Rebuilding Mathematics on a Quantum Logical Foundation

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THESIS

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To Richard J. DeJonghe, Jr.,

from him I learned that some things are worth doing,

and science is one of those things.

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LIST OF ABBREVIATIONS

ABP	atomic bisection property
def.	definition
iff	if and only if
EA	exchange axiom
NBG	von Neumann-Bernays-Gödel
OL	ortholattice
OML	orthomodular lattice
prop.	proposition
resp.	respectively
RZFC	Reduced Zermelo-Fraenkel with choice
sec.	section
SP	superposition principle
wff	well-formed formula
wlog	without loss of generality
w.r.t.	with respect to
ZFC	Zermelo-Fraenkel with choice

SUMMARY

In this work we construct a rich first-order quantum logic which generalizes the standard classical predicate logic used in the development of virtually all of modern mathematics, and we use this quantum logic to build the foundations of a new *quantum mathematics* which has the potential not only to provide insights into the nature of mathematical thought, but also to provide a completely new mathematical framework in which to develop the theory of quantum mechanics.

It is not unnatural to expect that the seemingly insurmountable difficulties that lie at the foundations of quantum mechanics can only be resolved by literally going back and rethinking the quantum theory from first principles (namely, the principles of *logic*) — if this is the case, then a fully developed quantum mathematics, including a quantum calculus and a theory of quantum Hilbert spaces, may be necessary to finally answer questions which have plagued quantum theorists for generations.

While we have not pushed the development of quantum mathematics anywhere near far enough to accomplish this goal, this work presents quite a few notable achievements. First, with regard to the first-order quantum logic developed herein, we present not only a proof of soundness and completeness,[1] but also prove a technically powerful new completeness result which heretofore had been known to hold for classical, but not quantum, first-order logic.

[1]As much has been accomplished before, using a different axiomatization of quantum logic (17).

We then proceed to use our quantum logic to develop multiple areas of mathematics, including abstract algebra, axiomatic set theory, and arithmetic. In some preliminary investigations into quantum mathematics, Dunn (15) found that (the usual first-order axiomatization of) Peano arithmetic is inherently a classical theory, in that the Peano axioms yield the same theorems using either classical or quantum logic. We prove a similar result for certain classes of abstract algebras. Of course, if Dunn's result were truly generic, the project of developing quantum mathematics would be a lost cause — generically we would simply reproduce classical mathematics. We show that this is not the case by presenting examples of quantum monoids, groups, lattices, vector spaces, and operator algebras, all of which differ from their classical counterparts. Moreover, we find natural classes of models of quantum lattices, vector spaces, and operator algebras which all have a beautiful interrelationship, and we make some preliminary investigations into using these models as a basis for a new mathematical formulation of quantum mechanics.

We also develop (using the aforementioned powerful completeness result) an axiomatic quantum set theory (equivalent to ZFC under classical logic) which is far more tractable than any previous attempt at a quantum set theory. We then use this set theory to construct a quantum version of the natural numbers, and develop an arithmetic of these numbers based upon an alternative to Peano's axioms (which avoids Dunn's theorem). Surprisingly, we find that these "quantum natural numbers" satisfy all of our arithmetical axioms *if and only if* the underlying truth values form a modular lattice, giving a new *arithmetical* characterization of this important lattice-theoretic property. Furthermore, these numbers have a natural interpretation in terms of quantum observables with whole number eigenvalues, and our quantum arithmetic of these numbers yields a new sum and product for such observables with some truly remarkable properties.

As with any decent intellectual investigation, this work raises more questions than it answers. We conclude by mentioning some of the more interesting problems we have uncovered that still await a solution.

CHAPTER 1

INTRODUCTION

As the title suggests, the goal of this work is to redevelop mathematics from its foundations within the framework of quantum logic. While axiomatic set theory is typically thought of as being a foundation for all mathematics, before one can begin to develop any mathematics at all, one needs a precise notion of the logical reasoning one will use in that development, which is to say that one needs (at the very least) a precise way to determine when some collection of statements entails another. Then, in order to develop any particular branch of mathematics (such as set theory), one posits a collection of axioms and proceeds to develop the mathematical theory by determining exactly what statements are entailed by the axioms.

To develop the *foundations* of mathematics, one needs a collection of axioms which is "universal", meaning that one can use these axioms to "describe" any other branch of mathematics completely within the original axiomatic theory. Axiomatic set theory fulfills this role effectively for classical mathematics.[1]

What we accomplish in this work is, succinctly stated, to develop a mathematically precise replacement (motivated by quantum mechanics) for the classical first-order logic usually used to develop mathematics, and then use this logic to develop various branches of quantum mathe-

[1]Some would argue that category theory may be a better choice for the foundations of classical mathematics, but this author humbly disagrees. For a short discussion, see (3).

matics, including abstract algebra, arithmetic, and, in particular, an axiomatic set theory which we hope will start us down the path toward a suitable foundation for quantum mathematics.[1]

Of course, everything concerning our formal quantum logic will be developed using only classical logic and mathematics *in the meta-language*[2] — however, if we are to take the idea of quantum mathematics seriously, we should ideally use some sort of "intuitive quantum reasoning" (in the meta-language) to develop and describe our formal quantum logic (which we can then use to make statements in the object language), just as one uses "intuitive classical reasoning" in the development of a formal first-order classical logic. Unfortunately for us macroscopic beings, this is no trivial task. Perhaps a good metaphor which describes the situation is that the quantum logic we construct is a "first-order perturbation" away from classical logic toward a "true" quantum logic.

Now, it would be natural for us to discuss the history and development of quantum mathematics[3] at this point, as to place this work in context. Unfortunately, there is not much to say. For whatever reason, this field of research consists of large tracts of fallow intellectual ground, with only a few small patches of cultivation. One verdant patch consists of the work of Dunn on quantum Peano arithmetic (15), although his investigations have not seen any further

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[1] We give a more detailed summary of the results of this work at the end of the chapter, after we have covered some basic concepts of classical and quantum logic.

[2] When we formalize logic, the formal statements under investigation are said to be in the *object language*, while statements *about* those formal statements are said to be in the *meta-language*.

[3] In our usage, areas of existing study which are commonly called "quantum groups" and "quantum algebras" are definitely *classical mathematics*, as they are developed using classical logic.

significant development. Another consists of the quantum set theory of Takeuti (42), which as been developed by a handful of other researchers (36; 37; 38; 47; 48). While certainly possessing a rich and intriguing structure, Takeuti's quantum set theory is far more complicated than the one we develop — so much so that it would be hard to imagine his quantum sets forming a practical foundation for all of quantum mathematics.[1] Finally, Tokuo has made some extremely preliminary investigations into quantum set theory and quantum number theory (45; 46). Other than these few explorations, we are in an undiscovered country.

The remainder of this introductory chapter proceeds as follows. First, in order to set the stage for the quantum logic developed in chapters 2 and 3, we will provide a brief introduction to classical propositional logic (section 1.1), which will allow us to introduce some important logical notions in the simplest possible context. We will then give a brief discussion of propositional quantum logic (section 1.2) which will help prepare us for the development of our first-order theory. The elementary lattice theory needed to understand our approach and results is included in these sections, while more technical lattice-theoretic results needed for proofs can be found in appendix C. The reader possessing a basic familiarity with orthomodular lattices and formal logic is invited to skim these sections, referring back as needed. Finally, in section 1.3, we give a brief discussion of logic in general, and present an overview of what is to come in the remaining chapters.

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[1] We will give a more detailed account of the differences between our quantum set theory and the one developed by Takeuti in section 5.3.

Before proceeding any further, the author would like to acknowledge that all of the work presented herein was done in collaboration Kimberly Frey and Tom Imbo. Also, we would like to set a little notation before continuing with the main thread of our discussion.

Conventions and Notation

In the sequel, we will use $:=$ to be the equality of definition. We will also let $\mathbb{N} := \{0, 1, 2, \dots\}$ be the natural (or whole) numbers, and use $\mathbb{R}$ to represent the real numbers, and $\mathbb{C}$ to represent the complex numbers. Also, we will use 'or' inclusively, so that for two statements p and q, the statement 'p or q' is true whenever either p is true, q is true, or both are true.

Major results of this work are set aside in numbered theorems. Less important, but still interesting results are set aside as propositions, and results which are of only technical interest are set aside as lemmas. All proofs and lemmas can be skipped without loss of continuity. In an effort to keep this work as self-contained as possible, (almost) all of the mathematics needed herein not covered in a standard physics graduate program is contained in the appendices or introduced when needed. Since we will be defining quite a few symbols and terms, both a glossary of notation and an index is provided following the appendices.

1.1 Basics of Classical Mathematical Logic

Formal logic is a precise way to encode reasoning and determine the truth of certain formal statements. For example, if we know that a given statement p is true, then from this we can deduce that (for any other statement q), the statement 'p or q' is true.[1] While classical logical reasoning is *intuitively* clear, in order to do *mathematical* logic, we need to have a formal characterization of logical reasoning precise enough so that we can hope to prove statements about said reasoning.

We begin by examining propositional logic, which is significantly simpler than first-order logic.[2] Also, interesting distinctions between quantum and classical reasoning already appear in the context of propositional logic, so it will behoove us to examine these differences in this simpler context before exploring first-order logic. However, before we can examine propositional logic we will first need a little lattice theory.

Since all of the material covered in this section is standard, for proofs as well as a more thorough discussion we refer the reader to (6; 13; 31) for the relevant lattice theory and (18) for the logic.

1.1.1 Posets and Lattices

The most primitive notion we will need is that of a poset.

Definition 1.1. Let P be a set and $\leq$ a binary relation on P. $\leq$ is called a *partial order* on P if $\leq$ satisfies (for all $a, b, c \in P$)

Reflexivity: $a \leq a$.

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[1] In using logic informally, the statement p might be something like 'the sky is blue', while q might be 'the sky is green'. In propositional logic we are not concerned in any way with anything going on 'inside' propositions, while in first-order logic we will start to characterize this internal structure.

[2] First-order logic is frequently called *predicate* logic, and we will use the two terms interchangeably.

Antisymmetry: If $a \leq b$ and $b \leq a$ then $a = b$.

Transitivity: If $a \leq b$ and $b \leq c$ then $a \leq c$.

If $\leq$ is a partial order on P, we call the pair $(P, \leq)$ a *partially ordered set* (or *poset*), and say P is *partially ordered* by $\leq$.

Note: We will sometimes refer to the set P as a poset rather than the pair $(P, \leq)$ when the partial order is clear from context. Also, when we mention a poset without explicitly giving a partial order, we will assume it to be denoted by the symbol '$\leq$'. For $a, b \in P$, we will also write $b \geq a$ to mean $a \leq b$.

Of course, the reader is no doubt familiar with many posets — the natural numbers $\mathbb{N}$ and the reals $\mathbb{R}$ form posets under the usual ordering, but these posets are not typical in that for every pair of elements a, b, we have either $a \leq b$ or $b \leq a$.[1] This is not the case in a generic poset.

Given a finite poset P, we can represent it graphically by a *Hasse diagram*, but we will need the following definition before we can describe such diagrams.

Definition 1.2. Let P be a lattice, with $a, b \in P$. b is said to *cover* a if for any $c \in P$ with $a \leq c \leq b$, we have either $a = c$ or $b = c$.

To obtain a Hasse diagram for a given finite poset P, we place a dot for each element of P, and for every $a, b \in P$ such that b covers a, we draw a line from b up to a.

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[1] Such a poset is called a *chain* — see definition C.3 in the appendix.

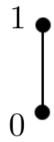

Figure 1. The Hasse diagram for $B_2 = \{0, 1\}$ from example 1.1.

Example 1.1. Let $B_2 = \{0, 1\}$ with the usual partial ordering. Then the Hasse diagram for B_2 is given in Figure 1.

We can also construct a slightly more complicated example.

Example 1.2. Let $D = \{0, x, y, 1\}$ be a poset with $0 \leq x \leq 1$ and $0 \leq y \leq 1$ and no other orderings.[1] Then the Hasse diagram for D is given in Figure 2.

One simple connection posets have to logic is this: given a poset P one can think of P as a set of formal propositions,[2] where for $p, q \in P$, the relation '$p \leq q$' is interpreted as 'p implies q'. Of course the structure of a partial order will not be enough to allow us to interpret P as a set of propositions — if we have some formal statement s, we want a formalization of the assertion 'not s', and similarly for two statements p and q, we would like formal assertions 'p and q' as well as 'p or q'. This motivates the following definitions:

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[1]Other than the ones required in order that $\leq$ be a partial order.

[2]In the most general case, we will use posets to represent not propositions, but rather equivalence classes of propositions under a natural equivalence to be defined in the sequel.

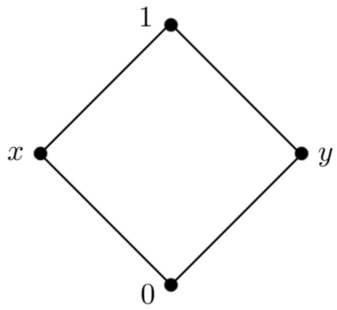

Figure 2. The Hasse diagram for $D = \{0, x, y, 1\}$ from example 1.2.

Definition 1.3. Let P be a poset, with $A \subseteq P$. Then the *meet of* A (or *greatest lower bound of* A), denoted $\bigwedge A$, is defined to be the element $z \in P$ (if it exists) such that

(i) $z \leq a$ for all $a \in A$

(ii) For any $b \in P$ such that $b \leq a$ for all $a \in A$, we have $b \leq z$.

We then also define $a \wedge b := \bigwedge\{a, b\}$.

Definition 1.4. Let P be a poset, with $A \subseteq P$. Then the *join of* A (or *least upper bound of* A), denoted $\bigvee A$, is defined to be the element $y \in P$ (if it exists) such that

(i) $a \leq y$ for all $a \in A$

(ii) For any $b \in P$ such that $a \leq b$ for all $a \in A$, we have $y \leq b$.

We then also define $a \vee b := \{a, b\}$.

Note: It is easy to see that the meet and join of any subset of a given poset is unique, if it exists, although meets and joins of such subsets do not always exist, as is shown in example 1.3 below. Also, for a given poset P, class[1] $\mathfrak{A}$, and class function f with dom $f = \mathfrak{A}$ and ran $f \subseteq P$ we define (when the relevant meets and joins exist)

$$\bigwedge_{a \in \mathfrak{A}} f(a) := \bigwedge f(\mathfrak{A}) \quad \text{and} \quad \bigvee_{a \in \mathfrak{A}} f(a) := \bigvee f(\mathfrak{A}).$$

For $f : \mathbb{N} \to P$ we also define, and $k, n \in \mathbb{N}$

$$\bigwedge_{i=k}^{n} f(i) := \bigwedge_{i \in \{k, \ldots, n\}} f(i) \quad \text{and} \quad \bigvee_{i=k}^{n} f(i) := \bigvee_{i \in \{k, \ldots, n\}} f(i).$$

Example 1.3. Consider the poset given by the Hasse diagram in Figure 3. The set $\{a, b\}$ does not have either a join or a meet. To see this, note that $\{a, b\}$ does not have a join because it does not have an upper bound, and $\{a, b\}$ does not have a meet because even though it has (two) lower bounds, neither of them is a *greatest* lower bound.

Since meets and joins of pairs are natural candidates to represent logical 'and' and 'or', it will behoove us to name posets where all such meets and joins exist[2].

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[1]See appendix A for a discussion of classes. However, the reader will lose nothing by assuming all classes are sets (and class functions are regular set functions) until we consider models of set theory in chapter 5.

[2]Posets where *all* meets and join exist are called *complete* lattices. See def. C.1.

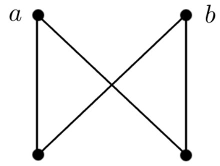

Figure 3. The Hasse diagram for a poset in which the set $\{a, b\}$ has neither a meet or a join.

Definition 1.5. Let L be a poset such that for every $a, b \in P$ both $a \vee b$ and $a \wedge b$ exist. Then L is called a *lattice under* $\leq$ (or simply a *lattice* if the partial order is clear from the context).

Note: Both B_2 of example 1.1 and D of example 1.2 are lattices. Also $\mathbb{N}$ and $\mathbb{R}$ form lattices under their usual orderings.

Since meets and joins are unique, a lattice as defined above is an algebra[1] with binary operations '$\vee$' and '$\wedge$'. Using this algebraic structure, one can completely characterize lattices:

Proposition 1.4. Let L be a set and let $\vee$ and $\wedge$ be binary operations on L satisfying (for all $a, b, c \in L$)

Commutativity: $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$.

Associativity: $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

Absorption: $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$.

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[1]Specifically, a $(2, 2)$-algebra — see appendix B.

Further define the relation $\leq$ on L by $a \leq b$ iff $a \wedge b = a$. Then $\leq$ is a partial order and L is a lattice under $\leq$. Conversely, if L is a lattice under $\leq$, then the meet and join (of pairs) satisfy the three properties above.

As mentioned before, when representing a collection of propositions as a lattice we can use the operations $\vee$ and $\wedge$ to represent logical 'or' and 'and', respectively. But we will need to add more structure to form a new proposition 'not p' from a given p.

Definition 1.6. Let L be a lattice under $\leq$, and let $\neg$ be a unary operation on L satisfying (for all $a, b \in L$).

Double-negation: $\neg(\neg a) = a$

Order-reversing: If $a \leq b$, then $\neg b \leq \neg a$.

Then $\neg$ is called an *involution on* L.

We would also like some way of representing propositions which are necessarily true (as well as propositions that are necessarily false) in a lattice. One property of a necessarily true statement is that *everything* must imply it. This motivates the following definition.

Definition 1.7. Let L be a lattice, and let $a \in L$ be such that $z \leq a$ for all $z \in L$. Then a is called the *top element of* L (denoted 1). Further let $b \in L$ such that $b \leq z$ for all $z \in L$. Then b is called the *bottom element of* L (denoted 0). If L has both a top and bottom element then L is called *bounded*.

<u>Note</u>: Top and bottom elements are clearly unique, and so we can think of them algebraically as constant functions. Every finite lattice is bounded — D from example 1.2 and B_2 from

example 1.1 already have their top and bottom elements labelled using our new notation. $\mathbb{N}$ has a bottom element, but is not bounded, and $\mathbb{R}$ has neither a top nor a bottom element.

Of course, one key fact of logic is that both a statement p and its negation 'not p' cannot both be true at the same time, and so we have the following definition.

Definition 1.8. Let L be a bounded lattice with involution '$\neg$' such that for every $a \in L$ we have that

$$a \wedge \neg a = 0 \quad \text{and} \quad a \vee \neg a = 1.$$

Then L is called an *ortholattice* (or *OL*). If $0 = 1$ then L is called *trivial.* Otherwise, L is called *non-trivial.*

Note: A trivial ortholattice has only one element. When not otherwise mentioned, we will assume that the symbol '$\neg$' represents the involution of any given ortholattice. Using the language of universal algebra from appendix B, for any ortholattice L, letting $F = \{\vee, \wedge, \neg, 1, 0\}$, we have that (L, F) is a $(2,2,1,0,0)$-algebra, and we will implicitly consider ortholattices to be such algebras.

We also have the following useful statement.

Proposition 1.5. *DeMorgan's Law:* Let L be an ortholattice. Then for any $a, b \in L$ we have that both

$$\neg(\neg a \wedge \neg b) = a \vee b \quad \text{and} \quad \neg(\neg a \vee \neg b) = a \wedge b. \tag{1.1}$$

<u>Note</u>: Using this, and the fact that $0 = \neg 1$ in any ortholattice, we can actually define ortholattices in terms of only the operations $\wedge$, $\neg$ and 0, and then define the operations $\vee$ and 1 in terms these. This technique will have some technical use for us later.

The most important ortholattice in the universe we have already seen as a lattice.

Example 1.6. Let $B_2 \coloneqq \{0, 1\}$ with the usual ordering (recall example 1.1 illustrated in Figure 1), and define $\neg$ so that $\neg 0 = 1$ and $\neg 1 = 0$. Then B_2 is an ortholattice. In the sequel, we will always take B_2 to be endowed with this involution.

Some comments concerning the ortholattice B_2 are in order. If a (gifted) child were to read the discussion up to this point, no doubt that child would tell you that B_2 is an example of a "universe of discourse" consisting of only a statement p and its negation $\neg p$ — one of these statements would be true (and correspond to '1'), while the other would be false (and correspond to '0'). Then (by double negation) we would have that $\neg\neg p$ is just p, which is all very nice. Of course, if you give a child a pair of oranges, that child will tell you that those oranges, in and of themselves, are an example of "twoness".

Now, when I was a child, I spake as a child, I understood as a child, I thought as a child: but when I became a mathematical physicist, I put away childish things. And as a mathematical physicist, if I look at a pair of oranges, I see their "twoness" as a result of their 1-1 correspondence with *any other pair*, rather than something which that pair possesses in isolation. Let us take a moment and try to apply this more sophisticated reasoning to "truth".

Just as one might think of "twoness" as the totality of all pairs, one can think of "truth" as the totality of all true statements — we call this totality 1, suggestively. Then "falsity" becomes the totality of all false statements (which we call 0). Now, consider this "truth" (1) and "falsity" (0) as comprising the algebra B_2 above. It is immediately apparent (under the interpretation of $\wedge$ as 'and', $\neg$ as 'not', etc.) that the algebraic operations of B_2 are also natural with this reading of '1' and '0'. To wit, if two statements p and q are true, then so is $p \wedge q$ $(1 \wedge 1 = 1)$, while if p is true, and q is false, then $p \wedge q$ must be false $(1 \wedge 0 = 0)$, etc. Thus, if we interpret $B_2 = \{0, 1\}$ as representing "truth" and "falsity" rather than any particular propositions, we see that these "truth values" for classical logic carry a natural algebraic structure. We will return to a discussion of the algebraic properties of this structure later (section 1.1.3), but first we wish to begin an examination of logic, and the role of this ortholattice therein.

1.1.2 Propositional Logic

With the above algebraic preliminaries out of the way, we can now begin a precise treatment of propositional logic.[1] We begin with a set A of *atomic propositions*. These may represent declarative statements such as 'cats are mammals' or 'cows have 10 legs', but as far as we are concerned we have no method with which to view their internal structure — atomic propositions are black boxes in propositional logic. The only property[2] that is relevant is that they may be either 'true' or 'false' (but never both). We also let the symbol '$\top$' represent an "essentially true" proposition, and the symbol '$\bot$' represent an "essentially false" proposition.

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[1] Our development will gloss over a large number of details. The reader is referred to (18) for a more thorough treatment.

[2] This is an informal notion which will be made precise below in the definition of a propositional truth function.

We can also form new propositions out of old ones. For any given proposition p, we write $\neg p$ to represent the assertion '*p is not the case*'. Also, for two propositions p and q we write $p \wedge q$ (read 'p and q') to mean the assertion that 'both p and q are the case', and we write $p \vee q$ (read 'p or q') to mean that 'either p or q or both are the case'. Then the set of all propositions we can construct in this way beginning from the atomic propositions form our *universe of discourse over A*. To be precise:

Definition 1.9. Let A be a set. Then the *propositional universe over A* (denoted $\mathcal{U}(A)$) is the free $(2, 2, 1, 0, 0)$-algebra[1] with operations $(\vee, \wedge, \neg, \top, \bot)$ over the set A. Elements in A are called *atomic propositions*, and elements in $\mathcal{U}(A)$ are called *propositions*.

With this definition, the propositions 'p' and '$\neg\neg p$' are considered to be different, even though intuitively they represent the same meaning. Also, we have not yet made precise what we mean for a proposition to be 'true' or 'false', or enforce the intuitive requirement that whenever 'p' is true, '$\neg p$' is false, and vice versa. We can remedy all these problems in the following way — the basic idea is that "truth" is represented by a function from the propositions to the "truth values" B_2 (which we have already seen naturally encodes the notions of 'truth' and 'falsity'), where this map must respect the "meaning" of the operations which represent 'and', 'not', etc. We call such a method of assigning truth values to propositions a *semantics* for that logic, which we proceed to formally define.

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[1]See appendix B for a definition of a free algebra. Intuitively speaking, the propositional universe consists of all "polynomials" in the atomic propositions constructed using the given operations.

Semantics for Classical Propositional Logic

Definition 1.10. Let A be a set. Then a map $\nu : \mathcal{U}(A) \to B_2$ is called a *propositional truth function on* $\mathcal{U}(A)$ if it is an (ortholattice) homomorphism[1] (considering $B_2 = \{0, 1\}$ to be a $(2, 2, 1, 0, 0)$-algebra as discussed above).

Note: From the above as well as the definition of $\mathcal{U}$, it is clear that a propositional truth function is entirely determined by its action on the atomic propositions, and (by the universal property of free algebras) that any set map from the atomic propositions to the set $\{0, 1\}$ can be extended uniquely to a propositional truth function.

Example 1.7. Take $A = \{p, q\}$ (with $p \neq q$ representing any two statements). Then we have 4 different propositional truth functions on $\mathcal{U}(A)$, given by

1. $\nu_1(p) = \nu_1(q) = 1$.
2. $\nu_2(p) = \nu_2(q) = 0$.
3. $\nu_3(p) = 1$, and $\nu_3(q) = 0$.
4. $\nu_4(p) = 0$, and $\nu_4(q) = 1$.

Then we can compute the action of any of the above maps on any proposition, for example

$$\nu_3(\neg p \vee (p \wedge q)) = \neg\nu(p) \vee (\nu(p) \wedge \nu(q)) = 0 \vee (1 \wedge 0) = 0,$$

---

[1] See definition B.3, recalling that $\mathcal{U}(A)$ is a $(2, 2, 1, 0, 0)$-algebra. Basically, a homomorphism f is just a "structure preserving map", so that $f(a \wedge b) = f(a) \wedge f(b)$, $f(\top) = 1$, etc.

while

$$\nu_4(\neg p \vee (p \wedge q)) = \neg\nu(p) \vee (\nu(p) \wedge \nu(q)) = 1 \vee (0 \wedge 1) = 1.$$

The symbols '0' and '1' above are to be interpreted as meaning 'false' and 'true' respectively, and a propositional truth function tells whether any given proposition in the universe of discourse is true or false.

There is then a natural equivalence relation on the universe of discourse with the following interpretation: two propositions have an "equivalent meaning" when one is true iff the other is true.

Definition 1.11. Let A be a set, and let $p, q \in \mathcal{U}(A)$. Then p and q are said to be *semantically equivalent* if, for every ν which is propositional truth function on $\mathcal{U}(A)$, we have $\nu(a) = \nu(b)$. If p is semantically equivalent to $\top$, then p is said to be a *tautology.*

Note: It is in this sense that for any proposition p, both p and $\neg\neg p$ "mean the same thing" — the two propositions p and $\neg\neg p$ are semantically equivalent. Also, for any proposition p, we have that $p \vee \neg p$ is a tautology, since $\nu(p \vee \neg p) = \nu(p) \vee \neg\nu(p) = 1$.

With the notions of a propositional truth function and semantical equivalence, we now have a formal handle on "truth" in classical propositional logic. Using this we can define what it means formally for one proposition to follow from a given collection of propositions.

Definition 1.12. Let A be a set, let $p \in \mathcal{U}(A)$, and let $\Gamma \subseteq \mathcal{U}(A)$. We say that Γ *semantically entails* p if, for every ν which is a propositional truth function on $\mathcal{U}(A)$, we have that $\nu(p) = 1$ whenever $\nu(q) = 1$ for all $q \in \Gamma$.

We then have the following interesting fact.

Proposition 1.8. Let A be a set, let $p, q \in \mathcal{U}(A)$, and let $\Gamma \subseteq \mathcal{U}(A)$. Then $\Gamma \cup \{p\}$ semantically entails q iff Γ semantically entails $\neg p \vee (p \wedge q)$.

Note: This gives us a formal proposition[1] which represents 'p implies q', as such we will define $p \rightarrow q := \neg p \vee (p \wedge q)$. '$\rightarrow$' so defined is called the *Sasaki hook.*

Now that we have a formal notion of truth and entailment within classical logic, in principle we're done. We have a fully formalized procedure for determining when a given proposition p follows from some set of propositions Γ. However, our criteria for determining when the statement 'p follows from Γ' is true is rather difficult to fulfill — we need to examine *all the possible* propositional truth functions in order to make the determination,[2] which is rather quite a lot of work. To remedy this situation, we introduce the notion of formal deductions.

Formal Provability in Classical Propositional Logic

Ideally, what we would like is a system of formal rules by which we can deduce when some proposition follows from others. While the details of such formal deductive systems are not

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[1]This differs from the *usual* formal proposition chosen for this role, namely the classical material conditional '$\neg p \vee q$', which also has the desired property given in the above proposition. We choose our conditional because the classical material conditional does not play well with quantum logic — see Hardegree (24). However, in classical logic both propositions are semantically equivalent.

[2]While this can be done fairly straightforwardly in propositional logic (for example, using truth tables), first-order logic is another story. The reader interested in more detail is referred to (18).

overly complicated,[1] the particulars of any such system will not concern us, and instead we will describe the general form of any deductive system for propositional logic.

Definition 1.13. Let A be a set. A *propositional rule of deduction over* A consists of a pair (Γ, p), where $\Gamma \subseteq \mathcal{U}(A)$ and $p \in \mathcal{U}(A)$. A *propositional deductive system over* A is a pair $(\mathcal{A}, \mathcal{D})$, where $\mathcal{A} \subseteq \mathcal{U}(A)$ and $\mathcal{D}$ is a set of propositional rules of deduction over A.

Note: One rule of deduction commonly found in classical propositional logic would be $(\{p, p \to q\}, q)$. The intuitive meaning of this rule is that if both the propositions p and $p \to q$ are true, then also the proposition q is true. This is an instance of modus ponens.

We can then use the notions above to define what it means for a proposition to be provable from a set of axioms.

