The Coarse Geometry of the Teichmüller Metric: A Quasiisometry Model and the Actions of Finite Groups

BY

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THESIS

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To my parents, James and Joan Durham.

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SUMMARY

Let S be an oriented surface of finite type, $\mathcal{MCG}(S)$ its mapping class group, and $\mathcal{T}(S)$ its Teichmüller space. We use the work of Masur-Minsky and Rafi to build an $\mathcal{MCG}(S)$-equivariant quasiisometry model for $\mathcal{T}(S)$ with the Teichmüller metric, to which we extend the Masur-Minsky hierarchy machinery.

As an application, we study the action of finite subgroups $H \leqslant \mathcal{MCG}(S)$ on $\mathcal{T}(S)$. Kerckhoff's solution to the Nielsen Realization Problem proves that any such H has a nonempty fixed point set, $\mathrm{Fix}(H) \subset \mathcal{T}(S)$. For any $R > 0$, we prove that the set of points whose H-orbits have diameter bounded by R, $\mathrm{Fix}_R^{\mathcal{T}}(H)$, lies in a bounded neighborhood of $\mathrm{Fix}(H)$. As a corollary, we use work of Tao to show that the orbit of any point $X \in \mathcal{T}(S)$ under the action of a finite mapping class has a fixed coarse barycenter. Both of these results are easily obtained in a negatively curved space, in which $\mathrm{Fix}_R(H)$ would also be convex. By contrast, we prove that $\mathrm{Fix}_R^{\mathcal{T}}(H)$ need not be quasiconvex with an explicit family of examples.

CHAPTER 1

INTRODUCTION

The Teichmüller space of a surface S, denoted $\mathcal{T}(S)$, is the space of marked hyperbolic structures on S up to isotopy, which we consider with both the Teichmüller (d_T) and Weil-Petersson metrics (d_{WP}). The mapping class group of S, $\mathcal{MCG}(S) = \text{Homeo}^+(S)/\text{Homeo}_0(S)$, is the group of orientation preserving homeomorphisms of S up to isotopy, which acts naturally on $\mathcal{T}(S)$ by isometries in both (d_T) and (d_{WP}) by changing the marking.

Both $\mathcal{T}(S)$ and $\mathcal{MCG}(S)$ are fundamental objects of study in the theory of hyperbolic 3-manifolds, Riemannian surfaces, the moduli space of curves, and geometric group theory, with our understanding of their interconnections revolutionized by Thurston ((FLP79), (Thu80), (Thu88), (Thu97)).

The Teichmüller and Weil-Petersson metrics have long been studied via analytic and fine geometric techniques, but many recent advances in the field, especially those of Brock, Masur, Minsky, and Rafi, have been achieved by adopting the coarse geometric perspective of geometric group theory. In particular, the introduction of coarse methods has helped to elucidate the complicated balance of positive and negative curvature characteristics inherent to the Teichmüler metric.

The goal of this thesis is to further investigate the coarse geometry of $(\mathcal{T}(S), d_T)$. The first part of this thesis describes the construction of a $\mathcal{MCG}(S)$-equivariant quasiisometry model for $(\mathcal{T}(S), d_T)$ called the *augmented marking complex.* The second part explores the geometry of sets naturally associated to the fixed point set of the action of a finite subgroup of $\mathcal{MCG}(S)$ acting on $\mathcal{T}(S)$.

1.1 The augmented marking complex

The study of various combinatorial complexes built from simple closed curves on surfaces has greatly advanced the state of knowledge of the geometry of Teichmüller space, $\mathcal{T}(S)$, the mapping class group, $\mathcal{MCG}(S)$, and hyperbolic 3-manifolds. In (Br03), Brock showed that $\mathcal{T}(S)$ with the Weil-Petersson metric is quasiisometric to the graph of pants decompositions on S, $\mathcal{P}(S)$, an insight which he used to prove that the Weil-Petersson distance between two points in $\mathcal{T}(S)$ is coarsely the volume of the convex core of the quasi-Fuchsian hyperbolic 3-manifold they simultaneously uniformize. Beginning with their proof of hyperbolicity of the curve complex, $\mathcal{C}(S)$, in (MM99), the hierarchy machinery Masur-Minsky developed in (MM00) was essential in the proof of the Ending Lamination Theorem (Min03; BCM11) for hyperbolic 3-manifolds. Moreover, in (MM00), Masur-Minsky built the marking complex, $\mathcal{M}(S)$, and prove it is quasiisometric to $\mathcal{MCG}(S)$ in any word metric, an analogy essential to the proofs of the rank ((BM08)) and quasiisometric rigidity ((BKMM)) theorems for the mapping class group.

Our first main result is the construction of a graph we call the *augmented marking complex*:

Theorem 1.1. *The augmented marking complex, $\mathcal{AM}(S)$, is $\mathcal{MCG}(S)$-equivariantly quasiisometric to $(\mathcal{T}(S), d_T)$.*

We show that the Masur-Minsky hierarchy machinery for $\mathcal{M}(S)$ and $\mathcal{P}(S)$ extends to $\mathcal{AM}(S)$ and we use it to build preferred families of quasigeodesics called *augmented hierar-*

chy paths, thereby completing the unification of the coarse geometries of $\mathcal{MCG}(S)$ and $\mathcal{T}(S)$ in the Weil-Petersson and Teichmüller metrics by a common framework begun in (Raf07).

Theorem 1.2. *Any two points $X, Y \in \mathcal{T}(S)$ can be connected by a* augmented hierarchy path*, each of which is a quasigeodesic with uniform constants.*

In a recent paper, Eskin-Masur-Rafi (EMR13) used $\mathcal{AM}(S)$ and augmented hierarchy paths, which they independently discovered, to prove the Brock-Farb Geometric Rank conjecture for $\mathcal{T}(S)$ (BF06) with the Teichmüller metric and recover the rank theorems for $\mathcal{MCG}(S)$ and $\mathcal{T}(S)$ with the Weil-Petersson metric from (BM08).

Our construction of $\mathcal{AM}(S)$ follows upon the work of Masur and Minsky on the curve and marking complexes (MM99; MM00) and Rafi's applications of their machinery to Teichmüller geometry (Raf05; Raf07). We now briefly discuss the context of these results.

The geometry of the thin part of $\mathcal{T}(S)$, those metrics for which the hyperbolic lengths of some curves on the surface are small, is fundamentally different from its complement, the thick part. One can see this in the completion of $\mathcal{T}(S)$ in the Weil-Petersson metric, where curves are pinched to nodes and the geometry of the boundary strata is that of a product of the Teichmüller spaces of the complements of the pinched curves. While this stark phenomenon does not exactly hold in the Teichmüller metric, Minsky proved in (Min96) that the Teichmüller metric on the thin part of $\mathcal{T}(S)$ is quasiisometric to the product of the Teichmüller spaces of the complements of the short curves and a product of horodisks, one for each short curve (see Theorem 2.13) with the sup metric; that is, the thin parts of $\mathcal{T}(S)$ coarsely have a product structure.

In (MM99), Masur and Minsky proved the curve graph of S, denoted by $\mathcal{C}(S)$, is δ-hyperbolic (see Section 2.2) and that the electrification of the thin parts of $\mathcal{T}(S)$ is quasiisometric to $\mathcal{C}(S)$ and thus hyperbolic. While this provides a substantial amount of control over the large-scale geometry of $\mathcal{C}(S)$ and the thick part of $\mathcal{T}(S)$, $\mathcal{C}(S)$ is locally infinite, whereas $\mathcal{T}(S)$ is proper with the Teichmüller metric, and thus hyperbolicity does little a priori to inform upon the local geometry of either. In (MM00), they introduced the machinery of *hierarchies* of tight geodesics which record the combinatorial information sufficient to gain a great deal of control over the local geometry of $\mathcal{C}(S)$, proving it shares some properties with locally finite complexes. These hierarchies also contain the information sufficient to build quasigeodesics in the associated marking complex, $\mathcal{M}(S)$, called *hierarchy paths.* They proved that the progress along a hierarchy path coarsely occurs in subsurfaces to which the end markings have heavily overlapping projections. Using the hierarchy machinery, they proved that $\mathcal{M}(S)$ is $\mathcal{MCG}(S)$-equivariantly quasiisometric to $\mathcal{MCG}(S)$ with any word metric and obtained a coarse distance formula for $\mathcal{MCG}(S)$ (Theorem 2.10 below).

The connection between the work of Masur-Minsky and the Teichmüller metric was largely developed by Rafi; see (Raf10) for a summary of the current state of this project. A Teichmüller geodesic is a path through a space of metrics on S and one may ask when a given curve $\alpha \in \mathcal{C}(S)$ is shorter than some fixed constant. In (Raf05), Rafi proved that the hyperbolic length of a curve along a Teichmüller geodesic, $\mathcal{G}$, is shorter than the constant from Minsky's Product Regions theorem (Theorem 2.13) at some point along $\mathcal{G}$ if the vertical and horizontal foliations which determine $\mathcal{G}$ heavily overlap on a subsurface of which that curve is a boundary component.

In its sibling paper, (Raf07), Rafi took this condition on foliations and translated it into the context of the curve complex. He proves $\mathcal{G}$ enters the thin part of $\mathcal{T}(S)$ of a subsurface $Y \subset S$ if and only if the curves which constitute ∂Y are short along $\mathcal{G}$, which happens if and only if Y is filled by subsurfaces to whose curve complexes the vertical and horizontal foliations have sufficiently large projections. In addition, he adapted the Masur-Minsky coarse distance formula for $\mathcal{MCG}(S)$ to obtain a coarse distance formula for $\mathcal{T}(S)$ with the Teichmüller metric (Theorem 2.15 below).

1.2 Elliptic actions on $\mathcal{T}(S)$

The Nielsen Realization Problem asks whether a finite subgroup $H \leqslant \mathcal{MCG}(S)$ of the mapping class group of a surface S can be realized as a subgroup $\widetilde{H} \leqslant \mathrm{Homeo}^+(S)$ which acts by isometries on some metric $\sigma \in \mathcal{T}(S)$ on S. While a finite subgroup of isometries of a negatively curved metric space always has fixed points, Masur (Mas75) showed that the Teichmüller metric is not negatively curved in his thesis. Kerckhoff (Ker83) proved that the problem in $\mathcal{T}(S)$ always has a solution by showing that the length functions of curves are convex along Thurston earthquake paths, a result later mirrored for Weil-Petersson geodesics by Wolpert (Wol87).

Kerckhoff's main theorem in (Ker83) was the following equivalent formulation:

Theorem 1.3 (Theorem 4 in (Ker83))**.** *Every finite subgroup $H \leqslant \mathcal{MCG}(S)$ fixes a point in $\mathcal{T}(S)$.*

A number of facts follow immediately from Kerckhoff's theorem. Let $X \in \mathrm{Fix}(H) \subset \mathcal{T}(S)$ be fixed by H. The quotient $X/H = \mathcal{O}$ is a hyperbolic 2-orbifold and any hyperbolic structure on $\mathcal{O}$ lifts to S, giving an embedding $i : \mathcal{T}(\mathcal{O}) \hookrightarrow \mathcal{T}(S)$ that is an isometry onto its image in the Teichmüller metric. Since $\mathrm{Fix}(H) = i(\mathcal{T}(\mathcal{O}))$, $\mathrm{Fix}(H) \subset \mathcal{T}(S)$ is a convex submanifold of $(\mathcal{T}(S), d_T)$.

In second half of this thesis, we investigate the structure of the set of points in $\mathcal{T}(S)$ which are moved a bounded Teichmüller distance $R > 0$ by the action of H:

$$\mathrm{Fix}_R^T(H) = \{X \in \mathcal{T}(S) | \mathrm{diam}_T(H \cdot X) < R\}$$

We call these points *almost fixed points*. These sets can be viewed as sublevel sets of the diameter map, $\mathrm{diam}_T : \mathcal{T}(S) \to \mathbb{R}$, given by $X \mapsto \mathrm{diam}_T(X)$. From this perspective, $\mathrm{Fix}(H) = \mathrm{diam}_T^{-1}(0)$ and $\mathrm{Fix}_R^T(H) = \mathrm{diam}_T^{-1}([0, R))$.

In a negatively curved space, the level sets of the diameter map would be convex regular neighborhoods of the set of fixed points. However, Masur (Mas75) showed that the Teichmüller metric is not negatively curved and Minsky (Min96) later showed that this assumption fails profoundly: in the thin parts of $\mathcal{T}(S)$, the Teichmüller metric is quasiisometric to a sup metric on a product space (See Theorem 2.13 below).

The results we obtain in this part of the thesis contrast the topological constraints coming from covering theory and the geometric flexibility coming from these product regions. Our second main Theorem 4.24 of this thesis proves that almost fixed points are uniformly close to fixed points:

Theorem 1.4 (Almost fixed points are close to fixed points)**.** *For any $R > 0$, there is a constant R' depending only on R and S such that the following holds. Let $H \leqslant \mathcal{MCG}(S)$ be a finite subgroup and* $\mathrm{Fix}(H) \subset \mathcal{T}(S)$ *its fixed point set. Then*

$$\mathrm{Fix}_R^T(H) \subset \mathcal{N}_{R'}^T(\mathrm{Fix}(H))$$

where $\mathcal{N}_{R'}^T(\mathrm{Fix}(H))$ *is the R'-neighborhood of* $\mathrm{Fix}(H)$*.*

In a CAT(0) space, a *barycenter* for a bounded set E with radius R is the unique point $b \in E$ around which a ball of radius R contains E, $E \subset B_R(b)$. A *coarse barycenter* for a set E is any point $x \in E$ invariant under the symmetries of E such that $E \subset B_{K\cdot\mathrm{diam}(E)+C}(x)$, where $K, C > 0$ are uniform constants and $\mathrm{diam}(E)$ is the diameter of E. Note that a coarse barycenter is a barycenter when $K = \frac{1}{2}$ and $C = 0$.

Using work of Tao (Tao13), we also prove that orbits of finite order elements of $\mathcal{MCG}(S)$ have coarse barycenters:

Theorem 1.5 (Coarse barycenters for $(\mathcal{T}(S), d_T)$)**.** *There are $K, C > 0$ such that for any $\sigma \in \mathcal{T}(S)$ and any finite order $f \in \mathcal{MCG}(S)$, there is a fixed point $X \in Fix(\langle f \rangle)$ such that*

$$d_T(\sigma, X) < K \cdot d_{\mathcal{T}(S)}(\sigma, f \cdot \sigma) + C$$

We note that while Theorem 1.4 follows from Theorem 1.5, the proof of the latter uses the former in an essential way (see Remark 4.28 below).

Both of Theorems 1.4 and 1.5 depend crucially on the fact that $\mathrm{Fix}(H)$ comes from a topological covering map, namely that the subsurfaces involved in the geometric considerations in $\mathcal{T}(S)$ are all lifts of suborbifolds of $\mathcal{O}$.

We say that a subset $Z \subset X$ of a metric space is *L-quasiconvex* if whenever $x, y \in Z$ and $\mathcal{G}_{x,y}$ is a geodesic between them, then $\mathcal{G}_{x,y} \subset \mathcal{N}_L(Z)$.

Recall that $\mathrm{Fix}(H) \subset \mathcal{T}(S)$ is convex in both the Teichmüller and Weil-Petersson metrics. In contrast with Theorems 1.4 and 1.5, the following theorem shows that relaxing the condition of being fixed to being almost-fixed dramatically changes convexity properties:

Theorem 1.6 (Nonquasiconvexity of $\mathrm{Fix}_R^T(H)$)**.** *Let $L > 0$. There exist a constant $R > 0$, a surface S, and a finite subgroup $H \leqslant \mathcal{MCG}(S)$ such that $\mathrm{Fix}_R^T(H)$ is not L-quasiconvex.*

The counterexamples built in Theorem 1.6 are based on work of Rafi (Raf10). See the discussion after the proof of Theorem 1.6 (Theorem 4.29 below) for how nonquasiconvexity of $\mathrm{Fix}_R^T(H)$ is a more general phenomenon.

Many of the tools and ideas in this thesis are motivated by ideas from geometric group theory and the theory surrounding the study of $\mathcal{MCG}(S)$. Quasiconvexity is a central notion in the theory of Gromov hyperbolic groups and is well-suited to this strong notion of negative curvature. Given the product structure on the thin parts, quasiconvexity, and thus convexity, in the Teichmüller metric are sensitive properties. The only known convex subsets of $(\mathcal{T}(S), d_T)$ are its (unique) geodesics, special isometrically embedded copies of $\mathbb{H}^2$ called Teichmüller disks, and the fixed point sets which are at the center of this discussion. As for quasiconvex subsets, the only known additional examples are bounded diameter subsets (LR11), the aforementioned product regions themselves, orbits of convex cocompact subgroups of $\mathcal{MCG}(S)$ (FM02), and certain subsets of metrics on pleated surfaces which fill the convex hull of a hyperbolic 3-manifold homeomorphic to $S \times \mathbb{R}$ (Min93). Theorem 1.6 (and its generalizations) suggest that it may be difficult to naturally enlarge $\mathrm{Fix}(H)$ to an H-invariant quasiconvex subset of $\mathcal{T}(S)$.

CHAPTER 2

PRELIMINARIES

In this chapter, we review the relevant foundational materials and collect the various results we need to invoke in the rest of the thesis.

2.0.1 Conventions and notation

Throughout this thesis, let $S = S_{g,n}$ denote an connected, oriented surface of finite complexity, $\xi(S) = 3g - 3 + n > 0$, with genus g and n punctures.

Our methods and calculations are frequently coarse and we introduce some notation for ease of the exposition. Given two quantities A, B, we write $A \prec B$ if there are constants $K, C > 0$ depending only on the topology of S such that $A \leqslant K \cdot B + C$. If $A \prec B$ and $B \prec A$, then we write $A \asymp B$.

Similarly, given a constant $R > 0$, we write $A \prec_R B$ if there are constants K' and C' depending only on R and the topology of S such that $A \leqslant K' \cdot B + C'$, and the same for $A \succ_R B$ and $A \asymp_R B$.

If we have X, Y, and Z such that $X \asymp_R Y$ and $Y \asymp_R Z$ (or $\prec_R$, $\asymp_R$), then we also have $X \asymp_R Z$, where the constants are worse for the latter coarse inequality. As long as we only make such estimates a uniformly bounded number of times depending only on R and S, the associated constants will still be uniform in R and S.

When we write $A = A(B,C) > 0$, we mean that A is a positive constant depending only on the objects B and C.

2.1 Teichmüller spaces and mapping class groups

The *Teichmüller space* of S, $\mathcal{T}(S)$, is the space of isotopy classes of marked hyperbolic structures on a surface. Formally, $\mathcal{T}(S) = \{(X,f) | f : S \to X\}/$, where S is the model topological surface, X is a hyperbolic surface, $f : S \to X$ is the marking homeomorphism, and the equivalence relationship is defined by (X,f) (Y,g) if and only if $f \cdot g^{-1}$ is isotopic to the identity.

Teichmüller space can be given coordinates called *Fenchel-Nielsen* coordinates roughly as follows. Given a point $X \in \mathcal{T}(S)$, choose a pants decomposition P on S. The curves $\gamma \in P$ determine length parameters $l_X(\gamma)$ simply by measuring the length of the geodesic representative of each curve on the surface. Decomposing the surface into pairs of pants, the lengths of the γ uniquely determine a hyperbolic metric on each pair of pants. There are twisting parameters which determine how the pairs of pants are then glued together to realize the metric X. These parameters give rise to a homeomorphism $\psi : \mathcal{T}(S) \to \mathbb{R}^{2\xi(S)}$, thus coordinatizing $\mathcal{T}(S)$ as a ball. See (FM12, Section 10.6) for more details.

Although Teichmüller space admits many natural metrics, the main two of interest in this thesis are the *Teichmüller* and *Weil-Petersson* metrics.

By the uniformization theorem, $\mathcal{T}(S)$ is the space of conformal classes of metrics on S up to isotopy. For two points $X, Y \in \mathcal{T}(S)$, the Teichmüller distance between X and Y is defined as

$$d_T(X,Y) = \frac{1}{2}\log \inf_h K_h$$

where the infimum is taken over all *quasiconformal* maps $h : X \to Y$ and K_h measures the *quasiconformal dilitation* of h. In (Tei40), Teichmüller proved that the infimum is always realized. As a consequence, Teichmüller geodesics are unique.

The Weil-Petersson metric on $\mathcal{T}(S)$ is a metric dual to a metric defined by an L^2-product on the cotangent bundle of $\mathcal{T}(S)$, the bundle of quadratic differentials. As a consequence of the convexity of length functions along Weil-Petersson geodesics, Wolpert (Wol86) proved that $(\mathcal{T}(S), d_{WP})$ has unique geodesics despite the fact that it is incomplete.

The mapping class group, $\mathcal{MCG}(S) = \mathrm{Homeo}^+(S)/\mathrm{Homeo}_0(S)$, is the group of orientation-preserving homeomorphisms modulo those which are isotopic to the identity. Teichmüller space admits a natural action of $\mathcal{MCG}(S)$ by changing the marking, namely given $g \in \mathcal{MCG}(S)$, $g \cdot (X, f) = (X, f \cdot g^{-1})$.

For the Teichmüller metric, see the books of Hubbard (Hub) and Papadopoulos (Pap07); see also the survey of Masur (Mas10). For the Weil-Petersson metric, see Ahlfors (Ahl61); see also the survey of Wolpert (Wol07). For the mapping class group, see the book of Farb-Margalit (FM12).

2.2 Curves, curve complexes, and subsurface projections

The *complex of curves* of S, denoted $\mathcal{C}(S)$, is a simplicial complex whose simplices consist of disjoint collections of isotopy classes of simple closed curves on S. In the case where S is a once-punctured torus or four-holed sphere, minimal intersection replaces disjointness as the adjacency relation. For Y_α an annulus in S with core curve α, $\mathcal{C}(Y_\alpha) = \mathcal{C}(\alpha)$ is the simplicial complex with vertices consisting of paths between the two boundary components of the metric compactification of $\widetilde{Y}_\alpha$, the cover of S corresponding to Y_α, up to homotopy relative to fixing the endpoints on the boundary; two paths are connected by an edge if they have disjoint interiors.

We will be considering only the 1-skeleton of $\mathcal{C}(S)$ with its path metric. Endowed with this metric, we have the following foundational theorem of Masur and Minsky (MM99):

Theorem 2.1. *$\mathcal{C}(S)$ is infinite-diameter and Gromov hyperbolic.*

Remark 2.2. *By now, there are many proofs of the latter fact (Bow03), several of which are recent and give a uniform hyperbolicity constant (Aou12), (Bow12),(CRS13), (HPW13). See (HPW13) for an especially nice proof.*

The curve complex is locally infinite, but the links of vertices are often (products of) Gromov hyperbolic graphs, which gives us a substantial amount of control over the global geometry of $\mathcal{C}(S)$, via the hierarchy machinery in (MM00).

Consider a curve $\alpha \in \mathcal{C}(S)$. Then the link of α is $\mathcal{C}(S \backslash \alpha)$, where $\mathcal{C}(S\backslash\alpha)$ is the join $\mathcal{C}(S_1) * \mathcal{C}(S_2)$ if α is separating and $S\backslash\alpha = S_1 \coprod S_2$. More generally, if $Y \subset S$ is any proper subsurface, then $\mathcal{C}(Y)$ lives in the 1-neighborhood of $\partial Y \subset \mathcal{C}(S)$.

We are often interested in understanding the combinatorial relationship between two curves or simplices of $\mathcal{C}(S)$ from the perspective of $\mathcal{C}(Y)$ for some subsurface $Y \subset S$. Let $\alpha \subset \mathcal{C}(S)$ be any simplex and let $Y \subset S$ be any subsurface of S which is not a pair of pants. The *subsurface projection* of α to Y is the canonical completion of the arcs in $\alpha \cap Y$ along the boundary of a regular neighborhood of $\alpha \cap Y$ and ∂Y to curves in Y. We denote this projection by $\pi_Y(\alpha)$ and remark that it is a simplex in $\mathcal{C}(Y)$. See Section 2 of (MM00) for more details.

For any two simplices $\alpha, \beta \subset \mathcal{C}(S)$ and subsurface $Y \subset S$, we use the shorthand $d_Y(\alpha, \beta) = d_Y(\pi_Y(\alpha), \pi_Y(\beta))$.

Subsurface projections are essential objects in the Masur-Minsky hierarchy machinery. One of the main outputs of that machinery is the distance formula for $\mathcal{M}(S)$, Theorem 2.10 below.

See (MM99), (MM00), and Schleimer's notes (Schleim) for basics on curve complexes.

2.3 Pants and markings

The section briefly introduces two fundamental players in the geometric-combinatorial approach of Brock, Masur-Minsky, and Rafi.

A *pair of pants* on S is a maximal simplex in $\mathcal{C}(S)$, whose complement in S is a disjoint collection of three-holed spheres. The *pants complex*, denoted $\mathcal{P}(S)$, is a simplicial complex whose vertices are pairs of pants and two pairs of pants P_1, P_2 are connected by an edge if there are two curves $\alpha \in P_1, \beta \in P_2$ such that $P_1 \backslash \alpha = P_2 \backslash \beta$, with α intersecting β minimally.

We frequently use the following insight of Brock (Br03):

Theorem 2.3. *The pants complex $\mathcal{P}(S)$ is $\mathcal{MCG}(S)$-equivariantly quasiisometric to Teichmüller space with the Weil-Petersson metric, $(\mathcal{T}(S), d_{WP})$.*

In (MM00), Masur-Minsky introduce a quasiisometry model for $\mathcal{MCG}(S)$ called the *marking complex*, denoted $\mathcal{M}(S)$.

A *complete marking*, μ, on a surface S is a collection of *transverse pairs*, (α, t_α), where the α form a pants decomposition of S, called the *base* of μ, denoted *base*(μ), and each t_α is a diameter-1 set of vertices in the annular complex $\mathcal{C}(\alpha)$ (see Section 2.4 of (MM00)), called the set of *transversals*. In addition, we assume that markings are *clean*, which means that the only base curve each transversal t_α intersects is its paired base curve, α.

We remark that, in any complete clean marking, each transversal intersects either one or two other transversals. Indeed, since the base curves form a pants decomposition, one can decompose S into a collection of pairs of pants where the base curves form the cuffs and the transverse curves are cut into essential arcs in the pairs of pants. In each pair of pants, each transverse arc must intersect exactly one other transverse arc. In the case that α is two cuffs in one pair of pants (that is, α and t_α fill a one-holed torus), t_α intersects only one other transverse curve; otherwise, each transverse curve intersects two others.

The *marking complex* of S, denoted $\mathcal{M}(S)$, is a graph whose vertices are complete clean markings and two markings are connected by an edge if they can be related by one of two types of *elementary moves*, called *twists* and *flips*, which we define now.

Given a marking μ and a pair (α, t_α) in μ, a *twist move* around α involves replacing μ with $T_\alpha(\mu)$, where T_α is a Dehn twist or half-twist around α, depending on whether $\alpha \cup t_\alpha$ fills a once-puncture torus or a four-holed sphere, respectively. By construction, t_α is the only curve in μ which intersects α, so this reduces to $(\alpha, t_\alpha) \mapsto (\alpha, T_\alpha(t_\alpha))$.

Given a pair (α, t_α), a *flip move* performed at α involves a flip $(\alpha, t_\alpha) \mapsto (t_\alpha, \alpha)$ and some extra changes to preserve cleanliness, which we now explain. As noted above, each transverse curve intersects (either one or two) others, so now that a transverse curve has become a base curve, at least one other transverse pair has been made unclean. In [Lemma 2.4, (MM00)], Masur and Minsky show that by choosing replacement transversals to minimize distance in the annular curve complexes of their bases, one has a finite number of possible new transversals which are all uniformly close to each other. The purpose of this cleaning is to preserve the twisting data around α while allowing for future flip moves to occur without the resulting base sets failing to be pants decompositions.

In the rest of the thesis, we assume that all markings are clean and complete.

Definition 2.4 (Subsurface projections for markings)**.** *We will be interested in subsurface projections for markings. For any $\mu \in \mathcal{M}(S)$ and $Y \subset S$ any subsurface which is not an annulus whose core is in base(μ), we define the* subsurface projection of μ to $\mathcal{C}(Y)$ *by $\pi_Y(\mu) = \pi_Y(base(\mu))$. In the case that Y is an annulus with core $\alpha \in base(\mu)$ with transversal t_α, then $\pi_Y(\mu) = t_\alpha$.*

We now define the projection of a marking on S to a marking on a subsurface, a construction we need in the construction the preferred paths for $\mathcal{AM}(S)$ in Subsection 3.2.2.

Definition 2.5 (Projections of markings to markings on subsurfaces)**.** *Let $\mu \in \mathcal{M}(S)$ and $Y \subset S$ be any subsurface. We build $\pi_{\mathcal{M}(Y)}(\mu)$ inductively as follows. Choose a curve $\alpha_1 \in \pi_Y(\mu)$, then build a pants decomposition on Y by choosing $\alpha_i \in \pi_{Y \setminus \bigcup_{j=1}^{i-1} \alpha_j}(\mu)$. From this pants decomposition, build a marking on Y by choosing transverse pairs $(\alpha_i, \pi_{\alpha_i}(\mu))$. We define $\pi_{\mathcal{M}(Y)}(\mu) \subset \mathcal{M}(Y)$ to be the collection of all markings resulting from varying the choices of the α_i.*

Lemma 2.4 in (MM00) and Lemma 6.1 of (Ber03) show that the freedom in this process builds a bounded diameter subset of $\mathcal{M}(Y)$. We remark however that if $\partial Y \subset base(\mu)$, then $\pi_{\mathcal{M}(Y)}(\mu)$ is a unique point in $\mathcal{M}(Y)$, since every curve in $base(\mu)$ either projects to itself in $\mathcal{C}(Y)$ or has an empty projection.

Remark 2.6. *The process of constructing $\pi_{\mathcal{M}(Y)}(\mu)$ preserves any curve $\alpha \in base(\mu)$ which happens to lie in Y, for $\alpha \in \pi_Y(\mu)$ and π_Y preserves disjointness. Otherwise, we could have chosen to build $\pi_{\mathcal{M}(Y)}(\mu)$ by first preferentially choosing curves in $base(\mu)$ which lie in Y.*

2.4 Hierarchies, hierarchy paths, and large links

We now briefly outline the features of the Masur-Minsky hierarchies we need. The main references for the hierarchy theory are (MM00) and (Min03), and we will point the reader to the corresponding sections when possible; the initial exposition begins in Section 4 of (MM00). See also the theses of Tao (Tao13) and Behrstock (Ber03) for nice introductions to the theory.

Given any two markings $\mu_1, \mu_2 \in \mathcal{M}(S)$, a *hierarchy*, H, between μ_1 and μ_2 is family of special geodesics $g_Y \subset \mathcal{C}(Y)$ with partial markings associated, denoted $\mathbf{I}(g_Y)$ and $\mathbf{T}(g_Y)$; see the lead up to the technical Proposition 3.28 below for more details on these markings. Each such geodesic is supported on a distinct subsurface $Y \subset S$, such that the geodesics satisfy a number of subordinancy relations among the g_Y determined by the associated partial markings; see Subsection 4.1 of (MM00). In particular, there is a distinguished *base geodesic* $g_H \subset \mathcal{C}(S)$ with $\mathbf{I}(g_H) = \mu_1$ and $\mathbf{T}(g_H) = \mu_2$. If $Y \subset S$ is a subsurface and there is some geodesic $g_Y \subset \mathcal{C}(Y)$ with $g_Y \in H$, then we say Y is the *domain of support* of g_Y and that Y supports a geodesic in H. Given any $g_Y \in H$ for Y nonannular and any curve $\alpha \in g_Y$, there are disjoint domains $\bigcup_{i=1}^{k} Y_i = S\backslash\alpha$ (including the annular domain Y_α) and geodesics $g_{Y_i} \in H$.

From any vertex $\alpha \in g_H$, one can can build markings, $\mu_\alpha \in \mathcal{M}(S)$, with $\alpha \in base(\mu_\alpha)$ by first choosing a vertex $\alpha_Y \in g_Y \in H$ for each component Y of $S\backslash\alpha$ for a curve in $base(\mu_\alpha)$. For each such α_Y, one can then obtain another base curve by choosing a vertex along any $g_Z \in H$ where Z is a component of $Y\backslash\alpha_Y$; recall that H contains such geodesics. One completes such a marking μ_α by proceeding inductively to the level of annuli, at which point one has assembled $base(\mu_\alpha)$ and can complete μ_α by choosing a vertex on the geodesics g_β for each $\beta \in base(\mu_\alpha)$.

While nearly all markings constructed in this manner will be unrelated to μ_1 and μ_2, one can use the subordinancy relations to piece together the geodesics in H into finite paths of markings in $\mathcal{M}(S)$ between μ_1 and μ_2 called a *hierarchy path* based on H. Loosely, the process of resolving a hierarchy involves progressing along geodesics in H, at bottom progressing along g_H and then at each vertex, progressing along geodesics whose domains live in the complement of that vertex, etc., conditional upon the subordinancy relations; see Section 5 of (MM00), especially Proposition 5.4 and Lemma 5.5. We remark that a hierarchy path is a path in $\mathcal{M}(S)$ and thus a sequence of elementary flip and twist moves connecting its endpoints.

