Puiseux series. For $\mathbf{v} \in \mathbb{R}^n$, let $M = \min\{\mathrm{val}(c_{\mathbf{a}}) + \langle \mathbf{v}, \mathbf{a}\rangle : \mathbf{a} \in A, c_{\mathbf{a}} \neq 0\}$. Then

$$\mathrm{in}_{\mathbf{v}}(f) = \sum_{\mathrm{val}(c_{\mathbf{a}})+\langle \mathbf{a},\mathbf{v}\rangle = M} \mathrm{lc}(c_{\mathbf{a}})\mathbf{x}^{\mathbf{a}} \in \mathbb{C}[\mathbf{x}], \tag{1.5}$$

where $\mathrm{lc}(\cdot)$ is the coefficient of the lowest order term. Note that for f and $f^{(x_1)}$ these definitions are very similar: if $f \in \mathbb{C}[x_1, \ldots, x_n]$ and $\mathbf{v} = (v_2, \ldots, v_n) \in \mathbb{R}^{n-1}$, then

$$\mathrm{in}_{\mathbf{v}}(f^{(x_1)}) = \mathrm{in}_{(1,v_2,\ldots,v_n)}(f)\big|_{x_1=1}. \tag{1.6}$$

We can extend these definitions naturally to polynomial systems and ideals. If $\mathbf{f} = (f_1, \ldots, f_m)$ is a polynomial system and $I = \langle \mathbf{f} \rangle$ the ideal generated by the f_i, then we define $\mathrm{in}_{\mathbf{v}}(\mathbf{f})$ as the polynomial system $(\mathrm{in}_{\mathbf{v}}(f_1), \ldots, \mathrm{in}_{\mathbf{v}}(f_m))$, and $\mathrm{in}_{\mathbf{v}}(I)$ as the ideal generated by $\{\mathrm{in}_{\mathbf{v}}(f) : f \in I\}$.

Now that we have this notation specified, we define the *tropical hypersurface* $\mathrm{trop}(V(f))$ of a polynomial f as the set of $\mathbf{v}$ for which $\mathrm{in}_{\mathbf{v}}(f)$ consists of at least two monomials. If $\mathbf{f} = (f_1, \ldots, f_m)$ is a polynomial system, the *tropical prevariety* of $\mathbf{f}$ is

$$\mathrm{trop}(V(\mathbf{f})) = \bigcap_{i\in\{1\ldots n\}} \mathrm{trop}(V(f_i)), \tag{1.7}$$

and the *tropical variety* of the ideal $I = \langle \mathbf{f} \rangle$ is

$$\operatorname{trop}(V(I)) = \bigcap_{f \in I} \operatorname{trop}(V(f)). \tag{1.8}$$

Elements of the tropical prevariety are called *pretropisms*, and elements of the tropical variety *tropisms*. In other words, the tropical prevariety is the set of $\mathbf{v}$ such that $\operatorname{in}_{\mathbf{v}}(f_i)$ is not a monomial for any f_i in the system $\mathbf{f}$, and the tropical variety is the set of $\mathbf{v}$ such that $\operatorname{in}_{\mathbf{v}}(f_i)$ is not a monomial for any $f \in I$. Since the polynomials in the system are a subset of the ideal, clearly $\operatorname{trop}(V(I)) \subseteq \operatorname{trop}(V(\mathbf{f}))$, i.e. every tropism is a pretropism.

Before moving on to an example, we record the following trivial but useful lemma, which follows from (1.6):

Lemma 1.2.4. *Let* $\mathbf{f}$ *be a system of polynomials in* $\mathbb{C}[\mathbf{x}]$ *or* $\mathbb{C}[\mathbf{x}^{\pm 1}]$*. Then*

$$\operatorname{trop}(V(\mathbf{f}^{(x_i)})) = \operatorname{trop}(V(\mathbf{f})) \cap \mathbb{V}(x_i - 1). \tag{1.9}$$

Example 2 (Example 1, continued)**.** *In Example 1 we computed the first terms of the Puiseux expansion of the polynomial*

$$f(x, y) = -2x^2 + x^2y + xy^2 + xy^3 + y^4 + xy^5 \in \mathbb{C}[x, y]. \tag{1.10}$$

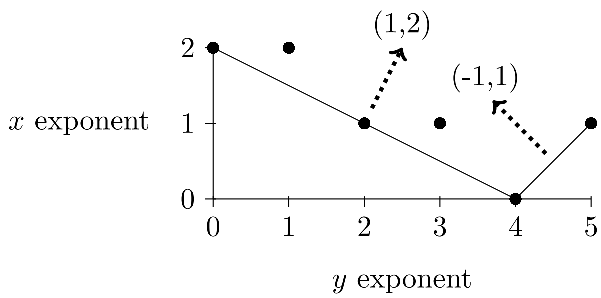


Figure 2: Newton polygon of $f = -2x^2+x^2y+xy^2+xy^3+y^4+xy^5$, with two rays of $\mathrm{trop}(V(f))$.

For $\mathbf{v} = (2,1)$, *we have* $M = 4$ *and* $\mathrm{in}_{\mathbf{v}}(f) = -2x^2 + xy^2 + y^4$, *so* $\mathbf{v}$ *is in the tropical prevariety of the system* (f)*; since the ideal* $I = \langle f \rangle$ *is principal,* $\mathbf{v}$ *must also be in the tropical variety* $\mathrm{trop}(\mathbf{v})(I)$*. The corresponding element of* $\mathrm{trop}(V(f^{(x)}))$ *is* $\frac{1}{2}$*, and as claimed in 1.6, we have*

$$\mathrm{in}_{(1,\frac{1}{2})}(f)\Big|_{x=1} = (-2x^2 + xy^2 + y^4)\big|_{x=1} = \mathrm{in}_{\frac{1}{2}}(f^{(x)}). \tag{1.11}$$

Figure 2 shows the tropical variety of f*, represented as two inward pointing normal vectors to edges of the Newton polygon. Note that we have flipped the coordinates* x *and* y *for consistency with the literature, so that the lower hull with respect to* x *is visually the lower hull in the picture.* $\triangle$

A *tropical basis* for an ideal I is a set $B \subseteq I$ such that B generates I and $\mathrm{trop}(V(B)) = \mathrm{trop}(V(I))$. In other words, for this generating set, the tropical prevariety equals the tropical variety. We can now state the two results from tropical algebraic geometry that are of greatest interest for our work, taken directly from [51]:

Theorem 1.2.5. *Let k be an arbitrary valued field. Every ideal I in the ring $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ has a finite tropical basis.*

Theorem 1.2.6 (Fundamental Theorem of Tropical Algebraic Geometry)**.** *Let k be an algebraically closed field with a nontrivial valuation, let I be an ideal in $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and let X be its variety in the algebraic torus $T^n \cong (k^*)^n$. Then the following three subsets of $\mathbb{R}^n$ coincide:*

1. *the tropical variety* $\mathrm{trop}(V(I))$*,*
2. *the set of all vectors $\mathbf{v} \in \mathbb{R}^n$ with $\mathrm{in}_{\mathbf{v}}(I) \neq \langle 1 \rangle$, and*
3. *the closure of the set of coordinatewise valuations of points in X,*

$$\mathrm{val}(X) = \{(\mathrm{val}(y_1), \ldots, \mathrm{val}(y_n)) : (y_1, \ldots, y_n) \in X\} \tag{1.12}$$

For proofs of these results, see [51]. Some general methods can be found in [16]. An algorithm for lifting points of $\mathrm{trop}(V(I))$ back to X is given in [44]; see also [46] and [59].

1.2.2.1 The Polyhedral View

Tropical hypersurfaces and their intersections have a nice geometric structure. For the trivial valuation, $\mathrm{trop}(V(f))$ is simply the $n-1$ dimensional skeleton of the *normal fan* of f's Newton polytope. To unpack this definition requires, of course, a few more definitions.

1. A *polytope* P is a bounded intersection of finitely many closed half-spaces.

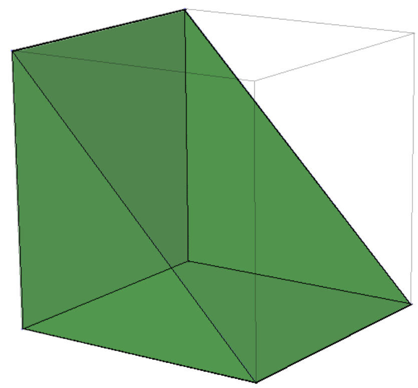


Figure 3: The Newton polytope of $f(x, y, z) = 1 + x + y + xy + z + zy \in \mathbb{C}[x, y, z]$.

2. The *initial form* of P with respect to a vector $\mathbf{v}$, denoted $\mathrm{in}_{\mathbf{v}}(P)$, is the set of points in P that are minimal with respect to $\langle \mathbf{v}, \cdot \rangle$.

3. Sets of the form $\mathrm{in}_{\mathbf{v}}(P)$ are *faces* of P.

4. The *normal cone* of a face $F \leq P$ is the closure in the Euclidean topology of the set $\{\mathbf{w} \in \mathbb{R}^n : \mathrm{in}_{\mathbf{w}}(P) = F\}$.

5. The *normal fan* of P is the set of its normal cones.

6. The *relative interior* of a cone C is its interior within its affine span.

In Figure 3 and Figure 4 we plot the Newton polytope and tropical hypersurface, respectively, of the polynomial

$$f(x, y, z) = 1 + x + y + xy + z + zy \in \mathbb{C}[x, y, z] \tag{1.13}$$

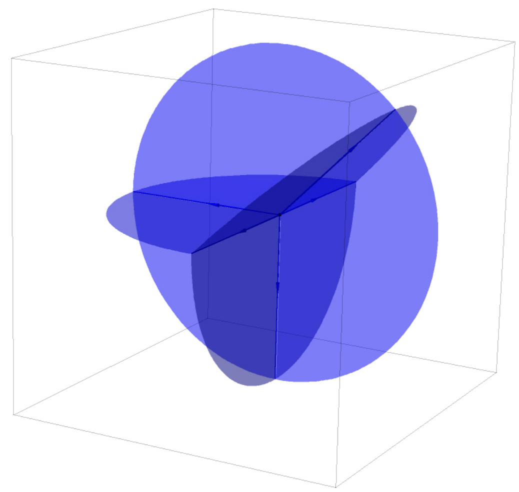


Figure 4: The tropical hypersurface $\mathrm{trop}(V(f))$ for $f(x,y,z) = 1+x+y+xy+z+zy \in \mathbb{C}[x,y,z]$ under the trivial valuation.

under the trivial valuation. The two-dimensional cones in Figure 4 are the sets of vectors giving edges as initial forms of the polytope. Under these definitions, the tropical prevariety of a system with the trivial valuation is just the intersection of polynomials' normal fans. Following Lemma 1.2.4, the tropical hypersurface of $f^{(x)}$ would simply be the intersection of the fan in Figure 4 with the plane $\mathbb{V}(x-1)$.

In view of how easy it is to switch between $\mathrm{trop}(V(f))$ and $\mathrm{trop}(V(f^{(x_i)}))$, for the most part we will work with the trivial valuation. This it matches the approach of both [43] and [40,68,69], which are the state-of-the-art for computing tropical prevarieties.

1.2.3 Polynomial Homotopy Continuation

Because one of our primary motivations is polynomial homotopy continuation, we describe it here in brief. Given a polynomial system $F(\mathbf{x})$ with finitely many solutions, one first constructs a "start system" $G(\mathbf{x})$ which is easier to solve. Using a homotopy system

$$H(\mathbf{x}, t) := \gamma(1-t)G + tF, \quad t \in [0,1],\ \gamma \in \mathbb{C}, \tag{1.14}$$

the solutions of G are tracked from $t = 0$ to 1 via predictor-corrector methods until solutions of F are obtained. The use of a random complex constant γ ensures that the solution paths are generically nonsingular.

Various methods for constructing the start system G exist, but for us the most relevant is the polyhedral method [10, 36, 78]. This approach is in many ways a precursor to tropical algebraic geometry and a generalization of the Newton-Puiseux algorithm. For roots in $(\mathbb{C}^*)^n$, it achieves a generically sharp minimal number of paths—the so-called mixed volume—which for sparse systems can be much smaller than other bounds. In the spirit of [1, 4], our work here can be seen as a generalization of the polyhedral homotopies, where instead of computing only zero-dimensional solutions sets, we study positive dimensional ones. For a general introduction to homotopy continuation, see one of [70, 71].

1.2.4 Related Work

In addition to the work mentioned above, the following is also relevant. Symbolic elimination techniques for sparse systems can be found in [33]. Tropical resultants are computed in [41].

Related polyhedral methods for sparse systems can be found in [36]. Bounds on the number of Puiseux terms necessary to determine if it is based at an isolated point are derived in [34]. The authors of [39] propose numerical methods for tropical curves. Polyhedral methods to compute tropical varieties are outlined in [16] and implemented in Gfan [42]. For computation of prevarieties see [42] or [68].

