Complex dynamics:

Schwarzian derivatives and measures of maximal entropy

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THESIS

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To my family:

My parents

Brother and sisters

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TABLE OF CONTENTS

LIST OF FIGURES

FIGURE **PAGE**

LIST OF ABBREVIATIONS

Crit	The set of critical points
Precrit	The set of critical points and backward orbits of the critical points
PrePer	The set of preperiodic points
Per	The set of periodic points
Rat_d	The set of rational functions with degree d

SUMMARY

We investigate the Schwarzian derivatives of a polynomial and its iterates, where the polynomial is defined over the field of complex numbers. The escape-rate function of the polynomial play an important role in the study of polynomial dynamics. The study of the sequence of Schwarzian derivatives $\{S_{f^n}\}$ leads to a connection with the escape-rate function. By using the cocycle property of the Schwarzian derivative, we show that S_{f^n}/d^{2n} converge locally uniformly to $-2(\partial G_f)^2$ on the complement of some compact subset of $\mathbb{C}$, where G_f is the escape-rate function of f.

The polynomial basin of infinity admits a natural metric, which keeps a lot of polynomial dynamics information; see [11] and [12]. The quadratic differential $S_{f^n}dz^2/d^{2n}$ determines a Riemannian metric on the complement of f^n's critical points. As $n \to \infty$, this sequence of metric spaces has an ultralimit, which is a complete geodesic space with non-positive curvature. And by the properties inherited from the the dynamics of the polynomial, we can naturally embed the basin of infinity isometrically to the ultralimit.

We also investigate rational functions with identical measure of maximal entropy. For a given rational function $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ with degree $d \geq 2$, there is a unique probability measure μ_f associated with it, which achieves maximal entropy $\log d$ among all the f-invariant probability measures. From the work of Beardon, Levin, Baker-Eremenko, Schmidt-Steinmetz, etc (1980s-90s), the set of polynomials with identical measure of maximal entropy has been characterized. We construct examples of non-exceptional rational functions with common measure of maximal entropy, and they won't share an iterate up to precomposition by any Möbius transformation. Following from Levin-Przytycki's result [29] (1997), we characterize the general sets of rational functions with identical measures of maximal entropy. Finally, we sum up some known results related to the set of preperiodic points and maximal entropy measure, and then provide some necessary and sufficient conditions for two rational functions sharing an iterate.

CHAPTER 1

INTRODUCTION

The study of the dynamics of rational functions on the Riemann sphere started in the early 20th century; see [17], [18], [19] and [25]. In the last three decades, it became popular partly due to the development of the modern computer graphics. In this thesis, we focus on the study of Schwarzian derivatives of polynomial iterates, and characterizing the set of rational functions with identical measure of maximal entropy. The contents of this thesis are contained in the papers [41] and [42].

1.1 Schwarzian derivatives

Recall that the Schwarzian derivative of a holomorphic function f on the complex plane is defined as

$$S_f(z) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

It is well known that $S_f \equiv 0$ if and only if f is a Möbius transformation. We can view the Schwarzian derivative as a measure of the complexity of a nonconstant holomorphic function.

Let $f : \mathbb{C} \to \mathbb{C}$ be a complex polynomial with degree $d \geq 2$. In this thesis we examine the sequence of Schwarzian derivatives of the iterates f^n (f composed with itself n times) of f. Specifically, we look at the sequence

$$\left\{\frac{S_{f^n}(z)}{d^{2n}}\right\}_{n\geq 1}$$

and view it as a sequence of meromorphic functions or quadratic differentials on the Riemann sphere. We are interested in understanding the limit as $n \to \infty$.

Before we address any statement of this sequence, let's take a look at one of the simplest examples.

Example 1. Let $f(z) = z^d$ with $d \geq 2$, then we get

$$\begin{aligned} S_{f^n}(z) &= \tfrac{2(d^n-1)(d^n-2)-3(d^n-1)^2}{2z^2} \\ &= \tfrac{1-d^{2n}}{2z^2} \end{aligned}$$

Since $d \geq 2$, the sequence of normalized Schwarzians converges,

$$\lim_{n\to\infty} \frac{S_{f^n}}{d^{2n}} = \lim_{n\to\infty} \frac{1-d^{2n}}{2d^{2n}z^2} = -\frac{1}{2z^2}$$

locally uniformly on $\mathbb{C}\backslash\{0\}$.

The normalized Schwarzians as quadratic differentials $\left\{\frac{S_{f^n}}{d^{2n}} dz^2\right\}$ converge to $-\frac{1}{2z^2}dz^2$. The associated conformal metric $ds = \frac{|dz|}{|\sqrt{2}z|}$ makes $\mathbb{C}\backslash\{0\}$ isometric to an infinite cylinder of radius $\frac{1}{\sqrt{2}}$. The cylinder's closed geodesics are the horizontal trajectories of the quadratic differential $-\frac{1}{2z^2}dz^2$.

Local convergence. Let G_f be the escape-rate function of f, which is defined as

$$G_f = \lim_{n\to\infty} \frac{\log^+ |f^n|}{d^n},$$

where $\log^+ |x| = \max(\log|x|, 0)$. Let $\text{Precrit}(f) = \cup_{n>0}\{c \in \mathbb{C} | (f^n)'(c) = 0\}$ be the union of the critical points of f and their backward orbits. Note that its closure $\overline{\text{Precrit}(f)}$ contains the Julia set $J(f)$ when f is not conjugate to z^d.

Theorem 1.1.1. *Let f be a polynomial with degree $d \geq 2$ and not conformally conjugate to z^d. Then the sequence of Schwarzian derivatives S_{f^n} satisfies*

$$\lim_{n\to\infty} \frac{S_{f^n}(z)}{d^{2n}} = -2\left(\frac{\partial G_f(z)}{\partial z}\right)^2,$$

locally uniformly on $\mathbb{C}\backslash\overline{\mathrm{Precrit}(f)}$.

Remark. The choice of normalization $\frac{1}{d^{2n}}$ allows us to focus on the basin of infinity. Other normalizations might detect interesting properties of $J(f)$. In Corollary 2.2.4, we show that $\left\{\frac{S_{f^n}}{d^{2n}}dz^2\right\}$ converge on the entire Fatou set, in the sense of $L^{\frac{1}{2}}_{loc}$ convergence.

Sometimes, people are also interested in the nonlinearity of a nonconstant holomorphic function on $\mathbb{C}$. Similar with Theorem 1.1.1, we have $\lim_{n\to\infty}\frac{N_{f^n}dz}{d^n} = \partial G_f$ locally uniformly on $\mathbb{C}\backslash\overline{\mathrm{Precrit}(f)}$, where $N_f = f'/f''$; see Theorem 2.2.3.

Metric space convergence. Let f be a polynomial with degree $d \geq 2$. Each S_{f^n} determines a conformal geodesic metric d_n on the complement of the critical points of f^n, given by $ds = \sqrt{\left|-\frac{S_{f^n}}{d^{2n}}dz^2\right|}$. From this sequence of geodesic spaces, we obtain an ultralimit $(X_\omega, d_\omega, a_\omega)$; see Chapter I §5 [8] for more details about the ultralimit. The limit space is a complete geodesic space.

The escape-rate function G_f also determines a conformal metric on the basin of infinity $X_o = \{z \in \mathbb{C}|f^n(z) \to \infty\}$ of f, given by $ds = \sqrt{2}|\partial G_f|$. Given a choice of base point $a \in X_o\backslash\mathrm{Precrit}(f)$, we denote this pointed metric space by (X_o, d_o, a); compare [11] or [12] where this metric already appeared.

When the Julia set is not connected, the ultralimit X_ω can be described as a “hairy" version of X_o.

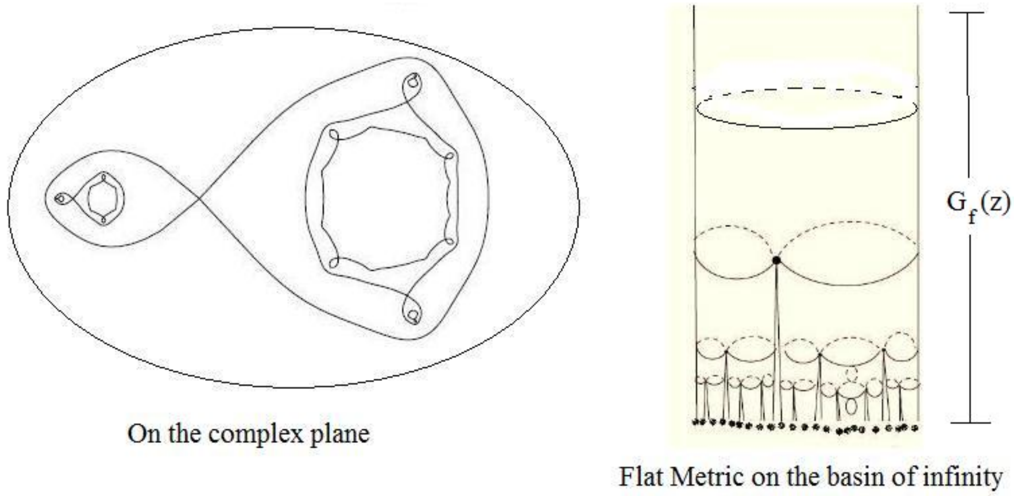

Figure 1.1.1. Level set structure of G_f for a cubic polynomial and flat metric structure on the basin of infinity

Theorem 1.1.2. *Let f be a polynomial with degree $d \geq 2$ and disconnected Julia set. There exists a natural embedding from $X_o(f) \setminus \mathrm{Precrit}(f)$ to X_ω which extends to the metric completion $(\overline{X}_o, d_o, a)$ as an isometric embedding.*

The metric space X_ω is obtained by attaching a real ray to $\overline{X}_o$ at each point in $\mathrm{Precrit}(f)$, and attaching some non trivial space (containing infinitely many real rays) to each connected component of $\overline{X}_o \setminus X_o$.

Remark. For $J(f)$ disconnected, we use the tree structure on the basin of infinity to show the embedding is an isometry. For $J(f)$ connected, we also have a locally isometric embedding of X_o to X_ω; this follows easily from the argument in the proof of the above theorem. But we do not expect the embedding to be a global isometry. The reason we use the ultralimit to study the limiting space is that the spaces are not compact and the metrics d_n are not uniformly proper, so more classical notions of convergence like Gromov-Hausdorff convergence won't work. We are not quite sure whether the ultralimit X_ω depends on the ultrafilter ω or not. But from the above theorem, the only things that might depend on the ultrafilter are the spaces attached to $\overline{X}_o \backslash X_o$.

Conjugacy classes. The study of $\{S_{f^n}\}$ has grown out of an attempt to better understand the moduli space M_d of polynomials (the space of conformal conjugacy classes). The geometric structure $(f, X_o, |\partial G_f|)$ has been studied in [12] and used to classify topological conjugacy classes. We can also use the Schwarzian derivative to classify polynomials with the same degree. We define an equivalence relation on the set of polynomials with degree $d \geq 2$ as: $f \sim g$ if $S_f dz^2 = A^*(S_g dz^2)$, for $A(z) = az + b$ some affine transformation. From this definition, polynomials f and g are equivalent to each other if and only if $f = B \circ g \circ A$ with A and B affine transformations, and if and only if f and g have the same critical set (counted with multiplicities) up to some affine transformation, i.e. $A(\mathrm{Crit}(f)) = \mathrm{Crit}(g)$ for some affine transformation A. See Lemma 2.1.2 for details. Note that affine conjugate polynomials are equivalent in this sense.

Theorem 1.1.3. *Let f and g be polynomials with the same degree $d \geq 2$. Then the following are equivalent:*

- *$f^n \sim g^n$ for infinitely many $n \in \mathbb{N}^*$.*
- *$f^n \sim g^n$ for all $n \in \mathbb{N}^*$.*

- *f and g have the same Julia set up to some affine transformation ($A(J(g)) = J(f)$ with A affine transformation).*

Polynomials f and g which satisfy the above conditions are called strongly equivalent. Each such strong equivalence class consists of finitely many affine conjugacy classes (no more than the order of the symmetry group of the Julia set).

Notes: the above theorem relies on the classification of polynomials with the same Julia set, and the proof uses the main result of [6] (see §2.1.2 for other references and historical context).

1.2 Measures of maximal entropy

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational function with degree $d \geq 2$. For each $z_o \in \mathbb{P}^1$, we can define a probability measure for $n \geq 1$,

$$\mu_{n,z_o} = \frac{1}{d^n} \sum_{f(z)=z_o} \delta_z,$$

here the sum is taken over all the roots of $f(z) = z_o$, counted with multiplicity. In 1965, Brolin [9] showed that for each polynomial f, with at most two exceptions for z_o, μ_{n,z_o} converges to some probability measure μ_f independent of the choice of z_o. Actually, when f is a polynomial, μ_f is the equilibrium measure of the filled Julia set of f (the set of points with bounded orbits under the iteration of f). In 1983, Lyubich [30] and independently Freire-Lopez-Mañé [21] generated this result to the the case of rational functions. They showed that this μ_f is the unique one supported on the Julia set, achieving the maximal entropy $\log d$ among all the f-invariant probability measures; this maximal measure is also discussed in [23].

We focus on characterizing the set of rational functions with identical measure of maximal entropy. It is well known that $\mu_f = \mu_{f^n}$ for all iterates f^n of f, and commuting rational functions have common measure of maximal entropy. For case of polynomials, having the same measure of maximal entropy is equivalent to having the same Julia set. During the 1980s and 90s, pairs of polynomials with identical Julia set were characterized; see [37], [7], [6] and [4]. The strongest result is: given any Julia set J of some non-exceptional polynomial, there is a polynomial p, such that the set of all polynomials with Julia set J is

$$\{\sigma \circ p^n | n \geq 1 \text{ and } \sigma \in \Sigma_J\}, \tag{1.2.1}$$

where Σ_J is the set of complex affine maps on $\mathbb{C}$ preserving J. By definition, a rational function is **exceptional** if it is conformally conjugate to either a power map, ±Chebyshev polynomial, or a Lattès map. From (1.2.1), if f and g are two non-exceptional polynomials with $\mu_f = \mu_g$, then there exists $\sigma(z) = az + b$ preserving μ_f with

$$f^n = \sigma \circ g^m \text{ for some } m, n \geq 2. \tag{1.2.2}$$

However, unlike the polynomial case, there exist non-exceptional rational functions with the same maximal measure but not related by the formula (1.2.2).

Theorem 1.2.1. *There exist non-exceptional rational functions f and g with degrees ≥ 2 and $\mu_f = \mu_g$, but*

$$f^n \neq \sigma \circ g^m \text{ for any } \sigma \in PSL_2(\mathbb{C}) \text{ and } n, m \geq 1. \tag{1.2.3}$$

Specifically, for R, S, T being rational functions with degrees ≥ 2 such that

- *For any $\sigma \in PSL_2(\mathbb{C})$, we have $R \neq \sigma \circ S$.*
- *$T \circ R = T \circ S$.*

we set $f = R \circ T$ and $g = S \circ T$, then $\mu_f = \mu_g$ and they satisfy (1.2.3).

The existence of the triples (R, S, T) in Theorem 1.2.1 is equivalent to the existence of an irreducible component of

$$V_T = \{(x, y) : T(x) = T(y)\} \subset \mathbb{P}^1 \times \mathbb{P}^1$$

with bidegree (r, r), $r \geq 2$, whose normalization is of genus 0. Explicit examples for such triples (R, S, T) are provided later in Chapter 3.

Let Rat_d be the set of all rational functions with degree $d \geq 2$. The space Rat_d sits inside $\mathbb{P}^{2d-1}(\mathbb{C})$, and it is the complement of the zero locus of an irreducible homogenous polynomial (the resultant) on $\mathbb{P}^{2d-1}$; therefore Rat_d is an affine variety. For any rational function $f \in \mathrm{Rat}_d$, denote by M_f the set of all rational functions with the same maximal entropy measure as f. As we discussed before, when f is non-exceptional and conjugate to some polynomial, M_f has very simple expression as in (1.2.1) by Corollary 3.2.2. However, from Theorem 1.2.1, we do not have the conclusion of (1.2.2) for all non-exceptional rational functions f and g with $\mu_f = \mu_g$, even we replace σ by any Möbius transformation. Levin [27], [28] showed $M_f \bigcap \mathrm{Rat}_n$ is a finite set unless f is conjugate to the power function $z^{\pm d}$.

For convenience, in the rest of this thesis, **generic** means with the exception of at most countably many proper Zariski closed subsets; **general** means with the exception of some proper Zariski closed subset.

We will show:

Theorem 1.2.2. *Let* Rat_d *be the set of all rational functions with degree* $d \geq 2$. *For generic rational functions* $f \in \mathrm{Rat}_d$, *we have*

- $M_f = \{f, f^2, f^3, \cdots\}$, *when* $d \geq 3$,
- $M_f = \{f, \sigma_f \circ f, f^2, \sigma_f \circ f^2, f^3, \sigma_f \circ f^3, \cdots\}$, *when* $d = 2$,

where σ_f *is the unique Möbius transformation permuting the fibers of* f.

The proof of Theorem 1.2.2 is mainly based on the next two theorems. The first, Theorem 1.2.3, asserts that for general rational functions with degree $d \geq 3$, having the same measure of maximal entropy is the same as sharing an iterate. We will say a critical value of $f \in \mathrm{Rat}_d$ is **simple** if its preimage contains exactly one critical point counted with multiplicity.

Theorem 1.2.3. *Let* f *be a rational function with degree* $d_f \geq 3$, *and* f *has at least three simple critical values. Then for any rational function* g *with degree* $d_g \geq 2$ *and* $\mu_f = \mu_g$, *we have*

$$f^n = g^m$$

for some integers $n, m \geq 1$.