Remark 2.7. *It follows from a careful understanding of the Masur-Minsky machinery that hierarchy paths are quasigeodesics in $\mathcal{M}(S)$, though it was not previously made explicit. The Masur-Minsky distance formula (Theorem 2.10 below) implies that hierarchy paths are globally efficient. However, a quasigeodesic must also be locally efficient. One can show that a restriction of a hierarchy path to a subsegment can be reconstructed as a hierarchy path based* *on a hierarchy obtained by restricting the original hierarchy. Another application of Theorem 2.10 then implies that hierarchy paths are locally efficient and hence quasigeodesics. This is the content of Proposition 3.28.*

For arbitrary distinct domains, Y and Y', one can often construct hierarchy paths where the orders of appearance of Y and Y' can be reversed. However, there is a partial ordering on the domains of geodesics in H, called the *time-order*, which is defined in terms of the subordinancy relations and determines when some domains must coarsely come before others along any hierarchy path; if Y is time-ordered before Y', we write $Y \prec_t Y'$. While two properly nested subsurfaces will not be time-ordered, a sufficient topological condition for two domains Y and Y' to be time-ordered is called *interlocking*, when neither Y nor Y' properly contains the other but $Y \cap Y' \neq \varnothing$; this is part of the content of Lemma 4.18 in (MM00). See subsection 4.6 of that paper for more details on time-ordering and Definition 4.16 for the precise meaning.

Remark 2.8 (Disjoint but time-ordered domains). *While interlocking is a sufficient condition for two surfaces to be time-ordered, it is not necessary. Indeed, disjoint but time-ordered domains in a hierarchy are a significant technical obstacle.*

Remark 2.9 (Time-order for curves). *As a convention, we say that two curves α, β are time-ordered in a hierarchy H if their are the core curves of annuli $Y_\alpha, Y_\beta \subset S$ and Y_α and Y_β are time-ordered in H.*

One of the main results of the hierarchy machinery is the inspirational Masur-Minsky distance formula for $\mathcal{M}(S)$:

Theorem 2.10 ($\mathcal{M}(S)$ distance formula; Theorem 6.12 of (MM00))**.** *There exists a $K > 0$ such that following holds: For any $k > K$, there are $E_1, E_2 > 0$, such that for any $\mu_1, \mu_2 \in \mathcal{M}(S)$*

$$\frac{1}{E_1}\sum_{Y\subset S}[d_Y(\mu_1,\mu_2)]_k - E_2 \leqslant d_{\mathcal{M}(S)}(\mu_1,\mu_2) \leqslant E_1\sum_{Y\subset S}[d_Y(\mu_1,\mu_2)]_k + E_2$$

where $[x]_k = x$ if $x > k$ and 0 otherwise.

Given any pair of markings $\mu_1, \mu_2 \in \mathcal{M}(S)$, we say that a subsurface $Y \subset S$ is a *large link* for μ_1 and μ_2 if $d_Y(\mu_1, \mu_2) > K$. Lemma 6.12 of (MM00) tells us large links are the main building blocks of hierarchy paths:

Lemma 2.11 (Lemma 6.12 in (MM00))**.** *Let $\mu_1, \mu_2 \in \mathcal{M}(S)$, let $Y \subset S$ a subsurface, and let K be as in Theorem 2.10. If $d_Y(\mu_1, \mu_2) > K$, then Y supports a geodesic $g_Y \in H$ for any hierarchy H between μ_1 and μ_2.*

Remark 2.12 (Large link)**.** *The intuition behind the term large link is as follows: If $Y \subset S$ is a large link for μ_1, μ_2, we know from Lemma 2.11 that Y supports some geodesic $g_Y \in H$; moreover, Y will necessarily appear as the component of some $Z\backslash\alpha$ where $Z \subset S$ is a subsurface supporting a geodesic $g_Z \in H$ and $\alpha \in g_Z$. While the length of g_Y in $\mathcal{C}(Y)$ is $d_Y(\mu_1, \mu_2) > K$, g_Y lives in the link of $\alpha \in g_Z$ as a path in $\mathcal{C}(Z)$, and hence the link of α is large from the viewpoint of μ_1 and μ_2.*

2.4.1 The thick part and Minsky's product regions

One of the main corollaries to the hyperbolicity of $\mathcal{C}(S)$ in (MM99) is Theorem 1.2, which states that the electrification of $(\mathcal{T}(S), d_T)$ is quasiisometric to $\mathcal{C}(S)$. In contrast, Minsky showed in Theorem 6.1 of (Min96) that the thin regions of $(\mathcal{T}(S), d_T)$, where at least one curve is short, are quasiisometric to a product space with its sup metric.

Let $\gamma = \gamma_1, \ldots, \gamma_n$ be a simplex in $\mathcal{C}(S)$, and let $Thin_\epsilon(S, \gamma) = \{\sigma \in \mathcal{T}(S) \big| l_\sigma(\gamma_i) \leqslant \epsilon\}$, where $l_\sigma(\gamma_i)$ is the hyperbolic length of γ_i in σ, for each i. Let

$$\mathcal{T}_\gamma = \mathcal{T}(S \backslash \gamma) \times \prod_{\gamma_i \in \gamma} \mathbb{H}_{\gamma_i} \tag{2.1}$$

be endowed with the sup metric, where $S \backslash \gamma$ a disjoint union of punctured surfaces and each $\mathbb{H}_{\gamma_i}$ is a *horodisk*, that is, a copy of the upper half-plane model of $\mathbb{H}^2$ with imaginary part $\geqslant 1$.

Theorem 2.13 (Product regions; Theorem 6.1 in (Min96))**.** *The Fenchel-Nielsen coordinates on $\mathcal{T}(S)$ give rise to a natural homeomorphism $\Pi : \mathcal{T}(S) \to \mathcal{T}_\gamma$, and for $\epsilon > 0$ sufficiently small, this homeomorphism restricted to $Thin_\epsilon(S, \gamma)$ distorts distances by a bounded additive amount.*

For the rest of this chapter, fix $\epsilon > 0$ to be sufficiently small so that 2.13 holds. When we say that a curve α is *short* for some $\sigma \in \mathcal{T}(S)$, we mean that $l_\sigma(\alpha) < \epsilon$.

Remark 2.14. *Up to quasiisometry, we may take the sup or product metric on the product space in Equation 2.1, though Minsky's version with the sup metric is finer and results in only an additive error.*

2.4.2 Rafi's combinatorial model

The main result of (Raf07) is an adaptation of the machinery in (MM00) to the setting of $(\mathcal{T}(S), d_T)$. In particular, Rafi obtains a distance estimate in Theorem 6.1 of (Raf07) analogus to the Masur-Minsky formula (Theorem 2.10 above), restated below in Theorem 2.15.

Given $\sigma \in \mathcal{T}(S)$, a *shortest marking* $\mu_\sigma \in \mathcal{M}(S)$ for σ is a marking inductively built by choosing a shortest curve in $\alpha_1 \in \mathcal{C}(S)$ on σ, then choosing a shortest curve $\alpha_2 \in \mathcal{C}(S \backslash \alpha_1)$, etc., until one has arrived at a shortest pants decomposition of S. One completes this to a shortest marking by choosing shortest curves β_i which intersect α_i but not α_j for $j \neq i$. The result is a complete, clean marking, of which there are finitely-many by [(MM00), Lemma 2.4]. We note that the collection of curves which are shorter in σ than the constant ϵ in Minsky's Theorem 2.13 form a simplex in $\mathcal{C}(S)$ by the Collar Lemma. Thus in the case that $\sigma \in Thin_\gamma$ for some simplex $\gamma \subset \mathcal{C}(S)$, we necessarily have $\gamma \subset base(\mu_\sigma)$.

Theorem 2.15 (Rafi's formula; Theorem 6.1 in (Raf07)). *Let $\epsilon > 0$ be as in Theorem 2.13. There exists $k > 0$ such the following holds:*

Let $\sigma_1, \sigma_2 \in \mathcal{T}(S)$, define Λ to be the set of curves short in both σ_1 and σ_2, and define Λ_i to be the set of curves short in σ_1 and not in Λ. Let μ_i be the shortest marking for σ_i. Then

$$d_{\mathcal{T}}(\sigma_1, \sigma_2) \asymp \sum_Y [d_Y(\mu_1, \mu_2)]_k + \sum_{\alpha \notin \Lambda} \log [d_\alpha(\mu_1, \mu_2)]_k + \max_{\alpha \in \Lambda} d_{\mathbb{H}_\alpha}(\sigma_1, \sigma_2) + \max_{\substack{\alpha \in \Lambda_i \\ i=1,2}} \log \frac{1}{l_{\sigma_i}(\alpha)} \quad (2.2)$$

In remarks after Corollary 3.32, we describe how to use Rafi's formula (Equation 2.2) to build a path in $\mathcal{AM}(S)$ (defined below in Section 3.1) which coarsely realizes (Equation 2.2) as its length. This path, while globally moving efficiently, can contain subpaths which involve arbitrary amounts of backtracking, coming from the fourth term on the right hand side. In Section 3.2, we build preferred quasigeodesic paths in $\mathcal{AM}(S)$, called *augmented hierarchy paths*, which coarsely have length Equation 2.2 and avoid this unbounded backtracking.

2.4.3 Combinatorial horoballs

Combinatorial horoballs were introduced by Groves and Manning in (GM08) in the context of relatively hyperbolic groups; see (CC92) for an earlier, similar construction. In particular, suppose that G is a finitely-generated group and $\mathcal{P} = \{P_1, \ldots, P_n\}$ is a finite collection of finitely-generated subgroups of G. Among other equivalences, in [Theorem 3.25, (GM08)] they showed that the augmentation of the Cayley graph of G by combinatorial horoballs along the subgroups in $\mathcal{P}$ is hyperbolic if and only if G is relatively hyperbolic to $\mathcal{P}$ in the sense of Gromov.

While $\mathcal{MCG}(S)$ is not relatively hyperbolic to any family of subgroups (BDM08, Theorem 8.1), the process of adding efficient paths to the marking complex via combinatorial horoballs to build the augmented marking complex is reminiscent of and indeed inspired by the relatively hyperbolic construction. We use combinatorial horoballs to model the hyperbolic upper half-planes which appear in the product structure of the thin parts discovered by Minsky (Min96) in Theorem 2.13. We fully explain the construction of $\mathcal{AM}(S)$ in the next section.

Definition 2.16 (Combinatorial horoball)**.** *Let X be any simplicial complex. The* combinatorial horoball based on X, $\mathcal{H}(X)$, *is the 1-complex with vertices $\mathcal{H}(X)^{(0)} = X^{(0)} \times (\{0\} \cup \mathbb{N})$ and edges as follows:*

- *If $x, y \in X^{(0)}$ and $n \in \{0\} \cup \mathbb{N}$ such that $0 < d_X(x,y) \leqslant e^n$, then (x,n) and (y,n) are connected by an edge in $\mathcal{H}(X)$.*
- *If $x \in X^{(0)}$ and $n \in \{0\} \cup \mathbb{N}$, then (x,n) is connected to $(x, n+1)$ by an edge.*

The metric on $\mathcal{H}(X)$ is the path metric, where each edge is isometric to $[0,1]$.

Remark 2.17. *X sits inside of $\mathcal{H}(X)$ as the full subgraph containing the vertices $X^{(0)} \times \{0\}$.*

As with horoballs in $\mathbb{H}^n$, combinatorial horoballs are uniformly hyperbolic:

Theorem 2.18 (Theorem 3.8 in (GM08))**.** *Let X be any simplicial complex. Then $\mathcal{H}(X)$ is δ-hyperbolic where δ is independent of X.*

Remark 2.19. *The combinatorial horoballs we use are a simple case of the above, for X is the orbit of a Dehn twist or half-twist and thus a copy of $\mathbb{Z}$.*

We need the understand geodesics in combinatorial horoballs. Fortunately, they have a nice description from Lemma 3.10 in (GM08):

Lemma 2.20 (Lemma 3.10 in (GM08))**.** *Let $\mathcal{H}(X)$ be a combinatorial horoball and $x, y \in \mathcal{H}(X)$ distinct vertices. Then there is a geodesic $\gamma(x,y) = \gamma(y,x)$ between x and y which consists of at most two vertical segments and a single horizontal segment of length at most 3. Moreover, any other geodesic between x and y is Hausdorff distance at most 4 from this geodesic.*

Following [(GM08), Section 5.1], we define preferred paths for $\mathcal{H}(X)$.

Suppose that $x, y \in X$ have $d_X(x,y) = C$. For any $(x,a), (y,b) \in \mathcal{H}(X)$, consider the path between these two points which consists of (at most) three segments: a vertical segment from (x,a) to $(x, \lceil \ln C \rceil)$, a horizontal segment of one edge from $(y, \lceil \ln C \rceil)$, and another vertical segment from $(y, \lceil \ln C \rceil)$ to (y,b). In the case that a or $b \geqslant \ln C$, then the respective vertical segment is not included and the horizontal segment connects at either height a or b, depending on whether or not $a \geqslant b$.

These paths are not geodesics (which are similar but will differ slightly in vertical height depending on the divisibility of C), but they are quasigeodesics which are a uniformly bounded distance from geodesics, with the bound independent of X. This can be seen from the easily verified fact that no geodesic can contain a horizontal segment of length greater than 5 (see Figure 3 in the proof of Lemma 3.11 in (GM08)). Because they are easy to define, these are the preferred paths through horoballs we consider in what follows.

CHAPTER 3

THE AUGMENTED MARKING COMPLEX OF A SURFACE

In this chapter, we prove our first Main Theorem 1.1, in which we construct our $\mathcal{MCG}(S)$-equivariant quasiisometry model for $(\mathcal{T}(S), d_T)$.

3.1 Construction of $\mathcal{AM}(S)$

The main idea of the construction of $\mathcal{AM}(S)$ is to model the product regions discovered by Minsky (Min96) using $\mathcal{M}(S)$ as the thick part. We begin by showing a combinatorial horoball over an orbit of a Dehn twist or half-twist in $\mathcal{M}(S)$ is quasiisometric to a horodisk. We then define $\mathcal{AM}(S)$ as a graph and make some observations about its structure. We finish the section by defining the maps identifying $\mathcal{AM}(S)$ with $\mathcal{T}(S)$ and prove some basic facts about the identification.

3.1.1 The horoballs $\mathcal{H}_\alpha$ are quasiisometric to horodisks

Let $\mathcal{H}_{(\alpha,t_\alpha)}$ be the combinatorial horoball over the orbit of the action of $\langle T_\alpha \rangle$ on μ, where μ contains a transverse pair (α, t_α). A typical point in $\mathcal{H}_{(\alpha,t_\alpha)}$ is of the form $(\alpha, T_\alpha^k(t_\alpha), n)$, where $T_\alpha^k(t_\alpha)$ records the horizontal position, n records the vertical position, and α and t_α identify the particular horoball. When the context is clear, we write $(\alpha, T_\alpha^k(t_\alpha), n) = (k, n)$. We also frequently suppress the transverse curve when referring to a horoball and simply write $\mathcal{H}_\alpha$ when the context is clear.

We begin this section with an elementary proof of the fact that horodisks are quasiisometric to combinatorial horoballs over orbits of Dehn twists or half-twists. In order to do this, we use a set of criteria for a map to be a quasiisometry from the lemma in Subsection 4.2 of (CC92):

Lemma 3.1. *Let X and Y be spaces with path metrics. In order for $\phi : X \to Y$ to be a quasiisometry, it suffices that*

1. *for some $L > 0$, $Y \subset N_L(\phi(X))$;*
2. *for some $K > 0$ and for all $x_1, x_2 \in X$, $d_Y(\phi(x_1), \phi(x_2)) \leqslant K \cdot d_X(x_1, x_2)$; and*
3. *for each $M > 0$ there exists an $N > 0$ such that if $d_X(x_1, x_2) > N$ then $d_Y(\phi(x_1), \phi(x_2)) > M$.*

Proposition 3.2 (Horoballs are quasiisometric to horodisks)**.** *Let $\mu \in \mathcal{M}(S)$, (α, t_α) a transverse pair in μ, and $\mathcal{H}_\alpha$ the combinatorial horoball over the orbit of the action of $\langle T_\alpha \rangle$ on μ. Then $\mathcal{H}_\alpha$ with the path metric is quasiisometric to a horodisk with the Poincaré metric.*

Proof of Proposition 3.2. Let Δ be the standard horodisk with the Poincaré metric. Define a map $\phi : \mathcal{H}_\alpha \to \Delta$ by $\phi(\alpha, T_\alpha^k(t_\alpha), n) = \phi(k, n) = (k, e^n)$. We verify that ϕ satisfies the conditions from Lemma 3.1.

To see that $\phi(\mathcal{H}_\alpha)$ is quasidense in Δ and thus satisfies condition 1, observe that $\phi(\mathcal{H}_\alpha)$ is all the points of the form (n, e^k), where $n, k \in \mathbb{Z}_{\geqslant 0}$. Since the Δ-distance between two horizontally adjacent vertices in $\phi(\mathcal{H}_\alpha)$ is uniformly bounded by the distance between two vertices at height 1, every point in Δ is at most distance 1 from a vertical geodesic line in $\phi(\mathcal{H}_\alpha)$. Similarly, the distance between two vertically adjacent vertices in $\phi(\mathcal{H}_\alpha)$ is bounded by $\frac{e-1}{e}$. Thus $\phi(\mathcal{H}_\alpha)$ is quasidense in Δ.

We now verify condition 2 on endpoints of edges of $\mathcal{H}_\alpha$. Vertical edges are geodesics in $\mathcal{H}_\alpha$ and ϕ sends them to vertical segments which are geodesics of the same length in Δ. Similarly, a horizontal edge in $\mathcal{H}_\alpha$, connecting (k_1, n) and (k_2, n) where $|k_1 - k_2| < e^n$, is a geodesic of length 1. A calculation verifies that the $d_\Delta\left((k_1, e^n), (k_2, e^n)\right)$ is bounded by $\frac{1}{\sqrt{2}}$, confirming condition 2.

Finally, we check condition 3. Suppose that we have $x_1 = (k_1, n_1), x_2 = (k_2, n_2) \in \mathcal{H}_\alpha$ such that $d_\Delta\left((k_1, e^{n_1}), (k_2, e^{n_2})\right)$ is bounded. We claim that implies $|k_1 - k_2|$ and $|n_1 - n_2|$ are bounded. From this, it follows immediately that $d_{\mathcal{H}_\alpha}\left((k_1, n_1), (k_2, n_2)\right)$ is bounded, confirming condition 3 for the vertices.

Now we check condition 3 for points in the interior of the edges. Assume that at least one of $|k_1 - k_2|, |n_1 - n_2|$ is large, for a contradiction. As noted above, ϕ sends vertical geodesics in $\mathcal{H}_\alpha$ to vertical geodesics in Δ of the same length, so if $k_1 = k_2$, then $d_{\mathcal{H}_\alpha}(x_1, x_2) = d_\Delta(\phi(x_1), \phi(x_2))$, so we may assume $k_1 \neq k_2$. Without loss of generality, assume that $k_1 < k_2$ and $n_1 \leqslant n_2$. Consider the Δ-geodesic triangle $\bigtriangledown$ with vertices $\bar{a} = [(k_1, e^{n_1}), (k_1, e^{n_2})], \bar{b} = [(k_1, e^{n_2}), (k_2, e^{n_2})], \bar{c} = [(k_1, e^{n_1}), (k_2, e^{n_2})]$; we note that $|\bar{c}|_\Delta = d_\Delta(\phi(x_1), \phi(x_2))$.

Since we are assuming that $|\bar{c}|$ is bounded, our assumption that one of $|k_1 - k_2|$ or $|n_1 - n_2|$ is large implies that one of $|\bar{a}|$ or $|\bar{b}|$ is large. It follows immediately the triangle inequality that both $|\bar{a}|$ and $|\bar{b}|$ are large. By δ-hyperbolicity of Δ, $\bigtriangledown$ is δ-thin. Note that angle in $\bigtriangledown$ at the vertex (k_1, e^{n_2}) where $\bar{a}$ and $\bar{b}$ meet is bigger than $\frac{\pi}{2}$. If we parametrize $\bar{a}$ and $\bar{b}$ moving away from (k_1, e^{n_2}) by $f_{\bar{a}} : [0, |\bar{a}|] \to \Delta$ and $f_{\bar{b}} : [0, |\bar{b}|] \to \Delta$, then $d_\Delta(f_{\bar{a}}(t), f_{\bar{b}}(t)) > \delta$ for $t > \delta$. Thus δ-thinness of $\bigtriangledown$ implies that $\bar{c}$ must be δ-close to $\bar{a}$ and $\bar{b}$ for almost their entire lengths. Since they were long , it implies that $\bar{c}$ must have been long, a contradiction. $\square$

3.1.2 Building $\mathcal{AM}(S)$ from $\mathcal{M}(S)$

We are now ready to define the augmented marking complex for a surface, denoted $\mathcal{AM}(S)$. $\mathcal{AM}(S)$ is a simplicial 1-complex with vertices and edges as follows.

A vertex $\tilde{\mu} \in \mathcal{AM}^{(0)}(S)$, called an *augmented marking*, is a complete clean marking, $\pi_{\mathcal{M}(S)}(\tilde{\mu}) = \mu \in \mathcal{M}(S)$ along with a collection of lengths for the curves in $base(\mu) = \{\alpha_1, \ldots, \alpha_n\}$:

$$\tilde{\mu} = (\mu, D_{\alpha_1}(\tilde{\mu}), \ldots, D_{\alpha_n}(\tilde{\mu}))$$

where the $D_{\alpha_i}(\mu)$ are nonnegative integers. The $D_{\alpha_i}(\tilde{\mu})$ are called the *length data* of $\tilde{\mu}$. When the context is clear, we shorten this to D_α. We also write $(\alpha, t_\alpha, D_\alpha) \in \tilde{\mu}$ if $\alpha \in base(\tilde{\mu})$ with transverse curve t_α and length D_α.

Remark 3.3 (Thick and thin)**.** *The integer D_{α_i} coarsely stands in for how short α_i is in a given augmented marking, in terms of extremal (or hyperbolic) length, with D_{α_i} positive implying α_i is short; this analogy is made explicit in the definition of the map $G : \mathcal{AM}(S) \to \mathcal{T}(S)$ in Subsection 3.1.3 below. When $D_{\alpha_i}(\tilde{\mu}) = 0$ for all $\alpha_i \in base(\mu)$, we say that $\tilde{\mu}$ is* in the thick part of $\mathcal{AM}(S)$*. Similarly, if $D_{\alpha_i}(\tilde{\mu}) > 0$, we say α_i is* short *in $\tilde{\mu}$ and $\tilde{\mu}$ is in the α_i*-thin part *of $\mathcal{AM}(S)$. More generally, let $\rho \subset \mathcal{C}(S)$ be a simplex. We say that $\tilde{\mu} \in \mathcal{AM}(S)$ is* *in the* ρ-thin part *of* $\mathcal{AM}(S)$ *if* $D_\alpha(\tilde{\mu}) > 0$ *for each* $\alpha \in \rho$*. If, in addition,* $D_\beta(\tilde{\mu}) = 0$ *for all* $\beta \in \mathcal{C}(S \backslash \rho)$*, we say that* $\tilde{\mu}$ *is* thick relative to ρ*.*

There are three types of edges in $\mathcal{AM}^{(1)}(S)$. The first type is the elementary *flip move* from $\mathcal{M}(S)$. The second type is a *twist move*, which comes from bundles of elementary twist moves from $\mathcal{M}(S)$ and corresponds to a horizontal edge in a combinatorial horoball. The last type is a *vertical move*, which involves adjusting the length data and corresponds to a vertical edge in a combinatorial horoball. We connect two augmented markings $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathcal{AM}^{(0)}(S)$ by an edge in each of the following cases:

- *Flip moves*: If $\mu_1, \mu_2 \in \mathcal{M}(S)$ differ by a flip move at a transverse pairing $(\alpha, t) \mapsto (t, \alpha)$, and if $\tilde{\mu}_1, \tilde{\mu}_2$ have the same base curves and length data, with $D_\alpha(\tilde{\mu}_1) = D_\alpha(\tilde{\mu}_2) = 0$.
- *Twist moves*: If $\alpha \in base(\mu_1) = base(\mu_2)$, $D_\alpha(\tilde{\mu}_1) = D_\alpha(\tilde{\mu}_2) = k > 0$, and $\tilde{\mu}_1 = T_\alpha^n \tilde{\mu}_2$ with $0 < n < e^k$.
- *Vertical moves*: If $\mu_1 = \mu_2$ and if $\tilde{\mu}_1, \tilde{\mu}_2$ only differ in length data by 1 in one component, say $D_\alpha(\tilde{\mu}_1) = D_\alpha(\tilde{\mu}_2) + 1$ and $D_\beta(\tilde{\mu}_1) = D_\beta(\tilde{\mu}_2)$ for all $\beta \in base(\mu_1) \backslash \alpha = base(\mu_2) \backslash \alpha$.

These D_α coordinates can be used to give a coarse measurement of the length of a curve in any augmented marking, regardless of whether the curve is in its base. We emphasize that this measurement records whether a curve is short in $\tilde{\mu}$ and, if so, coarsely how short it is. Given an augmented marking $\tilde{\mu} \in \mathcal{AM}(S)$ and a curve $\alpha \in \mathcal{C}(S)$. We define

$$D_\alpha(\tilde{\mu}) = \begin{cases} D_\alpha & \text{if } \alpha \in \text{base}(\tilde{\mu}) \\ 0 & \text{otherwise} \end{cases}$$

Since our above definition of D_α coincides with the length coordinate for any $\tilde{\mu} \in \mathcal{AM}(S)$ with $\alpha \in \text{base}(\tilde{\mu})$, we use the same notation for both going forward. For any $\tilde{\mu} \in \mathcal{AM}(S)$, we note that $D_\alpha(\tilde{\mu}) = 0$ for all but finitely many $\alpha \in \mathcal{C}(S)$. We also note that these coarse lengths coordinates, as with Fenchel-Nielsen length coordinates, behave nicely with respect to the action of $\mathcal{MCG}(S)$. In particular, if $\phi \in \mathcal{MCG}(S)$, then

$$D_\alpha(\phi \cdot \tilde{\mu}) = D_{\phi \cdot \alpha}(\tilde{\mu})$$

Remark 3.4 (No flipping a short curve)**.** *If $\tilde{\mu} \in \mathcal{AM}(S)$, $D_\alpha(\tilde{\mu}) > 0$ and (α, t) a transverse pair, then it is not possible, by construction, to perform a flip move $(\alpha, t) \mapsto (t, \alpha)$, for only base curves can be short. This is precisely to guarantee that the Teichmüller distance between the image under the map G of two augmented markings which differ by an elementary move is uniformly bounded; see Lemma 3.11 below.*

Since $\mathcal{M}(S)$ is locally finite and each augmented marking has at most 2 vertical edges for each base curve, we have the following immediately from the definition:

Lemma 3.5. *$\mathcal{AM}(S)$ is locally finite.*

The metric on $\mathcal{AM}(S)$ is the path metric, where each edge is given length 1. In the Section 3.2, we define preferred quasigeodesic paths. We close this subsection with a series of remarks.

Remark 3.6 ($\mathcal{M}(S) \hookrightarrow \mathcal{AM}(S)$)**.** *For any subsurface $Y \subset S$, there is a natural inclusion of $i_Y : \mathcal{M}(Y) \hookrightarrow \mathcal{AM}(Y)$ given by $i_Y(\mu) = (\mu, 0, \ldots, 0)$ and we call this embedded copy of $\mathcal{M}(S)$ the* thick part *of $\mathcal{AM}(Y)$ and points therein* thick points*. In particular, when $Y = S$, we think of $i_S(\mathcal{M}(S)) \subset \mathcal{AM}(S)$ as the thick part of $\mathcal{AM}(S)$. As we will see in Section 3.1.3, $i_S(\mathcal{M}(S))$ can be identified with the thick part of $\mathcal{T}(S)$, justifying our terminology.*

Remark 3.7 (Combinatorial horoballs in $\mathcal{AM}(S)$)**.** *Let $\mu \in \mathcal{M}(S)$ and (α, t) a transverse pair in μ. Consider the orbit, $X_\alpha \subset \mathcal{M}(S)$, of μ under $\langle T_\alpha \rangle \leqslant \mathcal{MCG}(S)$, the subgroup generated by the Dehn twist or half-twist about α. Consider the image of X_α in $\mathcal{AM}(S)$, namely $i_S(X_\alpha)$. Then $i_S(X_\alpha)$ lies at the base of the combinatorial horoball $\mathcal{H}_\alpha \subset \mathcal{AM}(S)$.*

Remark 3.8 (Shadows)**.** *Consider a path $\tilde{P} \subset \mathcal{AM}(S)$. Consider the unique path $P \subset \mathcal{M}(S)$ obtained from $\tilde{P}$ by releasing all short curves to have length coordinates 0 for each augmented marking in $\tilde{P}$ and replacing each horizontal move deep in a horoball with the corresponding string of twist moves along the boundary of that horoball. We call P the* shadow of $\tilde{P}$ in $\mathcal{M}(S)$.

Remark 3.9 (Thin parts and product regions)**.** *Let $\rho \subset \mathcal{C}(S)$ be a simplex. If we ignore the technical concerns about cleaning markings after flip moves, then the collection of ρ-thin points in $\mathcal{AM}(S)$, which we call* the ρ-thin part of $\mathcal{AM}(S)$, *coarsely has the structure of the 1-skeleton of $\prod_{\alpha \in \rho} \mathcal{H}_\alpha \times \mathcal{AM}(S \backslash \rho)$ (See Theorem 2.13 for comparison).*

3.1.3 Maps between $\mathcal{T}(S)$ and $\mathcal{AM}(S)$

We are now ready to define maps between $\mathcal{AM}(S)$ and $\mathcal{T}(S)$ which we later prove are quasiisometries in Theorem 3.31.

Let $\alpha \in \mathcal{C}(S)$ and $\sigma \in \mathcal{T}(S)$. Define a map $d_\alpha : \mathcal{T}(S) \to \mathbb{Z}_{\geqslant 0}$ by

$$d_\alpha(\sigma) = \begin{cases} \max\left\{ k \middle| \frac{\epsilon_0}{2^{k+1}} < \mathrm{Ext}_\sigma(\alpha) < \frac{\epsilon_0}{2^k} \right\} & \text{if } \mathrm{Ext}_\sigma(\alpha) < \epsilon_0 \\ 0 & \text{if} \mathrm{Ext}_\sigma(\alpha) \geqslant \epsilon_0 \end{cases}$$

For each $\sigma \in \mathcal{T}(S)$, let μ_σ be a shortest marking for σ as defined before Theorem 2.15.

Define $F : \mathcal{T}(S) \to \mathcal{AM}(S)$ by $F(\sigma) = (\mu_\sigma, d_{\alpha_1}(\sigma), \ldots, d_{\alpha_n}(\sigma))$ where $base(\mu_\sigma) = \{\alpha_1, \ldots, \alpha_n\}$. We think of F as choosing a *shortest augmented marking* for each $\sigma \in \mathcal{T}(S)$, and outside the context of the map F, we may write $\tilde{\mu}_\sigma$ for a shortest augmented marking for a point $\sigma \in \mathcal{T}(S)$.

We now construct an embedding $G : \mathcal{AM}(S) \to \mathcal{T}(S)$ in terms of Fenchel-Nielsen coordinates. Consider an augmented marking $\tilde{\mu} \in \mathcal{AM}(S)$ with $\tilde{\mu} = (\mu, D_{\alpha_1}, \ldots, D_{\alpha_n})$. In building coordinates for $G(\tilde{\mu})$, we are given a clear choice of a pants decomposition, $base(\mu)$, and bounds for the length coordinates, $\frac{\epsilon_0}{2^{D_{\alpha_i}+2}} < l_{\alpha_i} < \frac{\epsilon_0}{2^{D_{\alpha_i}+1}}$. Given a choice of length coordinates, say $l_{\alpha_i} = \frac{\epsilon_0}{2^{D_{\alpha_i}+\frac{3}{2}}}$, we can use the transverse curve data (α_i, t_i) to pick out a unique twisting numbers, $\tau_{\alpha_i}(t_i)$, and thus a unique metric on S, as follows.

For each i, α_i either bounds one or two pairs of pants, depending on whether α_i lives in a four-holed sphere or a one-holed torus. As we have chosen lengths for all the curves in the pants decomposition, the metrics on the pairs of pants are uniquely determined.

In the case of the four-holed sphere, consider the two unique essential geodesic arcs, β_1, β_2 in the pairs of pants connecting α_i to itself. Let $\tau_{\alpha_i}(t_i)$ be the unique twisting number associated to the gluing of the pairs of pants at α_i which connects β_1 to β_2 to realize t_i.

Similarly, for the case when α_i bounds two cuffs on one pair of pants which glue into a one-holed torus, there is a unique geodesic arc, β, connecting the two copies of α_i. Let $\tau_{\alpha_i}(t_i)$ be the unique twisting number associated to the gluing of the copies of α_i which connected the two ends of β to realize t_i.