Relevant to Chapter 3 is [11], which establishes a relationship between polynomials, power series, and Toeplitz matrices. A direct method to solve block banded Toeplitz systems is presented in [21]. The book [9] is a general reference for methods related to approximations and power series. Methods for efficient manipulation of truncated series are classical [18, 48]. Studies of deformation methods in symbolic computation appeared in [17], [20], and [31]. In particular, the use of power series and Padé approximants in [45] in the purely symbolic context stimulated our development of the methods of Chapter 3.

1.3 Problems and Solutions

1.3.1 Problem Statement

Now that the background framework has been established, we come to our problem statement. Generally speaking, the problem is to find series expansions of space curves. To be more precise, we start with a polynomial system $\mathbf{f}$ such that $\mathbb{V}(\mathbf{f})$ is one dimensional, and hope to find series expansions for the components of $\mathbb{V}(\mathbf{f})$. Our questions then become,

1. When is computing only the tropical prevariety sufficient?

2. If the prevariety is insufficient, how can this be detected, and what can be done about it?

3. Is there a more effective way of finding terms in the expansion than the term-by-term methodology of the Newton-Puiseux algorithm?

4. More specifically, can Newton's method be adapted to expand a series with quadratic convergence?

1.3.2 Our Contributions

Our answer to the first two questions in 1.3.1 is the content of Chapter 2. We argue that for generic coefficients, the prevariety contains sufficient information to apply a higher-dimensional analogue of the Newton-Puiseux method. When this does fail—i.e. when a tropism is in the relative interior of a cone of the prevariety, so that the leading Puiseux powers cannot be immediately determined— we provide a numerical alternative to symbolic approaches such as [16], using polyhedral end games to recover the hidden tropisms.

The latter two questions are answered in the affirmative in Chapter 3. We first apply linearization, which takes a matrix problem over $\mathbb{C}\{\{x\}\}$ and rewrites it as a system of linear equations over $\mathbb{C}$. We are then able to transform the problem so that applying the Gauss-Newton algorithm is reduced to structured linear algebra. Finally, we provide a simple and complete characterization of when tropical methods are necessary, versus when only a point in $\mathbb{C}^n$ is needed to start the Gauss-Newton algorithm.

CHAPTER 2

PRETROPISMS AND TROPISMS IN HIGHER DIMENSIONS

2.1 Introduction

In this chapter, we examine an obstacle to the generalization of the Newton-Puiseux algorithm, a so-called "hidden cone" of the tropical prevariety. This is a tropism that is not a one-dimensional cone of the prevariety. Our solution is based on material previously published in [13]. It involves applying polyhedral methods in order to complete the information lacking in the tropical prevariety. We are driven by the following questions:

1. If only the space curves are of interest, can we ignore the higher dimensional cones of pretropisms?
2. If some tropisms lie in the relative interior of higher dimensional cones of the prevariety, is it still possible to compute Puiseux series solutions for these space curves?

The layout of this chapter is as follows. In the first section we illustrate the general goal using Viviani's curve, which will be a running example throughout this thesis. We then lay out some of the assumptions necessary for the study. Next we illustrate the "hidden cone" problem with several examples, and provide a result about genericity conditions for when the hidden cone problem does not occur. For the non-generic case, we give an overview of current symbolic methods. And finally, we introduce polyhedral endgames for recovering tropisms contained in higher dimensional cones, and give some experimental results and timings.

2.2 Puiseux Series

When working with Puiseux series we apply a hybrid method, combining exact and approximate calculations. Figure 6 shows the plot, in black, of Viviani's curve, defined as the intersection of the sphere $f = x_1^2+x_2^2+x_3^2-4 = 0$ and the cylinder $g = (x_1-1)^2+x_2^2-1 = 0$ such that the two are mutually tangent at a point. Let $\mathbf{f}$ denote the polynomial system consisting of f and g, which we consider as polynomials in $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, x_2, x_3]$.

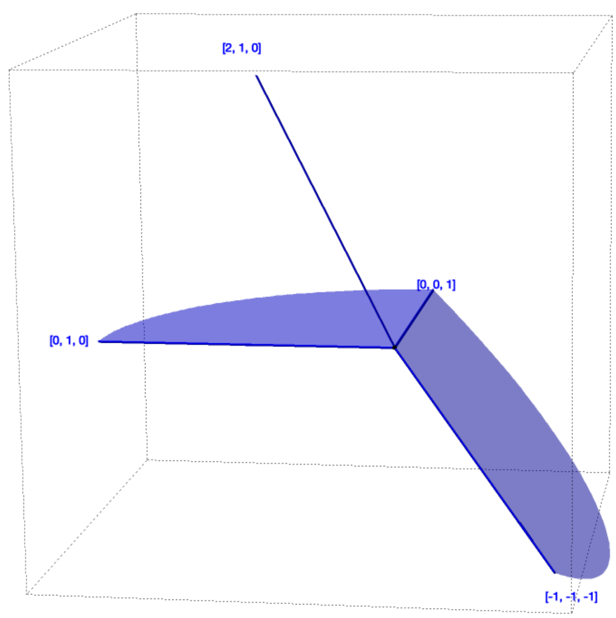


Figure 5: The tropical prevariety of Viviani's curve under the trivial valuation.

The tropical prevariety of $\mathbf{f}$ is shown in Figure 5. If we take x_1 as our series parameter, there is only one pretropism $(\frac{1}{2}, 0)$ in $\mathrm{trop}(V(\mathbf{f}^{(x_1)}))$. The corresponding ray $\mathbf{v} = (2, 1, 0) \in \mathrm{trop}(V(\mathbf{f}))$ gives initial forms of f and g respectively as $x_3^2 - 4$ and $x_2^2 - 2x_1$. For traditional Puiseux series, one would choose to set $x_1 = 1$, obtaining the four solutions $(1, \pm\sqrt{2}, \pm 2)$ and leading terms $(t^2, \pm\sqrt{2}t, \pm 2)$. If we instead use $x_1 = 2$, we obtain rational coefficients and the following partial expansion:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t^2 \\ 2t - t^3 - \frac{1}{4}t^5 - \frac{1}{8}t^7 - \frac{5}{64}t^9 \\ 2 - t^2 - \frac{1}{4}t^4 - \frac{1}{8}t^6 - \frac{5}{64}t^9 \end{bmatrix}. \tag{2.1}$$

The plot of several Puiseux approximations to Viviani's curve is shown in gray in Figure 6.

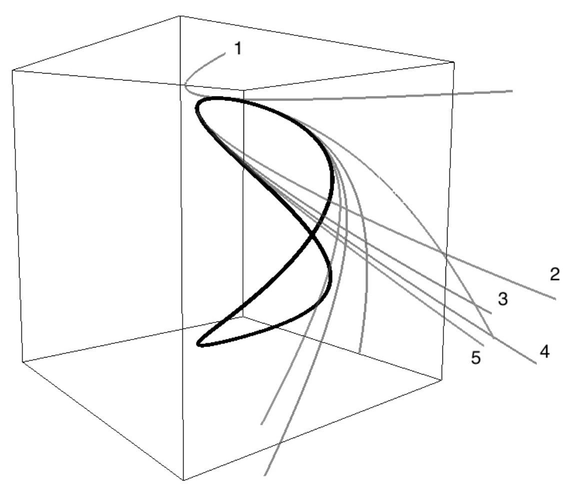


Figure 6: Viviani's curve with improving Puiseux series approximations, labeled with the number of terms used to plot each one.

If we shift the Viviani example so that its self-intersection is at the origin, we obtain the following:

$$\mathbf{f}(\mathbf{x}) = \begin{cases} x_1^2 + x_2^2 + x_3^2 + 4x_1 = 0 \\ x_1^2 + x_2^2 + 2x_1 = 0 \end{cases} \tag{2.2}$$

An examination of the first few terms of the Puiseux series expansion for this system, combined with the On-Line Encyclopedia of Integer Sequences [58] and some straightforward algebraic manipulation, allows us to hypothesize the following exact parameterization of the variety:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t^2 \\ 2\frac{t^3}{1+\sqrt{1-t^2}} - 2t \\ \pm 2t \end{bmatrix}. \tag{2.3}$$

We can confirm that this is indeed right via substitution. While this method is of course not possible in general, it does provide an example of the potential usefulness of Puiseux series computations for some examples.

2.3 Assumptions and Setup

Our object of study is space curves, by which we mean 1-dimensional varieties in $\mathbb{C}^n$. Because Puiseux series computations take one variable to be a free variable, we require that the curves not lie inside $V(\langle x_i \rangle)$ for some i; without loss of generality we choose to use the first variable. Some results require that the curve be in Noether position with respect to x_1, meaning that the degree of the variety is preserved under intersection with $x_1 = \lambda$ for a generic $\lambda \in \mathbb{C}$. It is of course possible to apply a random coordinate transformation to obtain Noether position, but we then lose the sparsity of the system's exponent support structure, which is what makes polyhedral methods effective.

2.4 Some Motivating Examples

In this section we illustrate the "hidden cone" problem with some simple examples, first in 3-space, and then with a family of curves in any dimensional space.

2.4.1 In 3-Space

Our first example is the system

$$\mathbf{f}(\mathbf{x}) = \begin{cases} x_1x_3 - x_2x_3 - x_3^2 + x_1 = 0 \\ x_3^3 - x_1x_2 - x_2x_3 - x_3^2 - x_1 = 0 \end{cases} \tag{2.4}$$

which has an irreducible quartic and the second coordinate axis $(0, x_2, 0)$ as its solutions. Because the line lies in the first coordinate plane $x_1 = 0$, the system is not in Noether position with respect to the first variable. Therefore, our methods will ignore this part of the solution set. The algorithms of [32] can be applied to compute components inside coordinate planes. Computing a primary decomposition yields the following alternative, which lacks the portion in the first coordinate plane:

$$\tilde{\mathbf{f}}(\mathbf{x}) = \begin{cases} x_1x_3 - x_2x_3 - x_3^2 + x_1 \\ x_1x_2 - x_2^2 - x_2x_3 + x_3^2 + x_1 - 2x_2 - 2x_3 \\ x_3^3 - x_2^2 - 2x_2x_3 - 2x_2 - 2x_3 \end{cases} \tag{2.5}$$

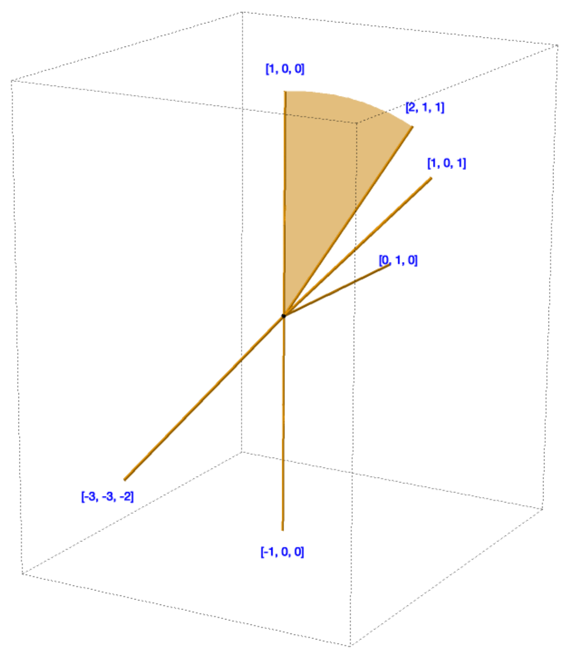


Figure 7: The tropical prevariety $\text{trop}(V(\tilde{\mathbf{f}}))$ of the system (2.5).

The tropical prevariety $\text{trop}(V(\tilde{\mathbf{f}}(\mathbf{x})))$, seen in Figure 7, contains the rays $(2, 1, 1)$, $(1, 0, 0)$, and $(1, 0, 1)$; because our Puiseux series start their development at $x_1 = 0$, rays that have a zero or negative value for their first coordinate have been discarded. The tropical variety however contains the ray $(3, 1, 1)$ instead of $(2, 1, 1)$, leading to the series expansion

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 108t^3 \\ t - 3t^2 - 15t^3 + 27t^4 + 36t^5 \\ -t - 3t^3 - 18t^4 + 18t^5 + 162t^6 \end{bmatrix}. \tag{2.6}$$

This ray is a positive combination of $(2,1,1)$ and $(1,0,0)$. In other words, it is possible for the 1-dimensional cones of the tropical prevariety to fail to be in the tropical variety, and for rays in the tropical variety to "hide" in the higher-dimensional cones of the prevariety.

