Asymptotic Methods Applied to an American Option

under the CEV Process

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THESIS

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To my mother Hui Xie,

for her unconditional love and support,

this is for you.

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SUMMARY

We consider an American put option under the Constant Elasticity of Variance (CEV) process. This corresponds to a free boundary problem for a partial differential equation (PDE). We show that this free boundary satisfies a nonlinear integral equation, and analyze it in the limit of small $\rho = 2r/\sigma^2$, where r is the interest rate and σ is the volatility. We find that the free boundary behaves differently for five ranges of time to expiry. We then analyze option price $P(S,t)$, as a function of the asset price S and time to expiry t. We obtain the asymptotic expansion of P as $\rho \to 0$, first via an integral equation formulation, and then using the PDE satisfied by P, and analyzing it by perturbation theory and matched asymptotic expansions.

CHAPTER 1

INTRODUCTION

Over the last thirty years, the rapid growth of financial derivatives has led practitioners and academics to develop more sophisticated models to match market behavior. The pricing and hedging of options has its origins in the Nobel prize winning work of Black, Scholes, and Merton (1), who assume that the price of an underlying asset $S(t)$ follows a geometric Brownian motion (GBM) with constant volatility. We let $C(S, t)$ be the price of a European call option at time t for an asset with initial price S, strike K, and expiry T with a payoff of $\max(S - K, 0)$. Then the Black-Scholes-Merton (BSM) theory leads to the well known formula

$$C(S, t; \sigma, K) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \tag{1.1}$$

where $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} \, dz$ is the cumulative normal distribution function and

$$d_1 = \frac{1}{\sigma\sqrt{T-t}} \left[\frac{S}{K} + \left(r + \frac{1}{2}\sigma^2 \right)(T-t) \right], \; d_2 = d_1 - \sigma\sqrt{T-t}. \tag{1.2}$$

Here r is the risk-free interest rate and σ is the constant volatility. The formula in 1.1 provides not only the price of the option but also a method for replicating the option in terms of the underlying asset and bonds. The parameters in 1.1 are all observable in the market except for the volatility σ, which must be estimated in some fashion. A natural approach would be

to use an estimate of the standard deviation based upon ex-post continuously compounded stock returns, measured over a specific sample period in the past with a specific frequency (e.g., daily). This estimate is known as the *realized volatility*. However, such an estimate may not be good enough to price options across all strikes K and expirations T.

A very useful aspect of exchange traded options is that the prices are listed for the public and quoted in the markets. Let $C_{Market}(K,T)$ be the observed market price of a call option with expiry T and strike K. If we solve the following inverse problem for $\sigma(K,T)$ to determine what the market implies for the value of the volatility

$$C(S,t;\sigma,K) = C_{Market}, \tag{1.3}$$

then we can think of $\sigma(K,T) = \sigma_{implied}(K,T)$ as the *implied volatility*. Another closely related concept is that of the term structure of volatility, which refers to how implied volatility differs for related options with different maturities. These two combined generate an implied volatility surface, where the implied volatility is plotted as a function of the strike price K and time t, as illustrated in Figure 1. Frequently, options are quoted in implied volatility directly. BSM theory predicts that the volatility surface is constant. However, since the crash of 1987 and the Long Term Capital Management (LTCM) demise in 1998, the market volatility has increased dramatically and the implied volatility surface is not constant, but instead has a "smile" and/or a skew. The in-the-money (ITM), for which $K < S$ for calls or $K > S$ for puts, and/or out of the money (OTM), for which $K > S$ for calls or $K < S$ for puts, options trade at higher implied
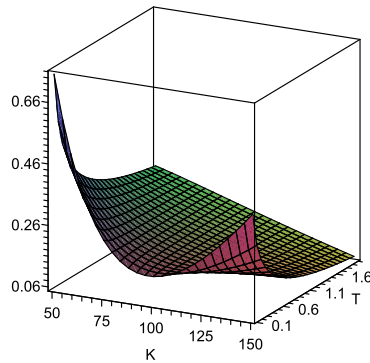
Figure 1. Implied Volatility Surface of S & P call options in Oct 2006. The function is $\sigma(K,T) = .1581 - .2777 \log(K/100) - 0.50/\sqrt{T} - .5011 \log(K/100)^2 + .1103 \log(K/100)/\sqrt{T} + .5909 \log(K/100)^2/\sqrt{T}$, where σ is the volatility that depends on strike price K and final time T.

volatilities than the at-the-money (ATM) option, for which $K = S$, which is also observable in Figure 1. In the financial world, traders and risk managers need to be able to better hedge volatility risk in an options portfolio, and BSM is deficient because changing implied volatility with a fixed strike essentially means that a different model is being applied for each strike. This means that the implied volatility also varies in a systematic fashion as the asset price changes, since asset price varies with strike. There is sufficient empirical evidence (2) to suggest that in many cases the assumption of a constant volatility is not consistent with the observed markets.

Due to the deficiency of the BSM assumption, pricing the options in a more realistic manner requires that we relax the constant volatility assumption. There have been various ideas as to

how to modify and extend the basic Black-Scholes framework, to better model the volatility. An early model that provided an alternative to GBM is the Constant Elasticity of Variance (CEV) diffusion model that was introduced, a few years after the seminal work by Black, Scholes, and Merton, in the context of European options. In this model, the local volatility is now a deterministic function of the underlying asset $\sigma(S,t) = \delta S^{\beta}$ and the asset price is described by the following risk-neutral stochastic partial differential equation (SDE)

$$dS = rS \ dt + \delta S^{\beta+1} \ dW \tag{1.4}$$

where β is the constant of elasticity parameter for the local volatility σ, r is the risk-free interest rate, and W is a standard Brownian motion. There are several reasons why the CEV model has received attention. First, it is consistent with the observation made by Black in (3) that volatility changes are inversely related to stock returns, often referred to as the leverage effect. Moreover, unlike the Black-Scholes model, the CEV model is potentially consistent with capturing the observed implied volatility skew for both equity and index options (2). As such, the CEV model is potentially an improvement over the original GBM model. It has been shown that under a standard GBM assumption, a financial institution would be exposed to significant pricing and hedging errors in comparison to the errors for the CEV model, which are much smaller (2).

The CEV model is derived using the same no arbitrage arguments as the BSM model, but it incorporates a nonlinear state-dependent (cf. (1.4)) volatility function. The model was first

introduced by Cox and Ross (4) for $-1 \le \beta < 0$, where they established the exact solution to the European option problem under a CEV Process in terms of an integral of a modified Bessel function. Later Emanuel and Macbeth (5) extended the model for $\beta > 0$. For $\beta = 0$, we recover our GBM model. A fundamental property for the CEV model, which differs from BSM, is the potential absorption of the process at the lower boundary. That is, the case of $S = 0$ (bankruptcy or default) occurs with positive probability. This feature of the model becomes attractive for equities in the case where bankruptcy is more often than not a recurring event (e.g., financial services or automobile industries). However, it fails to satisfy the positivity assumption ($S > 0$) of the underlying asset needed for the hedging and pricing of the realized variance (6). The pricing of variance swaps by replication using a log contract under the CEV process was studied in (7), and this is a major topic in the finance industry, since here a classical hedging argument can be made that relates the variance swap to the log contract of equity options. However, we do not consider this topic in the thesis, but only wish to make the reader aware that such pricing methods also exist.

For a more exotic type option, Davydov and Linetsky (2) and Lo, et al. (8) provided analytical formulas for barrier options under the CEV process. A barrier option is a path dependent option in which the existence of the option depends on whether the underlying asset price hits a critical value, termed the barrier, during the lifespan of the option. If the price reaches this barrier prior to expiry, the option becomes worthless. Barrier options are studied for their flexibility in tailoring portfolio returns, while lowering the cost of the option premiums. In (8), spectral expansion methods were used to reduce the pricing problem to

a simple computation of the roots of a confluent hypergeometric function. More recently, approximate formulas for the barrier option prices for small volatility have been derived using asymptotic methods (9) and shown to provide good approximations to the actual solution over the entire space-time domain. In general, the CEV model continues to be popular among practitioners due to the existence of asymptotic formulas to price European options while finding a good agreement between theoretical and observed smiles.

In this thesis, we deal with the CEV process in the case $\beta = -1/2$ for the American put option without any barriers. We analyze only put options, since by put-call parity, results for a call option are readily obtained. Because American options may be exercised anytime prior to the expiration dates, additional difficulties are presented in our analysis because we now to have to locate the optimal exercise boundary. Hence the American option problem can be thought of as a free boundary value problem involving a partial differential equation, and these can only rarely be solved exactly. For some historical perspective, free boundary problems (better known as Stefan problems, after Jožef Stefan) were first developed in the late 1800s by physicists to help study problems in ice formation (10). They are a class of boundary value problems in partial differential equations (PDE) in which a phase boundary evolves over time, and the solution to the Stefan Problem involves solving for the phase boundary in addition to solving the full PDE. As far as we know, there exists little or no analytic work for these types of free boundary problems in the valuation of American options under a CEV process. Indeed, we are unaware of a closed-form solution to the American option for even the original Black-Scholes-Merton model. The closest we have to a solution of that form is presented in

(11), where the author employs a highly original approach using homotopy analysis methods. The solution is presented in terms of a convergent infinite series of double integrals that uses the solution of the corresponding European option as an initial guess. However, the author mentions the long duration of time it takes for the computation of the aforementioned infinite series to reach a convergent numerical solution. This does not match our notion of a closed-form solution (analytical representation of the solution by a bounded number of "nice, well-behaved" functions) (12), nor would this formula be very efficient in pricing options in a real time market environment. As a result, we would like to obtain robust approximation formulas for our CEV model which would not only be readily accessible for usage and implementation in the financial markets, but would also stand on its own right as a mathematical result. We initially present some exact and asymptotic solutions for the free boundary over all times up to the expiry T. Then we extend our results to the general option pricing problem over the entire (S, t) domain, where S is the asset price and $t(< T)$ is time. Throughout, we employ various asymptotic techniques, including the method of matched asymptotic expansions, singular perturbation and boundary layer theory, and in particular the ray method of geometrical optics (13), which was first developed in the study of high frequency wave propagation problems. Recently such methods have also been applied to other option pricing problems in Addison, et al. (14), Evans, et al. (15), Fouque, et al. (16), Howison (17), Hu and Knessl (9), Knessl (18) and Widdicks, et al. (19).

CHAPTER 2

ASYMPTOTICS OF FREE BOUNDARY VIA INTEGRAL EQUATIONS

2.1 Problem Formulation and Results

We examine a complete market model in which we assume that the asset follows a one dimensional CEV process (4) where the local volatility is a deterministic function of the underlying asset. The original CEV model has the possibility of default for $\beta < 0$ since the process can be absorbed at 0 with positive probability. We assume that $\beta = -1/2$, and derive asymptotic formulas for the free boundary for all times to expiry.

To examine the behavior of the free boundary under different scaling regimes for the time to expiry, we look at the limit of small $\rho = 2r/\sigma^2$, where r is the interest rate, and σ is the volatility. This limit has a small interest rate and/or large volatility, and is of particular relevance to the financial status of the current economy. The main result in this section is the derivation of a nonlinear integral equation that is satisfied by the free boundary, from which we shall analyze its asymptotic structure for the different ranges of time. Then we extend our analysis by providing results for the option price over all ranges of the space-time domain, both from integral equations and from the underlying PDE by using singular perturbation theory in sections 3 and 4. Asymptotic and singular perturbation methods have been employed for the American put option, by Knessl $(18), (20)$ and Kuske et al. (21). In (18), analytic results in various limits were derived for both the free boundary and the option price, and the moving

boundary was analyzed for both small and large times to expiry. We shall show that the CEV model has a much richer asymptotic structure than the BSM.

We let $P(S, T_0)$ denote the price of an American put option for an asset with price S at some time T_0 prior to expiry T_F. We note that we could just as well have chosen the American call option, but by put-call parity, having a put price allows us to price the call. We assume that the dynamics of the price of the underlying asset S is governed by the following stochastic differential equation

$$dS = \mu S \ dt + \sigma \sqrt{S} \ dW_t. \tag{2.1}$$

where W_t is a standard Brownian motion, σ is the volatility of the underlying asset, and $\mu = r$ is the risk-free interest rate. Introducing the new variables

$$t = \frac{\sigma^2}{2}(T_F - T_0), \quad \rho = \frac{2r}{\sigma^2}, \tag{2.2}$$

and using Ito's Lemma and some arbitrage-free arguments, we find that P satisfies the following boundary value problem

$$P_t = SP_{SS} + \rho SP_S - \rho P; \ t > 0, \ S > \alpha(t) \tag{2.3}$$

$$P(S, 0) = \max(K - S, 0) \tag{2.4}$$

$$P(\alpha(t), t) = K - \alpha(t), \ P_S(\alpha(t), t) = -1, \ P(\infty, t) = 0, \tag{2.5}$$

where $\alpha(t)$ is the free boundary in the new time variable. We also have $P(S,t) = K - S$ for $0 < S < \alpha(t)$, and $\alpha(0) = K$. For $S \le \alpha(t)$ the option should be exercised, and for $S > \alpha(t)$ it should be held.

We convert 2.3 - 2.5 into an integral equation by first making a change in coordinates, letting

$$P(S,t) = K - S + \tilde{P}(V,t), \ V = S - \alpha(t) \tag{2.6}$$

where $V \ge 0$. Then $\tilde{P}$ satisfies the PDE

$$\tilde{P}_t - \alpha'(t)\tilde{P}_V = [V + \alpha(t)]\tilde{P}_{VV} + \rho[V + \alpha(t)]\tilde{P}_V - \rho K - \rho\tilde{P}; \ V, t > 0 \tag{2.7}$$

with the initial and boundary conditions

$$\tilde{P}(V,0) = V, \ \tilde{P}(0,t) = \tilde{P}_V(0,t) = 0. \tag{2.8}$$

We introduce the Laplace transform

$$Q(\theta,t) = \int_0^\infty e^{-\theta V}\tilde{P}(V,t) \ dV. \tag{2.9}$$

Using 2.9 in 2.7 and 2.8 then yields

$$Q_t + (\theta^2 + \rho\theta)Q_\theta = [\alpha'(t)\theta + (\theta^2 + \rho\theta)\alpha(t) - (2\theta + 2\rho)]Q - \frac{\rho K}{\theta} \tag{2.10}$$

with the initial condition $Q(\theta, 0) = 1/\theta^2$. Using the method of characteristics, (the characteristics are $c = \theta e^{-\rho t}/(\theta + \rho)$ where c is a constant) it can be shown that the only acceptable solution to 2.10 is

$$Q(\theta, t) = \frac{K\rho}{\theta^2} e^{\alpha(t)\theta} \int_{\theta/\rho}^{\infty} \frac{1}{z+1} \exp\left[-\rho z \alpha \left(t + \rho^{-1} \log\left(\frac{\theta + \rho}{\theta} \frac{z}{z+1}\right)\right)\right] dz. \qquad (2.11)$$

We note that the most general solution to 2.11 corresponds to replacing the upper limit on the integral by the arbitrary function $f(c)$ where c indexes the family of characteristics. But we must take $f(c) = \infty$ in order for the integral to decay as $\theta \to \infty$, which must happen to offset the exponentially growing factor $e^{\alpha(t)\theta}$ in 2.11. The next result readily follows, by taking the Laplace inversion of 2.11.

Theorem 1. The option price $P(S, t)$ for the CEV model has the integral representation

$$P(S, t) = K - S + \frac{1}{2\pi i} \int_{Br} e^{\theta V} Q(\theta, t) \, d\theta, \qquad (2.12)$$

where $\Re(\theta) > 0$ on the Bromwich contour, and $Q(\theta, t)$ is given by 2.11.

Moreover, after setting $t = 0$ and using $\alpha(0) = K$ in 2.11, it follows that $\alpha(t; \rho)$ satisfies the nonlinear integral equation (IE):

$$\frac{e^{-K\theta}}{K\rho} = \int_{\theta/\rho}^{\infty} \frac{1}{z+1} \exp\left[-\rho z \alpha \left(\rho^{-1} \log\left(\frac{\theta + \rho}{\theta} \frac{z}{z+1}\right)\right)\right] dz. \qquad (2.13)$$

In the next section we use asymptotic methods to analyze this IE for five different scales of time t, in the limit of small ρ. We let $\rho = e^{-\lambda}$ so that $\lambda = -\log \rho \to \infty$. The final results for the free boundary $\alpha(t; \rho)$ are listed below, and we sketch the derivations in section 3.

(i) $t = \omega/\lambda = O(\lambda^{-1})$, $0 < \omega < K$:

$$\alpha(t; \rho) = (\sqrt{\omega} - \sqrt{K})^2 + \frac{\log \lambda}{\lambda} \frac{\omega - \sqrt{K\omega}}{2} + \frac{1}{\lambda} \frac{\sqrt{K\omega} - \omega}{2} \log\left(\frac{4\pi K^2 \omega}{K - \sqrt{K\omega}}\right) + o(\lambda^{-1}), \quad (2.14)$$

(ii) $t = K/\lambda + O(\lambda^{-2})$, $\lambda^2 t - \lambda K = \lambda(\omega - K) = \Lambda$:

$$\alpha(t; \rho) \sim \frac{1}{\lambda^2} \mathcal{F}(\Lambda), \qquad (2.15)$$

where $\mathcal{F}(\cdot)$ satisfies the nonlinear IE

$$\frac{e^{-K\nu}}{K} = \int_0^\infty \frac{1}{\xi} \exp\left[-\xi \mathcal{F}\left(-\nu K^2 - \frac{1}{\xi}\right)\right] d\xi, \quad -\infty < \nu < \infty. \qquad (2.16)$$

For $\Lambda \to \pm\infty$, we have

$$\mathcal{F}(\Lambda) \sim \frac{\Lambda^2}{4K} + \frac{\Lambda}{4} \log(-\Lambda) - \frac{\Lambda}{4} \log(8\pi K^3), \quad \Lambda \to -\infty \qquad (2.17)$$

$$\mathcal{F}(\Lambda) \sim \Lambda e^{-\gamma} \exp\left[-\frac{1}{K} \exp\left(\frac{\Lambda}{K}\right)\right], \quad \Lambda \to +\infty \qquad (2.18)$$

where γ is the Euler constant.

(iii) $t = \omega/\lambda = O(\lambda^{-1})$, $K < \omega < \infty$:

$$\alpha(t;\rho) \sim \frac{\omega - K}{\lambda} e^{-\gamma} \exp\left(-\frac{1}{K}\rho^{K/\omega - 1}\right), \tag{2.19}$$

(iv) $t = O(1)$, $0 < t < \infty$:

$$\alpha(t;\rho) \sim t e^{-\gamma} \exp\left[-\left(\frac{1}{2} + \frac{1}{\rho K}\right) e^{-K/t}\right], \tag{2.20}$$

(v) $t = v/\rho = O(\rho^{-1})$, $v > 0$:

$$\alpha(t;\rho) \sim \frac{1}{\rho} e^{-\gamma} \exp\left(\frac{1}{e^v - 1}\right) (1 - e^{-v}) \exp\left(-\frac{1}{\rho K}\right). \tag{2.21}$$

We note that in four of the five cases the expression for $\alpha(t;\rho)$ is completely explicit, and only in case (ii) must we solve a nonlinear IE, which is somewhat simpler than the one in 2.13. We can easily compute $P(S,t)$ as $t \to \infty$, which corresponds to the perpetual American option, where the problem reduces to solving an ordinary differential equation. Setting $P(S,\infty) = P^\infty(S)$ and using $\alpha(\infty)$ to denote the limiting value of the free boundary, we obtain from 2.3- 2.5

$$P^\infty(S) = K e^{\rho\alpha(\infty)} \int_1^\infty \frac{1}{z^2} e^{-z\rho S} \, dz, \tag{2.22}$$

where $\alpha(\infty)$ satisfies

$$K\rho \int_1^\infty \frac{1}{z} e^{-\rho\alpha(\infty)z} \, dz = e^{-\rho\alpha(\infty)}. \tag{2.23}$$

For $\rho \to 0$ we have

$$\alpha(\infty) = \frac{1}{\rho}e^{-\gamma}\exp\left(-\frac{1}{\rho K}\right)[1 + O(\rho)], \tag{2.24}$$

which is exponentially small.