We can now define $G : \mathcal{AM}(S) \to \mathcal{T}(S)$ by $G(\tilde{\mu}) = \big(l_{\alpha_i}, \tau_{\alpha_i}(t_i)\big)_i$. Since G sends each augmented marking to a unique point for which each curve in the base of that marking is short, the shortest augmented marking for any point in the image of G is unambiguous; that is, $F \circ G(\tilde{\mu}) = \mu$. Thus

Lemma 3.10. *$F \circ G = id_{\mathcal{AM}(S)}$; in particular, G is an embedding and F is a surjection.*

It is important to have a uniform bound on the distance between $G(\tilde{\mu}_1)$ and $G(\tilde{\mu}_2)$, where $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are adjacent vertices in $\mathcal{AM}(S)$. We also need that $G(\mathcal{AM}(S))$ is quasidense in $\mathcal{T}(S)$. We record these fact in a series of lemmas.

Lemma 3.11. *There is a constant $L = L(S) > 0$ such that for any $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathcal{AM}(S)$ adjacent vertices in $\mathcal{AM}(S)$, $d_{\mathcal{T}(S)}(G(\tilde{\mu}_1), G(\tilde{\mu}_2)) < L$.*

Proof. Let $\epsilon > 0$ be as in Theorem 2.13. First, suppose that $\tilde{\mu}_1$ and $\tilde{\mu}_2$ differ by a vertical edge or horizontal edge in a horoball, $\mathcal{H}_\alpha$, where $\alpha \in base(\tilde{\mu}_1) \cap base(\tilde{\mu}_2)$. Recall that the length of α in both $G(\tilde{\mu}_1)$ and $G(\tilde{\mu}_2)$ is less than ϵ by the definition of G. By Minsky's Theorem 2.13, $G(\tilde{\mu}_1)$ and $G(\tilde{\mu}_2)$ coarsely live in the product $\mathbb{H}_\alpha \times \mathcal{T}(S\backslash\alpha)$. The projections of $G(\tilde{\mu}_1)$ and $G(\tilde{\mu}_2)$ to $\mathcal{T}(S\backslash\alpha)$ are identical, so $d_T(G(\tilde{\mu}_1), G(\tilde{\mu}_2))$ is (up to an additive constant) equal to the distance in $\mathbb{H}_\alpha$ of the projections of $G(\tilde{\mu}_1)$ and $G(\tilde{\mu}_2)$ to $\mathbb{H}_\alpha$, again by Minsky's Theorem 2.13.

This distance is coarsely the corresponding distance in a horodisk, via Proposition 3.2, which is precisely 1 by Lemma 3.10. Thus there is a uniform bound on $d_T(G(\tilde{\mu}_1), G(\tilde{\mu}_2))$.

Now suppose that $\tilde{\mu}_1$ and $\tilde{\mu}_2$ differ by a flip move. Then the statement follows immediately from [Lemma 5.6 in (Raf07)] and the local finiteness of $\mathcal{AM}(S)$, Lemma 3.5. $\square$

Before showing that $G(\mathcal{AM}(S)) \subset \mathcal{T}(S)$ is quasidense in Lemma 3.13, we need the following observation:

Lemma 3.12. *Every point in the ϵ-thick part of $\mathcal{T}(S)$ is a uniformly bounded distance away from the ϵ-thin parts of $\mathcal{T}(S)$. This bound depends only on the topology of S.*

Proof. If $\sigma \in \mathcal{T}(S)$ is in the ϵ-thick part of $\mathcal{T}(S)$ and $\mu_\sigma \in \mathcal{M}(S)$ is the shortest marking for σ with $base(\mu_\sigma) = \{\gamma_1, \ldots, \gamma_n\} = \gamma \in \mathcal{C}(S)$, then there is a uniform upper bound on the length of the γ_i, which depends only on the topology of S. Thus there is a uniform bound on the distance between σ and some point $\sigma_{thin} \in Thin_\gamma$, which is obtained by scaling the lengths of the curves in γ in σ to be less than ϵ. In fact, this holds for points in the ϵ-thick part of $\mathcal{T}(Y)$ for every subsurface $Y \subset S$, with the same constant bounding the distance to a uniformly thin part. $\square$

Lemma 3.13. *$G(\mathcal{AM}(S))$ is quasidense in $\mathcal{T}(S)$.*

Proof. We show by induction that $G(\mathcal{AM}(S))$ is quasidense in the ϵ-thin parts of $\mathcal{T}(S)$. Let $\sigma \in \mathcal{T}(S)$ and let $F(\sigma) = \tilde{\mu}_\sigma \in \mathcal{AM}(S)$ a shortest augmented marking for σ. It suffices to show that there is a uniform bound on the distance between σ and $G(\tilde{\mu}_\sigma)$. Suppose first that $\sigma \in Thin_\gamma$ where $\gamma = \{\gamma_1, \ldots, \gamma_n\} \subset \mathcal{C}(S)$ is a maximal simplex, i.e. pants decomposition, of

S. Then by Theorem 2.13, σ and $G(\tilde{\mu}_\sigma)$ coarsely live in $\prod_i \mathbb{H}_{\gamma_i}$ and have length coordinates which differ at most by $\frac{\epsilon}{2}$. As there is a uniform bound on the distance in each $\mathbb{H}_{\gamma_i}$ and on the dimension of the simplex γ, it follows that σ and $G(\tilde{\mu}_\sigma)$ are uniformly close.

Now suppose that $\sigma \in Thin_\gamma$ where $\gamma = \{\gamma_1, \ldots, \gamma_{n-1}\} \subset \mathcal{C}(S)$ is a simplex of dimension one less than maximal. Then σ and $G(\tilde{\mu}_\sigma)$ coarsely live in $\prod_i \mathbb{H}_{\gamma_i} \times \mathcal{T}(S\backslash\gamma)$. If μ_σ is the shortest marking for σ, with $base(\mu_\sigma) = \{\gamma_1, \ldots, \gamma_{n-1}, \alpha\}$, then α was the shortest curve in σ in $\mathcal{C}(S\backslash\gamma)$ and $G(\tilde{\mu}_\sigma)$ lives in $\prod_i \mathbb{H}_{\gamma_i} \times \mathbb{H}_\alpha$. By Lemma 3.12, there is a uniform bound on the distance between $\pi_{\mathcal{T}(S\backslash\gamma)}(\sigma)$ and $Thin_\alpha \subset \mathcal{T}(S\backslash\gamma)$. Thus there is a uniform bound on the distance between σ and $Thin_{\gamma\cup\{\alpha\}} \subset \mathcal{T}(S)$ by Theorem 2.13. Since $G(\mathcal{AM}(S))$ is quasidense in $Thin_{\gamma\cup\{\alpha\}}$, it follows by induction that $G(\mathcal{AM}(S))$ is quasidense in $\mathcal{T}(S)$, completing the proof.

$\square$

Remark 3.14 (Short curves are base curves)**.** *Let $\epsilon > 0$ be as in Theorem 2.13 and suppose $X \in \mathcal{T}(S)$ is such that $l_X(\alpha) < \epsilon$ for some $\alpha \in \mathcal{C}(S)$. It follows from the constuction that $\alpha \in \mathrm{base}(\tilde{\mu}_X)$, where $\tilde{\mu}_X = F(X)$ is a shortest augmented marking for X. That is, short curves are base curves.*

Remark 3.15 (Coarse naturality of F)**.** *It is clear from the construction that F is coarsely natural with respect to the action of $\mathcal{MCG}(S)$. More precisely, there is an $M_1 > 0$ depending only on S such that if $h \in \mathcal{MCG}(S)$ and $X \in \mathcal{T}(S)$, then $d_{\mathcal{AM}(S)}(h \cdot \tilde{\mu}_X, \tilde{\mu}_{h\cdot X}) < M_1$. This M_1 is precisely the diameter of the set of possible choices for $F(X) \in \mathcal{AM}(S)$.*

3.2 Augmented hierarchy paths

The preferred paths in $\mathcal{M}(S)$ constructed in (MM00) by Masur and Minsky are the hierarchy paths from Subsection 2.4, which are quasigeodesics in $\mathcal{M}(S)$ (as recorded in Lemma 3.28 below). The preferred paths in $\mathcal{AM}(S)$ are *augmented hierarchy paths*, which we define below in Theorem 3.21 and show are quasigeodesics in Proposition 3.29. The augmentation process involves adding pinching and releasing moves into the sequence of flip and twist moves prescribed by a given hierarchy path in $\mathcal{M}(S)$ to ensure that twisting is done efficiently. As with the process of resolving hierarchies (see Section 5 of (MM00)), this process is by no means canonical and may be adjusted to various purposes.

We first restrict ourselves to a particular type of hierarchy path in $\mathcal{M}(S)$, then we use these to build augmented hierarchy paths in $\mathcal{AM}(S)$. Finally, we prove some general facts about hierarchy paths and deduce distance estimates for their augmentations.

3.2.1 Bundling twists

We first need the right type of hierarchy path. In the proof of [Proposition 5.4,(Raf07)], Rafi uses a combinatorial argument to show that any hierarchy path may be rearranged to obtain a new hierarchy path based on the same hierarchy in which all twist moves around a given curve are bundled together by rearranging twist moves around a given base curve past flip and twist moves on disjoint curves. This argument uses [Lemma 5.16, (Min03)], which states that any curve appearing as a base curve in a marking in a hierarchy path appears in a contiguous interval. We remark that Rafi's rearrangement method can be used to shift twist moves forward or backward in a hierarchy path, the result is always a hierarchy path, and it depends only on the condition that the curve in question is a base curve during all the moves in consideration.

Let $\Gamma \subset \mathcal{M}(S)$ be a hierarchy path based on some hierarchy H. We say that Γ has *bundled twist moves* or simply *bundled twists* if, for each $\alpha \in \mathcal{C}(S)$, any twist moves around α in Γ occur consecutively. From now on, we will assume that all hierarchy paths in $\mathcal{M}(S)$ have this property of bundled twist moves.

Let $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathcal{AM}(S)$ be augmented markings as above. Let $\Lambda \subset \mathcal{C}(S)$ be the collection of curves contained in the bases of both $\tilde{\mu}_1$ and $\tilde{\mu}_2$, and $\Lambda_i \subset \mathcal{C}(S)$ the collection of curves short in $\tilde{\mu}_i$ but not $\tilde{\mu}_{3-i}$. Note that $\Lambda, \Lambda_1, \Lambda_2 \subset \mathcal{C}(S)$ are simplices. We summarize the above discussion in the following lemma:

Lemma 3.16. *Let Γ be a hierarchy path between μ_1 and μ_2 based on a hierarchy H. Let $\{\alpha_1, \ldots, \alpha_m\} = \Lambda_1$, $\{\beta_1, \ldots, \beta_m\} = \Lambda_2$, $\{\delta_1, \ldots, \delta_k\} = \Lambda$ be any orderings of Λ_1, Λ_2, and Λ which obey the time-order coming from H. Then there is a hierarchy path Γ' of H with bundled twist moves which is the concatenation $\Gamma' = \Gamma_{thin}\Gamma_1\Gamma_{thick}\Gamma_2$, where $\Gamma_{thin}, \Gamma_1, \Gamma_{thick}$, and Γ_2 are as follows:*

1. *Γ_{thin} consists entirely of twists move in the δ_i, done in order*
2. *Γ_1 consists entirely of twist moves in the α_i, done in order*
3. *Γ_2 consists entirely of twist moves in the β_i, done in order*
4. *Γ_{thick} consists of no twist moves around any curve in $\Lambda_1 \cup \Lambda_2 \cup \Lambda$*
5. *Λ is contained the base of every marking in Γ'*

Remark 3.17. *In the statement of Lemma 3.16, the orders on* Λ, Λ_1*, and* Λ_2 *are required to obey the time-ordering coming from* H*. Despite the fact that the curves in* Λ *and the* Λ_i *form simplices in* $\mathcal{C}(S)$*, the poorly-understood but seemingly unavoidable phenomenon of disjoint, time-ordered subsurfaces prevents uninhibited reordering of the curves. As* Γ *already obeys the time-order, this is an unproblematic technical assumption.*

Remark 3.18. *Since curves in* Λ *are base curves during the entirety of any hierarchy path* Γ *based on* H*, we are free to put twist moves around curves in* Λ *anywhere along* Γ*, but we choose group them together at the beginning in a segment* Γ_{thin} *for simplicity and (5) of Theorem 3.21 below.*

Let $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathcal{AM}(S)$, with base markings $\mu_1, \mu_2 \in \mathcal{M}(S)$. Let Γ be a hierarchy path in $\mathcal{M}(S)$ between μ_1 and μ_2, obtained by resolving a hierarchy H as in Lemma 3.16 above. We now sketch how to use Γ to construct an augmented hierarchy path $\widetilde{\Gamma}$ between $\tilde{\mu}_1$ and $\tilde{\mu}_2$. The technical details are contained in Theorem 3.21.

Γ is a sequence of flip and twist moves. The process of transforming Γ into $\widetilde{\Gamma}$ involves inserting pinching and releasing moves to ensure that twist moves are done as efficiently as possible, by moving through the combinatorial horoballs, and that flip moves are possible to do (for, as noted above in Remark 3.4, a flip move cannot be performed on a curve which is short). There is also the added concern of dealing with curves which are short in $\tilde{\mu}_1$, $\tilde{\mu}_2$, or both, but we show how to isolate these issues so that only the basic case in which $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are both thick relative to these collection of curves remains.

3.2.2 Augmenting hierarchy paths

In this subsection, we complete the augmentation of the hierarchy path Γ into an augmented hierarchy path, $\widetilde{\Gamma}$. We accomplish this by building the initial and terminal segments, $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ respectively, of $\widetilde{\Gamma}$ so that, excluding these segments, $\widetilde{\Gamma}$ is effectively a path between thick points in $\mathcal{AM}(S\backslash\Lambda)$; that is, during this middle segment of $\widetilde{\Gamma}$, no elementary moves involving curves in Λ are performed, where Λ is the set of curves short in both $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$. Using Γ_{thin}, Γ_1, and Γ_2 as built in Lemma 3.16, we can then build $\widetilde{\Gamma}_{thin}, \widetilde{\Gamma}_1$, and $\widetilde{\Gamma}_2$. The completion of the construction of $\widetilde{\Gamma}$ will be simply a matter of altering Γ_{thick} to pass through the combinatorial horoballs.

Before we state Theorem 3.21, we need some definitions.

As we did for markings in Definition 2.4, we want to be able to compare augmented markings on suburfaces. Definition 2.4 goes through except for projecting to annuli, which we replace with combinatorial horoballs to keep track of length data. Our horoballs $\mathcal{H}_{(\alpha,t_\alpha)}$ depend on the choice of transverse curve t_α, so we need a common horoball in order to compare the twisting and length coordinates of two augmented markings.

For each curve $\alpha \in \mathcal{C}(S)$, fix an arc $\beta_\alpha \in \mathcal{C}(\alpha)$. For $\gamma \in \mathcal{C}(\alpha)$, let $\gamma \cdot \beta$ denote the algebraic intersection number of γ with β. The map $\phi_{\beta_\alpha} : \mathcal{C}(\alpha) \to \mathbb{Z}$, given by $\phi_{\beta_\alpha}(\gamma) = \gamma \cdot \beta$ is a $(1,2)$-quasiisometry, independent of the choice of β. The map ϕ_{β_α} essentially records twisting around α relative to β. See Subsection 2.4 of (MM00) for more details.

For any curve $\alpha \in \mathcal{C}(S)$, let $\widehat{\mathcal{H}}_\alpha = \mathcal{H}(\mathbb{Z})$ be the combinatorial horoball over $\mathbb{Z}$. It follows from Proposition 3.2 that $\widehat{\mathcal{H}}_\alpha$ is quasiisometric to a horodisk.

Definition 3.19 (Subsurface projections for augmented markings)**.** *If $\tilde{\mu} \in \mathcal{AM}(S)$ is an augmented marking and $Y \subset S$ is any nonannular subsurface, we define the projection of $\tilde{\mu}$ to $\mathcal{C}(Y)$ as $\pi_Y(\tilde{\mu}) = \pi_Y(base(\mu))$. If Y is an annulus with core curve α, we define the projection of $\tilde{\mu}$ to $\widehat{\mathcal{H}}_\alpha$ as follow:*

$$\pi_{\widehat{\mathcal{H}}_\alpha}(\tilde{\mu}) = \begin{cases} (\phi_{\beta_\alpha}(t_\alpha), D_\alpha) & \textit{if } (\alpha, t_\alpha, D_\alpha) \in \tilde{\mu} \\ (\phi_{\beta_\alpha}(\pi_\alpha(\tilde{\mu})), 0) & \textit{otherwise} \end{cases}$$

Since $\pi_\alpha(\tilde{\mu}) \subset \mathcal{C}(\alpha)$ is a uniformly bounded set and ϕ_{β_α} is a quasiisometry, $\pi_{\widehat{\mathcal{H}}_\alpha}(\tilde{\mu})$ is also a uniformly bounded set, independent of the choice of $\beta \in \mathcal{C}(\alpha)$. Moreover, note that if $\tilde{\mu}, \tilde{\eta} \in \mathcal{AM}(S)$ live in the same horoball product $\mathcal{H}_\alpha \times \mathcal{AM}(S \backslash \alpha)$ for some $\alpha \in base(\tilde{\mu}) \cap base(\tilde{\mu})$, then $d_{\widehat{\mathcal{H}}_\alpha}(\tilde{\mu}, \tilde{\eta}) \asymp_{(1,2)} d_{\mathcal{H}_\alpha}(\tilde{\mu}, \tilde{\eta})$.

As we did with markings at the end of Subsection 2.3, we can also define the projection $\tilde{\mu}$ to $\mathcal{AM}(Y)$ for any subsurface $Y \subset S$. Recall that $\pi_{\mathcal{M}(Y)}(\mu)$ was defined by inductively building a pants decomposition from $\pi_Y(\mu)$ and choosing transverse curves from annular projections of μ to the chosen pants curves.

Definition 3.20 (Projecting an augmented marking to an augmented marking on a subsurface)**.** *For any augmented marking $\tilde{\mu} \in \mathcal{AM}(S)$ and nonannular subsurface $Y \subset S$, we similarly define the* projection of $\tilde{\mu}$ to $\mathcal{AM}(Y)$ *by setting $\pi_{\mathcal{M}(Y)}(\mu)$ to be the base marking of $\pi_{\mathcal{AM}(Y)}(\tilde{\mu})$ and, for each $\alpha \in base(\pi_{\mathcal{M}(Y)}(\mu))$, setting $D_\alpha(\pi_{\mathcal{AM}(Y)}(\tilde{\mu}))$ equal to $D_\alpha(\tilde{\mu})$ if $\alpha \subset Y$ and 0 otherwise. In the case that $Y \subset S$ is an annulus with core curve β, then $\pi_{\mathcal{AM}(Y)}(\tilde{\mu}) = \pi_{\widehat{\mathcal{H}}_\beta}(\tilde{\mu})$.*

By Remark 2.6, each such $\alpha \in base(\mu)$ with $\alpha \subset Y$ appears in $base(\pi_{\mathcal{M}(Y)}(\mu))$, so the $D_\alpha(\pi_{\mathcal{AM}(Y)}(\tilde{\mu}))$ are well defined.

We also need to know how to embed preferred paths into horoball products. Let $\tilde{\mu} \in \mathcal{AM}(S)$ and suppose $\alpha \in base(\widetilde{\mu})$ with (α, t_α) its transverse pair. Let P be any path in $\mathcal{AM}(S)$ beginning at $\tilde{\mu}$ which involves only moves which change the length of α, D_α, or involve a Dehn twist or half-twist around α. Consider the projection of $\tilde{\mu}$ to $\mathcal{H}_{(\alpha,t_\alpha)}$ and let P_α be the path in $\mathcal{H}_{(\alpha,t_\alpha)}$ beginning at $(\alpha, t_\alpha, D_\alpha)$ involving the same sequence of vertical and twist moves in P. Then there is a bijection between $P_\alpha \times \pi_{\mathcal{AM}(S\backslash\alpha}(\tilde{\mu}))$ and P, which embeds P_α into the unique copy of $\mathcal{H}_{(\alpha,t_\alpha)}$ in $\mathcal{AM}(S)$ which contains $\tilde{\mu}$, thereby assigning t_α as the transversal to α for the initial marking of P.

We say that such a path P lives in the *α-horoball product.* In the proof of Theorem 3.21, we do not distinguish between $P_\alpha \times \pi_{\mathcal{AM}(S\backslash\alpha}(\tilde{\mu}))$ and P.

Theorem 3.21 (Existence of augmented hierarchy paths)**.** *Let $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathcal{AM}(S)$, $\Lambda, \Lambda_1, \Lambda_2 \subset \mathcal{C}(S)$, and $\Gamma_{thin}, \Gamma_1, \Gamma_2$, and Γ_{thick} be as in Lemma 3.16. Then there are paths $\widetilde{\Gamma}_{thin}, \widetilde{\Gamma}_1, \widetilde{\Gamma}_2, \widetilde{\Gamma}_{thick} \subset \mathcal{AM}(S)$ such that the following hold:*

1. *The concatenation $\widetilde{\Gamma}_{thin}\widetilde{\Gamma}_1\widetilde{\Gamma}_{thick}\widetilde{\Gamma}_2 = \widetilde{\Gamma} \subset \mathcal{AM}(S)$ is a path between $\tilde{\mu}_1$ and $\tilde{\mu}_2$*
2. *$\widetilde{\Gamma}_{thin}$ consists entirely of moves in horoball products of curves in Λ*
3. *$\widetilde{\Gamma}_1$ consists entirely of moves in horoball products of curves in Λ_1, done in order*
4. *$\widetilde{\Gamma}_2$ consists entirely of moves in horoball products of curves in Λ_2, done in order*
5. *$\widetilde{\Gamma}_{thick}$ contains no moves involving curves in $\Lambda \cup \Lambda_1 \cup \Lambda_2$, and $\widetilde{\Gamma}_{thick}$ has endpoints which are in the thick part of $\mathcal{AM}(S)$ relative to Λ.*

Proof. Throughout all that follows, we use Γ as our guide to keep track of our progress from $\tilde{\mu}_1$ to $\tilde{\mu}_2$. Indeed, the path $\widetilde{\Gamma}$ we build has Γ as its shadow in $\mathcal{M}(S)$, by first projecting geodesics in combinatorial horoballs to their inefficient paths along the boundaries and then releasing all other length coordinates to 0 (see Remark 3.8).

By the assumption that Γ satisfies Lemma 3.16, Γ has bundled twists. For any curve $\gamma \in \mathcal{C}(S)$ which appears as a base curve in a twist move in Γ, let $\mu_{\gamma,1}$ and $\mu_{\gamma,2}$ be the initial and terminal markings in the γ twist bundle. As twisting around γ leaves all other components of a marking unchanged, we remark that $\pi_{\mathcal{M}(S\backslash\gamma)}(\mu') = \pi_{\mathcal{M}(S\backslash\gamma)}(\mu'')$ for any μ', μ'' between $\mu_{\gamma,1}$ and $\mu_{\gamma,2}$ in Γ. In particular, this means that the curve transverse to γ in μ' only differs by Dehn or half-Dehn twists around γ from the curve transverse to γ in μ''.

We complete the proof in three steps, in which we build $\widetilde{\Gamma}_{thin}$, $\widetilde{\Gamma}_1$, and $\widetilde{\Gamma}_{thick}$, respectively. The construction of $\widetilde{\Gamma}_2$ is nearly the same as $\widetilde{\Gamma}_1$.

<u>Step 1: Building $\widetilde{\Gamma}_{thin}$</u>

We first build $\widetilde{\Gamma}_{thin}$, beginning with the curves in $\Lambda = \{\delta_1, \ldots, \delta_k\}$. Let $\delta_1 \in \Lambda$ be the first curve appearing as the base curve for a bundle of twists in Γ_{thin} and let $\mu_{\delta_1,1}$ and $\mu_{\delta_1,2}$ be the initial and terminal vertices of the δ_1-twist bundle in Γ_{thin}. Since $\tilde{\mu}_1, \tilde{\mu}_2$ both have δ_1 in their bases, each of them has a δ_1 length coordinate, $D_{\delta_1}(\tilde{\mu}_1)$ and $D_{\delta_1}(\tilde{\mu}_2)$.

Let $\tilde{\mu}_{\delta_1,1}, \tilde{\mu}_{\delta_1,2} \in \mathcal{AM}(S)$ be such that $D_{\delta_1}(\tilde{\mu}_{\delta_1,1}) = D_{\delta_1}(\tilde{\mu}_1)$ and $D_{\delta_1}(\tilde{\mu}_{\delta_1,2}) = D_{\delta_1}(\tilde{\mu}_2)$ and, for each $\gamma \in base(\mu_{\delta_1,1})\backslash\gamma = base(\mu_{\delta_1,2})\backslash\gamma$, set $D_\gamma(\widetilde{\mu}_{\delta_1,1}) = D_\gamma(\widetilde{\mu}_{\delta_1,2}) = D_\gamma(\widetilde{\mu}_1)$. Thus $\widetilde{\mu}_{\delta_1,1}$ and $\widetilde{\mu}_{\delta_1,2}$ have the same length coordinates on the complement $S\backslash\delta_1$.

Since $\tilde{\mu}_{\delta_1,1}$ and $\tilde{\mu}_{\delta_1,2}$ live in the same δ_1-horoball product, we may project them down to $\mathcal{H}_{\delta_1}$ without any ambiguity. Let P_{δ_1} be the preferred path in $\mathcal{H}_{\delta_1}$, as in Lemma 2.20 which connects $\pi_{\mathcal{H}_{\delta_1}}(\tilde{\mu}_{\delta_1,1})$ to $\pi_{\mathcal{H}_{\delta_1}}(\tilde{\mu}_{\delta_1,2})$, and consider the path $\widetilde{P_{\delta_1}} = P_{\delta_1} \times \pi_{\mathcal{AM}(S\backslash\delta_1)}(\tilde{\mu}_{\delta_1,1})$, which is an embedded copy of P_{δ_1} into the unique copy of $\mathcal{H}_{\delta_1}$ in $\mathcal{AM}(S)$ containing $\tilde{\mu}_1$. The path $\widetilde{P}_{\delta_1}$ is a path from $\tilde{\mu}_1 = \tilde{\mu}_{\delta_1,1}$ to $\tilde{\mu}_{\delta_1,2}$.

By construction, the path $\widetilde{P}_{\delta_1}$ projects bijectively to P_{δ_1} in $\mathcal{H}_{\delta_1}$ and to the point $\pi_{\mathcal{AM}(S\backslash\delta_1)}(\tilde{\mu}_1)$ in $\mathcal{AM}(S\backslash\delta_1)$; in particular, $\pi_{\mathcal{H}_{\delta_1}}(\widetilde{P}_{\delta_1}) = P_{\delta_1}$ and $\pi_{\mathcal{AM}(S\backslash\delta_1)}(\widetilde{P}_{\delta_1}) = \pi_{\mathcal{AM}(S\backslash\delta_1)}(\tilde{\mu}_1)$. Moreover, $\pi_{\mathcal{H}_{\delta_1}}(\tilde{\mu}_{\delta_1,2}) = \mu_{\delta_1,2}$. Thus $\widetilde{P}_{\delta_1}$ makes progress along Γ past all twist moves around δ_1.

We then repeat this process by moving along the given order of bundled twists in Γ_{thin}. Setting $\tilde{\mu}_{\delta_1,2} = \tilde{\mu}_{\delta_2,1}$, we then similarly build $\widetilde{P}_{\delta_2}$ from the preferred path P_{δ_2} in $\mathcal{H}_{\delta_2}$ by $\widetilde{P}_{\delta_2} = P_{\delta_2} \times \pi_{\mathcal{AM}(S\backslash\delta_2)}(\tilde{\mu}_{\delta_2,1})$. As before, we label the terminal vertex of $\widetilde{P}_{\delta_2}$ with $\tilde{\mu}_{\delta_2,2}$ and note that its shadow in $\mathcal{M}(S)$ is $\mu_{\delta_2,2}$. We repeat this process for each $\delta_i \in \Lambda$, for $1 \leqslant i \leqslant k$ to obtain a path $\widetilde{\Gamma}_{thin} = \widetilde{P}_{\delta_1} \cdots \widetilde{P}_{\delta_k}$ from $\tilde{\mu}_1$ to $\tilde{\mu}_{\delta_k,2}$. We note that $\pi_{\mathcal{H}_{\delta_i}}(\tilde{\mu}_{\delta_k,2}) = \pi_{\mathcal{H}_{\delta_i}}(\tilde{\mu}_2)$ for each i and $\pi_{\mathcal{AM}(S\backslash\Lambda)}(\widetilde{\Gamma}_{thin}) = \pi_{\mathcal{AM}(S\backslash\Lambda)}(\tilde{\mu}_1)$. We remark that the shadow of $\widetilde{\Gamma}_{thin}$ in $\mathcal{M}(S)$ is precisely Γ_{thin}.

Step 2: Building $\widetilde{\Gamma}_1$

We now build $\widetilde{\Gamma}_1$ via segments which move exclusively through the horoballs of curves in $\Lambda_1 = \{\alpha_1, \cdots, \alpha_m\}$, those curves which are short in $\tilde{\mu}_1$ but not in $\tilde{\mu}_2$.

Let $\alpha_1 \in \Lambda_1$ be the first curve in the ordering on Λ_1. Recall that the terminal vertex of $\widetilde{\Gamma}_{thin}$ is $\tilde{\mu}_{\delta_k,2}$. Note that $\pi_{\mathcal{M}(S)}(\tilde{\mu}_{\delta_k,2}) = \mu_{\delta_k,2} = \mu_{\alpha_1,1}$, the initial vertex in the α_1-bundle of twists in Γ and set $\tilde{\mu}_{\delta_k,2} = \tilde{\mu}_{\alpha_1,1}$. Let $i_S : \mathcal{M}(S) \hookrightarrow \mathcal{AM}(S)$ be the canonical embedding from

Remark 3.6 and let P_{α_1} be the preferred path from Lemma 2.20 in $\mathcal{H}_{\alpha_1}$ between $\pi_{\mathcal{H}_{\alpha_1}}(\tilde{\mu}_{\alpha_1,1})$ and $\pi_{\mathcal{H}_{\alpha_1}}(i_S(\mu_{\alpha_1,2}))$. Set $\widetilde{P}_{\alpha_1} = P_{\alpha_1} \times \pi_{\mathcal{AM}(S\backslash\alpha_1)}(\tilde{\mu}_{\alpha_1,1})$.

Let $\tilde{\mu}_{\alpha_1,2}$ be the last vertex of $\widetilde{P}_{\alpha_1}$. Then $\widetilde{P}_{\alpha_1}$ projects bijectively to P_{α_1} in $\mathcal{H}_{\alpha_1}$. In particular, $\pi_{\mathcal{AM}(S\backslash\alpha_1}(\widetilde{P}_{\alpha_1}) = \pi_{\mathcal{AM}(S\backslash\alpha_1)}(\tilde{\mu}_{\alpha_1,1})$.

We obtain a path $\widetilde{\Gamma}_1 = \widetilde{P}_{\alpha_1}\cdots\widetilde{P}\alpha_m$, by performing the above operation for each $\alpha_i \in \Lambda_1$, with the initial vertex of $\widetilde{P}_{\alpha_i}$ coinciding with the terminal vertex of $\widetilde{P}_{\alpha_{i-1}}$, . Set $\widetilde{\Gamma}_1 = \widetilde{P}_{\delta_1}\cdots\widetilde{P}_{\delta_k}\widetilde{P}_{\alpha_1}\cdots\widetilde{P}_{\alpha_m}$. Note that $\widetilde{\Gamma}_1$ is a path beginning from $\tilde{\mu}_1$ with terminal vertex $\tilde{\mu}_{\alpha_m,2}$, whose only short curves are the curves in Γ, thus $\tilde{\mu}_{\alpha_m,2}$ is thick relative to Γ.

The path $\widetilde{\Gamma}_1$ has precisely Γ_1 as its shadow in $\mathcal{M}(S)$. By a nearly identical procedure, we can construct a sequence of preferred paths in the combinatorial horoballs of the curves, $\beta_i \in \Lambda_2$. Thinking of this sequence of paths beginning at $\tilde{\mu}_2$, we get $\widetilde{\Gamma}_2 = \widetilde{P}_{\beta_n}\cdots\widetilde{P}_{\beta_1}$, a path from $\tilde{\mu}_2$ to $\tilde{\mu}_{\beta_1,1}$, the terminal vertex of $\widetilde{P}_{\beta_1}$. In particular, $\widetilde{\Gamma}_2$ is a path from $\tilde{\mu}_{\beta_1,1}$ to $\tilde{\mu}_2$ whose shadow in $\mathcal{M}(S)$ is precisely Γ_2.