Theorem 1.2.3 only works for rational functions with degree $d \geq 3$. This is because that for degree $d = 2$ case, there is a special nontrivial symmetry $\sigma_f \in PSL_2(\mathbb{C})$ for each $f \in \mathrm{Rat}_2$. The symmetry σ_f is the unique Möbius transformation permuting the fibers of f with $\sigma_f^2 = Id$. As σ_f permutes the points in the fiber of f, we have $f \circ \sigma_f = f$ and then σ_f preserves μ_f. Hence it has $\mu_{\sigma_f \circ f} = \mu_f$. For any $f \in \mathrm{Rat}_2$, let $g = \sigma_f \circ f$. It satisfies $\mu_g = \mu_f$ and $g^n = \sigma_f \circ f^n \neq f^n$ for any $n \geq 1$. In other words, f and g have the same maximal measure but they never share an iterate. In all, for any $f \in \mathrm{Rat}_d$, we have obvious relations: $\{f, f^2, f^3, \cdots\} \subset M_f$, and when $d = 2$, $\{f, \sigma_f \circ f, f^2, \sigma_f \circ f^2, f^3, \sigma_f \circ f^3, \cdots\} \subset M_f$. Theorem 1.2.2 asserts that, generically, there is no other rational function in M_f. However, it is still not known whether we can replace “generic" in Theorem 1.2.2 by “general", which will greatly improve the result; at least it is clear from (1.2.1) that the statements in Theorem 1.2.2 are satisfied for general polynomials.

Let $d \geq 2$ and $n \geq 1$ be integers. There is a regular map between affine varieties:

$$\varphi_{d,n} : \mathrm{Rat}_d \to \mathrm{Rat}_{d^n},$$

defined by $\varphi_{d,n}(f) = f^n$. We call it the iteration map of rational functions.

The next result, Theorem 1.2.4, states that the iteration map is one-to-one for general points.

Theorem 1.2.4. *Let $\varphi_{d,n} : \mathrm{Rat}_d \to \mathrm{Rat}_{d^n}$ be the iteration map with $d \geq 2$ and $n \geq 2$. There is a Zariski closed set $A \subset \mathrm{Rat}_d$, which is the preimage of the singularities of the variety $\varphi_{d,n}(\mathrm{Rat}_d)$, such that*

$$\varphi_{d,n} : \mathrm{Rat}_d \backslash A \to \mathrm{Rat}_{d^n}$$

is injective. Moreover, A is a proper nonempty subset of Rat_d.

Finally, we characterize the condition that two non-exceptional rational functions share an iterate. Let $\mathrm{PrePer}(f) = \{x \in \mathbb{P}^1 \mid f^n(x) = f^m(x), n > m \in \mathbb{N}\}$ be the set of preperiodic points of rational function f and $\mathrm{Per}(f) = \{x \in \mathbb{P}^1 \mid f^n(x) = x, n \in \mathbb{N}^*\}$ be the set of periodic points of f.

Theorem 1.2.5. *Let f and g be non-exceptional rational functions with degrees ≥ 2. The following statements are equivalent:*

- *f and g share an iterate, i.e. $f^n = g^m$ for some $n, m \in \mathbb{N}^*$.*
- *There is some φ with degree ≥ 2, such that $f \circ \varphi = \varphi \circ f$ and $g \circ \varphi = \varphi \circ g$.*
- *$\mu_f = \mu_g$, and $J \cap \mathrm{Per}(f) \cap \mathrm{Per}(g) \neq \emptyset$.*
- $\mathrm{PrePer}(f) = \mathrm{PrePer}(g)$ *and* $J \cap \mathrm{Per}(f) \cap \mathrm{Per}(g) \neq \emptyset$.
- $\mathrm{Per}(f) = \mathrm{Per}(g)$.

The proof of Theorem 1.2.5 uses the following results: for non-exceptional rational functions, Levin-Przytycki [29] showed that two rational functions having the same maximal measure should have the same set of preperiodic points. And conversely, Yuan and Zhang [40] showed, via arithmetic methods, that rational functions having the same set of preperiodic points should have the same maximal measure.

A bit more historical background and related results. As a general question, what can we conclude from two rational functions with the same maximal measure? For any non-exceptional polynomial f, it is easy to read the symmetry group Σ_{J_f} from the expression of f. After changing coordinates, we can assume that f is a monic and centered polynomial ($f(z) = z^d + az^{d-2} + \cdots$). So we can write $f(z) = z^l g(z^n)$ with $g(0) \neq 0$ and maximal possible n. Then, whenever f is non-exceptional, we have $\Sigma_{J_f} = \{\sigma(z) = \zeta z | \zeta^n = 1\}$. From (1.2.1), the expression of M_f is clear for non-exceptional polynomials f.

For any rational function f, let $g \in M_f$ and $\sigma \in \Sigma_{\mu_f}$, it is clear that $\sigma \circ g$ and $g \circ \sigma$ are both in M_f. So from Levin's result that $M_f \bigcap \mathrm{Rat}_n$ is a finite set, Σ_{μ_f} has finite elements whenever f is not conjugate to $z^{\pm d}$. However, for rational function f, it is still not known how to get the symmetry group Σ_{μ_f} or Σ_{J_f} (the subgroup of $PSL_2(\mathbb{C})$ preserving J_f) from the expression of f; see Levin's paper [27] [28], and some other related results in [13] and [39]. And for rational functions, in 1997, Levin and Przytycki's paper [29] has the following result:

Theorem 1.2.6 (Levin-Przytycki [29]). *Let f and g be two non-exceptional rational functions. The following two are equivalent:*

- $\mu_f = \mu_g$;
- *There exist iterates F of f and G of g, integers $M, N \geq 1$, and locally defined branches of $G^{-1} \circ G$ and $F^{-1} \circ F$ such that*

$$(G^{-1} \circ G) \circ G^M = (F^{-1} \circ F) \circ F^N. \tag{1.2.4}$$

By analytic continuity, locally defined $G^{-1} \circ G$ and $F^{-1} \circ F$ can be extended to multi-valued functions, acting by permuting the fibers of G and F. Equation (1.2.4) implies that f and g have the same set of preperiodic points. Then as a consequence of Theorem 3 in Levin's paper [27] and Levin-Przytycki's theorem Theorem 1.2.6, we have the following theorem:

Theorem 1.2.7 (Levin-Przytycki)**.** *Let f and g be two non-exceptional rational functions with degrees* $\geq$ 2*. Then $\mu_f = \mu_g$ if and only if there are some iterates F and G of f and g such that*

$$F \circ F = F \circ G \text{ and } G \circ F = G \circ G. \tag{1.2.5}$$

Although the above theorem comes directly from Theorem 3 in [27] and Theorem 1.2.6, we will provide an easy proof later in Chapter 3 by just using Levin-Przytycki's theorem Theorem 1.2.6.

So far as we know, (1.2.5) is the strongest algebraic relation satisfied for all non-exceptional rational functions f and g with $\mu_f = \mu_g$.

In the writing of the paper [42], we learned of two related articles in preparation. Related to Theorem 1.2.4, Adam Epstein has shown that the iteration map $\varphi_{d,n}$ is an immersion for all d and $n \geq 2$; see Proposition 3.3.1. Related to Theorem 1.2.1, when T is assumed to be a polynomial, Avanzi, Zannier, Carney, Hortsch and Zieve has provided a complete list of such examples; See [2] and [10].

CHAPTER 2

THE SCHWARZIAN DERIVATIVE AND POLYNOMIAL ITERATION

2.1 Basic properties of Schwarzian derivatives

In this section, we give useful formulas for S_{f^n} and also basic definitions that we are going to use later in this chapter.

2.1.1 Basic formula for the Schwarzian derivative of f^n

In order to find the limit of $\frac{S_{f^n}}{d^{2n}}$ for a general polynomial with degree $d \geq 2$, we need to rewrite S_{f^n} in terms of S_f and then evaluate the limit. To do this, we use the formula for the Schwarzian derivative of the composition of two functions f and g. An easy calculation shows that (cocycle property)

$$S_{f\circ g}(z) = S_f(g(z))(g'(z))^2 + S_g(z) \tag{2.1.1}$$

From this relation we derive the following important formula:

$$\begin{aligned} S_{f^n}(z) &= S_f(f^{n-1}(z))((f^{n-1}(z))')^2 + S_{f^{n-1}}(z) \\ &= S_f(f^{n-1}(z))((f^{n-1}(z))')^2 + S_f(f^{n-2}(z))((f^{n-2}(z))')^2 + S_{f^{n-2}}(z) \\ &= \sum_{i=1}^{n-1} S_f \circ f^i(z)((f^i(z))')^2 + S_f(z) \end{aligned} \tag{2.1.2}$$

Proposition 2.1.1. *Let f be a polynomial with degree $d \geq 2$. For any point $z \in \mathbb{C} \backslash \mathrm{Precrit}(f)$ and any sequence $\{n_i\}_{i=1}^{\infty}$ of $\mathbb{N}$ with $n_i \to \infty$ as $i \to \infty$, the sequences $\left\{\frac{S_{f^{n_i}}(z)}{d^{2n_i}}\right\}$ and $\left\{\frac{S_{f^{n_i-1}}(f(z))}{d^{2(n_i-1)}}\right\}$ either both converge, diverge to infinity or diverge.*

Moreover, if both of them converge, then

$$d^2 \lim_{i\to\infty} \frac{S_{f^{n_i}}(z)}{d^{2n_i}} = (f'(z))^2 \lim_{i\to\infty} \frac{S_{f^{n_i-1}}(f(z))}{d^{2(n_i-1)}}.$$

Proof. From (2.1.1), we have:

$$\begin{aligned} \frac{S_{f^{n_i}}(z)}{d^{2n_i}} &= \frac{S_{f^{n_i-1}}(f(z))(f'(z))^2 + S_f(z)}{d^{2n_i}} \\ &= \frac{S_{f^{n_i-1}}(f(z))(f'(z))^2}{d^2 d^{2(n_i-1)}} + \frac{S_f(z)}{d^{2n_i}} \end{aligned}$$

By the assumption that z is not a critical point of f and $d \geq 2$, it is easy to see that this proposition is satisfied since $f'(z)$ is not equal to zero and $S_f(z)$ is finite. □

2.1.2 The Schwarzian derivative as a quadratic differential

In this subsection, we are not only considering polynomials, but also rational maps with degree $d \geq 2$.

Meromorphic quadratic differentials. For any Riemann surface S, a meromorphic quadratic differential Q on S is a section of the second tensor power of the cotangent bundle. In local coordinate, $Q = Q_1(z)dz^2$, where $Q_1(z)$ is a meromorphic function. And under changing of coordinate $w = w(z)$,

$$Q = Q_2(w)(dw)^2 = Q_2(w(z))(w'(z))^2 dz^2$$

i.e., $Q_2(w(z))(w'(z))^2 = Q_1(z)$.

Consider a non constant holomorphic map $f : \mathbb{P}^1 \to \mathbb{P}^1$. This is a rational map with finite degree. Let's look at the Schwarzian derivative of this rational map f, and view it as quadratic differential, i.e.

$$S_f dz^2 \text{ instead of } S_f(z)$$

From the definition of the Schwarzian derivative, it is not hard to show $S_g \equiv 0$ if and only if g is a Möbius transformation; see [14]. From the following identity

$$S_{f\circ g}(z)dz^2 = S_f(g(z))(dg)^2 + S_g(z)dz^2,$$

for any two Möbius transformations $g_\circ, g_1$, we have

$$S_{g_1\circ f\circ g_\circ} dz^2 = S_f \circ g_\circ (dg_\circ)^2$$

So the Schwarzian derivative $S_f dz^2$ as a quadratic differential is well defined on $\mathbb{P}^1$. More generally, for any non constant holomorphic map from some projective Riemann surface to another projective

Riemann surface, there is an unique quadratic differential (named as Schwarzian derivative) associated to it; see [14].

Recall that in the last part of the introduction, we defined an equivalence relation of polynomials: $f \sim g$ if the Schwarzian derivative of f is the same as the Schwarzian derivative of g up to some affine transformation. The Schwarzian derivative of a polynomial is determined by the locations and multiplicities of the critical points:

Lemma 2.1.2. *Let f and g be polynomials with degree $d \geq 2$. Then $f \sim g$ if and only if they have the same critical set (critical points are counted with multiplicity) up to some affine transformation ($A(\mathrm{Crit}(f)) = \mathrm{Crit}(g)$ with A some affine transformation).*

Proof. Assume $f \sim g$, then $S_f dz^2 = A^*(S_g dz^2)$, which means $f = B \circ g \circ A$ for some affine transformations A and B. Indeed, by the cocycle property, $A^*(S_g dz^2) = S_{g\circ A} dz^2$ and then $S_{f\circ(g\circ A)^{-1}} \equiv 0$ on some open subset of $\mathbb{C}$. So $B = f \circ (g \circ A)^{-1}$ is a Möbius transformation on some open subset of $\mathbb{C}$. By continuity, $f = B \circ g \circ A$ in $\mathbb{C}$, and so B is an affine transformation. This implies that A transforms the critical set of f to the critical set of g.

Conversely, assume that there is an affine transformation A that transforms the critical set of f to the critical set of g. Since $g \sim g \circ A$, it suffices to show that $f \sim g \circ A$. Because f and $g \circ A$ have the same critical set, so we can let the critical set be $\{c_i\}_{i=1}^{d-1}$. Then $f = ah(z) + b$ and $g \circ A = ch(z) + d$ with $a, c \neq 0$ and $h(z) = \int_0^z \prod_{i=1}^{d-1}(t - c_i)dt$. Which means $S_f = S_h = S_{g\circ A}$, i.e. $f \sim g \circ A \sim g$. □

Proof of Theorem 1.1.3. Let $\simeq$ be the notion of strong equivalence. Assume that the polynomial f with degree $d \geq 2$ is not conjugate to z^d, and there is a subsequence $\{n_i\}_{i=1}^{\infty} \subset \mathbb{N}^*$ such that $f^{n_i} \sim g^{n_i}$.

By Lemma 2.1.2, there are affine transformations $\{A_i = a_i z + b_i\}$ such that $A_i(\mathrm{Crit}(f^{n_i})) = \mathrm{Crit}(g^{n_i})$. Since f is not conjugate to z^d, so g is not conjugate to z^d. Indeed, for $n \geq 2$, $\mathrm{Crit}(f^n)$ has at least two distinct points, however, $\mathrm{Crit}(z^{d^n})$ has only one point. Set $M_1 = \mathrm{Diam}(\mathrm{Crit}(f^2)) > 0$, $M_1' = \mathrm{Diam}(\mathrm{Crit}(g^2)) > 0$, $M_2 = \mathrm{Diam}(\mathrm{Precrit}(f))$ and $M_2' = \mathrm{Diam}(\mathrm{Precrit}(g))$. Because $f(\mathbb{C}\backslash D(0,R)) \subset \mathbb{C}\backslash D(0,R)$ for R sufficiently large, so $\mathrm{Precrit}(f)$ is bounded and then $M_2 < \infty$. Similarly, $M_2' < \infty$. Moreover, since $f^{n+m} = f^n \circ f^m$, then $\mathrm{Crit}(f^m) \subset \mathrm{Crit}(f^{n+m})$. So for the diameters $\mathrm{Diam}(\mathrm{Crit}(f^{n_i}))$ and $\mathrm{Diam}(\mathrm{Crit}(g_i^n))$ of the critical sets, we have

$$0 < M_1 \leq Diam(\mathrm{Crit}(f^{n_i})) \leq M_2 < \infty,\ 0 < M_1' \leq Diam(\mathrm{Crit}(g^{n_i})) \leq M_2' < \infty,$$

for any $n_i \geq 2$. Consequently,

$$0 < M_3 \leq |a_i| \leq M_4 \leq \infty, |b_i| \leq M_5 < \infty$$

So after passing to a subsequence and without loss of generality, we can assume $A_i \to A$ as $i \to \infty$, where A is an affine transformation. Then it is easy to know $A(J(f)) = J(g)$. Indeed, for any $c \in \mathrm{Crit}(f^j) \subset \mathrm{Crit}(f^i)$, $A_i(c) \in \mathrm{Crit}(g^i) \subset \mathrm{Precrit}(g)$ with $j \leq i$. It indicates that $A(c) \in \overline{\mathrm{Precrit}(g)}$ and then $A(\overline{\mathrm{Precrit}(f)}) \subset \overline{\mathrm{Precrit}(g)}$. Similarly, by taking A_i^{-1} instead of A_i, we get $A^{-1}(\overline{\mathrm{Precrit}(g)}) \subset \overline{\mathrm{Precrit}(f)}$. Because f is not conjugate to z^d, the set of accumulating points of $\overline{\mathrm{Precrit}(f)}$ (respt. $\overline{\mathrm{Precrit}(g)}$) is $J(f)$ (respt. $J(g)$), which means that $A(J(f)) = J(g)$.

If f is conjugate to z^d, then by the above argument, g should also be conjugate to z^d. So there is an affine map A such that $A(J(f)) = J(g)$.

Conversely, assume f and g have the same degree $d \geq 2$ and $A(J(f)) = J(g)$ for some affine transformation A. Since $g_1 = A^{-1} \circ g \circ A \simeq g$ and $A(J(g_1)) = J(g) = A(J(f))$, so it is enough to prove that $g_1 \simeq f$ with the condition that they have the same Julia set. First, if the Julia set is a circle, then both of them are conjugate to z^d. Indeed, we can assume $J(f)$ is the closed unit disk. Let φ be the Boettcher function of f, such that $\varphi \circ f \circ \varphi^{-1} = z^d$ on the basin of infinity. Since the basin of infinity is the complement of the unit disk and φ is a conformal map that fixes the infinity, so φ should be a rotation. Then f is conjugate to z^d. So $f^n \sim z^{d^n} \sim g_1^n$ for any $n \in \mathbb{N}^*$. Second, since both f^n and g_1^n have the same degree and the same Julia set which is not a circle, then $f^n = \sigma_n \circ g_1^n$ with σ_n an affine transformation in the symmetry group of the Julia set; see [6] for details. So $S_{f^n} = S_{g_1^n}$ for any $n \in \mathbb{N}^*$. And moreover, when Julia set is not a circle, the order of symmetry group of the Julia set is finite; see Lemma 4 in [7]. Thus there are only finitely many conjugacy classes which belongs to a strong equivalence class. $\square$

Notes: the proof of Theorem 1.1.3 relies on the classification of polynomials with the same Julia set. When do two polynomials have the same Julia set? Historically, commuting polynomials have the same Julia set, as observed by by Julia in 1922 [24]. Later in 1987, Baker and Eremenko showed when the symmetric group of the Julia set is trivial, polynomials with this Julia set commute; see [4]. In 1989, Fernández showed there is at most one polynomial with given degree, leading coefficient and Julia set; see [20]. Finally in 1992, Beardon showed $\{g | \deg(f) = \deg(g), J(f) = J(g)\} = \{\sigma \circ f | \sigma \in$ symmetric group of $J(f)\}$; see [6].