2.2   <u>Analysis for $t = \omega/\lambda, \ \ 0 < \omega < K$</u>

We first examine 2.13 on the $t = O(\lambda^{-1})$ scale, for small ρ. Recalling that $\lambda = -\log\rho$, we

let

$$\theta = \lambda\beta, \ \ z = \frac{\lambda(\beta + x)}{\rho}, \ \ \alpha(t; \rho) \sim \alpha_0(\omega; \rho) \tag{2.25}$$

for $\omega = (-\log\rho)t = O(1)$. Then 2.13 can be approximated by

$$\frac{e^\lambda}{K} = \int_0^\infty \frac{1}{\beta + x}e^{\lambda\Phi(x;\beta,\rho)}[1 + O(e^{-\lambda})] \ dx, \tag{2.26}$$

where $\Phi(x; \beta, \rho) = K\beta - (\beta + x)\alpha_0(\frac{x}{\beta(x+\beta)}; \rho)$. For large λ and fixed β, we evaluate the right

hand side of 2.26 by an implicit form of the Laplace method, assuming for now that there is a

saddle point where

$$\frac{\partial\Phi}{\partial x} = -\alpha_0\left(\frac{x}{\beta(x + \beta)}\right) - \frac{1}{x + \beta}\alpha_0'\left(\frac{x}{\beta(x + \beta)}\right) = 0. \tag{2.27}$$

Let us denote $x = x_*(\beta)$ as the solution to 2.27. It follows that at $x = x_*$, $\Phi \sim 1$ so that

$$1 = K\beta - (\beta + x_*)\alpha_0\left(\frac{x_*}{\beta(x_* + \beta)}\right). \tag{2.28}$$

Now let $\omega = \frac{x_*}{\beta(x_*+\beta)}$. Then from 2.27 we have $\beta = \frac{\alpha_0'(\omega)}{\omega\alpha_0'(\omega)-\alpha_0(\omega)}$ which we use to eliminate β in 2.28 to obtain the ODE

$$[1 - \alpha_0'(\omega)][\omega\alpha_0'(\omega) - \alpha_0(\omega)] = K\alpha_0'(\omega). \tag{2.29}$$

Dividing 2.29 by $1 - \alpha_0'(\omega)$, we recognize this as the Clairaut equation. The solutions consist of a one-parameter family of lines and the singular solution

$$\alpha_0(\omega) = (\sqrt{K} - \sqrt{\omega})^2 \tag{2.30}$$

which is the envelope of this family. The linear solutions $\alpha_0(\omega) = \omega C - \frac{KC}{1-C}$ must be rejected, since these lead to $\alpha_0(0) \neq K$. The above analysis applies only for $0 < \omega < K$, since the solution 2.30 vanishes as ω approaches K. Hence we expect different asymptotics for $\omega \approx K$.

We next analyze some higher order terms in the expansion of α_0. We evaluate 2.26 by using the Laplace method, which gives

$$\frac{e^\lambda}{K} = \frac{1}{\beta + x_*}\sqrt{\frac{2\pi}{-\lambda\Phi_{xx}(x_*;\beta,\rho)}}e^{\lambda\Phi(x_*;\beta,\rho)}[1 + O(\lambda^{-1})], \tag{2.31}$$

and expand α_0 as

$$\alpha_0(\omega;\rho) = \alpha_0(\omega) + \frac{\log\lambda}{\lambda}\alpha_1(\omega) + \frac{1}{\lambda}\alpha_2(\omega) + o(\lambda^{-1}). \tag{2.32}$$

In order to balance the two sides of 2.31, we need α_1 to cancel the $\sqrt{1/\lambda}$ factor. Hence,

$$\frac{1}{\sqrt{\lambda}} \exp\left[-(\beta + x_*)(\log \lambda)\, \alpha_1\left(\frac{x_*}{\beta(x_* + \beta)}\right)\right] = 1. \tag{2.33}$$

Writing 2.33 in terms of ω we obtain

$$\alpha_1(\omega) = \frac{1}{2}(\omega - \sqrt{K\omega}). \tag{2.34}$$

To find the second order term α_2, we balance the $O(1)$ terms in 2.31, so that

$$\frac{1}{K} = \frac{1}{\beta + x_*}\sqrt{\frac{2\pi}{-\Phi_{xx}}} \exp\left[-(\beta + x_*)\alpha_2\left(\frac{x_*}{\beta(x_* + \beta)}\right)\right]. \tag{2.35}$$

It can be shown that $\Phi_{xx}(x_*; \beta, \rho) \sim -\frac{1}{2}K^{\frac{1}{2}}\beta^{\frac{3}{2}}x_*^{-\frac{3}{2}}(x_* + \beta)^{-\frac{3}{2}}$ and then

$$\alpha_2(\omega) = \frac{\sqrt{K\omega} - \omega}{2}\log\left(\frac{4\pi K^2\omega}{K - \sqrt{K\omega}}\right). \tag{2.36}$$

With 2.30, 2.32, 2.34, and 2.36 we have established 2.14.

2.3 Analysis for $t = \omega/\lambda$, $\omega \approx K$

We return to 2.13 and introduce the scaling

$$\theta = \lambda\beta, \;\; \beta = \frac{1}{K} + \frac{\nu}{\lambda}, \;\; \omega = \lambda t = K + \frac{\Lambda}{\lambda} \tag{2.37}$$

with

$$\alpha(t;\rho) = \frac{1}{\lambda^2}\mathcal{F}(\Lambda;\rho) = \frac{1}{\lambda^2}\mathcal{F}\left(\lambda^2\left(t - \frac{K}{\lambda}\right);\rho\right). \tag{2.38}$$

Then we have $e^{-\theta K} = \rho e^{-K\nu}$. By setting $z = (\theta + y)/\rho$ in 2.13 this equation becomes

$$\frac{1}{K}e^{-K\nu} = \int_0^\infty \frac{1}{\theta + y + \rho}\exp\left[-(\theta + y)\alpha\left(\frac{1}{\theta} - \frac{1}{\theta + y} + O(\rho);\rho\right)\right]\,dy. \tag{2.39}$$

We use 2.37 and 2.38, scale y as $y = \lambda^2\xi$ and note that $\rho = e^{-\lambda}$ is exponentially small, thus obtaining

$$\lambda^2\alpha\left(\frac{1}{\theta} - \frac{1}{\theta + y} + O(e^{-\lambda});\rho\right) = \lambda^2\alpha\left(\frac{K}{\lambda} - \frac{1}{\lambda^2}\left(\nu K^2 + \frac{1}{\xi}\right) + o(\lambda^{-2});\rho\right) \sim \mathcal{F}\left(-\nu K^2 - \frac{1}{\xi}\right) \tag{2.40}$$

where $\mathcal{F}(\Lambda)$ is the leading term in an expansion of $\mathcal{F}(\Lambda;\rho)$. Then scaling $y = \lambda^2\xi$ in 2.39 and letting $\rho \to 0$ ($\lambda \to \infty$) we obtain the limiting IE in 2.16. It does not seem possible to solve 2.16 explicitly for $\mathcal{F}(\Lambda)$. But we can infer the behavior as $\Lambda \to -\infty$ ($\nu \to \infty$) by evaluating the integral in 2.16 by an implicit Laplace type expansion, similarly to what we did in section

2.2. This will verify the asymptotic matching between the ω-scale (for $\omega < K$) and the Λ-scale, and lead to 2.17. Now consider the limit $\Lambda \to +\infty$. For $\nu < 0$, we rewrite 2.16 as

$$
\begin{aligned}
\frac{e^{K|\nu|}}{K} = & \int_{|\nu|K^2}^{\infty} \frac{1}{\eta} \exp\left[-\frac{1}{\eta}\mathcal{F}(|\nu|K^2 - \eta)\right] \, d\eta \\
& + \int_0^1 \frac{1}{u} \left\{ \exp\left[-\frac{1}{u}\frac{\mathcal{F}(|\nu|K^2(1-u))}{|\nu|K^2}\right] - \exp\left[-\frac{1}{u}\frac{\mathcal{F}(|\nu|K^2)}{|\nu|K^2}\right] \right\} \, du \\
& + \int_{\frac{\mathcal{F}(|\nu|K^2)}{|\nu|K^2}}^{\infty} \frac{e^{-v}}{v} \, dv.
\end{aligned}
\tag{2.41}
$$

Here we broke up the integral over $(0, \infty)$ into the two ranges $(0, |\nu|K^2)$ and $(|\nu|K^2, \infty)$ and made some elementary substitutions. Now, for $\Lambda \to -\infty$ we have $\mathcal{F}(\Lambda) \sim \frac{\Lambda^2}{4K}$ so that the first integral in the right hand side of 2.41 will vanish as $\nu \to -\infty$. If $\mathcal{F}(\Lambda) \to 0$ as $\Lambda \to +\infty$ the second integral in 2.41 will also vanish, and the third may be approximated by using

$$
\int_{\varepsilon}^{\infty} \frac{e^{-v}}{v} \, dv = -\log \varepsilon - \gamma + O(\varepsilon), \; \varepsilon \to 0^+.
\tag{2.42}
$$

Hence 2.41 can be replaced by the asymptotic relation

$$
\frac{e^{K|\nu|}}{K} \sim -\log\left[\frac{\mathcal{F}(|\nu|K^2)}{|\nu|K^2}\right] - \gamma
\tag{2.43}
$$

which upon exponentiation leads to the asymptotic result given in 2.18, for $\mathcal{F}(\Lambda)$ as $\Lambda \to \infty$.

2.4   Analysis for $t = \omega/\lambda, \ \ K < \omega < \infty$

In the remaining time ranges, $\alpha(t; \rho)$ will be exponentially small as $\rho = e^{-\lambda} \to 0$, and our analysis of 2.13 will rely heavily on the asymptotic form in 2.42. We let $z = Z/\rho$ in 2.13 to obtain

$$\frac{e^{-K\theta}}{K\rho} = \int_\theta^\infty \frac{1}{Z + \rho} \exp\left[-Z\alpha \left(\rho^{-1} \log \left(\frac{\theta + \rho}{\theta} \frac{Z}{Z + \rho} \right) \right) \right] \, dZ. \tag{2.44}$$

Now we scale $Z = \lambda z_*$ and $\theta = \lambda \theta_*$, let $\alpha(t; \rho) = \tilde{\alpha}(\lambda t; \rho)$ and note that in sections 3.1 and 3.2 we have already characterized $\tilde{\alpha}(\lambda t; \rho)$ for $\lambda t = \omega < K$ and $\omega \sim K$. We also simplify the argument of $\alpha(\cdot)$ in 2.44 using

$$\alpha \left(\frac{1}{\rho} \left[\log \left(1 + \frac{\rho}{\theta} \right) - \log \left(1 + \frac{\rho}{Z} \right) \right] \right) = \alpha \left(\frac{1}{\theta} - \frac{1}{Z} + O(\rho) \right) = \tilde{\alpha} \left(\frac{1}{\theta_*} - \frac{1}{Z_*} + O(e^{-\lambda}\lambda) \right). \tag{2.45}$$

When $Z = \theta$ we have $z_* = \theta_*$ and we rewrite the integral in 2.44 by splitting the range of integration into $z_* \in (\theta_*, \theta_*/[1 - K\theta_*])$ and $z_* \in (\theta_*/[1 - K\theta_*], \infty)$, thus obtaining

$$\frac{e^{-K\lambda\theta_*}}{K\rho} \sim \left(\int_{\theta_*}^{\frac{\theta_*}{1 - K\theta_*}} + \int_{\frac{\theta_*}{1 - K\theta_*}}^\infty \right) \left(\frac{1}{z_*} \exp\left[-\lambda z_* \tilde{\alpha} \left(\frac{1}{\theta_*} - \frac{1}{z_*} \right) \right] \right) \, dz_*. \tag{2.46}$$

In the first range $\tilde{\alpha}(\omega) \sim (\sqrt{K} - \sqrt{\omega})^2$ and the first integral will be $o(1)$ as $\lambda \to \infty$, since $\theta_*^{-1} - z_*^{-1} \le K$ when $z_* \le \theta_*/[1 - K\theta_*]$. In the second integral $\tilde{\alpha}$ will be exponentially small

and the main contribution will come from very large values of z_*, where roughly $z_* = O(\tilde{\alpha}^{-1})$.

Then we write $\tilde{\alpha}(\theta_*^{-1} - z_*^{-1}) \sim \tilde{\alpha}(\theta_*^{-1})$ and using 2.42 we conclude that

$$\frac{e^{-K\lambda\theta_*}}{K\rho} \sim \int_{\frac{\theta_*}{1-K\theta_*}}^{\infty} \frac{1}{z_*} \exp\left[-\lambda z_* \tilde{\alpha}\left(\frac{1}{\theta_*}\right)\right] dz_* \sim -\log\left[\lambda\tilde{\alpha}\left(\frac{1}{\theta_*}\right)\right] - \gamma - \log\left[\frac{\theta_*}{1 - K\theta_*}\right], \quad (2.47)$$

with an error that is $o(1)$ as $\lambda \to \infty$. Then exponentiating 2.47 and replacing θ_* by ω^{-1} we

obtain the asymptotic result in 2.19.

For $\omega \to K$ we note that $\rho^{\omega/K-1} = \rho^{-1}e^{-K/t} = \rho^{-1}\exp\left[-\lambda K/(K + \Lambda/\lambda)\right]$

$= \rho^{-1}\exp\left[-\lambda + \Lambda/K + O(\lambda^{-1})\right] \sim \exp\left(\Lambda/K\right)$ and $(\omega - K)/\lambda = \Lambda/\lambda^2$, which can be used

to verify the asymptotic matching between the Λ-scale and the ω-scale for $\omega > K$, in the

intermediate limit where $\omega \downarrow K$ and $\Lambda \to \infty$.

2.5   Analysis for $t = O(1)$, $0 < t < \infty$

Next we consider times $t = O(1)$. We scale $z = \theta w/\rho$. Since we again expect $\alpha(t; \rho)$ to be

very small we assume a "WKB-type" ansatz of the form

$$\alpha(t; \rho) \sim g(t)\exp\left[-\frac{1}{\rho}f(t)\right]. \quad (2.48)$$

Expanding $\alpha(t; \rho)$ in 2.13 for fixed θ and $\rho \to 0$, and noting that

$$\frac{1}{\rho}\left[\log\left(1 + \frac{\rho}{\theta}\right) - \log\left(1 + \frac{\rho}{\theta w}\right)\right] = \frac{1}{\theta} - \frac{1}{\theta w} - \frac{1}{2}\frac{\rho}{\theta^2} + O\left(\rho^2, \frac{\rho}{w^2}\right), \quad (2.49)$$

we have

$$-\theta w \alpha \left(\frac{1}{\rho} \left[\log \left(1 + \frac{\rho}{\theta} \right) - \log \left(1 + \frac{\rho}{\theta w} \right) \right] \right) = -\theta w g \left(\frac{1}{\theta} \right) \exp \left[-\frac{1}{\rho} f \left(\frac{1}{\theta} \right) + \frac{1}{2\theta^2} f' \left(\frac{1}{\theta} \right) + O(\rho) \right]$$

Here we also used $f(\theta^{-1} - (\theta w)^{-1}) \sim f(\theta^{-1})$, since w will be scaled to be exponentially large. Then setting $\varepsilon = \theta g \left(\frac{1}{\theta} \right) \exp \left[-\frac{1}{\rho} f \left(\frac{1}{\theta} \right) \right] \exp \left[\frac{1}{2\theta^2} f' \left(\frac{1}{\theta} \right) \right]$, scaling $w = \varepsilon^{-1} u$ and using 2.42, 2.13 asymptotically becomes

$$\frac{e^{-K\theta}}{K\rho} = \frac{1}{\rho} f \left(\frac{1}{\theta} \right) - \gamma - \log \left[\theta g \left(\frac{1}{\theta} \right) \right] - \frac{1}{2\theta^2} f' \left(\frac{1}{\theta} \right) + o(1). \tag{2.50}$$

From the $O(\rho^{-1})$ terms in 2.50 we conclude that $f(1/\theta) = K^{-1} e^{-K\theta}$ and then the $O(1)$ terms determine $g(\cdot)$ from

$$\theta g \left(\frac{1}{\theta} \right) = e^{-\gamma} \exp \left[-\frac{1}{2\theta^2} f' \left(\frac{1}{\theta} \right) \right].$$

The above along with 2.48 establishes the asymptotic result in 2.20. The asymptotic matching between 2.19 and 2.20 is immediate, since $\rho^{K/\omega - 1} = \rho^{-1} e^{-K/t}$, and $(\omega - K)/\lambda \sim \omega/\lambda = t$ as $\omega \to \infty$.

2.6   Analysis for $t = v/\rho = \mathbf{O}(\rho^{-1}), \ v > 0$

We assume that time to expiry for the option is large, with $t = v/\rho = O(\rho^{-1})$. On this time scale we assume that

$$\alpha(t; \rho) \sim \frac{1}{\rho} \exp \left(-\frac{1}{\rho K} \right) A(v), \tag{2.51}$$

where $A(\cdot)$ will be determined from 2.13. After scaling $\theta = \rho W$, 2.13 becomes

$$\frac{e^{-K\rho W}}{K\rho} \sim \int_W^\infty \frac{1}{z+1} \exp\left[-ze^{-\frac{1}{\rho K}} A\left(\log\left[\frac{W+1}{W}\frac{z}{z+1}\right]\right)\right] \, dz. \tag{2.52}$$

The major contribution to the integral in 2.52 will once more come from large values of z, so we approximate

$$A\left(\log\left[\frac{W+1}{W}\frac{z}{z+1}\right]\right) \sim A\left(\log\left[\frac{W+1}{W}\right]\right), \tag{2.53}$$

and then using 2.53 in 2.52 along with 2.42 leads to

$$\frac{e^{-K\rho W}}{K\rho} \sim -\log\left\{e^{-\frac{1}{\rho K}} A\left(\log\left[\frac{W+1}{W}\right]\right)\right\} - \gamma - \log(W+1) \tag{2.54}$$

$$= \frac{1}{K\rho} - \gamma - \log\left\{(W+1)A\left(\log\left[\frac{W+1}{W}\right]\right)\right\} + o(1). \tag{2.55}$$

Expanding $e^{-K\rho W} = 1 - K\rho W + O(\rho^2)$ we conclude that

$$A\left(\log\left[\frac{W+1}{W}\right]\right) = e^{-\gamma}\frac{1}{W+1}e^W \tag{2.56}$$

which determines the function $A(\cdot)$ and establishes 2.21.

Finally we verify the asymptotic matching between 2.20 and 2.21. For $v \to 0$ we have $(e^v - 1)^{-1} = v^{-1} - 1/2 + O(v)$ and $1 - e^{-v} \sim v = \rho t$. For $t \to \infty$ we have $e^{-K/t} = 1 - K/t + O(t^{-2}) = 1 - (K\rho)/v + O(\rho^2)$ so that $-(1/2 + 1/(\rho K))e^{-K/t} \sim -v^{-1} + 1/2$ and the matching follows. As $v \to \infty$ we have $A(v) \to e^{-\gamma}$ and thus the expansion in 2.51 agrees with the small ρ expansion of $\alpha(\infty; \rho)$, as given in 2.24.

CHAPTER 3

ASYMPTOTICS OF OPTION PRICE VIA INTEGRAL EQUATIONS

Thus far we have given asymptotic results for the free boundary $\alpha(t)$ for the CEV model for 5 ranges of time. We next give analogous asymptotic results for the option price $P(S, t)$. We shall again consider separately the 5 time ranges, but now within each time range it may be necessary to analyze several ranges of S.