Definition 1.14. Let A be a set, let $p \in \mathcal{U}(A)$, let $\Gamma \subseteq \mathcal{U}(A)$ and let $(\mathcal{A}, \mathcal{D})$ be a propositional deductive system. Then p is said to be *formally provable from* Γ *w.r.t.* $(\mathcal{A}, \mathcal{D})$ if there exists a finite tuple[2] $(p_1, \dots, p_n)$ such that $p_n = p$, and each p_i is either in $\mathcal{A}$, or if there is a rule $(\Gamma, p_i) \in \mathcal{D}$ such that $\Gamma \subseteq \{p_1, \dots, p_{i-1}\}$.

Soundness and Completeness

In order for formal provability to be of much use, a classical propositional deductive system must satisfy two properties. The first is *soundness*: *only* propositions which are semantically

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[1] See (18) for an example.

[2] This tuple is a formal notion which corresponds to a list whose items each consist of one step in a proof of a given statement.

entailed are formally provable. The second is *completeness*: *every* proposition which is semantically entailed is formally provable. More precisely we have:

Definition 1.15. Let A be a set, and let $(\mathcal{A}, \mathcal{D})$ be a propositional deductive system over A. Then $(\mathcal{A}, \mathcal{D})$ is said to be *sound* if for every set $\Gamma \subseteq \mathcal{U}(A)$ and every proposition p, whenever p is formally provable from Γ w.r.t. $(\mathcal{A}, \mathcal{D})$, we have that Γ semantically entails p. $(\mathcal{A}, \mathcal{D})$ is said to be *complete* if, whenever Γ semantically entails p, we have that p is formally provable from Γ w.r.t. $(\mathcal{A}, \mathcal{D})$.

Simply put, the content of soundness and completeness together is that 'semantic entailment' and 'formal provability' are entirely equivalent notions. This is certainly something we would want from any formal deductive system. Of course, there are many sound and complete deductive systems for classical logic, see e.g. (18).

We now state the "deduction theorem" for classical propositional logic, which demonstrates that the conditional '$\rightarrow$' defined above behaves as nicely as possible with regard to formal provability.

Proposition 1.9. *Deduction Theorem:* Let A be a set, let $(\mathcal{A}, \mathcal{D})$ be any sound and complete propositional deductive system over A, let $p, q \in \mathcal{U}(A)$, and let $\Gamma \subseteq \mathcal{U}(A)$. Then q is formally provable from $\Gamma \cup \{p\}$ w.r.t. $(\mathcal{A}, \mathcal{D})$ iff $p \rightarrow q$ is formally provable from Γ w.r.t. $(\mathcal{A}, \mathcal{D})$.

This deduction theorem is more evidence that the expression '$p \rightarrow q$' is a good choice for a proposition which "means" 'p implies q'.

Consistency

One more concept from mathematical logic that deserves discussion is that of *consistency* of a set of propositions; intuitively a set of propositions is consistent if it contains no logical contradictions. We make this precise in the following way.

Definition 1.16. Let A be a set, and $\Gamma \subseteq \mathcal{U}(A)$. Then Γ is said to be *inconsistent* if Γ semantically entails $\bot$. Otherwise Γ is said to be consistent.

Now it is easy to see that '$\bot \to p$' is always true for any proposition p. Computing (for any propositional truth function ν), we have

$$\nu(\bot \to p) = \neg\nu(\bot) \vee (\nu(\bot) \wedge \nu(p) = 1 \vee (0 \wedge \nu(p)) = 1.$$

What this shows (along with the deduction theorem) is that *every proposition* is provable (within a deduction system satisfying completeness) from an inconsistent set of axioms! We have following the nice test for consistency in propositional logic.

Proposition 1.10. Let A be a set, and $\Gamma \subseteq \mathcal{U}(A)$. Then Γ is consistent iff there exists a propositional truth function ν such that $\nu(\gamma) = 1$ for every $\gamma \in \Gamma$.

Intuitively, this says that a set of propositions is consistent if it is logically possible for all of the propositions in that set to be simultaneously true.

1.1.3 Boolean Algebras

If we blur our eyes, and treat semantically equivalent propositions as being equal,[1] then the resulting structure forms a Boolean Algebra.

Definition 1.17. Let B be an ortholattice which satisfies (for all $a, b, c \in B$)

Distributivity: $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Then B is called a *Boolean algebra*.

Note: B_2 from example 1.1 (with the natural involution) forms a Boolean algebra.

Then we have the following proposition which makes precise the notion of "blurring our eyes".

Proposition 1.11. Let A be a set, and define the binary relation $\sim$ by (for all $p, q \in \mathcal{U}(A)$)

$$a \sim b \quad \text{iff } a \text{ is semantically equivalent to } b.$$

Then $\sim$ is a congruence[2] on $\mathcal{U}(A)$, and $\mathcal{U}(A)/\sim$ is a Boolean algebra.

Note: The natural quotient homomorphism maps propositions which are semantically equivalent (in $\mathcal{U}(A)$) to the same element in the quotient algebra. See section B.5 of the

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[1] This will be made precise in proposition 1.11.

[2] See section B.5 in the appendix.

appendix for details. The quotient algebra is sometimes called the Lindenbaum-Tarski algebra; this notion will reappear later in a different context (section 2.4.2).

When defining propositional truth functions, we had a natural motivation for taking the Boolean algebra $B_2 = \{0, 1\}$ as the truth values. Interestingly, if we had instead defined propositional truth functions more generally, so that *any* Boolean algebra could serve as the codomain (i.e. as the set of truth values), then we would have arrived at the same notions of semantical equivalence and semantic entailment — more precisely, a set of propositions Γ would semantically entail a proposition q (by our previous definition) iff Γ semantically entailed q in a semantics defined over generic Boolean truth values. Moreover, this (more general) semantics would be complete w.r.t. the exact same deductive systems. Basically, taking the truth values to be a generic Boolean algebra yields a logic completely equivalent to the usual classical logic with truth values $\{0, 1\}$.[1] One way to understand this fact is to realize that B_2 is the only *irreducible*[2] Boolean algebra, and that irreducibles suffice to prove a completeness theorem.[3]

Proposition 1.12. Let B be a Boolean algebra. Then B is irreducible iff B is isomorphic to B_2.

---

[1]Using Boolean algebras for truth values has practical applications in axiomatic set theory — see section 5.1.3.

[2]See definition 2.27 in the appendix. Basically, an ortholattice is irreducible provided it is not (isomorphic to) a product of other ortholattices.

[3]We eschew a formal treatment of this claim since we give a proof within the broader context of quantum logic — see theorem 3.

1.1.4 First-Order Logic

The formulation of (classical) first-order logic is significantly more involved than for the propositional logic, and so rather than giving a full development, we will confine ourselves to some informal remarks.

As noted earlier, in propositional logic the atomic propositions are “black boxes”, with no way to get a handle on their internal structure. In first-order logic, on the other hand, atomic propositions are constructed from more fundamental pieces, namely *variables*, *predicate symbols*, and *function symbols*. One then uses these symbols to construct atomic propositions which (possibly) depend upon variables — for a proposition p which depends on a variable x, we write $p(x)$.

Then, just as in propositional logic, we can use the symbols ‘$\vee$’, ‘$\wedge$’, and ‘$\neg$’, to build more complicated propositions from the atomic propositions. We are also allowed to use two new logical symbols to build new propositions — ‘$\forall$’ (representing ‘for all’), and ‘$\exists$’ (representing ‘there exists’).[1] Then, if we have a proposition such as $p(x)$, $(\forall x)p(x)$ represents another proposition in first-order logic, as does $(\exists x)p(x)$.

Semantics is more complicated than in the propositional case. One needs to form *models* which include an underlying set M (which represent the objects that the variables run over) as well as a truth function which is defined similarly to the propositional case (with truth values

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[1]Similar to the way in which we eliminate ‘$\vee$’ and ‘1’ in terms of the other symbols ‘$\wedge$’, ‘$\neg$’, and ‘0’, we can replace ‘$\exists$’ with ‘$\neg\forall\neg$’. Intuitively, a given statement s holds for all x iff there does not exist an x for which s does not hold.

$B_2 = \{0,1\}$), except that it is required to play well with the new symbols — in particular (for a truth function ν) we require

$$\nu\big((\forall x)p(x)\big) = \bigwedge_{a\in M} \nu(p(a)) \quad \text{and} \quad \nu\big((\exists x)p(x)\big) = \bigvee_{a\in M} \nu(p(a)).$$

This makes intuitive sense — $(\forall x)p(x)$ is true iff $p(a)$ is true for every a in the model under consideration.

Classical first-order logic then shares some important properties with classical propositional logic:

1. Classical first-order logic has a deduction theorem.
2. There is a system of formal deductions with respect to which the classical first-order semantics (discussed above) is both sound and complete.
3. These formal deduction systems are sound and complete with respect to a semantics defined so the truth values may be any complete[1] Boolean algebra.
4. A set of axioms is consistent iff those axioms have a model.[2]

As we will see, the quantum logic we develop shares all of the above properties with classical logic except for the first — there is no deduction theorem for quantum logic.

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[1]This is a technical requirement on the truth value lattice (somewhat confusingly, this concept of lattice completeness is *completely* unrelated to the concept of soundness and completeness in the *logic*) that insures that certain meets and joins exist — namely, those needed to evaluate the truth values of statements involving the symbols '$\exists$' and '$\forall$'. See definition C.1.

[2]In the propositional case, we think of a model as simply a propositional truth function.

There is one bit of odd behavior in classical first-order systems when the underlying truth values are a general Boolean algebra rather than the usual $B_2 = \{0, 1\}$. In any (non-trivial) Boolean algebra $B \neq B_2$, we have some element $z \in B$ such that $z \neq 0$, $z \neq 1$, and (of course) $z \vee \neg z = 1$. Then, for a given proposition $p(x)$, we can show there exists a model over a set M such that $\nu(p(a)) = z$ for some $a \in M$, that $\nu(p(b)) = \neg z$ for some $b \in M$, and yet $\nu(p(c)) \neq 1$ for all $c \in M$. Then, the formal proposition representing the statement 'there exists an x such that $p(x)$' is true, since

$$\nu\big((\exists x)(p(x)\big) = \bigvee_{c \in M} \nu(p(c)) \geq \nu(p(a) \vee \nu(p(b)) = 1,$$

and yet there is no element of M such that $p(x)$ is true! This is quite odd behavior that appears as soon as we go beyond using the standard truth values $\{0, 1\}$ (and so will appear in our dealings with quantum logic), but this oddness should not be thought of as being particularly "quantum".[1]

1.2 Basics of Quantum Logic

The birth of quantum logic can be traced to a paper of Birkhoff and von Neumann (5) in which they observed that the collection of closed linear subspaces (equivalently, the projection operators) of any complex separable Hilbert space forms an ortholattice which, while not a

[1]In fact, as we will see in section 3.2.2, this oddness can be remedied when the truth values form an *irreducible* algebra, and so this oddness is best thought of as being about the reducibility, rather than the Booleanness, of the truth values.

Boolean algebra, still satisfies enough algebraic properties[1] that it has a natural interpretation as a collection of propositions, with the ortholattice operations $\vee$ representing 'or', $\wedge$ representing 'and', and $\neg$ representing 'not'.

Since this original paper, the discipline of quantum logic has grown so much that it would be impractical to attempt to summarize all the developments that have occurred since. The interested reader is referred to the review articles (11; 20; 25; 29) and references therein.

We now proceed to define the algebraic properties satisfied by the aforementioned subspace lattices — as we will now see, these subspace lattices are all *orthomodular*.

1.2.1 Orthomodular Lattices

The bible of orthomodular lattice theory is Kalmbach (31), to which the reader is referred for a more thorough discussion. We begin with the definition of an orthomodular lattice.

Definition 1.18. Let L be an ortholattice. If all $a, b \in L$ satisfy

Orthomodularity: $a \wedge b = a \wedge (\neg a \vee (a \wedge b))$,

then L is said to be an *orthomodular lattice* (or *OML*).

Note: Orthomodularity is a weakening of the distributivity property satisfied by Boolean algebras, and as such, every Boolean algebra is an OML. Recalling the definition of the Sasaki hook, note that we can rewrite orthomodularity as $a \wedge b = a \wedge (a \to b)$. Think about that.

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[1] See our discussion in section 1.2.2 for a detailed account.

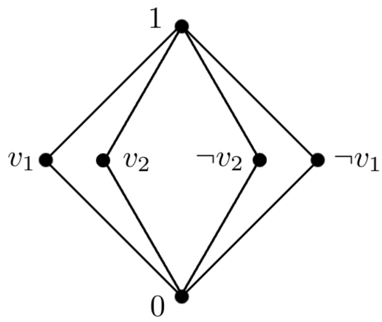

Figure 4. The Hasse diagram for $\mathrm{MO}_2 = \{0, 1, v_1, v_2, \neg v_1, \neg v_2\}$ of example 1.13.

Example 1.13. We define MO_2 to be the set $\{0, 1, v_1, v_2, \neg v_1, \neg v_2\}$ (with obvious involution), and with partial order given as in the Hasse diagram of Figure 4. Then MO_2 is an orthomodular lattice which is not a Boolean algebra.

There is a simple way to characterize when an OML is a Boolean algebra.

Definition 1.19. Let L be an OML, with $a, b \in L$. We say that *a commutes with b* (or *a and b commute*) (denoted $a\ C\ b$), if we have that

$$a = (a \wedge b) \vee (a \wedge \neg b).$$

Note: Saying that "a and b commute" is well-defined, since the relation of commuting is symmetric in an OML. See (31), p. 22. If a and b do not commute we write $a \not{C}\ b$.

Proposition 1.14. Let L be an orthomodular lattice. Then L is a Boolean algebra iff for every $a, b \in L$ we have $a\ C\ b$.

Proof. This is a trivial corollary of proposition C.9. □

1.2.2 Projection/Subspace Lattices

First we set a little notation. For any Hilbert space $\mathcal{H}$ we will use the Dirac bra-ket notation for the inner product, '$|0\rangle$' to represent the zero vector, and for any $S \subseteq \mathcal{H}$, we define

$$S^{\perp} := \{ \; |\psi\rangle \in \mathcal{H} \mid \langle\psi|\phi\rangle = 0 \text{ for all } |\phi\rangle \in S \; \},$$

(which is always a closed subspace). Also, for any set $S \subseteq \mathcal{H}$, we denote the linear span of S by $\mathrm{span}(S)$, and the closure of the span of S by $\overline{\mathrm{span}}(S)$. Also recall that an *orthogonal projection operator on* $\mathcal{H}$ is a Hermitian operator P such that $P^2 = P$. Finally, for any linear operator A on $\mathcal{H}$ we define $\ker(A) := \{|\psi\rangle \in \mathcal{H} : A|\psi\rangle = |0\rangle\}$.

Proposition 1.15. Let $\mathcal{H}$ be a separable Hilbert space. Then the set of closed linear subspaces (ordered under inclusion) form an orthomodular lattice with $S \mapsto S^{\perp}$ as the involution. This OML is called the *subspace lattice of* $\mathcal{H}$.

Note: For two closed subspaces $V, W \subseteq \mathcal{H}$, we have that $V \wedge W = V \cap W$ (the meet is the intersection), and $V \vee W = \overline{\mathrm{span}}(V \cup W)$ (the join is the closure of the span of the union). Also, the closed linear subspaces of a separable Hilbert space are in 1-1 correspondence with the orthogonal projection operators[1] on $\mathcal{H}$, and so the projection operators of $\mathcal{H}$ form an

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[1]An orthogonal projection operator P corresponds to its image (i.e. with $\ker(P)^{\perp}$) under this 1-1 correspondence.

OML (called the *projection lattice*) naturally isomorphic to the subspace lattice of $\mathcal{H}$. Given this correspondence, we will frequently blur the distinction between the two — no confusion should arise. See (28).

For a given quantum system described by some Hilbert space $\mathcal{H}$, the orthogonal projections on $\mathcal{H}$ have a natural interpretation as (equivalence classes of) propositions concerning that quantum system, since the measurement of such an operator always yields '1' or '0'. One may think of the relevant proposition for a projection operator P as being the declarative statement 'the state of the system is in the subspace $\ker(P)^{\perp}$'.

As Birkhoff and von Neumann noted in (5), when $\mathcal{H}$ is a finite dimensional Hilbert space, its corresponding projection lattice has even nicer properties.

Definition 1.20. Let L be a lattice. If every $a, b, c \in L$ satisfy

Modularity: If $a \leq c$, then $a \vee (b \wedge c) = (a \vee b) \wedge c$.

then L is said to be *modular*.

Note: This is easily seen to be a weakening of distributivity. Also, when L is an ortholattice, modularity implies orthomodularity (see (31)).

Proposition 1.16. Let $\mathcal{H}$ be a finite dimensional Hilbert space. Then the subspace lattice of $\mathcal{H}$ is a modular lattice.

von Neumann was strongly attached to the property of modularity — so much so that he developed his theory of von Neumann algebras to provide an alternative to standard quantum mechanics on an infinite dimensional Hilbert space. In this alternative, the underlying projection lattice is still modular.[1] Interestingly, modularity will make a surprise return to the stage in chapter 6 when we construct our quantum arithmetic.

1.2.3 Propositional Quantum Logic

For purposes of this work, by quantum logic we will mean the generalization of classical logic whereby one replaces the standard Boolean algebra $B_2 = \{0, 1\}$ that one uses as the truth values in classical logic with an arbitrary orthomodular lattice. The basic motivation for this replacement is this: in the classical case, the (equivalence classes of) propositions form a Boolean algebra, and also, the (equivalence classes of) propositions concerning quantum systems naturally form an orthomodular lattice (namely, the projection lattice). Since classical logic can be developed using any Boolean algebra as a set of truth values, we assume quantum logic can be developed using any OML for the truth values.[2]

We now give a brief treatment generalizing the propositional logic developed in section 1.1.2 to the quantum case, just to get the reader's feet wet. We will formalize first-order quantum logic in chapter 2.

In our consideration of quantum propositional logic, we utilize the same concept of 'proposition' and 'universe of discourse' previously defined.

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[1] For an introduction to the lattice-theoretic properties of von Neumann algebras, see (28).

[2] Of course, another natural candidate would be to only use *irreducible* truth values, as per the usual development of classical logic based on $B_2 = \{0, 1\}$. Just as in the classical case, these choices turn out to be equivalent, as we show in chapter 2.

Definition 1.21. Let A be a set. A *quantum propositional truth function* is a homomorphism $\nu : \mathcal{U}(A) \to L$, where L is an orthomodular lattice.

Since an OML L has, in general, lot more structure than $B_2 = \{0, 1\}$, there are different ways to define 'true'. For our purposes, we will say that a proposition p is 'true' when $\nu(p) = 1$.[1]

We then keep the same notion of semantic entailment as in the classical case, but now using quantum propositional truth functions — we will call this *Q-entailment*.

Definition 1.22. Let A be a set, let $p \in \mathcal{U}(A)$, and let $\Gamma \subseteq \mathcal{U}(A)$. If, for every quantum propositional truth function ν such that $\nu(\gamma) = 1$ for every $\gamma \in \Gamma$, we have that $\nu(p) = 1$ as well, then we say that Γ *semantically Q-entails* p.

Just as in the classical case, we can define a notion of semantical equivalence, and then use this to create an algebra of equivalence classes of propositions (as for the classical case, see proposition 1.11) — doing this would yield that the resultant algebra was an OML instead of a Boolean algebra, as the reader might expect.

Also, we can create formal deductive systems for this propositional quantum logic and prove that they are sound and complete with respect to the semantics given above — since we do exactly this for first-order quantum logic in the next chapter (and since propositional logic is a special case of first-order logic), we will not go through the details of the propositional case now.

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[1] In principle, rather than taking plain OMLs for the sets of truth values, one could take an OML along with some designated set, and every element of the designated set would count as true. For a further discussion of this approach, see (26).

However, if we assume the completeness theorem, we can then examine the deduction theorem 1.9 in quantum logic. The completeness theorem means that a proposition p will be formally provable (via our hypothetical formal deduction system for quantum logic) from a set of propositions Γ iff Γ semantically Q-entails p. Hence, the deduction theorem will hold in quantum logic iff (for any set of propositions Γ, and for any propositions p and q) we have that the following statement holds:

$$\Gamma \cup \{p\} \text{ semantically Q-entails } q \quad \text{iff} \quad \Gamma \text{ semantically Q-entails } p \to q,$$

recalling that $p \to q = \neg p \vee (p \wedge q)$ is the Sasaki hook. It is easy to construct a counterexample showing that the deduction theorem does *not* hold in quantum logic.

Example 1.17. Let p, q be propositions. Then $\varnothing \cup \{q\}$ semantically Q-entails $p \to q$. To see this consider any quantum propositional truth function ν such that $\nu(q) = 1$. Then we have

$$\nu(p \to q) = \neg\nu(p) \vee (\nu(p) \wedge \nu(q)) = \neg\nu(p) \vee (\nu(p) \wedge 1) = \neg\nu(p) \vee \nu(q) = 1.$$

However, we *do not have* that $\varnothing$ semantically Q-entails $q \to (p \to q)$. To see this, define a quantum propositional truth function ν_0 with codomain MO_2 (recall example 1.13) such that $\nu_0(p) = v_1$ and $\nu_0(q) = v_2$. Then

$$\nu_0(p \to q) = \neg v_1 \vee (v_1 \wedge v_2) = \neg v_1,$$

so that

$$\nu_0(q \to (p \to q)) = \neg v_2 \vee (v_2 \wedge \neg v_1) = \neg v_2 \neq 1.$$

Subclassicality of Quantum Propositional Logic

Our propositional quantum logic is easily seen to be subclassical — without going into a formal definition, essentially a logic is subclassical if everything that can be proven in that logic (via some formal deductive system satisfying completeness) can also be proven in classical logic. In terms of semantics, a logic is subclassical if every semantic entailment of that logic is also a semantic entailment of classical logic.

The subclassicality of quantum propositional logic is easy to see — if Γ is a set of propositions, and Γ semantically Q-entails a proposition p, then (since B_2 is an OML), we trivially have that Γ semantically entails p classically. As we will see, first-order quantum logic is also subclassical.

This concludes our discussion of propositional quantum logic. Before jumping into the first-order case, to conclude this chapter we briefly discuss logic in general and give an overview of the remainder of this work.

1.3 Overview of Logic and Preview of Remaining Chapters

The remainder of this work is dedicated to developing a formal first-order quantum logic, and then using this logic to begin a development of quantum mathematics. We have not, and will not, formally define what we mean by *logic* — but we will take a moment to elucidate the notions of a 'semantics' and 'formal deduction system' *for* a logic.

First, we should point out that before one can have a formal logic, one needs a formal syntax, i.e. rules for determining precisely what constitutes the object language and what formal statements are allowed. There is nothing “classical” or “quantum” concerning the syntax — it is a pre-logical notion. This has been made apparent in our examination of the propositional case, where the *universe of discourse* (over some set) constituted the allowed formal statements for both the classical and quantum propositional logic discussed.

Given a syntax, the job of the logic is then to provide a precise notion of entailment by determining what statements follow from others. For the classical propositional logic discussed previously, we mentioned two different sorts of entailment — one defined by the semantics, and another defined via a formal deductive system. While, in principle, these two entailments could have been different (in which case there would be no basis for saying they both represented the same classical logic), the content of the soundness and completeness theorems is that both of these entailments are, in fact, the same. Hence, it is soundness and completeness which allow us to think of a semantics and a formal deductive system as belonging to the same classical propositional logic.

As we have mentioned, first-order classical logic and propositional quantum logic also possess theorems of soundness and completeness, as does the first-order quantum logic we develop below, and for this reason we speak of these logics as possessing both a semantics and a syntax. For the first-order quantum logic, we will prove soundness and completeness in section 2.4 — given these theorems, we will henceforth identify this quantum logic with *both* its semantics as well as with its system of formal deduction even though, strictly speaking, they provide different formal routes to producing the same entailments.

We now proceed to summarize the remaining chapters of this work.

Summary of Chapter 2: First-Order Quantum Logic and Quantum Mathematics

In this chapter we formally define our first-order quantum logic by constructing both a semantics as well as a formal deduction system. We then prove that this deduction system is both sound and complete with respect to our quantum semantics. We conclude the chapter with a more powerful completeness result — namely that our deduction system is also complete with respect to a semantics where we restrict our truth values to be *irreducible* algebras. This result is the analog of the result from classical logic that any formal deductive system is sound and complete with respect to the standard $B_2 = \{0,1\}$ truth value semantics iff that deductive system is complete with respect to a semantics where any Boolean algebra is allowed for the truth values.

Summary of Chapter 3: Simple Developments in Quantum Logic

In this chapter we investigate the first-order quantum logic developed in the previous chapter. First, in section 3.1 we examine the connection of our quantum logic to the traditional classical first-order logic. As far as the semantics are concerned, just as in the propositional case, first-order logic is a generalization of classical logic obtained by using OMLs for the truth values rather than the usual $B_2 = \{0,1\}$ (or any other Boolean algebra), and as such is subclassical (since Boolean algebras are, in particular, OMLs). One consequence of this subclassicality is that systems of axioms which are equivalent classically may no longer be equivalent in our quantum logic — we investigate this property further is section 3.2. We conclude this chapter with a general discussion concerning the construction of models for axiom systems in first-order quantum logic, and prove some technical results which will be useful in constructing models in the sequel.

Summary of Chapter 4: Quantum Algebraic Systems

We begin this chapter with a discussion of equational languages, i.e. languages whose only predicate is 'equality'. These languages form the basis for axiomatizing abstract algebras. After proving that certain classes of axioms are inherently classical,[1] we then proceed to construct models of groups and monoids which exemplify certain behaviors which are new to quantum mathematics. We then present natural models of quantum OMLs, vector spaces, and operator algebras — all of which are beautifully interrelated — and make some preliminary investigations into developing the theory of quantum mechanics based on these natural quantum mathematical models. In particular, we examine both the Schrödinger and von Neumann equations within this context (although still assuming a classical mathematical theory of the derivative), and discuss how a quantum mathematical treatment of these equations may possibly yield solutions which yield dynamical evolutions which go beyond the usual unitary dynamics.

Summary of Chapter 5: Quantum Set Theory

In this chapter we begin by reviewing classical axiomatic set theory, and discuss both the ZFC axioms and the classical set theoretic universe. We then proceed to develop a modification

[1]By 'inherently classical', we mean that the theorems formally provable from those axioms are the same in classical logic as in quantum logic.

of these axioms of classical set theory, and develop a natural quantum model of these axioms. This model has a rich enough structure that it can serve as the basis for the quantum arithmetic we develop in the next chapter. Moreover, we find that a number of properties of projection lattices which are not shared by all OMLs are crucial to showing that some of the axioms of quantum set theory hold, and this suggests there may be multiple important logical features of the projection lattices besides the fact that they are orthomodular lattices. We conclude this chapter by comparing the set theory we develop to the (more technically unwieldy) quantum set theory developed by Takeuti (42).

Summary of Chapter 6: Quantum Arithmetic

After recalling a theorem of Dunn (15) which states that the usual Peano axioms are inherently classical, we use the quantum set theory from the previous chapter to develop "quantum natural numbers" in a fashion analogous to the construction of the ordinary natural numbers $\mathbb{N}$ within ordinary set theory. We then develop an addition and multiplication meant to satisfy an alternative axiomatization of arithmetic (so as to avoid Dunn's theorem) over these "quantum natural numbers". The development of this arithmetic yields two intriguing results.

First, we find that all of our arithmetical axioms hold on the natural models constructed iff the underlying truth values form a *modular lattice*, giving an arithmetical characterization of this important lattice-theoretic property. Second, when using projection lattices for the truth values, the "quantum natural numbers" which model our quantum arithmetic have a natural interpretation as bounded observables with whole number eigenvalues — and what is more, the sum and product operations given by our quantum arithmetic have a number of intriguing properties when interpreted in terms of observables. First, they reduce to the usual operator sum and product for the case of commuting observables. Second, this new sum and product are both commutative operations, and in particular, even the multiplication is commutative when the observables being multiplied *don't* commute. And finally, every eigenvalue of the quantum arithmetical sum (respectively product) of any two given observables A and B is equal to the sum (respectively product) of an eigenvalue of A with an eigenvalue of B. We conclude the chapter with a preliminary investigation into the physical relevance of this new sum and product.

Summary of Chapter 7: Conclusion

In the final chapter we summarize the major results, as well as identify new questions raised by our investigation of quantum mathematics.

Summary of Appendix

In appendix A we review the (classical) set theory needed for the rest of this work, including a treatment of classes, transfinite induction, and the ordinal numbers. In appendix B we review some basic universal algebra, such as free algebras, homomorphisms/isomorphisms, congruences, and product algebras along with the notion of irreducibility. In appendix C we provide a number of technical results in lattice theory which we will frequently use when proving various statements.

CHAPTER 2

FIRST-ORDER QUANTUM LOGIC AND QUANTUM MATHEMATICS

We now jump into the promised development of a first-order quantum logic which is rich enough to form a starting point for the construction of quantum mathematics. In this chapter we first begin by defining the syntax for our logic (section 2.1). This discussion closely parallels the development of syntax for classical first-order logic (see (18; 49), for example). In section 2.2 we then construct our system of formal deduction, and proceed to demonstrate a number of formal proofs within this system, both to help build intuition for our quantum logic, as well as to prove a number of useful technical results which we will find application for in the sequel. In the following section (2.3) we develop a semantics for our quantum logic. This semantics is similar to that for first-order classical logic except that it allows the truth values to consist of orthomodular lattices, rather than the usual $B_2 = \{0, 1\}$.