2.1.3 Conformal metric of quadratic differential.

For meromorphic quadratic differential Q on S, it determines a flat metric $ds^2 = |Q|$, with singularities at zeros and poles of Q.

Trajectories as a foliation. For any meromorphic quadratic differential Q on S, it determines a foliation structure on S with singularities at zeros and poles of Q. A smooth curve on S is a (horizontal) trajectory of Q, if it does not pass though any zero or pole of Q, and for any point p in the curve, the non zero vector dz tangent to this curve satisfy:

$$\arg(Q(p)dz^2) = 0$$

i.e $Q(p)dz^2$ is a positive real number. By a trajectory, we usually mean the trajectory that is not properly contained in another trajectory, i.e. a maximal trajectory.

Lemma 2.1.3. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map with degree $d \geq 2$. Then $S_f dz^2$ is a meromorphic quadratic differential with poles of order two at the critical points of f. For any critical point p of f with order k, near p*

$$S_f(z) = \frac{1-k^2}{2(z-p)^2} + O\left(\frac{1}{|z-p|}\right)$$

i.e., a neighborhood of p is an infinite cylinder with closed geodesics as trajectories of length $2\pi\sqrt{\frac{k^2-1}{2}}$.

Proof. The only thing we need to show here is that the coefficient of $\frac{1}{(z-p)^2}$ at p is $\frac{1-k^2}{2}$, and for other details of this lemma, please refer to the §6.3 [38]. The coefficient can be verified by a direct computation. □

2.1.4 $L^{\frac{1}{2}}_{loc}$ integrability of quadratic differential.

For any Riemann surface S, we consider the space $MQ(S)$ of meromorphic quadratic differentials on S . For any $\alpha \in MQ(S)$, we say that it is $L^{\frac{1}{2}}_{loc}$ integrable, if for any point $q \in S$, there is a local coordinate at q, and write α as $h(z)dz^2$ in this coordinate, such that the integration $\int\int \sqrt{|h(z)|}dxdy$ over some neighborhood of p is finite. We say that $\{\alpha_n\} \subset MQ(S)$ $L^{\frac{1}{2}}_{loc}$-converge to $\alpha \in MQ(S)$, if both α and α_n are $L^{\frac{1}{2}}_{loc}$ integrable, and for any point p, there is some local coordinate at q, and write α and α_n as $h(z)dz^2$ and $h_n(z)dz^2$ in this local coordinate, such that the integral $\int\int \sqrt{|h_n(z)-h(z)|}dxdy$ over some neighborhood of p converges to 0. Actually, the $L^{\frac{1}{2}}_{loc}$ integrable subset of $MQ(S)$ is a vector space.

Any meromorphic quadratic differential $\alpha \in MQ(S)$ with poles of order at most two is $L^{\frac{1}{2}}_{loc}$ integrable.

Lemma 2.1.4. *For any rational function f with degree $d \geq 2$, $S_{f^n}dz^2$ is $L^{\frac{1}{2}}_{loc}$ integrable.*

Proof. Since for any rational map f, the Schwarzian derivative $S_f dz^2$ has poles of order at most two, then $S_f dz^2$ is $L^{\frac{1}{2}}_{loc}$ integrable, and also $S_{f^n}dz^2$ is $L^{\frac{1}{2}}_{loc}$ integrable.

2.2 Local Convergence of S_{f^n}

In this section, our main goal is to prove Theorem 1.1.1, the local convergence of the normalized S_{f^n}.

2.2.1 Bounded Fatou components

In this subsection, we are trying to show that $\lim_{n\to\infty} \frac{S_{f^n}}{d^{2n}} = 0$ on the bounded Fatou components for any degree $d \geq 2$ polynomial f, which is not conformally conjugate to z^d.

Theorem 2.2.1. *For any* $z \notin \mathrm{Precrit}(f)$ *in a bounded Fatou component of a polynomial* f *with degree* $d \geq 2$ *, which is not conformally conjugate to* z^d*, then we have:*

$$\lim_{n\to\infty} \frac{S_{f^n}(z)}{d^{2n}} = 0.$$

Moreover, this is a local uniform convergence.

Proof. First, assume that z is attracted to some fix point z_1 (attracting or parabolic fix point), and z_1 is not a critical point. Then,

$$0 < \lambda = |f'(z_1)| \leq 1$$

For any fixed $0 < \epsilon < 1$, since we have $f^n(z)$ converges to z_1, there exists $N_o \in \mathbb{N}$ and $M < \infty$, such that for any $n > N_o$, we have:

$$|f'(f^n(z))| \leq 1 + \epsilon \text{ and } |S_f(f^n(z))| < M$$

By (2.1.2),

$$\begin{aligned} |S_{f^n}(z)| &= |\sum_{i=1}^{n-1} S_f \circ f^i(z) \cdot ((f^i)'(z))^2 + S_f(z)| \\ &\leq \sum_{i=1}^{n-1} |S_f \circ f^i(z) \cdot ((f^i)'(z))^2| + |S_f(z)| \end{aligned}$$

Since we have

$$|S_f(f^n(z))((f^n)'(z))^2| < M \cdot M_1 \cdot (1+\epsilon)^{2n},\ for\ any\ n > N_o.$$

where $M_1 = |(f^{N_o})'(z)|$, by the fact that $1+\epsilon < d$ for $d \geq 2$ and $\epsilon < 1$, it is obvious that $\lim_{n\to\infty} \frac{S_{f^n}(z)}{d^{2n}} = 0$ is satisfied.

Second assume z is attracted to a critical fix point z_1. Without loss of generality, we can assume $z_1 = 0$, so $f = az^r + bz^{r+1} + \cdots$, with $a \neq 0$ and $2 \leq r \leq d-1$. By Prop. 2.1.1, we can study the Schwarzian limit at forward iterate of z instead of the Swhwarzian limit at z. Then we can assume that z is close to 0. Let's conjugate f to z^r near 0 by a conformal map φ, such that $\varphi(0) = 0$ and $\varphi'(0) \neq 0$,

$$\varphi \circ f \circ \varphi^{-1} = z^r$$

By the cocycle property of Schwarzian and Example 1,

$$\begin{aligned} S_{f^n}(z) &= S_{\varphi^{-1}\circ z^{r^n}\circ\varphi}(z) \\ &= S_{\varphi^{-1}}\left((\varphi(z))^{r^n}\right)\cdot\left((\varphi^{-1})'(\varphi(z)^{r^n})\cdot(r^n-1)\varphi(z)^{r^n-1}\varphi'(z)\right)^2 \\ &\quad +\frac{1-r^{2n}}{2\varphi(z)^2}\cdot(\varphi'(z))^2 + S_\varphi(z) \end{aligned}$$

Since z is close to 0 and $2 \leq r \leq d-1$, then $\lim_{n\to\infty} \frac{S_{f^n}(z)}{d^{2n}} = 0$ is obviously true in this case by the above formula.

Third, assume that z is in some Siegel disc $\mathbb{D}_o$ fixed by f. Similar with the previous case, we can move the center of $\mathbb{D}_o$ to 0, and conjugate f by φ on this Siegel disc to a rotation map, i.e.

$$\varphi \circ f = \lambda \cdot \varphi, \text{ with } |\lambda| = 1$$

where we have $\varphi(0)=0$ and $\varphi'(0)=1$. And by (2.1.1),

$$\begin{aligned} S_{f^n}(z) &= S_{\varphi^{-1}(\lambda^n\varphi)}(z) \\ &= S_{\varphi^{-1}}(\lambda^n\varphi(z))\cdot((\varphi^{-1})'(\lambda^n\varphi(z)))^2+S_{\varphi}(z) \end{aligned}$$

Since $\lambda^n\varphi(z)$ is in φ's image of some compact subset of $\mathbb{D}_o$ for any n, then $S_{\varphi^{-1}}(\lambda^n\varphi(z))$ is uniformly bounded. So $\lim_{n\to\infty}\frac{S_{f^n}(z)}{d^{2n}}=0$ is obviously satisfied in this case by the above formula.

From above arguments, it is not hard to see the convergence is local uniform. For points in the periodic bounded Fatou components and not in Precrit(f), we can use similar arguments to show that the result of this theorem is satisfied. And for points attracted to periodic bounded Fatou components, Prop. 2.1.1 shows that the result is satisfied too. □

2.2.2 Basin of infinity.

In this subsection, we are going to prove the local convergence of S_{f^n} on the basin of infinity.

Consider the following escape-rate function of a polynomial f with degree $d\geq 2$:

$$G_f(z)=\lim_{n\to\infty}\frac{1}{d^n}\log^+|f^n(z)|,$$

where $\log^+|x|=\max(\log|x|,0)$. The escape-rate function $G_f(z)$ is the Green function of the basin of infinity. So it is harmonic on the basin of infinity. Actually, it is a subharmonic function on $\mathbb{C}$. By taking the partial derivative of $G_f(z)$, we get $g(z)=\frac{\partial G_f(z)}{\partial z}$ is a holomorphic function on the basin of infinity; the zeros of $g(z)$ are exactly the points in Precrit(f). Moreover, from the definition of $G_f(z)$, it is a limit of harmonic functions converging locally uniformly. The derivatives of these harmonic functions converge. Then we know that the partial derivative commutes with the limit, i.e.

$$\begin{aligned} g(z) &= \frac{\partial G_f(z)}{\partial z} = \lim_{n\to\infty} \frac{\partial \frac{1}{d^n} \log|f^n(z)|}{\partial z} \\ &= \lim_{n\to\infty} \frac{1}{2d^n} \frac{\partial \log(f^n(z)\bar{f}^n(z))}{\partial z} = \lim_{n\to\infty} \frac{1}{2} \frac{(f^n(z))'}{d^n f^n(z)} \end{aligned} \tag{2.2.1}$$

Theorem 2.2.2. *For any $z_o \notin \mathrm{Precrit}(f)$ in the basin of infinity of a polynomial f with degree $d \geq 2$, we have:*

$$\lim_{n\to\infty} \frac{S_{f^n}(z_o)}{d^{2n}} = -2(g(z_o))^2 = -2\left(\frac{\partial G_f(z_o)}{\partial z}\right)^2$$

Moreover, this is a local uniform convergence.

Proof. Since $g(z) = 0$ if and only if $z \in \mathrm{Precrit}(f)$, then we have $g(z_o) \neq 0$, because $z_o \notin \mathrm{Precrit}(f)$.

Let $r_n = \frac{(f^n(z_o))'}{2g(z_o)d^n f^n(z_o)}$, by (2.2.1), we get

$$\lim_{n\to\infty} r_n = 1$$

Moreover, let $f(z) = a_d z^d + a_{d-1}z^{d-1} + \cdots + a_0$ with $a_d \neq 0$, and an easy calculation shows that

$$S_f(z) = \frac{1-d^2}{2z^2} h(z), \ and \ \lim_{z\to\infty} h(z) = 1,$$

where $h(z)$ is a rational function. Let $s_n = h(f^n(z_o))$, since $\lim_{n\to\infty} f^n(z_o) = \infty$, so we get $\lim_{n\to\infty} s_n = 1$. Then

$$\begin{aligned} S_f(f^n(z_o))((f^n)'(z_o))^2 &= \tfrac{(1-d^2)s_n((f^n)'(z_o))^2}{2(f^n(z_o))^2} \\ &= 2(1-d^2)s_n r_n^2 (g(z_o))^2 d^{2n}, \end{aligned}$$

Substituting the above formula into (2.1.2), we get

$$\begin{aligned} \tfrac{S_{f^n}(z_o)}{d^{2n}} &= \tfrac{\sum_{i=1}^{n-1} S_f(f^i(z_o))((f^i)'(z_o))^2 + S_f(z_o)}{d^{2n}} \\ &= 2(g(z_o))^2(1-d^2)\tfrac{\sum_{i=0}^{n-1} s_i r_i^2 d^{2i}}{d^{2n}} \end{aligned}$$

Because both r_n and s_n converge to 1 as n tends to ∞, it follows that

$$\begin{aligned} \lim_{n\to\infty} \tfrac{S_{f^n}(z_o)}{d^{2n}} &= 2(g(z_o))^2(1-d^2)\lim_{n\to\infty}\tfrac{1-d^{2n}}{(1-d^2)d^{2n}} \\ &= -2(g(z_o))^2 = -2\left(\tfrac{\partial G_f(z_o)}{\partial z}\right)^2 \end{aligned}$$

The fact that this convergence is local uniform can be deduced from the fact that both $r_n(z)$ and $s_n(z)$ converge locally uniformly. □

Remark. Alternately, the result of Theorem 2.2.2 can be seen in the language of “Schwarzian between conformal metrics". On the complex plane $\mathbb{C}$, we define:

$$\widehat{S}(e^{\sigma_1}|dz|^2, e^{\sigma_2}|dz|^2) = \left(\sigma_{1zz} - \frac{1}{2}\sigma_{1z}^2 - (\sigma_{2zz} - \frac{1}{2}\sigma_{2z}^2)\right) dz^2,$$

where $\sigma_z = \frac{\partial\sigma}{\partial z}$. Easy to know, we have

- $S_f dz^2 = \widehat{S}(f^*|dz|^2, |dz|^2)$, for any non constant holomorphic map f.

- $\widehat{S}(c_1\rho_1|dz|^2, c_2\rho_2|dz|^2) = \widehat{S}(\rho_1|dz|^2, \rho_2|dz|^2)$, for any positive constant numbers c_1 and c_2.

- $\widehat{S}(\rho_1|dz|^2, \rho_3|dz|^2) = \widehat{S}(\rho_1|dz|^2, \rho_2|dz|^2) + \widehat{S}(\rho_2|dz|^2, \rho_3|dz|^2)$.

Then we have

$$
\begin{tikzcd}
\frac{1}{d^{2n}}\left(\widehat{S}(f^{n*}|dz|^2,|dz|^2)+\widehat{S}(|dz|^2,f^{n*}|\frac{dz}{z}|^2)\right) \arrow[dd, equal] & = & \frac{1}{d^{2n}}S_{f^n}dz^2 \arrow[ddrr, "n\to\infty"'] & + & \frac{1}{d^{2n}}\widehat{S}(|dz|^2,\frac{1}{d^{2n}}f^{n*}|\frac{dz}{z}|^2) \arrow[d, "n\to\infty"] \\
 & & & & 0 \\
\frac{1}{d^{2n}}\widehat{S}(f^{n*}|dz|^2,f^{n*}|\frac{dz}{z}|^2) & = & \frac{1}{d^{2n}}f^{n*}\widehat{S}(|dz|^2,|\frac{dz}{z}|^2) = \frac{1}{d^{2n}}f^{n*}(\frac{-dz^2}{2z^2}) & \longrightarrow & -2\left(\frac{\partial G_f(z)}{\partial z}\right)^2 dz^2
\end{tikzcd}
$$

For more properties about the Schwarzian of conformally equivalent Riemannian metrics, please refer to [35].

Nonlinearity. Similar with the Scharzian derivative, we can define nonlinearity N_f of a nonconstant holomorphic function f on the complex plane.

$$N_f = \frac{f''}{f'}$$

Nonlinearity $N_f \equiv 0$ if and only if f is an affine transformation. Sometimes we can view N_f as a one form $N_f dz$. We have the following cocycle property:

$$N_{f\circ g} dz = g^*(N_f dz) + N_g dz$$

Moreover, $N_f dz$ has a pole of order one at critical point of f. Using the same argument in Theorem 1.1.1, we have

Theorem 2.2.3. *Let f be an polynomial with degree $d \geq 2$ and not conformally conjugate to z^d, and X_o be its basin of infinity. Then we have:*

- $\lim_{n\to\infty} \frac{(f^n)' dz}{d^n f^n} = \partial G_f$, *on* X_o.
- $\lim_{n\to\infty} \frac{N_{f^n} dz}{d^n} = \partial G_f$, *on* $\mathbb{C} \setminus \overline{\mathrm{Precrit}(f)}$.
- $\lim_{n\to\infty} \frac{(f^n)''' dz^2}{d^{2n}(f^n)'} = -\frac{(\partial G_f)^2}{2}$, *on* $\mathbb{C} \setminus \overline{\mathrm{Precrit}(f)}$.

In each case, the convergence is locally uniform.

Proof of Theorem 1.1.1. Since we have $G_f \equiv 0$ on the bounded Fatou components of f, then this theorem is an easy consequence of Theorem 2.2.1 and Theorem 2.2.2. □

2.2.3 Global convergence on the Fatou set.

Recall the definition of $L^{\frac{1}{2}}_{loc}$ integrability and $L^{\frac{1}{2}}_{loc}$ convergence of meromorphic quadratic differentials on the Riemann surface in §2.1.2. For any rational function f with degree $d \geq 2$, by Lemma 2.1.4, $S_{f^n} dz^2$ is $L^{\frac{1}{2}}_{loc}$ integrable.

Corollary 2.2.4. *Let $S_{f^n} dz^2$ be the meromorphic quadratic differential on $\mathbb{P}^1$ determined by the Schwarzian derivative of f^n, where f is a polynomial with degree $d \geq 2$ and not conformally conjugate to z^d. Then*

$$\lim_{n\to\infty} \frac{S_{f^n}(z)}{d^{2n}} dz^2 = -2\left(\frac{\partial G_f(z)}{\partial z}\right)^2 dz^2,$$

on the Fatou set (including ∞) of f, converging in the sense of $L^{\frac{1}{2}}_{loc}$.