2.4.2 In Any Dimensional Space

This problem can also occur in arbitrary dimensions, as seen in the class of examples

$$\mathbf{f}(\mathbf{x}) = \begin{cases} x_1^2 - x_1 + x_2 + x_3 + \cdots + x_n &= 0 \\ x_2^2 + x_1 + x_2 + x_3 + \cdots + x_n &= 0 \\ x_3^2 + x_1 + x_2 + x_3 + \cdots + x_n &= 0 \\ &\vdots \\ x_{n-1}^2 + x_1 + x_2 + x_3 + \cdots + x_n &= 0. \end{cases} \tag{2.7}$$

The support of the linear portions of the polynomials each span an $n-1$ dimensional hyperplane. Since the ray $\mathbf{1} = (1,1,1,\ldots,1)$ has the linear portions as its initial form, they must be facets and $\mathbf{1}$ must be a 1-dimensional cone of the prevariety. The ray $\mathbf{1}$ is not, however, in the tropical variety, since the initial form system $\mathrm{in}_{\mathbf{1}}(\mathbf{f})$ contains the monomial x_1. For $n \leq 12$ we computed that instead the ray $(2,1,\ldots,1)$ is in the tropical variety, hiding in the cone of the prevariety generated by $\mathbf{1}$ and $(1,0,0,\ldots,0)$. The tropical prevariety $\mathrm{trop}(V(\mathbf{f}))$ for $n=3$ can be seen in Figure 8.

2.5 The Generic Case

This hiding of tropisms in the higher dimensional cones of the prevariety is problematic, as finding the tropical variety may require more expensive symbolic computations. For a compar-

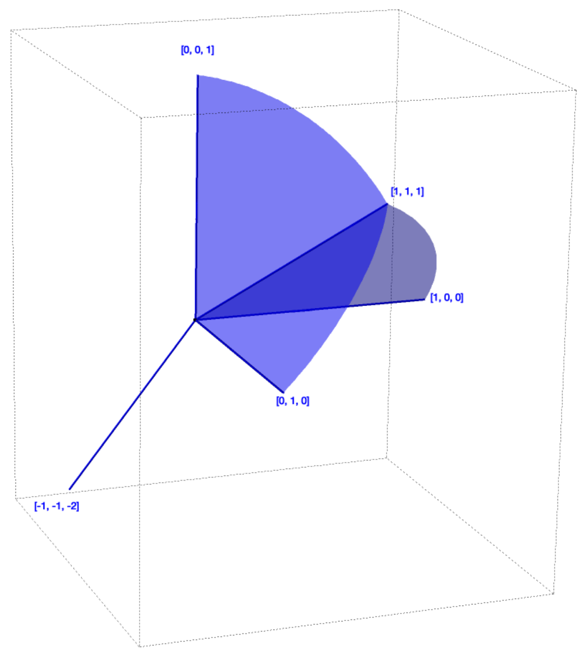


Figure 8: The tropical prevariety $\text{trop}(V(\mathbf{f}))$ of the system (2.7) for $n = 3$.

ison between various approaches to computing it, see Section 2.8. Fortunately, the hidden cone problem does not occur in general, as the next result will show.

Proposition 2.5.1. *For n equations in $n+1$ unknowns with generic coefficients, the set of ray generators of the tropical prevariety contains the tropical variety.*

A version of this was proved in [36] in the context of polyhedral homotopies; this was in turn strongly based on [10]. It should be noted that our use of generic here refers to the coefficients, and is not to be confused with generic tropical varieties as seen in [64], which are tropical varieties of ideals under a generic linear transformation of coordinates.

Proof. The tropical prevariety always contains the tropical variety. We simply want to show that all of the rays of the tropical variety show up in the prevariety as ray generators, and not as members of the higher-dimensional cones. Let $I = \langle p_1, \dots, p_n \rangle \subseteq \mathbb{C}[x_0, \dots, x_n]$, and let $\mathbf{w}$ be a ray in the tropical prevariety but not one of its ray generators. We want to show that $\mathbf{w}$ is not in the tropical variety, or equivalently that $\mathrm{in}_{\mathbf{w}}(I)$ contains a monomial. We will do so by showing that $I_{\mathbf{w}} := \langle \mathrm{in}_{\mathbf{w}}(p_1), \dots, \mathrm{in}_{\mathbf{w}}(p_n) \rangle$ contains a monomial, which suffices since this ideal is contained in $\mathrm{in}_{\mathbf{w}}(I)$.

Suppose $I_{\mathbf{w}}$ contains no monomial. Then $(x_0 x_1 \cdots x_n)^k \notin I_{\mathbf{w}}$ for any k. By Hilbert's Nullstellensatz $V := \mathbb{V}(I_{\mathbf{w}}) \nsubseteq \mathbb{V}(x_0 x_1 \cdots x_n)$, i.e. V is not contained in the union of the coordinate hyperplanes. Then there exists $a = (a_0, \dots, a_n) \in V$ such that all coordinates of a are all nonzero. Since $\mathbf{w}$ lies in the interior a cone of dimension at least 2, the generators of $I_{\mathbf{w}}$ are homogeneous with respect to at least two linearly independent rays $\mathbf{u}$ and $\mathbf{v}$. Thus $(\lambda^{\mathbf{u}_0} \mu^{\mathbf{v}_0} a_0, \dots, \lambda^{\mathbf{u}_n} \mu^{\mathbf{v}_n} a_n) \in V$ for all $\lambda, \mu \in \mathbb{C} \setminus \{0\}$ where the $\mathbf{u}_i, \mathbf{v}_i$ are the components of $\mathbf{u}$ and $\mathbf{v}$, and V contains a toric surface. However, if we intersect with a random hyperplane, Bernstein's theorem B [10] implies that the result will be a finite set of points, with the possibility of additional components that must be contained in the coordinate planes. Hence V cannot contain a component of dimension > 1 outside of the coordinate planes, and we have a contradiction. □

Remark 1. *The above result is reproduced directly from [13], but note that this result easily generalizes: for n equations in $n + d$ unknowns, the tropical variety is contained in the set of prevariety cones of dimension $\leq d$. This follows by a similar argument, where instead of ruling* *out cones of dimension ≥ 2 which have at least 2-dimensional homogeneity, one rules out cones of dimension greater than d.*

2.6 Current Approaches

In [16] a method is given for computing the tropical variety of an ideal I defining a curve. It involves appending witness polynomials from I to a list of its generators such that for this new set, the tropical prevariety equals the tropical variety. Recall that such a set is called a *tropical basis*. Each additional polynomial rules out one of the cones in the original prevariety that does not belong in the tropical variety. As stated in [16] only finitely many additional polynomials are necessary, since the prevariety has only finitely many cones.

The algorithm runs as follows. For each cone C in the tropical prevariety, we choose a generic element $\mathbf{w}$ in the relative interior of C. We check whether $\mathrm{in}_{\mathbf{w}}(I)$ contains a monomial by saturating with respect to m, the product of ring variables; the initial ideal contains a monomial if and only if this saturation ideal is equal to (1). If $\mathrm{in}_{\mathbf{w}}(I)$ does not contain a monomial, the cone C belongs in our tropical variety. If it does, we check whether $m^i \in I$ for increasing values of i until we find a monomial $m' \in \mathrm{in}_{\mathbf{w}}(I)$. Finally, we append $m' - h$ to our list of basis elements, where h is the reduction of m with respect to a Gröbner basis of I under any monomial order that refines $\mathbf{w}$. For $\mathbf{w}$ to define a global monomial order, and thus allow a Gröbner basis, it may be necessary to homogenize the ideal first.

Bounding the complexity of this algorithm is beyond the scope of this work, but for each cone it requires computing a Gröbner basis of I as well as another (possibly faster) basis when calculating the saturation to check if the initial ideal contains a monomial. In some cases we may only be concerned about tropisms hiding in a particular higher-dimensional cone of the prevariety, such as with our running example (2.7). Here it is reasonable to perform only one step of this algorithm, namely looking for a witness for a single cone, which could be significantly faster. However, this has the disadvantage of introducing more 1-dimensional cones into the prevariety. More details, including some timing comparisons, will be given in Section 2.8.

2.7 Polyhedral End Games

Polyhedral end games [37] use extrapolation methods to numerically estimate the winding number of solution paths defined by a homotopy. Their traditional purpose is to complete the tracking of solution paths towards a singular solution of the target system. The leading exponents of the Puiseux series can be recovered by taking differences of the logarithms of the magnitudes of the coordinates of the solution paths.

Even in cases such as our illustrative example, where the given polynomials have a prevariety that is insufficient to compute all tropisms, a polyhedral end game is capable of computing them. The setup is similar to that of [76], arising in a numerical study of the asymptotics of a space curve. We define the system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$:

$$\left\{ \begin{array}{c} \mathbf{f}(\mathbf{x}) = \mathbf{0} \\ tx_1 + (1-t)(x_1 - \gamma) = 0, \quad \gamma \in \mathbb{C} \setminus \{0\}. \end{array} \right. \tag{2.8}$$

As t moves from 0 to 1, the original variety $\mathbb{V}(\mathbf{f})$ is intersected with the hyperplane $\mathbb{V}(x_1 - \gamma - t)$, i.e. a plane perpendicular to the first coordinate axis.

It is important to note that we never allow t to actually reach 1; in the polyhedral end games of [37], the step size decreases in a geometric ratio. Another thing to note is that the constant γ in (2.8) is a randomly generated complex number. This implies that for $x_1 = \gamma$, the polynomial system in (2.8) for $t = 0$ has as many isolated solutions, counted with multiplicity, as the degree of the projection of the space curve onto the first coordinate axis. Because there can only be finitely many $c \in \mathbb{C}$ for which $\mathbb{V}(\mathbf{f}) \cap \mathbb{V}(x_1 - c)$ is singular, for $t < 1$ the introduction of γ assures that the points remain generic. However, the numerical condition numbers are expected to blow up as t approaches one.

This deteriorating numerical ill conditioning can be counteracted by the use of multiprecision arithmetic. For example, condition numbers larger than 10^8 make results unreliable in double precision. However in double double precision, much higher condition numbers can be tolerated, typically up to 10^{16}, and up to 10^{32} for quad double precision. Interpreting the inverse of the condition number as the distance to a singular solution, with multiprecision arithmetic we can compute more points more accurately as needed in the extrapolation to estimate winding numbers.

An additional difficulty arises when a path diverges to infinity, which manifests itself as a tropism with negative coordinates. This will already be detected in the prevariety, however, as a cone which contains rays with negative components. Reformulating the problem in weighted projective space using a unimodular coordinate transformation circumvents the problem and brings the cone into the finite case above.

2.8 Computational Experiments

2.8.1 Symbolic Methods

To substantiate the claim that finding the tropical variety is computationally expensive, we calculated tropical bases of the system (2.7) for various values of n. The symbolic computations of tropical bases was done with Gfan [42]. Times are displayed in Table I. The computations were executed on an Intel Xeon E5-2670 processor running RedHat Linux. As is clear from the table, as the dimension grows for this relatively simple system, computation time becomes prohibitively large.

TABLE I: Execution times, in seconds, of the computation of a tropical basis for the system (2.7); averages of 3 trials.

n	3	4	5	6	7
time	0.052	0.306	2.320	33.918	970.331

As mentioned in Section 2.6, an alternative to computing the tropical basis is to only calculate the witness polynomial for a particular cone of the tropical prevariety. We implemented this algorithm in Macaulay2 [28] and applied it to (2.7) to cut down the cone generated by the rays $(1, 1, \ldots, 1)$ and $(1, 0, 0, \ldots, 0)$. In all the cases we tried, the new prevariety contained the ray $(2, 1, \ldots, 1)$, as we expected.

From Table II it is clear that this has a significant speed advantage over computing a full tropical basis. However, it has the disadvantage of introducing many more rays into the prevariety. The number can vary depending on the random ray chosen in the cone, so we listed some of the values we obtained over several trials. We only computed up through dimension 10 because the prevariety computations were excessive for higher dimensions.

TABLE II: Execution times of the computation of a witness polynomial for the cone generated by $(1, 1, \ldots, 1), (1, 0, \ldots, 0)$ of the system (2.7); averages of 3 trials. The third column lists the number of rays in the fan obtained by intersecting the original prevariety with the normal fan of the witness polynomial.

dim	time (s)	#rays in new fan
3	0.004	4, 5
4	0.011	10, 11
5	0.004	13, 14
6	0.009	27, 49
7	0.033	13, 25, 102
8	0.170	124, 401, 504
9	0.963	758, 1076
10	10.749	514, 760, 1183, 2501
11	131.771	
12	1131.089	

2.8.2 Our Approach

The polyhedral end games were performed with version 2.4.10 of PHCpack [75], upgraded with double double and quad double arithmetic using QDlib [35]. For the first motivating example (2.4) in 3-space, there are four solutions when $x_1 = \gamma$. The tropism $(3, 1, 1)$, with winding number 3, is recovered when running a polyhedral end game, tracking four solution paths. Even in quad double precision (double precision already suffices), the running time is a couple of hundred milliseconds.