Step 3: Building $\widetilde{\Gamma}_{thick}$

Set $\tilde{\mu}_{thick,1} = \tilde{\mu}_{\alpha_m,2}$ and $\tilde{\mu}_{thick,2} = \tilde{\mu}_{\beta_1,1}$. Let $\mu_{thick,1}$ and $\mu_{thick,2}$ be the initial and terminal markings of Γ_{thick}, respectively. By construction, $\pi_{\mathcal{AM}(S\backslash\Lambda)}(\tilde{\mu}_{thick,1}) = i_{S\backslash\Lambda}(\pi_{\mathcal{M}(S\backslash\Lambda)}(\mu_{thick,1}))$ and $\pi_{\mathcal{AM}(S\backslash\Lambda)}(\tilde{\mu}_{thick,2}) = i_{S\backslash\Lambda}(\pi_{\mathcal{M}(S\backslash\Lambda)}(\mu_{thick,2}))$. By Lemma 3.16, each marking in Γ has Λ as part of its base. Moreover, $D_{\delta_i}(\tilde{\mu}_{thick,1}) = D_{\delta_i}(\tilde{\mu}_{thick,2})$ for each $\delta_i \in \Lambda$. Let $\widehat{\Gamma}_{thick}$ be the path obtained by setting $D_{\delta_i}(\tilde{\mu}) = D_{\delta_i}(\tilde{\mu}_{thick,1})$ for each $\tilde{\mu} \in i_S(\Gamma_{thick})$. Since $\tilde{\mu}_{thick,1}$ and $\tilde{\mu}_{thick,2}$ are thick relative to Λ, it follows that $\widehat{\Gamma}_{thick}$ is a path between $\tilde{\mu}_{thick,1}$ and $\tilde{\mu}_{thick,2}$, whose length is precisely that of Γ_{thick}.

We obtain $\widetilde{\Gamma}_{thick}$ from $\widehat{\Gamma}_{thick}$ as follows: Let $\mu_{\gamma,1} \to \cdots \to \mu_{\gamma,2}$ be a subsegment of Γ_{thick} which is the complete bundle of twist moves around some curve γ (that is, all twist moves around γ in Γ_{thick} occur during this subsegment). Then $\widehat{\Gamma}_{thick}$ has the same bundle of twist moves around γ. This sequence of moves, $\widehat{P}_{\gamma}$, projects to a path on the boundary of $\mathcal{H}_{\gamma}$. Let P_{γ} be the preferred path between $\pi_{\mathcal{H}_{\gamma}}(\widetilde{\mu}_{\gamma,1})$ and $\pi_{\mathcal{H}_{\gamma}}(\widetilde{\mu}_{\gamma,2})$ in $\mathcal{H}_{\gamma}$. Since $\pi_{\mathcal{AM}(S\setminus\gamma)}(\tilde{\mu}_{\gamma,1}) = \pi_{\mathcal{AM}(S\setminus\gamma)}(\tilde{\mu}_{\gamma,2})$, we can replace $\widehat{P}_{\gamma}$ in $\widehat{\Gamma}_{thick}$ by $\widetilde{P}_{\gamma} = P_{\gamma} \times \pi_{\mathcal{AM}(S\setminus\gamma}(\tilde{\mu}_{\gamma,1})$. The segment $\widetilde{P}_{\gamma}$ is a path between $\tilde{\mu}_{\gamma,1}$ and $\tilde{\mu}_{\gamma,2}$.

Define $\widetilde{\Gamma}_{thick}$ to be the result of performing this surgery to $\widehat{\Gamma}_{thick}$ for every curve γ which appears in a bundle of Dehn twists or half-twists in $\widehat{\Gamma}_{thick}$. Since this surgery does not alter the endpoints of the surgered subsegments, it follows that $\widetilde{\Gamma}_{thick}$ is a path from $\tilde{\mu}_{thick,1}$ and $\tilde{\mu}_{thick,2}$. What is more, $\widetilde{\Gamma}_{thick}$ involves no twist moves around any curves in $\Lambda \cup \Lambda_1 \cup \Lambda_2$.

Setting $\widetilde{\Gamma} = \widetilde{\Gamma}_{thin}\widetilde{\Gamma}_1\widetilde{\Gamma}_{thick}\widetilde{\Gamma}_2$ completes the proof.

□

Definition 3.22 (Augmented hierarchy path)**.** *We call* $\widetilde{\Gamma}$ *the* augmentation of Γ *and say that* $\widetilde{\Gamma}$ *is an* augmented hierarchy path *between* $\tilde{\mu}$ *and* $\tilde{\eta}$*.*

Remark 3.23. *Given any such* Γ *and augmentation* $\widetilde{\Gamma}$*,* $\widetilde{\Gamma}$ *casts a natural shadow on* $i_S(\mathcal{M}(S)) \subset \mathcal{AM}(S)$ *by first projecting preferred paths in combinatorial horoballs to their inefficient paths along the bottoms of these horoballs and then releasing all other length coordinates to 0, as in Remark 3.8. It is easy to see that this shadow is* $i_S(\Gamma)$*.*

Remark 3.24 (Uniqueness of augmentation)**.** *Once a base hierarchy path, Γ, and an order on common short curves are chosen, the augmentation process involves no auxiliary choices and thus $\widetilde{\Gamma}$ is uniquely determined. This follows from the uniqueness of a preferred path in a combinatorial horoball. We use this observation in the proof of Proposition 3.28 below.*

3.2.3 Length and efficiency of augmented hierarchy paths

In this subsection, we present a formula for the length of an augmented hierarchy path and observe that any subpath of an augmented hierarchy path also obeys this formula.

Theorem 3.25 (Length of an augmented hierarchy path)**.** *There are constants $M_1, M_2 > 0$ which depend only on S such that the following holds. Let $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathcal{AM}(S)$, $\widetilde{\Gamma}$ any augmented hierarchy path between them, and $k > 0$ the constant from Theorem 2.10. Then the length of $\widetilde{\Gamma}$ is given by the following formula:*

$$|\widetilde{\Gamma}| \asymp_{(M_1,M_2)} \sum_{Y \subset S} [d_Y(\tilde{\mu}_1, \tilde{\mu}_2)]_k \tag{3.1}$$

where if Y is an annulus with core curve α, then $d_Y = d_{\widehat{\mathcal{H}}_\alpha}$.

Proof. Theorem 3.21 gives a decomposition of $\widetilde{\Gamma}$ into four parts: $\widetilde{\Gamma}_{thin}, \widetilde{\Gamma}_1, \widetilde{\Gamma}_{thick}$, and $\widetilde{\Gamma}_2$. Connecting the lengths of these parts to formula (Equation 3.1) involves collecting properties of these segments from the proof of Theorem 3.21 and relating these properties to the formula.

We begin with $\widetilde{\Gamma}_{thin}$. Recall that $\widetilde{\Gamma}_{thin}$ consists of preferred paths in the δ_i-horoball products, $\widetilde{\Gamma}_{thin} = \widetilde{P}_{\delta_1} \cdots \widetilde{P}_{\delta_k}$. Let $\tilde{\mu}_{\delta_i,1}$ and $\tilde{\mu}_{\delta_i,2}$ be the initial and terminal verticies of $\widetilde{P}_{\delta_i}$. It follows from the construction of Γ in Lemma 3.16 and the proof of Theorem 3.21 that $\pi_{\widehat{\mathcal{H}}_{\delta_i}}(\tilde{\mu}_{\delta_i,1})$ and $\pi_{\widehat{\mathcal{H}}_{\delta_i}}(\tilde{\mu}_{\delta_i,2})$ are a bounded distance away from $\pi_{\widehat{\mathcal{H}}_{\delta_i}}(\tilde{\mu}_1)$ and $\pi_{\widehat{\mathcal{H}}_{\delta_i}}(\tilde{\mu}_2)$, respectively, for each i, where the bound only depends on the topology of S. In particular, if these distances were not bounded, then they must differ by some large number of twist moves around the δ_i, for they have the same δ_i-length components. However, Γ was constructed so that all twist moves around the δ_i are bundled together in a segment of Γ_{thin} which is the shadow of $\widetilde{P}_{\delta_i}$ for each i, giving us a contradiction. Thus $\sum_{\delta_i \in \Lambda} d_{\widehat{\mathcal{H}}_{\delta_i}}(\tilde{\mu}_1, \tilde{\mu}_2)$ is coarsely the length of $\widetilde{\Gamma}_{thin}$.

Now consider the length of $\widetilde{\Gamma}_1 = \widetilde{P}_{\alpha_1} \cdots \widetilde{P}_{\alpha,m}$. Let $\tilde{\mu}_{\alpha_i,1}$ and $\tilde{\mu}_{\alpha_i,2}$ be the initial and terminal vertices of $\widetilde{P}_{\alpha_i}$, respectively. It follows again from the construction of Γ in Lemma 3.16 and $\widetilde{\Gamma}_1$ in Theorem 3.21 that $\pi_{\widehat{\mathcal{H}}_{\alpha_i}}(\tilde{\mu}_{\alpha_i,1})$ and $\pi_{\widehat{\mathcal{H}}_{\alpha_i}}(\tilde{\mu}_{\alpha_i,2})$ are a bounded distance away from $\pi_{\widehat{\mathcal{H}}_{\alpha_i}}(\tilde{\mu}_1)$ and $\pi_{\widehat{\mathcal{H}}_{\alpha_i}}(\tilde{\mu}_2)$, respectively, where the bound depends only on S. Thus $\sum_{\alpha_i \in \Lambda_1} d_{\widehat{\mathcal{H}}_{\alpha_i}}(\tilde{\mu}_1, \tilde{\mu}_2)$ is coarsely the length of $\widetilde{\Gamma}_1$. Similarly, $\sum_{\beta_i \in \Lambda_2} d_{\widehat{\mathcal{H}}_{\beta_i}}(\tilde{\mu}_1, \tilde{\mu}_2)$ is coarsely the length of $\widetilde{\Gamma}_2$.

Now observe that there is a topological bound, $L = L(S) > 0$, on the number of curves in Λ_1, Λ_2, and Λ. By adding $L \cdot k$ to M_2, we can allow for the truncation by k in the third term of formula (Equation 3.1), for there are at most L terms of the sum which are less than k. This concludes proof that the third term of formula (Equation 3.1) is coarsely the length of the segments $\widetilde{\Gamma}_{thin}, \widetilde{\Gamma}_1$, and $\widetilde{\Gamma}_2$.

Finally, consider the length of $\widetilde{\Gamma}_{thick}$. In the proof of Theorem 3.21, the process of constructing $\widetilde{\Gamma}_{thick}$ from Γ_{thick} involved creating an intermediary path $\widehat{\Gamma}_{thick}$, which was an embedding of Γ_{thick} into the thick part of $\mathcal{AM}(S)$ relative to Λ, with the initial and terminal vertices of $\widehat{\Gamma}_{thick}$ coinciding with the terminal and initial vertices of $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$, respectively. In particular, the length of $\widehat{\Gamma}_{thick}$ is precisely that of Γ_{thick}. The path $\widetilde{\Gamma}_{thick}$ was constructed from $\widehat{\Gamma}_{thick}$ by a series of surgeries which replaced bundles of twist moves by preferred paths in combinatorial horoball products.

From the Masur-Minsky distance formula, Theorem 2.10, and the construction of Γ_{thick} in Lemma 3.16, it follows that the length of Γ_{thick} and $\widehat{\Gamma}_{thick}$ is coarsely

$$\sum_{Y \subset S\setminus\Lambda} [d_Y(\widetilde{\mu}_1, \widetilde{\mu}_2)]_k + \sum_{\gamma \in S\setminus\Lambda\cup\Lambda_1\cup\Lambda_2} [d_\gamma(\widetilde{\mu}_1, \widetilde{\mu}_2)]_k \tag{3.2}$$

where Y is taken over nonannular subsurfaces of $S\setminus\Lambda$.

The bundles of twists in $\widehat{\Gamma}_{thick}$ are replaced by preferred paths in the combinatorial horoballs $\mathcal{H}_\alpha$ to obtain $\widetilde{\Gamma}_{thick}$. By Lemma 2.20, preferred paths are uniform quasigeodesics, so the second term in equation (Equation 3.2) may be replaced by

$$\sum_{\gamma \in S\setminus\Lambda\cup\Lambda_1\cup\Lambda_2} [d_\gamma(\widetilde{\mu}_1, \widetilde{\mu}_2)]_k$$

by

$$\sum_{\gamma \in S\setminus\Lambda\cup\Lambda_1\cup\Lambda_2} \left[d_{\widehat{\mathcal{H}}_\gamma}(\widetilde{\mu}_1, \widetilde{\mu}_2)\right]_k$$

This completes the proof of the theorem. □

Remark 3.26. *Compare this formula with Rafi's formula, Theorem 2.15. As a consequence of the Main Theorem 3.31 of this chapter and Proposition 3.29, the formula in Theorem 3.25 gives us a distance estimate for points in $\mathcal{T}(S)$ and $\mathcal{AM}(S)$. It is fundamentally the same estimate* *as Rafi's, though more technically useful for the considerations in the current exposition. See the discussion after Corollary 3.32.*

Later when we prove that these augmented hierarchy paths move efficiently through $\mathcal{AM}(S)$, we need to know that subsegments of these paths also move efficiently. This follows from a careful reading of (MM00), but we include a proof for clarity of the exposition. In order to do so, we need some definitions from Subsection 4.1 of (MM00).

Let $Y \subset S$ be a nonannular subsurface and let $\mu \in \mathcal{M}(S)$ be a marking. The *restriction of μ to Y*, denoted $\mu|_Y$, is the set of transverse pairs (α, t_α) in μ whose base curve α meets Y essentially. If $Y \subset S$ is an annulus, then we set $\mu|_Y = \pi_Y(\mu)$.

Recall that each geodesic $g_Y \in H$ has two partial markings associated to it. They are called the *initial* and *terminal markings* of g_Y, respectively denoted $\mathbf{I}(g_Y)$ and $\mathbf{T}(g_Y)$.

Let $X, Y \subset S$ be subsurfaces. Let $g_Y \subset \mathcal{C}(Y)$ be a geodesic in the curve complex of Y. We say that a subsurface X is a *component domain* of g_Y if X is a component of $Y \backslash v$ for some $v \in g_Y$. Suppose that X is component domain for the i^{th} vertex of g_Y, namely $v_i \in g_Y$, $X \subset Y \backslash v_i$. We note that this determines v_i uniquely.

We define the *initial marking of Y relative to g_Y* to be

$$\mathbf{I}(X, g_Y) = \begin{cases} v_{i-1} & \text{if } v_i \text{ is not the first vertex} \\ \\ \mathbf{I}(g_Y)|_Y, & \text{if } v_i \text{ is the first vertex} \end{cases}$$

Similalry, we define the *terminal marking of Y relative to g_Y* to be

$$\mathbf{T}(X, g_Y) = \begin{cases} v_{i+1} & \text{if } v_i \text{ is not the last vertex} \\ \\ \mathbf{T}(g_Y)|_Y, & \text{if } v_i \text{ is the last vertex} \end{cases}$$

It follows from the definition of a hierarchy (Definition 4.4 in (MM00)) and Theorem 4.7 in (MM00) that each such geodesic g_Z is a component domain of precisely two geodesics $g_{X_1}, g_{X_2} \in H$, to which g_Y is a *direct forward or backward subordinate*, denoted $g_{X_1} \swarrow g_Y$ and $g_Y \nearrow g_{X_2}$. Masur-Minsky's constructive proof of the existence of hierarchies (Theorem 4.6 in (MM00)) shows that the initial and terminal markings of such a g_Y are the initial and terminal markings of Y relative to the geodesics to which it is backward and forward subordinate, respectively. Indeed, this is precisely the definition of subordinacy (Definition 4.3 in (MM00)). We record these facts in a lemma:

Lemma 3.27. *[Subordinancy] Given any geodesic $g_Y \in H$ with $g_Y \neq g_H$, there are unique geodesics $g_{X_1}, g_{X_2} \in H$ with $g_{X_1} \swarrow g_Y$ and $g_Y \nearrow g_{X_2}$, with $\boldsymbol{I}(g_Y) = \boldsymbol{I}(Y, g_{X_1})$ and $\boldsymbol{T}(g_y) = \boldsymbol{T}(Y, g_{X_2})$.*

We are now ready to prove the following proposition:

Proposition 3.28. *Any subpath of an augmented hierarchy path is itself an augmented hierarchy path.*

Proof. This follows from the facts that hierarchy paths themselves have this subpath property and that the augmentation process does not alter the underlying hierarchy. The latter fact is clear. To see the former, recall that a hierarchy, H, is a collection of geodesics in the curve complexes of various subsurfaces, $Y \subset S$, and a hierarchy path, Γ, is a sequence of collections of positions on these geodesics which fit together into a sequence of markings.

Let H be a hierarchy and Γ a hierarchy path between two markings $\mu = \mu_1 \to \cdots \to \mu_k = \eta$. For any geodesic $g_Y \in H$, let $g_{Y,int}$ and $g_{Y,ter}$ be the initial and terminal vertices of g_Y, respectively. Then there are two markings, $\mu_{Y,int}$ and $\mu_{Y,ter}$, the initial and terminal markings in which $g_{Y,int}$ and $g_{Y,ter}$ first and last appear, respectively. Having enumerated Γ, we may think of any marking τ appearing along Γ as coming before, after, or during some subsurface, $Y \subset S$, with $g_Y \in H$, depending on whether or not $\tau \in \Gamma_Y = [\mu_{Y,int}, \mu_{Y,ter}]$.

If μ_0 and η_0 are two markings in a hierarchy path Γ based on H, occurring in that order, then one can build a restricted hierarchy H_0 out of geodesics g_Y in H for which $\Gamma_Y \cap [\mu_0, \eta_0] \neq \emptyset$, with H_0 consisting of precisely the geodesics which form the overlaps. More precisely, if g_Y is such a geodesic and, for instance, μ_0 overlaps g_Y, then we can remove the initial segment of g_Y to obtain a geodesic g'_Y which begins at the vertex of g_Y which is a part of μ_0; we similarly truncate the end segment of g_Y if one of its vertices lies in η_0. Let H'_0 be the set of the g'_Y.

If g_Y has be truncated to obtain g'_Y, then we can attach new partial marking data. If either the initial or terminal vertex of g_Y remains in g'_Y, then set the initial or terminal marking of g'_Y to be that of g_Y. Otherwise we can build $\mathbf{I}(g'_Y)$ and $\mathbf{T}(g'_Y)$ inductively from μ_0 and η_0 as follows.

Let $g_{H'}$ be the truncation of the main geodesic g_H at μ_0 and η_0. Set $\mathbf{I}(g_{H'}) = \mu_0$ and $\mathbf{T}(g_{H'}) = \eta_0$. Now suppose that $g'_Y \in H'_0$ is supported in a subsurface $Y \subset S$. By Lemma 3.27, there are two geodesics $g_{X_1}, g_{X_2} in H'$ such that $g_{X_1} \swarrow g_Y$ and $g_Y \nearrow g_{X_2}$. Since Y is a component domain for some vertices of g_{X_1} and g_{X_2}, it follows that these vertices appear in the truncation of H. In particular, there are geodesics $g'_{X_1}, g_{X_2} \in H'$ which are truncated versions of g_{X_1} and g_{X_2}, respectively. Moreover, it is clear that Y appears as a component domain of each. Thus we define $\mathbf{I}(g'_Y) = \mathbf{I}(Y, g'_{X_1})$ and $\mathbf{T}(g'_Y) = \mathbf{T}(Y, g'_{X_2})$. Let H_0 be the collection of the geodesics from H' with their marking data

One can check both that H_0 is indeed a hierarchy and that one can build a hierarchy path, Γ_0, based on H_0 by resolving H_0 as H was resolved into Γ, and Γ_0 will be a restriction of Γ to the interval between μ_0 and η_0. That is, subpaths of hierarchy paths are hierarchy paths.

To show this subpath property lifts to augmented hierarchy paths, let $\widetilde{\Gamma}$ be an augmented hierarchy path between $\tilde{\mu}_1$ and $\tilde{\mu}_2$, with underlying hierarchy path Γ between μ_1 and μ_2. Let $\tilde{\mu}_0$ and $\tilde{\eta}_0$ be two vertices on $\widetilde{\Gamma}$ and let Γ_0 be the restriction of Γ to the interval between μ_0 and η_0.

Let $\Lambda_0, \Lambda_{0,1}$, and $\Lambda_{0,2}$ be the set curves short in both $\tilde{\mu}_0$ and $\tilde{\eta}_0$, those short in $\tilde{\mu}_0$ but not in $\tilde{\eta}_0$, and those short in $\tilde{\eta}_0$ but not in $\tilde{\mu}_0$, respectively. Each of these sets inherits an order from $\widetilde{\Gamma}$, so we assign them that order. Having chosen an order and an underlying hierarchy, it follows from Remark 3.24 that the corresponding augmentation of Γ_0, call it $\widetilde{\Gamma}_0$, coincides with the subpath of $\widetilde{\Gamma}$ betwee $\tilde{\mu}_0$ and $\tilde{\eta}_0$. This completes the proof.

□

Finally, we prove that augmented hierarchy paths are are uniform quasigeodesics in $\mathcal{AM}(S)$.

Proposition 3.29. *Augmented hierarchy paths are uniform quasigeodesics in $\mathcal{AM}(S)$.*

Proof. Let $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathcal{AM}(S)$ be augmented markings and let $\widetilde{\Gamma}$ be any augmented hierarchy path between them. Since all subpaths of augmented hierarchy paths are augmented hierarchy paths (Proposition 3.28), it follows that all subpaths of $\widetilde{\Gamma}$ obey the distance formula Equation 3.1 in Theorem 3.25. It thus suffices to show that Equation 3.1 is coarsely $d_{\mathcal{AM}(S)}(\tilde{\mu}_1, \tilde{\mu}_2)$. We do so by contradiction.

Let $L = L(S) > 0$ be the uniform bound on the distance in $\mathcal{T}(S)$ between the images of two points in $\mathcal{AM}(S)$ which differ by an edge from Lemma 3.11. Rafi's formula (Theorem 2.15) and Theorem 3.25 imply that $d_{\mathcal{T}(S)}(G(\tilde{\mu}), G(\tilde{\eta}))$ is $(P, Q)-$coarsely the length of $\widetilde{\Gamma}$, for appropriately chosen constants P, Q depending on L and the constants from Theorem 2.15 and Theorem 3.25. Suppose that there is a path, $\hat{\Gamma}$, in $\mathcal{AM}(S)$ between $\tilde{\mu}$ and $\tilde{\eta}$ whose length, R, is such that $R \cdot L < \frac{1}{P}(*) - Q$, where $(*)$ is the right-hand side Equation 2.2 of the equation from Theorem 2.15. In this case, it follows that $G(\hat{\Gamma})$ has length less than $d_{\mathcal{T}(S)}(G(\tilde{\mu}), G(\tilde{\eta}))$, a contradiction. Thus, $\widetilde{\Gamma}$ is globally efficient, completing the proof. □

As a corollary of Proposition 3.29 and Theorem 3.25, we have:

Corollary 3.30 (Distance formula for $\mathcal{AM}(S)$)**.** *There are constants $R_1, R_2 > 0$ which depend only on S such that for any $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathcal{AM}(S)$*

$$d_{\mathcal{AM}(S)}(\tilde{\mu}_1, \tilde{\mu}_2) \asymp_{(R_1,R_2)} \sum_{Y} [d_Y(\tilde{\mu}_1, \tilde{\mu}_2)]_k$$

where if $Y \subset S$ is an annulus with core curve α, then $d_Y = d_{\widehat{\mathcal{H}}_\alpha}$.

3.2.4 Proof of the quasiisometry theorem

We now are ready to prove the Main Theorem of this chapter:

Theorem 3.31. *$\mathcal{AM}(S)$ with the path metric is quasiisometric to $\mathcal{T}(S)$ with the Teichmüller metric.*

Proof. Since we have shown in Lemma 3.13 that $G(\mathcal{AM}(S) \subset \mathcal{T}(S)$ is quasidense, it suffices to show that G is a quasiisometric embedding. This is a direct consequence of Proposition 3.29 and various distance formulae.

Let $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathcal{AM}(S)$ and let Λ, Λ_1, and Λ_2 be as in Theorem 3.21. Corollary 3.30 gives that the distance in $\mathcal{AM}(S)$ between $\tilde{\mu}_1$ and $\tilde{\mu_2}$ is

$$d_{\mathcal{AM}(S)}(\tilde{\mu}_1, \tilde{\mu}_2) \asymp_{(R_1,R_2)} \sum_Y [d_Y(\tilde{\mu}_1, \tilde{\mu}_2)]_k \tag{3.3}$$

However, from the proof of Theorem 3.25, we can decompress this formula to

$$\begin{aligned} d_{\mathcal{AM}(S)}(\tilde{\mu}_1, \tilde{\mu}_2) \asymp_{(R_1',R_2')} \sum_Y [d_Y(\tilde{\mu}_1, \tilde{\mu}_2)]_k + \sum_{\alpha \notin \Lambda_1 \cup \Lambda_2 \cup \Lambda} \log [d_\alpha(\tilde{\mu}_1, \tilde{\mu}_2)]_k + \sum_{\alpha \in \Lambda} \left[d_{\widehat{\mathcal{H}}_\alpha} (\tilde{\mu}_1, \tilde{\mu}_2) \right]_k & \quad (3.4) \\ + \sum_{\alpha \in \Lambda_1 \cup \Lambda_2} \left[d_{\widehat{\mathcal{H}}_\alpha} (\tilde{\mu}_1, \tilde{\mu}_2) \right]_k & \quad (3.5) \end{aligned}$$

where $R_1', R_2' > 0$ are constants depending only on S. We now explain how.

First of all, observe that we can decompose the summand on the right side of equation (Equation 3.3) into

$$\sum_Y [d_Y(\tilde{\mu}_1, \tilde{\mu}_2)]_k + \sum_{\alpha \notin \Lambda_1 \cup \Lambda_2 \cup \Lambda} \left[d_{\widehat{\mathcal{H}}_\alpha}(\tilde{\mu}_1, \tilde{\mu}_2) \right]_k + \sum_{\alpha \in \Lambda} \left[d_{\widehat{\mathcal{H}}_\alpha}(\tilde{\mu}_1, \tilde{\mu}_2) \right]_k \tag{3.6}$$

$$+ \sum_{\alpha \in \Lambda_1 \cup \Lambda_2} \left[d_{\widehat{\mathcal{H}}_\alpha}(\tilde{\mu}_1, \tilde{\mu}_2) \right]_k \tag{3.7}$$

An elementary calculation shows that if a bundle of twists along the base of a combinatorial horoball has length n, then the preferred path between the endpoints of the bundle has length coarsely $\log n$. Thus removing the bundles of twists in $\widehat{\Gamma}_{thick}$ and inserting preferred paths to obtain $\widetilde{\Gamma}_{thick}$ introduces a logarithm into the annulur sum, thus replacing

$$\sum_{\alpha \notin \Lambda \cup \Lambda_1 \cup \Lambda_2} \left[d_{\widehat{\mathcal{H}}_\alpha}(\tilde{\mu}_1, \tilde{\mu}_2) \right]_k$$

by

$$\sum_{\alpha \notin \Lambda \cup \Lambda_1 \cup \Lambda_2} \log \left[d_\alpha(\tilde{\mu}_1, \tilde{\mu}_2) \right]_k$$

Rafi's formula Equation 2.2 from Theorem 2.15 gives us that

$$d_{\mathcal{T}}(G(\tilde{\mu}_1), G(\tilde{\mu}_2)) \asymp \sum_Y [d_Y(\tilde{\mu}_1, \tilde{\mu}_2)]_k + \sum_{\alpha \notin \Lambda} \log \left[d_\alpha(\tilde{\mu}_1, \tilde{\mu}_2) \right]_k + \max_{\alpha \in \Lambda} d_{\mathbb{H}_\alpha}(\sigma_1, \sigma_2) + \max_{\substack{\alpha \in \Lambda_i \\ i=1,2}} \log \frac{1}{l_{\sigma_i}(\alpha)} \tag{3.8}$$

The first terms on the right hand sides of equations (Equation 3.4) and (Equation 3.8) are coarsely the same. Changing the max in the third term of equation (Equation 3.8) to a sum only introduces a bounded multiplicative constant, so they are coarsely the same by Proposition 3.2. Finally, changing the max to a sum in the fourth term of equation (Equation 3.8) only introduces a bounded multiplicative constant. For any curve α common to both the second and fourth terms of (Equation 3.8), the sum

$$\log \frac{1}{l_{\sigma_i}(\alpha)} + \log d_\alpha(\mu_1, \mu_2)$$

is at most three times bigger than

$$d_{\widehat{\mathcal{H}}_\alpha}(\tilde{\mu}_1, \tilde{\mu}_2)$$

by Proposition 3.2.

Thus equations (Equation 3.3) and (Equation 3.8) coarsely coincide, up to constants depending only on S. It follows that the image in $\mathcal{T}(S)$ of an augmented hierarchy path under G is a quasigeodesic. Since we showed that augmented hierarchy paths are quasigeodesics in $\mathcal{AM}(S)$ in Proposition 3.29, we have shown that G is a quasiisometric embedding. This completes the proof. $\square$

As a corollary to Theorems 3.25 and 3.31, we have a new distance formula for Teichmüller space, though as is made clear in the remarks that follow, it is fundamentally a combinatorial restatement of Rafi's formula, Theorem 2.15.

Let $\sigma_1, \sigma_2 \in \mathcal{T}(S)$ and $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathcal{AM}(S)$ their shortest augmented markings. Let Λ_i be the collections of curves shorter than ϵ_0 in σ_i and not in σ_{3-i}. Let Λ be the set of curves short in both σ_1 and σ_2.

Corollary 3.32. *The distance in $\mathcal{T}(S)$ between σ_1 and σ_2 is given by the following formula:*

$$d_T(\sigma_1, \sigma_2) \asymp_{(N_1,N_2)} \sum_Y [d_Y(\tilde{\mu}_1, \tilde{\mu}_2)]_k$$

where if Y is an annulus with core curve α, then $d_Y = d_{\widehat{\mathcal{H}}_\alpha}$. The constants N_1 and N_2 depend only on S.

We now make some remarks on the difference between Rafi's distance estimate, Theorem 2.15, and that in Corollary 3.32.

Recall Rafi's distance formula (Theorem 2.15) and the aforementioned fourth term from the right hand side. This term is the distance moved by releasing all the base curves of the end points to the relative thick part. It takes advantage of sup metric on the product structure of the thin regions of $\mathcal{T}(S)$ from Theorem 2.13 to only take on the distance from the base curve deepest in its respective combinatorial horoball. If this curve, α, happens to also be a large link for the hierarchy (see 2.12), then it will appear in the second term and correspond to a full twisting path through the α-combinatorial horoball, $\mathcal{H}_\alpha$, which would require pinching down α again, thus potentially involving an arbitrary amount of backtracking.

By contrast, an augmented hierarchy path between the same two endpoints will take advantage of the initial short length α and move along the twisting path through $\mathcal{H}_\alpha$ without backtracking. The result is a subpath at most three times shorter than the one above. Since there are finitely-many curves which can start short, this difference is easily consumed by the constants (N_1, N_2) in Corollary 3.32.

Since the derivation of the formula in Corollary 3.32 depends on Rafi's formula (Theorem 2.15), we remark the new formula is merely a simplification thereof.

We conclude this section by compiling the work of Masur-Minsky (MM00), Brock (Br03), Rafi (Raf07), and Corollary 3.32 in coarse distance estimates for the marking complexes in terms of subsurface projections. As one can build the $\mathcal{AM}(S)$ from $\mathcal{M}(S)$ and $\mathcal{M}(S)$ from $\mathcal{P}(S)$ by adding additional layers of data, the distance formulae increase in complexity to account for the additional information.

Theorem 3.33. *There is a $K > 0$ such that the following holds. For any $X_1, X_2 \in \mathcal{T}(S)$, let $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathcal{AM}(S)$ be their shortest augmented markings, $\mu_1, \mu_2 \in \mathcal{M}(S)$ be the unique underlying markings and $P_1, P_2 \in \mathcal{P}(S)$ be the unique underyling pants decompositions.*

In (MM00, Theorem 6.12), Masur and Minsky develop a coarse distance formula for $\mathcal{M}(S)$:

$$d_{\mathcal{M}(S)}(\mu_1, \mu_2) \asymp \sum_{Y \subset S} [d_Y(\mu_1, \mu_2)]_K \tag{3.9}$$

An application of (Br03, Theorem 1.1) gives:

$$d_{WP}(X_1, X_2) \asymp \sum_{Y \subset S} [d_Y(P_1, P_2)]_K \tag{3.10}$$

where the $Y \subset S$ are nonannular.

Corollary 3.32 gives:

$$d_T(X_1, X_2) \asymp \sum_{Y \subset S} [d_Y(\tilde{\mu}_1, \tilde{\mu}_2)]_K \tag{3.11}$$

where if $Y \subset S$ is an annulus with core curve α, then $d_Y = d_{\widehat{\mathcal{H}}_\alpha}$.

As the subsurface projections π_Y are defined in terms of the projections of the bases of markings (i.e., pants decompositions) to $\mathcal{C}(Y)$, it follows that the sum appearing in (Equation 3.10) is precisely a proper subsum of (Equation 3.11). It follows that Weil-Petersson distance is (coarsely) shorter than Teichmüller distance, $d_{WP} \prec d_T$.