Proof. The proof of this corollary follows easily from the arguments in the proof of Theorem 1.1.1, together with the triangle inequality. So we omit the details here. □

2.3 Geometric Limit of Metric Spaces

In this section, we discuss the possible limit of metric spaces coming from Schwarzian derivatives.

2.3.1 Ultrafilter and ultralimit

A non-principal ultrafilter ω is a set consisting of a collection of subsets of $\mathbb{N}$, satisfying

- If $A \subset B \subset \mathbb{N}$ and $A \in \omega$, then $B \in \omega$.
- For any disjoint union $\mathbb{N} = A_1 \cup A_2 \cdots \cup A_n$, there exists one and only one $A_i \in \omega$.
- For any finite set $A \subset \mathbb{N}$, $A \not\in \omega$,

We can view a non-principal ultrafilter ω as a finitely additive measure on $\mathbb{N}$, only taking values in $\{0,1\}$, where any finite subset of $\mathbb{N}$ has measure zero and $\mathbb{N}$ has measure 1. By Zorn's lemma there exists some non-principal ultrafilter, and non-principal ultrafilter on $\mathbb{N}$ is not unique. Hereafter, we fix a non-principal ultrafilter ω. For more details about the ultrafilter, please refer to Chapter I §5 [8].

Let Y be a compact Hausdorff space. For any sequence of points $\{y_i\}_{i=1}^{\infty} \subset Y$, there is an unique point $y_o \in Y$ such that $\{i | y_i \in U\} \in \omega$ for any open set U containing y_o. This $y_o = \lim_{\omega} y_i$ is denoted as the ultralimit of $\{y_i\}_{i=1}^{\infty}$.

Let $\{(Y_n, d_n, b_n)\}_{n=1}^{\infty}$ be a sequence of pointed metric spaces. Let $\widetilde{Y}_{\omega}$ be the set of all the sequences (y_n) with $y_n \in Y_n$ satisfying:

$$\lim_{\omega} d_n(y_n, b_n) < \infty;$$

here, the ultralimit is taken in the space of $[0,+\infty]$. Set

$$\widetilde{d_\omega}((x_n),(y_n)) := \lim_\omega d_n(x_n,y_n) < \infty,$$

with (x_n) and $(y_n) \in \widetilde{Y_\omega}$. This is a pseudo-distance on $\widetilde{Y_\omega}$. Let $(Y_\omega, d_\omega, b_\omega) := (\widetilde{Y_\omega}, \widetilde{d_\omega}, (b_n))/\sim$, where we identify points with zero $\widetilde{d_\omega}$-distance. The point metric space $(Y_\omega, d_\omega, b_\omega)$ is called the ultralimit of $\{(Y_n, d_n, b_n)\}_{n=1}^\infty$; see Chapter I §5 [8].

2.3.2 The ultralimit of the Schwarzian metrics

Let f be a polynomial with degree $d \geq 2$. The Schwarzian derivative S_{f^n} determines a metric space (X_n, d_n), where $X_n = \mathbb{C} \setminus \mathrm{Crit}(f^n)$ and d_n is the arc length metric $ds = \sqrt{\left|\frac{S_{f^n}}{d^{2n}}\right|}|dz|$. This is a complete geodesic space with non positive curvature. Fix a point $a \notin \mathrm{Precrit}(f)$ on the basin of infinity. In this section we are considering the ultralimit $(X_\omega, d_\omega, a_\omega)$ of the pointed metric spaces (X_n, d_n, a).

Proposition 2.3.1. *$(X_\omega, d_\omega, a_\omega)$ is a complete geodesic space.*

Proof. The metric space $(X_\omega, d_\omega, a_\omega)$ is complete, since the ultralimit of any sequence of pointed metric spaces is complete; see §1 Lemma 5.53 [8]. Moreover, the geodesic property is also preserved by passing to the ultralimit. The pointed metric space $(X_\omega, d_\omega, a_\omega)$ is a geodesic space. Indeed, $\{(X_n, d_n, a)\}$ are pointed geodesic spaces. For any two points $x_\omega = (x_i)$ and $y_\omega = (y_i)$ in X_ω, there is (z_i) such that $d_i(x_i, z_i) = d_i(z_i, y_i) = \frac{1}{2} d_i(x_i, y_i)$. From this fact, we know that $z_\omega = (z_i) \in X_\omega$ and also

$$\lim_\omega \frac{1}{2} d_i(x_i, y_i) = \lim_\omega d_i(x_i, z_i) = \lim_\omega d_i(z_i, y_i),$$

i.e., $d_\omega(x_\omega, z_\omega) = d_\omega(z_\omega, y_\omega) = \frac{1}{2} d_\omega(x_\omega, y_\omega)$. So x_ω and y_ω have a midpoint in X_ω. Which means X_ω is a geodesic space, since it is complete. □

2.3.3 The flat metric on the basin of infinity

Let X_o be the basin of infinity of f and d_o be the arc length metric from $ds = \sqrt{2}\left|\frac{\partial G_f}{\partial z}\right||dz|$; see Figure Figure 1.1.1. The metric space (X_o, d_o) is not complete. If the filled Julia set $K(f)$ of f is disconnected, then we can complete X_o as follows

Lemma 2.3.2. *For polynomial f with degree $d \geq 2$ and $K(f)$ disconnected, the metric completion $(\overline{X}_o, d_o)$ of (X_o, d_o) is a quotient of $\mathbb{C}$, obtained by collapsing each connected component of $K(f)$ to a point. The completion $\overline{X}_o$ is homeomorphic to $\mathbb{R}^2$. Moreover, $(\overline{X}_o, d_o)$ is a geodesic space.*

Proof. Since $K(f)$ is disconnected, the first two conclusions of this lemma follow immediately from the flat structure of the metric d_o; compare [11]. A complete metric space is geodesic if there exists a midpoint for any two points on this metric space. It is obvious that $(\overline{X}_o, d_o)$ has this property. So it is a geodesic space. □

Let E be the set of $\overline{X}_o \setminus X_o$. The set E is totally disconnected. So we call E the *ends* of X_o. Each point $e \in E$ corresponds to a connected component of $K(f)$. Let $C = X_o \cap \mathrm{Precrit}(f)$. For any *end* in E, there is a sequence of annuli on $X_o \backslash C$ such that they nest down to this *end* with d_o-diameters tending to zero; see the flat metric structure of X_o in [11]. The following lemma follows easily from Theorem 1.1.1.

Lemma 2.3.3. *For any piece wise smooth and compact curve $\gamma \subset X_o \setminus C$, the d_n-length $d_n(\gamma)$ converge to the d_o-length $d_o(\gamma)$. Moreover, for any point $p \in X_o \setminus C$, there is a small neighborhood $U \subset X_o \setminus C$ of p, such that $\lim_{n\to\infty} d_n(x, y) = d_o(x, y)$, uniformly for $x, y \in U$.*

The following proposition is an essential ingredient in the proof of Theorem 1.1.1.

Proposition 2.3.4. *Let f be a polynomial with degree $d \geq 2$ and $J(f)$ disconnected. For any geodesic metric $\tilde{d}$ on $\overline{X}_o$ with $(X_o, \tilde{d})$ locally isometric to (X_o, d_o) under the identity map, we have $\tilde{d} = d_o$ on $\overline{X}_o$.*

To prove this lemma, we need the tree structure of (X_o, d_o); see [11]. Specifically, this is a quotient $\pi : X_o \longrightarrow T(f)$, defined by collapsing each connected component of the level set of $G_f(z)$ to a point. There is a canonical map F on $T(f)$ induced from f.

$$\begin{tikzcd} X_o \arrow[r, "f"] \arrow[d, "\pi"'] & X_o \arrow[d, "\pi"'] \\ T(f) \arrow[r, "F"] & T(f) \end{tikzcd}$$

The space $T(f)$ has a simplicial structure, and is equipped with a metric from G_f. The set of vertices is $V = \cup_{n,\ m\in\mathbb{Z}} F^m(F^n(\text{branch points}))$. The distance between two points in the same edge is given by the difference of their G_f values. Let $S = \pi^{-1}(V)$. Then $X_o \setminus S$ consists of countably many connected components. Each of them is an annulus. So we view $X_o \setminus S$ as a set of annuli. The map π is a one to one map from the annuli to the edges of $T(f)$. For any annulus $A \in X_o \setminus S$, it is a cylinder with finite height in the d_o-metric.

The height of A, denoted as $H(A)$, is equal to the length of the edge $\pi(A)$. Also, we define the level of A as $L(A)$ to be the G_f value of the middle point of $\pi(A)$. The closed geodesics of A are the level sets of G_f. They have the same arc length in the d_o-metric, denoted as $C(A)$.

The map f sends annulus in $X_o \setminus S$ to annulus. For each annulus $A \in X_o \setminus S$, there is a well defined local degree d_A, defined as the topological degree of $f|_A$. Moreover, let $N(f) = \max\{G_f(c)\,|c$ is a critical point of f}. We have the following properties:

- $$\sum_{B \in X_o \setminus S, L(B)=L(A)} C(B) = \sqrt{2}\pi \tag{2.3.1}$$

- $$L(f(A)) = d \cdot L(A),\ H(f(A)) = d \cdot H(A),\ C(f(A)) = \frac{d_A}{d} C(A) \tag{2.3.2}$$

- d_A is equal to one plus the number of critical points (counted with multiplicity) enclosed in A.

- The points enclosed in A should have G_f value less than $L(A)$. If $L(A) < N(f)$, then the annulus A can not enclose the critical point(s) with G_f value equals $N(f)$. And because f has $d-1$ critical points counted with multiplicity, then $d_A \le d-1$ when $L(A) < N(f)$. From (2.3.1) and (2.3.2), for any annulus A satisfying $d^n L(A) < N(f)$, we have $L(f^i(A)) = d^i L(A) < N(f)$, i.e. $d_{f^i(A)} \le d-1$ for any $1 \le i \le n$. Consequently,

$$C(A) \le (\frac{d-1}{d})^n \cdot C(f^n(A)) \le (\frac{d-1}{d})^n \cdot \sqrt{2}\pi \tag{2.3.3}$$

- Let $|f^{-1}(A)|$ be the number of annuli in the set $f^{-1}(A)$. We have $|f^{-1}(A)|$ equals to d minus the total number of critical points (counted with multiplicity) enclosed in the annuli in $f^{-1}(A)$.
- Let $A \in X_o \setminus S$. Any point $x \in \overline{X_o}$ with $G_f(x) \leq L(A)/d$ is enclosed in one of the annulus $B \in X_o \setminus S$ with $L(B) = L(A)$.
- Any two annuli $A, B \in X_o \setminus S$ with $L(A) = L(B)$ have the same height.

For more details about these annuli and the tree structure of the basin of infinity, please see [11].

Proof of Proposition 2.3.4. Since X_o is dense in $\overline{X}_o$ for both the metrics $\widetilde{d}$ and d_o, then it suffices to prove that $d_o(x, y) = \widetilde{d}(x, y)$ for any two points $x, y \in X_o$.

One direction: $d_o(x, y) \geq \widetilde{d}(x, y)$. From the definition of d_o, for any $\epsilon > 0$, there is an arc $\gamma \subset X_o$ connecting x and y with d_o length $d_o(\gamma) < d_o(x, y) + \epsilon$. Since these two metrics are locally isometric on X_o, for the length $\widetilde{d}(\gamma)$ of γ in the $\widetilde{d}$-metric, we have

$$\widetilde{d}(x, y) \leq \widetilde{d}(\gamma) = d_o(\gamma) < d_o(x, y) + \epsilon$$

Let $\epsilon \to 0$, then we get $\widetilde{d}(x, y) \leq d_o(x, y)$.

The other direction: $d_o(x, y) \leq \widetilde{d}(x, y)$. Choose a geodesic $\widetilde{g} : [0, r] \mapsto \overline{X}_o$ in the $\widetilde{d}$-metric, with $\widetilde{g}(0) = x$ and $\widetilde{g}(r) = y$. Our goal is to construct an arc $\gamma \subset X_o$ from $\widetilde{g}$, connecting x and y with d_o length $d_o(\gamma) < r + \epsilon$ for any fixed $\epsilon > 0$.

From (2.3.3), there is some small level $l > 0$ such that the set of annuli $T = \{A \in X_o \setminus S \mid L(A) = l\}$ is not empty and $C(A) < \frac{\epsilon}{3}$ for any $A \in T$. Since all the annuli with the same level have the same height, so $h = H(A)$ for $A \in T$ is well defined. We order the elements of T as $A_1, A_2, \cdots, A_k$, with $C(A_i) \leq C(A_j)$ for any $1 \leq i < j \leq k$. Let $T_s = f^{-s}(T)$ and decompose T_s into T_s^- and T_s^+, with T_s^+ the set of annuli crossed by $\widetilde{g}$. So the length of $\widetilde{g}$ in each annulus belonging to T_s^+ is at least the height of this annulus. Let $|\cdot|$ be the number of elements of a set. Then we have:

$$\sum_{s \in \mathbb{N}} |T_s^+| \cdot \frac{h}{d^s} \leq \widetilde{d}\text{-length of } \widetilde{g} = r < \infty$$

Form the above formula, there is some big n, such that $|T_n^+| \cdot \frac{h}{d^n} < h$, i.e., $|T_n^+| < d^n$. As $|f^{-1}(A)|$ is equal to d minus the total number of critical points (counted with multiplicity) enclosed in the annuli belonging to $f^{-1}(A)$, and f has $d-1$ critical points, then

$$|T_{s+1}| = |f^{-1}(T_s)| > d \cdot |T_s| - d,\ |f^{-n}(A)| \leq d^n$$

since any two annuli with the same level won't enclose each other, i.e., they won't share a common critical point inside. Consequently,

$$|T_n| \rangle\ d^n \cdot |T| - d^n - d^{n-1} - \cdots - d\ \rangle\ d^n \cdot k - 2d^n$$

Combining the above formula with $|T_n^+| < d^n$, we have $|T_n^-| = |T_n| - |T_n^+| > d^n \cdot (k-3)$. From (2.3.2), for any $1 \leq i \leq k$,

$$|f^{-n}(A_i)| \leq d^n,\ \frac{C(A_i)}{d^n} \leq C(B) \text{ any } B \in f^{-n}(A_i)$$

We order T_n as $B_1, B_2, \cdots, B_{|T_n|}$, such that $\beta_j = C(B_j) \le \beta_{j+1} = C(B_{j+1})$ for any $1 \le j \le |T_n| - 1$. From the above formula, we have

$$\beta_{d^n\cdot(i-1)+j} \ge \frac{C(A_i)}{d^n},\ 1 \le i \le k-3 \text{ and } 1 \le j \le d^n,$$

Then we get,

$$\begin{aligned}
\textstyle\sum_{B\in T_n^+} C(B) &= \textstyle\sum_{B\in T_n} C(B) - \sum_{B\in T_n^-} C(B) \\
&\le \sqrt{2}\pi - \textstyle\sum_{i=1}^{k-3}\sum_{j=1}^{d^n} \beta_{d^n\cdot(i-1)+j} \\
&\le \textstyle\sum_{i=1}^{k} C(A_i) - \sum_{i=1}^{k-3} d^n \cdot \frac{C(A_i)}{d^n} \\
&= C(A_{k-2}) + C(A_{k-1}) + C(A_k) \\
&< \epsilon.
\end{aligned}$$

Now, we can construct an arc $\gamma \subset X_o$ from $\widetilde{g}$ as follows (without loss of generality, we can always assume that both $G_f(x)$ and $G_f(y)$ are greater than l, and $n > 1$):

- Choose a minimal $r_1 \in [0, r]$ such that $\widetilde{g}(r_1)$ lying on the outer boundary of some annulus $B^1 \in T_n^+$.
- Choose a maximal $r_1' \in [0, r]$ such that $\widetilde{g}(r_1')$ lying on the outer boundary of the annulus B^1.
- Replace the arc $\widetilde{g}([r_1, r_1']) \subset \widetilde{g}$ with a shortest curve γ_1 on the outer boundary of B^1 connecting $\widetilde{g}(r_1)$ and $\widetilde{g}(r_1')$.
- Do the same thing as the previous three steps, we can find a minimal r_2 and maximal r_2' in $[r_1', r]$, such that $\widetilde{g}(r_2)$ and $\widetilde{g}(r_2')$ lying on the outer boundary of some $B^2 \in T_n^+$ and $r_2' - r_2 > 0$.

Replace $\widetilde{g}([r_2, r_2']) \subset \widetilde{g}$ with a shortest curve γ_2 on the outer boundary of B^2 connecting $\widetilde{g}(r_2)$ and $\widetilde{g}(r_2')$.

- Keep doing the same thing as previous step by step, we can replace sub-arcs of $\widetilde{g}$ by arcs γ_i on the boundary of $B^i \in T_n^+$. This process will stop under finite steps, since $B^i \neq B^j$ for any $i < j$ and $|T_n^+| < \infty$.