Table III shows execution times for the family of polynomial systems in (2.7). The computations were executed on one core of an Intel Xeon E5-2670 processor, running RedHat Linux.

TABLE III: Execution times on tracking d paths in n-space with a polyhedral end game. The reported time is the elapsed CPU user time, in seconds. The last column represents the average time spent on one path.

n	d	time	time/d
4	4	0.012	0.003
5	8	0.035	0.006
6	16	0.090	0.007
7	32	0.243	0.010
8	64	0.647	0.013
9	128	1.683	0.016
10	256	4.301	0.017
11	512	7.507	0.015
12	1024	27.413	0.027

All directions computed with double precision at an accuracy of 10^{-8}. For this family of systems, double precision sufficed to accurately compute the tropism $(2, 1, \dots, 1)$. Clearly, these times are significantly smaller than the time required to compute a full tropical basis.

2.9 Conclusions

The tropical prevariety provides candidate tropisms for Puiseux series expansions of space curves. As shown in [1, 4] on the cyclic n-root problems, the pretropisms may directly lead to series developments for the positive dimensional solution sets. In this chapter we studied cases where tropisms are in the relative interior of higher-dimensional cones of the tropical prevariety. If the tropical prevariety contains a higher dimensional cone and Puiseux series expansion fails at one of the cone's generating rays, then a polyhedral end game can recover the tropisms in the interior of that higher dimensional cone of pretropisms. As our example shows, this takes drastically less time than computing the tropical variety via a tropical basis, especially as dimension grows. It is also faster than finding a witness polynomial for just that particular cone, and avoids the issue of adding rays to the tropical prevariety.

CHAPTER 3

GAUSS-NEWTON FOR POWER SERIES

3.1 Introduction

In this chapter, we seek to define an efficient, numerically stable, and robust algorithm to apply the Gauss-Newton algorithm [12, 57] over power or Laurent series. We do so with an eye toward computing power series expansions for space curves, particularly those given as solution curves of polynomial homotopies. This chapter is based upon work published in [14]. We begin this section with some motivating examples, before stating the problem in more detail and giving an overview of the rest of the chapter.

3.1.1 Motivating Example: Padé Approximant

One motivation for finding a series solution is that once it is obtained, one can directly compute the associated Padé approximant, which often has much better convergence properties. Padé approximants [9] are applied in symbolic deformation algorithms [45]. In this section we reproduce [9, Figure 1.1.1] in the context of polynomial homotopy continuation. Consider the homotopy

$$(1-t)(x^2-1) + t(3x^2-3/2) = 0. \tag{3.1}$$

The function $x(t) = \left(\dfrac{1+t/2}{1+2t}\right)^{1/2}$ is a solution of this homotopy.

Its second order Taylor series at $t = 0$ is $s(t) = 1 - 3t/4 + 39t^2/32 + O(t^2)$. The Padé approximant of degree one in numerator and denominator is $q(t) = \dfrac{1 + 7t/8}{1 + 13t/8}$. In Figure 9 we see that the series approximates the function only in a small interval and then diverges, whereas the Padé approximant is more accurate.

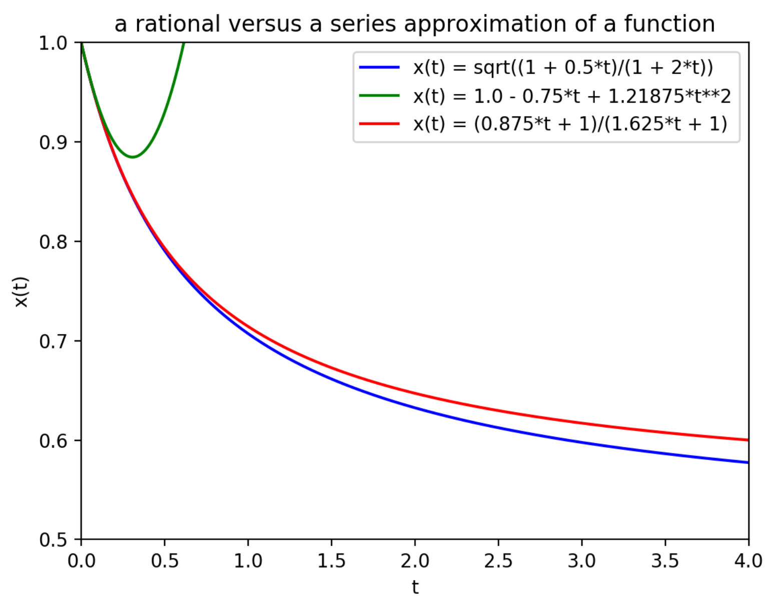


Figure 9: Comparing a Padé approximant to a series approximation shows the promise of applying Padé approximants as predictors in numerical continuation methods.

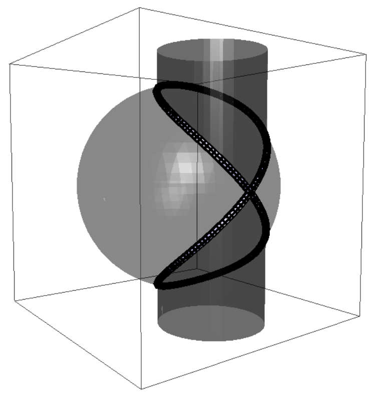


Figure 10: Viviani's curve as the intersection of a sphere with a cylinder.

3.1.2 Motivating Example: Viviani's Curve

We revisit Viviani's curve as seen in 2.2. Here we will define it as the intersection of the sphere $(x_1+2)^2+x_2^2+x_3^2=4$ and the cylinder $(x_1+1)^2+x_2^2=1$ such that the self-intersection is at the origin; see Figure 10. Our methods will allow us to find a series expansion around any point on a 1-dimensional variety, assuming we have suitable starting information. For example, the origin $(0,0,0)$ is a singular point of the curve. If we apply our methods at this point, we obtain the following series solution for x_1, x_2, x_3:

$$\begin{cases} -2t^2 \\ 2t - t^3 - \frac{1}{4}t^5 - \frac{1}{8}t^7 - \frac{5}{64}t^9 - \frac{7}{128}t^{11} - \frac{21}{512}t^{13} - \frac{33}{1024}t^{15} \\ 2t \end{cases} \tag{3.2}$$

This solution is plotted in Figure 11 for a varying number of terms. To check the correctness, we can substitute (3.2) into the original equations, obtaining series in $O(t^{18})$. The vanishing of the lower-order terms confirms that we have indeed found an approximate series solution. Such a solution, possibly transformed into an associated Padé approximant, would allow for path tracking starting at the origin.

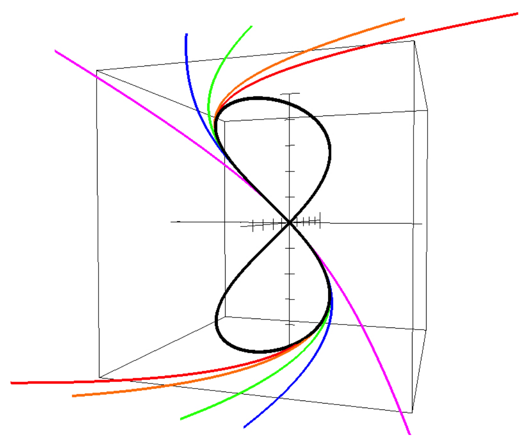


Figure 11: Viviani's curve, with improving series approximations and thus more accurate predictions for points on the curve.

3.1.3 Problem Setup

For a polynomial system $\mathbf{f} = (f_1, f_2, \ldots, f_m)$ where each $f_i \in \mathbb{C}[t, x_1, \ldots, x_n]$ or $\mathbb{C}[t^{\pm 1}, x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, recall that the solution variety $\mathbb{V}(\mathbf{f})$ is the set of points $\mathbf{p} \in \mathbb{C}^{n+1}$ such that $f_1(\mathbf{p}) = \cdots = f_m(\mathbf{p}) = 0$. Let $\mathbf{f}$ be a system such that the solution variety is 1-dimensional over $\mathbb{C}$ and is not contained in the $t = 0$ coordinate hyperplane. We seek to understand $\mathbb{V}(\mathbf{f})$ by studying $\mathbf{f}^{(t)}$, i.e. the associated system of polynomials in $x_1 \dots x_n$ with coefficients in the field of Puiseux series in t. In fact, we will consider all calculations to occur in $\mathbb{C}((t))$, since the exponents of the f_i (and therefore the $f_i^{(t)}$) are all integers.

Our approach is to use Newton iteration on the system $\mathbf{f}^{(t)}$. Namely, we find some starting $\mathbf{z} \in \mathbb{C}((t))^n$ and repeatedly solve

$$J_{\mathbf{f}^{(t)}}(\mathbf{z})\Delta\mathbf{z} = -\mathbf{f}^{(t)}(\mathbf{z}) \tag{3.3}$$

for the update $\Delta\mathbf{z}$ to $\mathbf{z}$, where $J_{\mathbf{f}^{(t)}}$ is the Jacobian matrix of $\mathbf{f}^{(t)}$ with respect to $x_1, \dots, x_n$. This is a system of equations that is linear over $\mathbb{C}((t))$, so the problem is well-posed. Computationally speaking, one approach to solving it would be to overload the operators on (truncated) power series and apply basic linear algebra techniques. A main point of the chapter is that this method can be improved upon.

Of course, applying Newton's method requires a starting guess. In most cases this can just be a point $\tilde{\mathbf{p}} = (p_1, \dots, p_n)$ such that $\mathbf{p} = (0, p_1, \dots, p_n)$ is in $\in \mathbb{V}(\mathbf{f})$. However, if $\mathbf{p}$ is a singular point, this is insufficient. In addition, $\mathbf{p}$ could be a branch point (which we discuss in Section 3.2.2), in which case it is also not enough to use as the starting guess for Newton's method.

We solve two problems in this chapter. First, we find an effective way to perform the Newton step; the framework is established in Section 3.2.1, and our solution is laid out in Section 3.2.3.

And second, we discuss the prelude to Newton's method in Section 3.2.2, characterizing when techniques from tropical geometry, such as those in Chapter 2, are needed to transform the problem and obtain the starting guess.

3.2 Our Solution

3.2.1 The Newton Step

Solving the Newton step (3.3) amounts to solving a linear system

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{3.4}$$

over the field $\mathbb{C}((t))$. Our first step is linearization, which turns a vector of series into a series of vectors, and likewise for a matrix series. In other words, we refactor the problem and think of $\mathbf{x}$ and $\mathbf{b}$ as in $\mathbb{C}^n((t))$ instead of $\mathbb{C}((t))^n$, and $\mathbf{A}$ as in $\mathbb{C}^{n\times n}((t))$ instead of $\mathbb{C}((t))^{n\times n}$.

Suppose that a is the lowest order of a term in $\mathbf{A}$, and b the lowest order of a term in $\mathbf{b}$. Then we can write the linearized

$$\mathbf{A} = A_0 t^a + A_1 t^{a+1} + \dots, \tag{3.5}$$

$$\mathbf{b} = \mathbf{b}_0 t^b + \mathbf{b}_1 t^{b+1} + \dots, \text{ and} \tag{3.6}$$

$$\mathbf{x} = \mathbf{x}_0 t^{b-a} + \mathbf{x}_1 t^{b-a+1} + \dots \tag{3.7}$$

where $A_i \in \mathbb{C}^{n\times n}$ and $\mathbf{b}_i, \mathbf{x}_i \in \mathbb{C}^n$. Expanding and equating powers of t, the linearized version of (3.4) is therefore equivalent to solving

$$\begin{aligned} A_0\mathbf{x}_0 &= \mathbf{b}_0 \\ A_0\mathbf{x}_1 &= \mathbf{b}_1 - A_1\mathbf{x}_0 \\ A_0\mathbf{x}_2 &= \mathbf{b}_2 - A_1\mathbf{x}_1 - A_2\mathbf{x}_0 \\ &\vdots \\ A_0\mathbf{x}_d &= \mathbf{b}_d - A_1\mathbf{x}_{d-1} - A_2\mathbf{x}_{d-2} - \cdots - A_d\mathbf{x}_0 \end{aligned} \tag{3.8}$$

for some d. This can be written in block matrix form as

$$\begin{bmatrix} A_0 & & & & \\ A_1 & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ \vdots & \vdots & \vdots & \ddots & \\ A_d & A_{d-1} & A_{d-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_d \end{bmatrix} = \begin{bmatrix} \mathbf{b}_0 \\ \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_d \end{bmatrix}. \tag{3.9}$$

For the remainder of this chapter, we will use $\mathbf{z}$ and $\Delta\mathbf{z}$ to denote vectors of series, while $\mathbf{x}$ and $\Delta\mathbf{x}$ will denote their linearized counterparts, that is, series which have vectors for coefficients.