After our semantics has been constructed, we then prove both soundness and completeness of our formal deductive system relative to this semantics in section 2.4. After proving these theorems, we then establish the promised more powerful completeness result which allows us to focus on models with a particular property[1] which will be quite useful in the sequel.

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[1]Namely that the truth value algebras of the models are irreducible OMLs. Models and truth value algebras will be defined later in this chapter.

2.1 Formal Syntax

In this section we set out to define precisely what we mean by first-order quantum mathematical statements, and direct the reader to pay special attention to the notion of *language* defined below. The discussion will mirror the discussion of syntax for classical first-order logic (see (18) or (49), for example) and will formalize both the object language (in which we can make mathematical statements) and some important portions of the metalanguage (in which we make statements about mathematical statements).

2.1.1 The Object Language

We begin by defining the basic symbols with which we will construct statements in our object language. The most significant difference from the propositional logic discussed previously is that the atomic propositions are no longer black boxes — instead these basic propositions are constructed from variables, function symbols and predicate symbols. We then use logical symbols to concatenate atomic propositions as in the propositional case, but with the added symbols '$\forall$' and '$\exists$'.

Syntactical Symbols

Definition 2.1. Define the set $\mathcal{B}_S := \{\wedge, \neg, \forall\}$ to be the set of *basic logical symbols.*[1] Then define the *variables* to be the set $\mathcal{B}_V := \{\upsilon_1, \upsilon_2, \ldots\}$. Finally, $\mathcal{B} := \mathcal{B}_S \cup \mathcal{B}_V$ will be the set of *basic symbols.*

---

[1]The logical symbols mentioned in the introduction which do not appear here ('$\vee$', '$\perp$', etc.) will occur later as *defined* notions — this is a technical trick which will save us a bit of work later.

These basic symbols are common to all the mathematical systems we consider. The symbols $\{\wedge, \neg, \forall\}$ should be interpreted as logical 'and', 'not' and 'for all', respectively. In the sequel we will use the letters x, y, z to stand for arbitrary variables (so they will be 'metavariables' — i.e. variables in the metalanguage which stand for arbitrary elements of $\mathcal{B}_V$). We now define the symbols which will vary depending on the mathematical system we are considering.[1]

Definition 2.2. A *language* is a class of symbols $\mathcal{L}$ such that

1. $\mathcal{L}$ is divided into two disjoint subclasses: $\mathcal{L}^{\mathcal{P}}$ (called the class of *predicate symbols*) which must be nonempty, and $\mathcal{L}^{\mathcal{F}}$ (called the class of *function (or operation) symbols*).
2. $\mathcal{L}$ and $\mathcal{B}$ contain no elements in common.
3. There is a class function $\alpha : \mathcal{L} \to \mathbb{N}$, and for any $s \in \mathcal{L}$, $\alpha(f)$ is called the *arity of* s.

If $f \in \mathcal{L}^{\mathcal{F}}$ with $\alpha(f) = n$, we will call f an *n-ary operation*,[2] and for $P \in \mathcal{L}^{\mathcal{P}}$ with $\alpha(P) = n$ we will call P an *n-ary predicate*.[3] Any language where $\mathcal{L}$ is a set is called a *typical language*. If $\mathcal{L}$ is a proper class, then $\mathcal{L}$ is called a *huge language*.

Consider the following example of a typical language.

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[1]In contrast to the usual development of classical first-order logic, we allow our language to be a proper class — this will be necessary for some technical reasons (namely the development of extended languages) when we construct models of set theory. The reader unfamiliar with classes is referred to appendix A, but little will be lost by assuming all classes are sets.

[2]We will call a 0-ary operation symbol a *constant*.

[3]A 0-ary predicate behaves like a proposition (as in section 1.1.2), and one essentially obtains propositional logic as a special case of first-order logic by simply taking a language which consists entirely of 0-ary predicates.

Example 2.1. Defining the sets $\mathcal{L}^{\mathcal{F}}_{Mon} := \{e, *\}$ and $\mathcal{L}^{\mathcal{P}}_{Mon} = \{\approx\}$, with $\alpha(e) := 0$, $\alpha(*) := 2$, and $\alpha(\approx) := 2$, gives a language $\mathcal{L}_{Mon} := \mathcal{L}^{\mathcal{P}}_{Mon} \cup \mathcal{L}^{\mathcal{F}}_{Mon}$ for monoids[1] (with $\approx$ representing the predicate of equality, 'e' representing the identity element, and '$*$' the multiplication).

Each different mathematical system — such as monoids, groups, lattices, etc. — will have a language (which is not necessarily unique) associated with it.

Formal Statements

Now that we have the notion of a language defined, we need to develop a notion of formal statements in that language.[2] We begin with a notion of a term, which is defined inductively below in the standard way. Informally, a term is what is obtained by applying a function (symbol) to variables.

Definition 2.3. Let $\mathcal{L}$ be a language. A string of symbols t is called an $\mathcal{L}$-*term* if either $t = x$ for $x \in \mathcal{B}_V$ or $t = f(t_1, \ldots, t_{\alpha(f)})$ for each t_i an $\mathcal{L}$-term and $f \in \mathcal{L}^{\mathcal{F}}$. We denote the class[3] of $\mathcal{L}$-terms by $\mathfrak{T}(\mathcal{L})$.

Note: For a binary operation such as '$*$', and terms t, u, we will abuse notation and write $(t * u)$ instead of $*(t, u)$. Recalling example 2.1, in the language $\mathcal{L}_{Mon}$, both $(e * x)$ and $((y * z) * e)$ with $x, y, z \in \mathcal{B}_V$ would be $\mathcal{L}_{Mon}$-terms, while $(e \approx e)$ would not.

---

[1]For those unfamiliar, a monoid is just a set with an associative binary operation and an identity — basically a group except that inverses are not guaranteed to exist.

[2]Of course, different languages will have different allowed formal statements.

[3]The $\mathcal{L}$-terms $\mathfrak{T}(\mathcal{L})$ form a set if $\mathcal{L}$ is a typical language.

Now that we have the notion of a 'term', we can (inductively) define a 'well formed formula' — this is similar to a proposition from propositional logic except that it may contain 'free' variables. For example, rather than a statement like 'the sky is green', a proposition with a free variable would be a statement such as 'x is green' (where x is the variable).

Definition 2.4. Let $\mathcal{L}$ be a language. A string of symbols s is called a *well-formed formula of* $\mathcal{L}$ (or $\mathcal{L}$*-wff*) if s is of the following form

1. $P(t_1, \ldots, t_{\alpha(P)})$ where $P \in \mathcal{L}^{\mathcal{P}}$ and $t_1, \ldots, t_{\alpha(P)}$ are $\mathcal{L}$-terms (such an $\mathcal{L}$-wff is called *atomic*)
2. $\neg A$ for A an $\mathcal{L}$-wff
3. $(A \wedge B)$ for A, B both $\mathcal{L}$-wffs
4. $(\forall x)A$ for $x \in \mathcal{B}_V$ and A an $\mathcal{L}$-wff.

Also, for P a binary predicate, we will define $t_1 P t_2 := P(t_1, t_2)$. We denote the class[1] of $\mathcal{L}$-wffs by $\mathfrak{W}(\mathcal{L})$.

Considering $\mathcal{L}_{Mon}$ from example 2.1 again, both the strings[2] $e \approx e$ and $(x \approx e \wedge e \approx e)$ would be $\mathcal{L}_{Mon}$-wffs, while $(\forall y)(y * e)$ would not.

---

[1]The $\mathcal{L}$-wffs $\mathfrak{W}(\mathcal{L})$ form a set if $\mathcal{L}$ is typical.

[2]Recall $e \approx e$ is (by definition) the same as $\approx (e, e)$.

We use the usual notion of free and bound variables.[1] For an $\mathcal{L}$-wff B ($\mathcal{L}$-term t, respectively), we write $B(x_1, \ldots, x_n)$ ($t(x_1, \ldots, x_n)$, respectively) to indicate that the only free variables occurring in B (t, respectively) are $x_1, \ldots, x_n$. Then, for any $\mathcal{L}$-term t and $\mathcal{L}$-wff $B(x, x_1, \ldots, x_n)$, we write $B(t, x_1, \ldots, x_n)$ to represent the formula B with all free occurrences of x replaced by t (and similarly in other slots). Finally, we define the notion of a sentence, which is the direct analog of a proposition from propositional logic.

Definition 2.5. An $\mathcal{L}$-*sentence* is an $\mathcal{L}$-wff with no free variables.

We now introduces some notation which will be useful in the sequel — for $\mathcal{L}$-wffs A and B, we define[2]

$$A \vee B \;:=\; \neg((\neg A) \wedge (\neg B)), \quad (\exists x)B := \neg(\forall x)(\neg B), \quad A \to B := \neg A \vee (A \wedge B),$$

and then define $A \leftrightarrow B := (A \to B) \wedge (B \to A)$. Also, for some fixed $\mathcal{L}$-wff X, define $\bot := X \wedge \neg X$, and $\top := \neg\bot$.[3] To reduce notational clutter, we take $\neg$ to bind tighter than $\wedge$ and $\vee$, which bind tighter than $\to$, which binds tighter than $\exists$ and $\forall$. We warn the reader that we may occasionally

[1]We eschew a formal definition here, but give an informal description. Any wff A is constructed (as per the inductive definition), and so a variable x which occurs in a wff A must have been added at some step in that construction. If, after the step where x is added, some further step adds $(\forall x)$ (via (4) in def. 2.4), then x *occurs bound in* A. Otherwise x *occurs free in* A. For a formal treatment see (49).

[2]The symbol '$\to$' takes the same form as the Sasaki hook — see def. C.8.

[3]We treat '$\vee$', '$\bot$', etc. as defined notions here (rather than as primitive in our treatment of propositional logic in section 1.1.2) only for the technical reason that this treatment will simplify a large number of proofs in the sequel.

either omit or add parentheses to make things clearer — ideally this will only reduce confusion and never further it.

As can be seen from the above, the syntax involved in first-order logic is quite a bit more complicated than for propositional logic. As we will see later, the semantics is quite a bit more involved as well. For this reason, we examine formal deductions in quantum logic before proceeding to define our semantics.

2.2 Formal Deduction in First-Order Quantum Logic

Now that we have defined our object language, we have all the tools we need to construct a formal deductive system for quantum logic. As in the propositional case, a deductive system consists of axioms and rules of deduction. We first state the axioms for our quantum logic.

Definition 2.6. Let $\mathcal{L}$ be a language. We define the *quantum logical axioms for* $\mathcal{L}$ to be the class of instances of the axiom schema[1] Q1-Q6 below, which we denote by $\mathcal{Q}_A(\mathcal{L})$.

(Q1) $A \to \top \wedge A$ for A any $\mathcal{L}$-wff.

(Q2) $(\neg\neg A) \to A$ and $A \to (\neg\neg A)$ for A any $\mathcal{L}$-wff.

(Q3) $A \wedge B \to A$ and $A \wedge B \to B$ for any $\mathcal{L}$-wffs A and B.

(Q4) $[A \wedge (A \to B))] \to B$ for any $\mathcal{L}$-wffs A and B.

(Q5) $(A \wedge \neg A) \to B$ for any $\mathcal{L}$-wffs A and B

---

[1]Axiom schema are just *forms* for wffs, and an instance of any such schema is a wff created by replacing the symbols A, B with one particular $\mathcal{L}$-wff each, and the symbol t by a particular $\mathcal{L}$-term. For a formal treatment (when $\mathcal{L}$ is typical) we refer the reader to (49). Also, when $\mathcal{L}$ is typical, these axioms form a set.

(Q6) $(\forall x)B(x, x_1, \ldots, x_n) \to B(t, x_1, \ldots, x_n)$ for any $\mathcal{L}$-wff B, and $x, x_1, \ldots, x_n \in \mathcal{B}_V$, and t any $\mathcal{L}$-term.

Note: In general, when presenting an axiom schema like those above, we will use the name of that schema (for example 'Q1') to refer to the entire class of axioms which are instances of that schema.

Having stated the axioms for quantum logic, we can now define the quantum rules of inference for our deductive system.

Definition 2.7. Let $\mathcal{L}$ be a language. We define an *$\mathcal{L}$-rule* to be a pair (Γ, A) where Γ is a finite set of $\mathcal{L}$-wffs and A is an $\mathcal{L}$-wff.

Note: We will also write '$\Gamma \Longrightarrow A$' to denote the $\mathcal{L}$-rule (Γ, A).

Definition 2.8. We then define our *quantum rules of inference* to be the schema

(R1) $\{A \to B, B \to C\} \Longrightarrow A \to C$ for any $\mathcal{L}$-wffs A, B, and C.

(R2) $\{A \to B\} \Longrightarrow \neg B \to \neg A$ for any $\mathcal{L}$-wffs A and B.

(R3) $\{A \to B, A \to C\} \Longrightarrow A \to (B \wedge C)$ for any $\mathcal{L}$-wffs A, B and C.

(R4) $\{A(x_1, \ldots, x_n) \to B(z, x_1, \ldots, x_n)\} \Longrightarrow A(x_1, \ldots, x_n) \to (\forall z)B(z, x_1, \ldots, x_n)$ for any $\mathcal{L}$-wffs A and B, with $z, x_1, \ldots, x_n \in \mathcal{B}_V$ such that z does not occur free in A.

(R5) $\{A, A \to B\} \Longrightarrow B$ for any $\mathcal{L}$-wffs A and B.

We denote the class[1] of all quantum rules of inference[2] for the language $\mathcal{L}$ by $\mathcal{Q}_R(\mathcal{L})$, and define $\mathcal{Q}(\mathcal{L}) := \mathcal{Q}_A(\mathcal{L}) \cup \mathcal{Q}_R(\mathcal{L})$.

Note: R5 is just the standard modus ponens.

Our quantum axioms Q1-Q6 and quantum rules of inference R1-R4 appear in Dunn (15) numbered 1-10, and he refers to this system as OM#, although his framework for treating first-order logic is slightly different.[3]

Now that we have established our quantum axioms and rules of inference, we can precisely define formal deductions within quantum logic.

Definition 2.9. Let $\mathcal{L}$ be a language, and Γ a class of $\mathcal{L}$-wffs. An $\mathcal{L}$-wff A is said to be *derivable from* Γ (denoted $\Gamma \vdash A$) if there exists a finite sequence $A_1, A_2, \ldots, A_n$ (called a *formal proof of* A *by* Γ) such that $A_n = A$ and for each $i < n$, either $A_i \in \Gamma \cup \mathcal{Q}_A(\mathcal{L})$ or there exists a subset $\Gamma_0 \subseteq \{A_1, \ldots A_{i-1}\}$ such that $\Gamma_0 \Longrightarrow A_i$ by some $\mathcal{L}$-rule (R1-R5). A class $\mathcal{A}$ of $\mathcal{L}$-wffs is then said to be *derivable from* Γ (denoted $\Gamma \vdash \mathcal{A}$) if every $A \in \mathcal{A}$ is derivable from Γ.

---

[1]Again, when $\mathcal{L}$ is typical, the quantum inferences for a set rather than a proper class.

[2]By which we mean all instances of the rules schema.

[3]We have transferred his 'relational logic' approach into a more standard treatment of axioms and derivability. Slight differences in how the two approaches handle deduction has forced us to add rule R5 and modify Q1 slightly.

<u>Note</u>: As a notational shorthand, given a class of $\mathcal{L}$-wffs Γ, and two $\mathcal{L}$-wffs A, B, we will write $\Gamma, A \vdash B$ to mean $\Gamma \cup \{A\} \vdash B$ and $A \vdash B$ to mean $\{A\} \vdash B$. Also, we will write $\vdash A$ to mean $\varnothing \vdash A$.

The (meta-language) statement 'A is derivable from Γ' informally means that one can construct a proof of A (within quantum logic) from the class of statements Γ. It is easy to see that this deduction system is monotonic — i.e. for two sets of $\mathcal{L}$-wffs Γ, Γ' with $\Gamma \subseteq \Gamma'$, whenever $\Gamma \vdash A$ then also $\Gamma' \vdash A$ (for any $\mathcal{L}$-wff A).

Another important notion is that of logical equivalence.

Definition 2.10. Let $\mathcal{L}$ be a language, let A and B be $\mathcal{L}$-wffs A and B, and let Γ be a class of $\mathcal{L}$-wffs. We say that A and B are *logically equivalent with respect to* Γ (denoted $A \sim_\Gamma B$) if $\Gamma \vdash A \leftrightarrow B$. If $\Gamma = \varnothing$, we simply say that A and B are *logically equivalent* (denoted $A \sim B$).

<u>Note</u>: Since our logic is monotonic, given two classes of $\mathcal{L}$-wff Γ, Γ' with $\Gamma \subseteq \Gamma'$, if two $\mathcal{L}$-wffs A and B are logically equivalent w.r.t. Γ, then A and B are also logically equivalent w.r.t. Γ'.

We present examples of derivability and logical equivalence in section 2.2.1, and we note that logical equivalence is, indeed, an equivalence relation (see proposition 2.5). Finally, we define a formal notion of a theorem in a given axiom system.

Definition 2.11. Let $\mathcal{L}$ be a language, and Γ a class of $\mathcal{L}$-wffs. The *quantum theorems of* Γ (or simply *theorems of* Γ), are defined to be the class of all $\mathcal{L}$-wffs derivable from Γ.

We conclude our discussion of formal deductions in quantum logic by defining a formal notion of a mathematical system.

Definition 2.12. A *mathematical system* (or *M-system*) is a pair $(\mathcal{L}, \mathcal{A})$ where $\mathcal{L}$ is a language, and $\mathcal{A}$ is a class of $\mathcal{L}$-wffs.

Note: For a given M-system, we will informally refer to the set $\mathcal{A}$ above as the *axioms* of that M-system.

2.2.1 Some Formal Arguments

In this section we will provide some examples utilizing the formal deduction system discussed above. Other than helping the reading build their intuition as regards formal proofs, the only purpose of this section is to develop a number of technical results which will be useful in the proof of various statements, most notably the completeness theorem. The reader less interested in these technical details can skip ahead to the next section without loss of continuity.

For the remainder of this section, let $\mathcal{L}$ be some fixed language.

Proposition 2.2. Let A be any $\mathcal{L}$-wff. Then $\vdash A \rightarrow A$.

Proof. We construct the following formal proof —

$s_1 := A \rightarrow \top \wedge A$ (Q1)

$s_2 := \top \wedge A \rightarrow A$ (by Q3)

$s_3 := A \rightarrow A$ (by R1 from s_1 and s_2)

□

Proposition 2.3. Let A, B, and C be $\mathcal{L}$-wffs. Then

$$[A \rightarrow (B \rightarrow C)] \vdash [(A \wedge B) \rightarrow C]. \tag{2.1}$$

Proof.

$s_1 := (A \wedge B) \;\rightarrow\; A$ (by Q3)

$s_2 := (A \wedge B) \;\rightarrow\; B$ (by Q3)

$s_3 := A \;\rightarrow\; (B \rightarrow C)$ (by assumption)

$s_4 := (A \wedge B) \rightarrow (B \rightarrow C)$ (by R1 from s_1 and s_3)

$s_5 := (A \wedge B) \rightarrow B \wedge (B \rightarrow C)$ (by R3 from s_2 and s_4)

$s_6 := [B \wedge (B \rightarrow C)] \;\rightarrow\; C$ (by Q4)

$s_7 := (A \wedge B) \;\rightarrow\; C$ (by R1 from s_5 and s_6)

□

Typically we will be more informal in demonstrating that an $\mathcal{L}$-wff is derivable. The following proposition collects a number of formal deductions we will use in the sequel.

Proposition 2.4. Let A, B, and C be $\mathcal{L}$-wffs, and let Γ be any class of $\mathcal{L}$-wffs. Then

1. $\vdash \top$
2. $\top \rightarrow A \vdash A$
3. $\vdash A \rightarrow B \vee A$ and $\vdash A \rightarrow A \vee B$
4. $A \vdash \top \rightarrow A$
5. $\{A, B\} \vdash A \wedge B$
6. If $\Gamma \vdash A \rightarrow B$ then $\Gamma, A \vdash B$.
7. $\vdash A \wedge B \leftrightarrow B \wedge A$
8. $\vdash A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C$

9. $\vdash \neg\neg A \leftrightarrow A$.

10. $\vdash A \wedge \bot \leftrightarrow \bot$.

11. $\vdash A \wedge (A \vee B) \leftrightarrow A$

12. $\vdash A \wedge B \leftrightarrow A \wedge (\neg A \vee (A \wedge B))$

Proof.

1. Let $Z := (\neg\neg A) \to A$ for some fixed $\mathcal{L}$-wff A, so we have Z by Q2. Then $\bot \to \neg Z$ by Q5. R2 then gives $\neg\neg Z \to \top$. But $Z \to \neg\neg Z$ by Q2, so R1 yields $Z \to \top$. Modus ponens (R5) then gives $\top$.

2. From (1) above, we have $\top$. By assumption we have $\top \to A$, and so R5 yields A.

3. First $(\neg B \wedge \neg A) \to \neg A$ by Q3. This then yields $\neg\neg A \to B \vee A$ by R2. By Q2 we have $A \to \neg\neg A$, and so R1 then gives $A \to B \vee A$. The other statment follows similarly.

4. Q1 gives $A \to \top \wedge A$. By assumption we have A, so R5 gives $\top \wedge A$, and so by (3) above using R5 we have $\neg\top \vee (\top \wedge A) = \top \to A$.

5. By (4) above, we have $\top \to A$ and $\top \to B$. Then by R3, this gives $\top \to (A \wedge B)$, which by (2) above gives $A \wedge B$.

6. First assume Γ, A. From Γ we have $A \to B$, and we have A by assumption, so that R5 yields B.

7. We have $A \wedge B \to B$, as well as $A \wedge B \to A$ by Q3. R3 then gives $A \wedge B \to B \wedge A$. In similar fashion, we have $B \wedge A \to A \wedge B$. Then (5) above yields $A \wedge B \leftrightarrow B \wedge A$.

8. We have $A \wedge (B \wedge C) \to A$ and $A \wedge (B \wedge C) \to (B \wedge C)$ by Q3. Then by Q3 again we have $B \wedge C \to B$ and $B \wedge C \to C$, so R1 gives $A \wedge (B \wedge C) \to B$ and $A \wedge (B \wedge C) \to C$. Then, using R3 we obtain $A \wedge (B \wedge C) \to (A \wedge B)$ and R3 again yields $A \wedge (B \wedge C) \to (A \wedge B) \wedge C$. A similar argument yields the other arrow, and (5) Above gives the desired conclusion.

9. Follows trivially from Q2 and (5) above.

10. By Q3 we have $A \wedge \perp \to \perp$. For the other implication, we have $\perp \to A \wedge \perp$ by Q5 (and the definition of '$\perp$'), and so (5) above yields $A \wedge \perp \leftrightarrow \perp$.

11. By Q3 we have $A \wedge (A \vee B) \to A$. For the other implication, we have $A \to A \vee B$ by (3) above. By (1) above we also have $A \to A$, and so these two together give $A \to A \wedge (A \vee B)$ by R3. The result then follows from (5) above.

12. First, we have $A \wedge B \to (\neg A \vee (A \wedge B)$ by (3) above, and $A \wedge B \to A$ by Q3, so R3 yields $A \wedge B \to A \wedge (\neg A \vee (A \wedge B))$, establishing the one implication. For the other, we first that by definition $\neg A \vee (A \wedge B) = A \to B$, so what we need to show is $A \wedge (A \to B) \to A \wedge B$. Now, we have $A \wedge (A \to B) \to B$ directly by Q4, and $A \wedge (A \to B) \to A$ by Q3. These two then give $\vdash A \wedge (A \to B) \to A \wedge B$ by R3. Both implications then give

$$\vdash A \wedge B \leftrightarrow A \wedge (\neg A \vee (A \wedge B))$$

by (5) above.

$\square$

We can use the results of the above proposition to show that logical equivalence is, indeed, an equivalence relation.

Proposition 2.5. Let $\mathcal{L}$ be a language, and let Γ be a class of $\mathcal{L}$-wffs. Then (for any $\mathcal{L}$-wffs A, B, and C), we have

Reflexivity: $A \sim_\Gamma A$.

Symmetry: If $A \sim_\Gamma B$, then $B \sim_\Gamma A$.

Transitivity: If $A \sim_\Gamma B$ and $B \sim_\Gamma C$, then $A \sim_\Gamma C$.

Proof. Considering first reflexivity, let A be any wff. Since $\vdash A \to A$ follows from Q1, we have that $\vdash A \leftrightarrow A$ by proposition 2.4, and hence $\Gamma \vdash A \leftrightarrow A$. Moving on to symmetry, consider wffs A and B such that $A \sim_\Gamma B$, i.e.

$$\Gamma \vdash A \leftrightarrow B = (A \to B) \wedge (B \to A),$$

so by Q3 and R5 we have $\Gamma \vdash A \to B$ and $\Gamma \vdash B \to A$, and then using (5) in proposition 2.4 gives that $\Gamma \vdash B \leftrightarrow A$. Finally, we consider transitivity of $\sim_\Gamma$, so let A, B and C be wffs such that $\Gamma \vdash A \leftrightarrow B$ and $\Gamma \vdash B \leftrightarrow C$. By the same reasoning as above we have both $\Gamma \vdash A \to B$ and $\Gamma \vdash B \to C$, so that R1 yields $\Gamma \vdash A \to C$. Similarly, we have $\Gamma \vdash C \to A$, and so $A \sim_\Gamma C$, again by (5) in by proposition 2.4. $\square$

We also prove the following technical lemma which shows that bound variables are 'dummy variables' — this will be useful in the proof of the completeness theorem.

Lemma 2.6. Let $\mathcal{L}$ be a language, let Γ be a set of $\mathcal{L}$-wffs, and let A be an $\mathcal{L}$-wff such that $x \in \mathcal{B}_V$ *only* occurs free (i.e. there are no bound occurrences of x) in A, and let $y \in \mathcal{B}_V$ such that y does not occur free in A. Then, defining A_y to be the $\mathcal{L}$-wff obtained by replacing every instance of x in A by y, we have

$$(\forall x)A \sim_\Gamma (\forall y)A_y.$$

Proof. $A = A(x, x_1, \ldots, x_n)$ where $x_i \neq y$ for all $i \in \{1, \ldots, n\}$ by assumption. Also by assumption, $A_y = A_y(y, x_1, \ldots, x_n)$. Then by R4 we have $\Gamma \vdash A \to (\forall y)A_y$, and since $\Gamma \vdash (\forall x)A \to A$ by Q6, R1 then gives that $\Gamma \vdash (\forall x)A \to (\forall y)A_y$.

For the other implication, we have $\Gamma \vdash (\forall y)A_y \to A$ by Q6, and so R4 yields $\Gamma \vdash (\forall y)A_y \to (\forall x)A$. Hence, by (5) in proposition 2.4, we have $\Gamma \vdash (\forall x)A \leftrightarrow (\forall y)A_y$. $\square$

We will prove one more technical result useful in the sequel.

Lemma 2.7. Let A, B, C, and D be $\mathcal{L}$-wffs. Then

$$\{A \to C, B \to D\} \vdash (A \vee B) \to (C \vee D).$$

Proof. First, $\neg C \to \neg A$ and $\neg D \to \neg B$ by R2, and then $(\neg C \wedge \neg D) \to \neg A$ and $(\neg C \wedge \neg D) \to \neg B$ by Q3 and R1. Then $(\neg C \wedge \neg D) \to (\neg A \ \wedge \neg B)$ by R3 and then, by R2 and the definition of $\vee$, this gives $(A \vee B) \to (C \vee D)$. $\square$

We conclude this section with an extremely useful technical result whose proof is similar to that for the classical first-order predicate calculus.

Lemma 2.8. *Replacement:* Let $(\mathcal{L}, \mathcal{A})$ be any M-system, and let A and B be $\mathcal{L}$-wffs which are logically equivalent with respect to $\mathcal{A}$. Further let C_A and C_B be $\mathcal{L}$-wffs which are identical except that one instance of A in C_A is replaced by B in C_B. Then C_A and C_B are logically equivalent with respect to $\mathcal{A}$.

Proof. Assume that A and B are logically equivalent, so that $\mathcal{A} \vdash A \leftrightarrow B$. We proceed by induction on the construction of $\mathcal{L}$-wffs, so let ψ_A and ψ_B are $\mathcal{L}$-wffs such as in the hypothesis with $\mathcal{A} \vdash \psi_A \leftrightarrow \psi_B$. Then $\psi_A \to \psi_B$ by Q3, and so $\neg\psi_B \to \neg\psi_A$ by R2. Similarly we have $\neg\psi_A \to \neg\psi_B$, and so (5) in proposition 2.4 gives $\neg\psi_A \leftrightarrow \neg\psi_B$. Let D be some other $\mathcal{L}$-wff. $\psi_A \wedge D \to \psi_A$ and $\psi_A \to D$ by Q3, and so $\psi_A \wedge D \to \psi_B$ by R1 and the inductive hypothesis, and then R3 gives $\psi_A \wedge D \to \psi_B \wedge D$. Similarly we have $\psi_B \wedge D \to \psi_A \wedge D$, and so (5) in proposition 2.4 gives $\psi_A \wedge D \leftrightarrow \psi_B \wedge D$. By (7) in the same proposition we have $D \wedge \psi_A \leftrightarrow D \wedge \psi_B$. Finally, assume all free variables in ψ_A, ψ_B are listed $(x, y_1, \ldots, y_n)$. Then by Q6,

$$\psi_A(x, y_1, \ldots, y_n) \to (\forall x)(\psi_A(x, y_1, \ldots, y_n)),$$

and also $\psi_B \to \varphi_A$ by inductive hypothesis and Q3. R1 then gives

$$\psi_B(x, y_1, \ldots, y_n) \to (\forall x)(\varphi_A(x, y_1, \ldots, y_n)).$$

Q6 gives

$$(\forall x)(\psi_B(x, y_1, \ldots, y_n)) \to \psi_B(x, y_1, \ldots, y_n),$$

and so R1 again gives

$$(\forall x)(\psi_B(x, y_1, \ldots, y_n)) \to (\forall x)(\psi_A(x, y_1, \ldots, y_n)).$$

The other arrow is derived similarly, so (5) in proposition 2.4 gives the desired conclusion. $\square$

Note: Using induction this can be extended to any number of replacements of A by B.