Our approach will be to use the representation in 2.11 - 2.13. Then we shall use the results for $\alpha(t)$ in 2.14 - 2.21 to asymptotically evaluate the integrals in 2.11 and 2.12. We shall begin by deriving an alternate representation to 2.12 (with 2.11), by using the integral equation for $\alpha(t)$ in 2.13.

3.1   Alternate representations

Since 2.13 holds for all $\theta > 0$ we can set

$$\theta = \frac{\rho U}{e^{\rho t}(U + \rho) - U} \tag{3.1}$$

and then 2.13 holds for $U > 0$ with

$$\frac{1}{K\rho} = \int_{\frac{U}{e^{\rho t}(U+\rho)-U}}^{\infty} \frac{1}{z+1} \exp\left(\frac{\rho U K}{(U+\rho)e^{\rho t} - U}\right) \exp\left[-\rho z\alpha\left(\frac{1}{\rho}\log\left(\frac{U+\rho}{U}\frac{z}{z+1}\right)\right)\right] \, dz. \tag{3.2}$$

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Next we split the range of integration $[\theta/\rho, \infty)$ for Q in 2.11 into the subintervals

$$\left[\frac{\theta}{(\theta + \rho)e^{\rho t} - \theta}, \infty\right) \text{ and } \left[\frac{\theta}{(\theta + \rho)e^{\rho t} - \theta}, \frac{\theta}{\rho}\right]. \tag{3.3}$$

We thus have

$$P(S,t) = K - S + Q_1 - Q_2, \tag{3.4}$$

where

$$Q_1 = \frac{1}{2\pi i} \int_{Br} \frac{K\rho}{\theta^2} e^{S\theta} \left(\int_{\frac{\theta}{(\theta + \rho)e^{\rho t} - \theta}}^{\infty} \frac{1}{z+1} \exp\left[-\rho z\alpha\left(t + \rho^{-1}\log\left(\frac{\theta + \rho}{\theta}\frac{z}{z+1}\right)\right)\right] dz\right) d\theta,$$

$$Q_2 = \frac{1}{2\pi i} \int_{Br} \frac{K\rho}{\theta^2} e^{S\theta} \left(\int_{\frac{\theta}{(\theta + \rho)e^{\rho t} - \theta}}^{\frac{\theta}{\rho}} \frac{1}{z+1} \exp\left[-\rho z\alpha\left(t + \rho^{-1}\log\left(\frac{\theta + \rho}{\theta}\frac{z}{z+1}\right)\right)\right] dz\right) d\theta.$$

$$\tag{3.5}$$

Applying 3.2 with U replaced by θ we can evaluate Q_1 explicitly as

$$Q_1 = \frac{1}{2\pi i} \int_{Br} \frac{e^{S\theta}}{\theta^2} \exp\left[-\frac{\rho K\theta}{(\theta + \rho)e^{\rho t} - \theta}\right] d\theta. \tag{3.6}$$

We note that $\theta_0 = -(\rho e^{\rho t})/(e^{\rho t} - 1)$ is an essential singularity of the integrand in 3.6, so we may rewrite 3.6 as

$$Q_1 = \frac{1}{2\pi i} \int_{Br} \frac{e^{S\theta}}{\theta^2} \exp\left[-\frac{\rho K\theta}{(e^{\rho t} - 1)(\theta - \theta_0)}\right] d\theta. \tag{3.7}$$

Letting $\theta = \theta_0 + \xi$, we obtain

$$Q_1 = \frac{1}{2\pi i} \int_{Br} \frac{e^{S\theta_0}}{(\theta_0 + \xi)^2} e^{A/\xi} \, d\xi, \text{ where } A = -\frac{\rho K \theta_0}{e^{\rho t} - 1} = \frac{\rho^2 K e^{\rho t}}{(e^{\rho t} - 1)^2} > 0, \qquad (3.8)$$

where now $\Re(\xi) > -\theta_0$ on the Bromwich contour Br. The contour integral in 3.8 may be evaluated as

$$Q_1 = S \exp\left(-\frac{\rho K}{e^{\rho t} - 1}\right) + \exp\left(-\frac{\rho K}{e^{\rho t} - 1}\right) \sqrt{A} \int_0^S \frac{I_1(2\sqrt{AV})(S - V)}{\sqrt{V}} e^{\theta_0 V} \, dV, \qquad (3.9)$$

where $I_1(\cdot)$ represents the modified Bessel function of first kind.

Asymptotic expansions for Q_1, are readily obtained using the saddle point method or singularity analysis, applied to 3.6. To obtain expansions for Q_2, however, will require the asymptotic results for $\alpha(t)$ in 2.14 - 2.21. We shall show that for the ranges $t = O(\lambda^{-1})$, Q_2 will be asymptotically important only near the free boundary. For the time range $t = O(1)$ with $S > 0$, Q_2 will be smaller than Q_1 by a factor of ρ, but the two are comparable for large space/time scales $S = O(\rho^{-1}), t = O(\rho^{-1})$. But for $t = O(1)$ and $t = O(\rho^{-1})$, Q_1 and Q_2 will again be comparable for values of S close to the free boundary.

3.2   Analysis for $t = \omega/\lambda = O(\lambda^{-1})$

We shall analyze separately Q_1 and Q_2 in 3.6 and 3.5 for the time range $t = \omega/\lambda = O(\lambda^{-1})$. We shall derive various asymptotic approximations for these two functions, and then obtain $P(S, t)$ from 3.4. We will see that the behavior of Q_2 depends on whether $\omega < K$, $\omega \sim K$, or $\omega > K$, while that of Q_1 depends on whether $S < K$, $S \sim K$, or $S > K$.

Consider first Q_1 in 3.6. The asymptotics we will obtain will be in powers of λ^{-1}, and thus $\rho = e^{-\lambda}$ will correspond to an exponentially small correction. We thus write

$$\frac{\rho K\theta}{(\theta + \rho)e^{\rho t} - \theta} = \frac{K\theta}{\theta t + 1} + O(\rho) \sim \frac{\lambda K\theta}{\theta \omega + \lambda} \tag{3.10}$$

and approximate the integrand in 3.6. Then scaling $\theta \to \lambda\theta$ we obtain

$$Q_1 \sim \frac{1}{\lambda} \frac{1}{2\pi i} \int_{Br} \exp\left[\lambda\left(S\theta - \frac{K\theta}{\theta\omega + 1}\right)\right] \frac{1}{\theta^2} \, d\theta, \tag{3.11}$$

with an error that is roughly $O(e^{-\lambda})$. To evaluate the integral in 3.11 for $\lambda \to \infty$ we note that there is a double pole at $\theta = 0$ and saddle(s) where

$$\frac{\partial}{\partial\theta}\left(S\theta - \frac{K\theta}{\theta\omega + 1}\right) = 0 \tag{3.12}$$

so that there is a saddle point at

$$\theta = \theta^* = \frac{1}{\omega}\left(\sqrt{\frac{K}{S}} - 1\right). \tag{3.13}$$

The saddle at $\theta = -(\sqrt{K/S} + 1)/\omega$ will play no role in the asymptotics. By shifting the Bromwich contour Br in 3.13 which has $\Re(\theta) > 0$ to the vertical line $\Re(\theta) = \theta^*$ and noting

that for $S < K$ the saddle lies to the right of the pole, we obtain, via a standard saddle point calculation,

$$Q_1 \sim \frac{\lambda^{-3/2}}{\sqrt{2\pi}} \frac{1}{(\theta^*)^2} \frac{e^{\lambda\phi(\theta^*)}}{\sqrt{-\phi''(\theta^*)}}, \ S < K \tag{3.14}$$

where

$$\phi(\theta^*) = S\theta^* - \frac{K\theta^*}{1 + \omega\theta^*} = -\frac{(\sqrt{S} - \sqrt{K})^2}{\omega} \tag{3.15}$$

and then

$$\phi''(\theta^*) = \frac{2K\omega}{(1 + \omega\theta^*)^3} = \frac{2\omega S^{3/2}}{\sqrt{K}}. \tag{3.16}$$

In terms of S and ω 3.14 becomes

$$Q_1 \sim \frac{\lambda^{-3/2}}{2\sqrt{\pi}} \frac{(SK)^{1/4}\omega^{3/2}}{(\sqrt{S} - \sqrt{K})^2} \exp\left[-\frac{\lambda}{\omega}(\sqrt{S} - \sqrt{K})^2\right], \ S < K. \tag{3.17}$$

When $S > K$ the saddle at $\theta = \theta^*$ is to the left of the pole and the residue at $\theta = 0$ is precisely $S - K$. Thus when $S > K$ the difference $Q_1 - (S - K)$ is again asymptotically given by the formula in 3.17.

When $S \sim K$ the saddle point and pole are close to one another and the expansion in 3.17 no longer holds. We thus analyze two other limits of 3.11, namely $S - K = O(\lambda^{-1/2})$ with $\omega > 0$, and then $S - K = O(\lambda^{-1})$ with $\omega = O(\lambda^{-1})$ (hence $t = O(\lambda^{-2})$). For the first we set

$$S = K + \frac{\zeta}{\sqrt{\lambda}}, \ \theta = \frac{\theta'}{\sqrt{\lambda}} \tag{3.18}$$

in 3.11 and expand $\phi(\theta) = \phi(\theta'/\sqrt{\lambda})$ for small θ which leads to

$$Q_1 \sim \frac{1}{\sqrt{\lambda}} \frac{1}{2\pi i} \int_{Br'} e^{\zeta\theta' + K\omega(\theta')^2} \frac{1}{(\theta')^2} \, d\theta', \tag{3.19}$$

where now $\Re(\theta') > 0$ on Br'. The integral in 3.19 may be recognized as a parabolic cylinder function of order -1, and an alternate expression for 3.18 is

$$Q_1 \sim \frac{1}{\sqrt{\lambda\pi}} \left[\sqrt{\omega K} \exp\left(-\frac{\zeta^2}{4\omega K}\right) - \frac{\zeta}{2} \int_{\frac{\zeta}{\sqrt{\omega K}}}^{\infty} e^{-\frac{u^2}{4}} \, du \right], \ S - K = O(\lambda^{-1/2}). \tag{3.20}$$

Finally, when $S - K = O(\lambda^{-1})$ and $\omega = O(\lambda^{-1})$ we set

$$S = K + \frac{V}{\lambda}, \ \omega = \frac{\tilde{t}}{\lambda} \tag{3.21}$$

in 3.11 and obtain the limiting result

$$\lambda Q_1 \to \frac{1}{2\pi i} \int_{Br} \frac{1}{\theta^2} e^{V\theta} e^{K\tilde{t}\theta^2} \, d\theta, \ \lambda \to \infty, \tag{3.22}$$

since then

$$\lambda \left[S\theta - \frac{K\theta}{1 + \theta\omega} \right] = \frac{\lambda K\theta^2\omega}{1 + \theta\omega} + \theta V = \frac{K\theta^2\tilde{t}}{1 + \theta\tilde{t}/\lambda} + \theta V. \tag{3.23}$$

The expression in the right side of 3.22 is essentially the same as that in 3.19, and thus on the $(V, \tilde{t})$ scale we have

$$Q_1 \sim \frac{1}{\lambda} \left[\frac{\sqrt{K\tilde{t}}}{\sqrt{\pi}} \exp\left(-\frac{V^2}{4K\tilde{t}}\right) + \frac{V}{\sqrt{\pi}} \int_{-\frac{V}{2\sqrt{K\tilde{t}}}}^{\infty} e^{-\xi^2}\, d\xi \right], \tag{3.24}$$

This concludes our analysis of Q_1 and we note that thus far the free boundary $\alpha(t)$ played no role. It will arise when we expand Q_2. Also, for Q_1, there is no change in the asymptotics when ω passes through the critical value K.

We will show below that as long as we are not close to the free boundary $\alpha(t)$, Q_2 is asymptotically small compared to Q_1, so that $P(S,t) - (K - S) \sim Q_1$. Thus we have

(i) $\alpha(t) < S < K$:

$$P(S,t) \sim K - S + \frac{\lambda^{-3/2}}{2\sqrt{\pi}} \frac{(SK)^{1/4}\omega^{3/2}}{(\sqrt{S} - \sqrt{K})^2} \exp\left[-\frac{\lambda}{\omega}(\sqrt{S} - \sqrt{K})^2\right] \tag{3.25}$$

(ii) $S > K$:

$$P(S,t) \sim \frac{\lambda^{-3/2}}{2\sqrt{\pi}} \frac{(SK)^{1/4}\omega^{3/2}}{(\sqrt{S} - \sqrt{K})^2} \exp\left[-\frac{\lambda}{\omega}(\sqrt{S} - \sqrt{K})^2\right] \tag{3.26}$$

(iii) $S - K = O(\lambda^{-1/2})$, $\omega > 0$:

$$P(S,t) \sim \frac{1}{\sqrt{\lambda}} \left[\frac{\sqrt{\omega K}}{\sqrt{\pi}} \exp\left(-\frac{\zeta^2}{4\omega K}\right) - \frac{\zeta}{2} \frac{1}{\sqrt{\pi}} \int_{\frac{\zeta}{\sqrt{\omega K}}}^{\infty} e^{-\frac{u^2}{4}}\, du \right] \tag{3.27}$$

(iv) $S - K = O(\lambda^{-1})$, $\omega = O(\lambda^{-1})$:

$$P(S,t) \sim \frac{1}{\lambda}\left[\frac{\sqrt{K\tilde{t}}}{\sqrt{\pi}}\exp\left(-\frac{V^2}{4K\tilde{t}}\right) + \frac{V}{\sqrt{\pi}}\int_{-\frac{V}{2\sqrt{K\tilde{t}}}}^{\infty} e^{-\xi^2}\,d\xi\right]. \tag{3.28}$$

The analysis of Q_2 and of Q_1 near the free boundary will differ accordingly as $\omega < K$, $\omega \sim K$, $\omega > K$. We first consider $\omega < K$. We have already shown that $\alpha(t) \sim (\sqrt{K} - \sqrt{\omega})^2$ in this time range. Thus 3.25 holds in the range $(\sqrt{K} - \sqrt{\omega})^2 < S < K$. We set $t = \omega/\lambda$ in Q_2, use the approximation in 3.5, and scale $z = u/\rho$ to obtain

$$Q_2 \sim \frac{K\rho}{2\pi i}\int_{Br}\frac{e^{S\theta}}{\theta^2}\left\{\int_{\frac{\theta\lambda}{\lambda+\theta\omega}}^{\theta}\frac{1}{u}\exp\left[-u\alpha\left(\frac{\omega}{\lambda} + \frac{1}{\theta} - \frac{1}{u} + O(e^{-\lambda})\right)\right]\,du\right\}\,d\theta. \tag{3.29}$$

Here we also used

$$\log\left(\frac{\theta+\rho}{\theta}\frac{z}{z+1}\right) = \log\left(\frac{\theta+\rho}{\theta}\frac{u}{u+\rho}\right) = \frac{1}{\theta} - \frac{1}{u} + O(\rho). \tag{3.30}$$

For $S > 0$ we know that α is $O(1)$, and the region of integration in the u-integral will be small for $\lambda \to \infty$, since $\theta\lambda/(\lambda + \theta\omega) \sim \theta$. This shows that $Q_2 = o(\rho)$ or $Q_2 = o(e^{-\lambda})$. But from 3.17 we know that (roughly) $Q_1 = O\left(e^{\lambda\phi(\theta^*)}\right)$ and $\phi(\theta^*) = -(\sqrt{S} - \sqrt{K})^2/\omega > -1$. Thus for $S > \alpha(t) \sim (\sqrt{K} - \sqrt{\omega})^2 \equiv \alpha_*^{(0)}(\omega)$ we have $Q_1 \gg Q_2$. However, when $S \sim \alpha_*^{(0)}(\omega)$ we have $\phi(\theta^*) \sim -(\sqrt{\alpha_*^{(0)}(\omega)} - \sqrt{K})^2/\omega = -1$ and then we must evaluate Q_2 more carefully.

We introduce the spatial scale $S - \alpha(t) = O(\lambda^{-1})$ and scale θ and u in 3.29 to be $O(\lambda)$.

Thus we set

$$S - \alpha(t) = \frac{v}{\lambda}, \ \theta = \lambda\theta_1, \ u = \lambda u_1. \tag{3.31}$$

It is important to note that in the definition of v in 3.31, $\alpha(t) = \alpha(t; \rho)$ represents the exact free boundary. For $t = \omega/\lambda$ and $\omega < K$ we write $\alpha(t; \rho) = \alpha_*(\omega; \lambda)$ where a three-term asymptotic approximation to α_* is given by 2.14. With the scaling in 3.31, Q_2 in 3.5 becomes (again using 3.10)

$$Q_2 \sim \frac{K\rho}{\lambda} \frac{1}{2\pi i} \int_{Br} \frac{e^{\lambda\theta_1 S}}{\theta_1^2} \left\{ \int_{\frac{\theta_1}{1+\omega\theta_1}}^{\theta_1} \frac{1}{u_1} \exp\left[-\lambda u_1 \alpha_* \left(\omega + \frac{1}{\theta_1} - \frac{1}{u_1} \right) \right] du_1 \right\} d\theta_1. \tag{3.32}$$

For $\lambda \to \infty$ we evaluate the integral over u_1 in 3.32 by the Laplace method, as the main contribution will come from the lower limit, where $u_1 \approx \theta_1/(1 + \omega\theta_1)$. We thus introduce u_2 by

$$u_2 = \lambda(\theta_1 - u_1) \text{ or } u_1 = \theta_1 - \frac{u_2}{\lambda} \tag{3.33}$$

and note that

$$
\begin{aligned}
&\lambda\theta_1 S - \lambda u_1 \alpha_* \left(\omega + \frac{1}{\theta_1} - \frac{1}{u_1} \right) \\
&= \lambda\theta_1 S - (\lambda\theta_1 - u_2)\alpha_* \left(\omega + \frac{1}{\theta_1} - \frac{1}{\theta_1 - \frac{u_2}{\lambda}} \right) \\
&= \lambda\theta_1 \left[S - \alpha_*(\omega) + \alpha_*'(\omega)\frac{u_2}{\lambda} + O(\lambda^{-2}) \right] + u_2\alpha_*(\omega) \\
&= \theta_1 v + u_2[\alpha_*(\omega) + \theta_1\alpha_*'(\omega)] + o(1).
\end{aligned}
\tag{3.34}
$$

With the approximation in 3.34, the limiting form for Q_2 in 3.32 is

$$Q_2 \sim \frac{K\rho}{\lambda^2} \frac{1}{2\pi i} \int_{Br} \frac{e^{\theta_1 v}}{\theta_1^3} \left\{ \int_0^\infty \exp\left[\left(\alpha_*(\omega) + \frac{\alpha_*'(\omega)}{\theta_1} \right) u_2 \right] du_2 \right\} d\theta_1. \tag{3.35}$$

Here $\Re(\theta_1) > 0$ on Br and we must also choose the Bromwich contour so that $\Re(\alpha_*(\omega) + \alpha_*'(\omega)/\theta_1) < 0$. This is possible since $\alpha_*(\omega) \sim (\sqrt{K} - \sqrt{\omega})^2$ so that $\alpha_*'(\omega) < 0$. Then the integral over u_2 in 3.35 converges and we have

$$\begin{aligned} Q_2 &\sim \frac{K\rho}{\lambda^2} \frac{1}{2\pi i} \int_{Br} \frac{e^{\theta_1 v}}{\theta_1^2} \frac{1}{\alpha_*'(\omega) + \theta_1 \alpha_*(\omega)} d\theta_1 \\ &= \frac{K\rho}{\lambda^2} \left[\frac{v}{\alpha_*'(\omega)} - \frac{\alpha_*(\omega)}{(\alpha_*'(\omega))^2} \right], \end{aligned} \tag{3.36}$$

as only the residue at $\theta_1 = 0$ contributes.