Remark 3.34 ($d_{WP} \prec d_T$)**.** *It is a theorem of Linch (Lin74) that one only needs a multiplicative constant.*

Remark 3.35 (Bounded d_{WP} implies a bounded number of annular large links)**.** *A key observation we use in the proof of Theorem 1.4 of the next chapter is that points that are a bounded d_{WP} distance apart can only have a uniformly bounded number of large projections to horoballs between their respective shortest augmented markings. This is because a bound on projections to nonannular subsurfaces places a bound on the number of flip moves and thus the number of base curves which can appear along any augmented hierarchy path. See Lemma 4.23 below for more details.*

3.3 Coarse product regions in $\mathcal{AM}(S)$

In this section, we analyze subgraphs of $\mathcal{AM}(S)$ which coarsely behave like the Minsky's product regions. We follow and build on work of Behrstock-Minsky (BM08) for $\mathcal{M}(S)$. The main goal of this section is Proposition 3.47, which is crucial for the distance estimates at the end of the proof of the Theorem 1.4 from the Introduction.

In Section 2 of (BM08), Behrstock-Minsky derive a distance estimate for two points of $\mathcal{M}(S)$ or $\mathcal{P}(S)$ whose base markings have curves in common. We need an analogous statement for $\mathcal{AM}(S)$, which gives a coarse distance estimate for two points in the same Minsky product region (Theorem 2.13). We also need to understand how to project to these regions.

Let $\Delta \subset \mathcal{C}(S)$ be a simplex and consider the subset $Q(\Delta) = \{\tilde{\mu} \in \mathcal{AM}(S) | \Delta \subset \text{base}(\tilde{\mu})\}$. Let $\sigma(\Delta) \subset S \backslash \Delta$ be the collection of complementary subsurfaces which are not pairs of pants. Subsurface projections give a map

$$\Phi : Q(\Delta) \to \prod_{Y \in \sigma(\Delta)} \mathcal{AM}(Y)$$

The following is the $\mathcal{AM}(S)$ analogue of [Lemma 2.1, (BM08)] and it appears in (EMR13) without proof, for it follows quickly from the distance formula in Theorem 3.33:

Lemma 3.36. *The map Φ is a $Stab_{\mathcal{MCG}(S)}(\Delta)$-equivariant quasiisometry.*

There are a couple of immediate corollaries. First, we have a coarse distance estimate for $Q(\Delta)$:

Corollary 3.37. *For $\tilde{\mu}_1, \tilde{\mu}_2 \in Q(\Delta)$, we have that $d_Y(\tilde{\mu}_1, \tilde{\mu}_2) \asymp 1$ for any $Y \pitchfork \Delta$ and thus*

$$d_{\mathcal{AM}(S)}(\tilde{\mu}_1, \tilde{\mu}_2) \asymp \sum_{Y \subset \sigma(\Delta)} [d_Y(\tilde{\mu}_1, \tilde{\mu}_2)]_K$$

In particular, $Q(\Delta)$ is quasiconvex with constants only depending on S.

Second, we have a coarse characterization of Minsky's product regions Theorem 2.13, which is well-known to the experts:

Corollary 3.38. *Let $\epsilon > 0$ be as in Theorem 2.13. Let $\Delta \subset \mathcal{C}(S)$ be a simplex and let $X_1, X_2 \in Thin_{\epsilon,S}(\Delta)$, with $\tilde{\mu}_{X_1}, \tilde{\mu}_{X_2} \in \mathcal{AM}(S)$ their shortest augmented markings. Then $\tilde{\mu}_{X_1}, \tilde{\mu}_{X_2} \in Q(\Delta)$ and there is a string of $\mathcal{MCG}(S)$-equivariant quasiisometries*

$$Thin_{\epsilon,S}(\Delta) \stackrel{+}{\asymp} \prod_{\alpha \in \Delta} \mathbb{H}_\alpha \times \mathcal{T}(S \backslash \Delta) \asymp \prod_{\alpha \in \Delta} \mathcal{H}_\alpha \times \mathcal{AM}(S \backslash \Delta) \asymp Q(\Delta)$$

where $\mathcal{AM}(S \backslash \Delta) = \prod_{Y \subset \sigma(\Delta)} \mathcal{AM}(Y)$ for Y nonannular.

The first quasiisometry is that of Minsky's Theorem 2.13. The second quasiisometry comes from applying Lemma 3.2 and Theorem 3.31 to the appropriate components, in the latter case by choosing a shortest augmented marking on each nonhorodisk component. The third quasiisometry is from Lemma 3.36. We remark that, up to quasiisometry, the metric on a product is unimportant.

In [Lemma 2.2, (BM08)], Behrstock-Minsky give a coarse estimate from any marking $\mathcal{M}(S)$ to $Q(\Delta)$. The following is the analogue for $\mathcal{AM}(S)$ whose proof we omit for it is essentially the same.

Lemma 3.39. *Distance to $Q(\Delta)$ Let $\tilde{\mu} \in \mathcal{AM}(S)$ and $\Delta \subset \mathcal{C}(S)$ and simplex. Then we have*

$$d_{\mathcal{AM}(S)}(\tilde{\mu}, Q(\Delta)) \asymp \sum_{Y \pitchfork \Delta} [[d_Y(\tilde{\mu}, \Delta)]]_K$$

where if Y is an annulus with core curve α, then $d_Y = d_{\mathcal{H}_\alpha}$.

In the proof of the Main Theorem 4.24 of the next chapter, we need to understand how to project any $\tilde{\mu} \in \mathcal{AM}(S)$ to a coarse nearest point in $Q(\Delta)$. This involves projecting $\tilde{\mu}$ to $\mathcal{H}_\alpha$ for each $\alpha \in \Delta$ and then completing those projections to an augmented marking by projecting $\tilde{\mu}$ to $\mathcal{AM}(S\backslash\Delta)$.

Before we proceed, we need to show that $\pi_{\mathcal{H}_\alpha}$ and $\pi_{\mathcal{AM}(Y)}$ are Lipschitz. Since both of these are entirely built out of subsurface projections, Lemmas 3.41 and 3.42 are easy consequences of the following result from (MM00):

Lemma 3.40 (Lipschitz projection; Lemma 2.4 in (MM00)). *Let $Z \subset Y \subset S$ be subsurfaces. For any simplex $\rho \in \mathcal{C}(Y)$, if $\pi_Z(\rho) \neq \varnothing$, then $diam_Z(\rho) \leqslant 3$. If Z is an annulus, then the bound is 1.*

Lemma 3.41 (Horoball projections are Lipschitz). *For any nonannular subsurface $Y \subset S$ and $\alpha \in \mathcal{C}(Y)$, if $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathcal{AM}(Y)$ have $d_{\mathcal{AM}(Y)}(\tilde{\mu}_1, \tilde{\mu}_2) = 1$, then $d_{\mathcal{H}_\alpha}(\tilde{\mu}_1, \tilde{\mu}_2) \asymp 1$.*

Lemma 3.42 (Marking projections are Lipschitz). *Let $Z \subset Y \subset S$ be subsurfaces. For any $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathcal{AM}(Y)$ with $d_{\mathcal{AM}(S)}(\tilde{\mu}_1, \tilde{\mu}_2) = 1$, we have $d_{\mathcal{AM}(Z)}\left(\pi_{\mathcal{AM}(Z)}(\tilde{\mu}_1), \pi_{\mathcal{AM}(Z)}(\tilde{\mu}_1)\right) \asymp 1$.*

Proof. The result follows easily from the distance formula in Theorem 3.33 and Lemmas 3.40 and 3.41 after the following observation.

For any $\alpha \in \text{base}(\tilde{\mu}_1)$ with $D_\alpha(\tilde{\mu}_1) > 0$, it follows that $\alpha \in \text{base}(\tilde{\mu}_2)$ and $d_{\mathcal{H}_\alpha}(\tilde{\mu}_1, \tilde{\mu}_2) \leqslant 1$. If in addition $\alpha \in \mathcal{C}(Z)$, it follows that $\alpha \in \text{base}(\pi_{\mathcal{AM}(Z)}(\tilde{\mu}_1)) \cap \text{base}(\pi_{\mathcal{AM}(Z)}(\tilde{\mu}_2))$. Thus the transversal and length data of α in $\tilde{\mu}_1$ and $\tilde{\mu}_2$ also descend to $\pi_{\mathcal{AM}(Z)}(\tilde{\mu}_1)$ and $\pi_{\mathcal{AM}(Z)}(\tilde{\mu}_2)$, and $d_{\mathcal{H}_\alpha}\left(\pi_{\mathcal{AM}(Z)}(\tilde{\mu}_1), \pi_{\mathcal{AM}(Z)}(\tilde{\mu}_2)\right) \asymp 1$. Thus $\pi_{\mathcal{AM}(Z)}(\tilde{\mu}_1)$ has a short curve if and only if $\pi_{\mathcal{AM}(Z)}(\tilde{\mu}_2)$ has that same short curve.

As all other parts of $\pi_{\mathcal{AM}(Z)}(\tilde{\mu}_1)$ and $\pi_{\mathcal{AM}(Z)}(\tilde{\mu}_2)$ are built from horoball and subsurface projections, the conclusion of the lemma follows from Theorem 3.33 above.

□

We can now define the coarse closest point projection to $Q(\Delta)$.

Definition 3.43 (Coarse closest point projection to $Q(\Delta)$)**.** *For any $\tilde{\mu} \in \mathcal{AM}(S)$ and any simplex $\Delta \subset \mathcal{C}(S)$, define $\phi_\Delta : \mathcal{AM}(S) \to Q(\Delta)$ by*

$$\phi_\alpha(\tilde{\mu}) = \left((\widehat{\pi}_\alpha(\tilde{\mu}))_{\alpha \in \Delta}, \pi_{\mathcal{AM}(S \setminus \Delta)}(\tilde{\mu}) \right)$$

It follows immediately from the definition that $d_{\mathcal{H}_\alpha}(\tilde{\mu}, \phi_\Delta(\tilde{\mu})) \asymp 1$ for any $\alpha \in \Delta$.

We now prove a number of properties of ϕ_Δ, culminating in Proposition 3.47, which we need for the proof of the Main Theorem 4.24 of the next chapter. The first lemma states that, for any $\tilde{\mu} \in \mathcal{AM}(S)$, the choices involved in building $\phi_\Delta(\tilde{\mu})$ result in a uniformly bounded set:

Lemma 3.44. *For any simplex $\Delta \subset \mathcal{C}(S)$ and $\tilde{\mu} \in \mathcal{AM}(S)$, we have*

$$\operatorname{diam}_{\mathcal{AM}(S)}(\phi_\Delta(\tilde{\mu})) \asymp 1$$

Proof. This follows from the facts that $\widehat{\pi}_\alpha$ and $\pi_{\mathcal{AM}(Y)}$ are uniformly bounded for any $\alpha \in \mathcal{C}(S)$ and subsurface $Y \subset S$. □

The following lemma proves that ϕ_Δ is indeed a coarse closest point projection to $Q(\Delta)$. More precisely, the lemma shows that $\phi_\Delta(\tilde{\mu})$ records the combinatorial data of any augmented marking $\tilde{\mu}$ relative to the complementary components of $S\backslash\Delta$. In particular, any augmented hierarchy path from $\tilde{\mu}$ to its projection $\phi_\Delta(\tilde{\mu})$ moves mainly through subsurfaces which interlock Δ:

Lemma 3.45. *For any $\tilde{\mu} \in \mathcal{AM}(S)$ and simplex $\Delta \subset \mathcal{C}(S)$, we have $d_Y(\tilde{\mu}, \phi_\Delta(\tilde{\mu})) \asymp 1$ for any $Y \subset \sigma(\Delta)$. In particular,*

$$d_{\mathcal{AM}(S)}(\tilde{\mu}, \phi_\Delta(\tilde{\mu})) \asymp d_{\mathcal{AM}(S)}(\tilde{\mu}, Q(\Delta))$$

Proof. For any $\alpha \in \Delta$, $d_{\mathcal{H}_\alpha}(\tilde{\mu}, \phi_\Delta(\tilde{\mu}))$ is bounded by definition of $\widehat{\pi}$. Similarly, for any nonannular subsurface $Y \subset \sigma(\Delta)$, $d_Y(\tilde{\mu}, \phi_\Delta(\tilde{\mu}))$ is also bounded by definition of $\pi_{\mathcal{AM}(S\backslash\Delta)}$. Thus all projections to subsurfaces disjoint from Δ are bounded and it follows from Theorem 3.33 and Lemma 3.39 that

$$d_{\mathcal{AM}(S)}(\tilde{\mu}, \phi_\Delta(\tilde{\mu})) \asymp \sum_{Y \subset S} [d_Y(\tilde{\mu}, \phi_\Delta(\tilde{\mu}))]_K = \sum_{Y \pitchfork \Delta} [d_Y(\tilde{\mu}, \phi_\Delta(\tilde{\mu}))]_K \asymp d_{\mathcal{AM}(S)}(\tilde{\mu}, Q(\Delta))$$

□

The next lemma proves that ϕ_Δ is Lipschitz:

Lemma 3.46. *For any simplex $\Delta \subset \mathcal{C}(S)$ and any $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathcal{AM}(S)$ with $d_{\mathcal{AM}(S)}(\tilde{\mu}_1, \tilde{\mu}_2) = 1$, we have $d_{\mathcal{AM}(S)}(\phi_\Delta(\tilde{\mu}_1), \phi_\Delta(\tilde{\mu}_2)) \asymp 1$.*

Proof. Let $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathcal{AM}(S)$ be such that $d_{\mathcal{AM}(S)}(\tilde{\mu}_1, \tilde{\mu}_2) = 1$. Then

$$\begin{aligned} d_{\mathcal{AM}(S)}(\phi_\Delta(\tilde{\mu}_1)\phi_\Delta(\tilde{\mu}_2)) &\asymp \sum_{Y \subset \sigma(\Delta)} [d_Y(\phi_\Delta(\tilde{\mu}_1), \phi_\Delta(\tilde{\mu}_2))]_K \\ &= \sum_{\alpha \in \Delta} [d_{\mathcal{H}_\alpha}(\phi_\Delta(\tilde{\mu}_1), \phi_\Delta(\tilde{\mu}_2))]_K + \sum_{Y \subset (\sigma(\Delta) \setminus \Delta)} [d_Y(\phi_\Delta(\tilde{\mu}_1), \phi_\Delta(\tilde{\mu}_2))]_K \\ &\asymp \sum_{\alpha \in \Delta} [d_{\mathcal{H}_\alpha}(\tilde{\mu}_1, \tilde{\mu}_2)]_K + d_{\mathcal{AM}(S \setminus \Delta)}(\tilde{\mu}_1, \tilde{\mu}_2) \\ &\asymp 1 \end{aligned}$$

$\square$

Finally, the following proposition proves that the composition of closest point projections to disjoint collections of curves coarsely commute.

Proposition 3.47. *For any pair of noninterlocking simplices $\Delta_1, \Delta_2 \subset \mathcal{C}(S)$ and any $\tilde{\mu} \in \mathcal{AM}(S)$, we have*

$$d_{\mathcal{AM}(S)}(\phi_{\Delta_1 \cup \Delta_2}(\phi_{\Delta_1}(\tilde{\mu})), \phi_{\Delta_1 \cup \Delta_2}(\tilde{\mu})) \asymp 1$$

Proof. First of all, note that since Δ_1 and Δ_2 do not interlock, equivalently $\mathrm{diam}_{\mathcal{C}(S)}(\Delta_1 \cup \Delta_2) \leqslant 1$, it follows from the definitions that $\phi_{\Delta_1 \cup \Delta_2}(\phi_{\Delta_1}(\tilde{\mu})) \in Q(\Delta_1 \cup \Delta_2)$.

By definition we have

$$\phi_{\Delta_1 \cup \Delta_2}(\phi_{\Delta_1}(\tilde{\mu})) = \left(\widehat{\pi}_\beta\left(\phi_{\Delta_1}(\tilde{\mu})\right)_{\beta \in \Delta_2}, \pi_{\mathcal{AM}(S \setminus \Delta_2)}\left(\left(\widehat{\pi}_\alpha(\tilde{\mu})\right)_{\alpha \in \Delta_1 \triangle \Delta_2}, \pi_{\mathcal{AM}(S \setminus \Delta_1)}(\tilde{\mu})\right)\right)$$

where $\Delta_1 \triangle \Delta_2 = \Delta_1 \backslash (\Delta_1 \cap \Delta_2)$ is the symmetric difference and

$$\phi_{\Delta_1 \cup \Delta_2}(\tilde{\mu}) = \Big(\big(\widehat{\pi}_\alpha(\tilde{\mu})\big)_{\alpha \in \Delta_1}, \big(\widehat{\pi}_\beta(\tilde{\mu})\big)_{\beta \in \Delta_2}, \pi_{\mathcal{AM}(S \backslash (\Delta_1 \cup \Delta_2)}(\tilde{\mu}) \Big)$$

Since $\phi_{\Delta_1 \cup \Delta_2}(\phi_{\Delta_1}(\tilde{\mu})), \phi_{\Delta_1 \cup \Delta_2}(\tilde{\mu}) \in Q(\Delta_1 \cup \Delta_2)$, Lemma 3.39 implies that

$$d_{\mathcal{AM}(S)}\left(\phi_{\Delta_1 \cup \Delta_2}(\phi_{\Delta_1}(\tilde{\mu})), \phi_{\Delta_1 \cup \Delta_2}(\tilde{\mu})\right) \asymp \sum_{Y \subset \sigma(\Delta_1 \cup \Delta_2)} \left[d_Y\left(\phi_{\Delta_1 \cup \Delta_2}(\phi_{\Delta_1}(\tilde{\mu})), \phi_{\Delta_1 \cup \Delta_2}(\tilde{\mu})\right)\right]_K$$

Thus we need only to compare projections to the components of $\sigma(\Delta_1 \cup \Delta_2)$.

By definition of $\pi_{\mathcal{AM}(Y)}$, if any $\alpha \in \Delta_1$ or $\beta \in \Delta_2$ lies in base$(\tilde{\mu})$, then the transversal and length data of such a curve descends to both $\phi_{\Delta_1 \cup \Delta_2}(\phi_{\Delta_1}(\tilde{\mu}))$ and $\phi_{\Delta_1 \cup \Delta_2}(\tilde{\mu})$. On the other hand, if $\alpha \in \Delta_1$ is not in base$(\tilde{\mu})$, then the length data of α in $\phi_{\Delta_1}(\tilde{\mu})$ is $(\alpha, \pi_\alpha(\tilde{\mu}), 0)$. Since Δ_1 and Δ_2 do not interlock, $\alpha \in$ base$(\phi_{\Delta_1 \cup \Delta_2}(\phi_{\Delta_1}(\tilde{\mu})))$ and, by definition of $\widehat{\pi}_\alpha$ and $\pi_{\mathcal{AM}(S \backslash \Delta_2)}$, its transversal data is the same as the transversal data of α in $\phi_{\Delta_2}(\tilde{\mu})$, namely $\pi_\alpha(\tilde{\mu})$. It follows in both cases that the distance between the projections of $\phi_{\Delta_1 \cup \Delta_2}(\phi_{\Delta_1}(\tilde{\mu}))$ and $\phi_{\Delta_1 \cup \Delta_2}(\tilde{\mu})$ to any horoball over a curve in $\Delta_1 \cup \Delta_2$ is uniformly bounded.

It remains to show that $d_{\mathcal{AM}(S \backslash (\Delta_1 \cup \Delta_2))}\left(\phi_{\Delta_1 \cup \Delta_2}(\phi_{\Delta_1}(\tilde{\mu})), \phi_{\Delta_1 \cup \Delta_2}(\tilde{\mu})\right) \asymp 1$. This follows from the definition and the fact that marking projections are Lipschitz, Lemma 3.42.

$\square$

CHAPTER 4

ELLIPTIC ACTIONS ON $\mathcal{T}(S)$

In this chapter, we study the action of a finite subgroup $H \leqslant \mathcal{MCG}(S)$ on $\mathcal{T}(S)$ and prove Theorems 1.2, 1.4, and 1.5 from the Introduction 1.

4.1 Pants and the Weil-Petersson Metric

We frequently pass back and forth between a point in $\mathcal{T}(S)$ and its coarse representatives in both $\mathcal{P}(S)$ and $\mathcal{AM}(S)$. To aid the clarity of the exposition, we recall the definitions of Brock's quasiisometry between $\mathcal{P}(S)$ and $(\mathcal{T}(S), d_{WP})$ (Br03) and comment on the difference between the quasiisometry between $\mathcal{AM}(S)$ and $(\mathcal{T}(S), d_T)$ we defined in Chapter 3.

We begin with Brock's theorem by recalling a theorem of Bers:

Theorem 4.1 (Bers)**.** *There is a constant $L > 0$ depending only on the topology of S, such that for any point $X \in \mathcal{T}(S)$, there is a $P_X \in \mathcal{P}(S)$ with $l_X(\alpha) < L$ for each $\alpha \in P_X$.*

For any $X \in \mathcal{T}(S)$, any $P_X \in \mathcal{P}(S)$ as in Theorem 4.1 is called a *Bers pants decomposition.*

For any $P \in \mathcal{P}(S)$, define

$$V_L(P) = \{X \in \mathcal{T}(S) | \max_{\alpha \in P} \{l_X(\alpha)\} < L\}$$

Using the convexity of the length functions l_X along Weil-Petersson geodesics (Wol87) and the augmented Teichmüller space, $\overline{\mathcal{T}(S)}$, in (Br03, Proposition 2.2), Brock proves that $V_L(P)$

is convex and has uniformly bounded diameter independent of P, a fact we later prove for the orbifold setting in Proposition 4.18 below:

Proposition 4.2 (Proposition 2.2 in (Br03)). *There is a $D > 0$ depending only on S such that for $L > 0$ as above and any $P \in \mathcal{P}(S)$*

$$\mathrm{diam}_{WP}(V_L(P)) < D$$

Define a map $\phi : \mathcal{P}(S) \to \mathcal{T}(S)$ by $\phi(P) = X_L(P)$, where $X_L(P) \in V_L(P)$. The content of (Br03, Theorem 1.1) is that this map is a quasiisometry. The difficulty of the proof is showing that the reverse identification is coarsely independent of the choice of P.

Let $\tilde{\mu} \in \mathcal{AM}(S)$ be any augmented marking. Recall that $\mathrm{base}(\tilde{\mu}) \in \mathcal{P}(S)$. For any $P \in \mathcal{P}(S)$, there are infinitely many augmented markings $\tilde{\mu} \in \mathcal{AM}(S)$ for which $\mathrm{base}(\tilde{\mu}) = P$. Indeed, for each curve $\alpha \in P$, there is a horoball's worth of choices one could make for a transversal, t_α, and length coordinate, D_α, thus it follows from the distance formula Theorem 3.33 that $\mathrm{diam}_{\mathcal{T}}(V_L(P)) = \infty$. In particular, the identification $\tilde{\mu} \mapsto V_L(\mathrm{base}(\tilde{\mu}))$ is far from a quasiisometry in the Teichmüller metric.

4.2 Fixed and almost-fixed points

In this section, we collect some of the basic properties of the naturally defined subsets of Teichmüller space coming from finite orbifold coverings which are at the heart of this chapter of the thesis. We also describe coarse analogues in the combinatorial setting of $\mathcal{AM}(S)$ and adapt some related work of Tao (Tao13).

4.2.1 Orbifold Teichmüller spaces

For the rest of the thesis, fix a finite subgroup $H \leqslant MCG(S)$. We note that there is a bound on the order of any such finite subgroup $H \leqslant \mathcal{MCG}(S)$ and the number of its conjugacy classes depending only on S (see [Section 7.1, (FM12)). As such, it suffices to consider a single such H.

Fix also a hyperbolic 2-orbifold $\mathcal{O}$ coming from a covering $\pi : X \to \mathcal{O}$ with deck transformation group H, where $X \in T(S)$ is fixed by H, the existence of which is guaranteed by the Nielsen Realization Theorem for an example of such a covering). Recall that $\mathcal{O}$ is essentially a smooth manifold with a finite number of singular neighborhoods. Because we are assuming that S is oriented and that H preserves that orientation, all such singular neighborhoods are quotients of discs by finite rotations which come from H. As H preserves the metric on X , the hyperbolic metric on X descends to $\mathcal{O}$ and we may consider its Teichmüller space, $\mathcal{T}(\mathcal{O})$. See [Section 7, (FM02)] for a formal definition of $\mathcal{T}(\mathcal{O})$.

In this subsection, we analyze the coarse geometry of $\mathcal{T}(\mathcal{O})$ and observe in Theorem 4.9 that a version of Minsky's product regions theorem holds for the thin parts of $\mathcal{T}(\mathcal{O})$.

Let Δ_i be a disjoint collection of small disks around each cone point of $\mathcal{O}$. In what follows, we only consider essential, nonperipheral simple closed curves on $\mathcal{O} \setminus \coprod \Delta_i$. In particular, we define the *orbifold curve graph of* $\mathcal{O}$, $\mathcal{C}(\mathcal{O})$, to be the graph whose vertices are homotopy classes of simple closed curves on $\mathcal{O}$ up to homotopies that do not pass through the Δ_i and whose edges are given by disjointness. We note that this is the same condition we impose on curves when S has marked points or punctures. These assumptions guarantee that any curve $\alpha \in \mathcal{C}(\mathcal{O})$

lifts uniquely to a simplex, $\pi^{-1}(\alpha) \subset \mathcal{C}(S)$ which is invariant under the action of H; we call the lift of any such curve H-*symmetric*. The covering map $\pi : S \to \mathcal{O}$ induces a covering relation $\Pi : \mathcal{C}(\mathcal{O}) \to \mathcal{C}(S)$ given by $\Pi(\beta) = \pi^{-1}(\beta)$. In (RS07, Theorem 8.1), Rafi-Schleimer show that Π is a quasiisometric embedding.

It is well-known that $\mathcal{T}(\mathcal{O})$ can be isometrically embedded $i : \mathcal{T}(\mathcal{O}) \hookrightarrow \mathcal{T}(S)$ into $\mathcal{T}(S)$ with the Teichmüller metric as a convex smooth submanifold (see (RS07) for a brief explanation of the former) and that $i(T(\mathcal{O})) = \mathrm{Fix}(H) \subset \mathcal{T}(S)$ is the fixed set of the action of H on $\mathcal{T}(S)$.

Consider a maximal simplex $A \subset \mathcal{C}(\mathcal{O})$. The complement $\mathcal{O}\backslash A$ is a collection of thrice-punctured spheres and spheres with one, two, or three cone points (the latter being the degenerate case when $\mathcal{O}$ is itself a tricornered pillow), which we call an *orbipants decomposition*. We define the orbipants graph of $\mathcal{O}$, $\mathcal{P}(\mathcal{O})$, in the same way as $\mathcal{P}(S)$. As with a genuine pair of pants, fixing the lengths of the boundary curves in a pair of orbipants uniquely determines a hyperbolic metric thereon, where the order of any cone point plays a fixed role, similar to that of fixing the length of a boundary curve. By fixing curve lengths and twisting factors when reglueing along the curves in A, one arrives at Fenchel-Nielsen coordinates for any point $X \in \mathcal{T}(\mathcal{O})$, $(l_\alpha(X), t_\alpha(X))_{\alpha \in A}$, in nearly the same manner as when $\mathcal{O}$ is a genuine surface. We now describe how to induce Fenchel-Nielsen coordinates on $\mathrm{Fix}(H)$ from those on $\mathcal{T}(\mathcal{O})$.

The simplex $A \subset \mathcal{C}(\mathcal{O})$ lifts to a simplex $\Pi(A) \subset \mathcal{C}(S)$. In order to obtain a pants decomposition on S, complete $\Pi(A)$ to a maximal simplex $P \subset \mathcal{C}(S)$, where $\Pi(A) \subset P$. The following lemma follows almost immediately from the fact that $i : \mathcal{T}(\mathcal{O}) \to \mathcal{T}(S)$ is an embedding:

Lemma 4.3 (Lifted coordinates)**.** *Let $X \in \mathcal{T}(\mathcal{O})$ and consider its image $i(X) \in \mathrm{Fix}(H) \subset \mathcal{T}(S)$. For any maximal simplex $A \subset \mathcal{C}(\mathcal{O})$ and completion of its lift $\Pi(A) \subset P \subset \mathcal{C}(S)$ to a maximal simplex, the following hold:*

1. *For each $\gamma \in P$, the coordinate pair $(l_\gamma(i(X)), t_\gamma(i(X)))$ is uniquely determined by the coordinates $(l_\alpha(X), t_\alpha(X))_{\alpha \in A}$.*
2. *For each $\alpha \in A$, there is a number N_α such that $l_{i(X)}(\beta) = N_\alpha \cdot l_X(\alpha)$ for each lift $\beta \in \Pi(\alpha)$.*

Moreover, the number N_α is uniformly bounded by a constant depending only on S

Proof. (1) follows from the fact that $i : \mathcal{T}(\mathcal{O}) \to \mathcal{T}(S)$ is a bijection. (2) follows from basic covering theory and the fact that $\pi : S \to \mathcal{O}$ is a local isometry away from preimages of the cone points. The constant N_α is bounded in terms of S because $|H|$ is and $N_\alpha \leqslant |H|$. □

Remark 4.4 (Convention for curves and metrics on $\mathcal{O}$, and their lifts)**.** *From now on, we adopt a bar notation, $\bar{\alpha} \in \mathcal{C}(\mathcal{O})$, for curves on $\mathcal{O}$ and denote their lifts by $\alpha = \Pi(\bar{\alpha}) \subset \mathcal{C}(S)$. Similarly, $\bar{X} \in \mathcal{T}(\mathcal{O})$ lifts uniquely to $X \in \mathrm{Fix}(H) \subset \mathcal{T}(S)$.*

Finally, we remark that Bers' Theorem 4.1 holds in the setting of $\mathcal{T}(\mathcal{O})$:

Corollary 4.5. *There is a constant $L' > 0$ depending only on $\mathcal{O}$ so that for any $X \in \mathcal{T}(\mathcal{O})$, there exists $\bar{P}_X \in \mathcal{P}(\mathcal{O})$ of $\mathcal{O}$ with $l_X(\bar{\alpha}) < L$ for each $\bar{\alpha} \in \bar{P}$.*

4.2.2 Short curves and Minsky's theorem for orbifolds

We are interested in passing back and forth between $\mathcal{T}(S)$ and $\mathcal{T}(\mathcal{O})$ while keeping track of short curves. The following lemma follows easily from the Collar Lemma (see (FM12, Lemma 13.4)) and states that all short curves of any point $\sigma \in \mathrm{Fix}(H)$ are H-symmetric:

Lemma 4.6. *Let $\sigma \in \mathrm{Fix}(H)$ and suppose that $l_\sigma(\gamma) < \epsilon$ for some $\gamma \in \mathcal{C}(S)$. Then γ is H-symmetric and the length and twisting coordinates for any curve in the H-orbit of γ equal those of γ.*

Bers-Greenberg (BG71) studied maps between Teichmüller spaces and, in particular, studied maps coming from coverings. Let $\mathcal{O}^*$ be the surface obtained by replacing each cone point of $\mathcal{O}$ with a puncture. The following is their main theorem:

Theorem 4.7 (Bers-Greenberg Isomorphism (BG71))**.** *There exists a conformal homeomorphism*

$$\theta : \mathcal{T}(\mathcal{O}) \to \mathcal{T}(\mathcal{O}^*)$$

It follows by definition that θ is an isometry between $\mathcal{T}(\mathcal{O}^*)$ and $\mathcal{T}(\mathcal{O})$ with the Kobayashi metric. In (Roy71), Royden proved that the Teichmüller and Kobayashi metrics coincide. Thus we have

Corollary 4.8 (Bers-Greenberg, Royden)**.** *$\mathcal{T}(\mathcal{O})$ and $\mathcal{T}(\mathcal{O}^*)$ with the Teichmüller metric are isometric.*

It follows immediately that Minsky's product regions theorem holds for $\mathcal{T}(\mathcal{O})$. For any simplex $\bar{\gamma} \subset \mathcal{C}(\mathcal{O})$, we have:

Theorem 4.9 (Minsky's product regions for $\mathcal{T}(\mathcal{O})$)**.** *There is an $\epsilon > 0$ depending only on S, such that the Fenchel-Nielsen coordinates on $\mathcal{T}(\mathcal{O})$ give rise to a natural homeomorphism $\Pi_{\mathcal{O}} : \mathcal{T}(\mathcal{O}) \to \mathcal{T}_{\bar{\gamma}}(\mathcal{O})$, whose restriction to $Thin_{\epsilon}(\mathcal{O}, \bar{\gamma})$ distorts distances by a bounded additive amount.*

Theorem 4.9 is an essential ingredient in the proof of the Main Theorem 4.24 of this chapter.

Remark 4.10 (Shortness defined)**.** *In what follows, fix $\epsilon_0 > 0$ to be sufficiently small so*

- *Both versions of Minsky's Product Regions Theorems 2.13 and 4.9 hold,*
- *If $\bar{X} \in \mathcal{T}(\mathcal{O})$ has $l_{\bar{X}}(\bar{\alpha}) < \epsilon_0$ for some $\bar{\alpha} \in \mathcal{C}(\mathcal{O})$ and $\bar{X}$ lifts to $X \in \mathrm{Fix}(H)$, then $l_X(\beta) < \epsilon$ for each $\beta \in \Pi(\alpha)$, where $\epsilon > 0$ is as in Theorem 2.13,*
- *If L is Bers' constant from Theorem 4.1, then $\epsilon_0 < L \cdot N_H$, where $|H| < N_H$ depends only on S, and if $l_X(\gamma) < \epsilon_0$ for some $\gamma \in \mathcal{C}(S)$, then $l_X(\delta) > L$, for any $\delta \in \mathcal{C}(S)$ with $i(\delta, \gamma) \geqslant 1$.*

Note that such an ϵ_0 depends only on the topology of S by Lemma 4.3 and the Collar Lemma. When we say that a curve α is short *for some $\sigma \in \mathcal{T}(S)$, we mean that $l_\sigma(\alpha) < \epsilon_0$.*

It follows from Remark 3.14 that if $l_{\bar{X}}(\bar{\alpha}) \leqslant \epsilon_0$, then $\alpha \subset \mathrm{base}(\tilde{\mu}_X)$, where $\tilde{\mu}_X$ is a shortest augmented marking for X.