2.3 Semantics for First-Order Quantum Logic

We now transition to constructing a semantics for our first-order quantum logic, that is, we will build formal structures which can serve as models for quantum axiom systems. Since this construction is somewhat involved, we will a give a brief preview. A structure for a given language will have a number of moving pieces — first, there is an underlying class (which the variables are taken to "run over"), and this class is equipped with an n-ary operation for each n-ary operation symbol in the associated language. Next, a structure will also have an OML which will serve as the truth values (which we will call the 'truth value algebra'). Finally, a structure has a truth valuation which assigns, to each sentence[1] in the language, an element of the truth value algebra — like the propositional truth functions seen in section 1.1.2, these truth valuations are required to play well with the logical symbols. For the reader familiar with a development of the semantics of classical first-order logic, we note that our construction is

[1]It would be a little unnatural to expect a truth valuation to assign truth values to wffs which contain free variables — to know if 'x is green' is a true statement, one must know what x is.

identical to the semantics used in that case except that our truth value algebra is not constrained to be $B_2 = \{0,1\}$, and is instead allowed to be any OML. (See (18) or (49), for example.)

2.3.1 Extended Languages and Structures

We begin with some technical preliminaries about extending languages, after which we define, for any language $\mathcal{L}$, the notion of an $\mathcal{L}$-structure.

Extending Languages

In constructing models below, we will begin with some class[1] $\mathfrak{A}$. It will be useful to have a method of referring to the elements of $\mathfrak{A}$ formally — to this end we need a construction which extends the language $\mathcal{L}$ we are working over by adding symbols which refer to the elements of the class $\mathfrak{A}$.

Definition 2.13. Let $\mathcal{L}$ be a language, and $\mathfrak{A}$ a class such that $\mathcal{L} \cup \mathcal{B}$ and $\mathfrak{A}$ have no elements in common. Then the *extension of* $\mathcal{L}$ *by* $\mathfrak{A}$ (denote $\mathcal{L}^{+\mathfrak{A}}$) is defined by

$$\mathcal{L}^{+\mathfrak{A}} := \mathcal{L} \cup \mathfrak{A} = \{s \ : \ s \in \mathcal{L} \text{ or } s \in \mathfrak{A}\},$$

where the arity function α is extended such that $\alpha(a) = 0$ for every $a \in \mathfrak{A}$. We then say that any $\mathcal{L}^{+\mathfrak{A}}$-wff is an $\mathfrak{A}$-*extended* $\mathcal{L}$ wff, and similarly $\mathcal{L}^{+\mathfrak{A}}$-sentences are called $\mathfrak{A}$-*extended* $\mathcal{L}$-sentences.

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[1] As with all our shenanigans involving classes, we will only need proper classes in the context of axiomatic set theory.

Note: When the class $\mathfrak{A}$ (as well as the language $\mathcal{L}$) is clear from the context, we will sometimes refer to $\mathfrak{A}$-extended wffs (resp. sentences) simply as *extended wffs* (resp. *extended sentences*). It is clear from the above definition that any $\mathcal{L}$-wff is also an extended $\mathcal{L}$-wff.

Interpreting Function Symbols

In order to construct a model, we will also need a way to interpret the function symbols in our language as actual functions on some underlying class $\mathfrak{A}$, which leads us to the following definition.

Definition 2.14. Let $\mathcal{L}$ be a language and $\mathfrak{A}$ a class. An *interpretation of* $\mathcal{L}^{\mathcal{F}}$ *in* $\mathfrak{A}$ is a class function $\mathfrak{F}$ with domain $\mathcal{L}^{\mathcal{F}}$ such that $\mathfrak{F}(f)$ is an $\alpha(f)$-ary operation on $\mathfrak{A}$ for every $f \in \mathcal{L}^{\mathcal{F}}$.

Note: Typically for the action of $\mathfrak{F}$ on $\mathcal{L}^{\mathcal{F}}$, we will take $f \mapsto \hat{f}$ or $f \mapsto f$ (using the same symbol for the function symbol and its interpretation) rather than the more cumbersome $f \mapsto \mathfrak{F}(f)$.

We can now define a notion of 'evaluation' of functions. The induction used in this definition makes it seem more complicated than it is, as the example following the definition illustrates.

Definition 2.15. Let $\mathcal{L}$ be a language, let $\mathfrak{A}$ a class, and let $\mathfrak{F}$ be an interpretation of $\mathcal{L}^{\mathcal{F}}$ in $\mathfrak{A}$ (represented by $f \mapsto \hat{f}$ for all $f \in \mathcal{L}^{\mathcal{F}}$). Further let $t(x_1, \ldots, x_n)$ be an $\mathfrak{A}$-extended $\mathcal{L}$-term (with $x_1, \ldots, x_n \in \mathcal{B}_V$), and let $a_1, \ldots, a_n \in \mathfrak{A}$. Then the $\mathfrak{F}$*-evaluation of* $t(x_1, \ldots, x_n)$ *at* $(a_1, \ldots, a_n)$ (denoted $\tilde{t}(a_1, \ldots, a_n)$) is defined (inductively on the construction of $\mathcal{L}$-terms) by

1. $\tilde{t}(a_1, \ldots, a_n) \coloneqq a_i$ if $t(x_1, \ldots, x_n) = x_i$
2. $\tilde{a} \coloneqq a$ if $a \in \mathfrak{A}$.

3. $\tilde{t}(a_1,\ldots,a_n) \coloneqq \hat{f}(\tilde{t}_1(a_1,\ldots,a_n),\ldots,\tilde{t}_m(a_1,\ldots,a_n))$ if, for some $f \in \mathcal{L}^{\mathcal{F}}$ and some terms $t_1(x_1,\ldots,x_n),\ldots,t_m(x_1,\ldots,x_n)$, we have $t(x_1,\ldots,x_n) = f(t_1(x_1,\ldots,x_n),\ldots,t_m(x_1,\ldots,x_n))$.

If $\tilde{u}$ is an $\mathfrak{F}$-evaluation of some term u, we call $\tilde{u}$ an *evaluated $\mathcal{L}$-term.*

Example 2.9. Consider the language $\mathcal{L}_{Mon}$ from example 2.1, and let $\mathfrak{A} = \{a,b\}$ with the interpretation of the function symbols defined by $\hat{e} \coloneqq a$, and $x \hat{*} y = a$ for all $x, y \in \mathfrak{A}$. Consider the terms $u(z) \coloneqq z$ and $t(z) \coloneqq u(z) * e$. Then considering evaluations of these terms, we have $\tilde{u}(b) = b$, and also $\tilde{t}(b) = b \hat{*} a = a$. Note that the set $\{a,b\}$ with this multiplication does *not* form a monoid! We still have not introduced any connection between evaluation of terms and axioms which could force any sort of monoidal behavior.[1]

Note that a term evaluated in $\mathfrak{A}$ is always an element of $\mathfrak{A}$.

Truth Valuations

While in classical logic the truth values were restricted to the set $\{0,1\}$, in first-order quantum logic (just as in the propositional case) we generically have a much larger set of possibilities for the truth values. Unlike propositional quantum logic, where we could use any OML L for the truth values, for first-order quantum logic we will also require that enough meets exist to evaluate any statement in the object language involving the symbol '$\forall$'. We make this precise in the following definitions.

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[1]The bumper-sticker phrase to remember will be (after we have all the definitions and terminology sorted out) "Structures don't satisfy axioms — models do".

Definition 2.16. Let $\mathcal{L}$ be a language, $\mathfrak{A}$ a class, and L an orthomodular lattice.[1] Further let $\mathfrak{F}$ be an interpretation of $\mathcal{L}^{\mathcal{F}}$ in $\mathfrak{A}$, let $\mathfrak{S}$ be the class of $\mathfrak{A}$-extended $\mathcal{L}$-sentences, and let $[\![\cdot]\!]$ be a class function from $\mathfrak{S}$ to L. Then $[\![\cdot]\!]$ is called a *truth valuation w.r.t.* $(\mathcal{L}, \mathfrak{A}, \mathfrak{F}, L)$ if it satisfies (for every n-nary predicate P, all $\mathfrak{A}$-extended $\mathcal{L}$-terms $t_1, \ldots, t_n$ with no variables, and all $\mathfrak{A}$-extended $\mathcal{L}$-sentences B, C, and $\mathfrak{A}$-extended $\mathcal{L}$-wffs $D(x)$.)

1. $[\![P(t_1, \ldots, t_n)]\!] = [\![P(\tilde{t}_1, \ldots, \tilde{t}_n)]\!]$, where for each $i \in \{1, \ldots, n\}$, $\tilde{t}_i$ is the evaluation of t_i in $\mathfrak{A}$ with respect to $\mathfrak{F}$.
2. $[\![\neg B]\!] = \neg[\![B]\!]$.
3. $[\![B \wedge C]\!] = [\![B]\!] \wedge [\![C]\!]$.
4. Whenever $\bigwedge_{a \in \mathfrak{A}} [\![D(a)]\!]$ exists, $[\![(\forall x)D(x)]\!] = \bigwedge_{a \in \mathfrak{A}} [\![D(a)]\!]$.
5. $\bigwedge_{a \in \mathfrak{A}} [\![D(a)]\!]$ exists.

If $[\![\cdot]\!]$ only satisfies (1-4) above, then $[\![\cdot]\!]$ is called a *quasi-valuation w.r.t.* $(\mathcal{L}, \mathfrak{A}, \mathfrak{F}, L)$.

Note: We will occasionally drop the 'w.r.t.' clauses and simply refer to the notions defined above as 'truth valuations' and 'quasi-valuations' when the extra information is either clear or irrelevant. Note that a truth valuation is determined by its action on (a certain subclass of) the extended atomic sentences, which follows easily by induction on the construction of wffs (see prop. 3.12). This is illustrated in the following example.

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[1] We will be using '$\neg$' and '$\wedge$' in two different ways — first, as logical symbols in our object language, and second, as operations in the OML L. Context should always serve to distinguish the two meanings.

Example 2.10. Let $\mathcal{L}_{Mon}$ be the language of example 2.1, let $\mathfrak{A}$ be any set, let $\mathfrak{F}$ be any interpretation of $\mathcal{L}^{\mathcal{F}}_{Mon}$ in $\mathfrak{A}$ such that $e \mapsto \hat{e}$ and let $[\![\cdot]\!]$ be a truth valuation w.r.t. $(\mathcal{L}_{Mon}, \mathfrak{A}, \mathfrak{F}, L)$. Then we can compute

$$\begin{aligned}[\![(\forall x)[(\forall y)(x \approx y \wedge x \approx e)]]\!] &= \bigwedge_{a \in A} [\![(\forall y)(a \approx y \wedge a \approx e)]\!] \\ &= \bigwedge_{a \in \mathfrak{A}} \Big[\bigwedge_{b \in \mathfrak{A}} [\![a \approx b \wedge a \approx e]\!] \Big] \\ &= \bigwedge_{a \in \mathfrak{A}} \Big[\bigwedge_{b \in \mathfrak{A}} \big([\![a \approx b]\!] \wedge [\![a \approx \hat{e}]\!] \big) \Big],\end{aligned}$$

where the first two equalities follow by properties (4) and (5) of a truth valuation, and the last equality follows by properties (1) and (3).

By the same inductive reasoning which shows a truth valuation is determined by its action on the extended atomic sentences, it is easy to see that any map from (a certain subclass of) the extended atomic sentences into an orthomodular lattice gives a quasi-valuation. Showing that the quasi-valuation is indeed a truth valuation is the tricky part, which motivates the following definition and lemma.

Definition 2.17. Let $\mathcal{L}$ be a language, let $\mathfrak{A}$ a class, let $\mathfrak{F}$ be an interpretation of $\mathcal{L}^{\mathcal{F}}$ in $\mathfrak{A}$, let L be an OML, and let $[\![\cdot]\!]$ be a quasi-valuation w.r.t. $(\mathcal{L}, \mathfrak{A}, \mathfrak{F}, L)$. If $[\![\cdot]\!]$ also satisfies (5) from definition 2.16, then L is said to be *complete enough for* $[\![\cdot]\!]$.

Lemma 2.11. Let L be a complete lattice, and let $\mathfrak{G}$ be a class function with ran $\mathfrak{G} \subseteq L$ and with any domain. Then

$$\bigwedge_{a \in \mathrm{dom}\ \mathfrak{G}} \mathfrak{G}(a) \quad \text{exists.}$$

Proof. Since L is complete, it suffices to show that ran $\mathfrak{G}$ is a set. But ran $\mathfrak{G} = \{b \in L \ : \ (a, b) \in \mathfrak{G}\}$, which is a set by the (classical) axiom of separation. □

The above lemma is sufficient to prove that any complete lattice is complete enough for any quasi-valuation, so that *any* map defined on a certain subclass of the extended atomic sentences into a *complete* OML will extend uniquely to a truth valuation. This will all be stated and proved formally in proposition 3.12.

$\mathcal{L}$-structures

We are now ready to put together interpretations, evaluations, and truth valuations to construct the structures which will serve as models of mathematical systems.

Definition 2.18. Let $\mathcal{L}$ be a language. An *$\mathcal{L}$-structure* is a sequence $\mathfrak{M} := (\mathfrak{A}, L, \nu, \mathfrak{F})$ where

1. $\mathfrak{A}$ is a class (called the *underlying class*, or *underlying set* if $\mathfrak{A}$ is a set, of $\mathfrak{M}$).

2. $\mathfrak{F}$ is an interpretation of $\mathcal{L}^{\mathcal{F}}$ in $\mathfrak{A}$.

3. L is a orthomodular lattice (called the *truth value algebra of* $\mathfrak{M}$).

4. ν is a truth valuation w.r.t. $(\mathcal{L}, \mathfrak{A}, \mathfrak{F}, L)$ such that ran $\nu = L$ (ν is called the *truth valuation of* $\mathfrak{M}$).[1]

Note: The requirement that ν be a truth valuation forces L to be complete enough for ν. Also, as in the case of wffs and sentences, we will drop the '$\mathcal{L}$' and refer to an $\mathcal{L}$-structure as simply a structure if the language is clear from the context or irrelevant.

Structures are designed to be candidates for models, the only ingredient remaining is a formal notion of what it means for a formal statement to 'hold' in a given structure.

Definition 2.19. Let $\mathcal{L}$ be a language, let $\mathfrak{M} := (\mathfrak{A}, L, [\![\cdot]\!], \mathfrak{F})$ be an $\mathcal{L}$-structure, and let $A(x_1, \ldots, x_n)$ (with $x_1, \ldots, x_n \in \mathcal{B}_V$) be an extended $\mathcal{L}$-wff. Then

$$\bigwedge_{a_1, \ldots, a_n \in \mathfrak{A}} [\![A(a_1, \ldots, a_n)]\!]$$

is called the *truth value of* A *in* $\mathfrak{M}$, and if the truth value of A is 1, we say that A *holds in* $\mathfrak{M}$ (or, equivalently, *is satisfied in* $\mathfrak{M}$), denoted $\mathfrak{M} \vDash A$. If $\mathcal{A}$ is a class of $\mathcal{L}$-wffs which hold in $\mathfrak{M}$, we write $\mathfrak{M} \vDash \mathcal{A}$.

Note: When A is an (extended) sentence, the truth value of A is just $[\![A]\!]$. In effect, the above definition treats any wff with free variables as if there were really 'for all' statements

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[1] We require ran $\nu = L$ to avoid cases where L is non-Boolean but ran ν is a Boolean algebra contained in L. Such cases are, in some sense, classical even though L is not Boolean. See section 2.3.2 for a brief discussion.

preceding the wff. We will abuse notation in the sequel, and for any $\mathcal{L}$-wff A, use $[\![A]\!]$ to denote its truth value, even if it *does* contain free variables.

We conclude the discussion of structures with a simple technical lemma that will be useful when proving that certain axioms hold in models.

Lemma 2.12. Let $\mathcal{L}$ be a language, let $\mathfrak{M}$ be an $\mathcal{L}$-structure with underlying class $\mathfrak{A}$, and let $A(x)$ be an extended $\mathcal{L}$-wff (with $x \in \mathcal{B}_V$). Then $(\forall x)A(x)$ holds in $\mathfrak{M}$ iff $A(a)$ holds in $\mathfrak{M}$ for every $a \in \mathfrak{A}$.

Proof. First, letting $[\![\cdot]\!]$ be the truth valuation of $\mathfrak{M}$, we assume that $A(a)$ holds in $\mathfrak{M}$ for every $a \in \mathfrak{A}$, i.e. that for any $a \in \mathfrak{A}$, we have $[\![A(a)]\!] = 1$. But then we have

$$[\![(\forall x)A(x)]\!] = \bigwedge_{a \in \mathfrak{A}} [\![A(a)]\!] = \bigwedge_{a \in \mathfrak{A}} 1 = 1,$$

so that $(\forall x)A(x)$ holds in $\mathfrak{M}$.

Conversely, assume $(\forall x)A(x)$ holds in $\mathfrak{M}$. Then we have

$$1 = [\![(\forall x)A(x)]\!] = \bigwedge_{a \in \mathfrak{A}} [\![A(a)]\!],$$

which means that $[\![A(a)]\!] = 1$ for all $a \in \mathfrak{A}$. □

The extension of the above result to an arbitrary number of variables follows as a simple corollary.

Corollary 2.13. Let $\mathcal{L}$ be a language, let $\mathfrak{M}$ be an $\mathcal{L}$-structure with underlying class $\mathfrak{A}$, and let $A(x_1, \ldots, x_n)$ (with $x_1, \ldots, x_n \in \mathcal{B}_V$) be an extended $\mathcal{L}$-wff. Then $A(x_1, \ldots, x_n)$ holds in $\mathfrak{M}$ iff $A(a_1, \ldots, a_n)$ holds in $\mathfrak{A}$ for every $a_1, \ldots, a_n \in \mathfrak{A}$.

Proof. This follows trivially from lemma 2.12 by induction. □

2.3.2 Quantum Models

Now that we have a notion of what it means for a wff to be 'satisfied' in a given structure, we are ready to give a formal definition of a model for an M-system in quantum logic.

Definition 2.20. Let $(\mathcal{L}, \mathcal{A})$ be an M-system, and let $\mathfrak{M} := (\mathfrak{A}, L, [\![\cdot]\!], \mathfrak{F})$ be an $\mathcal{L}$-structure. Then we say that $\mathfrak{M}$ is a *model for* $(\mathcal{L}, \mathcal{A})$ (or simply a model for $\mathcal{A}$ if $\mathcal{L}$ is clear from the context) if every $\mathcal{L}$-wff in $\mathcal{A}$ holds in $\mathfrak{M}$.

Of course, since every Boolean algebra is an OML, we can see that the semantics for first-order classical logic is just a special case of the quantum first-order logic we have just developed. This motivates the following definition.

Definition 2.21. Let $\mathfrak{M}$ be a model of an M-system $(\mathcal{L}, \mathcal{A}$ with truth value algebra L. We say that $\mathfrak{M}$ is *standard* if L is a Boolean algebra, otherwise we say that $\mathfrak{M}$ is *non-standard.*

In the usual approach to classical logic, the only admissible models are ones in which the truth value algebra is $B_2 = \{0, 1\}$, the two element Boolean algebra. From this we see that our notion of a standard model goes beyond models typically considered in classical logic. However, as we mentioned in section 1.1.4, an $\mathcal{L}$-wff s holds in all models where L is a Boolean algebra iff

s is true in all models with $L = \{0, 1\}$ (this also follows from our theorem 3), and so in this sense all standard models behave classically, which is why we will focus on non-standard models in this document.[1]

2.4 Soundness and Completeness of Quantum Logic

Having defined both a semantics and a formal deductive system for first-order quantum logic, the first things to look for are soundness and completeness.

In this section, we will first show that our quantum axioms and rules of inference are sound, i.e. for every language $\mathcal{L}$, every $\mathcal{L}$-wff which is derivable from a class of wffs Γ holds in every model of Γ. This intuitively says that our rules of inference will only lead to statements which are true (provided Γ is true). After proving soundness, we consider completeness. A first-order logic is complete when for any language $\mathcal{L}$ and any set of $\mathcal{L}$-wffs Γ, a $\mathcal{L}$-wff A holds in every model of Γ if it is derivable from Γ.

Just as in the propositional case, completeness and soundness are converse implications. To wit, soundness means that if $\Gamma \vdash A$, then in any model $\mathfrak{M}$ of Γ, we have $\mathfrak{M} \vDash A$. Completeness means that if, for any model $\mathfrak{M}$ of Γ, we have $\mathfrak{M} \vDash A$, then $\Gamma \vdash A$. The significant difference from the propositional case is that for first-order logic we must prove soundness and completeness *for any language*.

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[1] Recall the discussion of some peculiarities of models with Boolean truth values in section 1.1.3. We discuss Boolean-valued models of set theory in section 5.1.3.

Most of the remainder of this chapter consists of technical developments which are needed only to prove the soundness and completeness theorems — the reader interested only in the results is referred to theorems 1, 2, and 3 below.

2.4.1 Soundness

First, for a language $\mathcal{L}$, and any given $\mathcal{L}$-structure with truth value algebra L, it is the requirement that L be an OML which forces the quantum axioms $\mathcal{Q}_A(\mathcal{L})$ to hold.

Lemma 2.14. Let $\mathcal{L}$ be a language, and let $\mathfrak{M}$ be an $\mathcal{L}$-structure. Then $\mathfrak{M} \vDash \mathcal{Q}_A(\mathcal{L})$.

Proof. Let $[\![\cdot]\!]$ be the truth valuation of $\mathfrak{M}$. First, to show that Q1 holds by corollary 2.13, it suffices to consider any extended $\mathcal{L}$-sentence A, and show that $A \to (T \wedge A)$ holds in $\mathfrak{M}$. But we have

$$[\![A \to (T \wedge A)]\!] = [\![A]\!] \to \big([\![T]\!] \wedge [\![A]\!]\big) = [\![A]\!] \to [\![A]\!] = 1,$$

where the final equality follows from lemma C.14, and the second to last equality follows from the fact that $[\![T]\!] = 1$, as can be seen by

$$[\![\top]\!] = \neg[\![\bot]\!] = [\![X]\!] \wedge \neg[\![X]\!] = 0$$

where X is the fixed $\mathcal{L}$-wff chosen to define '$\bot$'.

To show Q2, we need to show that both $\neg\neg A \to A$ and $A \to \neg\neg A$ hold in $\mathfrak{M}$ for every extended $\mathcal{L}$-sentence A (again by corollary 2.13), i.e. we need to show that

$$[\![\neg\neg A \to A]\!] = 1 = [\![\neg\neg A \to A]\!],$$

but this follows trivially from lemma C.14 (and the fact that '¬' is an involution in any OML), since

$$[\![\neg\neg A]\!] = \neg[\![\neg A]\!] = \neg\neg[\![A]\!] = [\![A]\!].$$

Moving on to Q3, we need to show that $A \wedge B \rightarrow A$ and $A \wedge B \rightarrow B$ both hold in $\mathfrak{M}$ for all extended $\mathcal{L}$-sentences A, B. We have that

$$[\![A \wedge B \rightarrow A]\!] = \big([\![A]\!] \wedge [\![B]\!]\big) \rightarrow [\![A]\!],$$

and so by lemma C.14, we need to show $[\![A]\!] \wedge [\![B]\!] \leq [\![A]\!]$. But this is trivially true in any OML, and similarly for when A is replaced by B.

To show that Q4 is satisfied, we need to demonstrate that, for every pair of extended $\mathcal{L}$-sentences A and B, we have

$$[\![\big(A \wedge (A \rightarrow B)\big) \rightarrow B]\!] = 1,$$

and by lemma C.14, this is equivalent to

$$[\![A \wedge (A \rightarrow B)]\!] \leq [\![B]\!],$$

but

$$[\![A \wedge (A \rightarrow B)]\!] = [\![A]\!] \wedge (\neg[\![A]\!] \vee ([\![A]\!] \wedge [\![B]\!])) = [\![B]\!],$$

where the last equality is just the orthomodular law, and so Q4 is satisfied since the truth value algebra of $\mathfrak{M}$ is an OML.

Considering Q5, we need to show for any extended $\mathcal{L}$-sentences A and B (again by corollary 2.13 and lemma C.14), that

$$[\![A]\!] \wedge \neg[\![A]\!] \leq [\![B]\!],$$

but this follows trivially since $[\![A]\!] \wedge \neg[\![A]\!] = 0 \leq [\![B]\!]$, again since the truth value algebra is an OML.

Finally, let $\mathfrak{A}$ be the underlying class of $\mathfrak{M}$. Then, for Q6, we consider any $\mathcal{L}$-wff $B(x, x_1, \ldots, x_n)$ (with $x \in \mathcal{B}_V$), and we need to show that (for any $a_1, \ldots, a_n \in \mathfrak{A}$)

$$[\![[(\forall x)B(x, a_1, \ldots, a_n)] \to B(a, a_1, \ldots, a_n)]\!] = 1,$$

for any $a \in \mathfrak{A}$ (by corollary 2.13), and so by lemma C.14, this is equivalent to showing that

$$[\![(\forall x)B(x, a_1, \ldots, a_n)]\!] \leq [\![B(a, a_1, \ldots, a_n)]\!].$$

for any $a \in \mathfrak{A}$. Computing, we have

$$[\![(\forall x)B(x, a_1, \ldots, a_n)]\!] = \bigwedge_{b \in \mathfrak{A}} [\![B(b, a_1, \ldots, a_n)]\!] \leq [\![B(a, a_1, \ldots, a_n)]\!],$$

by definition of the meet in an OML. $\square$

The requirement that the truth value algebra be an OML also insures that the quantum rules of inference produce true statements from true statements.

Lemma 2.15. . Let $\mathcal{L}$ be a language, and let $\mathfrak{M}$ be an $\mathcal{L}$-structure. Further let A, B, and C be $\mathcal{L}$-wffs. Then

1. If $\mathfrak{M} \vDash A \to B$ and $\mathfrak{M} \vDash B \to C$, then $\mathfrak{M} \vDash A \to C$.
2. If $\mathfrak{M} \vDash A \to B$, then $\mathfrak{M} \vDash \neg B \to \neg A$.
3. If $\mathfrak{M} \vDash A \to B$ and $\mathfrak{M} \vDash A \to C$, then $\mathfrak{M} \vDash A \to (B \wedge C)$.
4. If (for $z, x_1, \ldots, x_n \in \mathcal{B}_V$) z does not occur free in $A(x_1, \ldots, x_n)$, and $\mathfrak{M} \vDash A(x_1, \ldots, x_n) \to B(z, x_1, \ldots, x_n)$, then $\mathfrak{M} \vDash A(x_1, \ldots, x_n) \to (\forall x) B(x, x_1, \ldots, x_n)$.
5. If $\mathfrak{M} \vDash A$ and $\mathfrak{M} \vDash A \to B$, then $\mathfrak{M} \vDash B$

Proof. Let $\mathfrak{A}$ be the underlying class of $\mathfrak{M}$, and $[\![\cdot]\!]$ the corresponding truth valuation. Further let $x_1, \ldots, x_n$ be all the variables which occur free in A, B and C.

Considering (1) above, assume $\mathfrak{M} \vDash A \to B$ and $\mathfrak{M} \vDash B \to C$. Then we have

$$1 = \bigwedge_{a_1, \ldots, a_n \in \mathfrak{A}} [\![A(a_1, \ldots, a_n) \to B(a_1, \ldots, a_n)[\![= \bigwedge_{a_1, \ldots, a_n \in \mathfrak{A}} [\![B(a_1, \ldots, a_n) \to C(a_1, \ldots, a_n)]\!],$$

and hence for every $a_1, \ldots, a_n \in \mathfrak{A}$, we have

$$1 = [\![A(a_1, \ldots, a_n) \to B(a_1, \ldots, a_n)]\!] = [\![\Big(B(a_1, \ldots, a_n) \to C(a_1, \ldots, a_n)]\!],$$

so by lemma C.14, this means

$$[\![A(a_1,\ldots,a_n)]\!] \leq [\![B(a_1,\ldots,a_n)]\!] \quad \text{and} \quad [\![B(a_1,\ldots,a_n)]\!] \leq [\![C(a_1,\ldots,a_n)]\!]$$

for every $a_1,\ldots,a_n \in \mathfrak{A}$. So by transitivity of '$\leq$', as well as lemma C.14 and corollary 2.13, this means that $\mathfrak{M} \vDash A \to C$.

Moving on to (2) above, now assume that $\mathfrak{M} \vDash A \to B$, which means that (by corollary 2.13 and lemma C.14)

$$[\![A(a_1,\ldots,a_n)]\!] \leq [\![B(a_1,\ldots,a_n)]\!]$$

for every $a_1,\ldots,a_n \in \mathfrak{A}$, and since the truth value algebra is an OML, this means that (for every $a_1,\ldots,a_n \in \mathfrak{A}$)

$$\neg[\![B(a_1,\ldots,a_n)]\!] \leq \neg[\![A(a_1,\ldots,a_n)]\!],$$

which yields the desired result (again by corollary 2.13 and lemma C.14).