Example 3. *Let*

$$\mathbf{f} = (2t^2 + tx_1 - x_2 + 1, x_1^3 - 4t^2 + tx_2 + 2t - 1). \tag{3.10}$$

Starting with $\mathbf{z} = (1,1)$, *the first Newton step* $J_{\mathbf{f}^{(t)}}(\mathbf{z})\Delta\mathbf{z} = -\mathbf{f}^{(t)}(\mathbf{z})$ *can be written:*

$$\begin{bmatrix} t & -1 \\ 3 & t \end{bmatrix} \Delta\mathbf{z} = - \begin{bmatrix} t + 2t^2 \\ 3t - 4t^2 \end{bmatrix}. \tag{3.11}$$

To put in linearized form, we have $a = 0$, $b = 1$,

$$A_0 = \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{3.12}$$

$$\mathbf{b}_0 = \begin{bmatrix} -1 \\ -3 \end{bmatrix}, \text{ and } \mathbf{b}_1 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}. \tag{3.13}$$

Since A_0 *is regular, we can solve in staggered form as in (3.8), which yields the next term:*

$$\Delta\mathbf{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t. \tag{3.14}$$

After another iteration, our series solution is

$$\begin{bmatrix} 1 - t \\ 1 + t + t^2 \end{bmatrix}. \tag{3.15}$$

In fact this is the entire series solution for $\mathbf{f}$ *— substituting (3.15) into* $\mathbf{f}$ *causes both polynomials to vanish completely.* $\triangle$

Remark 2. *When the series solution of a system is finite, as above, solving (3.4) is equivalent to applying Hermite interpolation at* 0*. To see this, observe that a series*

$$\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{x}_1 t + \mathbf{x}_2 t^2 + \mathbf{x}_3 t^3 + \cdots + \mathbf{x}_k t^k + \cdots \tag{3.16}$$

can be trivially rewritten via its Maclaurin expansion as

$$\mathbf{x}(t) = \mathbf{x}(0) + \mathbf{x}'(0)t + \frac{1}{2}\mathbf{x}''(0)t^2 + \frac{1}{3!}\mathbf{x}'''(0)t^3 + \cdots + \frac{1}{k!}\mathbf{x}^{(k)}(0)t^k + \cdots, \tag{3.17}$$

where $\mathbf{x}^{(k)}(0)$ *denotes the* k*-th derivative of* $\mathbf{x}(t)$ *evaluated at zero. This implies that*

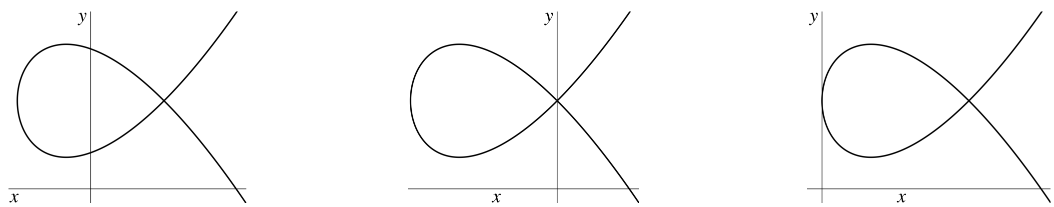
$$\mathbf{x}_k = \frac{1}{k!}\mathbf{x}^{(k)}(0), \quad k = 0, 1, \ldots, \tag{3.18}$$

so solving (3.4) up to degree d *is equivalent to saying that all derivatives up to degree* d *of the parameterization* $\mathbf{x}(t)$ *match the solution at* $t = 0$*; this is precisely Hermite interpolation. If the solution has finitely many terms, then it will be obtained if (3.4) is solved up to its degree.*

3.2.2 The Starting Guess, and Related Considerations

Our hope is that a solution $\mathbf{z}(t)$ of $\mathbf{f}^{(t)}$ parameterizes the curve in some neighborhood of a point $\mathbf{p} \in \mathbb{V}(\mathbf{f})$. In other words, if π is the projection map of $\mathbb{V}(\mathbf{f})$ onto the t-coordinate axis, then $\mathbf{z}(t)$ should be a branch of π^{-1}.

It is natural to think that there are two scenarios for the starting point $\mathbf{p} \in \mathbb{V}(\mathbf{f})$, namely that it is a regular point or it is singular. And indeed, when $\mathbf{p}$ is singular, tropical methods



a general point a singularity a branch point

Figure 12: Lifting $x = 0$ to three different types of point. In general, the line $x = 0$ intersects the curve at regular points. If the curve intersects itself for $x = 0$, we are at a singular point. The curve turns at a branch point.

are required. Intuitively speaking, when at a singular point, knowing just the point itself is insufficient to determine the series; higher-derivative information is required. Observe the second frame of Figure 12.

The point $\mathbf{p}$ being regular, however, is not enough. Consider the third frame of Figure 12. Here $x = 0$ cannot be lifted because the origin is a *branch point* of the curve. In other words, the derivative at $\mathbf{p}$ in terms of t is undefined, so a Taylor series in t is impossible without a transformation of the problem.

The proper way to check if Newton's method can be applied directly to $\mathbf{p}$, or whether tropical methods are needed, is by checking if $\mathbf{p}$ is a singular point of $\mathbb{V}(\mathbf{f}) \cap \mathbb{V}(t)$. Setting $\mathbf{f}_{aug} = (t, f_1, \ldots, f_n)$, we have $\mathbb{V}(\mathbf{f}_{aug}) = \mathbb{V}(\mathbf{f}) \cap \mathbb{V}(t)$. We can thus use $\mathbb{V}(\mathbf{f}_{aug})$ to distinguish the first frame of Figure 12 from the latter two. This is summarized and proven in the following.

Proposition 3.2.1. *Let* $\mathbf{p} = (0, p_1, \ldots, p_n) \in \mathbb{V}(\mathbf{f})$*, and set* $\tilde{\mathbf{p}} = (p_1, \ldots, p_n)$*. Then* $\mathbf{p}$ *is a regular point of* $\mathbb{V}(\mathbf{f}_{aug})$ *if and only if for every step of Newton's method applied to* $\mathbf{x}(t) := \tilde{\mathbf{p}}$*,* $a = 0$ *and* A_0 *has full rank.*

Proof. ($\Rightarrow$) By definition, $\mathbf{p}$ is a regular point of $\mathbf{f}_{aug}$ if and only if $J_{\mathbf{f}_{aug}}(\mathbf{p})$ has full rank. But note that $J_{\mathbf{f}_{aug}}$ is

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ df_1/dt & df_1/dx_1 & \cdots & df_1/dx_n \\ df_2/dt & df_2/dx_1 & \cdots & df_2/dx_n \\ \vdots & \vdots & & \vdots \\ df_m/dt & df_m/dx_1 & \cdots & df_m/dx_n \end{bmatrix}. \tag{3.19}$$

and $J_{\mathbf{f}^{(t)}}$ is

$$\begin{bmatrix} df_1/dx_1 & \cdots & df_1/dx_n \\ df_2/dx_1 & \cdots & df_2/dx_n \\ \vdots & & \vdots \\ df_m/dx_1 & \cdots & df_m/dx_n \end{bmatrix}. \tag{3.20}$$

So $J_{\mathbf{f}_{aug}}$ has full rank at $\mathbf{p}$ if and only if $J_{\mathbf{f}^{(t)}}|_{t=0}$ has full rank at $\tilde{\mathbf{p}}$. Thus it suffices to show that after each Newton step, $a = 0$ and $\mathbf{x}(0) = \tilde{\mathbf{p}}$ remain true, so that $A_0 = J_{\mathbf{f}^{(t)}}(\mathbf{x}(0)) = J_{\mathbf{f}^{(t)}}(\tilde{\mathbf{p}})|_{t=0}$ continues to have full rank.

We clearly have $a \geq 0$ at every step, since the Newton iteration cannot introduce negative exponents. At the beginning, $a = 0$ and $\mathbf{x}(0) = \tilde{\mathbf{p}}$ hold trivially. Inducting on the Newton steps, if $a = 0$ and $\mathbf{x}(0) = \tilde{\mathbf{p}}$ at some point in the algorithm, then the next A_0, namely $J_{\mathbf{f}^{(t)}}(\mathbf{x}(0)) = J_{\mathbf{f}^{(t)}}(\tilde{\mathbf{p}})|_{t=0}$, is the same matrix as in the last step, hence it is again regular and a is 0. Since $\mathbf{f}^{(t)}(\mathbf{x}(0)) = \mathbf{f}^{(t)}(\tilde{\mathbf{p}})|_{t=0} = 0$, b must be strictly greater than 0. Thus the next Newton update $\Delta\mathbf{x}$ must have positive degree in all components, leaving $\mathbf{x}(0) = \tilde{\mathbf{p}}$ unchanged.

($\Leftarrow$) If $\mathbf{p}$ is a singular point of $\mathbb{V}(\mathbf{f}_{aug})$, then on the first Newton step $A_0 = J_{\mathbf{f}^{(t)}}(\tilde{\mathbf{p}})|_{t=0}$ must drop rank by the same argument as above comparing (3.19) and (3.20). □

To summarize the cases:

Lemma 3.2.2. *There are three possible scenarios for* $\mathbb{V}(\mathbf{f}_{aug})$*:*

1. $\exists \mathbf{p} \in \mathbb{V}(\mathbf{f}_{aug})$ *regular,*
2. $\exists \mathbf{p} \in \mathbb{V}(\mathbf{f}_{aug})$ *singular, or*
3. $\nexists \mathbf{p} \in \mathbb{V}(\mathbf{f}_{aug})$

In the first case, we can simply use $\tilde{\mathbf{p}} = (p_1, p_2, \ldots, p_n)$ to start the Newton iteration. In the second, we must defer to tropical methods. If $\mathbf{p}$ is a branch point, the tropical methods will imply a substitution $t \to t^k$ where k is the winding number, and will provide a starting $\mathbf{z}$, which will lie in $\mathbb{C}[[t]]^n$. In the final case, we also defer to tropical methods, which provide a starting $\mathbf{z}$ that will have negative exponents. A change of coordinates brings the problem back into one of the first two cases, and we can apply our method directly. It is important to reiterate that $\mathbf{p}$ may be a regular point of $\mathbb{V}(\mathbf{f})$ but a singular point of $\mathbb{V}(\mathbf{f}_{aug})$, as is the case in the third frame of Figure 12. The following example also demonstrates this behavior.

Example 4 (Viviani, continued)**.** *We return to the example of Viviani's curve. We will use the formulation from Section 2.2, where setting $x_1 = 0$ gives the highest and lowest (real) points on the curve; the system is*

$$\mathbf{f} = (x_1^2 + x_2^2 + x_3^2 - 4, (x_1 - 1)^2 + x_2^2 - 1). \tag{3.21}$$

When $x_1 = 0$ we obtain the two points $(0, 0, 2)$ and $(0, 0, -2)$, which are both regular points. For the augmented system $\mathbf{f}_{aug}$, the Jacobian $J_{\mathbf{f}_{aug}}$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 2x_1 & 2x_2 & 2x_3 \\ 2x_1 - 2 & 2x_2 & 0 \end{bmatrix} \tag{3.22}$$

which at the point $\mathbf{p} = (0, 0, 2)$ becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 4 \\ -2 & 0 & 0 \end{bmatrix}. \tag{3.23}$$

This matrix drops rank, hence $\mathbf{p}$ is a singular point of $\mathbf{f}_{aug}$ and we are in the second case of Lemma 3.2.2. Following the lemma, we defer to tropical methods to begin, obtaining the *transformation* $x_1 \rightarrow 2t^2$ *and the starting term* $\mathbf{z} = (2t, 2)$*. Now the first Newton step can be written:*

$$\begin{bmatrix} 4t & 4 \\ 4t & 0 \end{bmatrix} \Delta\mathbf{z} = - \begin{bmatrix} 4t^2 + 4t^4 \\ 4t^4 \end{bmatrix}. \tag{3.24}$$

Note that $J_{\mathbf{f}(t)}(\mathbf{z})$ *is now invertible over* $\mathbb{C}((t))$*. Its inverse begins with negative exponents of* t*:*

$$\begin{bmatrix} 0 & 1/4 \\ 1/4\ t^{-1} & -1/4\ t^{-1} \end{bmatrix}. \tag{3.25}$$

To linearize, we first observe that $a = 0$ *and* $b = 2$*, so* $\mathbf{x}$ *will have degree at least* $b - a = 2$*. The linearized block form of (3.24) is then*

$$\left[\begin{array}{cc|cc|cc} 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 4 & 0 & 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 4 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 & 0 & 0 \end{array}\right] \Delta\mathbf{x} = \begin{bmatrix} -4 \\ 0 \\ 0 \\ 0 \\ -4 \\ -4 \end{bmatrix}. \tag{3.26}$$

Whether we solve (3.24) over $\mathbb{C}((t))$ *or solve (3.26) in the least squares sense, we obtain the same Newton update*

$$\Delta\mathbf{x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} t^2 + \begin{bmatrix} -1 \\ 0 \end{bmatrix} t^3, \tag{3.27}$$

or in non-linearized form,

$$\Delta \mathbf{z} = \begin{bmatrix} -t^3 \\ \\ -t^2 \end{bmatrix}. \tag{3.28}$$

Substituting $\mathbf{z} + \Delta\mathbf{z} = (2t - t^3, 2 - t^2)$ *into (3.21) produces* $(x_1^6 + x_1^4, x_1^6)$*, and we have obtained the desired cancellation of lower-order terms.* △

We will refer to the matrix in (3.26) as a Hermite-Laurent matrix because of its correspondence with Hermite-Laurent interpolation; see Remark 2.