4.2.3 Almost-fixed points, symmetric large links, and Tao's Lemma

Recall that for any finite $H \leqslant \mathcal{MCG}(S)$, $\mathrm{Fix}(H) \subset \mathcal{T}(S)$ is a totally geodesic submanifold, but less is understood if we relax the condition of being fixed by H to being almost-fixed by H, that is, having a bounded H-orbit. Our main theorem shows that these almost-fixed points are uniformly close to $\mathrm{Fix}(H)$. In order to find a fixed point near an almost-fixed point, we need to understand how efficient paths between almost-fixed points and fixed points move through $\mathcal{T}(S)$. Using $\mathcal{AM}(S)$, we reduce this to understanding the large links which appear along augmented hierarchy paths between almost-fixed augmented markings and certain almost-fixed augmented markings coming from fixed points in $\mathcal{T}(S)$.

In (Tao13), Tao shows that there is an exponential-time algorithm to solve the conjugacy problem for $\mathcal{MCG}(S)$. The bulk of the work in (Tao13) is proving a number of technical results about hierarchies in the setting of the action of a finite order element of $\mathcal{MCG}(S)$ on $\mathcal{M}(S)$.

Our first step is an easy extension of some of her results in our context to finite order subgroups. Let $H \leqslant \mathcal{MCG}(S)$ be a finite order subgroup. For any $R > 0$, we define the set of *R-almost-fixed points* of H in $\mathcal{M}(S)$ to be

$$\mathrm{Fix}_R^{\mathcal{M}}(H) = \{\mu \in \mathcal{M}(S) | \mathrm{diam}_Y(H \cdot \mu) \leqslant R, \forall Y \subset S\}$$

For any $R > 0$, we define the set of R-almost-fixed points of H in $\mathcal{T}(S)$ in the Teichmüller metric to be

$$\mathrm{Fix}_R^{\mathcal{T}}(H) = \{\sigma \in \mathcal{T}(S) | \mathrm{diam}_{\mathcal{T}}(H \cdot \sigma) \leqslant R\}$$

Throughout the rest of this thesis, we work with the coarse version of $\mathrm{Fix}_R^T(H)$, namely $\widetilde{\mathrm{Fix}_R}(H) \subset \mathcal{AM}(S)$ which we define as

$$\widetilde{\mathrm{Fix}_R}(H) = \{\tilde{\mu} \in \mathcal{AM}(S) | \mathrm{diam}_{\mathcal{AM}(S)}(H \cdot \mu) \leqslant R\}$$

For $\tilde{\mu} \in \widetilde{\mathrm{Fix}_R}(H)$, it follows from Theorem 2.15 that $d_Y(\mu, h \cdot \mu) \leqslant K_R$, for each $h \in H, Y \subset S$, and K_R depends on R and S.

For the rest of the subsection, fix an arbitrary augmented marking $\widetilde{X} \in \mathcal{AM}(S)$ and an arbitrary almost-fixed augmented marking $\tilde{\mu} \in \widetilde{\mathrm{Fix}_R}(H)$.

We say a subsurface $Y \subset S$ is *symmetric under the action of* H or simply H*-symmetric* if each component of $H \cdot Y$ is either Y or disjoint from Y.

Recall from Lemma 2.11 that we call a subsurface $Y \subset S$ a K-large link for two augmented markings $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathcal{AM}(S)$ if $d_Y(\tilde{\mu}_1, \tilde{\mu}_2) > K$.

The following lemma tells us that there is a large link constant $\widetilde{K}$, which depends on $\mathrm{diam}_T(H \cdot \widetilde{X})$, such that any $\widetilde{K}$-large link is H-symmetric. It is an easy adaptation to our purposes of (Tao13, Lemma 3.3.4). We give a proof of the adaptation starting from the basis of her lemma, which is the following lemma in which $H = \langle f \rangle$ for a finite order $f \in \mathcal{MCG}(S)$ and $\mathcal{M}(S)$ replaces $\mathcal{AM}(S)$.

Lemma 4.11 (Symmetric large links; Tao's lemma). *Let* $K > 0$ *be fixed as above. There is a* $\widehat{K} = \widehat{K}(\widetilde{X}, R, S) > 0$ *such that the following hold:*

1. *If $\tilde{\mu} \in \widetilde{\mathrm{Fix}}_R(H)$ and $Y \subset S$ is any subsurface satisfying $d_Y(\tilde{\mu}, \widetilde{X}) > \widehat{K}$, then the orbit $H \cdot Y$ is disjoint, $d_Z(\tilde{\mu}, \widetilde{X}) > \widehat{K}$ for each component $Z \subset H \cdot Y$, and none of the components of $H \cdot Y$ is time-ordered with respect to any other.*

2. *For any horoball $\mathcal{H}_\alpha$, if $d_{\mathcal{H}_\alpha}(\tilde{\mu}, \widetilde{X}) > \widehat{K}$, then α is H-symmetric.*

Proof. (Tao13, Lemma 3.3.4) implies (1) for each $f \in H$ and any $Y \subset S$ which is not horoball. We first extend the result to all of H. It suffices to show that none of the components of $H \cdot Y$ is time-ordered with respect to any other. Suppose that $Y \subset S$ is a subsurface such that for some $f, g \in H$ we have $f \cdot Y \prec_t g \cdot Y$. Since $f \cdot Y$ is contained in the orbit of $g \cdot Y$ under the action of $f \cdot g^{-1} \in H$, (Tao13, Lemma 3.3.4) implies that $f \cdot Y$ and $g \cdot Y$ cannot be time-ordered, which is a contradiction.

Let $\mathcal{H}_\alpha$ be a $\bar{K}$-large link, where $\bar{K} > K$ and K is the constant from (Tao13, Lemma 3.3.4) which depends on $\widetilde{X}$. If $d_\alpha(\tilde{\mu}, \widetilde{X}) > K$, then α is H-symmetric and we are done. Otherwise, it must be the case that the α-length coordinates of $\tilde{\mu}$ and $\widetilde{X}$ are bounded away from each other, that is $|D_\alpha(\tilde{\mu}) - D_\alpha(\widetilde{X})| > 2R$, for $\bar{K}$ sufficiently large and R is the almost-fixed constant for $\tilde{\mu}$. If $D_{h\cdot\alpha}(\widetilde{X}) = 0$ for some $h \in H$, then $D_{h\cdot\alpha}(\tilde{\mu}) > R$, and thus $D_{g\cdot\alpha}(\tilde{\mu}) > 0$ for each $g \in H$ because $\tilde{\mu} \in \widetilde{\mathrm{Fix}}_R(H)$, proving that α is H-symmetric. Similarly, if $D_{h\cdot\alpha}(\widetilde{X}) > 0$ for each $h \in H$, then we must also have that α is H-symmetric. This completes the proof of (2). □

Thus $\widehat{K}$-large links between any augmented marking and an almost-fixed augmented marking partition into H-invariant *symmetric families.*

Remark 4.12 (Bad domains)**.** *For the remainder of the thesis, fix* $\widehat{K}$ *as in Lemma 4.11. In (Tao13), subsurfaces in* $\mathcal{L}_{\widehat{K}}(\widetilde{X}, \tilde{\mu})$ *were called* bad domains*, though we do not use this terminology here.*

Remark 4.13 (Dependence of $\widehat{K}$)**.** *The dependence of* $\widehat{K}$ *on* $\operatorname{diam}_T(\widetilde{X})$ *in Lemma 4.11 means that* $\widehat{K}$ *depends only on* R *and* S *when* $\widetilde{X} \in \operatorname{Fix}^T_R(H)$*. In particular, the constant* R' *in the Main Theorem 4.24 below is independent of the choice of* R*-almost-fixed point. Similarly, the constants in the coarse barycenter Theorem 4.25 are independent of the choice of* $X \in \mathcal{T}(S)$*.*

While the hierarchical time-ordering is generally not preserved by the action of $\mathcal{MCG}(S)$, the following lemma gives an important exception:

Lemma 4.14. *Let* $\widetilde{X} \in \mathcal{AM}(S)$*,* $\tilde{\mu} \in \widetilde{\operatorname{Fix}_R}(H)$*. Suppose* $Y, Z \in \mathcal{L}_{\widehat{K}}(\widetilde{X}, \tilde{\mu})$ *are* $\widehat{K}$*-large links with distinct symmetric families and that* $Y \pitchfork Z$*. If* $Y \prec_t Z$ *and* $g \cdot Y \pitchfork Z$ *for some* $g \in H$*, then* $g \cdot Y \prec_t Z$*.*

Proof. Since $g \cdot Y \pitchfork Z$, (MM00, Lemma 4.18) implies that either $g \cdot Y \prec_t Z$ or $Z \prec_t g \cdot Y$. In the latter case, transitivity of $\prec_t$ implies $Y \prec_t Z \prec_t g \cdot Y$, a contradiction of Lemma 4.11. □

Remark 4.15. *Recall that an* H*-symmetric subsurface* Z *may have* $h \cdot Z = Z$ *for each* $h \in H$*. If* $Y \pitchfork Z$*, it is possible that* $h \cdot Y \pitchfork Z$ *for all* $h \in H$*. In this case, Lemmas* **??** *and 4.14 tell us that the active segment of* Z *either comes entirely before or entirely after the active segments of each subsurface in* $H \cdot Y$*.*

Another immediate consequence of the finite order of H is that subsurface projections within a symmetric family are all coarsely equal, with constants depending on $\widehat{K}$:

Lemma 4.16 (Subsurface projections for symmetric families)**.** *Let $\widetilde{X} \in \mathcal{AM}(S)$ and $\tilde{\mu} \in \widetilde{\mathrm{Fix}}_R(H)$. If $Y \in \mathcal{L}_{\widehat{K}}(\widetilde{X}, \tilde{\mu})$, then for all $h, g \in H$*

$$d_{h \cdot Y}(\widetilde{X}, \tilde{\mu}) \asymp_{\widehat{K}} d_{g \cdot Y}(\widetilde{X}, \tilde{\mu})$$

where $d_Y = d_{\mathcal{H}_\alpha}$ if Y is an annulus with core curve α.

4.2.4 Adjusting lengths of short curves for fixed points

In this subsection, we prove that adjusting the lengths of short curves in a fixed point only results in a bounded change in the Weil-Petersson metric and does not introduce any other short curves, an observation which is crucial for the proof of Proposition 4.22 below. We obtain this fact as a consequence of a version of Brock's Proposition 4.2 for our setting.

Before introducing Proposition 4.18 below, we recall some facts about $\mathcal{T}(S)$ in d_{WP}. In the Weil-Petersson metric, $\mathcal{T}(S)$ is an incomplete CAT(0) space (Wol87) and its completion, the augmented Teichmüller space $\overline{\mathcal{T}(S)}$, is obtained as a union of Teichmüller spaces of noded surfaces (Mas76), where disjoint collections of simple closed curves on S have been pinched down to points. This layers $\overline{\mathcal{T}(S)}$ into strata, with the combinatorics of the adjacency of the strata determined by $\mathcal{C}(S)$. Importantly, each stratum is WP-geodesically convex (Wol86). The incompleteness of $\mathcal{T}(S)$ in d_{WP} comes from the fact that there are Weil-Petersson geodesic rays which converge to metrics on noded surfaces in finite time. See (MW02) and (Br05) for more details.

We now recall a theorem of Wolpert (Wol05). Let $\alpha_1, \dots, \alpha_k \in \mathcal{C}(S)$ be a collection of disjoint curves. Let $X \in \mathcal{T}(S)$ and consider the length sum

$$l = l_X(\alpha_1) + \dots + l_X(\alpha_k)$$

Theorem 4.17 (Corollary 21, (Wol05))**.** *For any $X \in \mathcal{T}(S)$, the minimal distance from X to a surface, Z, noded along $\alpha_1, \dots, \alpha_k$ is*

$$d_{\overline{WP}}(X, Z) = \sqrt{2\pi l} + O(l^2)$$

Let $A \in \mathcal{P}(S)$ be any simplex and recall from Section 4.1 that

$$V_L(A) = \{X \in \mathcal{T}(S) | l_X(\alpha) < L, \forall \alpha \in A\}$$

is the set of all metrics on S for which A is included in any Bers pants decomposition, where L is the Bers constant from Theorem 4.1.

For any simplex $A \subset \mathcal{C}(S)$, let $\mathcal{T}(S, A) \subset \overline{\mathcal{T}(S)}$ be the stratum of marked noded surfaces which are noded along A. Recall that each point in $\mathcal{T}(S, A)$ is defined by a point in $\mathcal{T}(Y)$ for each nonpants component of $S \backslash A$. Since length functions are convex along Weil-Petersson geodesics (Wol87), each stratum $\mathcal{T}(S, A)$ is convex in d_{WP}. We also note that it follows from Wolpert's Theorem 4.17 that $d_{\overline{WP}}(X, \mathcal{T}(S, A)) \asymp_L 1$ for any $X \in V_L(A)$.

Proposition 4.18. *Let $\bar{P} \subset \mathcal{C}(\mathcal{O})$ be any orbipants decomposition of $\mathcal{O}$ and $P \subset \mathcal{C}(S)$ its lift. For any $\delta > 0$,* $\mathrm{Fix}(H) \cap V_\delta(P) \subset \mathcal{T}(S)$ *satisfies*

$$\mathrm{diam}_{WP}\left(\mathrm{Fix}(H) \cap V_\delta(P)\right) \asymp_\delta 1$$

Proof. Consider the lift $P \subset \mathcal{C}(S)$ of $\bar{P}$ to S. While $\bar{P}$ is an orbipants decomposition of $\mathcal{O}$, P need not be a pants decomposition of S. Observe, however, that any curve $\alpha \subset S \backslash P$ is not H-symmetric, otherwise it would descend to a curve on $\mathcal{O}$ disjoint from $\bar{P}$.

By the above observation, the components of $S \backslash P$ are pairs of pants and subsurfaces, $Y \subset S \backslash P$, which are stabilized by H. For any such Y, the action of H restricts to an action on Y. Since Y supports no symmetric curves, we must have that the quotient of Y by $H|_Y$, $Y/(H|_Y)$, is a pair of orbipants, which we note has a unique hyperbolic structure once the lengths of any pants curves are chosen. In particular, this means that the fixed point set in each such $\mathcal{T}(Y)$ is a single point.

Let $X \in \mathrm{Fix}(H) \cap V_\delta(P) \subset \mathcal{T}(S)$. Consider the stratum $\mathcal{T}(S,P) \subset \overline{\mathcal{T}(S)}$, where all curves in P have been pinched to nodes. Since $\mathcal{T}(S,P)$ is convex and $(\overline{\mathcal{T}(S)}, d_{\overline{WP}})$ is a complete CAT(0) space, it follows from (BH99, Proposition II.2.4) that there is a unique closest point $X_P \in \mathcal{T}(S,P)$ to X in $\mathcal{T}(S,P)$.

Recall that the action of $\mathcal{MCG}(S)$ extends to $(\overline{\mathcal{T}(S)}, d_{\overline{WP}})$ and observe that H stabilizes $\mathcal{T}(S,P)$ because its defining curves are H-symmetric. Since $X \in \mathrm{Fix}(H)$ and X_P is the closest point to X in $\mathcal{T}(S,P)$, it follows that X_P must also be fixed by H.

We claim that X_P is the only point in $\mathcal{T}(S,P)$ fixed by H. To see this, observe that X_P is defined by a point in $\mathcal{T}(Y)$ for each nonpants component $Y \subset S \backslash P$. Since X_P is fixed by the action of H, it follows that the points in the $\mathcal{T}(Y)$ which define X_P must also be fixed by H. As observed above, each such $\mathcal{T}(Y)$ has a unique point fixed by H. As such, X_P is the unique point in $\mathcal{T}(S,P)$ fixed by H.

Wolpert's Theorem 4.17 implies that

$$d_{\overline{WP}}(X, X_P) = d_{\overline{WP}}(X, \mathcal{T}(S,P)) \asymp_\delta 1$$

as X_P was the closest point in $\mathcal{T}(S,P)$ to X.

Let $X' \in \mathrm{Fix}(H) \cap V_\delta(P)$ be different from X. Since our choice of X was arbitrary, it follows that X_P is also the closest point to X' in $\mathcal{T}(S,P)$ and so

$$d_{\overline{WP}}(X', X_P) \asymp_\delta 1$$

Thus the triangle inequality implies that

$$d_{WP}(X, X') = d_{\overline{WP}}(X, X') \asymp_\delta 1$$

$\square$

4.3 Almost-fixed points are close to fixed points

This section is devoted to proving the Main Theorem 4.24 of this chapter, Theorem 1.4 of the Introduction.

The outline of the proof of Theorem 4.24 is as follows: Beginning with any almost-fixed point $\sigma \in \mathrm{Fix}_R(H) \subset \mathcal{T}(S)$, we first use the nonpositive curvature of $\mathcal{T}(S)$ with the Weil-Petersson metric and work of Wolpert to find a fixed point, $X \in \mathcal{T}(S)$. Applying results of Brock, Masur-Minsky, Rafi, and the author, we deduce that the Teichmüller distance of X to σ is coarsely determined by large projections to horoballs. Using a characterization of the short curves for the barycenter developed in Lemma 4.21, we apply Proposition 4.18 and results of Minsky, Rafi, Wolpert, and the author to show in Proposition 4.22 that the large projections to horoballs can be reduced to large projections to annuli. It follows from Tao's Lemma 4.11 that these annular large links can be grouped into symmetric families which come with an ordering from the hierarchy machinery. The proof of Theorem 4.24 describes how to leap across the symmetric families one at a time by applying H-symmetric multitwists, while staying in $\mathrm{Fix}(H)$ at each step. This process ends with new fixed point whose distance to σ is bounded as a function of R and the topology of S, thus completing the proof.

4.3.1 The Teichmüller geometry of Weil-Petersson barycenters

In this subsection, we analyze the short curves of the Weil-Petersson barycenter of an H-orbit of an almost-fixed point. First, we recall a basic result of coarse geometry, as recorded in (BH99, Proposition II.2.7):

Lemma 4.19. *Let X be a complete* CAT(0) *space. If $Y \subset X$ is a bounded set of radius R, then there exists a unique point $C \in X$, the* barycenter *of Y, such that $Y \subset \bar{B}(C, R)$.*

Fix $R_0 > 0$ and let $\tau \in \mathrm{Fix}^{T}_{R_0}(H)$. It follows from Theorem 3.31 that there is an $\widetilde{R} > 0$ depending only on R_0 and S such that $\tilde{\mu}_\tau \in \widetilde{\mathrm{Fix}}_{\widetilde{R}}(H)$. Since the Weil-Peterrson metric is coarsely dominated by the Teichmüller metric ((Lin74); see Remark 3.34), it follows that there is an $R = R(R_0) > 0$ for which $\tau \in \mathrm{Fix}^{WP}_{R}(H)$, where R only differs from R_0 by a multiplicative constant. Since the augmented Teichmüller space, $\overline{\mathcal{T}(S)}$, is a complete CAT(0) space, it follows from Lemma 4.19 that the H-orbit of σ has a barycenter $X_0' \in \overline{\mathrm{Fix}(H)} \subset \overline{\mathcal{T}(S)}$ in the Weil-Petersson metric, where $\overline{\mathrm{Fix}(H)}$ is the completion of $\mathrm{Fix}(H)$ to $\overline{\mathcal{T}(S)}$, namely marked noded surfaces which are preserved by the action of H.

In the case that $X_0' \in \overline{\mathcal{T}(S)} \backslash \mathcal{T}(S)$, the next lemma produces a new fixed point $X_0 \in \mathrm{Fix}(H)$ arbitrarily close to $X_0' \in \mathrm{Fix}(H)$ in $d_{\overline{WP}}$, the extension of the Weil-Petersson metric to $\overline{\mathcal{T}(S)}$:

Lemma 4.20. *For any $\delta > 0$, there is a point $X_0 \in \mathrm{Fix}(H) \subset \mathcal{T}(S)$ with $d_{WP}(X_0, X_0') \leqslant \delta$.*

Proof. If $X_0' \in \mathrm{Fix}(H) \subset \mathcal{T}(S)$, then we may choose $X_0 = X_0'$.

If not, then X_0' has some simplex of curves $\alpha \subset \mathcal{C}(S)$, each of whose constituent curves has been pinched down to a node. Since $X_0' \in \overline{\mathrm{Fix}(H)}$, it follows that H preserves α. That is, α is H-symmetric. Let $\bar{\alpha} \subset \mathcal{C}(\mathcal{O})$ be the simplex which lifts to α.

Let $Y \in \mathrm{Fix}(H)$ be any other fixed point and consider the unique, finite Weil-Petersson geodesic ray emanating from Y and terminating at X_0', which we denote by $\mathcal{G}$. Since the action of $\mathcal{MCG}(S)$ extends to the completion $\overline{\mathcal{T}(S)}$, it follows that $\mathcal{G}$ is fixed by H. Since $\mathcal{G}$ has finite length, we can let $X_0 \in \mathcal{G}$ be any point satisfying $d_{\overline{WP}}(X_0, X_0') < \delta$, completing the proof.

□

For any $\epsilon' > 0$, denote by $\Lambda_{\epsilon',\tau}$ the set of curves for which $l_\tau(\lambda) < \epsilon'$. Recall that in Remark 4.10 we fixed $\epsilon_0 > 0$ so that both versions of Minsky's Product Regions Theorems 2.13 and 4.9 hold. The following lemma says that if τ has a really short curve, then each curve in the H-orbit of given curve must have τ-length less than ϵ_0. In particular, the whole orbit must be in the base of $\tilde{\mu}_\tau$, a shortest augmented marking for τ.

Lemma 4.21 (Almost-fixed points have symmetric short curves)**.** *There exists $\epsilon'' > 0$ sufficiently small, so that if $\lambda \in \Lambda_{\epsilon'',\tau}$, then λ is H-symmetric and $H \cdot \lambda \subset \Lambda_{\epsilon_0,\tau}$.*

Proof. Consider a shortest augmented marking $\tilde{\mu}_\tau \in \mathcal{AM}(S)$. Since $\tau \in \mathrm{Fix}^T_{R_0}(H)$, recall that Theorem 3.31 implies that there is an $\widetilde{R} > 0$ depending only on R_0 and S such that $\tilde{\mu}_\tau \in \widetilde{\mathrm{Fix}}_{\widetilde{R}}(H)$.

Recall from Subsection 3.1.2 that to each curve $\alpha \in \mathrm{base}(\tilde{\mu}_\tau)$, we assign a length $D_\alpha(\tilde{\mu}_\tau)$, the coordinate which coarsely represents how short α is in $\tilde{\mu}_\tau$.

Let $\epsilon''_1 > 0$ be small enough so that if $\lambda \in \Lambda_{\epsilon''_1,\tau}$, then $D_\lambda(\tilde{\mu}_\tau) > \widetilde{R} + M_1$, where M_1 is the constant from Remark 3.15 (see Subsection 3.1.2 for why short curves have large length coordinates). If λ is not H-symmetric, then there is some $h \in H$ such that $i(\lambda, h \cdot \lambda) \geqslant 1$. Since $D_{h\cdot\lambda}(\tilde{\mu}_{h\cdot\tau}) > \widetilde{R}$, it follows that

$$d_{\mathcal{AM}(S)}(\tilde{\mu}_\tau, \tilde{\mu}_{h\cdot\tau}) \geqslant d_{\mathcal{AM}(S)}(\tilde{\mu}_\tau, h\cdot\tilde{\mu}_\tau) + M_1 \geqslant d_{\mathcal{H}_\lambda}(\tilde{\mu}_\tau, h\cdot\tilde{\mu}_\tau) + d_{\mathcal{H}_{h\cdot\lambda}}(\tilde{\mu}_\tau, h\cdot\tilde{\mu}_\tau) + M_1 > 2\widetilde{R} + M_1$$

a contradiction of $\tilde{\mu}_\tau \in \widetilde{\mathrm{Fix}}_{\tilde{R}}(H)$. The first inequality follows from Remark 3.15. The second inequality follows from the fact that any path from any augmented marking with $D_{\gamma_1} > 0$ to one with $D_{\gamma_2} > 0$ for $i(\gamma_1, \gamma_2) > 0$ must completely exit $\mathcal{H}_{\gamma_1}$ before entering $\mathcal{H}_{\gamma_2}$, at a cost of at least $D_{\gamma_1} + D_{\gamma_2}$.

Now suppose there is an $h \in H$ such that $h \cdot \lambda \notin \Lambda_{\epsilon_0,\tau}$. It follows that $\lambda \notin \Lambda_{\epsilon_0,h^{-1}\cdot\tau}$ and $D_\lambda(\tilde{\mu}_{h^{-1}\tau}) = 0$. For sufficiently small $\epsilon''_2 > 0$, we have $d_{\mathcal{H}_\lambda}(\tilde{\mu}_\tau, \tilde{\mu}_{h^{-1}\tau}) > A \cdot \tilde{R} + B + M_1$, where A, B are the constants depending only on S from Theorem 3.33 and $M > 0$ is again the constant from Remark 3.15. Theorem 3.33 implies that $d_{\mathcal{AM}(S)}(\tilde{\mu}_\tau, \tilde{\mu}_{h^{-1}\tau}) > \tilde{R}$, a contradiction of the fact that $\tilde{\mu}_\tau \in \widetilde{\mathrm{Fix}}_{\tilde{R}}(H)$.

Choose $\epsilon'' < \min\{\epsilon''_1, \epsilon''_2\}$ satisfies both of above arguments, completing the proof. $\square$

Consider the subset of $\mathcal{T}(S)$ of metrics in which all curves in $\Lambda_{\epsilon'',\tau}$ are shorter than ϵ_0:

$$V_{\epsilon_0}(\Lambda_{\epsilon'',\tau}) = \{Y \in \mathcal{T}(S) | l_Y(\lambda) < \epsilon_0, \forall \lambda \in \Lambda_{\epsilon'',\tau}\}$$

Equivalently, $V_{\epsilon_0}(\Lambda_{\epsilon'',\tau})$ contains all points in $\mathcal{T}(S)$ whose shortest augmented markings contain $\Lambda_{\epsilon'',\tau}$ in their bases. By WP-convexity of length functions, $V_{\epsilon_0}(\Lambda_{\epsilon'',\tau})$ is WP-convex. Lemma 4.21 implies that $H \cdot \tau \in V_{\epsilon_0}(\Lambda_{\epsilon'',\tau})$. Since $H \cdot \tau \subset \bar{B}(X_0, R_0)$ (see Lemma 4.19), it follows the convexity of length functions that $X_0 \in V_{\epsilon_0}(\Lambda_{\epsilon'',\tau})$. This implies that $\Lambda_{\epsilon'',\tau} \subset \Lambda_{\epsilon_0,X_0}$ and, in particular, that $\Lambda_{\epsilon'',\tau} \subset \mathrm{base}(\tilde{\mu}_{X_0})$. As $X_0 \in \mathrm{Fix}(H)$, it follows that $H \cdot \Lambda_{\epsilon'',\tau} \subset \mathrm{base}(\tilde{\mu}_{X_0})$. That is, the full H-orbits of all of τ's really short curves are also short in X_0.

Our goal in Proposition 4.22 below is to remove the combinatorial complexity between X_0 and τ coming from the short curves of X_0, which can come in the form of both the length of and twisting about these curves.

Let $\tilde{\mu}_1, \tilde{\mu}_2 \in \mathcal{AM}(S)$ be any two augmented markings. For each α, let $n_\alpha = d_\alpha(\tilde{\mu}_1, \tilde{\mu}_2)$, so that $d_\alpha(\tilde{\mu}_1, T_\alpha^{\pm n_\alpha}) < C$, where T_α denote the right Dehn (half)twist about α and C depends only on S. Then

$$d_{\widehat{\mathcal{H}}_\alpha}(\tilde{\mu}_1, T_\alpha^{\pm n_\alpha}\tilde{\mu}_2) < |D_\alpha(\tilde{\mu}_1) - D_\alpha(\tilde{\mu}_2)| + 2C$$

Now suppose there is a constant $D > 0$ such that $|D_\alpha(\tilde{\mu}_1) - D_\alpha(\tilde{\mu}_2)| < D$, for $\alpha \in \mathcal{C}(S)$. Then there is a D' which depends only on D and S such that

$$d_{\widehat{\mathcal{H}}_\alpha}(\tilde{\mu}_1, T_\alpha^{\pm n_\alpha}\tilde{\mu}_2) < D'$$

We are now ready to state and prove Proposition 4.22, a key technical step on the way to the proof of the Theorem 4.24. In it, we produce a new fixed point, $X \in \mathrm{Fix}(H)$, whose Weil-Petersson distance to τ is still uniformly bounded, but whose Teichmüller distance has decreased in two significant ways: X and τ have uniformly bounded projections to horoballs coming from the short curves X inherits from τ, $H \cdot \Lambda_{\epsilon'',\tau}$, and X and τ have uniformly bounded projections to horoballs coming from the short curves of X which it does not inherit from τ, $\Lambda_{\epsilon,X_0} \backslash \left(H \cdot \Lambda_{\epsilon'',\tau}\right)$. In the proof, we create a new, preliminary fixed point $X' \in \mathrm{Fix}(H)$, whose coarse lengths for curves short in X_0 are coarsely equal. Then we apply a carefully chosen combination of multitwists to X' to obtain a new fixed point $X \in \mathrm{Fix}(H)$, whose twisting coordinates about the short curves of X_0 are coarsely equal to those of τ. As we show in Lemma 4.23 below, the end result is that the Teichmüller distance between X and τ is coarsely determined by projections to a uniformly bounded number of annuli, which is a significant reduction of the combinatorial complexity between X_0 and τ.

Proposition 4.22 (Reducing short curves)**.** *There is a fixed point $X \in \mathrm{Fix}(H)$ with shortest augmented marking $\tilde{\mu}_X \in \mathcal{AM}(S)$ which has the following properties:*

1. *For every $\alpha \in \mathcal{C}(S)$, we have $D_\alpha(\tilde{\mu}_X) \stackrel{+}{\asymp}_R D_\alpha(\tilde{\mu}_\tau)$*
2. *For any $\alpha \in \Lambda_{\epsilon_0,X_0}$, we have $d_{\widehat{\mathcal{H}}_\alpha}(\tilde{\mu}_X, \tilde{\mu}_\tau) \asymp_R 1$*
3. *For any nonannular $Y \subset S$, we have $d_Y(\tilde{\mu}_X, \tilde{\mu}_\tau) \asymp_R 1$, and so $d_{WP}(X, \tau) \prec \widetilde{R}$*

Proof. Let $\bar{\Lambda}_{\epsilon_0,X_0} \subset \mathcal{C}(\mathcal{O})$ be the curves which lift to $\Lambda_{\epsilon_0,X_0} \subset \mathcal{C}(S)$. The comments following Lemma 4.21 imply that $\bar{\Lambda}_{\epsilon'',\tau} \subset \bar{\Lambda}_{\epsilon_0,X_0}$. The key initial observation, which follows from Lemmas 4.3 and 4.21 and the remarks which follow the latter, is that $\tau, X_0 \in V_{\epsilon_0}(H \cdot \Lambda_{\epsilon'',\tau}) \subset V_L(H \cdot \Lambda_{\epsilon'',\tau})$, with the latter inclusion following from our choice of ϵ_0 in Remark 4.10.

It follows from the proof of Lemma 4.21 that $D_\alpha(\tilde{\mu}_\tau) \asymp_R 0$ for all curves $\alpha \subset S \backslash \left(H \cdot \Lambda_{\epsilon'',\tau}\right)$ (see Subsection 3.1.2 for the definition of D_α). Since $H \cdot \Lambda_{\epsilon'',\tau} \subset \Lambda_{\epsilon_0,X_0}$, in order to build a fixed point which satisfies conclusion (1), it suffices to adjust the $D_\lambda(\tilde{\mu}_{X_0})$ to within bounded distance from $D_\lambda(\tilde{\mu}_\tau)$ for $\lambda \in H \cdot \Lambda_{\epsilon'',\tau}$, and to adjust $D_\lambda(\tilde{\mu}_{X_0})$ to 0 for $\lambda \in \Lambda_{\epsilon_0}(\epsilon_0, X_0)$. We can make both of these adjustments directly in the appropriate Minsky product regions in $\mathcal{T}(\mathcal{O})$. We arrive at conclusion 2 by applying appropriate multitwists to the new point we build. Finally, Proposition 4.18 then will imply conclusion 3.