For (3) above, assume that $\mathfrak{M} \vDash A \to B$ and $\mathfrak{M} \vDash A \to C$. Using corollary 2.13 and lemma C.14 again, we know both that for every $a_1,\ldots,a_n \in \mathfrak{A}$, we have

$$[\![A(a_1,\ldots,a_n)]\!] \leq [\![B(a_1,\ldots,a_n)]\!] \quad \text{and} \quad [\![A(a_1,\ldots,a_n)]\!] \leq [\![C(a_1,\ldots,a_n)]\!],$$

as well as that it suffices to show (for every $a_1,\ldots,a_n \in \mathfrak{A}$)

$$[\![A(a_1,\ldots,a_n)]\!] \leq [\![B(a_1,\ldots,a_n)]\!] \wedge [\![C(a_1,\ldots,a_n)]\!],$$

but this follows directly by lemma C.1.

Now considering (4) we can assume[1] (wlog) $x_1 = z$, so that $A = A(x_2,\ldots,x_n)$ and $B = B(x_1,x_2,\ldots,x_n)$ and we also have that $\mathfrak{M} \vDash A(x_2,\ldots,x_n) \to B(z,x_2,\ldots,x_n)$. Again using corollary 2.13 and lemma C.14, we have

$$\llbracket A(a_2,\ldots,a_n)\rrbracket \leq \llbracket B(a_1,a_2,\ldots,a_n)\rrbracket$$

for all $a_1,\ldots,a_n \in \mathfrak{A}$. Meeting over all a_1 gives, again by lemma C.1, that

$$\llbracket A(a_2,\ldots,a_n)\rrbracket \leq \bigwedge_{a\in\mathfrak{A}} \llbracket B(a,a_2,\ldots,a_n)\rrbracket = \llbracket (\forall x)B(x,a_2,\ldots,a_n)\rrbracket,$$

which gives the desired result (by corollary 2.13 and lemma C.14).

Finally, consider (5) we assume that $\mathfrak{M} \vDash A$ and $\mathfrak{M} \vDash A \to B$, which means that

$$\llbracket A(a_1,\ldots,a_n) = 1 \quad \text{and} \quad A(a_1,\ldots,a_n) \leq B(a_1,\ldots,a_n)$$

for every $a_1,\ldots,a_n \in \mathfrak{A}$ (by corollary 2.13 and lemma C.14). But this clearly gives that $B(a_1,\ldots,a_n) = 1$ for every $a_1,\ldots,a_n \in \mathfrak{A}$, so by corollary 2.13 we have $\mathfrak{M} \vDash B$. □

The soundness theorem then follows directly from the above two lemmas.

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[1] This is a slight change in notation from the statement of the lemma which is simply a relabeling for convenience.

Theorem 1. *Soundness of Quantum Logic:* Let $\mathfrak{M}$ be a model of an M-system $(\mathcal{L}, \mathcal{A})$, and let A be an $\mathcal{L}$-wff such that $\mathcal{A} \vdash A$. Then $\mathfrak{M} \vDash A$.

Proof. If A is derivable from $\mathcal{A}$, then let $A_1, \ldots, A_n$ be a formal proof of A from $\mathcal{A}$. We will prove by induction that $\mathfrak{M} \vDash A_i$ for all $i \in \{1, \ldots, n\}$. By definition of a formal proof we must have either $A_1 \in \mathcal{A}$, so that A_1 holds by assumption, or $A_1 \in \mathcal{Q}_A(\mathcal{L})$, so that A_1 holds by lemma 2.14.

For the inductive step, assume that A_j holds for $j < i$ (and $i \in \{2, \ldots, n\}$). Then if $A_j \in \mathcal{A}$ or $A \in \mathcal{Q}_A(\mathcal{L})$, it holds by the same reasoning that A_1 holds. Otherwise, by definition of a formal proof, there must be some quantum rule of inference and some subset of $\{A_1, \ldots, A_{j-1}\}$ from which A_j is derivable. But then A_j holds in $\mathfrak{M}$ by lemma 2.15 (where each quantum rule corresponds to an enumerated item in said lemma). □

2.4.2 Completeness

Our proof of completeness proceeds by means of constructing one particular model for each set of wffs Γ (which we call the Lindenbaum-Tarski model of Γ) in which a given a given wff A holds iff $\Gamma \vdash A$. This basic method has been used to prove completeness for a different set of quantum axioms and rules of inference in (17). Also, we restrict ourselves to typical languages here, although it seems likely the same procedure would work for huge languages with minimal changes.

Note that for a typical language $\mathcal{L}$, the set $\mathfrak{W}(\mathcal{L})$ of $\mathcal{L}$-wffs naturally form a $(2,1,0)$-algebra with operations $(\wedge, \neg, \bot)$. We will take $\mathfrak{W}(\mathcal{L})$ to be endowed with this algebraic structure in the sequel. Also recall, for a given set of $\mathcal{L}$-wffs Γ and $\mathcal{L}$-wffs A and B, that $A \sim_\Gamma B$ iff $\Gamma \vdash A \leftrightarrow B$.

Lemma 2.16. Let $\mathcal{L}$ be a typical language, let Γ be a set of $\mathcal{L}$-wffs, and let $\mathfrak{W}(\mathcal{L})$ be the $(2,1,0)$-algebra of $\mathcal{L}$-wffs. Then $\sim_\Gamma$ is a congruence on F.

Proof. First, $\sim_\Gamma$ is an equivalence relation by proposition 2.5.

Next, we wish to prove that $\sim_\Gamma$ is a congruence for '$\perp$', '$\neg$' and '$\wedge$'. Since '$\perp$' is a constant operation, any '$\sim_\Gamma$' is automatically a congruence for this operation. Next, consider wffs A and B with $A \sim_\Gamma B$, so that $\Gamma \vdash A \leftrightarrow B$, i.e. $\Gamma \vdash (A \to B) \wedge (B \to A)$. By Q3 and R5 we have $\Gamma \vdash A \to B$ and $\Gamma \vdash B \to A$, and so R2 gives $\Gamma \vdash \neg B \to \neg A$ and $\Gamma \vdash \neg A \to \neg B$, which gives $\Gamma \vdash \neg A \leftrightarrow \neg B$ by (5) in proposition 2.4.

Finally, we consider the operation '$\wedge$', so let A_1, A_2, B_1, and B_2 be wffs with $A_1 \sim_\Gamma B_1$ and $A_2 \sim_\Gamma B_2$. As above, we have $\Gamma \vdash A_i \to B_i$ and $\Gamma \vdash B_i \to A_i$ for $i \in \{1,2\}$. Then Q3 gives $\Gamma \vdash A_1 \wedge A_2 \to B_i$ as well as $\Gamma \vdash B_1 \wedge B_2 \to A_i$ for $i \in \{1,2\}$, so that R3 yields both $\Gamma \vdash A_1 \wedge A_2 \to B_1 \wedge B_2$ and $\Gamma \vdash B_1 \wedge B_2 \to A_1 \wedge A_2$, whereby (5) in proposition 2.4 gives that $A_1 \wedge A_2 \sim_\Gamma B_1 \wedge B_2$. $\square$

Now that we have established that for any set of wffs Γ the relation '$\sim_\Gamma$' is a congruence, we next show that the quotient algebra is an OML. Since this algebra will be important in the sequel, we name it.[1]

Definition 2.22. Let $\mathcal{L}$ be a typical language, and let Γ be a set of $\mathcal{L}$-wffs. Then the quotient (2,1,0)-algebra $L_\Gamma := \mathfrak{W}(\mathcal{L})/\sim_\Gamma$ is called the *Lindenbaum-Tarski algebra of* Γ. For a given $A \in \mathfrak{W}(\mathcal{L})$, we denote the equivalence class of A in L_Γ by $[A]_\Gamma$.

---

[1] The notion of a Lindenbaum-Tarski algebra is an old one, originating in (44).

Lemma 2.17. Let $\mathcal{L}$ be a typical language, and let Γ be a set of $\mathcal{L}$-wffs. Then the Lindenbaum-Tarski algebra L_Γ is an orthomodular lattice (as a $(2,1,0)$-algebra).

Proof. By proposition B.5 and lemma 2.16, we know that L_Γ is a (2,1,0)-algebra, we just need to show that it is an OML (see the introduction for definitions).

For this proof (and any $\mathcal{L}$-wff A), we use $[A]$ as shorthand notation for $[A]_\Gamma$. Also, since $\sim_\Gamma$ is a congruence (lemma 2.16), we have that $[\neg A] = \neg[A]$ and $[A \wedge B] = [A] \wedge [B]$.

First, we need to show that for any $A, B \in \mathfrak{W}(\mathcal{L})$, we have $[A] \wedge [B] = [B] \wedge [A]$, which reduces to showing that

$$\Gamma \vdash A \wedge B \leftrightarrow B \wedge A,$$

since $\sim_\Gamma$ is a congruence. But this follows directly from (7) in proposition 2.4.

Next, we need to show that, for any $A, B, C \in \mathfrak{W}(\mathcal{L})$, we have

$$[A] \wedge ([B] \wedge [C]) = ([A] \wedge [B]) \wedge [C],$$

and this reduces to showing

$$\Gamma \vdash A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C.$$

This follows directly from (8) in proposition 2.4.

We now show that for any $A \in \mathfrak{W}(\mathcal{L})$, we have $\neg\neg[A] = [A]$. Again since $\sim_\Gamma$ is a congruence, this reduces to showing that

$$\Gamma \vdash \neg\neg A \leftrightarrow A,$$

but this follows directly from (9) in proposition 2.4.

Next, we will show that for any $A \in \mathfrak{W}(\mathcal{L})$, we have $[A] \wedge [\bot] = [\bot]$. Similar to the above cases, this reduces to showing that

$$\Gamma \vdash A \wedge \bot \leftrightarrow \bot,$$

which follows directly from (10) in proposition 2.4.

Moving on, for $A, B \in \mathfrak{W}(\mathcal{L})$, we will now show that $[A] \wedge ([A] \vee [B]) = [A]$. Again, we reduce this to a statement involving formal deductions:

$$\Gamma \vdash A \wedge (A \vee B) \leftrightarrow A,$$

which follows from (11) in proposition 2.4.

Finally, for $A, B \in \mathfrak{W}(\mathcal{L})$, we need to show that $[A] \wedge [B] = [A] \wedge (\neg[A] \vee ([A] \wedge [B]))$, or equivalently that

$$\Gamma \vdash A \wedge B \leftrightarrow A \wedge (\neg A \vee (A \wedge B)),$$

but this follows from (12) in proposition 2.4. $\square$

We need some technical results before we can use the Lindenbaum-Tarski Algebra to construct our desired model which we will use to prove our completeness theorem.

Lemma 2.18. Let $\mathcal{L}$ be a typical language and Γ a set of $\mathcal{L}$-wffs, and let A, B be $\mathcal{L}$-wffs. Then $\Gamma \vdash A \to B$ iff $[A]_\Gamma \leq [B]_\Gamma$.

Proof. Assume that $\Gamma \vdash A \to B$. Note that we also have $\Gamma \vdash B \to B$ (by proposition 2.2). Together, these give that $\Gamma \vdash A \vee B \to B$, using R2, R3 and the definition of '$\vee$'. Also, we then have that $\Gamma \vdash B \to A \vee B$ (from Q3, R2, Q2 and R1). These two (using (5) from proposition 2.4) give $\Gamma \vdash B \leftrightarrow A \vee B$, i.e. that $[B]_\Gamma = [A]_\Gamma \vee [B]_\Gamma$, which is equivalent to $[A]_\Gamma \leq [B]_\Gamma$ by lemma C.1.

To establish the other implication, now assume that $[A]_\Gamma \leq [B]_\Gamma$, which is equivalent to $[A]_\Gamma \wedge [B]_\Gamma = [A]_\Gamma$ (by lemma C.1), and since $\sim_\Gamma$ is a congruence, this is equivalent to $\Gamma \vdash A \wedge B \leftrightarrow A$, from which we deduce (by Q3 and R5) that $\Gamma \vdash A \to A \wedge B$. Since $\Gamma \vdash A \wedge B \to B$ by Q3, applying R1 gives that $\Gamma \vdash A \to B$. $\square$

In order to construct an appropriate model (for a given set of $\mathcal{L}$-wffs Γ), we now need to create an appropriate underlying set, as well as a truth valuation. For the underlying set, we will use the free algebra (with operations exactly corresponding to the operation symbols in our language) over the variables $\mathcal{B}_V$, which then will carry a natural interpretation of the function symbols (using the correspondence between the operations on the free algebra and the operation symbols of the language).

We will then have a natural candidate for a truth valuation, which we will construct in a straightforward way out of the quotient map from $\mathfrak{W}(\mathcal{L})$ to the quotient algebra $\mathfrak{W}(\mathcal{L})/\sim_\Gamma$. We now begin to introduce this formally.

Definition 2.23. Let $\mathcal{L}$ be a typical language, let T be the free algebra with operations $\{\hat{f} : f \in \mathcal{L}^{\mathcal{F}}\}$ (with the arity of $\hat{f}$ equal to the arity of f) on the set $\{\hat{x} : x \in \mathcal{B}_V\}$.[1] Then T is called the *Lindenbaum-Tarski universe over* $\mathcal{L}$.

The next step in the construction of the aforementioned natural truth valuation involves a technical definition of a map. Unfortunately, the definition is more than a little convoluted — the whole mess is to ensure that we don't accidentally bind variables which would otherwise be free. A close examination of example 2.19 following the next definition will reveal how this might happen.

Definition 2.24. Let $\mathcal{L}$ be a typical language, let Γ be a set of $\mathcal{L}$-wffs, and let T be the Lindenbaum-Tarski universe over $\mathcal{L}$. Let j_t be the map from the set of T-extended $\mathcal{L}$-terms to the set of $\mathcal{L}$-terms which is defined inductively (on the construction of terms *and* elements of the free algebra) as follows.

1. $j_t(\hat{x}) := x$ for any $x \in \mathcal{B}_V$.
2. $j_t(\hat{f}(a_1, \ldots, a_n)) := f(j_t(t_1), \ldots, j_t(t_n))$ for any $\hat{f}$ which is an n-ary operation on T and any $a_1, \ldots, a_n \in T$.
3. $j_t(x) := x$ for any $x \in \mathcal{B}_V$.
4. $j_t(f(t_1, \ldots, t_n)) := f(j_t(t_1), \ldots, j_t(t_n))$ for any $f \in \mathcal{L}^{\mathcal{F}}$ with arity n and any extended terms $t_1, \ldots, t_n$.

---

[1] As a set, we need T to be a *copy* of the set of $\mathcal{L}$-terms so that T is disjoint from $\mathcal{L}$, which is in order to comply with our earlier stipulations on the definitions of models. This is the reason everyone is wearing hats.

Then define the map j_s from the set of T-extended $\mathcal{L}$-wffs to the set of $\mathcal{L}$-wffs inductively (on the construction of wffs) as follows:

1. $j_s(P(t_1, \ldots, t_n)) := P(j_t(t_1), \ldots, j_t(t_n))$ for P an n-ary predicate and $t_1, \ldots, t_n$ extended terms.
2. $j_s(A \wedge B) := j_s(A) \wedge j_s(B)$ for any extended wffs A and B.
3. $j_s(\neg A) := \neg j_s(A)$ for any extended wff A.
4. $j_s((\forall x)A) := (\forall x) j_s(A)$ for any extended wff A and any variable x.

For a given T-extended $\mathcal{L}$-wff A, let $\{x_1, \ldots, x_n\}$ be a listing the set of variables occurring in A (in the order which they occur, left-to-right), and let $\{y_1, \ldots, y_m\}$ be the set of basic elements of T occurring in A, and let k be the largest number such that $y_l = \hat{v}_k$ (for $l \in \{1, \ldots, m\}$).[1] Then let $\hat{A}$ be the sentence A with each x_i replaced by x_{i+k}.[2] Then j is defined to be the map from the set of T-extended $\mathcal{L}$ sentences to the set of $\mathcal{L}$-wffs given by (for any T-extended $\mathcal{L}$-sentence A)

$$j(A) := j_s(\hat{A}).$$

This map j is called the *Lindenbaum-Tarski embedding for* Γ.

Note: The complicated definition belies a simple concept, which is best illustrated in the context of an example.

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[1]Recall that our set of variables is $\mathcal{B}_V = \{v_1, v_2, \ldots, \}$.

[2]This convoluted replacement prevents us from accidentally binding any variables in applying j_s defined above.

Example 2.19. Using the language $\mathcal{L}_{Mon}$ of example 2.1, consider the T-extended $\mathcal{L}$-sentence (where $x, y \in \mathcal{B}_V$) $A := (\forall x)(\hat{x} * e \approx x * \hat{y})$. With notation as in the above definition, we have $j(s) = (\forall z)(x \star e \approx z \star y)$ (where z stands for a variable besides x or y). Examining the action of j on s shows that if we had not shifted the (dummy) variable x to z, then we would have taken $\hat{x}$ and turned it into a bound variable accidentally.

With the above technical machinery, we can now construct the truth valuation we will use for the Lindenbaum-Tarski model.

Definition 2.25. Let $\mathcal{L}$ be a typical language, let Γ be a set of $\mathcal{L}$-wffs, and let T be the Lindenbaum-Tarski universe over $\mathcal{L}$. Then let $[\![\cdot]\!] := q \circ j$, where $q : \mathfrak{W}(\mathcal{L}) \to \mathfrak{W}((L))/\sim_\Gamma$ is the quotient map, and j is Lindenbaum-Tarski embedding for Γ. Then $[\![\cdot]\!]$ is called the *Lindenbaum-Tarski truth valuation of* Γ.

The following lemma shows that the Lindenbaum-Tarski truth valuation is indeed a truth valuation according to our previous definition.

Lemma 2.20. Let $\mathcal{L}$ be a typical language, let Γ be a set of $\mathcal{L}$-wffs, let T be the Lindenbaum-Tarski universe over $\mathcal{L}$, and let $[\![\cdot]\!]$ be the Lindenbaum-Tarski truth valuation of Γ (with the natural interpretation $\mathfrak{F}$ of $\mathcal{L}^{\mathcal{F}}$). Then $[\![\cdot]\!]$ is a truth valuation w.r.t. $(\mathcal{L}, T, \mathfrak{F}, L_\Gamma)$.

Proof. We use notation as defined in definition 2.24, and we use the decomposition $[\![\cdot]\!] = f \circ j$ from definition 2.25. Now, by definition, $[\![\cdot]\!]$ is a map from the T-extended $\mathcal{L}$-sentences into L_Γ, and so we only need to show that $[\![\cdot]\!]$ satisfies the five conditions of definition 2.16.

First, for any extended term u with no variables, $j_t(\tilde{u}) = j_t(u)$ since we are using the natural interpretation[1] of the function symbols. Then condition (1) is trivially satisfied, since if we have an n-ary predicate P, and extended terms $t_1, \ldots, t_n$ with no free variables, $j_t(t)_i = j_t(\tilde{t})_i$ for $i \in \{1, \ldots, n\}$, so

$$[\![P(t_1, \ldots, t_n)]\!] = [P(j_t(t_1), \ldots, j_t(t_n))]_\Gamma = [P(j_t(\tilde{t}_1), \ldots, j_t(\tilde{t}_n))]_\Gamma = [\![P(\tilde{t}_1, \ldots, \tilde{t}_n)]\!].$$

Conditions (2) and (3) obviously hold, since j and f are both OL homomorphisms, and hence so is $[\![\cdot]\!] = f \circ j$.

To show that conditions (4) and (5) both hold, it will suffice to show that (for any extended wff $A(x)$)

$$[\![(\forall x)A(x)]\!] = \bigwedge_{a \in T} [\![A(a)]\!],$$

i.e. that $[\![(\forall x)A(x)]\!]$ is a greatest lower bound for the set of all $[\![A(a)]\!]$ such that $a \in T$.

Let $\hat{x}_1, \ldots, \hat{x}_n$ be the basic elements of T occurring in $A(x)$. First, we compute

$$[\![(\forall x)A(x)]\!] = q \circ j(A(x)) = [(\forall z)j_s(\hat{A}(z))]_\Gamma = [(\forall z)j_s(\hat{A})(z, x_1, \ldots, x_n)]_\Gamma, \qquad (2.2)$$

[1]Recall that we have defined the operations in T to correspond exactly to the set $\mathcal{L}^{\mathcal{F}}$, so that for any $f \in \mathcal{L}^{\mathcal{F}}$, we have $f \mapsto \hat{f}$ when evaluated, and then $\hat{f} \mapsto f$ under j_t.

where $z \in \mathcal{B}_V$ satisfies $z \neq x_i$ for $i \in \{1, \ldots, n\}$. We then also have (letting $\hat{y}_1, \ldots, \hat{y}_m$ be the basic elements occurring in the term a)

$$[\![A(a)]\!] = q \circ j(A(a)) = [j_s(\widehat{A(a)})]_\Gamma = [j_s(\hat{A})(j_t(a), x_1, \ldots, x_n)]_\Gamma,$$

where the last equality follows from[1] lemma 2.6.

That $[\![(\forall x)A(x)]\!]$ is a lower bound for $[\![A(a)]\!]$ then follows from Q6, which gives that

$$\Gamma \vdash (\forall z) j_s(\hat{A})(z, x_1, \ldots, x_n) \to j_s(\hat{A})(j_t(a), x_1, \ldots, x_n).$$

To see that $[\![(\forall x)A(x)]\!]$ is the greatest lower bound for the set of all $[\![A(a)]\!]$ such that $a \in T$, consider some $[B]_\Gamma$ (with B an $\mathcal{L}$-wff) such that $[B]_\Gamma \leq [\![A(a)]\!]$ for all $a \in T$, i.e.

$$[B]_\Gamma \leq j_s(\hat{A})(j_t(a), x_1, \ldots, x_n)$$

for all $a \in T$. Let $x_{n+1}, \ldots, x_m$ be all the variables occurring in B not already included in $x_1, \ldots, x_n$, so that $B = B(x_1, \ldots, x_m)$ and also $j_s(\hat{A})(j_t(a), x_1, \ldots, x_n) = j_s(\hat{A})(j_t(a), x_1, \ldots, x_m)$. Then, by lemma 2.18, we have that

$$\Gamma \vdash B(x_1, \ldots, x_m) \to j_s(\hat{A})(j_t(a), x_1, \ldots, x_m)$$

---

[1]While $\widehat{A(a)}$ is not necessarily equal to $\hat{A}(a)$, they are logically equivalent w.r.t. Γ, since they differ only by a charge of bound variables (see def. 2.24).

for all $a \in T$. In particular, for $a = \hat{y}$, where $y \in \mathcal{B}_V$ is some variable such that $y \neq x_i$ for $i \in \{1, \ldots, m\}$. R4 then gives that

$$\Gamma \vdash B(x_1, \ldots, x_m) \to (\forall y) j_s(\hat{A})(y, x_1, \ldots, x_m).$$

Then by lemma 2.18, this gives that

$$\begin{aligned} [B]_\Gamma &\leq [(\forall y) j_s(\hat{A})(y, x_1, \ldots, x_m)]_\Gamma \\ &= [(\forall y) j_s(\hat{A})(y, x_1, \ldots, x_n)]_\Gamma \\ &= [(\forall z) j_s(\hat{A})(z, x_1, \ldots, x_n]_\Gamma \\ &= [\![(\forall x) A(x)]\!]. \end{aligned}$$

where the second to last equality follows form lemma 2.6, and the final equality is equation 2.2, which shows that $[\![(\forall x) A(x)]\!]$ is indeed the greatest lower bound for the set of all $[\![A(a)]\!]$ such that $a \in T$. □

We are now (finally) ready to define the Lindenbaum-Tarski model.

Definition 2.26. Let $\mathcal{L}$ be a typical language, let Γ be a set of $\mathcal{L}$-wffs, let T be the Lindenbaum-Tarski universe over $\mathcal{L}$, and let $[\![\cdot]\!]$ be the Lindenbaum-Tarski truth valuation of Γ (with the natural interpretation $\mathfrak{F}$ of $\mathcal{L}^{\mathcal{F}}$). Then $\mathfrak{L}_\Gamma := (T, L_T, [\![\cdot]\!], \mathfrak{F})$ is called the *Lindenbaum-Tarski model of* Γ.

By lemma 2.20 $\mathfrak{L}_\Gamma$ is indeed an $\mathcal{L}$-structure[1] for any set of $\mathcal{L}$-wffs Γ. The following lemma shows that $\mathfrak{L}_\Gamma$ is aptly named, since it is indeed a model of Γ.

Lemma 2.21. Let $\mathcal{L}$ be a typical language, and let Γ be a set of $\mathcal{L}$-wffs. Then $\mathfrak{L}_\Gamma$ of definition 2.26 is a model of Γ.

Proof. We only need to show that every wff in Γ holds in $\mathfrak{L}_\Gamma$, so let $A \in \Gamma$. Since $\sim_\Gamma$ is a congruence, $[\bot]_\Gamma = 0$ in L_Γ, and so $1 = \neg[\bot]_\Gamma = [\neg\bot]_\Gamma = [\top]_\Gamma$. By (4) in proposition 2.4, we have that $\Gamma \vdash \top \to A$, and so by lemma 2.18, we have $[\top]_\Gamma \leq [A]_\Gamma$, so that $1 = [A]_\Gamma$, and hence every $A \in \Gamma$ holds in $\mathfrak{L}_\Gamma$. $\square$

The following simple lemma is all that remains before we prove completeness.

Lemma 2.22. Let $\mathcal{L}$ be a typical language and Γ a set of $\mathcal{L}$-wffs, and let A be an $\mathcal{L}$-wff with $[A]_\Gamma = 1$. Then $\Gamma \vdash A$.

Proof. If $[A]_\Gamma = 1$, then $\Gamma \vdash A \leftrightarrow \top$, so by Q3 and R5, $\Gamma \vdash \top \to A$. By by (2) in proposition 2.4, $\top \to A \vdash A$, and hence $\Gamma \vdash A$. $\square$

Theorem 2. *Completeness of Quantum Logic:* Let $\mathcal{L}$ be a typical language, and Γ a set of $\mathcal{L}$-wffs. Further let A be an $\mathcal{L}$-wff such that for any model $\mathfrak{M}$ of Γ, we have $\mathfrak{M} \vDash A$. Then $\Gamma \vdash A$.

---

[1] The range of the Lindenbaum-Tarski truth valuation is clearly L_Γ, since any wff A can be obtained from the Lindenbaum-Tarski embedding by replacing any x which occurs free in A by $\hat{x}$ to obtain an extended sentence which the embedding sends to A.

Proof. First, assume the theorem holds for any $\mathcal{L}$-sentence, then if $A = A(x_1, \ldots, x_n)$, we have $[\![A]\!] = [\![(\forall x_1)\cdots(\forall x_n)A]\!]$, so that if A holds in every model, then so does the sentence $(\forall x_1)\cdots(\forall x_n)A$, in which case $\Gamma \vdash A$ by repeated uses of Q6 and R5. Hence, we can restrict ourselves to the case where A is a sentence.

Now, let T be the Lindenbaum-Tarski universe over $\mathcal{L}$, and let A be an $\mathcal{L}$-sentence. Now since A holds in every model of Γ, in particular $\mathfrak{L}_\Gamma \vDash A$, i.e.

$$1 = [\![A]\!] = q \circ j(A) = q(A) = [A]_\Gamma$$

since $j(A) = A$ for any $\mathcal{L}$-sentence A (this follows trivially from the definition). Then, by lemma 2.22 we have that $\Gamma \vdash A$. □

2.4.3 Completeness with respect to Irreducibles

Now that we have a completeness theorem for quantum logic with respect to the semantics defined above, a natural question is whether we can restrict this semantics to a natural subclass of models — perhaps one with some "nice" properties — and still have a completeness theorem. In fact, this is possible for the subclass of models with irreducible truth value algebras, and these models do have one very useful property which we will exploit in section 3.2.2. We will prove this with the aid of one technical lemma, which first requires a short definition.

Definition 2.27. Let $\mathcal{L}$ be a language, and let $\mathfrak{M}$ be an $\mathcal{L}$-structure. If the truth value algebra of $\mathfrak{M}$ is irreducible (as an OML), then $\mathfrak{M}$ is called irreducible.

Lemma 2.23. Let $\mathcal{L}$ be a language, let Γ be a class of $\mathcal{L}$-wffs, and let $\mathfrak{M} = (\mathfrak{A}, L_1 \times L_2, \nu, \mathfrak{F})$ (where L_1, L_2 are both OMLs) be a model of Γ. Let $p_1 : L_1 \times L_2 \to L_1$ be the natural projection map onto L_1. Then $\mathfrak{M}' := (\mathfrak{A}, L_1, p_1 \circ \nu, \mathfrak{F})$ is a model of Γ.

Proof. For any $\gamma \in \Gamma$, $\nu(\gamma) = 1$ (since $\mathfrak{M}$ models Γ), and so $p_1 \circ \nu(\gamma) = 1$. Also, since ν and p_1 are both surjective, ran $(p_1 \circ \nu)$ trivially generates L_1. Also, (1-4) of def. 2.16 are satisfied since ν is a truth valuation and p_1 a continuous homomorphism (lemma C.2). It remains to show that L_1 is complete enough for $p_1 \circ \nu$. So consider a wff $B(x)$, and let $(\gamma, \beta) = \bigwedge_{a \in A} \nu(B(a))$, i.e. (γ, β) is the greatest lower bound of the set of all $\nu(B(a)) =: (\gamma_a, \beta_a)$ where $a \in A$. Hence $\gamma \leq \gamma_a$ for all $a \in A$, so γ is a lower bound for the set of all $\gamma_a = p_1 \circ \nu(B(a))$. Also, for any $\gamma' \in L_1$ which is a lower bound for all the $p_1 \circ \nu(B(a))$, we have that $(\gamma', 0)$ is a lower bound for all the $\nu(B(a))$ (and since ν is surjective $(\gamma', 0)$ is in the subalgebra generated by ran (ν)), and so by definition $(\gamma', 0) \leq (\gamma, \beta)$, and so $\gamma' \leq \gamma$, showing that

$$\gamma = \bigwedge_{a \in \mathfrak{A}} p_1 \circ \nu(B(a)) = p_1 \circ \nu\big((\forall x)(B(x))\big),$$

and so L_1 is complete enough for ν, and hence $\mathfrak{M}'$ is a model for Γ. □

Lemma 2.24. Let $\mathcal{L}$ be a typical language, let Γ be a set of $\mathcal{L}$-wffs, and A an $\mathcal{L}$-wff which holds in every irreducible model of Γ. Then A holds in every model of Γ.