3.2.3 A Lower Triangular Echelon Form

When we are in the regular case of Lemma 3.2.2 and the condition number of A_0 is low, we can simply solve the staggered system (3.8). When this is not possible, we are forced to solve (3.9). Figure 13 shows the structure of the coefficient matrix (3.9) for the regular case, when A_0 is regular and all block matrices are dense. The essence of this section is that we can use column operations to reduce the block matrix to a lower triangular echelon form as shown at the right of Figure 13, solving (3.9) in the same time as (3.8).

The lower triangular echelon form of a matrix is a lower triangular matrix with zero elements above the diagonal. If the matrix is regular, then all diagonal elements are nonzero. For a singular matrix, the zero rows of its echelon form are on top (have the lowest row index) and the zero columns are at the right (have the highest column index). Every nonzero column has one pivot element, which is the nonzero element with the smallest row index in the column. All elements at the right of a pivot are zero. Columns may need to be swapped so that the row indices of the pivots of columns with increasing column indices are sorted in decreasing order.

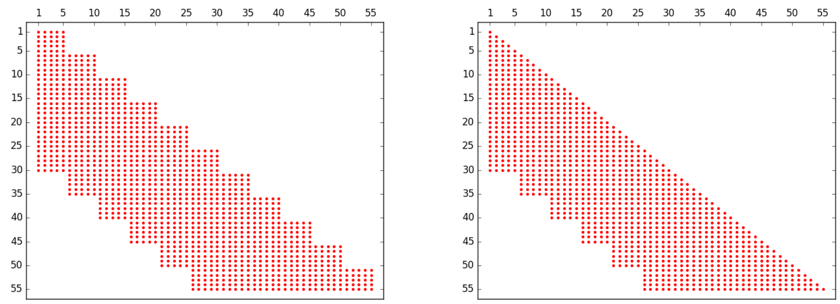


Figure 13: On the left is the banded block structure of a generic Hermite-Laurent matrix for $n = 5$. On the right is its lower triangular echelon form.

Example 5. *(Viviani, continued). For the matrix series in (3.26), we have the following reduction:*

$$\begin{bmatrix} 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 4 & 0 \end{bmatrix}. \tag{3.29}$$

Because of the singular matrix coefficients in the series, we find zeros on the diagonal in the echelon form. $\triangle$

Given a general n-by-m dimensional matrix A, the lower triangular echelon form L can be described by one n-by-n row permutation matrix P which swaps the zero rows of A, and a sequence of m column permutation matrices Q_k (of dimension m) and multiplier matrices U_k

(also of dimension m). The matrices Q_k swap columns in order to bring the pivots with lowest row indices to the lowest column indices. The matrices U_k contain the multipliers necessary to reduce what is at the right of the pivots to zero. From these, the construction of the lower triangular echelon form can be summarized by the following matrix equation:

$$L = PAQ_1U_1Q_2U_2 \cdots Q_mU_m. \tag{3.30}$$

Solving the matrix equation is similar to solving a linear system with LU factorization—the multipliers are applied to the solution of the lower triangular system which has L as its coefficient matrix.

3.3 Some Basic Cost Estimates

Working with truncated power series is somewhat similar to working with extended precision arithmetic. In this section we make some observations regarding the cost overhead.

3.3.1 Cost of one step

First we compare the cost of computing a single Newton step using the various methods introduced. We let d denote the degree of the truncated series in $\mathbf{A}(t)$, and n the dimension of the matrix coefficients in $\mathbf{A}(t)$ as before.

The staggered system. In the case that $a \geq 0$ and the leading coefficient A_0 of the matrix series $\mathbf{A}(t)$ is regular, the equations in (3.8) can be solved with $O(n^3) + O(dn^2)$ operations. The cost is $O(n^3)$ for the decomposition of the matrix A_0, and $O(dn^2)$ for the back substitutions using the decomposition of A_0 and the convolutions to compute the right hand sides.

The big block matrix. Ignoring the triangular matrix structure, the cost of solving the larger linear system (3.9) is $O((dn)^3)$.

The lower triangular echelon version. If the leading coefficient A_0 in the matrix series is regular (as illustrated by Figure 13), we may copy the lower triangular echelon form $L_0 = A_0 Q_0 U_0$ of A_0 to all blocks on the diagonal and apply the permutation Q_0 and column operations as defined by U_0 to all other column blocks in $\mathbf{A}$. The regularity of A_0 implies that we may use the lower triangular echelon form of L_0 to solve (3.9) with substitution. Thus with this quick optimization we obtain the same cost as solving the staggered system (3.8).

In general, A_0 and several other matrix coefficients may be rank deficient, and the diagonal of nonzero pivot elements will shift towards the bottom of L. We then find as solutions vectors in the null space of the upper portion of the matrix $\mathbf{A}$.

3.3.2 Cost of computing D terms

Assume that $D = 2^k$. In the regular case, assuming quadratic convergence, it will take k steps to compute 2^k terms. We can reuse the factorization of A_0 at each step, so we have $O(n^3)$ for the decomposition plus

$$O(2n^2 + 4n^2 + 8n^2 + \cdots + 2^{k-1}n^2) = O(2^k n^2) \tag{3.31}$$

for the back substitutions. Putting these together, we find the cost of computing D terms to be $O(n^3) + O(Dn^2)$.

3.4 Conclusion

In this chapter, we considered the extension of the Gauss-Newton algorithm from complex floating-point arithmetic to the field of truncated power series with complex floating-point coefficients. Using linearization, we formulated a linear system where the coefficient matrix is a series with matrix coefficients, and provided a characterization for when the matrix series is regular based on the algebraic variety of an augmented system. The structure of the linear system leads to a block triangular system, which can solved effectively with a lower triangular echelon form. We show that this solution has cost cubic in the problem size. In general, at singular points, we rely on methods of tropical algebraic geometry to provide the starting guess for the algorithm as well as possibly a transformation of the system. In the next chapter, we will provide some illustrative examples and demonstrate our method's application to polynomial homotopy continuation.

CHAPTER 4

EXAMPLES AND APPLICATIONS

4.1 Introduction

In this chapter, we tie together Chapters 2 and 3 with several examples that illustrate their use in finding series solutions for space curves. To set up the problems we used the computer algebra system SageMath [74], for tropical computations we used Gfan [15] unless otherwise noted, and for commutative algebra calculations we used Singular [23] via the SageMath interface. Our power series methods have been implemented in PHCpack [75] and are available in Python via the interface phcpy [77]. We provide four examples of our methods at work, applying them to the circle problem of Apollonius, the tangents to four spheres problem, and the cyclic 8- and 16-roots systems. The first three examples were published in [14], while the cyclic 16-roots example is new.

4.2 The Problem of Apollonius

The classical problem of Apollonius consists in finding all circles that are simultaneously tangent to three given circles. A special case is when the three circles are mutually tangent and have the same radius; see Figure 14. Here the solution variety is singular – the circles themselves are double solutions. In this figure, all have radius 1, and centers $(0, 0)$, $(2, 0)$, and $(1, \sqrt{3})$. We can study this configuration with power series techniques by introducing a parameter t to

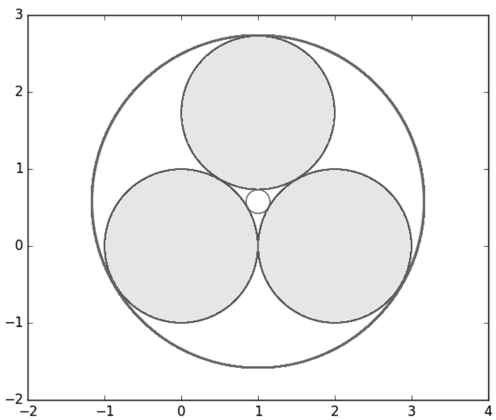


Figure 14: Singular configuration of Apollonius circles. The input circles are filled in, the solution circles are dark gray. Because the input circles mutually touch each other, three of the solution circles coincide with the input circles.

represent a vertical shift of the upper circle. We then examine the solutions as we vary t. This is represented algebraically as a solution to

$$\left\{\begin{array}{rcl} x_1^2 + x_2^2 - r^2 - 2r - 1 & = & 0 \\ x_1^2 + x_2^2 - r^2 - 4x_1 - 2r + 3 & = & 0 \\ t^2 + x_1^2 - 2tx_2 + x_2^2 - r^2 + 2\sqrt{3}t - 2x_1 - 2\sqrt{3}x_2 + 2r + 3 & = & 0. \end{array}\right. \tag{4.1}$$

Because we are interested in power series solutions of (4.1) near $t = 0$, we use t as our free variable. To simplify away the $\sqrt{3}$, we substitute $t \to \sqrt{3}t$, $x_2 \to \sqrt{3}x_2$, and the system becomes

$$\begin{cases} x_1^2 + 3x_2^2 - r^2 - 2r - 1 = 0 \\ x_1^2 + 3x_2^2 - r^2 - 4x_1 - 2r + 3 = 0 \\ 3t^2 + x_1^2 - 6tx_2 + 3x_2^2 - r^2 + 6t - 2x_1 - 6x_2 + 2r + 3 = 0. \end{cases} \tag{4.2}$$

Call this system $\mathbf{f}$. Now we examine the system at $(t, x_1, x_2, r) = (0, 1, 1, 1) = \mathbf{p}$. The Jacobian $J_{\mathbf{f}}$ at $\mathbf{p}$ is

$$\begin{bmatrix} 0 & 2 & 6 & -4 \\ 0 & -2 & 6 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.3}$$

so $\mathbf{f}$ — and by extension $\mathbf{f}_{aug}$ — is singular at $\mathbf{p}$, and we are in the second case of Lemma 3.2.2.

Applying Gfan to the system provides the tropical prevariety F of $\mathbf{f}$ under the trivial valuation. Keeping the variable order as (t, x_1, x_2, r), F is generated by the five rays (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), and (-1, -1, -1, -1). Our series parameter is t, so we only care about rays with positive t-component, and choose $\mathbf{v} = (1, 0, 0, 0)$. Solving $\text{in}_{\mathbf{v}}\mathbf{f}$ gives two possible starting solutions, which rounded for readability are $(t, 1, 1 + 0.536t, 1 + 0.804t)$ and

$(t, 1, 1 + 7.464t, 1 + 11.196t)$. We will continue with the second; call it $\mathbf{z}$. For the first step of Newton's method, $\mathbf{A}$ is

$$\begin{bmatrix} 2 & 6 & -4 \\ -2 & 6 & -4 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 44.785 & -22.392 \\ 0 & 44.785 & -22.392 \\ 0 & 38.785 & -22.392 \end{bmatrix} t \tag{4.4}$$

and $\mathbf{b}$ is

$$\begin{bmatrix} 41.785 \\ 41.785 \\ 0 \end{bmatrix} t^2. \tag{4.5}$$

From these we can construct the linearized system

$$\begin{bmatrix} A_0 & & \\ A_1 & A_0 & \\ & A_1 & A_0 \end{bmatrix} \Delta\mathbf{x} = \begin{bmatrix} \mathbf{b}_0 \\ 0 \\ 0 \end{bmatrix}. \tag{4.6}$$

Solving in the least squares sense, we obtain two more terms of the series, so in total we have

$$\begin{cases} x_1 &= 1 \\ x_2 &= 1 + 7.464t + 45.017t^2 + 290.992t^3 \\ r &= 1 + 11.196t + 77.971t^2 + 504.013t^3. \end{cases} \tag{4.7}$$

By comparison, the series we obtain from the other possible starting solution is

$$\begin{cases} x_1 &= 1 \\ x_2 &= 1 + 0.536t - 0.017t^2 + 0.0077t^3 \\ r &= 1 + 0.804t + 0.029t^2 - 0.013t^3. \end{cases} \tag{4.8}$$

From these, we get a good idea of what happens near $t = 0$: the first solution circle grows rapidly (corresponding to the larger coefficients in (4.7)), while the other stays small (corresponding to the smaller coefficient in (4.8)). This is illustrated in Figure 15, which shows the solutions of the system at $t = 0.13$.