From the above construction, we get a new arc γ from $\widetilde{g}$. We have $\gamma \subset X_o$. In fact, for any point $p \in \gamma$, p is not enclosed by any annulus $B \in T_n$. So we have $G_f(p) > \frac{h}{d^{n+1}}$. And

$$\begin{aligned} d_o(x,y) &\leq d_o(\gamma) \leq r + d_o(\gamma_1) + d_o(\gamma_2) + \cdots \\ &\leq r + C(B^1) + C(B^2) + \cdots \\ &\leq r + \textstyle\sum_{B \in T_n^+} C(B) < r + \epsilon. \end{aligned}$$

Let $\epsilon \to 0$, we get $d_o(x,y) \leq r = \widetilde{d}(x,y)$. □

Proof of Theorem 1.1.2. First, we construct a natural map ρ

$$\rho : X_o \backslash C \mapsto X_\omega,$$

as: for any $p \in X_o \backslash C$, $\rho(p) = p_\omega = (p_n = p) \in X_\omega$. This map is well defined, since $\{d_n(p,a)\}$ is uniformly bounded by Lemma 2.3.3. Let $p,\ q$ be two points in $X_o \backslash C$, and $p_\omega,\ q_\omega$ the ρ image of $p,\ q$ in X_ω. We want to show that $d_\omega(p_\omega, q_\omega) \leq d_o(p,q)$. For any $\epsilon > 0$, we can choose a smooth curve on the basin of infinity with d_o-length less than $d_o(p,q) + \epsilon$. By a small perturbation, we can assume this curve does not pass any point in C, and this curve is also on the basin of infinity. Since this curve is compact, by Lemma 2.3.3, the d_n-length of this curve converges to the d_o-length of this curve. So for n big enough, the d_n-length of this curve is less than $d_o(p,q)+2\epsilon$. Then $d_\omega(p_\omega, q_\omega) \le d_o(p,q)+2\epsilon$. Let $\epsilon \to 0$, we get $d_\omega(p_\omega, q_\omega) \le d_o(p,q)$. Also, from Lemma 2.3.3, this is a locally isometric and distance non-increasing embedding. Then we can extend the map ρ from $X_o \backslash C$ to $\overline{X_o}$:

$$\rho : \overline{X_o} \longrightarrow X_\omega$$

For any *end* $e \in E$, let K_e be the corresponding connected component of $K(f)$ and $X_\omega^e = \{(x_i) \in X_\omega | \lim_\omega x_i \in K_e\}$. And for any $c \in C$, let $X_\omega^c = \{(x_i) \in X_\omega | \lim_\omega x_i = c\}$. Obviously, the set

$$X_\omega = \left(\cup_{\alpha \in C \cup E} X_\omega^\alpha\right) \cup \rho(X_o \backslash C),$$

is a disjoint union, i.e.

$$X_\omega^{\alpha_1} \cap X_\omega^{\alpha_2} = \emptyset \text{ and } X_\omega^{\alpha_1} \cap \rho(X_o \backslash C) = \emptyset, \text{ for any } \alpha_1 \neq \alpha_2 \in C \cup E$$

Indeed, there is a closed annulus A in $X_o \backslash C$ with the corresponding parts of α_1 and α_2 in the two different components of $\mathbb{C} \setminus A$. Let h be the distance between the two boundaries of A in the d_o-metric. we have $h > 0$. Since any arc connecting two points in distinct components of $\mathbb{C} \setminus A$ should across A. By Lemma 2.3.3, the d_n-distance of the two boundaries of A converge to h. So we have

$d_\omega(X_\omega^{\alpha_1}, X_\omega^{\alpha_2}) \geq h > 0$. Consequently, $X_\omega^{\alpha_1} \cap X_\omega^{\alpha_2} = \emptyset$. Similarly, any point $x_\omega \in \rho(X_o \backslash C)$, we have $d_\omega(X_\omega^{\alpha_1}, x_\omega) > 0$, then $X_\omega^{\alpha_1} \cap \rho(X_o \backslash C) = \emptyset$.

From above, for any two distinct points α, β in $C \cup E$ and $p_\omega \in \rho(X_o \backslash C)$, we have $d_\omega(\rho(\alpha), X_\omega^\beta) > 0$ and $d_\omega(\rho(\alpha), p_\omega) > 0$. Moveover, since $\overline{X}_o$ is homeomorphic to $\mathbb{R}^2$ by Lemma 2.3.2, so $\rho : \overline{X}_o \mapsto \rho(\overline{X}_o)$ is a distance non-increasing homeomorphism. We want to show $(\rho(\overline{X}_o), d_\omega)$ is a geodesic space and ρ is locally isometric at the points in C. Then by Proposition 2.3.4, we may conclude that ρ is an isometric embedding.

For any $x_\omega \in X_\omega^\alpha$ and $y_\omega \notin X_\omega^\alpha$, with $\alpha \in C \cup E$, we want to show any geodesic g_ω connecting x_ω and y_ω should pass through $\rho(\alpha)$. For any closed annulus $A \subset X_o \backslash C$ with the points $\lim_\omega(x_i)$ and $\lim_\omega(y_i)$ lying in the different components of $\mathbb{C} \setminus A$, as previous, $X_\omega \setminus \rho(A)$ consists two connected components, with distance at least the distance of the two boundaries of A in the d_o-metric. And since x_ω and y_ω are in different components of $X_\omega \setminus \rho(A)$, the geodesic connecting them should intersect $\rho(A)$. Let $\{A_i\}_{i=1}^\infty$ be a sequence of such annuli, and they nest down to α in the sense that $\lim_{i\to\infty}\text{diameter}(\alpha \cup A_i) = 0$ in the d_o-metric. Choose some x_ω^i in $\rho(A_i) \cap g_\omega$. Since $\{A_i\}_{i=1}^\infty$ nest down to α and the map ρ does not increase the distance, then we have that $\{x_\omega^i\}$ converges to $\rho(\alpha)$. As the geodesic is compact, we know that $\rho(\alpha)$ should in the geodesic g_ω.

If there is a geodesic $g_\omega \subset X_\omega$ with two end points in $\rho(\overline{X}_o)$ such that it has some point $p_\omega \in g_\omega \cap (X_\omega \setminus \rho(\overline{X}_o))$. Assume p_ω belongs to X_ω^α with $\alpha \in C \cup E$. Then p_ω divides g_ω in to two parts. Each of these two parts should pass though $\rho(\alpha)$. Which means g_ω can not be the shortest curve connecting the two end points. In all, any geodesic with two ends in $\rho(\overline{X}_o)$ should be contained in

$\rho(\overline{X}_o)$. So $(\rho(\overline{X}_o), d_\omega)$ is a geodesic space, since X_ω is a geodesic space. Plus $\rho|_{X_o \setminus C}$ is locally isometric and distance non-increasing embedding, we get $\rho|_{X_o}$ is locally isometric.

For any $\alpha \in E \cup C$, since there always exists a sequence of $\{c_i\}_{i=1}^{\infty} \subset C$ converging to α, then for any $l > 0$ big enough, we can always choose a sequence of points $\{x_i\}_{i=1}^{\infty} \subset X_o \setminus C$, such that $d_i(x_i, a_i) = l$ and $\lim_\omega x_i$ in α's corresponding subset of $\mathbb{C}$. Then we have $x_\omega = (x_i) \in X_\omega^\alpha$ and $d_\omega(x_\omega, a_\omega) = l$.

For any $c \in C$, to prove that X_ω^c is a real ray, it suffices to prove that for any two sequence $\{x_n\}$ and $\{y_n\}$ converging to c, with

$$\lim_{n\to\infty} \left[d_n(x_n, a) = d_n(y_n, a)\right] = l > l_o = d_o(a, c)$$

then, we have $\lim_{n\to\infty} d_n(x_n, y_n) = 0$. This follows easily from Lemma 2.3.7 proved below. □

2.3.4 Real rays attached to C

In this subsection, we are going to complete the proof of Theorem 1.1.2 by showing Lemma 2.3.7. What remains is to show that the extra pieces of X_ω are real rays attached to X_o. The basic idea is to show that X_n has no "bulb" near the critical points of f^n. For doing this, we need to use the fact that X_n is a metric space with non-positive curvature.

Let S be a closed Riemann surface with genus $g \geq 2$ and Q be a holomorphic quadratic differential on S. Then Q determines a flat metric $ds^2 = |Q|$ with finite singularities on S at zeros of Q. At non-singular points, it is flat, so it has curvature 0; at singular points, it's a cone with angle $k\pi$ for $3 \leq k \in \mathbb{N}$. So at the singular points, it has negative curvature. Then in this metric, S is a complete arc length metric space with non-positive curvature. For the definition and properties of the curvature, please refer to p. 159 [8]. Lift this metric to the universal cover $\widetilde{S}$ of S, we also get a metric on $\widetilde{S}$ with non-positive curvature.

Lemma 2.3.5. *$\widetilde{S}$ is a complete CAT(0) unique geodesic space, any geodesic locally is a straight line at non singular point.*

Proof. By [1], $\widetilde{S}$ is an unique geodesic space (any two points are connected by an unique geodesic) with geodesic locally straight line, and $\widetilde{S}$ is also complete. Moreover, since it is a complete and simply connected metric space with non positive curvature, by Cartan-Hadamard Theorem, such space is a CAT(0) space; see p. 193 [8]. $\square$

Fix a point $c \in C$ and some very small $\epsilon > 0$. Let $c_\epsilon \subset X_o \backslash C$ be the closed curve at ϵ-distance from c in the d_o-metric. For each n, choose some closed geodesic (in the d_n-metric) $c_n \subset X_n$ such that c_n is sufficiently close to c; see Lemma 2.1.3. By Lemma 2.3.3 and Lemma 2.1.3, there is some $M < \infty$, such that the d_n-length of c_ϵ $d_n(c_\epsilon) < M \cdot \epsilon$ and $d_n(c_n) < M/d^n$. Choose $x_n \in c_\epsilon$ and $y_n \in c_n$, such that $r_n = d_n(x_n, y_n) = d_n(c_\epsilon, c_n)$. We can do this is because both c_ϵ and c_n are compact. Choose a geodesic $g_n : [0, r_n] \mapsto X_n$ with $g_n(0) = x_n$ and $g_n(r_n) = y_n$. Let A_n be the open annulus bounded by c_ϵ and c_n. We have that $g_n((0, r_n)) \subset A_n$. Otherwise it won't be the shortest curve connecting the two boundaries of A_n.

We can construct closed Riemann surface S_n with genus $g \geq 2$ from X_n. The metric space (X_n, d_n) has finitely many infinite cylinders (cylinder with infinite height). Each such infinite cylinder lying in some neighborhood of a critical point (including $\{\infty\}$) of f^n. So cut the infinite cylinders off X_n along some of the closed geodesics inside the cylinders, such that the closed geodesic is much closer than

c_n to the critical point. Then we get X_n'. Double X_n', and glue them together along the corresponding boundaries to get a closed surface S_n. The metric on S_n is the obvious metric induced from X_n'. Consider the universal cover $\widetilde{S}_n$ of S_n with the induced metric $\widetilde{d}_n$ from S_n. Topologically, $\widetilde{S}_n$ is a unit disk. Let $\widetilde{A}_n \subset \widetilde{S}_n$ be one of the connected components of the preimage of $A_n \subset X_n'$. Then $\widetilde{A}_n$ is a strip on $\widetilde{S}_n$ separating $\widetilde{S}_n$ into two connected components. Since the projection of any curve connecting this two components should be some curve across $A_n \subset X_n'$, the distance of these two components is $d_n(x_n, y_n)$ obtained by some lift $\widetilde{g}_n$ of g_n connecting these two components.

The boundaries $\partial\widetilde{A}_n$ of $\widetilde{A}_n$ are two curves in the preimages of $c_\epsilon, c_n \subset X_n'$. The preimage of g_n on $\widetilde{A}_n$ cuts $\widetilde{A}_n$ into quadrilaterals. All of them can be mapped into each other by some isometry of $\widetilde{S}_n$. Pick one of these quadrilateral $\widetilde{B}_n$ with $\widetilde{g}_n \subset \partial\widetilde{B}_n$. Then $\widetilde{B}_n$ is a copy of the lift of $A_n \backslash g_n$. Denote $\widetilde{c}_\epsilon$ and $\widetilde{c}_n$ as the lift of c_ϵ and c_n on $\partial\widetilde{B}_n$, and $\widetilde{g}_n^o$ the other lift of g_n on $\partial\widetilde{B}_n$.

Lemma 2.3.6. *For any two points $\widetilde{z}_1 \in \widetilde{c}_\epsilon$ and $\widetilde{z}_2 \in \widetilde{c}_n$, there is an unique geodesic $\widetilde{g}$ connecting these two points, and if we varies $\widetilde{z}_1$ and $\widetilde{z}_2$ continuously, then $\widetilde{g}$ varies continuously. In particular, any point $\widetilde{q} \in \widetilde{B}_n$, there is some geodesic $\widetilde{g}_q$ passing though $\widetilde{q}$ with two ends in $\widetilde{c}_\epsilon$ and $\widetilde{c}_n$.*

Proof. By Lemma 2.3.5, there is an unique geodesic $\widetilde{g}$ connecting $\widetilde{z}_1$ and $\widetilde{z}_2$. Because $\widetilde{S}_n$ is a complete and simply connected CAT(0) metric space, by Cartan-Hadamard theorem in p. 193 [8], geodesic varies continually with respect to the two end points.

Assume there is some point $\widetilde{q} \in \widetilde{B}_n$ such that any geodesic with two ends in $\widetilde{c}_\epsilon$ and $\widetilde{c}_n$ won't pass though it. Choose $\widetilde{z}_1(t) \in \widetilde{c}_\epsilon$ and $\widetilde{z}_2(t) \in \widetilde{c}_n$ varies from the ends of $\widetilde{g}_n$ to the ends of $\widetilde{g}_n^o$. Then the corresponding geodesics varies from $\widetilde{g}_n$ to $\widetilde{g}_n^o$ without touching $\widetilde{q}$. From this we get that, in $\widetilde{S}_n \backslash \widetilde{q}$, $\partial\widetilde{B}_n$ is homotopic to a point. This is impossible since $\widetilde{S}_n$ is topologically a disc. □

Lemma 2.3.7. *Let $\widetilde{q}_1$ and $\widetilde{q}_2$ be two points in $\overline{\widetilde{B}}_n \backslash (\widetilde{c}_n \cup \widetilde{c}_\epsilon)$, with $l_1 = \widetilde{d}_n(\widetilde{q}_1, \widetilde{c}_\epsilon)$ and $l_2 = \widetilde{d}_n(\widetilde{q}_2, \widetilde{c}_\epsilon)$. Then $\widetilde{d}_n(\widetilde{q}_1, \widetilde{q}_2) \leq |l_1 - l_2| + 9r_3 + 7r_4$, where r_3 and r_4 are $\widetilde{d}_n$-lengths of $\widetilde{c}_\epsilon$ and $\widetilde{c}_n$.*

Proof. As in Lemma 2.3.6, we can choose geodesics $\widetilde{g}_i : [0, r_i] \to \widetilde{S}_n$ of length r_i passing though $\widetilde{q}_i$, with $\widetilde{g}_i(0) \in \widetilde{c}_\epsilon$ and $\widetilde{g}_i(r_i) \in \widetilde{c}_n$ for $i = 1, 2$. Also, we have geodesic $\widetilde{g}_o : [0, r_o] \to \widetilde{S}_n$ with $\widetilde{g}_o(0) = \widetilde{g}_1(0)$ and $\widetilde{g}_o(r_o) = \widetilde{g}_2(r_2)$.

For $1 \leq i \leq 2$, there is r_i' such that $\widetilde{q}_i = \widetilde{g}_{s_i}(r_i')$ with $0 < r_i' < r_i$. And since $l_i = \widetilde{d}_n(\widetilde{p}_i, \widetilde{c}_\epsilon)$ and $r_i' = \widetilde{d}_n(\widetilde{g}_i(0), \widetilde{p}_i)$, then $l_i \leq r_i' \leq l_i + r_3$.

First, assume that we have $r_o \leq r_1 \leq r_2$. In the isosceles triangle with three vertices $\widetilde{g}_1(0), \widetilde{g}_2(r_2)$ and $\widetilde{g}_1(r_o)$, since $r_1' = \widetilde{d}_n(\widetilde{q}_1, \widetilde{g}_1(0)) = \widetilde{d}_n(\widetilde{g}_o(r_1'), \widetilde{g}_1(0))$, by CAT(0) property of $\widetilde{S}_n$, we have

$$\widetilde{d}_n(\widetilde{q}_1, \widetilde{g}_o(r_1')) \leq \widetilde{d}_n(\widetilde{g}_1(r_o), \widetilde{g}_2(r_2))$$

$$\leq \widetilde{d}_n(\widetilde{g}_1(r_o), \widetilde{g}_1(r_1)) + \widetilde{d}_n(\widetilde{g}_1(r_1), \widetilde{g}_2(r_2)) \leq (r_1 - r_o) + r_4$$

In the isosceles triangle with three vertices $\widetilde{g}_2(r_2), \widetilde{g}_1(0)$ and $\widetilde{g}_2(r_2 - r_o)$, since $r_2' - (r_2 - r_o) = \widetilde{d}_n(\widetilde{q}_2, \widetilde{g}_2(r_2 - r_o)) = \widetilde{d}_n(\widetilde{g}_o(r_2' - (r_2 - r_o)), \widetilde{g}_1(0))$, by CAT(0) property of $\widetilde{S}_n$, we have

$$\widetilde{d}_n(\widetilde{q}_2, \widetilde{g}_o(r_2' - (r_2 - r_o))) \leq \widetilde{d}_n(\widetilde{g}_1(0), \widetilde{g}_2(r_2 - r_o))$$

$$\leq \widetilde{d}_n(\widetilde{g}_1(0), \widetilde{g}_2(0)) + \widetilde{d}_n(\widetilde{g}_2(0), \widetilde{g}_2(r_2 - r_o)) \leq r_3 + (r_2 - r_o)$$

Moreover, since r_3 and r_4 are $\widetilde{d}_n$-lengths of $\widetilde{c}_\epsilon$ and $\widetilde{c}_n$, so we have

$$|r_1 - r_o| \leq r_3 + r_4 \text{ and } |r_2 - r_o| \leq r_3 + r_4$$

Consequently,

$$\begin{aligned}
\widetilde{d}_n(\widetilde{q}_1, \widetilde{q}_2) \quad &\leq \widetilde{d}_n(\widetilde{q}_1, \widetilde{g}_o(r_1')) + \widetilde{d}_n(\widetilde{g}_o(r_1'), \widetilde{g}_o(r_2' - (r_2 - r_o))) + \widetilde{d}_n(\widetilde{g}_o(r_2' - (r_2 - r_o)), \widetilde{q}_2) \\
&\leq ((r_1 - r_o) + r_4) + |(r_2' - (r_2 - r_o)) - r_1'| + (r_3 + (r_2 - r_o)) \\
&\leq |r_1 - r_o| + |r_2' - r_1'| + 2|r_2 - r_o| + r_3 + r_4 \\
&\leq 2(r_3 + r_4) + (2r_3 + |l_1 - l_2|) + 2 \cdot 2(r_3 + r_4) + r_3 + r_4 \\
&= |l_1 - l_2| + 9r_3 + 7r_4,
\end{aligned}$$

Second, for all other cases, similarly, we can always get:

$$\widetilde{d}_n(\widetilde{q}_1, \widetilde{q}_2) \leq |l_1 - l_2| + 9r_3 + 7r_4.$$

□

CHAPTER 3

RATIONAL FUNCTIONS WITH IDENTICAL MEASURE OF MAXIMAL ENTROPY

3.1 Graph of the multi-valued functions $G^{-1} \circ G$

In this section, we study the geometry of the algebraic curves defined by

$$V_G = \{(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1 | G(x) = G(y)\} \subset \mathbb{P}^1 \times \mathbb{P}^1$$

for rational functions G. The irreducible components of V_G correspond to multi-valued functions $G^{-1} \circ G$, as appearing in equation (1.2.4). We prove Theorem 3.1.2 , 3.1.3 and 3.1.4 allowing us to estimate the genus of the irreducible components of V.