Proof. We prove the contrapositive, so let A be an $\mathcal{L}$-wff and assume that there is some model $\mathfrak{M}$ (with truth valuation ν) of Γ in which $\nu(A) \neq 1$. By the completeness theorem (theorem 2), we must have that $\nu(A) \neq 1$ in the Lindenbaum-Tarski algebra of Γ, i.e. $\Gamma \nvdash \top \leftrightarrow A$. Now define $\mathcal{A}$ to be the set which consists of exactly those sets of $\mathcal{L}$-wffs Φ in which (i) $\Gamma \subseteq \Phi$, and (ii) $\nu_\Phi(A) \neq 1$, where ν_Φ is the truth valuation of the Lindenbaum-Tarski model of Φ. That $\mathcal{A}$ is indeed a set follows from the fact that $\mathcal{L}$ is a set (since this means the $\mathcal{L}$-wffs form a set).

$\mathcal{A}$, like any set, is partially ordered by inclusion, and we will use Zorn's lemma (lemma A.9) to prove that $\mathcal{A}$ contains some maximal element Ω. Let $\mathcal{C} \subseteq \mathcal{A}$ be linearly ordered under inclusion, and define $U := \bigcup \mathcal{C}$. Clearly U is an upper bound for $\mathcal{C}$ under inclusion, so it suffices to show that $U \in \mathcal{A}$. Property (i) is clear, since trivially $\Gamma \subseteq U$. We prove property (ii) by contradiction, so assume $\nu_U(A) = 1$, i.e. $U \vdash \top \leftrightarrow A$. The formal proof of $\top \leftrightarrow A$ is finite by definition, and so may only contain a finite number of wffs $A_1, A_2, \ldots, A_n \in U$. Each $A_i \in \Phi_i$ for some $\Phi_i \in \mathcal{C}$. Since $\mathcal{C}$ is linearly ordered by inclusion, we must have that $\bigcup_{i=1}^n \Phi_i = \Phi_j$ for some $j \in \{1, \ldots, n\}$, and since $\{A_1, \ldots, A_n\} \subseteq \Phi_j$, we have that $\Phi_j \vdash \top \leftrightarrow A$, so $\nu_{\Phi_j}(A) = 1$, which is a contradiction since $\Phi_j \in \mathcal{A}$. Hence $U \in \mathcal{A}$, and since $\mathcal{C}$ was a generic chain in $\mathcal{A}$, Zorn's lemma gives that $\mathcal{A}$ has a maximal element Ω.

We claim that the Lindenbaum-Tarski algebra L corresponding to Ω is irreducible. Assume not, so that $L \simeq L_1 \times L_2$. By the lemma 2.23, this produces two new models of Ω with corresponding truth values L_1 and L_2 (and truth valuations ν_1 and ν_2, respectively). Since $\nu_\Omega(A) \neq 1$, we must have (wlog) $\nu_1(A) \neq 1$. Let B be some wff such that $\nu_1(B) = 1$ and $\nu_2(B) = 0$ (which exists since ν is surjective). Clearly, $B \notin \Omega$, since $\nu_\Omega(B) \neq 1$. Also, we cannot have $\Omega \cup \{B\} \vdash \top \leftrightarrow A$, since then we would have $\nu_1(A) = 1$. This means that $\Omega \cup \{B\} \in \mathcal{A}$, and also that Ω is a proper subset of $\Omega \cup \{B\}$, which contradicts the maximality of Ω in $\mathcal{A}$. Hence

L must be irreducible, and so we have constructed an irreducible model in which $\nu_\Omega(A) \neq 1$, which establishes the contrapositive. □

Together with theorem 2, this gives the following powerful result which is a strengthening of our earlier completeness result, and will be quite useful in the sequel (see, for example, section 3.2.2).

Theorem 3. Let $(\mathcal{L}, \mathcal{A})$ be an M-system with $\mathcal{L}$ a typical language, and let A be an $\mathcal{L}$-wff such that, for every model $\mathfrak{M}$ of $\mathcal{A}$ with irreducible truth value algebra, $\mathfrak{M} \vDash A$ Then $\mathcal{A} \vdash A$.

Proof. By lemma 2.24, A holds in every model of $\mathcal{A}$, so that $\mathcal{A} \vdash A$ by theorem 2. □

We conclude our discussion of completeness by demonstrating some useful consequences that will immediately be useful in the next chapter.

Consequences of Completeness

As a direct consequence of the completeness theorem, we have the following useful proposition which will allow us to utilize the full breadth of lattice theory in order to derive facts about (for any given typical language $\mathcal{L}$) our deductive system for quantum logic $\mathcal{Q}(\mathcal{L})$.

Proposition 2.25. Let $\mathcal{L}$ be a typical language, let Γ be a set of $\mathcal{L}$-wffs, and let $p(x_1, \ldots, x_n)$ and $q(x_1, \ldots, x_n)$ be any ortholattice polynomials (in the operations $\wedge$, $\vee$, $\neg$, 1, and 0, with lattice variables $x_1, \ldots, x_n$) such that $p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n)$ in any OML L which is a truth value algebra for a model of Γ. Then, for any $\mathcal{L}$-wffs $A_1, \ldots, A_n$, we have that the $\mathcal{L}$-wffs $p(A_1, \ldots, A_n)$ and $q(A_1, \ldots, A_n)$ are logically equivalent w.r.t. Γ.

Proof. Let $\mathfrak{M}$ be any model of Γ, and let $[\![\cdot]\!]$ be the corresponding truth valuation. Then

$$[\![p(A_1,\ldots,A_n) \leftrightarrow q(A_1,\ldots,A_n)]\!] = p([\![A_1]\!],\ldots,[\![A_n]\!]) \leftrightarrow q([\![A_1]\!],\ldots,[\![A_n]\!]) = 1,$$

where the final equality follows from lemma C.14 and the fact that the truth value algebra of $\mathfrak{M}$ is an OML in which the corresponding lattice identity holds by assumption. Since $p(A_1,\ldots,A_n) \leftrightarrow q(A_1,\ldots,A_n)$ holds in any model, by the completeness theorem (theorem 2) we have

$$\Gamma \vdash p(A_1,\ldots,A_n) \leftrightarrow q(A_1,\ldots,A_n),$$

i.e. $p(A_1,\ldots,A_n)$ and $q(A_1,\ldots,A_n)$ are logically equivalent. □

Corollary 2.26. Let $\mathcal{L}$ be a typical language, and let $p(x_1,\ldots,x_n)$ and $q(x_1,\ldots,x_n)$ be any ortholattice polynomials (in the operations $\wedge$, $\vee$, $\neg$, 1, and 0, with lattice variables $x_1,\ldots,x_n$) such that $p(x_1,\ldots,x_n) = q(x_1,\ldots,x_n)$ in any OML. Then, for any $\mathcal{L}$-wffs $A_1,\ldots,A_n$, we have that the $\mathcal{L}$-wffs $p(A_1,\ldots,A_n)$ and $q(A_1,\ldots,A_n)$ are logically equivalent.

Proof. This is just a special case of the previous proposition for $\Gamma = \varnothing$. □

This concludes the construction of the essential elements of our quantum logic. The soundness and completeness theorems above connect the final dots — we have established that the formal deduction system we have developed represents the same underlying logic as the quantum semantics we constructed, and so we are now justified in speaking of *first-order quantum logic*. In the next chapter, we proceed to investigate some properties of this logic.

CHAPTER 3

SIMPLE DEVELOPMENTS IN FIRST-ORDER QUANTUM LOGIC

In this chapter we begin to investigate some important properties of the first-order quantum logic developed in chapter 2. In section 3.1 we discuss the relationship between our first-order quantum logic and the traditional classical first-order logic. While we have already seen that the semantics of our quantum logic is a generalization of that for classical first-order logic, we have not yet seen how this subclassicality plays out in terms of our formal deduction system. In particular, we take the opportunity to examine how to alter our quantum axioms so that our formal deduction system becomes classical. We then examine some basic questions that the subclassicality of our quantum logic raises, and point out some simple consequences of this subclassicality for establishing consistency and independence of axiom systems.

In section 3.2 we investigate some general properties of axiom systems in quantum logic. In particular, since our logic is subclassical, axiom systems which are classically equivalent may no longer be equivalent[1] in our first-order quantum logic. We examine some general approaches to taking axiom systems commonly used in classical mathematics and modifying them to obtain axiomatizations which are classically equivalent to the usual ones, but which are more "suitable" for quantum logic. Specifically, we examine the status of statements involving '$\exists$' in the context of our powerful completeness result (theorem 3).

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[1]By equivalent we mean that they have the same theorems.

Finally, in section 3.3, we discuss models where the functions are constrained to behave classically, but the truth values of the predicates are still allowed to behave in a completely "quantum" fashion. We also discuss models which are rather natural from the quantum perspective — namely those whose truth value algebras are the projection lattice of a Hilbert space. We finish this chapter by proving a number a technical results which we will find useful for constructing models in later chapters.

3.1 Relationship to Classical Predicate Logic

As we briefly mentioned earlier, one can easily see from examining any development of classical first-order logic (e.g. (18; 49)), that our semantics developed for first-order quantum logic generalizes the usual classical semantics only[1] in that we allow our truth value algebra to be an OML, rather than forcing it to be $B_2 = \{0, 1\}$ (or, more generally, Boolean). From this fact, it is already apparent that our first-order quantum logic is subclassical,[2] since models in first-order classical logic are simply special cases of our notion of a model.

We can also obtain classical logic as a special case of quantum logic in terms of axiomatics and formal deduction simply by restricting ourselves to certain M-systems — namely those M-systems which (for $\mathcal{L}$ the language of the M-system) prove a certain $\mathcal{L}$-wff schema.[3] Or, what

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[1]We also allow models to have underlying classes, rather than just underlying sets. While this aspect of our approach differs from a usual treatment of classical logic, there is nothing particularly "quantum" about it.

[2]Recall that, due to our theorems of soundness and completeness, the formal deduction system we have developed represents the *same* first-order quantum logic as our quantum semantics.

[3]We provide two choices for such an axiom schema; there are many more that would also do the trick.

amounts to the same thing, we can expand our axioms $\mathcal{Q}_A(\mathcal{L})$ by such a schema to arrive at an axiomatization of classical logic. We first demonstrate how this can be accomplished, and then proceed to examine some consequences of the subclassicality of our quantum logic, discuss the status of the deduction theorem, and finally, make some comments concerning the existence of non-standard models.

3.1.1 Compatibility and Distributivity

We recall that a Boolean algebra is just an OML in which every pair of elements commute[1] (prop. 1.14). It is not surprising then that the following wff schema is relevant to a discussion of classicality within quantum logic.

Definition 3.1. Let $\mathcal{L}$ be a language, and let A and B be $\mathcal{L}$-wffs. We define the $\mathcal{L}$-wff $A\,\widetilde{C}\,B$ (read A *is compatible with* B) by

$$A\,\widetilde{C}\,B := A \leftrightarrow [(A \wedge B) \vee (A \wedge \neg B)], \tag{3.1}$$

and for a given M-system $(\mathcal{L}, \mathcal{A})$, we say that A *is compatible with* B *w.r.t.* $(\mathcal{L}, \mathcal{A})$ if $\mathcal{A} \vdash A\,\widetilde{C}\,B$.

Note: As can be seen from the definition (and (7) in proposition 2.4), $A\,\widetilde{C}\,B$ is logically equivalent to $A\,\widetilde{C}\,(\neg B)$ for any wffs A and B.

As we will now see, we can formally deduce, for any wffs A and B, that $A\,\widetilde{C}\,B$ is logically equivalent to $B\,\widetilde{C}\,A$.

[1] Recall that for an OML L with $a, b \in L$, a commutes with b if $a = (b \wedge a) \vee (\neg b \wedge a)$. See def. 1.19.

Proposition 3.1. Let $\mathcal{L}$ be a typical language, and Γ a set of $\mathcal{L}$-wffs. Further let A and B be $\mathcal{L}$-wffs such that $\Gamma \vdash A \widetilde{C} B$. Then $\Gamma \vdash B \widetilde{C} A$.

Proof. We use the completeness theorem, so let $\mathfrak{M}$ be any model of Γ with truth valuation $[\![\cdot]\!]$, and let $a = [\![A]\!]$ and $b = [\![B]\!]$. Now since $\Gamma \vdash A \widetilde{C} B$, we have that

$$a = (a \wedge b) \vee (a \wedge \neg b),$$

so that $a \; C \; b$. Recall that commuting is a symmetric relationship in any OML, so that we have $b \; C \; a$, i.e.

$$[\![B]\!] = b = (b \wedge a) \vee (b \wedge \neg a) = [\![(B \wedge A) \vee (B \wedge \neg A)]\!].$$

By lemma C.14, this gives that $[\![B \widetilde{C} A]\!] = 1$, and so the desired result follows directly from the completeness theorem. □

As such, if A is compatible with B, we will say that A and B are *compatible*. As mentioned above, a Boolean algebra can be defined to be an OML in which any pair of lattice elements commute. From this fact (along with the completeness theorem) it is easy to see that we can derive an axiomatization of classical logic from the quantum logical axioms along with the additional axiom schema (for any given language $\mathcal{L}$)

(CL) $A \widetilde{C} B$ for any $\mathcal{L}$-wffs A and B,

as we now demonstrate.

Proposition 3.2. Let $\mathcal{L}$ be any language, and let $\mathfrak{M}$ be an $\mathcal{L}$-structure. Then $\mathfrak{M}$ is a model of the axiom schema CL iff the truth value algebra for $\mathfrak{M}$ is a Boolean algebra.

Proof. Let L be the truth value algebra of $\mathfrak{M}$ and $[\![\cdot]\!]$ the corresponding truth valuation. First, we assume $\mathfrak{M}$ is a model of CL. Then, since the range of the truth valuation is all of L (by definition), for any $a, b \in L$ there are $\mathcal{L}$-wffs A and B such that $a = [\![A]\!]$ and $b = [\![B]\!]$, and so the fact that $\mathfrak{M}$ models CL gives

$$1 = [\![A \widetilde{C} B]\!] = [\![A \leftrightarrow (A \wedge B) \vee (A \wedge \neg B)]\!] = a \leftrightarrow (a \wedge b) \vee (a \wedge \neg b),$$

and so by lemma C.14, we have that

$$a = (a \wedge b) \vee (a \wedge \neg b),$$

i.e. that a and b commute. Then since a, b were arbitrary, by proposition 1.14 we have that L is Boolean.

Next, assume that L is a Boolean algebra. Again by proposition 1.14 and lemma C.14, we have that for any $a, b \in L$, that

$$a \leftrightarrow (a \wedge b) \vee (a \wedge \neg b) = 1,$$

and so for any wffs A and B we have that

$$[\![A \widetilde{C} B]\!] = \big([\![A]\!] \leftrightarrow [([\![A]\!] \wedge [\![B]\!]) \vee ([\![A]\!] \wedge \neg[\![B]\!])]\big) = 1,$$

so that $\mathfrak{M} \vDash \mathrm{CL}$. □

As mentioned in the introduction (section 1.2), another way to characterize the difference between Boolean algebras and a general OML is the presence of distributivity (def. 1.17) in the former. Hence, by the completeness theorem we can also use distributivity to distinguish between classical and quantum logic. For any language $\mathcal{L}$, we define the following axiom schema

(DL) $[A \wedge (B \vee C)] \leftrightarrow [(A \wedge B) \vee (A \wedge C)]$ for any $\mathcal{L}$-wffs A, B and C.

Proposition 3.3. Let $\mathcal{L}$ be any language, and let $\mathfrak{M}$ be an $\mathcal{L}$-structure. Then $\mathfrak{M}$ is a model of the axiom schema DL iff truth value algebra for $\mathfrak{M}$ is a Boolean algebra.

Proof. Let L be the truth value and $[\![\cdot]\!]$ the truth valuation of $\mathfrak{M}$. First, if $\mathcal{M}$ is a model for DL, then by definition for any $a, b, c \in L$, there exists wffs A, B and C such that $a = [\![A]\!]$, $b = [\![B]\!]$ and $c = [\![C]\!]$. Then we have

$$a \wedge (b \vee c) \leftrightarrow (a \wedge b) \vee (a \wedge c) = [\![A \wedge (B \vee C) \leftrightarrow (A \wedge B) \vee (A \wedge C)]\!] = 1$$

since $\mathfrak{M} \vDash \mathrm{DL}$, so by lemma C.14, this gives that

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

for every $a, b, c \in L$. Hence, by definition 1.17, L is a Boolean algebra.

Next, if L is a Boolean algebra, by definition 1.17, we know that, for any $a, b, c \in L$ we have (using lemma C.14)

$$1 = \big([a \wedge (b \vee c)] \leftrightarrow [(a \wedge b) \vee (a \wedge c)]\big).$$

Then, for any wffs A, B and C we have

$$[\![A \wedge (B \vee C) \leftrightarrow (A \wedge B) \vee (A \wedge C)]\!] = \Big([\![A]\!] \wedge ([\![B]\!] \vee [\![C]\!]) \leftrightarrow ([\![A]\!] \wedge [\![B]\!]) \vee ([\![A]\!] \wedge [\![C]\!])\Big) = 1,$$

so that $\mathfrak{M} \vDash \mathrm{DL}$. $\square$

By the above, adding either the axiom schema DL or CL to $\mathcal{Q}_A(\mathcal{L})$ gives an axiomatization of classical logic. In what follows, however, we will focus on the axiom schema CL, as compatibility often proves to be an easier property to work with (in general) than distributivity. For convenience we denote[1] $\mathcal{Q}(\mathcal{L}) \cup \mathrm{CL}$ by $\mathcal{C}(\mathcal{L})$. Additionally, for an M-system $(\mathcal{L}, \mathcal{A})$ we will say that an $\mathcal{L}$-wff is *classically derivable* if it is derivable from $\mathcal{A} \cup \mathrm{CL}$.

The fact that an axiomatization of classical logic (for a given language $\mathcal{L}$) can be obtained from $\mathcal{Q}_A(\mathcal{L})$ by simply adding another axiom schema provides another avenue to see that the first-order quantum logic we have developed is subclassical (which, of course, it had to be, given our earlier results of soundness and completeness along with the subclassicality of our quantum semantics).

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[1]Recall that we take CL to represent the class of instances of the axiom schema of $\mathcal{L}$-wffs given above.

Finally, we note that in most developments of classical mathematical logic, some different set of axiom schema and inference rules (for any given language $\mathcal{L}$) are used (e.g. as in (18)). By our completeness theorem of the previous chapter, $\mathcal{C}(\mathcal{L})$ is entirely equivalent to any other axiomatization which can be proven to be sound and complete.

3.1.2 Implications of Subclassicality for our Quantum Logic

We now discuss some simple consequences of the subclassicality of the first-order quantum logic we have developed.

Consistency and Independence

As noted previously, since our first order quantum logic is subclassical, every classical model of a given M-system $(\mathcal{L}, \mathcal{A})$ will *still* be a model of $(\mathcal{L}, \mathcal{A})$ within our quantum semantics. Equivalently, every wff which is derivable from $\mathcal{A}$ will be classically derivable. This has a couple of simple consequences.

First, recall that a set of axioms is *consistent* provided it cannot prove $\perp$. From this, it is trivial to see that any mathematical system $(\mathcal{L}, \mathcal{A})$ which is *consistent* in first-order classical logic is still consistent in our first-order quantum logic. The converse does not generically hold, however — there are M-systems which are classicaly inconsistent but which are consistent under quantum logic! Another consequence of the subclassicality of our quantum logic is that any set of axioms which are independent in first-order classical logic remain so in our first-order quantum logic, where we recall that set of axioms is said to be *independent* if no axiom in that set is derivable from the others.

The Lack of a Deduction Theorem in First-Order Quantum Logic

In first-order classical logic there is a clear connection between (the meta-linguistic) notion of derivability ($\vdash$) and the (object language) conditional ($\rightarrow$) in the form of a 'deduction theorem' which states:[1] for any set of $\mathcal{L}$-wffs Γ and $\mathcal{L}$-wffs A and B, we have that $\Gamma, A \vdash B$ iff $\Gamma \vdash A \rightarrow B$. Just as in the propositional case, only one direction[2] of this holds for our first-order quantum logic, even when $\Gamma = \varnothing$. The same example (1.17) which shows that the deduction theorem fails in quantum propositional logic can easily be used to create a simple model showing the deduction theorem still fails in first-order quantum logic; we leave the details to the reader.

Existence of Non-standard Models and Inherent Classicality

Of course, some axiom systems will only admit standard models, for example any M-system $(\mathcal{L}, \mathcal{A})$ such that the axiom schema CL is contained in $\mathcal{A}$. One may expect that an M-system $(\mathcal{L}, \mathcal{A})$ where the axioms $\mathcal{A}$ are only "mathematical"[3] would *always* admit non-standard models. An example of a "mathematical axiom" would be something like (in the language $\mathcal{L}_{Mon}$ of example 2.1)

$$(\forall x)(\forall y)(x * e = x).$$

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[1]See also proposition 1.9 in section 1.1.2.

[2]Namely $\Gamma \vdash A \rightarrow B$ implies that $\Gamma, A \vdash B$ — see (6) in prop. 2.4.

[3]As opposed to 'logical', such as the schema CL or DL.

It is difficult to make this notion of a "mathematical" axiom precise, but hopefully the reader can see a qualitative difference between an axiom such as the one just presented and a schema such as CL.

Surprisingly, as first shown by Dunn in (15), this is not always the case. It turns out that certain M-systems with only "mathematical axioms"[1] will admit *no* non-standard models. Thus, for for such M-systems there is no difference, as far as the mathematics is concerned, between using the first-order quantum logic we have developed or a traditional classical first-order logic — i.e. quantum mathematics and classical mathematics are the same for such M-systems. It will be useful to have a formal definition for this concept.

Definition 3.2. Let $(\mathcal{L}, \mathcal{A})$ be an M-system. If $\mathcal{A} \vdash \text{CL}$, then $(\mathcal{L}, \mathcal{A})$ (or simply $\mathcal{A}$ if $\mathcal{L}$ is clear from the context or irrelevant) is said to be *inherently classical.*

Note: It is easy to see that $\mathcal{A}$ is inherently classical iff any classically derivable wff of such an M-system is also a quantum theorem.

3.2 Axiomatizations of Mathematical Theories

Now, for any given area of mathematics, there are often alternative but classically equivalent formulations or presentations of the axioms (as an M-system). Even fixing a language $\mathcal{L}$, two different collections of axioms may present the 'same' classical mathematics. We will investigate

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[1]Such M-systems will be discussed in sections 4.1.3 and 6.1.2, and include the usual first-order axiomatization of Peano arithmetic as well as certain axiomatizations of group and OMLs.

the consequences of this phenomena (namely, that the same classical mathematics results from different axiomatizations in the same language) for the development of quantum mathematics.

3.2.1 Classically Equivalent Axiom Systems

We begin by defining two different notions of what it means for different axiomatizations (i.e. classes of wffs) to be equivalent, one for quantum logic, and the other for classical.

Definition 3.3. Let $\mathcal{L}$ be a language, and let $\mathcal{A}$ and $\mathcal{B}$ be classes of $\mathcal{L}$-wffs. If $\mathcal{A} \vdash \mathcal{B}$ and $\mathcal{B} \vdash \mathcal{A}$, then $\mathcal{A}$ and $\mathcal{B}$ are said to be *logically equivalent.*[1] If $\mathcal{A} \cup \mathrm{CL}$ and $\mathcal{B} \cup \mathrm{CL}$ are logically equivalent, then we say that $\mathcal{A}$ and $\mathcal{B}$ are *classically equivalent.*

Note: Of course, since quantum logic is subclassical, two different classes of wffs may be classically equivalent, but *not* logically equivalent. However, logically equivalent classes of wffs are always classically equivalent.

In providing M-systems for various branches of mathematics, we will frequently be interested in finding an axiomatization of that mathematics which is classically equivalent to some more commonly used set of axioms, but more "suitable" for quantum logic. As such, for some class of wffs $\mathcal{A}$ frequently used in axiomatizing an area of classical mathematics, we will call any other class of wffs $\mathcal{A}'$ which is classically equivalent to $\mathcal{A}$ a *reduction* of $\mathcal{A}$. In the same vein, for

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[1] This notion of logical equivalence is easily seen to be a generalization (to classes of wffs) of the notion of logical equivalence of wffs. See def. 2.10.

a single wff A, we will say that A' is a *reduction* of A when A and A' are logically equivalent w.r.t. CL.[1]

Using our new terminology, one obvious question to ask is this: for a given M-system $(\mathcal{L}, \mathcal{A})$ and an $\mathcal{L}$-structure $\mathfrak{M}$, if we know that there exists some reduction $\mathcal{A}'$ of $\mathcal{A}$ such that $\mathfrak{M} \models \mathcal{A}'$, is there any sense in which $\mathfrak{M}$ has to be "close" to being a model of $\mathcal{A}$? We will need a few technical developments before we can answer this question, the first of which is the following definition.

Definition 3.4. Let $\mathcal{L}$ be a language. For $\mathcal{L}$-wffs A, B, define the $\mathcal{L}$-wff $c(A, B)$ by

$$c(A,B) := [(A \wedge B) \vee (A \wedge \neg B)] \vee [(\neg A \wedge B) \vee (\neg A \wedge \neg B)]. \tag{3.2}$$

$c(A, B)$ is called the *commutator of A and B.*

It well known from orthomodular lattice theory (see the appendix, prop. C.11) that the expression in equation 3.2, when interpreted as a ortholattice polynomial, has the property that $c(a, b) = 1$ iff $a \; C \; b$ for any a, b in a given OML. We then have the following useful proposition which shows that we can utilize the commutator to characterize the classicality of a given structure.

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[1] While the concept of reduction is symmetric insofar as we have defined it, we will always consider some usual classical formulation of a wff (or class of wffs) as the unreduced statement, and consider its classically equivalent alteration to be the reduction.

Proposition 3.4. Let $\mathcal{L}$ be a language, and let $\mathfrak{M}$ be an $\mathcal{L}$-structure. Then $\mathfrak{M} \vDash c(A, B)$ for every pair of $\mathfrak{A}$-extended $\mathcal{L}$-wffs A and B iff L is Boolean.

Proof. Let $[\![\cdot]\!]$ be the truth valuation of $\mathfrak{M}$. First we prove the 'only if' direction. By definition, for any $a, b \in L$ there exists wffs A and B such that $a = [\![A]\!]$ and $b = [\![B]\!]$. Then for any $a, b \in L$, we have

$$c(a, b) = c([\![A]\!], [\![B]\!]) = [\![c(A, B)]\!] = 1,$$

so that L is Boolean by propositions C.11 and 1.14.

Conversely, if L is Boolean, then for any wffs A, B, we have

$$[\![c(A, B)]\!] = c([\![A]\!], [\![B]\!]) = 1$$

by the same proposition, i.e. $\mathfrak{M} \vDash c(A, B)$. □

This result, along with proposition 3.2, gives that CL $\vdash c(A, B)$ for any two $\mathcal{L}$-wffs A and B, so that $c(A, B)$ is a tautology of classical logic for any language $\mathcal{L}$. Whether or not it is also a tautology of our quantum logic depends upon the specific A and B — for example, in the language $\mathcal{L}_{Mon}$, taking $A = (x \approx y)$ and $B = (y \approx z)$ gives that $c(A, B)$ is not a tautology in quantum logic, while $c(A, A)$ is a quantum tautology (for any wff A in any language).

The commutator then gives us a natural way to reduce axioms of any piece of classical mathematics, which we will present in the form of an example, rather than as a general formulation.

Example 3.5. Consider the language $\mathcal{L}_{Mon}$ from example 2.1. One common axiom in the theory of monoids is

$$(\forall x)(x * e \approx x).$$

A natural way to reduce this axiom involving the commutator would be

$$\Big((\forall x)(\forall y)c(x \approx y, x \approx e)\Big) \to (\forall x)(x * e \approx x)).$$

Since the truth value of $c(x \approx y, x \approx e)$ is always 1 when the truth value algebra L is Boolean, and since (for any $a \in L$), $1 \to a = a$ by lemma C.14, this is clearly a reduction of the given axiom.

Using the above reduction strategy, we can answer the question posed earlier. For a given M-system $(\mathcal{L}, \mathcal{A})$, one can ensure that there is some reduction $\mathcal{A}'$ of $\mathcal{A}$ such that *any* $\mathcal{L}$-structure satisfying one (rather innocuous) condition on its truth value algebra will model $\mathcal{A}'$. In particular, the following proposition shows that such $\mathcal{L}$-structures model a reduction of $\mathcal{A}$ even though they are not constrained by the particular choice of wffs $\mathcal{A}$ *in any way.*

Proposition 3.6. Let $(\mathcal{L}, \mathcal{A})$ be an M-system, and let $\mathfrak{M}$ be an $\mathcal{L}$-structure with truth value algebra L such that there exists $a, b \in L$ with $c(a, b) = 0$. Then there exists a class of $\mathcal{L}$-wffs $\mathcal{A}'$ which is classically equivalent to $\mathcal{A}$ such that $\mathfrak{M} \vDash \mathcal{A}'$.