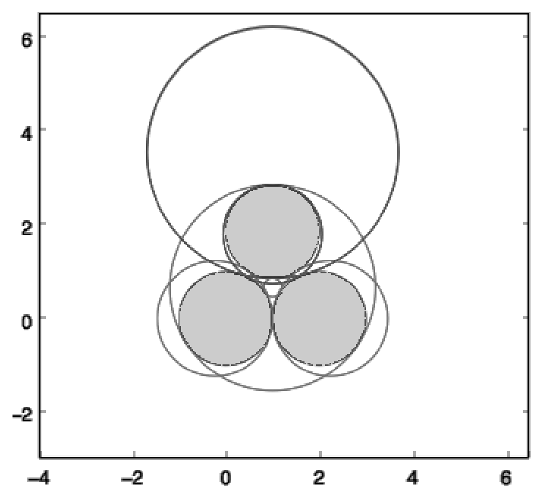


Figure 15: Solution to (4.1) for $t = 0.13$. The largest circles correspond to power series solutions with larger coefficients than the coefficients of the power series solutions for the smaller circles.

This example demonstrates the application of power series solutions in polynomial homotopies. Current numerical continuation methods cannot be applied to track the solution paths defined by the homotopy in (4.1), because at $t = 0$, the start solutions are double solutions. The power series solutions provide reliable predictors to start tracking the solution paths defined by (4.1).

4.3 Tangents to Four Spheres

Our next example is that of finding all lines mutually tangent to four spheres in $\mathbb{R}^3$; see [24], [50], [72], and [73]. If a sphere S has center $\mathbf{c}$ and radius r, the condition that a line in $\mathbb{R}^3$ is tangent to S is given by

$$\|\mathbf{m} - \mathbf{c} \times \mathbf{t}\|^2 - r^2 = 0, \tag{4.9}$$

where $\mathbf{m} = (x_0, x_1, x_2)$ and $\mathbf{t} = (x_3, x_4, x_5)$ are the moment and tangent vectors of the line, respectively. For four spheres, this gives rise to four polynomial equations; if we add the equation $x_0x_3 + x_1x_4 + x_2x_5 = 0$ to require that $\mathbf{t}$ and $\mathbf{m}$ are perpendicular and $x_3^2 + x_4^2 + x_5^2 = 1$ to require that $\|\mathbf{t}\| = 1$, we have a system of 6 equations in 6 unknowns which we expect to be 0-dimensional.

If we choose the centers to be $(+1, +1, +1)$, $(+1, -1, -1)$, $(-1, +1, -1)$, and $(-1, -1, +1)$ and the radii to all be $\sqrt{2}$, the spheres all mutually touch and the configuration is singular; see Figure 16. In this case, the number of solutions drops to three, each of multiplicity 4.

Next we introduce an extra parameter t to the equations so that the radii of the spheres are $\sqrt{2} + t$. This results in a 1-dimensional system F, which we omit for succinctness. F is singular

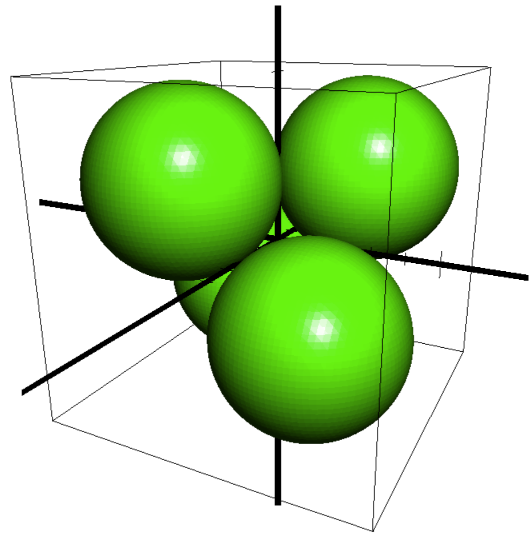


Figure 16: A singular configuration of four spheres. The input spheres mutually touch each other and the tangent lines common to all four input spheres occur with multiplicity.

at $t = 0$, so we are once again in the second case of Lemma 3.2.2. Computing a tropical basis in Gfan and the primary decomposition in Singular, we decompose F into three systems, one of which is

$$\mathbf{f} = \begin{cases} x_0 &= 0 \\ x_3 &= 0 \\ x_4^2 + x_2x_5 + x_5^2 &= 0 \\ x_1x_4 + x_2x_5 &= 0 \\ x_1x_2 - x_2x_4 + x_1x_5 &= 0 \\ x_1^2 + x_2^2 - 1 &= 0 \\ 2t^4 + 4t^2 + x_2x_5 &= 0 \\ x_2^2x_4 - x_2x_4x_5 + x_1x_5^2 - x_4 &= 0 \\ x_2^3 - x_2 - x_5 &= 0. \end{cases} \tag{4.10}$$

Using our methods we can find several solutions to this, one of which is

$$\begin{cases} x_0 = 0 \\ x_1 = 2t + 4.5t^3 + 30.9375t^5 + 299.3906t^7 + 3335.0889t^9 + 40316.851t^{11} \\ x_2 = 1 - 2t^2 - 11t^4 - 94t^6 - 986.5t^8 - 11503t^{10} \\ x_3 = 0 \\ x_4 = 2t - 3.5t^3 - 23.0625t^5 - 193.3594t^7 - 2019.3486t^9 - 23493.535t^{11} \\ x_5 = -4t^2 - 10t^4 - 64t^6 - 614t^8 - 6818t^{10} - 82283t^{12} \end{cases}.$$

Substituting back into $\mathbf{f}$ yields series in $O(t^{12})$, confirming the calculations. This solution could be used as the initial predictor in a homotopy beginning at the singular configuration.

In contrast to the small Apollonius circle problem, this example is computationally more challenging, as covered in [24], [50], [72], and [73]. It illustrates the combination of tropical methods in computer algebra with symbolic-numeric power series computations to define a polynomial homotopy to track solution paths starting at multiple solutions.

4.4 Series Developments for Cyclic 8-Roots

A vector $\mathbf{u} \in \mathbb{C}^n$ of a unitary matrix A is biunimodular if for $k = 1, 2, \ldots, n$: $|u_k| = 1$ and $|v_k| = 1$ for $\mathbf{v} = A\mathbf{u}$. The following system arises in the study [26] of biunimodular vectors:

$$\mathbf{f}(\mathbf{x}) = \begin{cases} x_0 + x_1 + \cdots + x_{n-1} = 0 \\ i = 2, 3, 4, \ldots, n-1 : \displaystyle\sum_{j=0}^{n-1} \prod_{k=j}^{j+i-1} x_{k \ mod \ n} = 0 \\ x_0 x_1 x_2 \cdots x_{n-1} - 1 = 0. \end{cases} \tag{4.11}$$

Cyclic 8-roots has solution curves not reported by Backelin [8]. Note that because of the last equation, the system has no solution for $x_0 = 0$, or in other words $\mathbb{V}(\mathbf{f}_{aug}) = \emptyset$. Thus we are in the third case of Lemma 3.2.2.

In [1,4], the vector $\mathbf{v} = (1, -1, 0, 1, 0, 0, -1, 0)$ gives the leading exponents of the series. The corresponding unimodular coordinate transformation $\mathbf{x} = \mathbf{z}^M$ is

$$M = \begin{array}{cl} \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} & \begin{array}{l} x_0 \rightarrow z_0 \\ x_1 \rightarrow z_1 z_0^{-1} \\ x_2 \rightarrow z_2 \\ x_3 \rightarrow z_3 z_0 \\ x_4 \rightarrow z_4 \\ x_5 \rightarrow z_5 \\ x_6 \rightarrow z_6 z_0^{-1} \\ x_7 \rightarrow z_7. \end{array} \end{array} \tag{4.12}$$

Solving the transformed system with z_0 set to 0 gives the leading coefficient of the series.

After 2 Newton steps, invoked in PHCpack with `phc -u`, the series for z_1 is

```
(-1.25000000000000E+00 + 1.25000000000000E+00*i)*z0^2
+( 5.00000000000000E-01 - 2.37676980513323E-17*i)*z0
+(-5.00000000000000E-01 - 5.00000000000000E-01*i);
```

After a third step, the series for z_1 is

```
( 7.12500000000000E+00 + 7.12500000000000E+00*i)*z0^4
+(-1.52745512076048E-16 - 4.25000000000000E+00*i)*z0^3
+(-1.25000000000000E+00 + 1.25000000000000E+00*i)*z0^2
+( 5.00000000000000E-01 - 1.45255178343636E-17*i)*z0
```

```
+(-5.00000000000000E-01 - 5.00000000000000E-01*i);
```

4.5 A Note on Cyclic 16-Roots

Lastly we apply power series methods to the cyclic 16-roots F, the 16-dimensional version of the polynomial system in 4.11, for which the tropical prevariety was computed recently [40]. In [8] a result by Backelin establishes that there is an $(m-1)$-dimensional component of cyclic n-roots whenever $n = m^2$. One result from [1,4] is an explicit construction of such a component in general, which we reproduce below:

Proposition 4.5.1 (Proposition 4.31 in [2]). *For $n = m^2$, there is an $(m-1)$-dimensional set of cyclic n-roots, represented exactly as*

$$\begin{aligned}
x_{km+0} &= u_k t_0 \\
x_{km+1} &= u_k t_0 t_1 \\
x_{km+2} &= u_k t_0 t_1 t_2 \\
&\vdots \\
x_{km+(m-2)} &= u_k t_0 t_1 t_2 \cdots t_{m-2} \\
x_{km+(m-1)} &= u_k t_0^{-m+1} t_1^{-m+2} \cdots t_{m-3}^{-2} t_{m-2}^{-1}
\end{aligned} \tag{4.13}$$

for $k = 0, 1, 2, \ldots, m-1$ and $u_k = e^{2\pi i k/m}$.

For $n = 16$, this says that there must exist a component of dimension 3. From the formulation (4.13) we can write down the corresponding cone of the prevariety, as generated by the exponents of the t_i's. This gives the following three rays:

$$\begin{gathered}(1, 1, 1, \text{-}3, 1, 1, 1, \text{-}3, 1, 1, 1, \text{-}3, 1, 1, 1, \text{-}3),\\ (0, 1, 1, \text{-}2, 0, 1, 1, \text{-}2, 0, 1, 1, \text{-}2, 0, 1, 1, \text{-}2),\\ (0, 0, 1, \text{-}1, 0, 0, 1, \text{-}1, 0, 0, 1, \text{-}1, 0, 0, 1, \text{-}1).\end{gathered} \tag{4.14}$$

However, using the prevariety computed in [40], we calculated that the cone (4.14) is actually contained in a larger cone of the tropical prevariety, which is generated by the following four rays:

$$\begin{aligned}&(\ 1, 1, 1, \text{-}3, 1, 1, 1, \text{-}3, 1, 1, 1, \text{-}3, 1, 1, 1, \text{-}3)\\ &(\text{-}1, \text{-}1, 3, \text{-}1, \text{-}1, \text{-}1, 3, \text{-}1, \text{-}1, \text{-}1, 3, \text{-}1, \text{-}1, \text{-}1, 3, \text{-}1),\\ &(\ 1, \text{-}1, 1, \text{-}1, 1, \text{-}1, 1, \text{-}1, 1, \text{-}1, 1, \text{-}1, 1, \text{-}1, 1, \text{-}1),\\ &(\text{-}1, 1, 1, \text{-}1, \text{-}1, 1, 1, \text{-}1, \text{-}1, 1, 1, \text{-}1, \text{-}1, 1, 1, \text{-}1),\end{aligned} \tag{4.15}$$

Using the same coefficients as in (4.13), this yields the following formulation, which vanishes entirely when substituted into F:

$$\begin{aligned} x_{4k} &= u_k t_0^1 t_1^1 t_2^1 t_3^{-3} \\ x_{4k+1} &= u_k t_0^{-1} t_1^{-1} t_2^3 t_3^{-1} \\ x_{4k+2} &= u_k t_0^1 t_1^{-1} t_2^1 t_3^{-1} \\ x_{4k+3} &= u_k t_0^{-1} t_1^1 t_2^1 t_3^{-1} \end{aligned} \tag{4.16}$$

for $k = 0,\ 1,\ 2,\ 3$ and $u_k = e^{2\pi ik/4}$. To visualize these two cones, we can project them into the x_0, x_1, x_2 exponent-hyperplane and intersect with the plane $x_2 = 1$; see Figure 17. No proof is necessary to show the containment, other than noting that the initial form of $\mathbf{f}$ with respect to any ray in the interior of these cones is the same.