3.1.1 Multi-valued functions $G^{-1} \circ G$

For a rational function G with degree $d \geq 2$, we consider the correspondence $V_G = \{(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1 | G(x) = G(y)\} \subset \mathbb{P}^1 \times \mathbb{P}^1$ of function G. Obviously, V_G is a projective variety of bidegree (d, d), consisting of finitely many irreducible curves. Let V_o be an irreducible component of V_G with bidegree (r_1, r_2). Geometrically, for $i = 1$ and 2, r_{2-i} is the topological degree of the coordinate projection map $\pi_i : V_o \to \mathbb{P}^1$. The diagonal $\triangle \subset \mathbb{P}^1 \times \mathbb{P}^1$ is an irreducible component of V_G with bidegree $(1, 1)$.

For any two distinct and noncritical points z_1, z_2 with $G(z_1) = G(z_2)$, there is a unique holomorphic germ $b(z)$ locally defined at z_1, such that $b(z_1) = z_2$ and $G \circ b = G$. By analytic continuity, we can extend this germ $b(z)$ to a multi-valued function from the function on all of $\mathbb{P}^1$, denoted as $G^{-1} \circ G$. To be clear, throughout this chapter, $G^{-1} \circ G$ will refer to a particular multi-valued functions defined in this way. Any such multi-valued function $G^{-1}\circ G$ corresponds to an irreducible component V_o of V_G. Conversely, each irreducible component V_o corresponds to exactly one multi-valued function $G^{-1}\circ G$. We call V_o the **graph** of its corresponding multi-valued function $G^{-1}\circ G$. And then V_G is the union of the graphes for all the multi-valued functions $G^{-1}\circ G$.

Let $\mathrm{Crit}(G)\subset\mathbb{P}^1$ be the set of critical points of G and $\widetilde{\mathrm{Crit}}(G)$ be the preimage of the set of critical values of G. Let $S_G=\mathbb{P}^1\setminus\widetilde{\mathrm{Crit}}(G)$ and $\widetilde{S}_G$ be the universal cover of S_G, i.e. we have the covering map

$$\widetilde{S}_G\longrightarrow S_G=\widetilde{S}_G/H, \tag{3.1.1}$$

where H is a subgroup of the automorphism group of $\widetilde{S}_G$, and H is isomorphic to the fundamental group of S_G.

Fix a non-diagonal irreducible component V_o of V_G and its corresponding multi-valued function $G^{-1}\circ G$. Let (r_1,r_2) be the bidegree of V_o in $\mathbb{P}^1\times\mathbb{P}^1$. $G^{-1}\circ G$ is a multi-valued function from S_G to S_G. Although it may not be single-valued, we can lift it to the universal cover, and get a single valued function h_o from $\widetilde{S}_G$ to $\widetilde{S}_G$. h_o is an automorphism of $\widetilde{S}_G$ without fixed point, since V_o is not the diagonal. Moreover, we have $h_o\not\in H$, otherwise $G^{-1}\circ G$ would be identity map, here H is the group in (3.1.1). Now, we can use the index of the fundamental groups to interpret r_1 and r_2. For any $\tilde{x}\in\widetilde{S}_G$, $H\tilde{x}\in S_G=\widetilde{S}_G/H$. For any $h_i,h_j\in H$, $Hh_oh_i\tilde{x}=Hh_oh_j\tilde{x}$ if and only if $Hh_oh_i=Hh_oh_j$, if and only if $h_ih_j^{-1}\in h_o^{-1}Hh_o$. Since V_o is of bidegree (r_1,r_2), each coset $H\tilde{x}$ splits into $\{H_i\tilde{x}\}$, where H_i are the cosets of $H\cap h_o^{-1}Hh_o$ in H, for $i=1,2,\cdots,r_2$. As a consequence, we can write $r_2=[H:H\cap h_o^{-1}Hh_o]$. In order to find r_1, we can use h_o^{-1} instead of h_o. Similarly, we have $r_1=[H:H\cap h_oHh_o^{-1}]$. The map

$G: S_G \to \mathbb{P}^1 \setminus CV(G)$ is a covering map of degree d, where d is the degree of G and $CV(G)$ is the set of critical values of G.

$$\widetilde{S}_G \longrightarrow S_G = \widetilde{S}_G/H \longrightarrow \mathbb{P}^1 \setminus CV(G) = \widetilde{S}_G/\widetilde{H}$$

Because $G^{-1} \circ G$ permutes the points in each fiber of G, we have $h_o \in \widetilde{H}$ and $[\widetilde{H} : H] = d$.

$$\begin{aligned}
r_1 d \quad &= [\widetilde{H} : H \cap h_o H h_o^{-1}] \\
&= [h_o^{-1} \widetilde{H} h_0 : h_o^{-1}(H \cap h_o H h_o^{-1}) h_o] \\
&= [\widetilde{H} : h_o^{-1} H h_o \cap H] \\
&= r_2 d
\end{aligned}$$

Then we have $r_1 = r_2$.

Proposition 3.1.1. *Let G be a rational function with degree $d \geq 2$, $V_G = \{(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1 | G(x) = G(y)\}$. The graph V_G is of bidegree (d, d), and any irreducible component (as a variety) of V_G is of bidegree (r, r) with $1 \leq r \leq d - 1$. Moreover, the sum of bidegrees r of the irreducible components of V_G is d.*

Let V_o be an irreducible component of V_G with bidegree (r, r). It may contain singularities, but we can normalize it and get a smooth curve $\widetilde{V_o}$ as its normalization. Then we have the following natural projections:

$$\widetilde{V_o} \xrightarrow{\pi} V_o \xrightarrow{\pi_i} \mathbb{P}^1 \ ,$$

where, π_i is the coordinate projection, for $i = 1, 2$.

We use $d_{h,x}$ to denote the local degree of a holomorphic map $h(z)$ at point x. For any $(\widetilde{x},\widetilde{y}) \in \widetilde{V}_o$ with $\pi((\widetilde{x},\widetilde{y})) = (x,y) \in V_o$, we can express the local degree of the map $\pi_1 \circ \pi$ at $(\widetilde{x},\widetilde{y})$ in terms of $d_{G,x}$ and $d_{G,y}$. From the local behavior of G at points x and y, it has

$$d_{\pi_1 \circ \pi,(\widetilde{x},\widetilde{y})} = \frac{d_{G,y}}{\gcd(d_{G,x}, d_{G,y})}. \tag{3.1.2}$$

3.1.2 Genus zero components of the graph V_G

Theorem 3.1.2. *Let $G^{-1} \circ G$ be a multi-valued function with corresponding irreducible component $V_o \subset V_G$. The following two are equivalent:*

- *The normalization $\widetilde{V}_o$ of V_o has genus zero.*
- *There exist rational functions $\widetilde{G}$ and $\widetilde{F}$ such that $(G^{-1} \circ G) \circ \widetilde{G} = \widetilde{F}$.*

Proof: Assume that there are rational functions $\widetilde{G}$ and $\widetilde{F}$, such that $(G^{-1} \circ G) \circ \widetilde{G} = \widetilde{F}$. Then we have a well defined map

$$\rho : \mathbb{P}^1 \to V_o$$

with $\rho(z) = (\widetilde{G}(z), \widetilde{F}(z))$. The map ρ can be lifted to V_o's normalization $\widetilde{V}_o$, and denote the lifting map as $\widetilde{\rho}$,

$$\widetilde{\rho} : \mathbb{P}^1 \to \widetilde{V}_o.$$

The lifting map $\widetilde{\rho}$ is holomorphic from $\mathbb{P}^1$ to the smooth curve $\widetilde{V}_o$. And by Riemann-Hurwitz formula, there is no nonconstant holomorphic map from $\mathbb{P}^1$ to a curve with genus greater than zero. Then the genus of $\widetilde{V}_o$ should be zero.

Conversely, if the genus of the smooth curve $\widetilde{V}_o$ is zero. We can parameterize $\widetilde{V}_o$ by $\mathbb{P}^1$ using some parametrization $\widetilde{\rho}$,

$$\widetilde{\rho} : \mathbb{P}^1 \to \widetilde{V}_o.$$

After projecting it down to V_o, we get the following map:

$$\rho = \widetilde{\rho} \circ \pi : \mathbb{P}^1 \to V_o,$$

where $\rho(z) = (\widetilde{G}(z), \widetilde{F}(z))$ with $\widetilde{G}$ and $\widetilde{F}$ being rational functions of degree r. From this parametrization, it is clear that

$$(G^{-1} \circ G) \circ \widetilde{G} = \widetilde{F}.$$

□

Example. Let $T(z) = z^3 - 3z$ be a degree 3 polynomial. It is easy to check that the graph V_T has two irreducible components. One of them is the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$ with bidegree $(1,1)$ corresponding to the identity map, and the other one V_o is of bidegree $(2,2)$. From Riemann-Hurwitz formula, it can be computed that the genus of V_o's normalization $\widetilde{V}_o$ is zero. □

3.1.3 Functions G without nontrivial genus zero components of V_G

Theorem 3.1.3. *Let G be a rational function with degree $d_G \geq 3$. Assume that there are at least three simple critical values of G. Then for any irreducible component $V_o \subset V_G$ with bidegree $(r, r \geq 2)$, its normalization $\widetilde{V}_o$ has genus≥ 1.*

Proof. By assumption, $V_o \subset V_G$ is an irreducible component of V_G with bidegree $(r, r \geq 2)$, corresponding to some multi-valued function $G^{-1} \circ G$. Let $\{y_1, y_2, y_3\}$ be three simple critical values of the rational function G. Consider the sets:

$$\{x_{i,1}, x_{i,2}, \cdots, x_{i,d_G-1}\} = G^{-1}(y_i),$$

where x_{i,d_G-1} is a critical point of G and $i = 1,2,3$. Since V_o has bidegree (r,r), there are at least $r-1$ noncritical points in $G^{-1}(y_i)$, which can be assumed to be $\{x_{i,1}, x_{i,2}, \cdots, x_{i,r-1}\}$, such that

$$(x_{i,j}, x_{i,d_G-1}) \in V_o, \text{ for } j = 1,2,\cdots,r-1.$$

For each such point $(x_{i,j}, x_{i,d_G-1}) \in V_o$, by equation (3.1.2), any point in $\pi^{-1}((x_{i,j}, x_{i,d_G-1})) \subset \widetilde{V}_o$ is a critical point of the projection map $\pi_1 \circ \pi$:

$$\widetilde{V}_o \xrightarrow{\pi} V_o \xrightarrow{\pi_1} \mathbb{P}^1 .$$

So we have at least $3*(r-1)$ critical points for the map $\pi_1 \circ \pi$. By Riemann-Hurwitz formula, we have

$$\begin{aligned} 2 - 2\text{genus}(\widetilde{V}_o) &= 2r - \#\text{ of critical points of } \widetilde{V}_o \\ &\leq 2r - 3*(r-1) \\ &\leq 3 - r. \end{aligned}$$

Since $r \geq 2$, we have

$$\text{genus}(\widetilde{V_o}) \geq (r-1)/2 > 0.$$

□

For degree 2 case, we have the following:

Theorem 3.1.4. *Let g be a degree two rational function with two critical orbits and none of them is preperiodic. Let $G = g^n$ for some $n \geq 1$. Assume V_o is an irreducible component of V_G such that the corresponding $G^{-1} \circ G$ is not the identity map or σ_f. Then its normalization $\widetilde{V_o}$ has genus≥ 1.*

Proof. Let (r, r) be the bidegree of $V_o \subset V_G$. Let $x_{1,1}, x_{2,1}$ be the two critical points of f and

$$\{x_{i,1}, x_{i,2}, \cdots, x_{i,d_G-1}\} = G^{-1}(G(x_{i,1})), \text{ for } i = 1, 2.$$

As the multi-valued function $G^{-1} \circ G$ does not correspond to the identity map or σ_f, there are r noncritical points in $\{x_{i,2}, \cdots, x_{i,d_G-1}\}$, which can be assumed to be $\{x_{i,2}, x_{i,3}, \cdots, x_{i,r+1}\}$, such that

$$(x_{i,j}, x_{i,1}) \in V_o, \text{ for } j = 2, 3, \cdots, r+1.$$

For each such point $(x_{i,j}, x_{i,1}) \in V_o$, by equation (3.1.2), any point in $\pi^{-1}((x_{i,j}, x_{i,d_G-1})) \subset \widetilde{V_o}$ is a critical point of the projection map $\pi_1 \circ \pi$:

$$\widetilde{V_o} \xrightarrow{\pi} V_o \xrightarrow{\pi_1} \mathbb{P}^1 .$$

So we have at least $2r$ critical points for the map $\pi_1 \circ \pi$. By Riemann-Hurwitz formula, we have

$$\begin{aligned} 2 - 2\text{genus}(\widetilde{V}_o) &= 2r - \#\text{ of critical points of } \widetilde{V}_o \\ &\le 2r - 2r \\ &= 0. \end{aligned}$$

Then we have

$$\text{genus}(\widetilde{V}_o) \ge 1.$$

□

3.2 Rational functions with common measure of maximal entropy

In this section, we study the relation of two rational functions f and g with $\mu_f = \mu_g$, and then prove Theorem 1.2.3, 1.2.1 and 1.2.7. Moreover, we give examples of non-exceptional functions f and g with $\mu_f = \mu_g$ and they do not satisfy (1.2.2).

Theorem 3.2.1. *Assume f and g are rational functions with degrees≥ 2, satisfying*

$$(g^{-1} \circ g) \circ g = (f^{-1} \circ f) \circ f \tag{3.2.1}$$

for some multi-valued functions $g^{-1} \circ g$ and $f^{-1} \circ f$. Then there are some iterates F and G of f and g, such that:

$$F \circ F = F \circ G,\ G \circ F = G \circ G.$$

Proof. Choose a point $a_0 \in \mathbb{P}^1$ such that a_0 is neither in f's critical orbits nor in g's critical orbits. Let $a_0, a_1, a_2, \cdots$ be a sequence of points such that $g(a_i) = a_{i-1}$. From equation (3.2.1), for any $i \geq 1$, after composing each side i times with themself, we have

$$(g^{-1} \circ g) \circ g^i = (f^{-1} \circ f) \circ f^i \text{ or } f \circ (g^{-1} \circ g) \circ g^i = f \circ f^i. \tag{3.2.2}$$

Then for each $i \geq 1$, there is function germ $(g^{-1} \circ g)_i$ of $g^{-1} \circ g$, locally defined at a_0, such that functions germs

$$f \circ (g^{-1} \circ g)_i \circ g^i|_{\text{near } a_i} = f \circ f^i|_{\text{near } a_i},$$

which are locally defined near a_i. Let $b_i = (g^{-1} \circ g)_i(a_0)$. Since $g(b_i) = g(a_0)$ for any $i \geq 1$, there are only finitely many distinct b_i. Choose some $j > 2i_1 > 2$ such that $b_{i_1} = b_j$. Then we have germs $(g^{-1} \circ g)_{i_1} = (g^{-1} \circ g)_j$ locally defined near a_0. From equation (3.2.2), we have locally defined germs

$$f \circ (g^{-1} \circ g)_{i_1} \circ g^{i_1}|_{\text{near } a_{i_1}} = f \circ f^{i_1}|_{\text{near } a_{i_1}}$$

$$f \circ (g^{-1} \circ g)_j \circ g^j|_{\text{near } a_j} = f \circ f^j|_{\text{near } a_j}$$

As $(g^{-1} \circ g)_{i_1} = (g^{-1} \circ g)_j$, combining above two equations, it follows

$$f \circ f^{i_1} \circ g^{j-i_1}|_{\text{near } a_j} = f \circ (g^{-1} \circ g)_{i_1} \circ g^{i_1} \circ g^{j-i_1}|_{\text{near } a_j} = f \circ f^j|_{\text{near } a_j}$$

Because both sides of the above equation are germs of rational functions, we have

$$f \circ f^{i_1} \circ g^{j-i_1} = f \circ f^j.$$

Since $j \geq 2i_1 \geq 2$, we can post compose some iterate of f to both sides of the above equation:

$$f^{j-i_1} \circ g^{j-i_1} = f^{j-i_1} \circ f^{j-i_1}.$$

Let $f_o = f^{j-i_1}$ and $g_o = g^{j-i_1}$. The above equation shows $(f_o^{-1} \circ f_o) \circ f_o = g_o$. Consequently, $(f_o^{-1} \circ f_o) \circ f_o^i = g_o^i$ and $f_o^i \circ f_o^i = f_o^i \circ g_o^i$ for any $i \geq 1$.