Proof. Let $[\![\cdot]\!]$ be the truth valuation of $\mathfrak{M}$. By the definition of an $\mathcal{L}$-structure, we know that there are $\mathcal{L}$-sentences ψ_a and ψ_b such that $[\![\psi_a]\!] = a$ and $[\![\psi_b]\!] = b$. Then, for every $A \in \mathcal{A}$, define

$$A' := \left[c(\psi_a, \psi_b)\right] \to A,$$

And let $\mathcal{A}' := \{A' \;:\; A \in \mathcal{A}\}$. From the previous discussion, we know that $\mathcal{A}'$ is indeed classically equivalent to $\mathcal{A}$. Also, from our assumption that $c(a,b) = 0$ it is trivial to see that $\mathfrak{M} \vDash \mathcal{A}'$. $\square$

Note: One can show that the assumption that $c(a,b) = 0$ for some $a, b \in L$ is satisfied is equivalent to L having subortholattice isomorphic to MO_2 — in particular, *many* OMLs L will satisfy this condition.

This shows that, for any given class of wffs $\mathcal{A}$, some reductions of $\mathcal{A}$ will be absolutely trivial to satisfy. As such, the fact that a particular structure models *some* reduction of $\mathcal{A}$ means virtually nothing — one must pay close attention to the particular *form* of that reduction.

Also, the above considerations immediately show that for any class of $\mathcal{L}$-wffs $\mathcal{A}$ which is inherently classical, there is always a reduction of $\mathcal{A}$ which is not. In particular, when proving certain branches of mathematics only admit standard models, we must always do so with respect to a particular axiomatization — there can be no such result for a more general notion of mathematical system which encompasses all classically equivalent axiomatizations.

3.2.2 Classicality Operators

In this section we now capitalize on the powerful completeness result proved in the previous chapter (theorem 3). We construct two 'operators on wffs' (i.e. two 'wff schema') which, due to the aforementioned completeness result, will be extremely useful in producing reductions of axioms, especially those involving the symbol '∃'.[1]

First, recall that the center of an OML L consists of those elements with commute with everything in L(def. C.6), and the center of any OML forms a Boolean subalgebra (proposition C.9), so that the center represents the "classical" elements of the OML. Clearly, it would be useful to have a wff schema which would inform us when a given wff ψ only took "classical" truth values in a given structure. The first schema we construct does precisely this, at least when we have only a finite number of predicates — in particular this schema will have a truth value of 1 iff the truth value of ψ is in the center of the given model's truth value algebra. Let $\mathcal{L}^P = \{P_1, \ldots, P_n\}$, and let $\mathcal{L} = \mathcal{L}^P \cup \mathcal{L}^F$ be our language (for some function symbols $\mathcal{L}^{\mathcal{F}}$. Then, for any $\mathcal{L}$–wff ψ define

$$\begin{aligned}\mathbf{C}(\psi) := (\forall s_1^1)\cdots(\forall s_{\alpha(P_1)}^1)[\varphi_{P_1(s_1,\ldots,s_{\alpha(P_1)})}(\psi) \to \psi] \wedge \cdots \\ \wedge (\forall s_1^n)\cdots(\forall s_{\alpha(P_n)}^n)[\varphi_{P_n(s_1,\ldots,s_{\alpha(P_n)})}(\psi) \to \psi)],\end{aligned} \tag{3.3}$$

where $\varphi_x(y) = x \wedge (\neg x \vee y)$ is the Sasaki projection.

Proposition 3.7. Let $\mathcal{L}$ be a language with a finite number of predicates, let $\mathfrak{M}$ be an $\mathcal{L}$-structure with truth valuation $[\![\cdot]\!]$, and let ψ be an $\mathfrak{A}$-extended $\mathcal{L}$-sentence. Then $[\![\mathbf{C}(\psi)]\!] = 1$ iff $[\![\psi]\!]$ is in the center of the truth value algebra of $\mathfrak{M}$.

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[1] We will need the technical requirement that the languages we consider have only a finite number of predicates.

Proof. Let L be the truth value algebra of $\mathfrak{M}$. By definition of $\mathbf{C}(\psi)$, and since $[\![\mathbf{C}(\psi)]\!] = 1$ iff every term in the meet equals one (recalling that '$\forall$' statements evaluate to meets in the structures), we have that $[\![\mathbf{C}(\psi)]\!] = 1$ iff

$$\varphi_{[\![P(a_1,\ldots,a_{\alpha(P)})]\!]}([\![\psi]\!]) \to [\![\psi]\!] = 1$$

for every predicate P in $\mathcal{L}$, and for every $a_1, \ldots, a_{\alpha(P)} \in L$. But by lemma C.14, the above statement holds iff $\varphi_{[\![P(a_1,\ldots a_{\alpha(P)})]\!]}([\![\psi]\!]) \leq [\![\psi]\!]$. By lemma C.13, this is true iff

$$[\![P(a_1, \ldots, a_{\alpha(P)})]\!] \; C \; [\![\psi]\!]$$

for every $a_1, \ldots, a_{\alpha(P)} \in L$ and $P \in \mathcal{L}^P$. Clearly if $[\![\psi]\!]$ is in the center of L, the previous statement is satisfied. Conversely, since by the definition of an $\mathcal{L}$-structure the set of all $[\![P(a_1, \ldots a_{\alpha(P)})]\!]$ generate L, by proposition C.10, $[\![\psi]\!]$ must be in the center of L. □

Just as it is useful to have an 'operator' which tells us when a given sentence takes only "classical" truth values in a given model, it will also be useful to have an 'operator' which "classicizes" the truth value of any wff — more specifically, this 'operator' would take a given wff ψ and yield a new wff with truth value 1 if the truth value of ψ were 1, and truth value 0 otherwise. As the following shows, we can construct just such an 'operator' for models with irreducible truth value algebras. Define, for a given $\mathcal{L}$-wff ψ, (where $\mathcal{L}, \mathcal{L}^P$ are as above)

$$\begin{aligned}\mathbf{T}(\psi) \;:=\; & (\forall s_1^1)\cdots(\forall s_{\alpha(P_1)}^1)(P_1(s_1,\ldots,s_{\alpha(P_1)}) \to \psi) \\ & \cdots \wedge (\forall s_1^n)\cdots(\forall s_{\alpha(P_n)}^n)(P_n(s_1,\ldots,s_{\alpha(P_n)}) \to \psi).\end{aligned} \tag{3.4}$$

We will now show that, in a certain class of models, this '**T** operator' behaves exactly as promised. First, we prove a short technical lemma.

Lemma 3.8. Let $\mathcal{L}$ be a language with a finite number of predicates, and let

$\mathfrak{M}$ be an $\mathcal{L}$-structure with truth valuation $[\![\cdot]\!]$. Further let ψ be any $\mathfrak{A}$-extended $\mathcal{L}$-wff such that $[\![\psi]\!] = 1$. Then $[\![\mathbf{T}(\psi)]\!] = 1$.

Proof. Assuming $[\![\chi]\!] = 1$, then $[\![\mathbf{T}(\chi)]\!] = 1$ trivially by lemma C.14, which gives that in any OML L with $a \in L$, we have $(a \to 1) = 1$. □

Proposition 3.9. Let $\mathcal{L}$ be a language with a finite number of predicates, and let

$\mathfrak{M} := (\mathfrak{A}, L, [\![\cdot]\!], \mathfrak{F})$ be an $\mathcal{L}$-structure such that L is irreducible and which also satisfies (for all $\mathfrak{A}$-extended $\mathcal{L}$-wffs ψ)

1. $\mathfrak{M} \vDash \mathbf{T}(\psi) \to \psi$
2. $\mathfrak{M} \vDash \mathbf{C}(\mathbf{T}(\psi))$

Then for any $\mathfrak{A}$-extended $\mathcal{L}$-wff χ,

$$[\![\mathbf{T}(\chi)]\!] = \begin{cases} 1 & \text{if } [\![\chi]\!] = 1 \\ 0 & \text{if } [\![\chi]\!] \neq 1 \end{cases} \tag{3.5}$$

Proof. First, assume $[\![\chi]\!] = 1$. Then $[\![\mathbf{T}(\chi)]\!] = 1$ by lemma 3.8. Next, assume $[\![\chi]\!] \neq 1$. By (2) above, $[\![\mathbf{C}(\mathbf{T}(\chi))]\!] = 1$, and so by proposition 3.7, $[\![\mathbf{T}(\chi))]\!]$ is in the center of L. Since L is irreducible, the center of L is just $\{0, 1\}$. Then by (1) above, $[\![\mathbf{T}(\chi)]\!] \to [\![\chi]\!] = 1$, so that

$$[\![\mathbf{T}(\chi)]\!] \leq [\![\chi]\!] \neq 1$$

by lemma C.14, and hence we must have that $[\![\mathbf{T}(\chi)]\!] = 0$. □

Since the aforementioned completeness result (theorem 3) allows us to restrict ourselves to irreducible truth value algebras for our semantics without loss of generality, the '**T** operator' defined above will prove to be powerful indeed. It will now behoove us to determine some conditions under which the hypotheses (1) and (2) of the above proposition are satisfied.

Lemma 3.10. Let $\mathcal{L}$ be a language with a finite number of predicates, let $\mathfrak{M} := (\mathfrak{A}, L, [\![\cdot]\!], \mathfrak{F})$ be an $\mathcal{L}$-structure, and let ψ be any $\mathfrak{A}$-extended $\mathcal{L}$-wff. Further assume that there exists some $a_1, \ldots, a_n \in \mathfrak{A}$ and some predicate P with arity n such that $[\![P(a_1, \ldots, a_n)]\!] = 1$. Then

$$\mathfrak{M} \vDash \mathbf{T}(\psi) \to \psi.$$

Proof. Let ψ be any extended sentence. By lemma C.14, we only need to show that $[\![\mathbf{T}(\psi)]\!] \leq [\![\psi]\!]$. But by the definition of $\mathbf{T}$ (equation 3.4) and lemma C.14 we have

$$[\![\mathbf{T}(\psi)]\!] \leq \bigwedge_{a_1,\ldots,a_n \in A} \big([\![P(a_1,\ldots a_n)]\!] \to [\![\psi]\!]\big) \leq \big(1 \to [\![\psi]\!]\big) = [\![\psi]\!].$$

□

The above lemma says that (for any wff ψ) the wff $\mathbf{T}(\psi) \to \psi$ is "almost" a tautology — the only requirement for $\mathbf{T}(\psi)$ to hold in any given model is that there is *one* atomic sentence in the model which is true. Also, we have the following lemma concerning the other hypothesis in proposition 3.9.

Lemma 3.11. Let $\mathcal{L}$ be a language with a finite number of predicates, and let $\mathfrak{M} := (\mathfrak{A}, L, [\![\cdot]\!], \mathfrak{F})$ be an $\mathcal{L}$-structure such that

1. L satisfies the *relative center property*,[1]
2. For every $z \in L$, there is some $P \in \mathcal{L}^P$ and $a_1, \ldots, a_{\alpha(P)} \in \mathfrak{A}$ with $[\![P(a_1, \ldots, a_{\alpha(P)})]\!] = z$.

Then for any $\mathfrak{A}$-extended $\mathcal{L}$-sentence ψ, we have $\mathfrak{M} \vDash \mathbf{C}(\mathbf{T}(\psi))$.

[1]See definition C.13.

Proof. Since L has the relative center property, by lemma C.22, we have, for any extended sentence ψ, that $\bigwedge_{b\in L}(b \to [\![\psi]\!])$ is in the center of L. Then by (2) above and the definition of '**T**' (equation 3.4) we have

$$\bigwedge_{z\in L}(z \to [\![\psi]\!]) = \bigwedge_{\substack{P\in\mathcal{L}^{\mathcal{P}} \\ a_1,\ldots,a_{\alpha(P)}\in\mathfrak{A}}} \big([\![P(a_1,\ldots,a_{\alpha(P)})]\!] \to [\![\psi]\!]\big) = [\![\mathbf{T}(\psi)]\!],$$

so that $[\![\mathbf{C}(\mathbf{T}(\psi))]\!] = 1$ by proposition 3.7. □

Finally, note that for any model with truth values $B_2 = \{0,1\}$ and truth valuation $[\![\cdot]\!]$, we have that $[\![\mathbf{C}(\psi)]\!] = 1$ for any extended wff ψ, and also $[\![\psi \to \mathbf{T}(\psi)]\!] = 1$. Now that we have nice criteria for demonstrating that a given structure $\mathfrak{M}$ satisfies both $\mathfrak{M} \vDash \mathbf{C}(\mathbf{T}(\psi))$ and $\mathfrak{M} \vDash \mathbf{T}(\psi) \to \psi$, we are ready to examine reductions of axioms involving the above operators.

Reduction of Statements Involving Existential Quantifiers

The '**T** operator' defined above is extremely useful when attempting to reduce statements from classical mathematics involving '$\exists$'. To wit, suppose we have a statement (for some $\mathcal{L}$-wff ψ) such as

$$(\exists x)\psi(x)$$

which holds in a given model. In contrast to classical logic with the standard truth values $\{0,1\}$, we cannot conclude that there actually exists some object a in this model for which $\psi(a)$ holds. (Recall the discussion in section 1.1.4.)

However, we can consider instead the statement

$$(\exists x)\mathbf{T}(\psi(x)).$$

If this statement holds in a given irreducible model[1] in which both $\mathbf{C}(\mathbf{T}(\psi))$ and $\mathbf{T}(\psi) \to \psi$ hold, then we *are* guaranteed the existence of some object a in the model for which $\psi(a)$ holds. For this reason, we consider this method of reduction to be a "suitable" reduction for certain quantified statements in quantum logic, as it will allow us to retain the full power of axioms involving '$\exists$'.

3.3 General Properties of Models and Structures in Quantum Logic

We have already defined two types of models, namely *standard* models (def. 2.21), which are those models whose truth value algebra is Boolean, as well as *irreducible* models (def. 2.27), which are those whose truth value algebra is irreducible. In this section we discuss two other interesting classes of models.

We begin by discussing *conservative* models, which are "built" out of models whose truth value algebra is $B_2 = \{0, 1\}$. We will also discuss models whose truth value algebra is a projection lattice of a Hilbert space — in some sense, these are the most natural models in quantum logic since their truth value algebras are the very OMLs which *inspired* the semantics for our first-

[1]As mentioned previously, irreducibility is no serious restriction given our completeness result (theorem 3).

order quantum logic in the first place. In the final portion of this section, we prove some technical results on general model construction which will be useful in the sequel.

3.3.1 Conservative Structures and Models

We begin with the formal definition of a conservative structure.

Definition 3.5. Let $(\mathcal{L}, \mathcal{A})$ be an M-system, and let $\mathfrak{M} = (\mathfrak{A}, L, \nu, \mathfrak{F})$ be an $\mathcal{L}$-structure. If there is a model $\mathfrak{B}$ of $\mathcal{A}$ with $\mathfrak{B} = (\mathfrak{A}, B_2, \nu', \mathfrak{F})$ such that for any $P \in \mathcal{L}^{\mathcal{P}}$, and any $a_1, \ldots, a_{\alpha(P)}$, we have

$$\nu(P(a_1, \ldots, a_{\alpha(P)})) = 1 \quad \text{iff} \quad \nu'(P(a_1, \ldots, a_{\alpha(P)})) = 1,$$

then we say that $\mathfrak{M}$ is *conservative.* If $\mathfrak{M}$ is not conservative, we call $\mathfrak{M}$ *non-conservative.*

Note: As models are examples of structures, we will also use the above terminology in regard to models.

Essentially, the above definition states that a structure $\mathfrak{M}$ (with underlying class $\mathfrak{A}$) is conservative if each n-ary operation $f \in \mathcal{L}^{\mathcal{F}}$ has the same interpretation in $\mathfrak{M}$ as it does in some classical model $\mathfrak{B}$ with the same underlying class $\mathfrak{A}$, and moreover, the truth valuations of $\mathfrak{M}$ and $\mathfrak{B}$ agree on which elements of $\mathfrak{A}$ the predicates are 'true' on.

Clearly every standard irreducible model is conservative.[1] We do not know, however, if *every* standard model is conservative, but this is really just a question of classical logic which has nothing to do with quantum logic whatsoever.

[1] Recall by proposition 1.12, that $\{0, 1\}$ is the only irreducible Boolean algebra.

Examples of non-standard conservative models can be found in sections 4.2, 4.3, and 4.4; examples of non-conservative models can be found in section 4.2.4 as well as chapter 6. We will see that non-conservative models allow wild possibilities for the interpretations of the operation symbols $\mathcal{L}^{\mathcal{F}}$ as well as the truth valuations. However, conservative models are equally interesting in that they (in the non-standard case) allow for new, non-classical mathematical properties to hold while still retaining many essential properties of whatever classical model the conservative model is based on.

3.3.2 Models on Projection Lattices

As we have seen, using the first-order quantum logic as the foundational logic with which to build mathematics opens up new possibilities for models, as now any complete OML L (and more besides) is a candidate for the truth value algebra of a model. Of course, there are certain OMLs which are very natural with respect to quantum theory, namely the projection lattices of Hilbert spaces. Now, despite the wide variation in what is considered quantum logic, there is no doubt that the odd, non-classical behavior of subatomic systems is the ultimate motivator for the questioning of classical logic, and the very reason one might be tempted to replace it when developing mathematics. As such, it is clear that the projection lattices of Hilbert spaces are the most interesting OMLs to consider as truth value algebras when doing quantum mathematics.

This will have two interesting consequences for us. First, we expect that models built naturally from projection lattices of Hilbert spaces will shed light on what constitutes a "suitable" reduction of a given axiom systems. In the sequel, we construct a natural class of structures as candidates for models of (a choice of) the axioms for orthomodular lattices (section 4.3) where, in the construction of any one of these structures, we use a given (classical) OML both as the underlying set as well as for the truth value algebra of that structure. In doing so we find that such structures, when based on projection lattices, satisfy a particular natural reduction of one of the classical axioms not satisfied by these structures generally. We also develop a natural class of models of (a choice of) operator algebra axioms (section 4.4), where we use the bounded operators on a Hilbert space as the underlying set and the associated projection lattice of that Hilbert space as the truth value algebra. For these natural models, we again find that a particular classical axiom does not hold, but a reduction (similar to that for OML case) does. It may not be surprising that certain classical properties no longer hold in these natural quantum models, but since we expect such models to be the most "in touch" with quantum reality, we suspect that the precise classical properties which fail to hold in these natural models may have some deep meaning with regard to applying mathematics to describing the physical world, and even possibly for the foundations of mathematics itself.[1]

Second, when developing our axiomatic quantum set theory (chapter 5), and our quantum arithmetic (chapter 6), we find certain properties of projection lattices (which are not common to all OMLs) playing a prominent role in forcing certain desirable mathematical axioms to hold. This certainly suggests that our first-order quantum logic may be "too general", and there may

[1]The precise nature of this deep meaning is, however, a question beyond the pay grade of this simple author.

be important logical properties possessed by the projection lattices which are missing in our more general treatment. Only time[1] will tell.

3.3.3 Construction of Models

In the following chapters we will be constructing explicit models of various axiom systems, and so it will behoove us to examine conditions under which, for a fixed language $\mathcal{L}$, a given $\mathcal{L}$-structure is a model of some M-system $(\mathcal{L}, \mathcal{A})$. Before we present any propositions, we will make the following useful definition.

Definition 3.6. Let $\mathcal{L}$ be a language, and let $\mathfrak{M}$ be an $\mathcal{L}$-structure with underlying class $\mathfrak{A}$. For any atomic $\mathfrak{A}$-extended $\mathcal{L}$-sentence ψ, the *evaluation of* ψ *in* $\mathfrak{M}$ is the atomic $\mathfrak{A}$-extended $\mathcal{L}$-sentence obtained by replacing each $\mathcal{L}$-term in ψ with its evaluation. If an $\mathfrak{A}$-extended atomic $\mathcal{L}$-sentence χ is an evaluation of some $\mathfrak{A}$-extended atomic $\mathcal{L}$-sentence ψ in $\mathfrak{M}$, then χ is said to be an $\mathfrak{M}$-*evaluated atomic sentence.*

Note: Any atomic extended sentence ψ is of the form $\psi = P(t_1, \ldots, t_n)$ where P is an n-ary predicate, and $t_1, \ldots, t_n$ are extended terms with no variables. To obtain the evaluation of ψ, we first evaluate each term in the usual fashion so that each t_i evaluates to some $a_i \in \mathfrak{A}$. Then, the evaluation of ψ is just $P(a_1, \ldots, a_n)$.

We begin with a proposition which states that any map from the evaluated atomic sentences to a complete OML L always extends to a truth valuation.

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[1]Time, as well as an awful lot of hard work and the accompanying crushed dreams and aspirations of youth.

Proposition 3.12. Let $\mathcal{L}$ be a language, let $\mathfrak{A}$ be a class, let $\mathfrak{F}$ be an interpretation of $\mathcal{L}^{\mathcal{F}}$ in $\mathfrak{A}$, and let L be a complete OML. Define $\mathfrak{S}_0$ to be the subclass of atomic $\mathfrak{A}$-extended $\mathcal{L}$-sentences (with α the arity function on $\mathcal{L}^{\mathcal{P}}$) given by

$$\mathfrak{S}_0 \coloneqq \{P(a_1, \ldots, a_{\alpha(P)}) \ : \ P \in \mathcal{L}^{\mathcal{P}} \text{ and } a_1, \ldots, a_{\alpha(P)} \in \mathfrak{A}\},$$

and let ν be a class function from $\mathfrak{S}_0$ to L. Then there exists a unique $\hat{\nu}$ which is a truth valuation w.r.t. $(\mathcal{L}, \mathfrak{A}, \mathfrak{F}, L)$ which agrees with ν on $\mathfrak{S}_0$.

Proof. We simply define $\hat{\nu}$ inductively to satisfy the conditions (1-4) of definition 2.16, and then condition (5) of that same definition holds by lemma 2.11. □

Often, we will already have some model (namely a common model from classical mathematics) $\mathfrak{M}_0$ of an M-system $(\mathcal{L}, \mathcal{A})$, and we will then want to construct a (conservative) model from $\mathfrak{M}_0$. The following proposition will be useful in this regard.

Proposition 3.13. Let $(\mathcal{L}, \mathcal{A})$ be an M-system with model $\mathfrak{M}_0 = (\mathfrak{A}, L_0, [\![\cdot]\!]_0, \mathfrak{F})$, and let μ be a map from the $\mathfrak{M}_0$-evaluated atomic sentences to any complete OML L satisfying (for any $m \in \mathbb{N}$, any m-ary $P \in \mathcal{L}^{\mathcal{P}}$, and any $a_1, \ldots, a_m \in \mathfrak{A}$)

$$[\![P(a_1, \ldots, a_m)]\!]_0 = 1 \quad \text{iff} \quad \mu(P(a_1, \ldots, a_m)) = 1.$$

Then let $\mathfrak{M} \coloneqq (\mathfrak{A}, L, [\![\cdot]\!], \mathfrak{F})$ be the $\mathcal{L}$-structure with $[\![\cdot]\!]$ the unique extension of μ to a truth valuation (which exists by proposition 3.12), and let A be any $\mathcal{L}$-wff which can be written as to not contain the symbol '$\neg$' (using '$\vee$' and '$\exists$' is still allowed), and such that $\mathfrak{M}_0 \vDash A$. Then $\mathfrak{M} \vDash A$.

Proof. The proof is by induction on the definition of $[\![\cdot]\!]$ (the extension of μ to all evaluated $\mathcal{L}$-wffs). First, we consider the base case of an atomic $\mathcal{L}$-wff (for any $P \in \mathcal{L}^{\mathcal{P}}$, with arity m) $P(t_1(y_1,\ldots,y_n),\ldots,t_m(y_1,\ldots,y_n))$ such that $\mathfrak{M}_0 \vDash P(t_1,\ldots,t_m)$. Then for any $a_1,\ldots,a_n \in \mathfrak{A}$, we have

$$[\![P(t_1(a_1,\ldots,a_n),\ldots,t_m(a_1,\ldots,a_n))]\!]_0 = 1$$

because $P(t_1,\ldots,t_m)$ holds in $\mathfrak{M}_0$. Then, by assumption, we must have that

$$[\![P(t_1(a_1,\ldots,a_n),\ldots,t_m(a_1,\ldots,a_n))]\!] = 1$$

for every $a_1,\ldots,a_n \in A$, and hence $\mathfrak{M} \vDash P(t_1,\ldots,t_m)$.

For the inductive step, first assume that for extended $\mathcal{L}$-wffs B and C that both $\mathfrak{M}_0 \vDash B$ and $\mathfrak{M}_0 \vDash C$. Then for any evaluations $\tilde{B}$ of B and $\tilde{C}$ of C, $[\![\tilde{B}]\!] = [\![\tilde{C}]\!] = 1$ by induction, and so

$$[\![\tilde{B} \wedge \tilde{C}]\!] = [\![\tilde{B}]\!] \wedge [\![\tilde{C}]\!] = 1,$$

so $\mathfrak{M} \vDash B \wedge C$ (and similarly $\mathfrak{M} \vDash B \vee C$). Next, assume that $\mathfrak{M}_0 \vDash B(x,y_1,\ldots,y_n))$, so that $[\![B(a,a_1,\ldots,a_n)]\!]_0 = 1$ for all $a,a_1,\ldots,a_n \in \mathfrak{A}$. Then we have

$$[\![(\forall x)B(x,a_1,\ldots,a_n)]\!] = \bigwedge_{a\in\mathfrak{A}} [\![B(a,a_1,\ldots,a_n)]\!] = \bigwedge_{a\in\mathfrak{A}} [\![B(a,a_1,\ldots,a_n)]\!]_0 = 1$$

where the second to last inequality is by induction. Hence, we have $\mathfrak{M} \vDash B(x, y_1, \ldots, y_n)$. For the final case (using '$\exists$'), assume that $\mathfrak{M}_0 \vDash B(x, y_1, \ldots, y_n)$. Then for any $a_1, \ldots, a_n \in \mathfrak{A}$ we have

$$[\![(\exists x)B(x, a_1, \ldots, a_n)]\!] = \neg \bigwedge_{a \in \mathfrak{A}} \neg [\![B(x, a_1, \ldots, a_n)]\!] = \bigvee_{a \in \mathfrak{A}} [\![B(a, a_1, \ldots, a_n)]\!]_0 = 1,$$

so that $\mathfrak{M} \vDash (\exists x)B$. □

Note: For wffs A and B, recall that $A \to B := \neg A \vee (A \wedge B)$. Since this involves the symbol '$\neg$', we *cannot* use this proposition concerning wffs involving '$\to$'.

We conclude this section with one more technical lemma which we will find useful in our consideration of quantum set theory in chapter 5.

Lemma 3.14. Let $\mathcal{L}$ be a language, and let $\mathfrak{M}$ be an $\mathcal{L}$-structure with truth valuation $[\![\cdot]\!]$ and underlying class $\mathfrak{A}$. Further let $a, b, a_1, \ldots a_n \in \mathfrak{A}$ such that

$$[\![P(a_1, \ldots, a_{m-1}, a, a_{m+1}, a_{\alpha(P)})]\!] = [\![P(a_1, \ldots, a_{m-1}, b, a_{m+1}, a_{\alpha(P)})]\!]$$

for every $P \in \mathcal{L}^{\mathcal{P}}$ and for every $m \in \{1, \ldots, \alpha(P)\}$. Then for any $\mathfrak{A}$-extended $\mathcal{L}$-wff $\psi(x, x_1, \ldots, x_n)$ (with $x \in \mathcal{B}_V$), we have that $[\![\psi(a, a_1, \ldots, a_n)]\!] = [\![\psi(b, a_1, \ldots, a_n)]\!]$.

Proof. The proof is by induction on the construction of extended wffs. By assumption, the result holds for the atomic wffs. Assuming the result holds for some wff ψ, we have

$$[\![\neg\psi(a, a_1, \ldots, a_n)]\!] = \neg[\![\psi(a, a_1, \ldots, a_n)]\!] = \neg[\![\psi(b, a_1, \ldots, a_n)]\!] = [\![\neg\psi(b, a_1, \ldots, a_n)]\!].$$

Next, assuming the result holds for ψ_1 and ψ_2 we have

$$\begin{aligned}
[\![(\psi_1 \wedge \psi_2)(a, a_1, \ldots, a_n)]\!] &= [\![\psi_1(a, a_1, \ldots, a_n) \wedge \psi(a, a_1, \ldots, a_n)]\!] \\
&= [\![\psi_1(a, a_1, \ldots, a_n)]\!] \wedge [\![\psi_2(a, a_1, \ldots, a_n)]\!] \\
&= [\![\psi_1(b, a_1, \ldots, a_n)]\!] \wedge [\![\psi_2(b, a_1, \ldots, a_n)]\!] \\
&= [\![\psi_1(b, a_1, \ldots, a_n) \wedge \psi_1(b, a_1, \ldots, a_n)]\!] \\
&= [\![(\psi_1 \wedge \psi_2)(b, a_1, \ldots, a_n)]\!].
\end{aligned}$$

Finally, assume the result holds for some $\psi(x, x_1, \ldots, x_n, y)$. Then

$$\begin{aligned}
[\![(\forall y)\psi(a, a_1, \ldots, a_n, y)]\!] &= \bigwedge_{c \in \mathfrak{A}} [\![\psi(a, a_1, \ldots, a_n, c)]\!] \\
&= \bigwedge_{c \in \mathfrak{A}} [\![\psi(b, a_1, \ldots, a_n, c)]\!] \\
&= [\![(\forall y)\psi(b, a_1, \ldots, a_n, y)]\!].
\end{aligned}$$

This completes the induction. $\square$

Now that we have established these useful technical results and characterized some simple classes of M-systems and structures, in the following chapters we will proceed to consider particular M-systems and construct models of these systems within the framework of our first-order quantum logic.