Complete $\bar{\Lambda}_{\epsilon_0,X_0}$ to a Bers orbipants decomposition for X_0, $\bar{P}_{\bar{\Lambda}_{\epsilon_0,X_0}} \in \mathcal{P}(\mathcal{O})$; that is, $l_{\bar{X}_0}(\alpha) < L'$ for all $\bar{\alpha} \in \bar{P}_{\bar{\Lambda}_{\epsilon_0,X_0}}$, where $L' > 0$ is the constant from Corollary 4.5. Recall from Lemma 4.3 that $\bar{P}_{\bar{\Lambda}_{\epsilon_0,X_0}}$ lifts to an H-symmetric partial pants decomposition on S, $P_{\Lambda_{\epsilon_0,X_0}}$, which we can extend to a full pants decomposition $P_0 \in \mathcal{P}(S)$. Fix Fenchel-Nielsen coordinates for $\mathcal{T}(S)$ based on P_0.

Observe that X_0 lives in the Minsky product region:

$$Thin_{\epsilon_0}(S, \Lambda_{\epsilon_0,X_0}) \stackrel{+}{\asymp} \left(\prod_{\lambda \in \Lambda_{\epsilon_0,X_0}} \mathbb{H}_\lambda \right) \times \mathcal{T}\left(S \backslash \Lambda_{\epsilon_0,X_0}\right) \tag{4.1}$$

where the quasiisometry is given by the Fenchel-Nielsen coordinates chosen above.

By symmetric of Λ_{ϵ_0,X_0}, this descends to a Minsky product region

$$Thin_{\epsilon_0}(\mathcal{O}, \bar{\Lambda}_{\epsilon_0,X_0}) \stackrel{+}{\asymp} \left(\prod_{\bar{\lambda} \in \bar{\Lambda}_{\epsilon_0,X_0}} \mathbb{H}_{\bar{\lambda}} \right) \times \mathcal{T}\left(S \backslash \bar{\Lambda}_{\epsilon_0,X_0}\right) \tag{4.2}$$

in which $\bar{X}_0$ lives. We remark that we may keep the same thinness constant, ϵ_0, by Lemma 4.3 and our choice of ϵ_0 in Remark 4.10.

For each orbit of curve in Λ_{ϵ_0,X_0}, fix a representative λ which lifts from $\bar{\lambda} \in \mathcal{C}(\mathcal{O})$. Let $\overline{X'} \in \mathcal{T}(\mathcal{O})$ be any point whose length coordinates with respect to $\bar{P}_{\bar{\Lambda}_{\epsilon_0,X_0}}$ satisfy the following conditions:

1. $l_{\bar{X}'}(\bar{\lambda}) = l_\tau(\lambda) \cdot \frac{1}{N_{\bar{\lambda}}} < \epsilon_0$ for each orbit representative $\lambda \in H \cdot \Lambda_{\epsilon'',\tau}$, where $N_{\bar{\lambda}}$ is the constant from Lemma 4.3

2. $l_{\bar{X}'}(\bar{\gamma}) = \epsilon_0$ for each orbit representative $\gamma \in \Lambda_{\epsilon_0,X_0} \backslash (H \cdot \Lambda_{\epsilon'',\tau})$

3. $l_{\bar{X}'}(\bar{\alpha}) = l_{\bar{X}_0}(\bar{\alpha})$ for every other $\bar{\alpha} \in \bar{P}_{\bar{\Lambda}_{\epsilon_0,X_0}} \backslash \bar{\Lambda}_{\epsilon_0,X_0}$.

We claim the lift $X' \in \mathrm{Fix}(H)$ of any such $\overline{X'} \in \mathcal{T}(\mathcal{O})$ satisfies conclusion (1).

To see this, first observe that condition (1) implies that $D_\alpha(\tilde{\mu}'_X) \asymp_R D_\alpha(\tilde{\mu}_\tau)$ for any $\alpha \in H \cdot \Lambda_{\epsilon'',\tau}$, as the $N_{\bar{\lambda}}$ are uniformly bounded by Lemma 4.3. Next, since Lemma 4.21 implies that $D_\alpha(\tilde{\mu}_\tau) \asymp 0$ for all curves $\alpha \subset S \backslash \left(H \cdot \Lambda_{\epsilon'',\tau}\right)$, conditions (2) and (3), and the fact that $X' \in \mathrm{Fix}(H)$ so that any $\alpha \in S \backslash P_{\bar{\Lambda}_{\epsilon_0,X_0}}$ are necessarily not H-symmetric and thus cannot be short in X', imply that $D_\alpha(\tilde{\mu}_{X'}) \asymp_R D_\alpha(\tilde{\mu}_\tau)$ for all such $\alpha \subset S \backslash \left(H \cdot \Lambda_{\epsilon'',\tau}\right)$. Finally, since $X', \tau \in V_{\epsilon''}(H \cdot \Lambda_{\epsilon'',\tau})$, we have $D_\alpha(\tilde{\mu}_{X'}) = D_\alpha(\tilde{\mu}_\tau) = 0$, for all $\alpha \pitchfork H \cdot \Lambda_{\epsilon'',\tau}$, by the Collar Lemma. Thus conclusion (1) holds for X'.

It follows from the definition that $\bar{X}' \in V_{L'}(\bar{P}_0)$. As $\bar{X}_0 \in V_{L'}(\bar{P}_0)$, conclusion (3) for X' follows from Proposition 4.18.

Generically, X' does not satisfy conclusion (2). To build a point which does, we apply some carefully chosen H-symmetric multitwists to reduce the annular projections between X' and τ. We then prove that the resulting point still satisfies conclusions (1) and (3).

Let $\bar{\Lambda}_{\epsilon_0,X_0} \subset \mathcal{C}(\mathcal{O})$ be the set of curves which lift to $H \cdot \Lambda_{\epsilon_0,X_0} \subset \mathcal{C}(S)$. Suppose that $\bar{\Lambda}_{\epsilon_0,X_0}$ consists of N_τ different H-orbits of curves and decompose it into these orbits,

$$\bar{\Lambda}_{\epsilon_0,X_0} = \{\lambda_{1,1}, \ldots, \lambda_{1,m_1}, \ldots, \lambda_{N_\tau,1}, \ldots, \lambda_{N_\tau,m_{N_\tau}}\}$$

Note that both the m_i and N_τ are uniformly bounded.

For each i, let $T_{\lambda_i} = \prod_{j=1}^{m_i} T_{\lambda_{i,j}}^{(-1)^{s_i}\cdot d_i}$, where $T_{\lambda_{i,j}}$ is the Dehn (half)twist around $\lambda_{i,j}$, $d_i = d_{\lambda_i,1}(\tilde{\mu}_{X'}, \tilde{\mu}_\tau)$, and the sign s_i depends on whether $\pi_{\lambda_{i,1}}(\tilde{\mu}_{X'})$ differs from $\pi_{\lambda_{i,1}}(\tilde{\mu}_\tau)$ by right or left Dehn (half)twists around $\lambda_{i,1}$.

Set $T_{\Lambda_{\epsilon_0,X_0}} = \prod_{i=1}^{N_\tau} T_{\lambda_i}$ and $X = T_{\Lambda_{\epsilon_0,X_0}} \cdot X'$. We claim that X' satisfies the conclusions of the proposition.

First, observe that since Λ_{ϵ_0,X_0} is an H-symmetric multicurve, $T_{\Lambda_{\epsilon_0,X_0}} \in C_{\mathcal{MCG}(S)}(H)$, the centralizer of H in $\mathcal{MCG}(S)$, which is contained in the normalizer of H, which stabilizes $\mathrm{Fix}(H)$. Thus $X \in \mathrm{Fix}(H)$.

Second, since $\Lambda_{\epsilon_0,X_0} \subset \mathrm{base}(\tilde{\mu}_{X'}) \cap \mathrm{base}(\tilde{\mu}_X)$, it follows that $d_Y(\tilde{\mu}_X, \tilde{\mu}_{X'}) \asymp 1$ uniformly for any $Y \subset S$ not an annulus over a curve in Λ_{ϵ_0,X_0}. Because $T_{\Lambda_{\epsilon_0,X_0}}$ preserves the curves in Λ_{ϵ_0,X_0} and any curves disjoint from them, namely P_0, conclusions (1) and (3) hold for X.

Finally, observe that Lemma 4.16 implies that $d_{\lambda_{i,j}}(\tilde{\mu}_{X_0}, \tilde{\mu}_\tau) \asymp_R d_{\lambda_{i,k}}(\tilde{\mu}_{X_0}, \tilde{\mu}_\tau)$ for any j, k. Thus the choice of $T_{\Lambda_{\epsilon_0,X_0}}$ and the triangle inequality imply that $d_\alpha(\tilde{\mu}_X, \tilde{\mu}_\tau) \asymp_R 1$ for each $\alpha \in \Lambda_{\epsilon_0,X_0}$. Since conclusion (1) also holds for X for each $\alpha \in \Lambda_{\epsilon_0,X_0}$, it follows that conclusion (2) holds for X. This completes the proof.

□

4.3.2 Proof of the main theorem

Recall our main goal of this section, achieved in Theorem 4.24 below, is to find a fixed point whose distance to $\tau \in \mathrm{Fix}_R^T(H)$ is bounded in terms of R and S. Proposition 4.22 produces a fixed point $X \in \mathrm{Fix}(H)$ which has the same very short curves as τ, whose distance to τ in any horoball over any of these short curves is uniformly bounded, and whose distance in any other nonhoroball subsurface is uniformly bounded. Before proceeding with the proof of Theorem 4.24, we analyze and organize the remaining large horoball projections.

Observe that X and τ have $H \cdot \Lambda_{\epsilon'',\tau}$ as short curves, so $X, \tau \in Thin_{\epsilon,S}(\Lambda_{\epsilon'',\tau}) \asymp Q(\Lambda_{\epsilon'',\tau})$. By Corollary 3.37,

$$d_{\mathcal{AM}(S)}(\tilde{\mu}_X, \tilde{\mu}_\tau) \asymp \sum_{\alpha \in \mathcal{C}(S \setminus H \cdot \Lambda_{\epsilon'',\tau})} [d_{\mathcal{H}_\alpha}(\tilde{\mu}_X, \tilde{\mu}_\tau)]_K$$

Since $d_{\mathcal{H}_\lambda}(\tilde{\mu}_X, \tilde{\mu}_\tau) \prec \tilde{R}$ for all $\lambda \in \Lambda_{\epsilon,X_0}$ by Proposition 4.22, we have

$$d_{\mathcal{AM}(S)}(\tilde{\mu}_X, \tilde{\mu}_\tau) \asymp \sum_{\alpha \in \mathcal{C}(S \setminus \Lambda_{\epsilon,X_0})} [d_{\mathcal{H}_\alpha}(\tilde{\mu}_X, \tilde{\mu}_\tau)]_K$$

Recall that the very short curves of τ, $\Lambda_{\epsilon'',\tau}$, are a subset of the short curves of X, $\Lambda_{\epsilon,X} = \Lambda_{\epsilon,X_0}$. Because there is a uniform bound on the distance between the projections of τ and X to any horoball over a curve in $\Lambda_{\epsilon,X}$, it follows that there is a lower bound on the τ- and X-lengths of any curve not in $\Lambda_{\epsilon,X}$. Thus the projections of $\tilde{\mu}_\tau$ and $\tilde{\mu}_X$ to any other combinatorial horoball have uniformly bounded length coordinates and the sum becomes

$$d_{\mathcal{AM}(S)}(\tilde{\mu}_X, \tilde{\mu}_\tau) \asymp \sum_{\alpha \in \mathcal{C}(S \setminus \Lambda_{\epsilon,X_0})} [\log d_\alpha(\tilde{\mu}_X, \tilde{\mu}_\tau)]_K \tag{4.3}$$

Lemma 4.23. *The number of terms which can appear in the sum of (Equation 4.3) is uniformly bounded.*

Proof. Let $\tilde{\Gamma}$ be an augmented hierarchy path between $\tilde{\mu}_X$ and $\tilde{\mu}_\tau$ based on a hierarchy J (see Subsection **??**). Observe that the number of curves appearing as base curves of augmented markings in $\widetilde{\Gamma}$ is determined by the number of flip moves in $\widetilde{\Gamma}$. Since each such flip move makes progress along some $g_Y \in J$, for some nonannular $Y \subset S$, it follows that if there is not a bound on the number of base curves appearing in $\widetilde{\Gamma}$, then there is not a bound on either the length of geodesics in J or the number of nonannular subsurfaces supporting geodesics in J. Both imply that $d_Y(\tilde{\mu}_X, \tilde{\mu}_\tau)$ is unbounded for some nonannular $Y \subset S$ (possibly S itself), which contradicts the fact that $\tilde{\mu}_X$ and $\tilde{\mu}_\tau$ have bounded nonannular subsurface projections. The bound on the number of curves appearing in the sum of (Equation 4.3) is uniform because the bound on the subsurface projections is uniform, depending only on S and the almost-fixed constant R. □

We are now ready to prove the main theorem.

Theorem 4.24 (Almost-fixed points are close to fixed points)**.** *For any $R > 0$, there is an $R' = R'(R, S) > 0$ such that the following holds. Let $H \leqslant \mathcal{MCG}(S)$ be a finite subgroup and* $\mathrm{Fix}(H) \subset \mathcal{T}(S)$ *its fixed point set. For any $\tau \in \mathrm{Fix}_R^{\mathcal{T}}(H)$, there is fixed point $\sigma \in \mathrm{Fix}(H)$ such that $d_T(\tau, \sigma) < R'$.*

Proof. Let $X \in \mathcal{T}(S)$ be as in Proposition 4.22. As the constant $\widetilde{R}$ in Proposition 4.22 was a constant depending on R, we have shown that

$$d_T(X, \tau) \asymp_R \sum_{\alpha \in \mathcal{C}(S \setminus \Lambda_{\epsilon, X_0})} [\log d_\alpha(\tilde{\mu}_X, \tilde{\mu}_\tau)]_K \tag{4.4}$$

More precisely, Proposition 4.22 states that $d_Y(\tilde{\mu}_X, \tilde{\mu}_\tau) < K$ for any nonannular subsurface $Y \subset S \setminus \Lambda_{\epsilon, X_0}$, where K is a constant depending only on R and S.

We now organize the α that have nonzero terms in equation (Equation 4.4). By Tao's Lemma 4.11, if we increase the large link threshold to $\widehat{K} = \widehat{K}(R, S) > 0$, then these annuli are H-symmetric and we can group them into their H-orbits, $\mathcal{A} = \{\mathcal{A}_1, \ldots, \mathcal{A}_N\}$, where $\mathcal{A}_i$ is the H-orbit of α_i.

We note that N is uniformly bounded because the number of annuli appearing in the sum is uniformly bounded, by Lemma 4.23.

Let $\Gamma_{X,\tau}$ be any augmented hierarchy path from $\tilde{\mu}_X$ to $\tilde{\mu}_\tau$. By rearranging, we may assume that the order of the indices of the α_i coincides with the order of appearance of the α_i along $\Gamma_{X,\tau}$. Note that Lemma 4.11 implies that the curves within each symmetric family, $\mathcal{A}_i$, are not time-ordered.

We now apply the tools developed in Subsection 3.3. Recall that for a simplex $\Delta \subset \mathcal{C}(S)$, $Q(\Delta) = \{\tilde{\mu} \in \mathcal{AM}(S) | \Delta \subset \mathrm{base}(\tilde{\mu})\}$ and $\phi_\Delta : \mathcal{AM}(S) \to Q(\Delta)$ was the closest point projection.

In what follows, we explain how to create a sequence of fixed points $X_1, \ldots, X_N \in \mathrm{Fix}(H)$, with $d_{\mathcal{T}(S)}(X_N, \tau) \asymp_R 1$, where N is again the number of symmetric families of annuli in $\mathcal{A}$. The $(i+1)$-step begins with projecting $\tilde{\mu}_{X_i}$, a shortest augmented marking for X_i, to $Q(\mathcal{A}_{i+1})$ and showing that this projection is uniformly close to $\tilde{\mu}_{X_i}$. We then apply a large H-symmetric multitwist around the curves in $\mathcal{A}_{i+1}$ to both X_i and its projection to $Q(\mathcal{A}_i)$, the latter of which we show has made the progress toward τ that we want, with the former coming along for the ride and whose image we call X_{i+1}. This multitwisting process is identical to the process at the end of the proof of Proposition 4.22, but now the X_i need not be in a obviously good place to apply the $(i+1)^{\text{th}}$-group of multitwists. The key observation is that $d_{\mathcal{AM}(S)}(\tilde{\mu}_{X_i}, \phi_{\mathcal{A}_{i+1}}(\tilde{\mu}_{X_i})) \asymp_R 1$ for all i, a fact which requires understanding the subsurfaces through which $\Gamma_{X,\tau}$ passes. Showing that $d_{\mathcal{AM}(S)}(X_N, \tilde{\mu}_\tau) \asymp_R 1$ then involves comparing subsurface projections and showing that projections to horoballs over curves in $\mathcal{A}$ have changed a significant amount, in particular moving them close to those for $\tilde{\mu}_\tau$.

Let $\tilde{\mu}_X \in \mathcal{AM}(S)$ be a shortest augmented marking for X. We begin by projecting $\tilde{\mu}_X$ to $Q(\mathcal{A}_1)$. Set $\tilde{\mu}_{\alpha_1} = \phi_{\mathcal{A}_1}(\tilde{\mu}_X)$.

Claim 1: $d_{\mathcal{AM}(S)}(\tilde{\mu}_X, \tilde{\mu}_{\alpha_1}) \asymp 1$.

Before we prove the claim, we introduce some notation to simplify our calculations. For each i, label the curves in $\mathcal{A}_1 = \{\alpha_{i,1}, \ldots, \alpha_{i,n_i}\}$. We note that each n_i satisfies $n_i \leqslant |H|$.

First, we prove that for all j, $d_{\mathcal{AM}(S)}\left(\tilde{\mu}_X, \phi_{\alpha_{1,j}}(\tilde{\mu}_X)\right) \asymp 1$. To see this, note that Lemma 3.45 implies that $\phi_{\alpha_{1,j}}$ is coarsely a closest point projection to $Q(\alpha_{1,j})$, so that

$$d_{\mathcal{AM}(S)}(\tilde{\mu}_X, \phi_{\alpha_{1,j}}(\tilde{\mu}_X)) \asymp \sum_{Y \pitchfork \alpha_{1,j}} \left[d_Y(\tilde{\mu}_X, \phi_{\alpha_{1,j}}(\tilde{\mu}_X))\right]_{L_1}$$

and

$$\sum_{Y \subset S \setminus \alpha_{1,j}} \left[d_Y(\tilde{\mu}_X, \phi_{\alpha_{1,j}}(\tilde{\mu}_X))\right]_{L_1} = 0$$

where L_1 is the uniform constant from Lemma 3.45.

In order to show that $d_{\mathcal{AM}(S)}(\tilde{\mu}_X, \phi_{\alpha_{1,j}}(\tilde{\mu}_X))$ is bounded, it suffices to exhibit a path from $\tilde{\mu}_X$ to a point in $Q(\alpha_{1,j})$ which makes only bounded progress in subsurfaces which interlock $\alpha_{1,j}$.

The augmented hierarchy path $\Gamma_{X,\tau}$ is precisely such a path. Recall that $\mathcal{A}$ consists of all the $\widehat{K}$-large links between $\tilde{\mu}_X$ and $\tilde{\mu}_\tau$, which we have ordered by their appearance along $\Gamma_{X,\tau}$, and that α_1 is the first curve in $\mathcal{A}$ to appear as a base curve along $\Gamma_{X,\tau}$. Since Lemma 4.14 implies that the orbits in $\mathcal{A}$ are time-ordered together, it follows that any other curve $\beta \in \mathcal{A}$ which intersects $\alpha_{1,j}$ can only appear as a base curve along $\Gamma_{X,\tau}$ *after* all progress through $\alpha_{1,j}$ has already been made. By Lemma **??**, $\Gamma_{X,\tau}$ makes a bounded amount of progress in subsurfaces which interlock $\alpha_{1,j}$ between $\tilde{\mu}_X$ and the first point along $\Gamma_{X,\tau}$ at which $\alpha_{1,j}$ appears in its base.

Thus $d_{\mathcal{AM}(S)}(\tilde{\mu}_X, \phi_{\alpha_{1,j}}(\tilde{\mu}_X)) \asymp 1$ for all j.

Since the $\phi_{\alpha_{1,j}}$ are Lipschitz (Lemma 3.46), it follows that

$$d_{\mathcal{AM}(S)}\left(\tilde{\mu}_X, \phi_{\alpha_{1,1}}(\tilde{\mu}_X)\right) \asymp d_{\mathcal{AM}(S)}\left(\phi_{\alpha_{1,2}}(\tilde{\mu}_X), \phi_{\alpha_{1,2}}(\phi_{\alpha_{1,1}}(\tilde{\mu}_X))\right) \asymp d_{\mathcal{AM}(S)}\left(\phi_{\alpha_{1,2}}(\tilde{\mu}_X), \phi_{\alpha_{1,1}\cup\alpha_{1,2}}(\tilde{\mu}_X)\right)$$

with the second coarse equality following from Proposition 3.47.

Since $d_{\mathcal{AM}(S)}\left(\tilde{\mu}_X, \phi_{\alpha_{1,2}}(\tilde{\mu}_X)\right) \asymp 1$, it follows from applying the triangle inequality that

$$d_{\mathcal{AM}(S)}\left(\tilde{\mu}_X, \phi_{\alpha_{1,1}\cup\alpha_{1,2}}(\tilde{\mu}_X)\right) \asymp 1$$

Applying this observation a uniformly bounded number of times (for $n_1 \leqslant |H|$), we obtain $d_{\mathcal{AM}(S)}(\tilde{\mu}_X, \tilde{\mu}_{\alpha_1}) \asymp 1$, proving Claim 1.

Let $T_{\alpha_1} = \prod_{j=1}^{n_1} T_{\alpha_{1,j}}^{(-1)^{s_1}\cdot d_1}$, where $T_{\alpha_{1,j}}$ is the Dehn (half)twist around $\alpha_{1,j}$, $d_1 = d_{\alpha_1,1}(\tilde{\mu}_X, \tilde{\mu}_\tau)$, and the sign s_1 depends on whether $\pi_{\alpha_{1,1}}(\tilde{\mu}_X)$ differs from $\pi_{\alpha_{1,1}}(\tilde{\mu}_\tau)$ by right or left Dehn (half) twists around $\alpha_{1,1}$. Set $X_1 = T_{\alpha_1}(X)$ and let $\tilde{\mu}_{X_1}$ be its shortest augmented marking.

First, note that since $T_{\alpha_1} \in C_{\mathcal{MCG}(S)}(H)$ centralizes H in $\mathcal{MCG}(S)$ and is thus contained in the normalizer, which stabilizes $\mathrm{Fix}(H)$, we have $X_1 \in \mathrm{Fix}(H)$. Moreover, we claim that the distance between X and X_1 is coarsely determined by the distance traveled in $\mathcal{A}_1$:

$$d_{\mathcal{T}(S)}(X, X_1) \asymp \sum_{\alpha \in \mathcal{A}_1} [\log d_\alpha(\tilde{\mu}_X, \tilde{\mu}_{X_1})]_{K_1} \tag{4.5}$$

and

$$\sum_{Y \subset S \setminus \mathcal{A}_1} [d_Y(\tilde{\mu}_X, \tilde{\mu}_{X_1})]_{K_1} = 0 \tag{4.6}$$

where K_1 is a constant depending only on R and S.

Recall that Lemma 4.16 implies that $d_{\alpha_{1,i}}(\tilde{\mu}_X, \tilde{\mu}_\tau) \asymp_R d_{\alpha_{1,j}}(\tilde{\mu}_X, \tilde{\mu}_\tau)$ for any i, j and since X is fixed and τ is has a bounded diameter orbit, it follows that $\pi_{\alpha_{1,i}}(\tilde{\mu}_X)$ differs from $\pi_{\alpha_{1,i}}(\tilde{\mu}_\tau)$ by coarsely the same number of right or left Dehn (half)twists for all i, where the handedness is independent of i. We immediately obtain $d_{\alpha_{1,i}}(\tilde{\mu}_{X_1}, \tilde{\mu}_\tau) \asymp_R 1$ for all i. Thus once we prove that (Equation 4.5) and (Equation 4.6) are true, it will follow from the triangle inequality that

$$d_T(X_1, \tau) \asymp \sum_{\alpha \in \mathcal{A} \setminus \mathcal{A}_1} [\log d_\alpha(\tilde{\mu}_X, \tilde{\mu}_\tau)]_{K_1} \tag{4.7}$$

and

$$\sum_{Y \subset S \setminus (\mathcal{A} \setminus \mathcal{A}_1)} [d_Y(\tilde{\mu}_{X_1}, \tilde{\mu}_\tau)]_{K_1} = 0 \tag{4.8}$$

By establishing (Equation 4.7) and (Equation 4.8), we will have shown that X_1 has removed the curves in $\mathcal{A}_1$ as combinatorial obstacles between X and τ, while all other projections remain coarsely unchanged. These equations are rephrased as the inductive hypothesis in (1) and (2) below.

To see (Equation 4.5) and (Equation 4.6), observe that $\phi_{\mathcal{A}_1}(\tilde{\mu}_X), \tilde{\mu}_{X_1} \in Q(\mathcal{A}_1)$. By Lemma 3.37, the distance between $\phi_{\mathcal{A}_1}(\tilde{\mu}_X)$ and $\tilde{\mu}_{X_1}$ is coarsely determined by projections to subsurfaces $Y \subset \sigma(\mathcal{A}_1)$, with all other subsurface projections being uniformly bounded. However, note that since $\phi_{\mathcal{A}_1}(\tilde{\mu}_X), \tilde{\mu}_{X_1} \in Q(\mathcal{A}_1)$, all base and transverse curves in $\phi_{\mathcal{A}_1}(\tilde{\mu}_X)$ and $\tilde{\mu}_{X_1}$ are disjoint from $\mathcal{A}_1$, and so T_{α_1} only acts nontrivially on the $\mathcal{A}_1$ coordinates of $\phi_{\mathcal{A}_1}(\tilde{\mu}_X)$ and $\tilde{\mu}_{X_1}$. Lemma 3.42 implies that $d_Y(\phi_{\mathcal{A}_1}(\tilde{\mu}_X), \tilde{\mu}_{X_1}) \asymp 1$ for all $Y \subset \sigma(\mathcal{A}_1) \setminus \mathcal{A}_1$, from which (Equation 4.5) and (Equation 4.6), and thus (Equation 4.7) and (Equation 4.8), follow for some choice of K_1 depending only on R and S.

In summary, we have produced a point $X_1 \in \mathrm{Fix}(H)$ whose distance to τ is determined by one less set of annuli, while the distances of projections to all other subsurfaces are coarsely unchanged.

We remark that in the above calculations, we repeatedly made coarse estimates to determine that the distance in (Equation 4.6) is bounded. Since we did so only finitely many times, where the number of times depended only on the topology of S and the almost-fixed constant R, it follows that the coarseness of our estimates is still uniformly bounded as a function of R and S.

In what follows, we make an inductive argument in which we perform a similar series of computations to create the sequence of fixed points $X_1, \ldots, X_N$. With the last point, X_N, we will have moved past each of the families in $\mathcal{A}$, at each step leaving all complementary subsurface projections coarsely fixed. Since N was a number which depended only on R and S, we find a bound for $d_T(X_N, \tau)$ that depends only on R and S. Since R was a fixed constant independent of τ, it follows that $d_T(X_N, \tau)$ and thus $d_T(\tau, \mathrm{Fix}(H))$ are uniformly bounded in terms of R and S, completing the proof.

We proceed by induction on the A_i. Suppose we have created a sequence of fixed points, $X_1, \ldots, X_i \in \mathrm{Fix}(H)$ with shortest augmented markings $\tilde{\mu}_{X_1}, \ldots, \tilde{\mu}_{X_i}$ and a sequence of constants, K_i depending only on R and S, such that for each $j \leqslant i$ the following properties hold:

1. For every subsurface $Y \subset S$ which is not an annulus with core curve $\alpha_{l,m} \in \mathcal{A}$ for $l \leqslant j$, we have $d_Y(\tilde{\mu}_X, \tilde{\mu}_{X_j}) < K_j$
2. For every subsurface $Y \subset S$ which is not an annulus with core curve $\alpha_{l,m} \in \mathcal{A}$ for $l \geqslant j$, we have $d_Y(\tilde{\mu}_{X_j}, \tilde{\mu}_\tau) < K_j$

We have already shown that the base case of $i = 1$ holds above in (Equation 4.7) and (Equation 4.8).

Note that (1) and the triangle inequality imply that $d_{\alpha_{l,m}}\left(\tilde{\mu}_{X_j}, \tilde{\mu}_X\right) \asymp_R 1$ for all $j \geqslant i$, $l \geqslant j$, and $m \leqslant n_j$. Similarly, (2) and the triangle inequality imply that $d_{\alpha_{l,m}}\left(\tilde{\mu}_{X_j}, \tilde{\mu}_\tau\right) \asymp_R 1$ for all $j \leqslant i$, $l \leqslant j$, and $m \leqslant n_j$.

Since $\mathcal{A}$ consisted of N orbits of curves with $N = N(R, S) > 0$, once the inductive step is proven, we will have constructed a fixed point $X_N \in \text{Fix}(H)$ which satisfies the inequality in (2). Since j in (2) is bounded by N, it will follow that $d_T(X_N, \tau) < K_N$, where K_N depends only on R and S, completing the proof.

We now proceed to prove the inductive step. The construction of X_{i+1} from X_i is similar to the construction of X_1 from X, but there are now are more quantities to manage. Let $\tilde{\mu}_{i+1} = \phi_{\mathcal{A}_{i+1}}(\tilde{\mu}_{X_i})$. As before, we begin with the following claim:

Claim $(i+1)$: $d_{\mathcal{AM}(S)}\left(\tilde{\mu}_{X_i}, \tilde{\mu}_{i+1}\right) \asymp 1$.

As with Claim 1, the proof of Claim $(i+1)$ involves showing that $d_{\mathcal{AM}(S)}\left(\tilde{\mu}_{X_i}, \tilde{\mu}_{i+1}\right) \asymp 1$ for $1 \leqslant j \leqslant n_{i+1}$ and then repeatedly applying Lemma 3.46 and Proposition 3.47 and the triangle inequality.

Let $1 \leqslant j \leqslant n_{i+1}$. By Lemma 3.45, $\phi_{\alpha_{i+1,j}}$ is coarsely the closest point projection to $Q(\alpha_{i+1,j})$, so Lemma 3.39 implies that

$$d_{\mathcal{AM}(S)}(\tilde{\mu}_{X_i}, \tilde{\mu}_{i+1}) \asymp \sum_{Y \pitchfork \alpha_{i+1,j}} \left[d_Y(\tilde{\mu}_{X_i}, \alpha_{i+1,j})\right]_{L_1}$$

and

$$\sum_{Y \subset S \setminus \alpha_{i+1,j}} \left[d_Y(\tilde{\mu}_{X_i}, \alpha_{i+1,j})\right]_{L_1} = 0$$

where L_1 is the uniform constant from Lemma 3.45.

Let $\tilde{\mu}_{\alpha_{i+1,j}} \in \Gamma_{X,\tau}$ be the first point along $\Gamma_{X,\tau}$ in which $\alpha_{i+1,j}$ appears as a base curve. If $\alpha_{l,m} \in \mathcal{A}$ is such that $\alpha_{l,m} \pitchfork \alpha_{i+1,j}$ and $l \leqslant i$ and $m \leqslant n_l$, then Lemma 4.14 implies that $\alpha_{l,m} \prec_t \alpha_{i+1,j}$. Lemma **??** implies that the active segment of $\alpha_{l,m}$ entirely precedes the active segment of $\alpha_{i+1,j}$, of which $\tilde{\mu}_{\alpha_{i+1,j}}$ is the first point. Thus $d_{\alpha_{l,m}}(\tilde{\mu}_{\alpha_{i+1,j}}, \tilde{\mu}_\tau) \asymp 1$ by Lemma **??**. Since $d_{\alpha_{l,m}}(\tilde{\mu}_{X_i}, \tilde{\mu}_\tau) \asymp 1$ by inductive assumption (2), the triangle inequality implies that $d_{\alpha_{l,m}}\left(\tilde{\mu}_{X_i}, \tilde{\mu}_{\alpha_{i+1,j}}\right) \asymp 1$ for all $l \leqslant i, m \leqslant n_l$ for which $\alpha_{l,m} \pitchfork \alpha_{i+1,j}$. Since $\alpha_{i+1,j} \in \operatorname{base}(\tilde{\mu}_{\alpha_{i+1,j}})$, it follows that $d_Y(\alpha_{i+1,j}, \tilde{\mu}_{\alpha_{i+1,j}}) \asymp 1$ for any $Y \pitchfork \alpha_{i+1,j}$ and thus $d_{\alpha_{l,m}}(\tilde{\mu}_{X_i}, \alpha_{i+1,j}) \asymp 1$.

On the other hand, for any $\beta \in \mathcal{A}$ with $\alpha_{i+1,j} \prec_t \beta$, Lemma **??** implies that $d_\beta(\tilde{\mu}_X, \tilde{\mu}_{\alpha_{i+1,j}}) \asymp 1$. Thus inductive assumption (1) and the triangle inequality imply that $d_\beta(\tilde{\mu}_{X_i}, \tilde{\mu}_{\alpha_{i+1,j}}) \asymp 1$ for any such β.