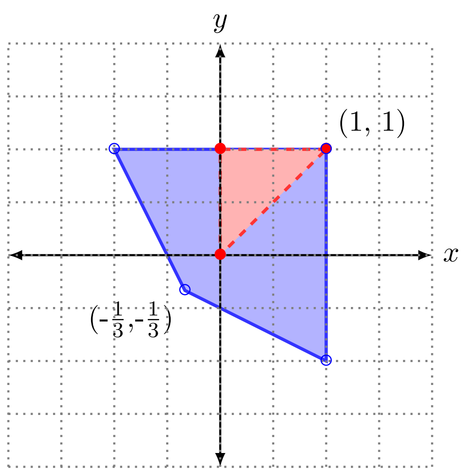


Figure 17: The Backelin cone for cyclic 16. Cone predicted by [1, 4], red. Cone found using computation from [40], blue.

CHAPTER 5

CONCLUSION

Polynomial systems are at the center of computational algebraic geometry. The majority of the focus of the field has been their solution in the zero dimensional case, although much work has certainly been done towards solving positive dimensional systems. In this work, we advance the understanding of the positive dimensional case in two important ways. First, we investigate the numerical side of tropical geometry, defining "hidden cone" behavior and giving an alternative approach to current symbolic methods for computing the tropical variety. And secondly, we give an efficient method for extending series information using a modified Gauss-Newton approach, and categorize situations where this method alone is sufficient. As our examples and applications demonstrate, these methods can provide great insight into space curve solutions of polynomial systems.

Our work begs several future avenues of exploration. The most obvious is to extend our methods to varieties of dimension greater than one. This would likely require a significant rethinking of the necessary data structures and algorithms, as well as a better understanding of the tropical geometry involved—our methods take advantage of the fact that the tropical shadows of one dimensional varieties are much simpler than those of higher dimensional varieties. Another, smaller avenue to explore is whether the Newton solve and substitution steps might be combined to save some cost; we made an attempt by applying techniques of algorithmic differentiation, but were ultimately unsuccessful. And finally, questions of convergence call for much investigation. We mean convergence in both senses—the regions of convergence of the series, and the rate of convergence of the Gauss-Newton method itself. Both questions are in their essence numerical. The first naturally falls in the realm of numerical and complex analysis, while the second is numerical in nature because polynomial coefficients, as well as the series coefficients obtained through our tropical approach, are generally approximate. These questions were outside the scope of our work here, but might well offer fruitful lines of inquiry for further research.

APPENDICES

Appendix A

THE NEWTON-PUISEUX ALGORITHM

For a polynomial $p \in k[x_1, x_2]$ where k is an algebraically closed field of characteristic 0, the Newton-Puiseux algorithm can be seen as a way of factoring p into $(x_2 - y_0)(x_2 - y_1)\cdots(x_2 - y_n)$ where $y_i \in k\{\{x_1\}\}$, i.e. the (algebraically closed) field of fractional power series. What follows is a rather straightforward pseudocode implementation of the algorithm. Recursion is used to overcome the difficulty of branching solutions, i.e. when two Puiseux roots of the polynomial have the same first few terms.

In the function RECURSE, the variable `curSol` is a list of the monomials in the solution currently being computed. The variable solutions is a list of Puiseux solutions, to which `curSol` is added when complete. cx^γ is an initial Puiseux term of p, which the function assumes is already in `curSol`.

function COMPUTE PUISEUX EXPANSION($p(x, y)$, number of terms n)

  if $\deg_y(p) = 0$ **then return** $\emptyset$

  end if

  toReturn $\leftarrow$ empty list

  initTerms $\leftarrow$ list of initial terms of Puiseux solutions of p

  for ${c_0 x_0}^{\gamma_0}$ in initTerms **do**

    RECURSE(p, ${c_0 x_0}^{\gamma_0}$, $[{c_0 x_0}^{\gamma_0}]$, toReturn, n)

end for

return toReturn

end function

function RECURSE($p(x, y)$,cx^{γ}, curSol, solutions, depth)

$p_0(x, y) \leftarrow p(x, x^{\gamma}(c + y))$

initTerms $\leftarrow$ list of initial terms of Puiseux solutions of p_0 with positive exponents

if depth=0 OR p_0 is a monomial OR initTerms is empty **then**

append curSol to solutions

return

end if

for $c_0{x_0}^{\gamma_0}$ in initTerms **do**

revisedList $\leftarrow$ curSol

$\gamma_1 \leftarrow$ exponent of last monomial in curSol

append $c_0{x_0}^{\gamma_0+\gamma_1}$ to revisedList

RECURSE(p_0, term, revisedList, solutions, depth-1)

end for

end function

Appendix B

SOURCE CODE DOCUMENTATION FOR THE GENERALIZED NEWTON-PUISEUX ALGORITHM

B.1 General Newton-Puiseux code

```
def reducePoly(p):
    """
    For a polynomial p, factors out any extra x_i's unless p is a
    monomial.
    """

def reduceIdeal(I):
    """
    For an ideal I, factors out any excess x_i's from its generators.
    """

def getRationalCoeffs(I, clockout = 2, height_bound = 0):
    """
    Searches for a rational solution to the ideal defined by I. If
    height_bound is specified and greater than 0, uses it as the
    numerator/denominator bound, otherwise increments the bound until
    points are found or clockout time is spent, whichever comes first.
    Clockout doesn't quite work--it will start a rational_points
    calculation as long as we haven't reached it, meaning it could start
    one just before the clockout and then take a while.
    WARNING: definitely might return an empty list even if variety is
    non-empty and contains rational points.
    """

def getCoeffs(I):
    """
    Looks for nonzero solutions to the initial form system defined by I.
    """

def npSubstitution(I, exps, coeffs):
```

```
    """
    Prepares for the next iteration of the Newton-Puiseux algorithm,
    using the exponent and coefficient tuples in exps and coeffs,
    respectively.
    In other words, does the higher-dim version of substituting
    x^gamma(c+y) for y as we would do in the planar version.
    """

def printConeInformation(inForm):
    """
    For the initial form system object in inForm, prints the following
    useful information:
     - the defining rays of the cone,
     - the initial form system itself,
     - the system with x=1 substituted, and
     - the system with a unimodular coordinate transformation applied.
    """

def getInput(s, myType):
    """
    Using the prompt in the string s, gets input from the user, coerces
    it to type myType, and returns it. If this fails, the process
    repeats until success or an entry of 'q' or 'Q'.
    """

def performStep(I, SOLUTION, showHigherCones=False):
    """
    Ties together the above methods to perform a step of the
    Newton-Puiseux method on the ideal I. The current solution must be
    given in SOLUTION, and will be modified according to the step
    performed here. showHigherCones is a boolean that gives the option
    to use a ray in a higher dimensional cone of the prevariety.
    Because each step (can) involve a choice of which ray and
    coefficient tuple to use, this function requires user input.
    The user is also given the option of looking for rational
    coefficients, which calls getRationalCoeffs. If the user indicates
    they are done, the function returns SOLUTION, otherwise it
    recurses.
    """

def newtonPuiseux(I):
    """
```

Appendix B (Continued)

```
    This function applies the Newton-Puiseux algorithm to the ideal I by
    starting the recursion of performStep() with an empty SOLUTION.
    """
```

B.2 Series tuple class

```python
class pSeriesTuple(object):
    """
    Class representing a tuple of power or Laurent series.
    """
    def __init__(self,expander_index=0):
        """
        Class constructor. expander_index is the index in the tuple of
        variables of the variable considered the series parameter.
        """

    def addTerm(self,coeffs,exps):
        """
        Extends the series with another term.
        """

    def seriesTuple(self):
        """
        Returns the associated tuple of power or Laurent series.
        """

    def __eq__(self,other):
        """
        Checks equality of series tuples.
        """

    def __repr__(self):
        """
        String representation of self.
        """

    def __str__(self):
        """
        String representation of self.
        """
```

```
    def __call__(self,value):
        """
        Substitutes value in each of the series of the tuple.
        """
```

B.3 gfan prevariety wrapper class

```
class inFormWrapper(object):
    """
    Wrapper for gfan's tropical computations. Allows us to compute
    tropical prevarieties for ideals whose base ring is not QQ.
    More specifically, this object is a wrapper for the gfan
    initial_form_systems of Sage.
    NOTE: takes the negatives of gfan's rays, since this is what we want
    for our algorithm
    """
    def __init__(self,forms,rayList,rationalVersion = []):
        """
        Constructor. forms is a list of polynomials (the initial form
        system), rayList is the cone of the prevariety that gives forms
        as its initial form system, and rationalVersion is a list of the
        polynomials with their coefficients set to \(1 \in QQ\). Does no
        sanity checking.
        """
    def rays(self):
        """
        Returns the list of rays of the cone of the prevariety that has
        self as its initial form system.
        """
    def changeRays(self,rays):
        """
        Returns an inFormWrapper with the current data but with rays
        substituted for the current rays; useful when specializing to an
        internal ray.
        """
    def initial_forms(self):
        """
        Accessor for the internally stored forms.
        """
```

Appendix B (Continued)

```
    def mixedVolume(self):
        """
        When self.forms is a square system, uses gfan to compute its
        mixed volume. If it fails, prints a message but does not raise
        an exception.
        """

def getInitialForms(I, justFan=False):
    """
    If the ideal is already over QQ, no need for anything fancy. We
    still return the above initial form wrapper for consistency, but do
    no more than call gfan. Otherwise, we convert the polynomials to
    dicts, construct new dicts with integer coefficients such that the
    coefficients are 1..n ordered to correspond to the order of the
    original polys in the original list. We pass these to gfan, then use
    the coeffs of the initial forms to figure out which of the original
    polys they correspond to, then make some new dicts and restore the
    original coefficents, finally wrapping as our objects.
    """
```

Appendix C

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Computing all Space Curve Solutions of Polynomial Systems by Polyhedral Methods. With Jan Verschelde. Proceedings of the 18th International Workshop on Computer Algebra in Scientific Computing (CASC 2016).

Solving Polynomial Systems in the Cloud with Polynomial Homotopy Continuation. With Jeff Sommars, Jan Verschelde, and Xiangcheng Yu. Proceedings of the 17th International Workshop on Computer Algebra in Scientific Computing (CASC 2015).

Strong Divisibility, Cyclotomic Polynomials, and Iterated Polynomials. With Ben Fulan, Stephen Lovett, and Jeff Sommars. American Mathematical Monthly, Summer 2013.

Talks

The Method of Gauss-Newton to Compute Power Series Solutions of Polynomial Homotopies. Numerical Linear Algebra Seminar, Purdue University. April 12, 2017.

The Method of Gauss-Newton to Compute Power Series Solutions of Polynomial Homotopies. ACMS Applied Math Seminar, University of Notre Dame. March 9, 2017.

Computing all Space Curve Solutions of Polynomial Systems by Polyhedral Methods. Proceedings of the 18th International Workshop on Computer Algebra in Scientific Computing. September 20, 2016.

Computing all Space Curve Solutions of Polynomial Systems by Polyhedral Methods. Fall Western Sectional Meeting of the American Mathematical Society. University of Denver, Denver, CO. October 8, 2016.

A Symbolic-Numeric Method for Higher-Dimensional Newton-Puiseux Expansions. Sectional Meetings of the American Mathematical Society. Loyola University, Chicago, IL. October 3rd, 2015.

Computers, Polynomials, Shapes: A Whiff of Computational Algebraic Geometry. Wheaton Mathematics Lecture Series. Wheaton College, Wheaton, IL. September 24th, 2015.

Travel Awards

1.	Event :	SIAM Conference on Applied Algebraic Geometry.
	Title :	SIAM Student Travel Award.
	Amount :	$650 and conference registration.
	Period :	7/31/17 – 8/4/17
2.	Event :	Macaulay2 Tutorial. Georgia Tech.
	Amount :	$150 and accomodation.
	Period :	7/27/17 – 7/29/17
3.	Event :	Macaulay2 Conference. UC Berkeley.
	Amount :	$700.
	Period :	7/17/17 – 7/21/17
4.	Event :	Joint Mathematics Meetings. Atlanta, GA
	Amount :	$950.
	Period :	1/4/17 – 1/4/17
5.	Event :	AMS Mathematics Research Community in Algebraic Statistics.
	Amount :	≈$700, accomodation, and meals.
	Period :	6/12/16 – 6/18/16
6.	Event :	Macaulay2 Workshop. University of Warwick
	Amount :	$750 and accomodation.
	Period :	5/23/16 – 5/26/16
7.	Event :	Macaulay2 Workshop. Boise State University
	Amount :	≈$800 and accomodation.
	Period :	5/27/15 – 5/30/15
8.	Event :	Macaulay2 Conference. University of Illinois
	Amount :	$400
	Period :	6/16/14 – 6/20/14

Professional Memberships

American Mathematical Society

Society for Industrial and Applied Mathematics (SIAM). Vice President of the Student Chapter at UIC, May 2014 to December 2015. Member of the Algebraic Geometry activity group.