For $S - \alpha(t) = O(\lambda^{-1})$ the expansion of Q_1 can be obtained simply by setting $S = \alpha(t) + v/\lambda$ in 3.25, expanding for $\lambda \to \infty$ and also using the three term approximation for $\alpha(t)$ in 2.14; this yields

$$Q_1 \sim \frac{K\rho}{\lambda^2} \frac{\alpha_*(\omega)}{(\alpha_*'(\omega))^2} \exp\left(-\frac{\alpha_*'(\omega)}{\alpha_*(\omega)} v \right). \tag{3.37}$$

Combining 3.36 and 3.37 we have thus shown that for $S = \alpha(t) + O(\lambda^{-1})$ and $\omega < K$

$$P(S, t) \sim K - S + \frac{K\rho}{\lambda^2} \left[\frac{\alpha_*}{(\alpha_*')^2} \left(e^{-\frac{\alpha_*'}{\alpha_*} v} - 1 \right) + \frac{v}{\alpha_*'} \right]. \tag{3.38}$$

We next examine the case $\omega \sim K$, recalling that the free boundary behaves (cf.(19)) as $\alpha(t) \sim \mathcal{F}(\Lambda)/\lambda^2$ when $\omega - K = \Lambda/\lambda = O(\lambda^{-1})$. The approximations in 3.25, 3.26 and 3.27 for $P(S,t)$ still hold, and we may replace ω by $K + \Lambda/\lambda$ and further simplify, to get

(i) $\alpha(t) < S < K$:

$$P(S,t) \sim K - S + \frac{\lambda^{-3/2}}{2\sqrt{\pi}} \frac{S^{1/4} K^{7/4}}{(\sqrt{S} - \sqrt{K})^2} \exp\left[\left(\frac{\Lambda}{K^2} - \frac{\lambda}{K}\right)(\sqrt{S} - \sqrt{K})^2\right] \qquad (3.39)$$

(ii) $S > K$:

$$P(S,t) \sim \frac{\lambda^{-3/2}}{2\sqrt{\pi}} \frac{S^{1/4} K^{7/4}}{(\sqrt{S} - \sqrt{K})^2} \exp\left[\left(\frac{\Lambda}{K^2} - \frac{\lambda}{K}\right)(\sqrt{S} - \sqrt{K})^2\right] \qquad (3.40)$$

(iii) $S - K = O(\lambda^{-1/2})$:

$$P(S,t) \sim \frac{1}{\sqrt{\lambda\pi}} \left[K \exp\left(-\frac{\zeta^2}{4K^2}\right) - \frac{\zeta}{2} \int_{\frac{\zeta}{K}}^{\infty} e^{-\frac{u^2}{4}} \, du\right]. \qquad (3.41)$$

We can again easily show that for $S > 0$, Q_2 is negligible compared to Q_1. But, for $S = O(\lambda^{-2})$ these two functions become asymptotically of the same order. To this end, we proceed to evaluate Q_1 and Q_2 by scaling

$$S = \frac{R}{\lambda^2}. \qquad (3.42)$$

To find Q_1, we use 3.42 and set $\theta = \lambda\Phi$ in 3.11, and for $\omega = K + \Lambda/\lambda$ we obtain

$$Q_1 \sim \frac{e^{-\lambda}}{\lambda^2} \frac{e^{\Lambda/K}}{2\pi i} \int_{Br} \frac{e^{R\Phi}}{\Phi^2} \exp\left(\frac{1}{\Phi K}\right) d\Phi \tag{3.43}$$

where we use the fact that

$$\frac{\lambda K \theta}{\theta\omega + 1} = \frac{\lambda^2 \Phi K}{\lambda \Phi K + \Lambda\Phi + 1} = \lambda - \frac{\Lambda}{K} - \frac{1}{K\Phi} + O(\lambda^{-1}). \tag{3.44}$$

The contour integral in 3.44 is a Bessel function, so that

$$Q_1 \sim \frac{e^{-\lambda}}{\lambda^2} e^{\Lambda/K} \sqrt{KR}\, I_1\left(\frac{2\sqrt{R}}{\sqrt{K}}\right). \tag{3.45}$$

For Q_2, we may use the approximation in 3.29 and introduce the scaling

$$S = \frac{R}{\lambda^2}, \ \theta = \lambda^2 \phi, \ u = \lambda^2 \eta, \ \alpha(\omega) \sim \frac{1}{\lambda^2}\mathcal{F}(\Lambda) = \frac{1}{\lambda^2}\mathcal{F}\left\{\lambda^2\left(t - \frac{K}{\lambda}\right)\right\}. \tag{3.46}$$

Then we note that

$$u\alpha\left(\frac{\omega}{\lambda} + \frac{1}{\theta} - \frac{1}{u}\right) = \lambda^2 \eta\alpha\left(\frac{K}{\lambda} + \frac{\Lambda}{\lambda^2} + \frac{1}{\lambda^2\phi} - \frac{1}{\lambda^2\eta}\right) \sim \eta\mathcal{F}\left(\Lambda + \frac{1}{\phi} - \frac{1}{\eta}\right). \tag{3.47}$$

Also, when $u = \theta$ we have $\eta = \phi$ and when $u = \theta\lambda/(\lambda+\theta\omega)$ we have $\eta = \phi/(\lambda\phi K+\phi\Lambda+1) \approx 0$.

Thus for $\omega = K + O(\lambda^{-1})$, 3.29 becomes

$$Q_2 \sim \frac{e^{-\lambda}}{\lambda^2}\frac{K}{2\pi i}\int_{Br}\frac{e^{R\phi}}{\phi^2}\left\{\int_0^\phi \frac{1}{\eta}\exp\left[-\eta\mathcal{F}\left(\Lambda+\frac{1}{\phi}-\frac{1}{\eta}\right)\right]d\eta\right\}d\phi. \tag{3.48}$$

To summarize, on the scale $S = R/\lambda^2$ and $\omega = K + O(\lambda^{-1})$, we have obtained

$$\begin{aligned}P(S,t) \sim K - S &+ \frac{e^{-\lambda}}{\lambda^2}e^{\Lambda/K}\sqrt{KR}\,I_1\left(\frac{2\sqrt{R}}{\sqrt{K}}\right)\\ &- \frac{e^{-\lambda}}{\lambda^2}\frac{K}{2\pi i}\int_{Br}\frac{e^{R\phi}}{\phi^2}\left\{\int_0^\phi\frac{1}{\eta}\exp\left[-\eta\mathcal{F}\left(\Lambda+\frac{1}{\phi}-\frac{1}{\eta}\right)\right]d\eta\right\}d\phi,\end{aligned} \tag{3.49}$$

which expresses $P(S,t)$ in terms of the solution $\mathcal{F}(\Lambda)$ of the integral equation in 2.16.

Next we consider the case where $t = \omega/\lambda$ for $\omega > K$. On this scale, the expansions in 3.25 and 3.26 still hold for $0 < S < K$ and $S > K$ respectively, and the free boundary $\alpha(t)$ is exponentially small in ρ, as given by 2.19. For $S > 0$, we again have $Q_1 \gg Q_2$ and only for S near the free boundary $\alpha(t)$ are Q_1 and Q_2 of comparable magnitude. Once more we shall separately analyze Q_1 and Q_2 in 3.11 and 3.5. It will be necessary to analyze several spatial scales, specifically $S = O(\lambda^{-1}), S = O(\lambda^{-2})$, and $S = O(\alpha(t))$.

For $S = O(\lambda^{-1})$ we can obtain the expansion of Q_1 as a limiting case of 3.17, by setting $S = \tilde{S}/\lambda$ and expanding for $\lambda \to \infty$, since Q_1 does not experience a transition. We thus obtain

$$Q_1 \sim \frac{\lambda^{-7/4}\tilde{S}^{1/4}}{2\sqrt{\pi}K^{3/4}}\omega^{3/2}\exp\left(-\frac{\lambda K}{\omega}\right)\exp\left(\frac{2\sqrt{\lambda}\sqrt{\tilde{S}K}}{\omega}\right)e^{-\tilde{S}/\omega}. \tag{3.50}$$

For Q_2, we scale $z = W/\rho$ and the integral in 3.5 becomes, in the limit as $\rho \to 0$,

$$Q_2 \sim \frac{K\rho}{2\pi i} \int_{Br} \frac{e^{S\theta}}{\theta^2} \int_{\frac{\theta}{1+\theta t}}^{\theta} \frac{1}{W+\rho} \exp\left[-W\alpha\left(t + \frac{1}{\theta} - \frac{1}{W} + O(\rho)\right)\right] \, dW \, d\theta. \tag{3.51}$$

With the further scaling

$$S = \frac{\tilde{S}}{\lambda}, \; \theta = \tilde{\theta}\lambda, \; W = \lambda U \tag{3.52}$$

and setting $\alpha(t) = \tilde{\alpha}(\lambda t) = \tilde{\alpha}(\omega)$, so that

$$\alpha\left(t + \frac{1}{\theta} - \frac{1}{W}\right) = \tilde{\alpha}\left(\omega + \frac{1}{\tilde{\theta}} - \frac{1}{U}\right), \tag{3.53}$$

we obtain

$$Q_2 \sim \frac{K\rho}{\lambda} \frac{1}{2\pi i} \int_{Br} \frac{e^{\tilde{S}\tilde{\theta}}}{\tilde{\theta}^2} \int_{\frac{\tilde{\theta}}{1+\tilde{\theta}\omega}}^{\tilde{\theta}} \frac{1}{U} \exp\left[-\lambda U \tilde{\alpha}\left(\omega + \frac{1}{\tilde{\theta}} - \frac{1}{U}\right)\right] \, dU \, d\tilde{\theta}. \tag{3.54}$$

We rewrite 3.54 by splitting the range of integration into $U \in (\tilde{\theta}/[1 + \tilde{\theta}\omega], \; \tilde{\theta}/[1 + \tilde{\theta}(\omega - K)])$ and $U \in (\tilde{\theta}/[1 + \tilde{\theta}(\omega - K)], \; \tilde{\theta})$, thus obtaining

$$Q_2 \sim \frac{K\rho}{\lambda} \frac{1}{2\pi i} \int_{Br} \frac{e^{\tilde{S}\tilde{\theta}}}{\tilde{\theta}^2} \left(\int_{\frac{\tilde{\theta}}{1+\tilde{\theta}\omega}}^{\frac{\tilde{\theta}}{1+\tilde{\theta}(\omega-K)}} + \int_{\frac{\tilde{\theta}}{1+\tilde{\theta}(\omega-K)}}^{\tilde{\theta}}\right) \left(\frac{1}{U} \exp\left[-\lambda U \tilde{\alpha}\left(\omega + \frac{1}{\tilde{\theta}} - \frac{1}{U}\right)\right] \, dU\right) \, d\tilde{\theta}. \tag{3.55}$$

We note that in the first range the integral will be $o(1)$ as $\lambda \to \infty$, since $\omega + \tilde{\theta}^{-1} + U^{-1} < K$ whenever $\tilde{\theta} \leq \tilde{\theta}/[1 + \tilde{\theta}(\omega - K)]$, and thus $\tilde{\alpha}(\omega) \sim (\sqrt{K} - \sqrt{\omega})^2$ may be used to approximate

the integrand. In the second integral $\tilde{\alpha}$ will be exponentially small, and we may approximate $\exp(-\lambda U \tilde{\alpha}) \sim 1$. Hence we obtain

$$
\begin{aligned}
Q_2 &\sim \frac{K\rho}{\lambda} \frac{1}{2\pi i} \int_{Br} \frac{e^{\tilde{S}\tilde{\theta}}}{\tilde{\theta}^2} \int_{\frac{\tilde{\theta}}{1+\tilde{\theta}(\omega-K)}}^{\tilde{\theta}} \frac{1}{U} \, dU \, d\tilde{\theta} \\
&= \frac{K\rho}{\lambda} \frac{1}{2\pi i} \int_{Br} \frac{e^{\tilde{S}\tilde{\theta}}}{\tilde{\theta}^2} \log\left[1 + \tilde{\theta}(\omega - K)\right] \, d\tilde{\theta}.
\end{aligned}
\tag{3.56}
$$

Putting 3.50 and 3.56 together, we hence obtain

$$
\begin{aligned}
P(\tilde{S}, \omega) &\sim K - \frac{\tilde{S}}{\lambda} + \frac{\lambda^{-7/4}\tilde{S}^{1/4}}{2\sqrt{\pi}K^{3/4}} \omega^{3/2} \exp\left(-\frac{\lambda K}{\omega}\right) \exp\left(\frac{2\sqrt{\lambda}\sqrt{\tilde{S}K}}{\omega}\right) e^{-S/\omega} \\
&\quad - \frac{e^{-\lambda}}{\lambda} \frac{K}{2\pi i} \int_{Br} \frac{e^{\tilde{\theta}\tilde{S}}}{\tilde{\theta}^2} \log[1 + \tilde{\theta}(\omega - K)] \, d\tilde{\theta}, \quad S = \frac{\tilde{S}}{\lambda} = O(\lambda^{-1}).
\end{aligned}
\tag{3.57}
$$

Note that for $\omega > K$ and $\tilde{S} > 0$ we have $P(S,t) - (K - S) \sim Q_1$, as the integral in 3.57 is roughly of order $O(e^{-\lambda})$ while roughly $Q_1 = O(\exp(-\lambda K/\omega))$.

For the case where $S = O(\lambda^{-2})$, we scale

$$
\theta = -\frac{1}{\omega} + U
\tag{3.58}
$$

in 3.11 to obtain

$$
Q_1 \sim \frac{1}{2\pi i} \exp\left(-\frac{\lambda S}{\omega} - \frac{\lambda K}{\omega}\right) \int_{Br} \frac{e^{\lambda S U} e^{\lambda K/(\omega^2 U)}}{(U - \frac{1}{\omega})^2} \, dU.
\tag{3.59}
$$

After further scaling

$$S = \frac{R}{\lambda^2}, \ U = \lambda\nu \tag{3.60}$$

we obtain

$$Q_1 \sim \frac{1}{\lambda^2} \exp\left(-\frac{K\lambda}{\omega}\right) \frac{1}{2\pi i} \int_{Br} \frac{e^{R\nu} e^{K/(\omega^2\nu)}}{\nu^2} \, d\nu. \tag{3.61}$$

Evaluating the contour integral in 3.61 as a modified Bessel function, we obtain

$$Q_1 \sim \exp\left(-\frac{K\lambda}{\omega}\right) \frac{1}{\lambda^2} \omega \frac{\sqrt{R}}{\sqrt{K}} \, I_1\left(2\sqrt{\frac{KR}{\omega^2}}\right), \ S = \frac{R}{\lambda^2} = O(\lambda^{-2}). \tag{3.62}$$

Note that if $\omega = K + \Lambda/\lambda$, then 3.62 agrees with 3.45, so that the latter is a special case of 3.62.

Now we proceed to analyze Q_2 for $S = O(\lambda^{-2})$. On this scale Q_2 does not experience a transition and the result can be obtained by simply expanding 3.56 for $S = \tilde{S}/\lambda \to 0$. By expanding the integrand in 3.56 for θ large we obtain

$$Q_2 \sim \frac{K\rho}{\lambda^2} R[\log(\omega - K) - R\log R + \log\lambda + 1 - \gamma]. \tag{3.63}$$

Combining 3.62 and 3.63, we obtain on the $S = O(\lambda^{-2})$ scale

$$P(S,t) \sim K - S + \exp\left(-\frac{K\lambda}{\omega}\right) \frac{1}{\lambda^2} \omega \frac{\sqrt{R}}{\sqrt{K}} \, I_1\left(2\sqrt{\frac{KR}{\omega^2}}\right)$$

$$- \frac{K\rho}{\lambda^2} R[\log(\omega - K) - R\log R + \log\lambda + 1 - \gamma], \ S = \frac{R}{\lambda^2} = O(\lambda^{-2}). \tag{3.64}$$

Note that for $R > 0$ we still have $P(S,t) - (K - S) \sim Q_1$ as $Q_1 \gg Q_2$. But when S (and hence R) becomes exponentially small, of the order of $\alpha(t)$ in 2.19, Q_1 and Q_2 will become of comparable magnitude.

Finally we need to analyze near the free boundary $\alpha(t)$, where $S = O(\alpha)$. We return to 2.12 and 2.13 and scale $z \to z/\rho$ to obtain

$$P(S,t) - (K - S) = \frac{K\rho}{2\pi i} \int_{Br} \frac{e^{S\theta}}{\theta^2} \int_{\theta}^{\infty} \frac{1}{z+\rho} \exp\left[-z\alpha\left(t + \frac{1}{\rho}\log\left(\frac{\theta+\rho}{\theta}\frac{z}{z+\rho}\right)\right)\right] \, dz \, d\theta.$$

$$(3.65)$$

In 3.65 we expand the argument α for small ρ using

$$\alpha\left[t + \frac{1}{\rho}\log\left(\frac{\theta+\rho}{\theta}\frac{z}{z+\rho}\right)\right] = \alpha\left(t + \frac{1}{\theta} - \frac{1}{z} + O\left(\frac{\rho}{\theta^2}, \frac{\rho}{z^2}\right)\right)$$

$$(3.66)$$

Then we scale S to be small, with $S = O(\alpha)$, and scale θ and z to be large, with $\theta = O(\alpha^{-1})$ and $z = O(\alpha^{-1})$. Thus we introduce (U, θ', z') where

$$S = \alpha(t)(U + 1), \ \theta = \frac{\theta'}{\alpha(t)}, z = \frac{z'}{\alpha(t)}.$$

$$(3.67)$$

Then using the asymptotic relation $\alpha(t + 1/\theta - 1/z + O(\rho)) \sim \alpha(t)$ we obtain from 3.66 and 3.67

$$P(S,t) - (K - S) \sim \alpha\rho K \frac{1}{2\pi i} \int_{Br} \frac{e^{(U+1)\theta'}}{(\theta')^2} \left(\int_{\theta'}^{\infty} \frac{e^{-z'}}{z'} \, dz'\right) \, d\theta'.$$

$$(3.68)$$

By setting $z' = \theta' W$ we can explicitly evaluate the integral in 3.68 as

$$
\begin{aligned}
P(S,t) - (K-S) &\sim \alpha\rho K \frac{1}{2\pi i} \int_{Br} \frac{e^{(U+1)\theta'}}{(\theta')^2} \left(\int_1^\infty \frac{e^{-\theta' W}}{W} \, dW \right) d\theta' \\
&= \alpha\rho K \int_1^\infty \frac{1}{W} \left[\frac{1}{2\pi i} \int_{Br} \frac{e^{(U+1-W)\theta'}}{(\theta')^2} \, d\theta' \right] dW \\
&= \alpha\rho K \int_1^\infty \frac{U+1-W}{W} \mathbf{1}_{\{U+1>W\}} \, dW \\
&= \alpha\rho K \int_1^{U+1} \left(\frac{U+1}{W} - 1 \right) dW \\
&= \alpha\rho K [(U+1)\log(U+1) - U].
\end{aligned}
$$
(3.69)

Here $\mathbf{1}_{\{\cdot\}}$ is the indicator function.

To summarize, on the scale $t = \omega/\lambda = O(\lambda^{-1})$, $\omega > K$, and $S = O(\alpha)$, we have obtained 3.69, which in terms of S becomes

$$
P(S,t) \sim K - S + \rho K S \log S - \rho K S \log \alpha - \rho K S + \alpha\rho K,
$$
(3.70)

where $\alpha(t)$ is now asymptotically given by 2.19.

3.3 Analysis for $t = O(1)$

We next consider the time scale $t = O(1)$. In this regime, the expansions in 3.25 and 3.26 are no longer valid, and we will need to consider the two spatial scales $S = O(1)$ and $S = O(\alpha)$.