To summarize, we have shown:

$$\begin{aligned}
d_{\mathcal{AM}(S)}\left(\tilde{\mu}_{X_i}, \phi_{\alpha_{i+1,j}}(\tilde{\mu}_{X_i})\right) &\asymp \sum_{Y \pitchfork \alpha_{i+1,j}} \left[d_Y(\tilde{\mu}_{X_i}, \phi_{\alpha_{i+1,j}}(\tilde{\mu}_{X_i}))\right]_{K'} \\
&\asymp \sum_{Y \pitchfork \alpha_{i+1,j}} \left[d_Y(\tilde{\mu}_{X_i}, \tilde{\mu}_{\alpha_{i+1,j}})\right]_{K'} \\
&\asymp \sum_{\substack{\alpha_{l,m} \pitchfork \alpha_{i+1,j} \\ l \leqslant i}} \left[d_{\alpha_{l,m}}(\tilde{\mu}_{X_i}, \tilde{\mu}_{\alpha_{i+1,j}})\right]_{K'} \\
&\asymp 1
\end{aligned}$$

where $K' = \max\{K_i, L_1\}$, which we note depends only on R and S.

Claim $(i+1)$ follows by applying Lemma 3.46 and Proposition 3.47 a uniformly bounded number of times, as in the proof of Claim 1.

We now proceed to create X_{i+1} from X_i as we did X_1 from X_0. Let $T_{\alpha_{i+1}} = \prod_{j=1}^{n_{i+1}} T_{\alpha_{i+1,j}}^{(-1)^{s_{i+1}} d_{i+1}}$, where $T_{\alpha_{i+1,j}}$ is the Dehn (half)twist around $\alpha_{i+1,j}$, $d_{i+1} = d_{\alpha_{i+1,1}}(\tilde{\mu}_X, \tilde{\mu}_\tau)$, and the sign s_{i+1} depends on whether $\pi_{\alpha_{i+1,1}}(\tilde{\mu}_X)$ differs from $\pi_{\alpha_{i+1,1}}(\tilde{\mu}_\tau)$ by right or left Dehn (half) twists around $\alpha_{i+1,1}$. Set $X_{i+1} = T_{\alpha_{i+1}}(X_i)$ and let $\tilde{\mu}_{X_{1+1}}$ be its shortest augmented marking.

Once again $T_{\alpha_{i+1}} \in C_{\mathcal{MCG}(S)}(H)$ centralizes H, so it stabilizes $\mathrm{Fix}(H)$ and $X_{i+1} \in \mathrm{Fix}(H)$. We claim that X_{i+1} satisfies the properties in the inductive assumptions (1) and (2) above.

Lemma 4.16 implies that $d_{\alpha_{i+1,j}}(\tilde{\mu}_X, \tilde{\mu}_\tau) \asymp_R d_{\alpha_{i+1,l}}(\tilde{\mu}_X, \tilde{\mu}_\tau)$ for any j, l, and since X is fixed and τ is has a bounded diameter orbit, it follows that $\pi_{\alpha_{i+1,j}}(\tilde{\mu}_X)$ differs from $\pi_{\alpha_{i+1,j}}(\tilde{\mu}_\tau)$ by coarsely the same number of right or left Dehn (half)twists for all j, where the handedness is independent of i. It follows immediately that $d_{\alpha_{i+1,j}}(\tilde{\mu}_{X_{i+1}}, \tilde{\mu}_\tau) \asymp_R 1$ for all j.

Observe that $\phi_{\mathcal{A}_{i+1}}(\tilde{\mu}_{X_i}), \tilde{\mu}_{X_{i+1}} \in Q(\mathcal{A}_{i+1})$. By Lemma 3.37, the distance between $\phi_{\mathcal{A}_{i+1}}(\tilde{\mu}_{X_i})$ and $\tilde{\mu}_{X_{i+1}}$ is coarsely determined by projections to subsurfaces $Y \subset \sigma(\mathcal{A}_{i+1})$, with all other subsurface projections being uniformly bounded. However, note that since $\phi_{\mathcal{A}_{i+1}}(\tilde{\mu}_{X_i}), \tilde{\mu}_{X_{i+1}} \in Q(\mathcal{A}_{i+1})$, all base and transverse curves in $\phi_{\mathcal{A}_{i+1}}(\tilde{\mu}_{X_i})$ and $\tilde{\mu}_{X_{i+1}}$ are disjoint from $\mathcal{A}_{i+1}$, and so $T_{\alpha_{i+1}}$ only acts nontrivially on the $\mathcal{A}_{i+1}$ coordinates of $\phi_{\mathcal{A}_{i+1}}(\tilde{\mu}_{X_i})$ and $\tilde{\mu}_{X_{i+1}}$. Thus $d_Y(\phi_{\mathcal{A}_{i+1}}(\tilde{\mu}_{X_i}), \tilde{\mu}_{X_{i+1}}) \asymp_R 1$ for all $Y \subset \sigma(\mathcal{A}_{i+1}) \backslash \mathcal{A}_{i+1}$. Equations (1) and (2) for $j = i+1$ follow immediately from the triangle inequality and the inductive assumptions that (1) and (2) hold for X_i.

This completes the inductive step and thus the proof.

□

4.4 Coarse barycenters for the Teichmüller metric

The goal of this section is to prove the following theorem:

Theorem 4.25 (Coarse barycenters for $(\mathcal{T}(S), d_T)$)**.** *There are $\widetilde{K}, \widetilde{C} > 0$ such that for any $\sigma \in \mathcal{T}(S)$ and any finite order $f \in \mathcal{MCG}(S)$, there is a fixed point $X \in Fix(\langle f \rangle)$ such that*

$$d_T(\sigma, X) < \widetilde{K} \cdot d_{\mathcal{T}(S)}(\sigma, f \cdot \sigma) + \widetilde{C}$$

The proof relies in an essential way on Tao's main technical result (Tao13, Theorem 4.0.2), from which the linearly bounded conjugator property for $\mathcal{MCG}(S)$ for finite order elements follows almost immediately. She proves that there are coarse barycenters in $\mathcal{M}(S)$ for finite order elements of $\mathcal{MCG}(S)$:

Theorem 4.26 (Coarse barycenters for $\mathcal{M}(S)$; Theorem 4.0.2 in (Tao13))**.** *There are $R, K, C > 0$ depending only on S, so that for any marking $\mu \in \mathcal{M}(S)$ and finite order $f \in \mathcal{MCG}(S)$, there is a $\mu_0 \in \mathcal{M}(S)$ with $\operatorname{diam}_{\mathcal{M}(S)}(\langle f \rangle \cdot \mu_0) < R$, such that*

$$d_{\mathcal{M}(S)}(\mu, \mu_0) < K \cdot d_{\mathcal{M}(S)}(\mu, f \cdot \mu) + C$$

The proof of Theorem 4.26 proceeds by choosing a marking in $\mathcal{M}(S)$ with uniformly bounded f-orbit and then step by step reducing the complexity of the large subsurface projections between μ and the chosen marking, each step resulting in a new marking with uniformly bounded

f-orbit, whose combinatorial relationship with μ is simpler. At the heart of the proof are two technical Propositions 4.2.1 and 4.2.2, which construct the new markings. We need some observations about the proof these propositions.

Let $\mu_0 \in \mathcal{M}(S)$ have a uniformly bounded f-orbit and suppose that $Y \in \mathcal{L}_{\widehat{K}}(\mu, \mu_0)$ is a $\widehat{K}$-large link with $Y \subsetneq S$ a proper subsurface, where $\widehat{K}$ is the constant from Tao's Lemma 4.11 and $\mathcal{L}_{\widehat{K}}(\mu, \mu_0)$ is the set of $\widehat{K}$ large links between μ and μ_0. Let $L_Y \in \mathbb{N}$ be the smallest natural number such that f^{L_Y+1} is the first return map of f to Y and set $\Lambda_Y = \partial Y \cup f \cdot \partial Y \cup \cdots \cup f^{L_Y} \partial Y$. We note that Lemma 4.11 implies that $f^i \cdot Y \in \mathcal{L}_{\widehat{K}}(\mu, \mu_0)$ for each i.

Proposition 4.2.2 of (Tao13) produces a new marking $\mu_1 \in \mathcal{M}(S)$ with uniformly bounded f-orbit with $\Lambda_Y \subset \text{base}(\mu_1)$, such that $f^i \cdot Y \notin \mathcal{L}_{\widehat{K}}(\mu, \mu_1)$ for each i and if moreover $Z \in \mathcal{L}_{\widehat{K}}(\mu, \mu_1)$ and $Z \notin \mathcal{L}_{\widehat{K}}(\mu, \mu_0)$, then $Z \subset Y$ is a proper subsurface and thus has lower complexity. The marking μ_1 is first constructed via marking projections. Namely, one chooses correct transversals on the curves in Λ_Y, then by builds pieces on $\mathcal{M}(f^i \cdot Y)$ for each i. To complete these pieces to a marking on all of S, one induces the structure of μ on $S \backslash \left(\coprod_{1 \leqslant i \leqslant L_Y} f^i \cdot Y\right)$ by projecting μ_0 to a marking on each component thereof. In particular, this means that μ_0 and μ_1 have uniformly bounded projections to any subsurface of $S \backslash \left(\coprod_{1 \leqslant i \leqslant L_Y} f^i \cdot Y\right)$.

The proof of Theorem 4.25 proceeds by analyzing the short f-symmetric curves of the arbitrary point $\sigma \in \mathcal{T}(S)$ and choosing an initial fixed point, X', via Minsky's Product Regions Theorem 2.13 whose length and twisting coordinates in these short curves are sufficiently close to those of σ. We then apply Tao's Theorem 4.26 to the marking, $\mu_{X'}$, underlying a shortest augmented marking for X'. By the above observations, the result is a new almost-fixed marking,

μ_X, whose base curves contain the short f-symmetric curves of σ and whose transversals to these curves have changed a uniformly bounded amount compared to those of $\mu_{X'}$. We may then build an almost-fixed augmented marking, $\tilde{\mu}_{X''}$, whose projections to the horoballs over the f-symmetric short curves of σ are the same as those of $\tilde{\mu}_{X'}$. After performing similar calculations to the proof of Theorem 4.24, we find that any point $X'' \in \mathcal{T}(S)$ whose shortest augmented marking is $\tilde{\mu}_{X''}$ is an R-almost-fixed barycenter for σ in $\mathcal{T}(S)$, for some R depending only on S. An application of Theorem 4.24 produces the desired fixed point, $X \in Fix(\langle f \rangle)$.

Proof of Theorem 4.25. Let $\sigma \in \mathcal{T}(S)$ be arbitrary and $f \in \mathcal{MCG}(S)$ finite order. Let $\epsilon_0 > 0$ be as in Remark 4.10 with $H = \langle f \rangle$.

Let $\Lambda_{\epsilon_0,sym}(\sigma) = \{\lambda | l_\sigma(f^k \cdot \lambda) < \epsilon, \forall k\}$, the set of f-symmetric short curves of σ. We note that it is possible that other f-symmetric curves will be short in σ, but we are only interested in those whose entire f-orbit is short in σ.

Theorem 2.13 implies that σ lives in

$$Thin_{\epsilon_0}(S, \Lambda_{\epsilon,sym}(\sigma)) \stackrel{+}{\asymp} \prod_{\lambda \in \Lambda_{\epsilon,sym}(\sigma)} \mathbb{H}_\lambda \times \mathcal{T}(S \backslash \Lambda_{\epsilon,sym}(\sigma))$$

Let $\mathcal{O}$ be the 2-orbifold covered by S with deck group $\langle f \rangle$ and $\mathcal{T}(\mathcal{O})$ its Teichmüller space. Since the above product decomposition is f-symmetric, it descends via Theorem 4.9 to a product decomposition on $\mathcal{T}(\mathcal{O})$:

$$Thin_{\epsilon'}(\mathcal{O}, \Lambda') \stackrel{+}{\asymp} \prod_{\lambda \in \Lambda'} \mathbb{H}_\lambda \times \mathcal{T}(\mathcal{O} \backslash \Lambda')$$

for some $\epsilon' < \epsilon_0$ by Lemma 4.3 and $\Lambda' \subset \mathcal{C}(\mathcal{O})$ lifts to $\Lambda_{\epsilon_0,sym}(\sigma)$.

Decompose the curves in $\Lambda_{\epsilon_0,sym}(\sigma)$ into their f-orbits, $\Lambda_1, \ldots, \Lambda_k$. For each i, we can identify the various copies of $\mathbb{H}_\lambda$ for $\lambda \in \Lambda_i$ into a common horodisk, $\mathbb{H}_{\Lambda_i}$. Let $b_i \in \mathbb{H}_{\Lambda_i}$ be the barycenter of $\pi_{\mathbb{H}_{\Lambda_i}}(\langle f \rangle \cdot \sigma)$, the projection of the f-orbit of σ to $\mathbb{H}_{\Lambda_i}$. Note that the b_i are f-invariant.

For each $1 \leqslant i \leqslant k$, let $\bar{\Lambda}_i \in \mathcal{C}(\mathcal{O})$ be the projection of Λ_i to $\mathcal{C}(\mathcal{O})$. Since the b_i are f-invariant, each projects to a distinct point $\bar{b}_i \in \mathbb{H}_{\bar{\Lambda}_i}$. Let $\bar{X} \in Thin_{\epsilon',\Lambda'}(\mathcal{O})$ be any point whose length and twisting coords of the curves in the $\bar{\Lambda}_i$ are the $\bar{b}_i$.

Let $X' \in \mathcal{T}(S)$ be the unique lift of $\bar{X}' \in \mathcal{T}(\mathcal{O})$. Let $\tilde{\mu}_\sigma, \tilde{\mu}_{X'} \in \mathcal{AM}(S)$ be shortest augmented markings for σ, X' respectively. Since both σ and X' are in $Thin_{\epsilon_0,\Lambda_{\epsilon,sym}(\sigma)}(S)$, it follows that both $\tilde{\mu}_\sigma, \tilde{\mu}_{X'} \in Q(\Lambda_{\epsilon_0,sym}(\sigma))$ by Remark 3.14. By the choice of X', all projections of $\tilde{\mu}_\sigma$ and $\tilde{\mu}_{X'}$ to horoballs over curves in $\Lambda_{\epsilon,sym}(\sigma)$ are linearly bounded in terms of $d_{\mathcal{T}(S)}(\sigma, f \cdot \sigma)$: that is, there exist $K', C' > 0$ depending only on S such that $d_{\mathcal{H}_\lambda}(\tilde{\mu}_{X'}, \tilde{\mu}_\sigma) \leqslant K' \cdot d_{\mathcal{T}(S)}(\sigma, f \cdot \sigma) + C'$ for all $\lambda \in \Lambda_{\epsilon_0,sym}(\sigma)$. Moreover, by virtue of the fact that $\tilde{\mu}_\sigma, \tilde{\mu}_{X'} \in Q(\Lambda_{\epsilon_0,sym}(\sigma))$, it follows that for any other subsurface Y to which $\tilde{\mu}_\sigma$ and $\tilde{\mu}_{X'}$ have a large projection, we must have $Y \subset S \backslash \Lambda_{\epsilon,sym}(\sigma)$.

Let $\widehat{K} > 0$ be the constant from Tao's Lemma 4.11 with $H = \langle f \rangle$ and let $\mathcal{L}_{\widehat{K}}(\tilde{\mu}_\sigma, \tilde{\mu}_{X'})$ be the collection of $\widehat{K}$-large links between $\tilde{\mu}_\sigma$ and $\tilde{\mu}_{X'}$. As noted at the end of the previous paragraph, each $Y \in \mathcal{L}_{\widehat{K}}(\tilde{\mu}_\sigma, \tilde{\mu}_{X'})$ satisfies $Y \subset S \backslash \Lambda_{\epsilon_0,sym}(\sigma)$.

Let $\mu_{X'} \in \mathcal{M}(S)$ be the marking underlying $\tilde{\mu}_{X'}$. We now apply Tao's Theorem 4.26 to $\mu_{X'}$. By the discussion of the proof of Proposition 4.2.2 of (Tao13), Tao's Theorem 4.26

produces an R-almost fixed marking $\mu_{X''} \in \mathcal{M}(S)$, which has the property that, for each $Y \subset S$, $d_Y(\mu_{X''}, \mu_\sigma) < K' \cdot d_Y(\mu_\sigma, f \cdot \mu_\sigma) + C'$, where $K', C' > 0$ depend only on S. Moreover, we have that $\Lambda_{\epsilon_0,sym}(\sigma) \subset \text{base}(\mu_{X''})$, so we may build an augmented marking $\tilde{\mu}_{X''} \in \mathcal{AM}(S)$ whose length coordinates for the curves in $\Lambda_{\epsilon_0,sym}(\sigma)$ are those of $\tilde{\mu}_{X'}$.

We have already shown that $d_Y(\tilde{\mu}_{X''}, \tilde{\mu}_\sigma) < K'' \cdot d_Y(\tilde{\mu}_\sigma, f \cdot \tilde{\mu}_\sigma) + C''$ for any subsurface $Y \subset S$ (including annuli), where $K'', C'' > 0$ depend only on S. It remains to show that we have a similar bound on projections to all horoballs.

By construction, we have such a bound on any projection to a horoball over one of the curves in $\Lambda_{\epsilon_0,sym}(\sigma)$. If $\lambda \in \mathcal{C}(S)$ and $\lambda \notin \Lambda_{\epsilon_0,sym}(\sigma)$, then it follows that at least one curve $f^i \cdot \lambda$ in the f-orbit of λ satisfies $l_\sigma(f^i \cdot \lambda) > \epsilon_0$. In particular, for such a curve $f^i \cdot \lambda$, the projection $\pi_{\mathcal{H}_{f^i \cdot \lambda}}(\tilde{\mu}_\sigma)$ must bounded coarse length coordinate equal to 0; the coarse length coordinate of $\pi_{\mathcal{H}_{f^i \cdot \lambda}}(\tilde{\mu}_{X''})$ is 0 by construction. As the twisting coordinate of $\pi_{\mathcal{H}_{f^i \cdot \lambda}}(\tilde{\mu}_\sigma)$ and $\pi_{\mathcal{H}_{f^i \cdot \lambda}}(\tilde{\mu}_{X''})$ satisfy the above desired bound, it follows that $d_{\mathcal{H}_{f^i \cdot \lambda}}(\tilde{\mu}_{X''}, \tilde{\mu}_\sigma) < K' \cdot d_{\mathcal{H}_{f^i \cdot \lambda}}(\tilde{\mu}_\sigma, f \cdot \tilde{\mu}_\sigma) + C'$.

Let $X'' \in \mathcal{T}(S)$ be any point whose shortest augmented marking is $\tilde{\mu}_{X''}$. Then there are $K''', C''' > 0$ depending only on S such that

$$d_{\mathcal{T}(S)}(X'', \sigma) < K''' \cdot d_{\mathcal{T}(S)}(\sigma, f \cdot \sigma) + C'''$$

Applying Theorem 4.24, it follows that there are $\widetilde{K}, \widetilde{C} > 0$ depending only on S and a fixed point $X \in Fix(\langle f \rangle) \subset \mathcal{T}(S)$ such that

$$d_{\mathcal{T}(S)}(X, \sigma) < \widetilde{K} \cdot d_{\mathcal{T}(S)}(\sigma, f \cdot \sigma) + \widetilde{C}$$

as desired. □

Remark 4.27 (Theorem 4.25 for arbitrary finite subgroups). *We expect that Theorem 4.25 can be generalized to hold for any finite subgroup $H \leqslant \mathcal{MCG}(S)$. This might be accomplished by generalizing Tao's Theorem 4.26, but this would require a nearly complete reworking of her proof.*

Remark 4.28 (Independence of Theorem 4.24 and Theorem 4.26). *At first glance, it may seem that one might derive Theorem 4.24 from Theorem 4.26 or vice versa. The former does not imply the latter, since the bound in Theorem 4.24 from the starting point to* $\mathrm{Fix}(H)$ *is not linear in terms of the diameter of the orbit of the starting point. On the other hand, the latter does not imply the former, for it can at best produce an almost fixed point, when a genuine fixed point is needed. What is more, Theorem 4.24 holds for any finite subgroup of $\mathcal{MCG}(S)$ and Theorem 4.26 is only known for finite order elements.*

4.5 Non-quasiconvexity of $\mathrm{Fix}_R^{\mathcal{T}}(H)$

This purpose of this section is to prove the following theorem:

Theorem 4.29. *There exist an $R > 0$, a surface S, and a finite subgroup $H \subset \mathcal{MCG}(S)$ such that* $\mathrm{Fix}_R^{\mathcal{T}}(H)$ *is not L-quasiconvex for any $L > 0$.*

The example built in Theorem 4.29 is based on Rafi's example in Theorem 7.3 of (Raf10) of two Teichmüller geodesic segments which start and end at a bounded distance from each other and yet do not fellow travel. These two geodesics segments necessarily live in a thin part of $\mathcal{T}(S)$, as Theorem 7.1 of (Raf10) proves that this phenomenon does not occur when the endpoints are thick. Our construction requires the techniques from Rafi's example in the proof of Theorem 4.29, so an unfamiliar reader may want to familiarize himself with Rafi's proof. After the proof, we remark on how this theorem could be generalized. Indeed, we expect that $\mathrm{Fix}_R^T(H)$ is typically not quasiconvex.

Proof of Theorem 4.29. Let $d > 0$. Let S_0 be the closed genus 2 surface and let $\gamma \in \mathcal{C}(S_0)$ be a separating curve on S_0. Let $Y, Z \subset S_0$ be the two once-punctured tori which are the complements of γ. In his construction, Rafi builds two Teichmüller geodesics $\mathcal{G}_1, \mathcal{G}_2 : [0, 2d] \to \mathcal{T}(S_0)$ such that $d_T(\mathcal{G}_1(0), \mathcal{G}_2(0)) \asymp 1$ and $d_T(\mathcal{G}_1(2d), \mathcal{G}_2(2d)) \asymp 1$, but $d_T(\mathcal{G}_1(d), \mathcal{G}_2(d)) \dot{\succ} d$, where $d > 0$ can be chosen to be a large as necessary.

Both $\mathcal{G}_1$ and $\mathcal{G}_2$ live in $Thin_\epsilon(S_0, \gamma)$ and both Y and Z become isolated along $\mathcal{G}_1$ and $\mathcal{G}_2$. The key to the construction is altering *when* Y is isolated. In particular, the active interval of Y along $\mathcal{G}_1$ is $[0, d]$ and along $\mathcal{G}_2$ is $[d, 2d]$, so that one can show $d_Y(\tilde{\mu}_{\mathcal{G}_1(d)}, \tilde{\mu}_{\mathcal{G}_2(d)}) \dot{\succ} d$, from which the conclusion follows after an application of the distance formula (Theorem 3.33).

In the proof, we first lift $\mathcal{G}_1$ and $\mathcal{G}_2$ to $\mathcal{T}(S)$, where S is an appropriate finite cover with deck group isomorphic to $H \leqslant \mathcal{MCG}(S)$, where the lifts of Y and Z fill S, and the lifts $\mathcal{G}_1'$ and $\mathcal{G}_2'$ are now geodesics between points in $\mathrm{Fix}(H)$. Then, using Rafi's construction, we build a new geodesic $\mathcal{G} : [0, 2d] \to \mathcal{T}(S)$ which starts and ends at almost-fixed points (with the almost-fixed constant to be determined below), and which performs the restriction of $\mathcal{G}_1$ to Y on one of the lifts of Y and the restriction of $\mathcal{G}_2$ to Y on the other lifts of Y. Consequently, the projections of $\tilde{\mu}_{\mathcal{G}(d)}$ to the various lifts of Y disagree by a factor of at least d, and so it follows that $\mathcal{G}(d)$ cannot be close to $\mathrm{Fix}(H)$.

Let S be the 5-genus, degree 4 cover of S_0. Note that each of Y and Z lifts to two disjoint subsurfaces of S, say Y_1, Y_2 and Z_1, Z_2, with $S = Y_1 \cup Y_2 \cup Z_1 \cup Z_2$. Let $QD(S_0)$ denote the space of holomorphic quadratic differentials on S_0. Let $q_0, \bar{q}_0 \in QD(S_0)$ be the quadratic differentials which define the geodesic segments $\mathcal{G}_1$ and $\mathcal{G}_2$ in $\mathcal{T}(S_0)$, which were glued together from quadratic differentials on Y and Z, $q_Y \in QD(Y)$ and $q_Z \in QD(Z)$. The quadratic differentials q_0 and $\bar{q}_0$ lift to pairs of quadratic differentials $q_0', \bar{q}_0' \in QD(S)$. Similarly, q_Y and q_Z lift to quadratic differentials $q_{Y_1}' \in QD(Y_1), q_{Y_2}' \in QD(Y_2)$ and $q_{Z_1}' \in QD(Z_1), q_{Z_2}' \in QD(Z_2)$, respectively. These are the building blocks of our desired geodesic in $\mathcal{T}(S)$.

We now closely follow Rafi's construction. Let ϕ be the same Anosov map on a torus and let T be the same flat structure thereon used to create $\mathcal{G}_1$ and $\mathcal{G}_2$. Recall that T was chosen so that the vertical direction on T matches the unstable foliation of ϕ. Instead of cutting one slit in T, cut open two parallel but not colinear slits in T of size $\rho = ce^{-\frac{d}{2}}$ and of angle $\frac{\pi}{4}$, where the constant $0 < c < 1$ is specified shortly. Fix a homeomorphism from Y_1 to this double-slit torus and called this marked flat surface $T_{Y_1,0}$. Set

$$T_{Y_1,t} = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} T_{Y_1,0}$$

For any t, $T_{Y_1,t}$ is still a marked surface and the slits have minimum length at $t = 0$, growing exponentially as $t \to \pm\infty$. For $-\frac{d}{2} \leqslant t \leqslant \frac{d}{2}$, the length of the slits is smaller than c, but since the stable and unstable foliations of q_{Y_1} are cobounded, the length of any curve in $Y_{1,t}$ is comparable with 1. As with Rafi's example, when c is sufficiently small, $T_{Y_1,t}$ is an isolated subsurface when we glue it with the other slit tori to form $\mathcal{G}$. Choose $\delta \ll \rho$ as does Rafi, and let q be the quadratic differential defined by gluing T to $\delta T_{Y_1,-\frac{d}{2}}$ to another copy of T to $\delta T_{Y_2,-\frac{3d}{2}}$ back onto the first copy of T'.

The details of this gluing are as follows. We first scale down the given slitted tori by a factor of δ. Then we cut two slits in each of the two copies of T: in the first, we cut two slits, one each the same sizes and angles as the sizes and angles of the slits in $\delta T_{Y_1,-\frac{d}{2}}$ and $\delta T_{Y_2,-\frac{3d}{2}}$ and glue the appropriate pairings along these slits; then we similarly cut two slits in the second torus, one of each size and angle as before, and then attach them to the remaining slits on $\delta T_{Y_1,-\frac{d}{2}}$ and $\delta T_{Y_2,-\frac{3d}{2}}$. Importantly, we glue them so that the twisting around each of the newly formed curves which bound these subsurfaces (and are the lifts of γ) is equal to that of the twisting around these curves in $\mathcal{G}_1'(0)$. In particular, the twisting around each of the curves lifted from γ is coarsely equal.

Fix homeomorphisms from Z_1 to each of the above double-slitted tori. This allows us to define a quadratic differential $q \in Q(S)$. Let $\mathcal{G} : [0, 2d] \to \mathcal{T}(S)$ be the Teichmüller geodesic segment defined by q. Let $h \in \mathcal{MCG}(S)$ be the involution which rotates S to switch Y_1 with Y_2 and Z_1 with Z_2. We claim the following hold:

1. $d_T(\mathcal{G}(0), \mathcal{G}_1'(0)) \asymp 1$
2. $d_T(\mathcal{G}(2d), \mathcal{G}_1'(2d)) \asymp 1$
3. $d_T(\mathcal{G}(d), h \cdot \mathcal{G}(d)) \asymp d$

where the constants subsumed by the symbol $\asymp$ depend only on S. We remark that claims (1) and (2) imply that $\mathcal{G}(0)$ and $\mathcal{G}(2d)$ are R-almost-fixed for some constant R, as $\mathcal{G}_1'(0)$ and $\mathcal{G}'(2d)$ are fixed. The content of (3) is that $\mathcal{G}(d)$ is not d-almost-fixed. The constant d is of our choosing, while R depends only on the topology of S. Thus, verification of (1), (2), and (3) completes the proof of the theorem.

The remainder of the proof follows Rafi's closely. We first show claims (1) and (2) by satisfying the conditions of Corollary 2.6 of (Raf10). Then we apply Theorem 4.2 of (Raf10) to conclude claim (3) holds.

First, note that, by construction, relative twisting around the lifts of γ to S with $\mathcal{G}_1'(0)$ is uniformly bounded. Second, we note that since the vertical and horizontal foliations of Y_1, Y_2, Z_1, and Z_2 are cobounded, no curve in any of them is ever short along $\mathcal{G}$, so the set of short curves of both $\mathcal{G}(0)$ and $\mathcal{G}_1'(0)$ are precisely the lifts of γ.

As for the aforementioned subsurfaces, the restrictions of q to each of Y_1, Z_1, and Z_2 are identical to q_{Y_1}, q_{Z_1}, and q_{Z_2}, which are the projections of q_0' to Y_1, Z_1, and Z_2, respectively; similarly, the projection of q to Y_2 is identical to q_{Y_2}, which is the projection of $\vec{q}_0'$ to Y_2. By construction, the active intervals along $\mathcal{G}$ of Y_1 and Y_2, which we denote I_{Y_1}, I_{Y_2}, are $[0, d]$ and $[d, 2d]$ respectively. By Theorem 4.2 of (Raf10), the projections of $\mathcal{G}$ to $\mathcal{T}(Y_1)$ during I_{Y_1} and to $\mathcal{T}(Y_2)$ during I_{Y_2} fellow-travel the geodesics defined by the restriction of q to Y_1 and Y_2, respectively, and outside of these intervals have uniformly bounded projections to $\mathcal{C}(Y_1)$ and $\mathcal{C}(Y_2)$. In particular:

1. For any $t \in [0, d]$, we have $d_{\mathcal{T}(Y_1)}(\mathcal{G}(t)\big|_{Y_1}, q_{Y_1}) \asymp 1$

2. For $t \in [d, 2d]$ we have $d_{\mathcal{T}(Y_2)}(\mathcal{G}(t)\big|_{Y_2}, q_{Y_2}) \asymp 1$

3. $d_{Y_1}(\mathcal{G}(0), \mathcal{G}_1'(0)) \asymp 1$ and $d_{Y_2}(\mathcal{G}(2d), \mathcal{G}_1'(2d)) \asymp 1$

To finish the proof of claim (1), it remains to show that $\mathrm{Ext}_{\mathcal{G}(0)}(\gamma') \asymp \mathrm{Ext}_{\mathcal{G}_1'(0)}(\gamma')$ for each lift γ' of γ. Of the four lifts of γ, the two bounding Y_1 have the same length in $\mathcal{G}(0)$ as they do in $\mathcal{G}_1'(0)$ for we have scaled them in the same fashion, whereas the two bounding Y_2 have the same lengths in $\mathcal{G}(0)$ as they do in $\mathcal{G}_2'(0)$. Thus, by the construction of $\mathcal{G}_1$ and $\mathcal{G}_2$ in Theorem 7.3 of (Raf10), they have the same length.

It remains to show that claim (3) holds. Since Y_1 is an isolated subsurface along $\mathcal{G}$ during $[0, d]$ and no curve in Y_1 becomes short, it follows from Theorem 6.1 of (Raf10) and Lemma 4.4 of (RS07) that the shadow of $\mathcal{G}$ in $\mathcal{C}(Y_1)$ during $[0, d]$ is a parametrized quasigeodesic. Thus it follows from (3) that

$$d_{\mathcal{C}(Y_1)}(\mathcal{G}(0), \mathcal{G}(d)) \stackrel{\cdot}{\asymp} d \qquad \text{and} \qquad d_{\mathcal{C}(Y_2)}((\mathcal{G}(0), \mathcal{G}(d)) \asymp 1$$

Since Y_1 and Y_2 are homeomorphic, $\mathcal{C}(Y_1)$ and $\mathcal{C}(Y_2)$ are isometric. Let $\Phi : \mathcal{C}(Y_1) \to \mathcal{C}(Y_2)$ be such an identification. Since $d_{\mathcal{C}(Y_2)}(\Phi(\mathcal{G}(d)), \mathcal{G}(d)) \asymp d$, claim (3) follows from the distance formula Theorem 3.33, completing the proof of the theorem.

□

Remark 4.30 (Generalizations of the counter-example)**.** *We expect that the counter-example constructed in Theorem 4.29 should be a common phenomenon. The construction takes advantage of a surface lifting to disjoint subsurfaces in the covering surface, after which a geodesic* *is made to move at different times through the subsurfaces. We expect that nonquasiconvexity should hold any time this phenomenon occurs. More generally, it would not be surprising if nonquasiconvexity holds any time* $\mathrm{Fix}(H)$ *has infinite diameter, that is when* $\mathcal{O}$ *is not an orbifold with three cone points.*

CHAPTER 5

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