Since we have $(f_o^{-1} \circ f_o) \circ f_o = g_o$, repeating the same process of the above proof, there is some $i_o \geq 1$ such that $g_o^{i_o} \circ f_o^{i_o} = g_o^{i_o} \circ g_o^{i_o}$.

Let $F = f_o^{i_o}$ and $G = g_o^{i_o}$. Then we have $F \circ F = F \circ G,\ G \circ F = G \circ G$. □

As a consequence of Theorem 3.2.2, we can easily prove Theorem 1.2.7 by just using Theorem 1.2.6.

Proof of Theorem 1.2.7. From Theorem 1.2.6, there are some iterates f_o and g_o of f and g and $M, N \geq 1$, such that

$$(g_o^{-1} \circ g_o) \circ g_o^M = (f_o^{-1} \circ f_o) \circ f_o^N.$$

Since for any multi-valued function $(g_o^{-1} \circ g_o)$, we can choose a multi-valued function $(g_o^{-M} \circ g_o^M)$ which the same as $(g_o^{-1} \circ g_o)$, from above equation, we have

$$(g_o^{-M} \circ g_o^M) \circ g_o^M = (f_o^{-N} \circ f_o^N) \circ f_o^N.$$

By Theorem 3.2.1, there are iterates F and G of f_o^N and g_o^M, such that

$$F \circ F = F \circ G, \ G \circ F = G \circ G.$$

□

Given any non-exceptional polynomial g with degree≥ 2, for any rational function f with $\mu_g = \mu_f$, it was known that f should also be a polynomial; see [34]. As a corollary of Theorem 1.2.7, here we give an easy proof of this result.

Corollary 3.2.2. *Let g be a non-exceptional polynomial with degree $d \geq 2$. Then any rational function f with $\mu_f = \mu_g$ should be a polynomial. Consequently, there exist some $m, n \geq 1$, s.t.*

$$f^n = \sigma \circ g^m,$$

where $\sigma(z) = az + b$ is an affine transformation preserving $\mu_g = \mu_f$.

Proof. From Theorem 1.2.7, there are some iterates F and G of f and g, such that

$$G \circ F = G \circ G.$$

Exception set of a rational function h is the maximal finite set, which is invariant under h. Exception set can only be an empty, one point or two points set. If the exception set is one point, then h is conjugate to a polynomial. Since G is a non-exceptional polynomial, its exception set is $\{\infty\}$. Then $\{\infty\}$ is also an invariant set of F, which means that F is a polynomial. If the exceptional set of F contains two points, then F is conjugate to polynomial z^{d_F}, which means F is an exceptional polynomial. And because $\mu_F = \mu_G$, G is exceptional. This contradicts to the assumption. Consequently, $\{\infty\}$ is the exceptional set of F. Since F is some iterate of f, they should have the same exceptional set. Then f should be a polynomial. The last statement comes from the main theorem of [37]. □

Proof of Theorem 1.2.3. Because any exceptional rational function has at most two simple critical values, f is non-exceptional. As $\mu_g = \mu_f$, then by Theorem 1.2.7, there are some integers $m, n \geq 1$, such that for $F = f^n$ and $G = g^m$,

$$F \circ F = F \circ G, \text{ or } (F^{-1} \circ F) \circ F = F \circ G \tag{3.2.3}$$

If $F = G$, f and g has a common iterate. Then the statement is satisfied. So we can assume that $F \neq G$. Let $k \geq 0$ be the smallest integer such that $f^k \circ F \neq f^k \circ G$ and $f^{k+1} \circ F = f^{k+1} \circ G$. Since

$$(f^{-1} \circ f) \circ f^k \circ F = f^k \circ G, \tag{3.2.4}$$

by Theorem 3.1.2 and 3.1.3, the corresponding irreducible component of the multi-valued function $(f^{-1} \circ f)$ in (3.2.4) should have bidegree $(1,1)$. It means the multi-valued function $\sigma = (f^{-1} \circ f)$ is a Möbius transformation and $f \circ \sigma = f$.

Under changing of coordinates, we can assume that $\sigma(z) = \zeta z$ where ζ is a k's primitive root of unit. If $k \geq 2$, then we can decompose $f(z)$ into

$$f(z) = f_o(z^k).$$

Since $k \geq 2$, from the above decomposition, f cannot have three simple critical values. This is a contradiction. So we have

$$\sigma(z) = (f^{-1} \circ f)(z) = z.$$

And by (3.2.4), finally we get $f^k \circ F = f^k \circ G$, which is a contradiction to the assumption that $f^k \circ F \neq f^k \circ G$. In all, it has $F = G$, i.e. f and g share an iterate. □

Remark. Theorem 1.2.3 asserts that for general $f \in \text{Rat}_d$ with degree $d \geq 3$, $\mu_f = \mu_g$ implies that f and g share an iterate. And as we discussed in the introduction, the existence of the special symmetry σ_f for any $f \in \text{Rat}_2$ prevents the same conclusion as in Theorem 1.2.3. Precisely, for any $f \in \text{Rat}_2$ and $g = \sigma_f \circ f$, we have $\mu_f = \mu_g$, but they never share an iterate. However, we can modify it a bit, and show that for generic $f \in \text{Rat}_2$ (see Theorem 3.1.4), $\mu_f = \mu_g$ implies that $g^m = \sigma_f \circ f^n$ or f^n; for details see the proof of Theorem 1.2.2.

However, $\mu_f = \mu_g$ does not always imply that f and g share an iterate. Even worse, Theorem 1.2.1 asserts that f and g may not even satisfy (1.2.2) for any Möbius transformation σ.

Proof Theorem 1.2.1. Since $T \circ R = T \circ S$, we have $f \circ f = f \circ g$ and $T \circ f^i(z) = T \circ g^i(z)$ for any $i \geq 1$. Consequently,

$$(f^{-1} \circ f) \circ f = g.$$

Then from Theorem 1.2.6, $\mu_f = \mu_g$.

Assume that there exist integers $n, m \geq 1$ and $\sigma \in PSL_2(\mathbb{C})$ such that

$$f^n = \sigma \circ g^m.$$

Since f and g have the same degree, we have $n = m$, i.e. $f^n = \sigma \circ g^n$.

$$f^n(z) = R \circ T \circ f^{n-1} = R \circ T \circ g^{n-1} = \sigma \circ g^n = \sigma \circ S \circ T \circ g^{n-1}$$

So we have $R = \sigma \circ S$, which contradicts to the assumption in this theorem.

For the first statement of this theorem, see the following example. □

Example. To illustrate Theorem 1.2.1, let $T(z) = z^3 - 3z$, $R(z) = az + \frac{1}{az}$ and $S(z) = a\omega z + \frac{1}{a\omega z}$, with $\omega^2 + \omega + 1 = 0$ and $a \in \mathbb{C}^*$. For any $a \in \mathbb{C}^*$, it easy to check that $T \circ R = T \circ S$ and there is no $\sigma \in PSL_2(\mathbb{C})$ such that $R = \sigma \circ S$. So from Theorem 1.2.1, we know that $f = R \circ T$ and $g = S \circ T$ have the same measure of maximal entropy. And for any $n, m \geq 1, \sigma \in PSL_2(\mathbb{C})$, we have

$$f^n \neq \sigma \circ g^m.$$

It is not hard to see that neither f nor g is exceptional rational function, since they are not critical finite. There are more such T, R and S satisfying assumptions in Theorem 1.2.1; see [2] and [10]. □

Remark. In the above example, rational functions $f_a(z) = a(z^3 - 3z) + \frac{1}{a(z^3-3z)}$ and $g_a(z) = f_{a\omega}(z)$ come from composition of rational functions $T(z) = z^3 - 3z, R(z) = az + \frac{1}{az}$ and $S(z) = a\omega z + \frac{1}{a\omega z}$.

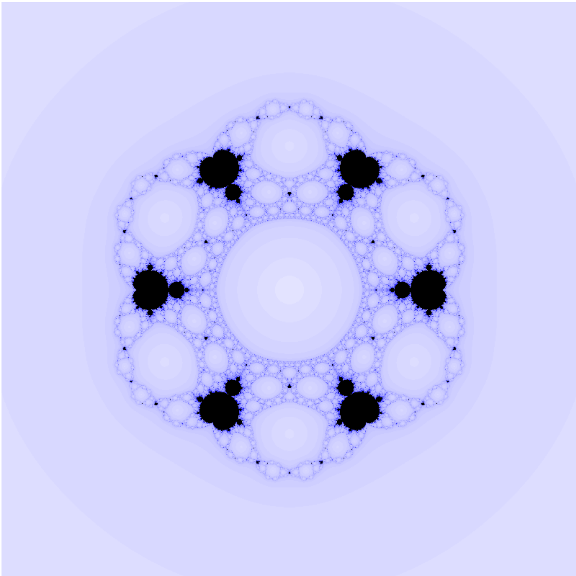

Figure 3.2.1. The parameter space of f_a

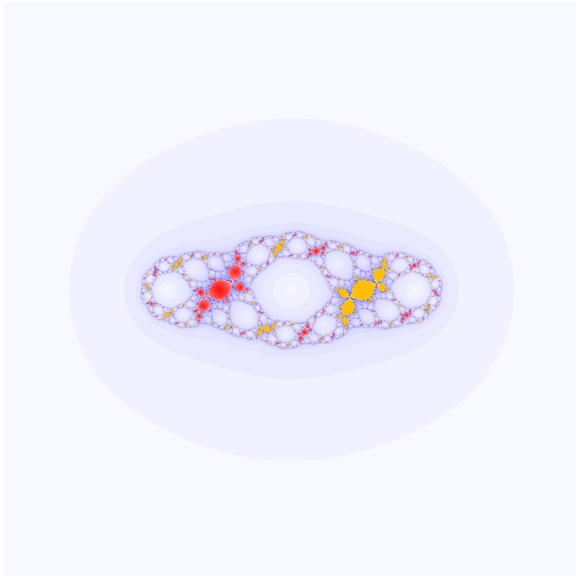

Figure 3.2.2. Julia set of f_a with

$a = 0.4843 + 0.07776i$

And T, R, R satisfy the assumptions in Theorem 1.2.1. Figure 3.2.1 is the parameter space of f_a which indicates that $\mu_{f_{\zeta a}} = \mu_{f_a}$ for any ζ with $\zeta^6 = 1$. Actually, by Theorem 1.2.1, we know that $\mu_{f_{\omega a}} = \mu_{f_a}$ for $\omega^2 + \omega + 1 = 0$ and $f_a^2 = f_{-a}^2$. So $\mu_{f_{\zeta a}} = \mu_{f_a}$ for any ζ with $\zeta^6 = 1$. Since ∞ is a supper attracting point and f_a is not a polynomial, there is a critical point attracted to ∞ and it is not periodic. As exceptional functions are all post-critical finite, f_a won't be exceptional. By Theorem 1.2.1, f_a and g_a has the same measure and there is no iterate of f_a conjugated to an iterate of g_a. However, for any non-exceptional polynomials f and g with $\mu_f = \mu_g$, there always exist iterates of f and g which are in the same conjugacy class.

There are more examples of such rational functions T, R and S as in Theorem 1.2.1. For example, let t, r, s be rational functions satisfying assumptions in Theorem 1.2.1. Then for any rational function h, $T = h \circ t, R = r, S = s$ satisfy the same assumptions. All such rational functions T, R and S have been classified, with the restriction that T is a polynomial. However, when T is not a polynomial, it is still not known how to classify it. For details, please refer to [2] and [10].

3.3 Generic rational function with identical measure

In this section, we are going to prove Theorem 1.2.4, which indicates the iteration map is one-to-one for general points. And then get to prove the main theorem, Theorem 1.2.2, which says: for generic rational functions $f \in \mathrm{Rat}_d$, we have $M_f = \{f, f^2, f^3, \cdots\}$ or $\{f, \sigma_f \circ f, f^2, \sigma_f \circ f^2, \cdots\}$, where M_f the set of rational functions with the same maximal entropy measure as f.

For $d, n \geq 2$, let x be a point in Rat_d. There is an induced map between the tangent spaces of $x \in \mathrm{Rat}_d$ and $\varphi_{d,n}(x) \in \mathrm{Rat}_{d,n}$:

$$\varphi_{d,n\,*} : T_x \to T_{\varphi_{d,n}(x)}.$$

The map $\varphi_{d,n}$ is singular at $x \in \mathrm{Rat}_d$ if the induced map between the tangent spaces T_x and $T_{\varphi_{d,n}(x)}$ is not injective. The map $\varphi_{d,n}$ is nonsingular if it is not singular at any point of Rat_d. For any $f \in \mathrm{Rat}_d$, we can express it as

$$f(z) = \frac{h(z)}{k(z)} = \frac{a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0}{b_d z^d + b_{d-1} z^{d-1} + \cdots + b_0}.$$

We can define a holomorphic map from $t \in \mathbb{C}$ to $f_t \in \mathrm{Rat}_d$ in a neighborhood of $0 \in \mathbb{C}$:

$$f_t(z) = \frac{(a_d + t\alpha_d) z^d + (a_{d-1} + t\alpha_{d-1}) z^{d-1} + \cdots + (a_0 + t\alpha_0)}{(b_d + t\beta_d) z^d + (b_{d-1} + t\beta_{d-1}) z^{d-1} + \cdots + (b_0 + t\beta_0)}.$$

It can be checked that this parametrization map is singular at $t = 0$ iff

$$(\alpha_d z^d + \alpha_{d-1} z^{d-1} + \cdots + \alpha_0) k(z) - (\beta_d z^d + \beta_{d-1} z^{d-1} + \cdots + \beta_0) h(z) = 0.$$

Take the derivative of f_t with respect to t,

$$\frac{d f_t(z)}{dt}|_{t=0} = \frac{(\alpha_d z^d + \alpha_{d-1} z^{d-1} + \cdots + \alpha_0) k(z) - (\beta_d z^d + \beta_{d-1} z^{d-1} + \cdots + \beta_0) h(z)}{(b_d z^d + b_{d-1} z^{d-1} + \cdots + b_0)^2}.$$

From the above expression, the map $t \to f_t$ is singular at 0 iff $\frac{d f_t(z)}{dt}|_{t=0} = 0$.

More generally, for any holomorphic map $(t \to f_t)$ from a neighborhood of $0 \in \mathbb{C}$ to Rat_d, we can express it as

$$f_t(z) = \frac{a_d(t) z^d + a_{d-1}(t) z^{d-1} + \cdots + a_0(t)}{b_d(t) z^d + b_{d-1}(t) z^{d-1} + \cdots + b_0(t)},$$

where $a_i(t)$ and $b_i(t)$ are holomorphic. Similar to the special case we discussed above, it can be checked that

$$f_t \text{ is singular at } t = 0 \Longleftrightarrow \frac{d f_t(z)}{dt}|_{t=0} = 0. \tag{3.3.1}$$

When the map $t \to f_t$ is nonsingular at $t = 0$, then $\frac{d f_t(z)}{dt}|_{t=0}$ is a nonzero rational function with degree at most $2d$.

The following proposition has been proved by Adam Epstein [15]. For completeness, we will give a proof here.

Proposition 3.3.1. *The map* $\varphi_{d,n} : \mathrm{Rat}_d \to \mathrm{Rat}_{d^n}$ *is nonsingular. In particular,* $\varphi_{d,n}$ *is an immersion from* Rat_d *to* Rat_{d^n}.

Proof. In order to prove that $\varphi_{d,n}$ is nonsingular, it suffices to prove that if a holomorphic map $t \to f_t$ from a neighborhood of $0 \in \mathbb{C}$ to Rat_d is nonsingular at $t = 0$, then the map $t \to f_t^n$ is nonsingular at $t = 0$.

First assume that the holomorphic map $t \to f_t$ is singular at $t = 0$. Let $z_0 \in \mathbb{P}^1$ be a periodic point of f_0 with period $p \geq 1$ and multiplier $\frac{df_0^p}{dz}(z_0) \neq 1$. Then there is a holomorphic motion z_t of the periodic point z_0 in $\mathbb{P}^1$ such that

$$f_t^p(z_t) = z_t. \tag{3.3.2}$$

We claim that $\frac{dz_t}{dt}|_{t=0} = 0$, i.e. the holomorphic motion z_t of the periodic point z_0 is singular at $t = 0$. Indeed, let

$$\psi_p(z_1, z_2) = f_{z_2}^p(z_1),$$

and then $\psi_p(z, t) = f_t^p(z)$. By taking the derivative of t for both sides of the equation (3.3.2),

$$\frac{d\psi_p(z_t, t)}{dt} = \frac{\partial \psi_p}{\partial z_1}\frac{\partial z_t}{\partial t} + \frac{\partial \psi_p}{\partial z_2} = \frac{dz_t}{dt} \tag{3.3.3}$$

Since $t \to f_t$ is singular at $t = 0$, the map $t \to f_t^p$ is singular at $t = 0$. As a consequence of (3.3.1), $\frac{\partial \psi_p}{\partial z_2}$ is zero at $(z_0, 0)$. Then from equation (3.3.3):

$$\frac{df_0^p}{dz}(z_0)\frac{\partial z_t}{\partial t}|_{t=0} = \frac{\partial \psi_p}{\partial z_1}(z_0, 0)\frac{\partial z_t}{\partial t}|_{t=0} = \frac{dz_t}{dt}|_{t=0}.$$

And by the assumption that $\frac{df_0^p}{dz}(z_0) \neq 1$, we have $\frac{dz_t}{dt}|_{t=0} = 0$.