For $t = O(1)$ and $S = O(1)$, we re-consider Q_1 in 3.6, expand the integrand for $\rho \to 0$, and keep the leading two terms to obtain

$$Q_1 = \frac{1}{2\pi i} \int_{Br} \frac{e^{S\theta}}{\theta^2} \exp\left(-\frac{K\theta}{1+t\theta}\right) \, d\theta + \rho \frac{Kt}{2\pi i} \int_{Br} \frac{e^{S\theta}}{\theta^2} \frac{1+\frac{t\theta}{2}}{(1+t\theta)^2} \exp\left(-\frac{K\theta}{1+t\theta}\right) \, d\theta + O(\rho^2),$$

(3.71)

For $t = O(1)$ the free boundary α is exponentially small, as given by 2.20, and thus in the integrand for Q_2 in 3.5 we may approximate $\exp(-\rho z \alpha) \sim 1$ to obtain

$$Q_2 \sim \frac{K\rho}{2\pi i} \int_{Br} \frac{e^{S\theta}}{\theta^2} \left[\log\left(\frac{\theta}{\rho}+1\right) - \log\left(\frac{\theta}{(\theta+\rho)e^{\rho t}-\theta}+1\right)\right] \, d\theta.$$

(3.72)

Using the relation

$$\log\left(\frac{\theta}{\rho}+1\right) - \log\left(\frac{\theta}{(\theta+\rho)e^{\rho t}-\theta}+1\right) = \log(1+t\theta) + O(\rho),$$

(3.73)

we obtain the leading term for Q_2 as

$$Q_2 \sim \frac{K\rho}{2\pi i} \int_{Br} \frac{e^{S\theta}}{\theta^2} \log(1+t\theta) \, d\theta.$$

(3.74)

We note that $Q_2 = O(\rho)$ and is thus comparable to the correction term in 3.71.

To summarize, by using 3.71 and 3.74 in 3.4, we have obtained for $S, t > 0$

$$
\begin{aligned}
P(S,t) \sim K - S &+ \frac{1}{2\pi i} \int_{Br} \frac{e^{S\theta}}{\theta^2} \exp\left(-\frac{K\theta}{1+t\theta}\right) d\theta \\
&+ \rho \frac{Kt}{2\pi i} \int_{Br} \frac{e^{S\theta}}{\theta^2} \frac{1+\frac{t\theta}{2}}{(1+t\theta)^2} \exp\left(-\frac{K\theta}{1+t\theta}\right) d\theta \\
&- \rho \frac{K}{2\pi i} \int_{Br} \frac{e^{S\theta}}{\theta^2} \log(1+t\theta) \, d\theta.
\end{aligned} \tag{3.75}
$$

The expansion in 3.75 ceases to be valid for small values of S, since the last terms behave as $O(\rho S \log S)$ and if S is sufficiently small this becomes comparable to the $O(1)$ leading term in 3.75. For $S = O(\alpha)$ we again set $S = \alpha(U+1)$ where now $\alpha(t)$ is given asymptotically by 2.20, and a calculation completely analogous to 3.69 shows that 3.69 and 3.70 remain valid as long as we use 2.20 to compute $\alpha(t)$.

3.4 Analysis for $t = v/\rho = \mathbf{O}(\rho^{-1})$

For the large time behavior of $P(S,t)$, on the scale $t = v/\rho = O(\rho^{-1})$, there are two spatial scales to consider, namely $S = O(\rho^{-1})$ and $S = O(\alpha)$. For the former, we scale

$$
t = \frac{v}{\rho}, \; S = \frac{\Omega}{\rho}, \; \theta = \rho\hat{\theta} \tag{3.76}
$$

and use 3.76 in 3.6 to obtain

$$
\begin{aligned}
Q_1 &= \frac{1}{\rho}\frac{1}{2\pi i}\int_{Br}\frac{e^{\Omega\hat\theta}}{\hat\theta^2}\exp\left[-\frac{\rho K\hat\theta}{(\hat\theta+1)e^v-\hat\theta}\right]\,d\hat\theta \\
&= \frac{1}{\rho}\frac{1}{2\pi i}\int_{Br}\frac{e^{\Omega\hat\theta}}{\hat\theta^2}\left[1-\frac{\rho K\hat\theta}{(\hat\theta+1)e^v-\hat\theta}+O(\rho^2)\right]\,d\hat\theta \\
&= \frac{1}{\rho}\frac{1}{2\pi i}\int_{Br}\frac{e^{\Omega\hat\theta}}{\hat\theta^2}\,d\hat\theta-\frac{1}{2\pi i}\int_{Br}\frac{e^{\Omega\hat\theta}}{\hat\theta^2}\frac{K\hat\theta}{(\hat\theta+1)e^v-\hat\theta}\,d\hat\theta+O(\rho) \\
&= \frac{\Omega}{\rho}-\frac{K}{2\pi i}\int_{Br}\frac{e^{\Omega\hat\theta}}{\hat\theta}\frac{1}{(\hat\theta+1)e^v-\hat\theta}\,d\hat\theta+O(\rho) \\
&= \frac{\Omega}{\rho}-Ke^{-v}+Ke^{-v}\exp\left(-\frac{\Omega}{1-e^{-v}}\right)+O(\rho).
\end{aligned}
\tag{3.77}
$$

For Q_2, we use the scaling in 3.76 in the integral in 3.5. The free boundary α is still exponentially small, as given in 2.21, and thus in the integrand for Q_2 we may again approximate $\exp(-\rho z\alpha)\sim 1$ to obtain

$$
\begin{aligned}
Q_2 &\sim \frac{K}{2\pi i}\int_{Br}\frac{e^{\Omega\hat\theta}}{\hat\theta^2}\left[\log(\hat\theta+1)-\log\left(\frac{\hat\theta}{(\hat\theta+1)e^v-\hat\theta}+1\right)\right]\,d\hat\theta \\
&= \frac{K}{2\pi i}\int_{Br}\frac{e^{\Omega\hat\theta}}{\hat\theta^2}\log\left[\frac{(\hat\theta+1)e^v-\hat\theta}{e^v}\right]\,d\hat\theta \\
&= \frac{K}{2\pi i}\int_{Br}\frac{e^{\Omega\hat\theta}}{\hat\theta^2}\log[(\hat\theta+1)-\hat\theta e^{-v}]\,d\hat\theta.
\end{aligned}
\tag{3.78}
$$

To summarize, by using 3.77 and 3.78 in 3.4, we have obtained on the scale $S=\Omega/\rho$, $t=v/\rho$ for ρ small,

$$
P(S,t)\sim K-Ke^{-v}+Ke^{-v}\exp\left(-\frac{\Omega}{1-e^{-v}}\right)-\frac{K}{2\pi i}\int_{Br}\frac{e^{\Omega\hat\theta}}{\hat\theta^2}\log[(\hat\theta+1)-\hat\theta e^{-v}]\,d\hat\theta.
\tag{3.79}
$$

Note that for $S, t = O(\rho^{-1})$, both $Q_1 - \Omega/\rho$ and Q_2 are $O(1)$, and so is $P(S, t)$. On the large/space time scales $S, t = O(\rho^{-1})$ the relation $P(S, t) \sim K - S$ no longer holds.

The expansion in 3.79 ceases to be valid for small values of Ω. For $S = O(\alpha)$ (thus $\Omega = O(\rho\alpha)$) we once more set $S = \alpha(U + 1)$ where now $\alpha(t)$ is given asymptotically in 2.21, and an argument analogous to that for the time scales $t = O(\lambda^{-1})$ with $\omega > K$ and $t = O(1)$ shows that 3.69 and 3.70 remain valid as long as 2.21 is used to asymptotically compute $\alpha(t) = \alpha(v/\rho)$.

CHAPTER 4

PERTURBATION METHODS

We next re-examine the free boundary $\alpha(t)$ and option price $P(S,t)$, by using singular perturbation methods to analyze the basic boundary value problem, which we repeat below:

$$P_t = SP_{SS} + \rho SP_S - \rho P; \ t > 0, \ S > \alpha(t) \tag{4.1}$$

$$P(S,0) = \max(K - S, 0), \ P(\infty, t) = 0 \tag{4.2}$$

$$P(\alpha(t), t) = K - \alpha(t), \ P_S(\alpha(t), t) = -1. \tag{4.3}$$

We shall analyze 4.1 - 4.3 for increasing ranges of time t, starting with $t = O(\lambda^{-1}) = O\left([\log(1/\rho)]^{-1}\right)$ and ending with the large time scale $t = O(\rho^{-1})$. Within each time range it will be necessary to analyze several different ranges of S, including boundary layers near the free boundary where $S \sim \alpha(t)$.

4.1 Analysis for $t = \omega/\lambda$, for $0 < \omega < K$

On this time scale, we make a change of variables

$$P(S,t) = K - S + \tilde{P}(S, \omega), \tag{4.4}$$

where $\tilde{P}(S,\omega)$ satisfies the following boundary value problem

$$\lambda \tilde{P_\omega} = S\tilde{P_{SS}} + e^{-\lambda}S\tilde{P_S} - e^{-\lambda}\tilde{P} - e^{-\lambda}K \tag{4.5}$$

$$\tilde{P}(S,0) = \max(S - K, 0). \tag{4.6}$$

We shall consider the boundary conditions in 4.3 later.

For $S > \alpha(t)$, we employ the ray method (13) by assuming an expansion of the form

$$\tilde{P}(S,\omega) \sim \lambda^\nu \mathcal{L}(S,\omega)e^{\lambda\phi(S,\omega)}. \tag{4.7}$$

We make the assumption that $-1 < \phi < 0$, so that $\tilde{P} \gg e^{-\lambda}$, the inhomogeneous term $(= -\rho K)$ in 4.5 will be negligible, and, up to an exponentially small error, 4.5 becomes $\lambda\tilde{P_\omega} \sim S\tilde{P_{SS}}$. Moreover, ν is a constant that will be determined by appropriate matching conditions. Using 4.7 in 4.5, we obtain the eikonal equation for ϕ

$$\phi_\omega = \phi_S^2 S \tag{4.8}$$

and the transport equation for $\mathcal{L}$

$$\mathcal{L}_\omega = 2S\phi_S\mathcal{L}_S + S\phi_{SS}\mathcal{L}. \tag{4.9}$$

We shall consider two solutions to 4.8, corresponding to two different ray families. The first family corresponds to rays that start from $\omega = 0$, and the corresponding solution must satisfy the boundary condition in 4.6. Then $\phi(S,0) = 0$ and $\mathcal{L}(S,0) = \max(S - K, 0)$, with $\nu = 0$. The rays from $\omega = 0$ are vertical lines and they lead to the asymptotic solution $\tilde{P}_I$

$$\tilde{P}_I(S,\omega) \sim \max(S - K, 0). \tag{4.10}$$

The second ray family corresponds to rays that start at the point $(S, \omega) = (K, 0)$. These rays are given by

$$b\omega^2 = (\sqrt{K} - \sqrt{S})^2, \tag{4.11}$$

where the parameter b indexes the family. They lead to the solution

$$\phi(S,\omega) = -\frac{(\sqrt{K} - \sqrt{S})^2}{\omega}. \tag{4.12}$$

Using 4.12 in 4.9, we find that the most general solution to 4.9 is

$$\mathcal{L}(S,\omega) = f\left(\frac{(\sqrt{S} - \sqrt{K})^2}{\omega^2}\right) \frac{S^{1/4}}{\sqrt{\omega}} \tag{4.13}$$

where $f(\cdot)$ will be determined later by asymptotic matching conditions. Superimposing the solutions for the two ray families, we have thus shown that

$$\tilde{P}(S,\omega) \sim S - K + \lambda^\nu f\left(\frac{(\sqrt{S} - \sqrt{K})^2}{\omega^2}\right) \frac{S^{1/4}}{\sqrt{\omega}} e^{\lambda\phi(S,\omega)}, \ S > K \tag{4.14}$$

and

$$\tilde{P}(S,\omega) \sim \lambda^\nu f\left(\frac{(\sqrt{S}-\sqrt{K})^2}{\omega^2}\right)\frac{S^{1/4}}{\sqrt{\omega}}e^{\lambda\phi(S,\omega)}, \ S < K. \tag{4.15}$$

We next determine $f(\cdot)$ and ν, by considering another scale, near the "corner" $(S,\omega) = (K,0)$. We consider the scaling

$$\omega = \frac{\tilde{t}}{\lambda}, \ S = K + \frac{V}{\lambda}, \ \tilde{P}(S,\omega) \sim \frac{1}{\lambda}P_*(V,\tilde{t}). \tag{4.16}$$

To leading order, 4.5 and 4.6 yield

$$\frac{\partial P_*}{\partial \tilde{t}} = K\frac{\partial^2 P_*}{\partial V^2} \tag{4.17}$$

$$P_*(V,0) = \max(V,0). \tag{4.18}$$

This is a standard heat equation, whose solution is given by

$$P_*(V,\tilde{t}) = \frac{1}{\sqrt{4\pi K\tilde{t}}}\int_0^\infty \exp\left[-\frac{(V-V')^2}{4K\tilde{t}}\right]V' \ dV'. \tag{4.19}$$

By setting

$$V' = V + 2\sqrt{K\tilde{t}}\,\xi \tag{4.20}$$

we can rewrite 4.19 as

$$P_*(V,\tilde{t}) = \frac{V}{\sqrt{\pi}}\int_{-\frac{V}{2\sqrt{K\tilde{t}}}}^\infty e^{-\xi^2}d\xi + \frac{\sqrt{K\tilde{t}}}{\sqrt{\pi}}\exp\left[-\frac{V^2}{4K\tilde{t}}\right]. \tag{4.21}$$

We match the expansion on the $(V, \tilde{t})$ scale to the ray expansion 4.15 for $S < K$. The matching to 4.14 for $S > K$ will then be automatically satisfied. We thus consider an intermediate limit where $S \to K$, $\omega \to 0$, but with $V \to -\infty$, $\tilde{t} \to \infty$. Expanding 4.15 for $(S, \omega) \to (K, 0)$ and setting $S = K + V/\lambda$ yields

$$\tilde{P}(S, \omega) \sim \lambda^{\nu} f\left(\frac{V^2}{4K\tilde{t}^2}\right) K^{1/4} \left(\frac{\lambda}{\tilde{t}}\right)^{1/2} \exp\left(-\frac{V^2}{4K\tilde{t}}\right). \tag{4.22}$$

For $(V, \tilde{t}) \to (-\infty, \infty)$, 4.21 shows that

$$\frac{1}{\lambda} P_*(V, \tilde{t}) \sim \frac{1}{4\lambda} e^{-\frac{V^2}{4K\tilde{t}}} \left(\frac{2\sqrt{K\tilde{t}}}{V}\right)^3 \frac{V}{\sqrt{\pi}}. \tag{4.23}$$

By comparing 4.22 to 4.23, we determine ν and $f(\cdot)$ as

$$\nu = -\frac{3}{2}, \quad f(z) = \frac{1}{z} \frac{K^{1/4}}{2\sqrt{\pi}}. \tag{4.24}$$

Hence our final expansions for $P = \tilde{P} + K - S$ are for $\omega > 0$

$$P(S, t) \sim K - S + \frac{\lambda^{-3/2}(SK)^{1/4}}{2\sqrt{\pi}} \frac{\omega^{3/2}}{(\sqrt{S} - \sqrt{K})^2} \exp\left[-\frac{\lambda}{\omega}(\sqrt{S} - \sqrt{K})^2\right], \quad S < K \tag{4.25}$$

and

$$P(S, t) \sim \frac{\lambda^{-3/2}}{2\sqrt{\pi}} \frac{(SK)^{1/4}\omega^{3/2}}{(\sqrt{S} - \sqrt{K})^2} \exp\left[-\frac{\lambda}{\omega}(\sqrt{S} - \sqrt{K})^2\right], \quad S > K. \tag{4.26}$$

For $\omega = O(\lambda^{-1})$ (thus $t = O(\lambda^{-2})$) and $S - K = O(\lambda^{-1})$ we must use 4.16 and 4.21. The above analysis does not apply in the case where $S = K$, since 4.25 and 4.26 develop singularities there. For $\omega > 0$ we construct a transition layer of thickness $O(\lambda^{-1/2})$. As such, we scale

$$S = K + \frac{\zeta}{\sqrt{\lambda}}, \ P(S,t) \sim \frac{1}{\sqrt{\lambda}}\mathcal{P}(\zeta,\omega), \tag{4.27}$$

and to leading order the PDE in 4.1 becomes

$$\mathcal{P}_\omega = K\mathcal{P}_{\zeta\zeta}. \tag{4.28}$$

The initial condition becomes $\mathcal{P}(\zeta,0) = -\zeta$ for $\zeta < 0$, with $\mathcal{P}(\zeta,0) = 0$ for $\zeta > 0$. We can construct a solution that matches to 4.25 as $S \uparrow K$ and to 4.26 as $S \downarrow K$ by using the similarity variable $u = \zeta/\sqrt{\omega K}$. We thus set

$$\mathcal{P}(\zeta,\omega) = \zeta f_0\left(\frac{\zeta}{\sqrt{\omega K}}\right) = \zeta f_0(u) \tag{4.29}$$

which leads to the second order ODE

$$f_0''(u) + \frac{2}{u}f_0'(u) + \frac{2}{u}f_0'(u) = 0. \tag{4.30}$$

Solving 4.30 for $f_0(\cdot)$, we obtain

$$f_0(u) = C\int_u^\infty \frac{1}{u^2}e^{-\frac{u^2}{4}} \ du, \ \zeta > 0, \tag{4.31}$$

and this can be continued to the range $\zeta < 0$. The constant C is fixed by matching 4.31 to 4.25 in the intermediate limit $\zeta \to \infty$ and $S \to K$. By expanding 4.25 for $S \to K$ we conclude that

$$\mathcal{P} \sim \frac{2K^{\frac{3}{2}}\omega^{\frac{3}{2}}}{\sqrt{\pi}\zeta^2} \exp\left(-\frac{\zeta^2}{4\omega K}\right). \tag{4.32}$$

By expanding 4.31 for $\zeta \to \infty$ we see that 4.32 holds provided that $C = 1/\sqrt{\pi}$. We thus have

$$\mathcal{P}(\zeta,\omega) = \frac{\zeta}{\sqrt{\pi}} \int_{\frac{S}{\sqrt{\omega K}}}^{\infty} \frac{1}{u^2} e^{-\frac{u^2}{4}} \, du, \ \zeta > 0, \tag{4.33}$$

and hence, after integrating by parts, the expansion for P for $S - K = O(\lambda^{-1/2})$ can be written as

$$P(S,t) \sim \frac{1}{\sqrt{\lambda}} \frac{1}{\sqrt{\pi}} \left[\sqrt{\omega K} e^{-\frac{\zeta^2}{4\omega K}} - \frac{\zeta}{2} \int_{\frac{\zeta}{\sqrt{\omega K}}}^{\infty} e^{-\frac{u^2}{4}} \, du \right], \tag{4.34}$$

and this applies for all ζ.

The ray expansion for $S < K$ does not satisfy the boundary conditions in 4.3 at the free boundary. Hence we construct a boundary layer correction to it. In this layer we use the scaling

$$\alpha(t) = \alpha_*(\omega), \ S - \alpha(t) = \frac{v}{\lambda}, \ P(S,t) \sim K - S + \frac{\rho}{\lambda^2} P_0(v,\omega). \tag{4.35}$$

To leading order we obtain from 4.5

$$0 = \alpha_*(\omega) P_{0,vv} + \alpha'_*(\omega) P_{0,v} - K. \tag{4.36}$$

Here we used $\partial_S \sim \lambda \partial_v$ and $\partial_\omega \sim \lambda \alpha_*'(\omega)\partial_v$ and we note that the forcing term $-\rho K = -e^{-\lambda}K$ now appears in 4.36, whereas this was negligible in the ray expansions. From 4.3 and 4.4 we obtain the boundary conditions

$$P_0(0,\omega) = P_{0,v}(0,\omega) = 0. \tag{4.37}$$

The solution to 4.36 and 4.37 is

$$P_0(v,\omega) = \frac{Kv}{\alpha_*'(\omega)} + \left\{ \exp\left[\frac{-\alpha_*'(\omega)v}{\alpha_*(\omega)}\right] - 1 \right\} \frac{\alpha_*(\omega)K}{(\alpha_*'(\omega))^2}. \tag{4.38}$$

For $v \to \infty$ and $\alpha_*'(\omega) < 0$ we obtain from 4.38

$$\frac{\rho}{\lambda^2} P_0 \sim \frac{\rho}{\lambda^2} \frac{\alpha_*(\omega)K}{(\alpha_*'(\omega))^2} \exp\left[\frac{-\alpha_*'(\omega)v}{\alpha_*(\omega)}\right]. \tag{4.39}$$

By matching $(\rho/\lambda^2)P_0$ for $v \to \infty$ to the expansion of $\tilde{P}(S,\omega)$ for $S > K$, with $S = \alpha_*(\omega)+v/\lambda$, we conclude that 4.39 must agree with

$$\frac{\lambda^{-3/2}(\alpha_*(\omega)K)^{1/4}}{2\sqrt{\pi}} \frac{\omega^{3/2}}{(\sqrt{\alpha_*(\omega)} - \sqrt{K})^2} \exp\left[-\frac{\lambda}{\omega}\left(\sqrt{\alpha_*(\omega) + \frac{v}{\lambda}} - \sqrt{K}\right)^2 \right]. \tag{4.40}$$

Using $\sqrt{\alpha_*(\omega) + u/\lambda} = \sqrt{\alpha_*(\omega)} + (1/2)\lambda^{-1}v/\sqrt{\alpha_*(\omega)} + O(\lambda^{-2})$ we compare the exponential orders of magnitude of 4.39 and 4.40 to conclude that

$$\rho = e^{-\lambda} \sim \exp\left[-\frac{\lambda}{\omega}(\sqrt{\alpha_*(\omega)} - \sqrt{K})^2 \right] \tag{4.41}$$

so that (for $\omega < K$)

$$\alpha_*(\omega) \sim (\sqrt{K} - \sqrt{\omega})^2. \tag{4.42}$$

This is consistent up to leading order with the asymptotic approximation for the free boundary $\alpha(t)$ on the $t = \omega/\lambda$ scale for $\omega < K$ in 2.14. Moreover, 4.42 is consistent with

$$\frac{\alpha_*'(\omega)}{\alpha_*(\omega)}v \sim \frac{1}{\omega}\frac{\sqrt{\alpha_*(\omega)} - \sqrt{K}}{\alpha_*(\omega)}u, \tag{4.43}$$

so that the functional dependence on v is asymptotically the same in 4.39 and 4.40. By refining the matching between 4.39 and 4.40 we obtain a three term approximation of the form

$$\alpha_*(\omega) = \alpha_*^{(0)}(\omega) + \frac{\log\lambda}{\lambda}\alpha_*^{(1)}(\omega) + \frac{1}{\lambda}\alpha_*^{(2)}(\omega) + o\left(\frac{1}{\lambda}\right), \tag{4.44}$$

where $\alpha_*^{(0)}$ is as in 4.42 and $\alpha_*^{(1)}$ and $\alpha_*^{(2)}$ are determined successively by substituting 4.44 in both 4.39 and 4.40 and matching like order terms in λ. By doing this, we obtain

$$\alpha_*^{(1)}(\omega) = \frac{1}{2}(\omega - \sqrt{K\omega}) \tag{4.45}$$

and

$$\alpha_*^{(2)}(\omega) = \frac{1}{2}\frac{\sqrt{K\omega} - \omega}{2}\log\left[\frac{4\pi K^2\omega}{K - \sqrt{K\omega}}\right]. \tag{4.46}$$

To summarize, for $S - \alpha(t) = v/\lambda = O(\lambda^{-1})$ we have obtained

$$P(S,t) \sim K - S + \frac{\rho}{\lambda^2}\left[Kv\frac{\sqrt{\omega}}{\sqrt{\omega} - \sqrt{K}} + K\omega\left\{\exp\left(\frac{v}{\sqrt{\omega}(\sqrt{K} - \sqrt{\omega})}\right) - 1\right\}\right]. \qquad (4.47)$$

This concludes the analysis of the time scale $t = O(\lambda^{-1})$.

4.2 $\quad$ Analysis for $t = K/\lambda + \mathbf{O}(\lambda^{-2})$, or $\omega = K + \mathbf{O}(\lambda^{-1})$

We next consider the ranges where $\omega \approx K$ (or $t \approx K/\lambda$). As $\omega \uparrow K$ the expansion in 4.44 breaks down since the leading term has a double zero at $\omega = K$ while the second term has a simple zero. When $\omega - K = O(\lambda^{-1})$ all of the three terms in 4.44 become of comparable magnitude, which suggests that the appropriate scaling is $\omega - K = O(\lambda^{-1})$ and that the free boundary should then be scaled as $\alpha = O(\lambda^{-2})$. We thus introduce the scaling

$$t = \frac{K}{\lambda} + \frac{\Lambda}{\lambda^2}, \ \omega = K + \frac{\Lambda}{\lambda} \qquad (4.48)$$

and then set

$$\alpha(t) \sim \frac{\mathcal{F}(\Lambda)}{\lambda^2}, \ S = \frac{R}{\lambda^2} = \frac{\mathcal{F}(\Lambda) + W}{\lambda^2}, P(S,t) \sim K - S + \frac{\rho}{\lambda^2}\tilde{Q}(W,\Lambda). \qquad (4.49)$$

We note that, as was the case when $\omega < K$ and $S < K$, $P(S,t) \sim K - S$ with a correction term that is $O(\rho/\lambda^2) = O(e^{-\lambda}/\lambda^2)$. The ray expansions in 4.25 and 4.26 remain valid for any $\omega > 0$, and we shall see that only the boundary layer near $S = \alpha(t)$ breaks down (cf. 4.47) as $\omega \uparrow K$.

Using 4.49 in 4.1 we obtain to leading order the following PDE for $\tilde{Q}$

$$\tilde{Q}_\Lambda - \mathcal{F}'(\Lambda)\tilde{Q}_W = [W + \mathcal{F}(\Lambda)]\tilde{Q}_{WW} - K, \tag{4.50}$$

with the boundary conditions

$$\tilde{Q}(0, \Lambda) = \tilde{Q}_W(0, \Lambda) = 0. \tag{4.51}$$

We introduce the Laplace transform

$$Q(\phi, \Lambda) = \int_0^\infty e^{-W\phi}\tilde{Q}(W, \Lambda)\ dW, \tag{4.52}$$

and apply 4.52 to 4.50 and 4.51 to obtain

$$Q_\Lambda + \phi^2 Q_\phi = [\phi\mathcal{F}'(\Lambda) + \phi^2\mathcal{F}(\Lambda) - 2\phi]Q - \frac{K}{\phi}. \tag{4.53}$$

Since 4.53 is a first order PDE, we employ the method of characteristics, and obtain the characteristic equations as

$$\dot{\Lambda} = 1, \ \dot{\phi} = \phi^2 \tag{4.54}$$

whose solution is given by

$$\phi = \frac{1}{c - u}, \ \Lambda = u. \tag{4.55}$$

Here u measures how far we are along a particular characteristic and c indexes the family of curves. Using 4.55 in 4.53, we rewrite the PDE in terms of u and c, and obtain

$$\frac{d}{du}Q = \left(\frac{\mathcal{F}'(u)}{c-u} + \frac{1}{(c-u)^2}\mathcal{F}(u) - \frac{2}{c-u}\right)Q - K(c-u). \tag{4.56}$$

After rewriting this as an exact differential

$$\frac{d}{du}\left[\exp\left(-\frac{\mathcal{F}(u)}{c-u}\right)\frac{1}{(c-u)^2}Q\right] = -\frac{K}{c-u}\exp\left(-\frac{\mathcal{F}(u)}{c-u}\right), \tag{4.57}$$

we solve 4.57 for Q to obtain

$$Q = (c-u)^2 \exp\left(\frac{\mathcal{F}(u)}{c-u}\right)\int_u^{h(c)} \frac{K}{c-\eta}\exp\left(-\frac{\mathcal{F}(\eta)}{c-\eta}\right) d\eta, \tag{4.58}$$

where $h(c)$ is an arbitrary function. From 4.55, we obtain $c = \Lambda + 1/\phi$. Hence using the original variables (ϕ, Λ) in 4.58, we finally obtain, after the change of variables $\eta \to c - 1/\eta$,

$$Q(\phi, \Lambda) = \frac{K}{\phi^2}e^{\phi\mathcal{F}(\Lambda)}\int_\phi^{(c-h(c))^{-1}} \frac{1}{\eta}\exp\left[-\eta\mathcal{F}\left(\Lambda + \frac{1}{\phi} - \frac{1}{\eta}\right)\right] d\eta. \tag{4.59}$$

To have acceptable behavior as $\phi \to \infty$ we must choose $h(c) = c$ so that

$$Q(\phi, \Lambda) = \frac{K}{\phi^2}e^{\phi\mathcal{F}(\Lambda)}\int_\phi^{\infty} \frac{1}{\eta}\exp\left[-\eta\mathcal{F}\left(\Lambda + \frac{1}{\phi} - \frac{1}{\eta}\right)\right] d\eta. \tag{4.60}$$

Inverting the Laplace transform in 4.52 using

$$\tilde{Q}(W, \Lambda) = \frac{1}{2\pi i} \int_{Br} e^{W\phi} Q(\phi, \Lambda) \, d\phi \tag{4.61}$$

and 4.59, and setting $R = W + \mathcal{F}(\Lambda)$ (recall that $S = R/\lambda^2$), we obtain

$$\tilde{Q}(R, \Lambda) = \frac{K}{2\pi i} \int_{Br} \frac{e^{R\phi}}{\phi^2} \left\{ \int_{\phi}^{\infty} \frac{1}{\eta} \exp\left[-\eta \mathcal{F}\left(\Lambda + \frac{1}{\phi} - \frac{1}{\eta} \right) \right] d\eta \right\} d\phi, \tag{4.62}$$

where we now view $\tilde{Q}$ as a function of R rather than W. After the shift $\eta \to \phi + \eta$, 4.62 becomes

$$\tilde{Q}(R, \Lambda) = \frac{K}{2\pi i} \int_{Br} \frac{e^{R\phi}}{\phi^2} \left\{ \int_{0}^{\infty} \frac{1}{\phi + \eta} \exp\left[-(\phi + \eta) \mathcal{F}\left(\Lambda + \frac{1}{\phi} - \frac{1}{\phi + \eta} \right) \right] d\eta \right\} d\phi. \tag{4.63}$$

It remains to determine the function $\mathcal{F}(\Lambda)$. To this end we shall use an asymptotic matching argument to characterize $\mathcal{F}$ as the solution of an integral equation. We first note that by setting $\omega = K + \Lambda/\lambda$ in 4.44 and expanding for $\lambda \to \infty$, we conclude that

$$\mathcal{F}(\Lambda) \sim \frac{\Lambda^2}{4K} + \frac{\Lambda}{4} \log(-\Lambda) - \frac{\Lambda}{4} \log(8\pi K^3), \quad \Lambda \to -\infty, \tag{4.64}$$

which is the asymptotic matching condition for the free boundary. This will ensure that the expansion 4.44 matches to $\mathcal{F}(\Lambda)/\lambda^2$ in the intermediate limit where $\omega \uparrow K$ and $\Lambda = \lambda(\omega - K) \to -\infty$.

We shall apply asymptotic matching to the difference $P(S,t) - (K - S)$ in the limit where $R \to \infty$ and $\Lambda \to -\infty$, with $\omega \uparrow K$ and $S \to 0$. In the transform domain this will correspond to the limit $\Lambda \to -\infty$ and $\phi \to 0^+$ with $\Lambda + 1/\phi$ fixed. By recalling that 4.25 can be written as $P(S,t) - (K - S) = \tilde{P}(S,\omega)$, and setting $S = R/\lambda^2$ and $\omega = K + \Lambda/\lambda^2$, in the matching region we have

$$\tilde{P}(S,\omega) \sim \frac{e^{-\lambda}}{\lambda^2} e^{\Lambda/K} \frac{R^{1/4} K^{3/4}}{2\sqrt{\pi}} \exp\left(\frac{2\sqrt{R}}{\sqrt{K}}\right) \tag{4.65}$$

and thus

$$\tilde{Q}(R,\Lambda) \sim e^{\Lambda/K} \frac{R^{1/4} K^{3/4}}{2\sqrt{\pi}} \exp\left(\frac{2\sqrt{R}}{\sqrt{K}}\right), \quad R \to \infty, \ \Lambda \to -\infty. \tag{4.66}$$

Next we apply a Tauberian-like argument to obtain the analogous matching condition in the (ϕ, Λ) variables. If a function $g(R)$ has as $R \to \infty$ the sub-exponential behavior

$$g(R) \sim C e^{A\sqrt{R}} R^B, \ R \to \infty \tag{4.67}$$

with $A > 0$ and B, C are constants, its Laplace transform $\hat{g}(\phi)$ behaves as

$$\hat{g}(\phi) = \int_0^\infty e^{-\phi R} g(R) \, dR \sim \frac{C\sqrt{\pi} A^{1+2B}}{4^B \phi^{2B+\frac{3}{2}}} \exp\left(\frac{A^2}{4\phi}\right), \ \phi \to 0^+. \tag{4.68}$$

This can be seen, for example, by using 4.67 in $\hat{g}(\phi) = \int_0^\infty e^{-\phi R} g(R) \, dR$ and evaluating the resulting integral for $\phi \to 0^+$ by the Laplace method. Applying this argument to 4.66 with

$$A = \frac{2}{\sqrt{K}}, \ B = \frac{1}{4}, \ C = \frac{e^{\Lambda/K} K^{3/4}}{2\sqrt{\pi}} \tag{4.69}$$

we conclude that

$$Q(\phi, \Lambda) \sim \frac{1}{\phi^2} \exp\left(\frac{1}{K}\left(\Lambda + \frac{1}{\phi}\right)\right), \tag{4.70}$$

for $\Lambda \to -\infty, \phi \to 0^+$ with $\Lambda + 1/\phi$ fixed. Using 4.63 we see that in this limit

$$e^{\frac{1}{K}(\Lambda+\frac{1}{\phi})} \sim K \int_0^\infty \frac{1}{\phi+\eta} \exp\left[-(\phi+\eta)\mathcal{F}\left(\Lambda + \frac{1}{\phi} - \frac{1}{\phi+\eta}\right)\right] d\eta. \tag{4.71}$$

Expression 4.71 can be further simplified by writing the integral over $0 < \eta < \infty$ as an integral over $-\phi < \eta < \infty$, minus an integral over $-\phi < \eta < 0$. Thus the right side of 4.71 becomes, after some elementary substitutions,

$$K \int_0^\infty \frac{1}{\eta} \exp\left[-\eta\mathcal{F}\left(\Lambda + \frac{1}{\phi} - \frac{1}{\eta}\right)\right] d\eta - K \int_0^\phi \frac{1}{\eta} \exp\left[-\eta\mathcal{F}\left(\Lambda + \frac{1}{\phi} - \frac{1}{\eta}\right)\right] d\eta. \tag{4.72}$$

But, for $\phi \to 0^+$ and $\Lambda + 1/\phi = O(1)$ the argument of $\mathcal{F}(\cdot)$ in the second integral is negative and large, so that 4.64 can be used to show that this integral is negligible compared to the first integral in 4.72. Thus from 4.71 and 4.72 we conclude that $\mathcal{F}(\cdot)$ must satisfy the integral equation

$$\frac{1}{K}e^{\frac{Z}{K}} = \int_0^\infty \frac{1}{\eta} \exp\left[-\eta\mathcal{F}\left(Z - \frac{1}{\eta}\right)\right] d\eta, \quad -\infty < Z < \infty. \tag{4.73}$$

Note that if we set $\eta = \xi, Z = -\nu K^2$, then 4.73 is consistent with the result in 2.16.

Using 4.73 and 4.72 in 4.63 we thus summarize our results on the scale $S = R/\lambda^2 = O(\lambda^{-2})$ and $\omega - K = \Lambda/\lambda = O(\lambda^{-1})$, as

$$
\begin{aligned}
P(S,t) \sim K - S &+ \frac{e^{-\lambda}}{\lambda^2} \frac{1}{2\pi i} \int_{Br} \frac{1}{\phi^2} \exp\left(R\phi + \frac{1}{K\phi} + \frac{\Lambda}{K}\right) d\phi \\
&- \frac{e^{-\lambda}}{\lambda^2} \frac{K}{2\pi i} \int_{Br} \frac{e^{R\phi}}{\phi^2} \left\{ \int_0^\phi \frac{1}{\eta} \exp\left[-\eta \mathcal{F}\left(\Lambda + \frac{1}{\phi} - \frac{1}{\eta}\right)\right] d\eta \right\} d\phi
\end{aligned}
\tag{4.74}
$$

with $\mathcal{F}(\cdot)$ satisfying 4.73. The first integral in 4.74 may be evaluated explicitly as a Bessel function, using

$$
\frac{1}{2\pi i} \int_{Br} \frac{1}{\phi^2} \exp\left(R\phi + \frac{1}{K\phi} + \frac{\Lambda}{K}\right) d\phi = e^{\Lambda/K} \sqrt{KR}\, I_1\left(\frac{2\sqrt{R}}{\sqrt{K}}\right).
\tag{4.75}
$$

Finally, we can use 4.74 to verify the asymptotic matching, now for a fixed Λ, between the ray expansion in 4.25 and the expansion in 4.74. By setting $\omega = K + \Lambda/\lambda$ in 4.25, expanding first for $\lambda \to \infty$ and then for $S \to 0$, we again obtain 4.65. For $R \to \infty$ with $\Lambda = O(1)$ the second integral in 4.74 is again negligible and the first integral can be expanded using 4.75 and the formula

$$
I_1(z) \sim \frac{1}{\sqrt{2\pi z}} e^z, \ z \to \infty.
\tag{4.76}
$$

This verifies the matching for $S \to 0$ and $\lambda^2 S = R \to \infty$ with Λ fixed, i.e, the matching in the spatial variables on the present time scale.

4.3 Analysis for $t = \omega/\lambda$, for $K < \omega < \infty$

Next we consider $t = \omega/\lambda = O(1/\lambda)$ with now $\omega > K$. We shall see that in this time range the free boundary $\alpha(t)$ will become exponentially small in ρ as $\rho \to 0$. We also note that for $\Lambda \to \infty$ we can evaluate asymptotically the integral in 4.73 to show that

$$\frac{1}{\lambda^2}\mathcal{F}(\Lambda) \sim \frac{1}{\lambda^2}\Lambda e^{-\gamma} \exp\left[-\frac{1}{K}e^{\Lambda/K}\right], \ \Lambda \to \infty, \tag{4.77}$$

and this will yield an asymptotic matching condition for the free boundary on the present time scale. The rapid, double exponential, decay of $\mathcal{F}(\Lambda)$ also suggests that, for $\omega > K$, $\alpha(t)$ will become very small (in ρ).

For the time range $K < \omega < \infty$ it will be necessary to analyze several spatial scales, specifically $S = O(1)$, $S = O(\lambda^{-1})$, $S = O(\lambda^{-2})$, and $S = O(\alpha(t))$. Asymptotic matching arguments relating the last two scales will be used to ultimately derive an equation for $\alpha(t) = \alpha(\omega/\lambda)$.

When $S = O(1)$ the ray expansions in 4.25 and 4.26 still apply, as does the transition layer for $S - K = O(\lambda^{-1/2})$ in 4.34. We recall that the ray expansion in 4.7 does not account for the inhomogeneous term $(= -\rho K = -e^{-\lambda}K)$ in 4.5, since $\phi(S, \omega) = -(\sqrt{S} - \sqrt{K})^2/\omega > -1$. In fact the inhomogeneous term would not affect higher order correction terms to 4.7, which would involve powers of λ^{-1}. However, for sufficiently small S the forcing term in 4.5 will become important, and must be considered to determine $\alpha(t)$.