Second, we prove that $\varphi_{d,n}$ is nonsingular by contradiction. Assume that the map $t \to f_t$ is nonsingular at $t = 0$, but $t \to f_t^n$ is singular at $t = 0$. Then for any repelling periodic point $z_0 \in \mathbb{P}^1$, $f_0(z_0)$ is a repelling periodic point of $f_0(z)$. Let z_t be the holomorphic motion of the periodic point z_0. Therefore, $f_t(z_t)$ is the holomorphic motion of the periodic point $f_0(z_0)$. Because z_t and $f_t(z_t)$ are in the repelling cycle of f_t and $t \to f_t^n$ is singular at $t = 0$. These two motions are singular at $t = 0$. Then by taking the derivative of $\psi_1(z_t, t) = f_t(z_t)$ with respect to t,

$$\frac{d\psi_1(z_t,t)}{dt}|_{t=0} = \frac{\partial\psi_1}{\partial z_1}(z_0,0)\frac{\partial z_t}{\partial t}|_{t=0} + \frac{\partial\psi_1}{\partial z_2}(z_0,0) = \frac{df_t(z_t)}{dt}|_{t=0}$$

Since the motions z_t and $f_t(z_t)$ are singular at $t = 0$, i.e. $\frac{dz_t}{dt}|_{t=0} = 0$ and $\frac{df_t(z_t)}{dt}|_{t=0} = 0$, we can reduce the above equation to

$$\frac{\partial\psi_1}{\partial z_2}(z_0,0) = 0.$$

As we know from the previous discussion and the assumption that the map $t \to f_t$ is nonsingular at $t = 0$, $\frac{\partial\psi_1}{\partial z_2}(z,0) = \frac{df_t}{dt}|_{t=0}(z)$ is a nonzero rational function with degree at most $2d$. It has finitely many zeros. However, the set of repelling periodic points of degree≥ 2 rational functions is infinite. This contradicts to the fact that $\frac{\partial\psi_1}{\partial z_2}(z,0)$ vanishes at any repelling periodic points of f_0. □

Proof of Theorem 1.2.4. As we know, any regular map from $\mathbb{P}^{2d-1}$ to $\mathbb{P}^{2d^n-1}$ is closed, i.e. the image of any Zariski closed set is Zariski closed. And since the map $\varphi_{d,n} : \mathrm{Rat}_d \to \mathrm{Rat}_{d^n}$ is the restriction of a regular map from $\mathbb{P}^{2d-1}$ to $\mathbb{P}^{2d^n-1}$, the image $\varphi_{d,n}(\mathrm{Rat}_d)$ is a subvariety of Rat_{d^n}. And by Theorem 5.3 in [22], the singularities $\mathrm{Sing}(\varphi_{d,n}(\mathrm{Rat}_d))$ of $\varphi_{d,n}(\mathrm{Rat}_d)$ is a proper Zariski closed subset of

$\varphi_{d,n}(\mathrm{Rat}_d)$. Because the map $\varphi_{d,n}$ is regular, we have that the preimage $A = \varphi_{d,n}^{-1}(\mathrm{Sing}(\varphi_{d,n}(\mathrm{Rat}_d)))$ of $\mathrm{Sing}(\varphi_{d,n}(\mathrm{Rat}_d))$ is a proper Zariski closed subset of Rat_d.

Choose a polynomial $p \in \mathrm{Rat}_d$ such that the symmetry group Σ_{J_p} of its Julia set is trivial. From the main theorem of [37] and Corollary 3.2.2, there is no other $f \in \mathrm{Rat}_d$ such that $\varphi_{d,n}(f) = \varphi_{d,n}(p)$. And since $\varphi_{d,n}$ is a proper (preimage of any compact set is compact) regular map and $\varphi_{d,n}$ is an immersion by Proposition 3.3.1, p is not in A.

The set $A \subset \mathrm{Rat}_d$ is $\varphi_{d,n}$'s preimage of some Zariski closed subset of Rat_{d^n}. After throwing away the singularities of $\varphi_{d,n}(\mathrm{Rat}_d)$, $\varphi_{d,n}(\mathrm{Rat}_d \setminus A)$ is a connected smooth submanifold of $\mathrm{Rat}_{d,n}$. Moreover, since $\varphi_{d,n}$ is a proper nonsingular map, the map $\varphi_{d,n}$ is a covering map with restricted to the following sets:

$$\varphi_{d,n} : \mathrm{Rat}_d \setminus A \to \varphi_{d,n}(\mathrm{Rat}_d \setminus A).$$

Because $p \notin A$ and there is no other $f \in \mathrm{Rat}_d$ such that $\varphi_{d,n}(f) = \varphi_{d,n}(p)$. The degree of the covering map should be 1, i.e. $\varphi_{d,n} : \mathrm{Rat}_d \setminus A \to \varphi_{d,n}(\mathrm{Rat}_d \setminus A)$ is injective.

Finally, since $\varphi_{d,n}(z^d) = \varphi_{d,n}(\zeta z^d)$ for any ζ with $\zeta^{d^{n-1}+d^{n-2}+\cdots+d^0} = 1$, $\varphi_{d,n}$ is not injective in Rat_d. Consequently, A is nonempty. $\square$

Theorem 1.2.4 states that $\varphi_{d,n}$ is injective at general points $f \in \mathrm{Rat}_d$. The next theorem indicates that for generic rational functions $f \in \mathrm{Rat}_d$, any rational function g sharing an iterate with f should be some iterate of f.

Theorem 3.3.2. *For the generic rational functions $f \in \mathrm{Rat}_d$ with degree $d \geq 2$, we have that any rational function g,*

$$f^n = g^m \Longleftrightarrow g = f^{n/m} \text{ and } m|n.$$

Proof. First, for any $d \geq 2$ and $n \geq 2$, $\varphi_{d,n}(\mathrm{Rat}_d)$ is a proper subvariety of Rat_{d^n}. Then the set of rational functions, with degree d and coming from the iteration of lower degree rational functions, is a proper Zariski closed subset of Rat_d. Then for general $f \in \mathrm{Rat}_d$, it is not an iterate of some lower degree rational function.

Second, for any $d \geq 2$ and $m, n \geq 2$, consider the iteration maps:

$$\mathrm{Rat}_d \to \mathrm{Rat}_{d^n} \to \mathrm{Rat}_{d^{mn}}$$

given by $\varphi_{d,n}$ and $\varphi_{d^n,m}$. Let $A_{d,mn} \subset \mathrm{Rat}_d$ and $A_{d^n,m} \subset \mathrm{Rat}_{d^n}$ be preimage of the singularities of $\varphi_{d,nm}(\mathrm{Rat}_d)$ and $\varphi_{d^n,m}(\mathrm{Rat}_{d^n})$. The set $B = A_{d,nm} \cup \varphi_{d,n}^{-1}(A_{d^n,m})$ is a Zariski closed subset of Rat_d. Choose a polynomial $p \in \mathrm{Rat}_d$ with trivial symmetry group Σ_{J_p}. From the proof of Theorem 1.2.4, p is not in $A_{d,nm}$, and $\varphi_{d,n}(p)$ is not in $A_{d^n,m}$. Then $p \in \mathrm{Rat}_d \backslash B$, i.e. B is a proper Zariski closed subset of Rat_d. And from the choice of B, we know for any $f \in \mathrm{Rat}_d \backslash B$,

$$f^{mn} = g^m \Longleftrightarrow g = f^n.$$

Third, let $d_1, d_2, n_1, n_2 \geq 2$ be integers with $d_1^{n_1} = d_2^{n_2}$ and n_2 is not divisible by n_1. We claim that

$$\varphi_{d_1,n_1}(\mathrm{Rat}_{d_1}) \not\subseteq \varphi_{d_2,n_2}(\mathrm{Rat}_{d_2}).$$

Actually, from the main theorem of [37] and Corollary 3.2.2, there is a polynomial $q \in \mathrm{Rat}_{d_1}$ such that $M_q = \{q, q^2, q^3, \cdots\}$. If there is an $h \in \mathrm{Rat}_{d_2}$ such that $h^{n_2} = q^{n_1}$, then h must be in M_f. Since $M_q = \{q, q^2, q^3, \cdots\}$, there is some i such that $q^i = h$. Consequently, $q^{n_1} = q^{i*n_2} = h^{n_2}$. So $n_2|n_1$, which contradicts to the assumption that $n_2 \nmid n_1$. Consequently, $q^{n_1} \in \varphi_{d_1,n_1}(\mathrm{Rat}_{d_1}) \nsubseteq \varphi_{d_2,n_2}(\mathrm{Rat}_{d_2})$. Then $\varphi^{-1}_{d_1,n_1}(\varphi_{d_2,n_2}(\mathrm{Rat}_{d_2}))$ is a proper Zariski closed subset of Rat_{d_1}. And for any $f \in \mathrm{Rat}_{d_1} \backslash \varphi^{-1}_{d_1,n_1}(\varphi_{d_2,n_2}(\mathrm{Rat}_{d_2}))$, there is no $g \in \mathrm{Rat}_{d_2}$ such that $f^{n_1} = g^{n_2}$.

From the above three statements, we can remove countably many proper Zariski closed subsets of Rat_d. The left rational functions $f \in \mathrm{Rat}_d$ satisfy the statement of this theorem. □

With all these preparations, we are ready to prove Theorem 1.2.2.

Proof of the Theorem 1.2.2. The first statement of Theorem 1.2.2 is just a consequence of Theorem 1.2.3 and 3.3.2.

Let $f \in \mathrm{Rat}_2$ be rational function with two critical orbits and none of them is preperiodic. Let g be any rational function with $\mu_g = \mu_f$. Since all exceptional functions are post-critical finite and f is not post-critical finite, f cannot be exceptional. So by Theorem 1.2.7, there are integers $m, n, k \geq 1$, such that

$$g^m = (f^{-k} \circ f^k) \circ f^n.$$

By Theorem 3.1.2 and 3.1.3, $f^{-k} \circ f^k$ should be σ_f or the identity map. So it indicates

$$g^m = \sigma_f \circ f^n \text{ or } f^n.$$

By Theorem 3.3.2, there is a generic subset $C \subset \mathrm{Rat}_2$, such that for any $f \in C$, $f^n = g^m$ implies that $m|n$ and $g = f^{n/m}$, and $\mu_f = \mu_g$ implies that $f^k = \sigma_f \circ g^l$ or g^l for some $k, l \geq 1$.

For any $f(z) = \frac{az^2+bz+c}{dz^2+ez+r} \in \mathrm{Rat}_2$, we can write σ_f down explicitly,

$$\sigma_f(z) = \frac{(ar-cd)z-(br-ce)}{(ae-bd)z+(ar-cd)}.$$

There is a free and order two automorphism ρ of Rat_2,

$$\rho : \mathrm{Rat}_2 \to \mathrm{Rat}_2,$$

given $\rho(f) = \sigma_f \circ f$.

Since ρ is an automorphism, $C \bigcap \rho^{-1}(C)$ is a generic subset of Rat_2. For any $f \in C \bigcap \rho^{-1}(C)$ and any g with $\mu_f = \mu_g$, we have $f^n = \sigma_f \circ g^m$ or g^m for some $m, n \geq 1$. If $f^n = g^m$, then $m|n$ and $g = f^{n/m}$. If $f^n = \sigma_f \circ g^m$, then $g^m = (\sigma_f \circ f)^n$. Since $f \in C \bigcap \rho^{-1}(C)$, it indicates that $\sigma_f \circ f \in C$. So we have $m|n$ and $g = (\sigma_f \circ f)^{n/m} = \sigma_f \circ f^{n/m}$. □

3.4 Rational functions with common iterates

In this section, we characterize the condition that two non-exceptional rational functions share an iterate.

Theorem 1.2.3 says that: generally, having the same measure of maximal entropy is the same as sharing an iterate. It is easy to see that two rational functions sharing an iterate should have the same set of periodic points. Conversely, for non-exceptional rational functions, having the same set of periodic points also guarantees that they share an iterate. However, this is not true for exceptional functions; see Proposition 3.4.2.

Theorem 3.4.1 (restatement of Theorem 1.2.5)**.** *Let f and g be non-exceptional rational functions with degrees ≥ 2. The following statements are equivalent:*

1. *f and g have some common iterate, i.e. $f^n = g^m$ for some $n, m \in \mathbb{N}^*$.*
2. *There is some φ with degree ≥ 2, such that $f \circ \varphi = \varphi \circ f$ and $g \circ \varphi = \varphi \circ g$.*
3. *The maximal entropy measures $\mu_f = \mu_g$, and $J \cap \mathrm{Per}(f) \cap \mathrm{Per}(g) \neq \emptyset$.*
4. $\mathrm{PrePer}(f) = \mathrm{PrePer}(g)$ *and* $J \cap \mathrm{Per}(f) \cap \mathrm{Per}(g) \neq \emptyset$.
5. $\mathrm{Per}(f) = \mathrm{Per}(g)$.

Proof Theorem 1.2.5. For (1), let $\varphi = f^n = g^m$. Since f and g are both commutable with φ, (1) $\Rightarrow$ (2). By Ritt's theorem, non-exceptional commutable rational functions share an iterate; see [36] and also [16]. Since f and g are non-exceptional, $f \circ \varphi = \varphi \circ f$ and $g \circ \varphi = \varphi \circ g$ implies that φ is non-exceptional, and f, g, φ share an iterate. So (2) $\Rightarrow$ (1).

By Yuan and Zhang's Theorem 1.6 in [40], PrePer(f)=PrePer(g) implies that $\mu_f = \mu_g$. Then we have (4)$\Rightarrow$(3). Two rational functions sharing an iterate must have the same set of periodic (preperiodic) points and also the same measure of maximal entropy. We have (1)$\Rightarrow$(4) and (1)$\Rightarrow$(5).

From Theorem 3.5 in [40] and Theorem 1.2 in [3], it shows that two rational functions sharing infinitely many preperiodic points guarantees they have common set of periodic points. Assume that Per(f)=Per(g). Since Per(f) is not a finite set, PrePer(f)=PrePer(g). This shows that (5)$\Rightarrow$(4).

It remains to show that (3)⇒ (1). The proof is inspired by [31]. Suppose $\mu_f = \mu_g$, and $J \cap \mathrm{Per}(f) \cap \mathrm{Per}(g) \neq \emptyset$. By passing to some iterates and changing of coordinates, we can assume that 0 is a fixed point of f and g, and 0 is in their Julia set. Since 0 is in the Julia set and it is fixed by both f and g, then $R(z) = f^{-1} \circ g^{-1} \circ f \circ g(z)$ is locally well defined near 0. We claim that R is the identity map. Otherwise, since R has multiplier equaling 1 at its fixed point 0. It determines attracting and repelling flowers near 0. Suppose that there is some point x near 0 in the Julia set and also in some attracting petal of the flowers determined by R; see Section 10 of [33]. Then there is some fundamental domain of R for this petal, which contains some neighborhood of this point x. As μ_f is supported in the Julia set, the fundamental domain won't have zero measure. Since R acts on this petal like a transformation (in appropriate coordinate) and R preserves the measure μ_f, the μ_f-measure of this petal cannot be finite. However, we know that the total mass of μ_f on $\mathbb{P}^1$ is 1, which is a contradiction. So there is no point in the Julia set which is in the attracting petals of the flowers of R. Replace R by R^{-1}, similarly we see that there is no point in the Julia set which is in the attracting petals of the flowers of R^{-1}. As the union of the attracting petals of R and R^{-1} contains a small disc punctured at 0. The point 0 should be an isolated point in the Julia set of f and g. This is impossible, since a Julia set cannot have isolated points. Therefore, R should be the identity map. Which means f and g are commutable. So the third statement implies the first statement.

□

For exceptional case, we would not have this nice result.

Proposition 3.4.2. *Let $f(z) = z^{d_f}$ and $g(z) = z^{d_g}$, with $d_f, d_g \geq 2$. Then*

$$\mathrm{Per}(f) = \mathrm{Per}(g) \Leftrightarrow \forall \text{ prime } p,\ p|d_f \text{ iff } p|d_g.$$

Proof. Assume $p \geq 2$ is a prime number such that $p|d_f$ and $p \nmid d_g$. There are integers $n \geq 1$ and m, with

$$d_g^n = mp + 1$$

Let $z_o = e^{2\pi i/p}$, we have $f(z_o) = 1$ and $g^n(z_o) = e^{2\pi i(mp+1)/p} = z_o$. So z_o is preperiodic and not periodic for f but it is periodic for g, i.e. $\mathrm{Per}(f) \neq \mathrm{Per}(g)$.

Conversely, assume that for any prime number p, $p|d_f$ iff $p|d_g$. Let $z_o = e^{2a\pi i/b}$ be a periodic point of f with period n, where a and b are coprime integers. Then we have $d_f^n a/b = a/b + m$ for some integer m.

$$d_f^n a = a + mb \Rightarrow (d_f^n - 1)a = mb$$

So $b|(d_f^n - 1)$, which means that b and d_f are coprime integers. Then by previous assumption, b and d_g are coprime integers. There is some integer $k \geq 1$ such that $d_g^k \equiv 1 (\mathrm{mod}\ b)$, i.e. $d_g^k a/b = a/b + t$ for some integer t. In all we have $g^k(z_o) = z_o$. So z_o is a periodic point of g. Consequently, $\mathrm{Per}(f) \subset \mathrm{Per}(g)$. Similarly, we have $\mathrm{Per}(g) \subset \mathrm{Per}(f)$. In all, $\mathrm{Per}(g) = \mathrm{Per}(f)$. □

Let $f(z) = z^{2*3}$ and $g(z) = z^{4*3}$. From the above proposition, they have the same set of periodic points. However, they do not share an iterate, which can be seen from the degrees of them.

There is one more thing I would like to mention here. As we know, for any two rational functions, if the intersection of their sets of preperiodic points has infinitely many points, then they have identical set of preperiodic points. For any two non-exceptional rational functions f and g, $|\text{Per}(f)\bigcap\text{Per}(g)|=\infty$ guarantees that $\text{Per}(f)=\text{Per}(g)$. However, for exceptional polynomials $f(z) = z^3$ and $g(z) = z^5$, we have $|\text{Per}(f) \cap \text{Per}(g)| = \infty$ and $\text{Per}(f) \neq \text{Per}(g)$ by Proposition 3.4.2, since it is not hard to see that $e^{2\pi i/2^k}$ is a periodic point for both f and g with any $k \geq 1$.

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