We thus consider the scale $S = O(\lambda^{-1})$, set $S = \tilde{S}/\lambda$ and write the solution as

$$P(S,t) = K - S + \bar{P}(\tilde{S}, \omega). \tag{4.78}$$

Using 4.78 in 4.5 we find that

$$\lambda \bar{P}_\omega = \lambda \tilde{S} \bar{P}_{\tilde{S}\tilde{S}} + e^{-\lambda}(\tilde{S} \bar{P}_{\tilde{S}} - \bar{P}) - K e^{-\lambda}. \tag{4.79}$$

Now we write

$$\bar{P}(\tilde{S}, \omega) = \bar{P}^{\text{hom}}(\tilde{S}, \omega) + \bar{P}^{\text{part}}(\tilde{S}, \omega) \tag{4.80}$$

where hom/part denote homogeneous and particular solutions to the PDE in 4.79, which asymptotically satisfy

$$\bar{P}^{\text{hom}}_\omega = \tilde{S} \bar{P}^{\text{hom}}_{\tilde{S}\tilde{S}} + O\left(\frac{e^{-\lambda}}{\lambda} \bar{P}^{\text{hom}}\right) \tag{4.81}$$

and

$$\bar{P}^{\text{part}}_\omega = \tilde{S} \bar{P}^{\text{part}}_{\tilde{S}\tilde{S}} - K \frac{e^{-\lambda}}{\lambda} + O\left(\frac{e^{-\lambda}}{\lambda} \bar{P}^{\text{part}}\right). \tag{4.82}$$

The analysis of 4.81 can again be done by using the ray method, but this will be the same as simply expanding 4.25 for $S \to 0$, so that

$$\bar{P}^{\text{hom}}(\tilde{S}, \omega) \sim \frac{\lambda^{-7/4} \tilde{S}^{1/4}}{2\sqrt{\pi} K^{3/4}} \omega^{3/2} \exp\left(-\frac{\lambda K}{\omega}\right) \exp\left(\frac{2\sqrt{\lambda}\sqrt{\tilde{S} K}}{\omega}\right) e^{-\tilde{S}/\omega}. \tag{4.83}$$

From 4.82 we scale the particular solution to be $O(e^{-\lambda}/\lambda)$ so we set

$$\bar{P}^{\text{part}}(\tilde{S}, \omega) \sim \frac{e^{-\lambda}}{\lambda} \mathcal{H}(\tilde{S}, \omega) \tag{4.84}$$

where $\mathcal{H}$ will satisfy

$$\mathcal{H}_\omega = \tilde{S} \mathcal{H}_{\tilde{S}\tilde{S}} - K; \ \ \tilde{S} > 0, \ \omega > K. \tag{4.85}$$

To solve 4.85 we need to impose initial and boundary conditions, or use asymptotic matching to the previous time range. We shall see that the two arguments lead to identical solutions for $\mathcal{H}$. First we argue that $\bar{P}$ in 4.80 should vanish as $\tilde{S} \to 0$, since it must vanish at the free boundary and the latter will be asymptotically small. Then, by 4.83, $\bar{P}^{\text{hom}} = 0$ at $\tilde{S} = 0$ so that $\bar{P}^{\text{part}}$ must also vanish and hence

$$\mathcal{H}(0, \omega) = 0, \ \omega > K. \tag{4.86}$$

Since the particular solution did not play a role in the time range $\omega \sim K$, we also argue that

$$\mathcal{H}(\tilde{S}, K) = 0, \ \tilde{S} > 0. \tag{4.87}$$

With 4.86 and 4.87 the solution to 4.85 is uniquely determined.

We solve 4.85 using a Laplace transform over the spatial variable $\tilde{S}$, setting

$$\widehat{\mathcal{H}}(\tilde{\theta}, \omega) = \int_0^\infty e^{-\tilde{\theta}\tilde{S}} \mathcal{H}(\tilde{S}, \omega) \ d\tilde{S}. \tag{4.88}$$

Using 4.88 along with 4.86 in 4.85 leads to

$$\widehat{\mathcal{H}}_\omega + \tilde{\theta}^2 \mathcal{H}_{\tilde{\theta}} + 2\tilde{\theta}\widehat{\mathcal{H}} = -\frac{K}{\tilde{\theta}}, \ \omega > K. \tag{4.89}$$

Using the method of characteristics we find that the most general solution to 4.89 is

$$\widehat{\mathcal{H}}(\tilde{\theta}, \omega) = \frac{K}{\tilde{\theta}^2} \left[\log\left(\frac{1}{\tilde{\theta}}\right) + f\left(\omega + \frac{1}{\tilde{\theta}}\right) \right] \tag{4.90}$$

where $f(\cdot)$ is an arbitrary function. Then the boundary condition given in 4.87 implies that $\widehat{H}(\tilde{\theta}, K) = 0$ so that $f(z) = -\log(z - K)$ and hence

$$\widehat{H}(\tilde{\theta}, \omega) = -\frac{K}{\tilde{\theta}^2} \log[1 + \tilde{\theta}(\omega - K)]. \tag{4.91}$$

Then inverting the transform and combining 4.84 with 4.83 we have, for $S = \tilde{S}/\lambda = O(\lambda^{-1})$,

$$P(S, t) \sim K - \frac{\tilde{S}}{\lambda} + \frac{\lambda^{-7/4}\tilde{S}^{1/4}}{2\sqrt{\pi}K^{3/4}} \omega^{3/2} \exp\left(-\frac{\lambda K}{\omega}\right) \exp\left(\frac{2\sqrt{\lambda}\sqrt{\tilde{S}K}}{\omega}\right) e^{-S/\omega}$$
$$- \frac{e^{-\lambda}}{\lambda} \frac{K\rho}{2\pi i} \int_{Br} \frac{e^{\tilde{\theta}\tilde{S}}}{\tilde{\theta}^2} \log[1 + \tilde{\theta}(\omega - K)] \ d\tilde{\theta}. \tag{4.92}$$

Here $\Re(\tilde{\theta}) > 0$ on the Bromwich contour Br.

Next we give an alternate argument for solving 4.85. We asymptotically match the expansion on the $(\tilde{S}, \omega)$ scale to that on the (R, Λ) scale, in an intermediate limit that has

$\tilde{S} \to 0, \omega \to K$ with $R = \lambda \tilde{S} \to \infty, \Lambda \to \infty$. On the (R, Λ) scale the expansion for $P - (K - S)$ follows from 4.74 and 4.75. As $R \to \infty$

$$\frac{e^{-\lambda}}{\lambda^2} e^{\Lambda/K} \sqrt{KR}\, I_1 \left(\frac{2\sqrt{R}}{\sqrt{K}} \right) = \frac{e^{-\lambda}}{\lambda^2} e^{\Lambda/K} \sqrt{\lambda K \tilde{S}}\, I_1 \left(\frac{2\sqrt{\lambda \tilde{S}}}{\sqrt{K}} \right)$$

$$\sim e^{-\lambda} e^{\Lambda/K} \frac{1}{2\sqrt{\pi}} \lambda^{-7/4} \tilde{S}^{1/4} K^{3/4} \exp \left(\frac{2\sqrt{\lambda \tilde{S}}}{\sqrt{K}} \right) \qquad (4.93)$$

so that the first integral in 4.74 will match to the ray expansion, or to $\bar{P}^{\mathrm{hom}}$, in 4.83. Thus the second integral in 4.74 must match to $\bar{P}^{\mathrm{part}}$ so that as $\tilde{S} \to 0$, $\omega \to K$

$$\mathcal{H}(\tilde{S}, \omega) \sim -\frac{1}{\lambda} \frac{K\rho}{2\pi i} \int_{Br} \frac{e^{R\phi}}{\phi^2} \left\{ \int_0^\phi \frac{1}{\eta} \exp \left[-\eta \mathcal{F} \left(\Lambda + \frac{1}{\phi} - \frac{1}{\eta} \right) \right] d\eta \right\} d\phi \qquad (4.94)$$

where the right side of 4.94 is to be expanded as $R \to \infty$ with $\Lambda \to \infty$.

Now consider the inner integral in 4.94 for $\phi \to 0$. We break up the integration range $[0, \phi]$ into $[0, \phi/(1 + \Lambda\phi)]$ and $[\phi/(1 + \Lambda\phi), \phi]$. In the first subinterval $\Lambda + 1/\phi - 1/\eta < 0$ and we can use the approximation in 4.64 to show that the integral over the first range is negligible. In the second range we will typically have $\Lambda + 1/\phi - 1/\eta$ large and positive, and then 4.77 shows that $\mathcal{F}(\cdot)$ will be exponentially small. Hence we have

$$\int_0^\phi \frac{1}{\eta} \exp \left[-\eta \mathcal{F} \left(\Lambda + \frac{1}{\phi} - \frac{1}{\eta} \right) \right] d\eta \sim \int_{\frac{\phi}{1+\Lambda\phi}}^\phi \frac{1}{\eta} \exp \left[-\eta \mathcal{F} \left(\Lambda + \frac{1}{\phi} - \frac{1}{\eta} \right) \right] d\eta$$

$$\sim \int_{\frac{\phi}{1+\Lambda\phi}}^\phi \frac{d\eta}{\eta} = \log(1 + \Lambda\phi). \qquad (4.95)$$

Using 4.95 in 4.94, scaling $\phi = \tilde{\theta}/\lambda$ and recalling that $\Lambda/\lambda = \omega - K$, 4.94 becomes

$$\mathcal{H}(\tilde{S}, \omega) \sim -\frac{K\rho}{2\pi i} \int_{Br} \frac{e^{\tilde{\theta}\tilde{S}}}{\tilde{\theta}^2} \log[1 + \tilde{\theta}(\omega - K)] \, d\tilde{\theta}; \ \tilde{S} \to 0, \ \omega \to K. \tag{4.96}$$

Thus we must solve 4.85 subject to 4.96, but the right side of 4.96 is an exact solution to 4.85 and is thus the solution we seek. This shows that the same answer is obtained whether we invoke the matching condition in 4.96 or the boundary/initial condition in 4.86 and 4.87.

Next we consider the scale $S = O(\lambda^{-2})$ and as before we let $S = R/\lambda^2$. We shall again decompose $P(S,t) - (K - S) = \check{P}(R, \omega)$ into a homogeneous solution $\check{P}^{\mathrm{hom}}$ and a particular solution $\check{P}^{\mathrm{part}}$. The latter can be obtained as a limiting case of $\bar{P}^{\mathrm{part}}$, for $\tilde{S} \to 0$. Letting $\tilde{S} \to 0$ in the integral in 4.92 corresponds to expanding the integrand for $\tilde{\theta} \to \infty$, so that

$$\begin{aligned}
\mathcal{H}(\tilde{S}, \omega) &= -\frac{K\rho}{2\pi i} \int_{Br} \frac{e^{\tilde{\theta}\tilde{S}}}{\tilde{\theta}^2} \left[\log \tilde{\theta} + \log(\omega - K) + O\left(\frac{1}{\tilde{\theta}}\right) \right] \\
&= -K[-\tilde{S}\log\tilde{S} + \tilde{S}(1 - \gamma) + \tilde{S}\log(\omega - K) + O(\tilde{S}^2)] \\
&= -\frac{K}{\lambda} R \left[1 - \gamma + \log(\omega - K) - \log\left(\frac{R}{\lambda}\right) \right] + O\left(\frac{R^2}{\lambda^2}\right),
\end{aligned} \tag{4.97}$$

where γ is the Euler constant.

The expansion of $\check{P}^{\mathrm{hom}}$ can no longer be obtained by the ray method. Let us set

$$\check{P}^{\mathrm{hom}}(R, \omega) = \exp\left(-\frac{\lambda K}{\omega}\right) \lambda^{-2} \mathcal{G}(R, \omega). \tag{4.98}$$

Then if 4.98 as $R \to \infty$ is to asymptotically match to 4.83 as $\tilde{S} \to 0$ we must have

$$\mathcal{G}(R,\omega) \sim \frac{R^{1/4}K^{-3/4}}{2\sqrt{\pi}}\omega^{3/2}\exp\left(\frac{2}{\omega}\sqrt{KR}\right), \ \ R \to \infty. \tag{4.99}$$

Using 4.98 we see that $\partial_\omega \check{P}^{\mathrm{hom}}(R,\omega) \sim \lambda K \omega^{-2}\check{P}^{\mathrm{hom}}(R,\omega)$ and hence on the (R,ω) scale 4.5 becomes to leading order

$$\frac{K}{\omega}\mathcal{G} = R\mathcal{G}_{RR}. \tag{4.100}$$

Note that again for $\omega > K$, $\exp(-\lambda K/\omega) \gg e^{-\lambda} = \rho$ so the forcing term in 4.5 does not appear to leading order. Its effect we have already estimated by 4.97.

In 4.100 the time variable ω appears only as a parameter, and this differential equation can be easily transformed into a Bessel equation, by setting

$$R = \frac{\omega^2}{K}\theta, \ V = 2\sqrt{\theta}, \ \mathcal{G} = \frac{V}{2}\mathcal{G}^*(V). \tag{4.101}$$

Using 4.101 in 4.100 we obtain

$$V^2\mathcal{G}^*_{VV} + V\mathcal{G}^*_V - (V^2 + 1)\mathcal{G}^* = 0 \tag{4.102}$$

which is a modified Bessel equation of order 1. We again expect that the free boundary will be small and $\mathcal{G}$ must vanish when R hits the free boundary, so that $\mathcal{G}(0,\omega) = 0$ and thus $V\mathcal{G}^*(V) \to 0$ as $V \to 0$. The solution to 4.102 is thus

$$\mathcal{G}^*(V) = C(\omega)I_1(V) \tag{4.103}$$

and hence

$$\mathcal{G}(R,\omega) = C(\omega)\frac{1}{\omega}\sqrt{RK}\ I_1\left(\frac{2}{\omega}\sqrt{RK}\right), \tag{4.104}$$

where $C(\cdot)$ is an arbitrary multiplicative function of ω. Using 4.76 to expand 4.104 for $R \to \infty$ and imposing the matching condition in 4.99 we obtain

$$C(\omega) = \omega^2 K^{-1}. \tag{4.105}$$

To summarize, for $S = R/\lambda^2 = O(\lambda^{-2})$ we have obtained

$$P(S,t) \sim K - \frac{R}{\lambda^2} + \exp\left(-\frac{\lambda K}{\omega}\right)\frac{1}{\lambda^2}\omega\sqrt{\frac{R}{K}}\ I_1\left(\frac{2}{\omega}\sqrt{KR}\right) \tag{4.106}$$

$$- \frac{Ke^{-\lambda}}{\lambda^2}R[1 - \gamma + \log(\omega - K) - \log R + \log \lambda]. \tag{4.107}$$

Since $\omega > K$ the term proportional to the Bessel function dominates the last term for all $R > 0$.

But as $R \to 0$ we use the fact that $I_1(z) \sim \frac{z}{2}$ as $z \to 0$ to approximate 4.106 and we obtain

$$P(S,t) - (K-S) \sim \exp\left(-\frac{\lambda K}{\omega}\right) S - K e^{-\lambda} S[1 - \gamma + \log(\omega - K) - \log S - \log \lambda], \quad (4.108)$$

which holds for $S = O(\lambda^{-2})$, and for convenience we wrote the answer in terms of S rather than R.

Up to now we have discussed the scales $S = O(1)$, $S = O(\lambda^{-1})$ and $S = O(\lambda^{-2})$, but have not determined the free boundary $\alpha(t)$. To this end we consider $S = O(\alpha)$ and introduce the variable U via

$$S = \alpha(U+1), \; P(S,t) = K - S + \acute{P}(U,\omega). \quad (4.109)$$

Then from 4.5 we obtain

$$\lambda \acute{P}_\omega - 2\frac{\alpha'}{\alpha} U \acute{P}_U = \frac{1}{\alpha}(U+1)\acute{P}_{UU} + e^{-\lambda}(U+1)\acute{P}_U - e^{-\lambda}\acute{P} - e^{-\lambda}K. \quad (4.110)$$

Since this expansion must hold when S is very small, we must satisfy both of the boundary conditions in 4.3 at the free boundary, which corresponds to $U = 0$. Thus,

$$\acute{P}(0,\omega) = \acute{P}_U(0,\omega) = 0. \quad (4.111)$$

Expecting α to be very small for $\rho = e^{-\lambda} \to 0$ and $\omega > K$, we replace 4.110 by the asymptotic relation

$$\frac{1}{\alpha}(U+1)\acute{P}_{UU} \sim e^{-\lambda}K, \tag{4.112}$$

as clearly the second derivative term, which contains the large factor α^{-1}, must be balanced by the forcing term in 4.110.

Solving 4.112 subject to 4.111 leads to

$$\acute{P} \sim e^{-\lambda}\alpha K[(U+1)\log(U+1) - U] \tag{4.113}$$

so that for $S = O(\alpha)$

$$P(S,t) \sim K - S + e^{-\lambda}\alpha K[(U+1)\log(U+1) - U]. \tag{4.114}$$

Finally we asymptotically match 4.114 as $U \to \infty$ to 4.106 as $R \to 0$, by comparing 4.108 to 4.114. Recalling that $U + 1 = S/\alpha$ and $U \sim U + 1$ for large U we thus have

$$e^{-\lambda}K[-S\log\alpha - S] \sim \exp\left(-\frac{\lambda K}{\omega}\right)S - Ke^{-\lambda}S[1 - \gamma + \log(\omega - K) - \log\lambda]. \tag{4.115}$$

Here we matched the difference $P - (K - S)$ and note that the terms proportional to $S\log S$ matched automatically. Dividing 4.115 by S, this relation yields

$$\exp\left(\lambda - \frac{\lambda K}{\omega}\right) \sim K[-\log\alpha + \log(\omega - K) - \log\lambda - \gamma] \tag{4.116}$$

and since $e^{-\lambda} = \rho$ we exponentiate 4.116 to obtain

$$
\begin{aligned}
\alpha &\sim \frac{\omega - K}{\lambda} e^{-\gamma} \exp\left[-\frac{1}{K} \exp\left(\lambda - \frac{\lambda K}{\omega} \right) \right] \\
&= \frac{\omega - K}{\lambda} e^{-\gamma} \exp\left[-\frac{1}{K} \rho^{K/\omega - 1} \right],
\end{aligned}
\tag{4.117}
$$

which shows that α is indeed exponentially small for $\omega > K$. Note that 4.117 is also consistent

with the balance we argued in 4.110. To summarize, for $S = \alpha(U + 1)$ we have obtained

$$
P(S,t) \sim K - S + K(\omega - K) e^{-\gamma} \frac{e^{-\lambda}}{\lambda} \exp\left(-\frac{1}{K} \rho^{K/\omega - 1} \right) [(U + 1)\log(U + 1) - U]. \tag{4.118}
$$

We shall proceed to analyze the larger time ranges $t = O(1)$ and $t = O(\rho^{-1})$, where once

again the free boundary α will be exponentially small, and the analysis for $S = O(\alpha)$ will

be essentially the same as that here. Finally we note that by setting $\omega = K + \Lambda/\lambda$ we have

$\rho^{K/\omega - 1} = \exp(-\lambda(K - \omega)/\omega) \sim \exp(\Lambda/K)$ so that 4.117 as $\omega \to K$ agrees with the right side

of 4.77, and hence the expressions for $\alpha(t)$ on the Λ and $\omega > K$ time scales asymptotically

match.

4.4   Analysis for $t = O(1)$

We next consider the time range where $t = O(1)$. On this scale, the ray expansions in 4.25

and 4.26 no longer hold. It will be necessary to consider two spatial scales, namely $S = O(1)$,

and $S = O(\alpha)$, where asymptotic arguments relating these two scales will once more be used

to derive an equation for $\alpha(t)$. We shall see that the free boundary will again be exponentially

small, but 4.117 is no longer valid.

For $t = O(1)$, $S = O(1)$, we assume a regular perturbation expansion for $P(S,t)$ of the form

$$P(S,t) = K - S + \dot{P}(S,t), \ \dot{P} = P_0(S,t) + \rho P_1(S,t) + O(\rho^2). \tag{4.119}$$

Applying this to 4.1, the PDE becomes to leading order

$$P_{0,t} = S P_{0,SS} \tag{4.120}$$

with the initial and boundary conditions given by

$$P_0(0,t) = 0, \ P_0(S,0) = \max(S - K, 0). \tag{4.121}$$

We will see that P_0 cannot satisfy the BC $P_{0,S}(0,t) = 0$, since we will have a boundary layer where $S = O(\alpha)$. The initial condition in 4.121 is equivalent to asymptotically matching 4.119 to the ray expansions on the ω time scale.

We once more introduce a Laplace transform over the spatial variable S, setting

$$\widehat{P}_0(\theta,t) = \int_0^\infty e^{-S\theta} P_0(S,t) \ dS. \tag{4.122}$$

Using 4.122 along with the boundary condition at $S = 0$ in 4.121 in 4.120 leads to

$$\widehat{P}_{0,t} + \theta^2 \widehat{P}_{0,\theta} = 2\theta \widehat{P}_0 \tag{4.123}$$

with the initial condition

$$\widehat{P}_0(\theta, 0) = \frac{1}{\theta^2} e^{-K\theta}. \tag{4.124}$$

Using the method of characteristics we find a most general solution to 4.124 as

$$\widehat{P}_0(\theta, t) = \frac{1}{\theta^2} g\left(t + \frac{1}{\theta}\right), \tag{4.125}$$

which, after applying the initial condition 4.124, we determine $g(\cdot)$ and then obtain

$$\widehat{P}_0(\theta, t) = \frac{1}{\theta^2} \exp\left(-\frac{K\theta}{t\theta + 1}\right). \tag{4.126}$$

We observe from 4.126 that $\widehat{P}_0 = O(\theta^{-2})$ as $\theta \to \infty$ so that P_0 will vanish at $S = 0$ but its derivative will not.

We next obtain the correction term P_1 in 4.119, as the $O(\rho)$ term will become important in our matching condition for the free boundary. From 4.1 and 4.119 we find that the PDE for the correction term P_1 is given by

$$P_{1,t} = SP_{1,SS} + SP_{0,S} - P_0 - K \tag{4.127}$$

with the initial and boundary conditions

$$P_1(0, t) = 0, \ P_1(S, 0) = 0. \tag{4.128}$$

Using a Laplace transform in 4.127, with $\widehat{P}_1(\theta, t) = \int_0^\infty e^{-\theta S} P_1(S, t)\, dS$, along with 4.128 and 4.126, we obtain

$$\widehat{P}_{1,t} + \theta^2 \widehat{P}_{1,\theta} + 2\theta \widehat{P} = \frac{K}{\theta(\theta t + 1)^2} \exp\left(-\frac{K\theta}{\theta t + 1}\right) - \frac{K}{\theta}. \tag{4.129}$$

Using the method of characteristics, we find that the most general solution of 4.129 is

$$\widehat{P}_1(\theta, t) = -\frac{K}{2(1 + t\theta)^2 \theta^2} \exp\left(-\frac{K\theta}{\theta t + 1}\right) - \frac{K}{\theta^2} \log\theta + \frac{1}{\theta^2} \widehat{P}\left(t + \frac{1}{\theta}\right), \tag{4.130}$$

and after using $\widehat{P}_1(\theta, 0) = 0$ we determine the function $\widehat{P}$ and obtain

$$\widehat{P}_1(\theta, t) = \frac{Kt\left(1 + \frac{t\theta}{2}\right)}{(1 + \theta t)^2 \theta} \exp\left(-\frac{K\theta}{\theta t + 1}\right) - \frac{K}{\theta^2} \log(1 + \theta t). \tag{4.131}$$

By combining 4.126 and 4.127 with 4.119 and inverting the transform, the expansion for P on this time scale $t = O(1)$ is given by

$$P(S, t) \sim K - S + \frac{1}{2\pi i} \int_{Br} \frac{e^{S\theta}}{\theta^2} \exp\left(-\frac{K\theta}{t\theta + 1}\right) d\theta$$
$$+ \rho \frac{Kt}{2\pi i} \int_{Br} \frac{e^{S\theta}}{\theta} \frac{\left(1 + \frac{t\theta}{2}\right)}{(1 + \theta t)^2} \exp\left(-\frac{K\theta}{\theta t + 1}\right) d\theta - \rho \frac{K}{2\pi i} \int_{Br} \frac{\log(1 + \theta t)}{\theta^2} e^{S\theta}\, d\theta. \tag{4.132}$$

Here $\Re(\theta) > 0$ on the Bromwich contour.

We need to determine the free boundary $\alpha(t)$ for times $t = O(1)$. As we did when $t = \omega/\lambda$ with $\omega > K$, we consider the spatial scale $S = O(\alpha)$ and again set $S/\alpha = U + 1$. The analysis leading up to 4.114 still applies and we obtain

$$P(S,t) - (K - S) \sim e^{-\lambda}\alpha K \left[\frac{S}{\alpha}\log\left(\frac{S}{\alpha}\right) - \frac{S}{\alpha} + 1\right] = \rho KS \log S - \rho KS \log \alpha - \rho KS + \rho K\alpha.$$

$$(4.133)$$

To obtain an expression for $\alpha(t)$ we match 4.132 and 4.133 in an intermediate limit where $S \to 0$ and $U + 1 = S/\alpha \to \infty$. To expand 4.132 for $S \to 0$ we expand the integrands for $\theta \to \infty$ to asymptotically invert the transform, and this yields, for $P - (K - S)$,

$$Se^{-K/t} + \frac{1}{2}\rho KSe^{-K/t} - \rho K[-S \log S + S \log t + (1 - \gamma)S] + O(S^2, \rho S^2). \qquad (4.134)$$

Now we compare 4.134 to the right hand side of 4.133. We note that the $S \log S$ terms agree, and then divide S and note that $\rho K\alpha/S$ is negligible in the matching region. We thus obtain

$$-\rho K \log \alpha \sim \left(1 + \frac{\rho K}{2}\right)e^{-K/t} - \rho K(\log t - \gamma) \qquad (4.135)$$

which upon exponentiation gives

$$\alpha(t) \sim te^{-\gamma}\exp\left(-\frac{1}{\rho K}e^{-K/t}\right)\exp\left(-\frac{1}{2}e^{-K/t}\right). \qquad (4.136)$$

Note that 4.136 is consistent with the result of the free boundary $\alpha(t; \rho)$ given in 2.20. Finally we note that as $t \to 0$, 4.136 matches to 4.117 as $\omega \to \infty$, since $(\omega - K)/\lambda \sim \omega/\lambda = t$ and $\rho^{K/\omega} = \exp(K/t)$.

4.5 Analysis for $t = v/\rho = \mathbf{O}(\rho^{-1})$, $v > 0$

Finally we assume the time to expiry for the option is large, with the scaling $t = v/\rho = O(\rho^{-1})$. Once again, there are two natural scales for the asset price S, namely $S = O(\rho^{-1})$ and $S = O(\alpha)$. For the latter scale we shall employ the result of 4.113, since it still holds in this time range, as α will be exponentially small.

With the scaling

$$t = \frac{v}{\rho}, \; S = \frac{\Omega}{\rho}, \; P(S,t) = K + \mathcal{P}(\Omega, v) = K + \mathcal{P}_0(\Omega, v) + O(\rho), \tag{4.137}$$

the PDE in 4.1 becomes

$$\mathcal{P}_{0,v} = \Omega \mathcal{P}_{0,\Omega\Omega} + \Omega \mathcal{P}_{0,\Omega} - \mathcal{P}_0 - K. \tag{4.138}$$

The boundary condition $P(\alpha(t), t) = K - \alpha(t)$ implies to leading order that

$$\mathcal{P}_0(0, v) = 0 \tag{4.139}$$

since α will be exponentially small. For $\Omega > 0$, $\Omega/\rho - K < 0$ so we use the initial condition

$$\mathcal{P}_0(\Omega, 0) = -K. \tag{4.140}$$

We shall see that the above is equivalent to asymptotically matching the present scale, for $v \to 0$, $\Omega \to 0$, to the results in section 7.4 on the (S, t) scale, for $t \to \infty$, $S \to \infty$.

We introduce the Laplace transform over the spatial variable Ω, setting

$$\widehat{\mathcal{P}}(\hat{\theta}, v) = \int_0^\infty e^{-\Omega \hat{\theta}} \mathcal{P}_0(\Omega, v) \, d\Omega. \tag{4.141}$$

Applying 4.141 along with 4.139 to 4.138, we obtain the first order PDE

$$\widehat{\mathcal{P}}_v + (\hat{\theta}^2 + \hat{\theta})\widehat{\mathcal{P}}_{\hat{\theta}} = -2(\hat{\theta} + 1)\widehat{\mathcal{P}} - \frac{K}{\hat{\theta}} \tag{4.142}$$

and from 4.140 we have the initial condition

$$\widehat{\mathcal{P}}(\hat{\theta}, 0) = -\frac{K}{\hat{\theta}}. \tag{4.143}$$

Using the method of characteristics, we find the most general solution for $\widehat{P}$ as

$$\widehat{\mathcal{P}}(\hat{\theta}, v) = -\frac{K \log(\hat{\theta} + 1)}{\hat{\theta}^2} + e^{-2v}\frac{1}{(\hat{\theta} + 1)^2} h \left\{ v - \log\left(\frac{\hat{\theta}}{\hat{\theta} + 1}\right) \right\}, \tag{4.144}$$

where $h(\cdot)$ is an arbitrary function. We determine $h(\cdot)$ by applying the condition in 4.143 to 4.144, and hence $h(\cdot)$ becomes

$$h(\xi) = Ke^{2\xi}\left[\xi - \frac{1}{e^\xi - 1} - \log(e^\xi - 1)\right] \tag{4.145}$$

and thus

$$\widehat{\mathcal{P}}(\hat\theta, v) = -K \left[\frac{e^{-v}}{\hat\theta} + \frac{e^{-v} - 1}{(\hat\theta + 1)e^v - \hat\theta} \right] - \frac{K}{\hat\theta^2} \log(\hat\theta + 1 - \hat\theta e^{-v}). \tag{4.146}$$

Then inverting the Laplace transform leads to the final result, for $S, t = O(\rho^{-1})$,

$$P(S,t) \sim K - Ke^{-v} + Ke^{-v} \exp\left(-\frac{\Omega}{1 - e^{-v}}\right) - \frac{K}{2\pi i} \int_{Br} \frac{e^{\Omega\hat\theta}}{\hat\theta^2} \log(\hat\theta + 1 - \hat\theta e^{-v}) \, d\hat\theta. \tag{4.147}$$

By expanding the result in 4.132 on the (S, t) scale, for $S \to \infty$ and $t \to \infty$ with θ scaled as $\theta = \rho\tilde\theta$ leads to

$$
\begin{aligned}
K - S &+ \frac{1}{2\pi i} \int_{Br} \frac{e^{S\theta}}{\theta^2} \left[1 - \frac{K\theta}{\theta t + 1} + O(\theta^2) \right] d\theta + \frac{\rho K t}{2\pi i} \int_{Br} \frac{1 + \frac{t\theta}{2}}{(1 + \theta t)^2} \frac{e^{S\theta}}{\theta} d\theta \\
&- \frac{\rho K}{2\pi i} \int_{Br} \frac{e^{S\theta}}{\theta^2} \log(1 + \theta t) \, d\theta \\
&= K + \frac{K}{2\pi i} \int_{Br} \frac{e^{\Omega\tilde\theta}}{\tilde\theta} \left[-\frac{1}{\tilde\theta v + 1} + \frac{v(1 + \frac{v\tilde\theta}{2})}{(1 + v\tilde\theta)^2} \right] d\tilde\theta - \frac{K}{2\pi i} \int_{Br} \frac{e^{\Omega\tilde\theta}}{\tilde\theta^2} \log(1 + \tilde\theta v) \, d\tilde\theta \\
&= Kv + Ke^{-\Omega/v} - Ke^{-\Omega/v} \left(\frac{\Omega}{2} + v \right) - \frac{K}{2\pi i} \int_{Br} \frac{e^{\Omega\tilde\theta}}{\tilde\theta^2} \log(1 + \tilde\theta v) \, d\tilde\theta
\end{aligned}
\tag{4.148}
$$

and the above agrees with the expansion of 4.147 as $(\Omega, v) \to (0, 0)$, with Ω/v fixed and $\hat\theta$ scaled to be $O(\Omega^{-1})$. This asymptotic matching argument can be used instead of the initial condition 4.140.

Lastly, we need to derive an asymptotic formula for the free boundary α. We will accomplish this by asymptotically matching 4.147 to the free boundary result in 4.113 in the intermediate limit where $\Omega \to 0$ and $U + 1 = \Omega/(\alpha\rho) \to \infty$. We rewrite 4.113 in terms of Ω to get

$$
\begin{aligned}
P &\sim e^{-\lambda}\alpha K \left[\frac{\Omega}{\rho\alpha} \log\left(\frac{\Omega}{\rho\alpha}\right) - \frac{\Omega}{\rho\alpha} + 1 \right] + K - \frac{\Omega}{\rho} \\
&= K\Omega \log\Omega - K\Omega \log(\rho\alpha) - K\Omega + e^{-\lambda}\alpha K + K - \frac{\Omega}{\rho}.
\end{aligned}
$$

(4.149)

Here α is now viewed as a function of the large time scale $v = \rho t$.

We next expand 4.147 for a fixed v and $\Omega \to 0$, which leads to

$$
K - \frac{K\Omega}{e^v - 1} - K\Omega \log(1 - e^{-v}) + K\Omega \log\Omega - K\Omega(1 - \gamma) + O(\Omega^2).
$$

(4.150)

By comparing 4.149 and 4.150 we see that $-K\Omega\log(\rho\alpha)$ must balance the term $-\Omega/\rho$ in 4.149. Hence α must again be exponentially small and we set

$$
\rho\alpha \sim \exp\left(-\frac{1}{\rho K}\right) H(v).
$$

(4.151)

Then matching 4.149 and 4.150 yields

$$
-K\Omega \log[H(v)] - K\Omega \sim -\frac{K\Omega}{e^v - 1} - K\Omega \log(1 - e^{-v}) - K\Omega(1 - \gamma)
$$

(4.152)

so that

$$
\log[H(v)] \sim \frac{1}{e^v - 1} + \log(1 - e^{-v}) - \gamma.
$$

(4.153)

Exponentiating the above we have thus obtained

$$\alpha \sim \frac{1}{\rho} \exp\left(-\frac{1}{\rho K}\right) \exp\left(\frac{1}{e^v - 1}\right) (1 - e^{-v})e^{-\gamma}. \tag{4.154}$$

The expansion of $P(S,t)$ on the scale $t = O(\rho^{-1})$ and $S = O(\alpha)$ is then given by

$$P(S,t) \sim K + \exp\left(-\frac{1}{\rho K}\right) \exp\left(\frac{1}{e^v - 1}\right) (1 - e^{-v})e^{-\gamma}$$
$$\times \left\{-\frac{1}{\rho}(U+1) + K[(U+1)\log(U+1) - U]\right\}. \tag{4.155}$$

We now let $v \to \infty$ in our results on the two spatial scales. Assuming that $P(S,t) \to P^\infty(S)$

we have thus obtained the expansions

$$P^\infty(S) \sim K - \frac{K}{2\pi i} \int_{Br} \frac{e^{\Omega\hat\theta}}{\hat\theta^2} \log(1 + \hat\theta) \, d\hat\theta$$
$$= K \int_1^\infty \frac{e^{-\xi\Omega}}{\xi^2} \, d\xi, \ \Omega > 0, \tag{4.156}$$

for $S = O(\rho^{-1})$, and for $S = O(\alpha_\infty)$,

$$P^\infty(S) \sim K + \exp\left(-\frac{1}{\rho K}\right) e^{-\gamma} \left\{-\frac{1}{\rho}(U_* + 1) + K(U_* + 1)\log(U_* + 1) - U_*\right\} \tag{4.157}$$

where

$$U_* + 1 = \frac{S}{\alpha_\infty} = \frac{\Omega}{\rho\alpha_\infty}, \ \alpha_\infty \sim \exp\left(-\frac{1}{\rho K}\right)\frac{1}{\rho}e^{-\gamma}. \tag{4.158}$$

These expressions agree with the exact results for the perpetual option in 2.22- 2.24.

CHAPTER 5

DISCUSSION AND EXTENSIONS

To summarize, we have given several asymptotic formulas for a free boundary problem in an options pricing model. In contrast to the Black-Scholes model in the same asymptotic limit, the free boundary $\alpha(t)$ moves from $S = K$ to $S \approx 0$ on logarithmically small time scales where $t = O((\log(1/\rho))^{-1})$. For the BS model this movement occurs on logarithmically *large* time scales, with $t = O(\log(1/\rho))$. Note however that the basic expression in 2.14, where the boundary decreases from $S = K$ to $S = 0$, is similar to the corresponding one for the BS model (see (14), (9)). For times where the free boundary approaches zero, the behavior of the CEV model is much different from the BS model, as for the former $\alpha(t)$ becomes exponentially small (see 2.19 - 2.21) while for BS $\alpha(t)$ becomes only $O(\rho)$ as $\rho \to 0$. This implies that for the CEV model it is advantageous to exercise the option only on short time to expiry scales. If such times are $O(1)$ or $O(\rho^{-1})$ then the put option should be exercised only if the stock price S is very small.

The present model, where S is governed by 2.1 is sometimes called the square root process and a more general CEV model corresponds to the SDE $dS = rS\ dt + \sigma S^{\beta+1} dW$. Here β is known as the elasticity of the local volatility σ, where $\sigma(S, t) = \delta S^\beta$, so that $\beta = 0$ for BS and $\beta = -1/2$ for the model here. It seems that the integral equation approach here works only for these two special cases. Using our asymptotic results for $\alpha(t)$ in 2.11 and by expanding the integrals in 2.11 and 2.12 we are able to obtain asymptotic results for the option value

$P(S, t)$ for $\rho \to 0$. Our results show that for each of the five time ranges we obtain different expansions for $P(S, t)$ for several ranges of S. Furthermore, we verify these expansions directly by using singular perturbation theory to analyze 2.3 - 2.5 for $\rho \to 0$, and subsequently reconcile our results with the IE approach; the perturbation method should extend to general elasticity factors β.

We comment that while here we took $\rho \to 0$, the behavior of $\alpha(t; \rho)$ for $t \to 0$ with $\rho = O(1)$ is essentially contained in formula 2.14. Then we would replace ω by λt and use $(\sqrt{\omega} - \sqrt{K})^2 \sim K - 2\sqrt{K\omega}$ and $\sqrt{K\omega} - \omega \sim \sqrt{K\omega}$. In this limit, however, the BS and CEV models behave similarly.

It is not immediately clear (at least to us) whether the CEV model call option leads to an interesting problem. For the American call option under the BS model, early exercise is never advantageous. An important difference between the European CEV ($\beta = -1/2$) and BS models is that the Green's function for the BS model is a proper probability distribution that integrates to one. In contrast the Green's function for the CEV model is deficient in mass, due to partial absorption at $S = 0$. For the put option analyzed here the basic PDE 2.3 does not apply near $S = 0$, since $\alpha > 0$. However, for a call option the partial absorption of the process may have to be considered.

We have not at this point done numerical studies to compare with the asymptotic results (i.e., determine how small ρ must be). However, since $\alpha(t)$ is exponentially small for $t > K/\log(1/\rho)$ it may become difficult to locate the free boundary by purely numerical methods, for small values of ρ. Knowing asymptotic properties of the type here may aid in the

development of, e.g, multi-scale numerical methods, in addition to providing qualitative information on the optimal exercise boundary.

Finally we would like to point out that the methods employed here could be applied to more complicated, exotic type American options under the CEV process. Such problems lead to even more interesting asymptotic results for both the option price and the free boundary. Barrier and lookback options would be a natural starting point. Dai and Kwok in (22) provided some analytical results for the option price for a knock-in American option under the Black-Scholes framework.

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