CHAPTER 4

QUANTUM ALGEBRAIC SYSTEMS

In this chapter, we proceed to use the first-order quantum logic developed in the previous chapters to do some quantum mathematics. In the first section (4.1), we will discuss *equational languages*,[1] which are simply languages where there is only one predicate symbol (for which we will use '$\approx$'), namely a binary predicate meant to represent "equality". We will use such languages in a formal treatment of abstract and linear algebra.

Before presenting particular examples of such quantum algebraic systems, we recall our discussion of section 3.1.2, in which we noted that some axiom systems are inherently classical. One inherently classical axiom system is the usual axiomatization of Peano arithmetic, as showed by Dunn (15). In section 4.1.3, we will prove two theorems which show that certain further classes of axiom systems (over equational languages) are inherently classical.

We then proceed to our examples of quantum algebraic systems. In section 4.2 we will construct non-standard models of both (common choices of) the monoid and group axioms, which will prove useful for building our quantum mathematical intuition, as well as serve to illustrate the general types of models discussed in the previous chapter (section 3.3.3). In the following section (4.3) we will get a little more "meta" and begin a quantum mathematical treatment of the mathematics which underlies the quantum logic we have developed — namely

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[1] See (8) for a treatment of equational languages in classical logic.

we will construct natural (non-standard) models of (a natural reduction of) the axioms for orthomodular lattices. In the final section (4.4), we will examine both linear algebra and operator algebras within quantum mathematics. After constructing natural models of axiom systems associated with both these areas of mathematics, we will show that these vector space and operator algebra models have not only a beautiful interrelationship with one another, but also with the models of the orthomodular lattice axioms previously discussed. We finally discuss how these natural models may be relevant for a quantum mathematical formulation of quantum mechanics itself — specifically, we will examine both the Schrödinger and von Neumann equations in the context of the quantum mathematics developed.

4.1 Equational Languages

In this chapter we will only be concerned with M-systems using a language with a single binary predicate (meant to represent equality). We begin with the formal definition of such a language.

Definition 4.1. Let $\mathcal{L}$ be a language. If $\mathcal{L}^{\mathcal{P}} = \{\approx\}$, where '$\approx$' is a binary predicate, then $\mathcal{L}$ is called an *equational language.*

4.1.1 Equality and Substitution Axioms

Of course, '$\approx$' is just a symbol. We also need the following axioms to insure that our 'equality' behaves as expected.

Definition 4.2. Let $\mathcal{L}$ be an equational language. We define the following $\mathcal{L}$-wffs (for $x, y, z \in \mathcal{B}_V$)

(E1) $(\forall x)(x \approx x)$ (Reflexivity)

(E2) $(\forall x)(\forall y)[(x \approx y) \to (y \approx x)]$ (Symmetry)

(E3) $(\forall x)(\forall y)(\forall z)[(x \approx y) \wedge (y \approx z)] \to (x \approx z)$ (Weak Transitivity)

(E3′) $(\forall x)(\forall y)(\forall z)\big((x \approx y) \to [(y \approx z) \to (x \approx z)]\big)$ (Strong Transitivity).

We then define the *equality axioms* (denoted $\mathcal{E}$) to be

$$\mathcal{E} := \{\mathrm{E1}, \mathrm{E2}, \mathrm{E3}\}.$$

The wffs E3 and E3′ are classically equivalent.[1] However, we will now show that E3′ is strictly stronger in our first-order quantum logic (see example 4.14 below for the *strictly* portion of this claim).

Proposition 4.1. Let $\mathcal{L}$ be an equational language. Then $\mathrm{E3}' \vdash \mathrm{E3}$.

Proof. Define $A := (x \approx y)$, $B := (y \approx z)$, and $C := (x \approx z)$ with $x, y, z \in \mathcal{B}_V$. By Q6 and R5 we have $\mathrm{E3}' \vdash A \to (B \to C)$. Then proposition 2.3 gives $\mathrm{E3}' \vdash (A \wedge B) \to C$. We will now

[1] This follows directly from the deduction theorem, which is discussed in section 1.1.2.

use soundness and completeness, so consider any model $\mathfrak{M}$ of E3$'$ with truth valuation $[\![\cdot]\!]$ and underlying class $\mathfrak{A}$. We have (by soundness)

$$\begin{aligned} 1 &= [\![(A(x,y) \wedge B(y,z)) \to C(x,z)]\!] \\ &= \bigwedge_{a,b,c \in \mathfrak{A}} [\![(A(a,b) \wedge B(b,c)) \to C(a,c)]\!] \\ &= [\![(\forall x)(\forall y)(\forall z)[(A \wedge B) \to C]]\!], \end{aligned}$$

and so completeness gives

$$\mathrm{E3}' \vdash (\forall x)(\forall y)(\forall z)[(A \wedge B) \to C],$$

i.e. E3$' \vdash$ E3. □

Since we expect our predicate symbol '$\approx$' to behave like an equality, we force that an equational M-system satisfy E1-E3.

Definition 4.3. Let $(\mathcal{L}, \mathcal{A})$ be an M-system. If $\mathcal{L}$ is equational, and $\mathcal{A} \vdash \mathcal{E}$, then we call $(\mathcal{L}, \mathcal{A})$ an *equational M-system.*

In classical logic, it is customary to enforce the following *substitution axioms for* $\mathcal{L}$, which involve, for each $f \in \mathcal{L}^{\mathcal{F}}$, $\alpha(f)$ axioms (one for each 'slot').

Definition 4.4. Let $\mathcal{L}$ be an equational language. Define the following wff (for $f \in \mathcal{L}^{\mathcal{F}}$, $m \in \{1, \ldots, \alpha(f)\}$, and variables $x, y, z_1, \ldots, z_{\alpha(f)}$)

(Sub$[f,m]$) *Substitution for f in the m^{th} slot:*

$$(\forall x)(\forall y)(\forall z_1)\cdots(\forall z_{\alpha(f)})\big((x \approx y) \to$$

$$[f(z_1,\ldots,z_{m-1},x,z_{m+1},\ldots,z_{\alpha(f)}) \approx f(z_1,\ldots,z_{m-1},y,z_{m+1},\ldots,z_{\alpha(f)})]\big).$$

We also define

(Sub$[f]$) *Substitution for f:* Sub$[f,1] \wedge \cdots \wedge$ Sub$[f,\alpha(f)]$,

as well as the *substitution axioms for $\mathcal{L}$* to be

$$(\mathrm{Sub}(\mathcal{L})) \coloneqq \{\mathrm{Sub}[f] \ : \ f \in \mathcal{L}^{\mathcal{F}}\}.$$

The motivation for the substitution axioms is clear — it is intuitive to expect that whenever the statement 'a equals b' is true, the statement '$f(a)$ equals $f(b)$' (for any unary operation f) is true as well. Now in quantum logic, the substitution axioms given above actually force *more* than the above intuitive requirement on the extended sentence $a \approx b$ (for a, b in a given model). The basic reason for this is that, for wffs A and B, we have that $A \to B$ holds in a given model if (for $[\![\cdot]\!]$ the truth valuation of that model) $[\![A]\!] \leq [\![B]\!]$. In the context of our unary f above, forcing substitution requires $[\![a \approx b]\!] \leq [\![f(a) \approx f(b)]\!]$, which is a significantly stronger requirement than $[\![f(a) \approx f(b)]\!] = 1$ whenever $[\![a \approx b]\!] = 1$. These considerations make it clear that the substitution axioms (as given above) are of questionable motivation in quantum mathematics, and suggest that a reduced version of these axioms may be more appropriate.

As further justification for this claim, our natural non-standard models of axiom systems for both orthomodular lattices and vector spaces do *not* satisfy the substitution axioms above for all their operations (see sections 4.3 and 4.4).

One natural choice for a reduced version of substitution is the following (where $f \in \mathcal{L}^{\mathcal{F}}$, $m \in \{1, \ldots, \alpha(f)\}$, and '$\mathbf{T}$' is the operator from section 3.2.2):

(RSub$[f, m]$) *Reduced substitution for f in the m^{th} slot:*

$$(\forall x)(\forall y)(\forall z_1)\cdots(\forall z_{\alpha(f)})\big(\mathbf{T}(x \approx y) \rightarrow$$
$$[f(z_1, \ldots, z_{m-1}, x, z_{m+1}, \ldots, z_{\alpha(f)}) \approx f(z_1, \ldots, z_{m-1}, y, z_{m+1}, \ldots, z_{\alpha(f)})]\big),$$

where we then define the axioms RSub$[f]$ and RSub$(\mathcal{L})$ in the same fashion as the unreduced case. One problem with this reduction is that the '$\mathbf{T}$' operator only behaves like we want it to when the models under consideration have certain properties, so that this approach is not entirely general. In particular it does not necessarily yield a reduction of substitution. However, the following proposition shows that it *will* be a reduction under a simple criteria.

Proposition 4.2. Let $(\mathcal{L}, \mathcal{A})$ be an equational M-system such that $\mathcal{A} \vdash \mathbf{T}(\psi) \rightarrow \psi$ for any $\mathcal{L}$-wff ψ, let $f \in \mathcal{L}^{\mathcal{F}}$, and $m \in \{1, \ldots \alpha(f)\}$. Then $\mathcal{A} \cup \{\text{Sub}[f, m]\}$ is classically equivalent to $\mathcal{A} \cup \{\text{RSub}[f, m]\}$.

Proof. First we assume Sub$[f, m]$. By assumption, $\mathcal{A} \vdash \mathbf{T}(x \approx y) \to (x \approx y)$ and also (using Q6 and R5 repeatedly)

$$\mathcal{A} \vdash (x \approx y) \to [f(z_1, \ldots, z_{m-1}, x, \ldots, z_{\alpha(f)}) \approx f(z_1, \ldots, x_{m-1}, y, \ldots z_{\alpha(f)})].$$

Then R1 gives

$$\mathcal{A} \vdash \mathbf{T}(x \approx y) \to [f(z_1, \ldots, z_{m-1}, x, \ldots, z_{\alpha(f)}) \approx f(z_1, \ldots, x_{m-1}, y, \ldots z_{\alpha(f)})].$$

so the completeness theorem gives that $\mathcal{A} \cup \{\mathrm{Sub}[f, m]\} \vdash \mathrm{RSub}[f, m]$, and hence trivially that

$$\mathcal{A} \cup \{\mathrm{Sub}[f, m], \mathrm{CL}\} \vdash \mathrm{RSub}[f, m].$$

For the other direction, in the presence of CL we use completeness with respect to irreducibles. This means we need to show that (for any truth valuation $[\![\cdot]\!]$, truth value algebra B_2, and underlying set A)

$$[\![a \approx b]\!] \leq [\![\mathbf{T}(a \approx b)]\!] \tag{4.1}$$

for any $a, b \in A$. Since $[\![a \approx b]\!] \in \{0, 1\}$, we consider the non-trivial case $[\![a \approx b]\!] = 1$. But then

$$[\![\mathbf{T}(a \approx b)]\!] = \bigwedge_{c,d \in A} \left([\![c \approx d]\!] \to [\![a \approx b]\!]\right) = 1$$

using lemma C.14, showing the inequality (equation 4.1) holds, establishing the desired result.

□

For the models we consider, with one exception (our natural model for the orthomodular lattice axioms), we will either impose the full substitution, or the reduced substitution described above. Before moving on to some general considerations in the construction of models of equational languages, we prove one technical (although intuitive) lemma concerning equality which will be useful later in propositions 4.13 and 4.19.

Lemma 4.3. Let $(\mathcal{L}, \mathcal{A})$ be an equational M-system such that $\mathcal{A} \vdash \mathcal{E}$, let A be an $\mathcal{L}$-wff, and let $t_1, \ldots, t_{n+1}$ be $\mathcal{L}$-terms such that, for each $i \in \{1, \ldots, n\}$, at least one of the following holds

1. $\mathcal{A} \vdash (t_i \approx t_{i+1})$
2. $\mathcal{A} \vdash (t_{i+1} \approx t_i)$
3. $\mathcal{A} \vdash A \to (t_i \approx t_{i+1})$
4. $\mathcal{A} \vdash A \to (t_{i+1} \approx t_i)$.

Then $\mathcal{A} \vdash A \to (t_1 \approx t_{n+1})$.

Proof. First, let $i \in \{1, \ldots, n\}$. Note that $\mathcal{A}, (t_{i+1} \approx t_i) \vdash t_i \approx t_{i+1}$ by Q6 and R5, since $\mathcal{A} \vdash$E2. Also, it is easy to see (by completeness and lemma C.14) that if $\mathcal{A} \vdash t_i \approx t_{i+1}$, that $\mathcal{A} \vdash A \to (t_i \approx t_{i+1})$. Finally, if $\mathcal{A} \vdash A \to (t_{i+1} \approx t_i)$, then (again since $\mathcal{A} \vdash$E2) using Q6 gives that $\mathcal{A} \vdash (t_{i+1} \approx t_i) \to (t_i \approx t_{i+1})$, so that by R1 $\mathcal{A} \vdash A \to (t_i \approx t_{i+1})$. Given these considerations, for each $i \in \{1, \ldots, n\}$, we have $\mathcal{A} \vdash A \to (t_i \approx t_{i+1})$.

We then complete the proof by induction, for which we only need show that if $\mathcal{A} \vdash A \to (t_1 \approx t_i)$, then $\mathcal{A} \vdash A \to (t_1 \approx t_{i+1})$ for $i \in \{1, \ldots, n\}$. Now if $\mathcal{A} \vdash A \to (t_1 \approx t_i)$, we then also have (given the above considerations), using R3, that

$$\mathcal{A} \vdash A \to \big[(t_1 \approx t_i) \wedge (t_i \approx t_{i+1})\big].$$

Also, since $\mathcal{A} \vdash$E3, we have by Q6 and R5 that

$$\mathcal{A} \vdash \big[(t_1 \approx t_i) \wedge (t_i \approx t_{i+1})\big] \to (t_1 \approx t_{i+1}),$$

so that R1 gives $\mathcal{A} \vdash A \to (t_1 \approx t_{i+1})$, completing the induction. □

4.1.2 Constructing Models for Equational Languages

We will now provide some example OMLs which will be useful as truth value algebras of our models. Following this, we will do a little bit of technical work which will help in proving that the structures we construct in the remainder of this chapter are, indeed, models.

Some Simple OMLs

Example 4.4. For $n \in \mathbb{N}\backslash\{0\}$, We define $\mathrm{MO}_n := \{v_i \;:\; i = \pm 1, \pm 2, \ldots, \pm n\} \cup \{0, 1\}$ with involution $\neg v_i := v_{-i}$, and where for every distinct $v, w \in \mathrm{MO}_n \backslash \{0, 1\}$ we have $v \wedge w = 0$ and $v \vee w = 1$. We note that MO_1 is, in fact, a reducible Boolean algebra (isomorphic to the product Boolean algebra $B_2 \times B_2$), and also that MO_n is irreducible and modular for every $n \geq 2$. These generalize MO_2 of example 1.13.

Example 4.5. One simple reducible modular ortholattice is the product $\mathrm{MO}_2 \times B_2$. Rather than write the elements in the usual product fashion, we simply label the atoms a, b, c, d, e. One can then see the ordering (and involution) from the labeled Hasse diagram in Figure 5.

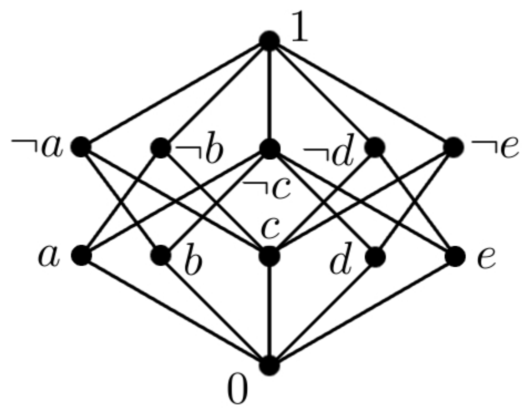

Figure 5. The Hasse diagram for $\mathrm{MO}_2 \times B_2$ from example 4.5.

We will refer back to these OMLs as needed when constructing models below.

Proving Structures are Models

The algebraic axiom systems we consider will all contain (at least) E1-E3, and so it will behoove us to determine simple criteria for showing that a given structure is a model for these axioms. We begin with a simple definition.

Definition 4.5. Let A be a set, L an OML, and $\nu : A^2 \to L$ a map satisfying

1. $\nu(a,b) = 1 \quad \text{iff} \quad a = b$

2. $\nu(a,b) = \nu(b,a)$

Then ν is called a *pre-truth map.*

We already know from proposition 3.12 that we can extend such a pre-truth map[1] to obtain a unique truth valuation. The following lemma shows such a truth valuation will always yield a model of E1-E2, and if the pre-truth map satisfies a simple criteria, that structure will model E3 as well.

Lemma 4.6. Let $\mathcal{L}$ be an equational language, let A be a set with interpretation $\mathfrak{F}$ of $\mathcal{L}^{\mathcal{F}}$ on A, let L be a complete OML, and let $\nu : A^2 \to L$ be a pre-truth map. Then there exists a unique truth valuation $\hat{\nu}$ w.r.t. $(\mathcal{L}, A, \mathfrak{F}, L)$ such that $\hat{\nu}(a \approx b) = \nu(a,b)$ for all $a, b \in A$. Furthermore, $\mathfrak{M} := (A, \operatorname{ran} \hat{\nu}, \hat{\nu}, \mathfrak{F})$ is a model for E1 and E2. If ν furthermore satisfies (for all $a, b, c \in A$)

$$\nu(a,b) \wedge \nu(b,c) \leq \nu(a,c) \tag{4.2}$$

Then $\mathfrak{M} \vDash \text{E3}$ as well.

Proof. For every atomic sentence of the form $a \approx b$ (for $a, b \in A$) we define $\sigma(a \approx b) := \nu(a,b)$. Then since L is complete, by proposition 3.13, let $\hat{\nu}$ be the unique truth valuation w.r.t. $(\mathcal{L}, A, \mathfrak{F}, L)$, so that, $\mathfrak{M}$ as above is an $\mathcal{L}$-structure.

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[1]Technically, in order to use the aforementioned proposition, we need to form a map $\hat{\nu}$ from ν such that $\hat{\nu}(a \approx b) = \nu(a,b)$ for each $a, b \in A$, and then extend the map $\hat{\nu}$.

Then $\mathfrak{M}$ is a model for E1-E2, since these sentences are satisfied by properties (1) and (2) of definition 4.5. Also if equation 4.2 is satisfied, then it is easy to see that $\mathfrak{M} \models$ E3. $\square$

In the coming sections, we will frequently construct structures beginning from standard models familiar from classical mathematics. The reason this procedure will be so effective in generating models is that the axioms for algebraic systems (besides the equality and substitution axioms) are all of the same simple form, motivating the following definition.

Definition 4.6. Let $\mathcal{L}$ be an equational language, and let ψ be an $\mathcal{L}$-wff given by

$$\psi = (\forall x_1)\cdots(\forall x_n)\,[t(x_1,\ldots,x_n) \approx u(x_1,\ldots,x_n)],$$

where t and u are $\mathcal{L}$-terms and where $x_1,\ldots,x_n \in \mathcal{B}_V$. Then ψ is said to be *algebraic.*

Lemma 4.7. Let $\mathcal{L}$ be an equational language, let Γ be a set of algebraic $\mathcal{L}$-sentences, and let $\mathfrak{M}_0 := (A_0, B_2, [\![\cdot]\!], \mathfrak{F}_0)$ be a model[1] of Γ. Further let L be a complete OML, let $\nu : A^2 \to L$ be a pre-truth map, and let $\mathfrak{M}$ be the structure given by lemma 4.6 with underlying set A_0 and the interpretation $\mathfrak{F}_0$. Then $\mathfrak{M} \models \Gamma$.

Proof. This is just a special case of proposition 3.13. $\square$

4.1.3 Inherent Classicality of Certain Axiom Systems

Recall that an M-system $(\mathcal{L},\mathcal{A})$ is said to be inherently classical if $\mathcal{A} \vdash$CL. As mentioned previously, Dunn's theorem (15) states that (for the usual first-order presentation of the ax-

[1]Note that $\mathfrak{M}_0$ is a classical model, and so we usually write '$a = b$' to mean '$[\![a \approx b]\!] = 1$' for $a, b \in A$.

ioms), the theorems of Peano arithmetic under quantum logic are exactly the same as those under classical logic, i.e. the usual Peano arithmetic axioms have only standard models (by the completeness theorem). In this section, we prove two similar theorems for algebraic systems which satisfy strong transitivity and substitution, and also meet some additional simple criteria. One surprising fact about these theorems is that they apply to M-systems with finite sets of axioms,[1] showing that (at least for some M-systems commonly used in algebra) one does not need to add wff schema to $\mathcal{Q}_A(\mathcal{L})$ to obtain classicality (i.e. derive CL).

The first such theorem will show that any M-system which has cancellative terms (and satisfies E3′) will be inherently classical — we now define what we mean for a term to be cancellative.[2]

Definition 4.7. Let ($\mathcal{L}$, $\mathcal{A}$) be an equational M-system. Define the following two $\mathcal{L}$-wffs (for $t(x,y)$ an $\mathcal{L}$-term)

(LC[t]) $t(x,y) \approx t(x,z) \to y \approx z$

(RC[t]) $t(x,z) \approx t(y,z) \to x \approx y$

If, for a given $t(x,y)$, both LC[t] and RC[t] are derivable from $\mathcal{A}$, we say that t is *cancellative in* $(\mathcal{L},\mathcal{A})$.[3]

---

[1]See, for example, propositions 4.13 and 4.19. This contrasts with the case of Peano arithmetic — see proposition 6.1 in section 6.1.2 for details.

[2]Although we only consider terms with two free variables for simplicity, we could easily generalize our notion of cancellativity to terms with an arbitrary number of free variables.

[3]We may omit the M-system $(\mathcal{L},\mathcal{A})$ when no confusion will arise.

In order to obtain our theorems of inherent classicality, we will need a number of technical lemmas. The reader interested only in the results and not their proof may skip over these lemmas and their proofs without loss of continuity.

Lemma 4.8. Let $(\mathcal{L}, \mathcal{A})$ be an M-system such that $\mathrm{Sub}(\mathcal{L})$ is derivable from $\mathcal{A}$. Then, for any $\mathcal{L}$-term $t(y_1, \ldots, y_n)$,

$$(y_1 \approx z_1) \wedge \cdots \wedge (y_n \approx z_n) \to t(y_1, \ldots, y_n) \approx t(z_1, \ldots, z_n) \tag{4.3}$$

is derivable from $\mathcal{A}$.

Proof. Since every operation in $\mathcal{L}$ satisfies $\mathrm{Sub}(\mathcal{L})$, the statement follows by a simple induction on the construction of $\mathcal{L}$-terms. $\square$

This allows us to define substitution for *terms* as well as functions.

Definition 4.8. Let $(\mathcal{L}, \mathcal{A})$ be an equational M-system, and let $t(y_1, \ldots, y_n)$ be an $\mathcal{L}$-term such that the $\mathcal{L}$-wff

$$(y_1 \approx z_1) \wedge \cdots \wedge (y_n \approx z_n) \to t(y_1, \ldots, y_n) \approx t(z_1, \ldots, z_n)$$

is derivable from $\mathcal{A}$. Then we will say that t *satisfies substitution in* $(\mathcal{L}, \mathcal{A})$.[1]

Lemma 4.9. Let $(\mathcal{L}, \mathcal{A})$ be an equational M-system and let A and B be $\mathcal{L}$-wffs. Then $\mathcal{A} \vdash A \,\widetilde{C}\, B$ iff $\mathcal{A} \vdash A \wedge (\neg A \vee B)) \to B$.

---

[1] And, as the reader has probably become accustomed to, we will frequently omit the extra baggage $(\mathcal{L}, \mathcal{A})$ when the M-system is clear from the context.

Proof. First we assume $\mathcal{A} \vdash A\widetilde{C}B$. Then $\mathcal{A} \vdash A \to [\neg(\neg A \vee B) \vee (A \wedge B)]$ by Q3, R1, and replacement (using prop. 2.4). Then

$$\mathcal{A} \vdash [A \wedge (\neg A \vee B)] \to [(\neg A \vee B) \wedge (\neg(\neg A \vee B) \vee (A \wedge B))]$$

by Q3 and R3. Then, since $\mathcal{A} \vdash A \wedge B \to B$ by Q3, and $\mathcal{A} \vdash B \to (\neg A \vee B) \wedge B$ by Q3, R2, Q1 and R3, we have

$$\mathcal{A} \vdash [A \wedge (\neg A \vee B)] \to [(\neg A \vee B) \wedge (\neg A \vee B) \to B)]$$

by R1 and lemma 2.7. R1 and Q4 then give the desired conclusion.

Next, assume $\mathcal{A} \vdash A \wedge (\neg A \vee B)) \to B$. By Q4, R2, and Q2 we have

$$\mathcal{A} \vdash A \to \Big((A \wedge \neg B) \vee [\neg(A \wedge \neg B) \wedge (A \vee (A \wedge \neg B))]\Big)$$

and then by Q3, Q2, and R1 this gives $\mathcal{A} \vdash A \to (A \wedge \neg B) \vee [(\neg A \vee B) \wedge A]$. Our assumption also gives $\mathcal{A} \vdash A \wedge (\neg A \vee B) \to (A \wedge B)$ by Q3 and R3, so then by R1, Q3 and R3 we have $\mathcal{A} \vdash A \to (A \wedge \neg B) \vee (A \wedge B)$, so that $A\widetilde{C}B$ (the other implication is trivial in any OML, and so follows for wffs by corollary 2.26). Statement (5) in proposition 2.4 then gives the desired conclusion. $\square$

As stated in the beginning of this section, the proof of our theorems of inherently classicality both require E3′. The following lemma is the reason.

Lemma 4.10. Let $(\mathcal{L}, \mathcal{A})$ be an equational M-system such that $\mathcal{A} \vdash \mathrm{E3}'$, and let t, u, v be $\mathcal{L}$-terms. Then

$$\mathcal{A} \vdash (t \approx u) \widetilde{C} (t \approx v).$$

Proof. Let $A \coloneqq (t \approx u)$, $B \coloneqq (t \approx v)$ and $C \coloneqq (u \approx v)$. From E3′, and Q3, along with E2 and replacement, we have $\mathcal{A} \vdash (B \wedge C) \to (B \to A)$, and so then by R1 and lemma 2.7, we have $\mathcal{A} \vdash (B \to C) \to (B \to A)$. Also by E3′ and E2, we have $\mathcal{A} \vdash A \to (B \to C)$, and hence by R1 we have $\mathcal{A} \vdash A \to (B \to A)$, and then by R2 and Q2 we have $\mathcal{A} \vdash B \wedge (\neg B \vee \neg A) \to \neg A$, which gives $\mathcal{A} \vdash B \widetilde{C} \neg A$ by lemma 4.9. Recalling the note following def. 3.1, this gives $\mathcal{A} \vdash A \widetilde{C} B$. ☐

We need only one more technical lemma to prove our theorems.

Lemma 4.11. Let $(\mathcal{L}, \mathcal{A})$ be an equational M-system such that

$$\mathcal{A} \vdash (t \approx u) \widetilde{C} (v \approx w)$$

for all $\mathcal{L}$-terms t, u, v, w. Then the axiom schema CL is derivable from $\mathcal{A}$.

Proof. The hypothesis of this lemma is that all atomic $\mathcal{L}$-wffs are compatible. Hence, for any atomic extended sentences A and B, and any model $\mathfrak{M}$ with truth valuation $[\![\cdot]\!]$ and truth value algebra L, we have $[\![A]\!]\ C\ [\![B]\!]$, i.e. the corresponding truth values commute in L. Proposition C.10 along with the fact that such truth values generate L then give that L is Boolean. Since $\mathfrak{M}$ was a generic model, the result follows from the completeness theorem. ☐

We are now ready to prove our main theorems. The first says that any M-system which (a) satisfies strong transitivity, and (b) has some cancellative term satisfying substitution is inherently classical. For an example see proposition 4.13.

Theorem 4. Let $(\mathcal{L}, \mathcal{A})$ be an M-system such that $\mathcal{A} \vdash \mathrm{E3}'$, and also let there exist an $\mathcal{L}$-term $t(x, y)$ which is cancellative and which satisfies substitution. Then CL is derivable from $\mathcal{A}$.

Proof. By lemma 4.11, it suffices to prove that any two atomic $\mathcal{L}$-wffs $u_1 \approx v_1$ and $u_2 \approx v_2$ are compatible. Then by lemma 4.8 and since $t(x, y)$ is cancellative, it follows that

$$\mathcal{A} \vdash u_1 \approx v_1 \leftrightarrow t(u_1, v_2) \approx t(v_1, v_2) \quad \text{and} \quad \mathcal{A} \vdash u_2 \approx v_2 \leftrightarrow t(v_1, u_2) \approx t(v_1, v_2).$$

Further using modus ponens (R5) we have that $u_1 \approx v_1$ is logically equivalent to $t(u_1, v_2) \approx t(v_1, v_2)$ and $u_2 \approx v_2$ is logically equivalent to $t(v_1, u_2) \approx t(v_1, v_2)$ (with respect to $\mathcal{A}$).

Then, by lemma 4.10, we have that

$$\mathcal{A} \vdash [t(u_1, v_2) \approx t(v_1, v_2)] \, \widetilde{C} \, [t(v_1, u_2) \approx t(v_1, v_2)].$$

and then, using replacement (lemma 2.8), this gives that $\mathcal{A} \vdash (u_1 \approx v_1) \, \widetilde{C} \, (u_2 \approx v_2)$. Since this holds for arbitrary $\mathcal{L}$-terms u_1, u_2, v_1, v_2, the conclusion is established. $\square$

Our second theorem in this vein states that any M-system which (a) satisfies strong transitivity, and (b) has an appropriate constant term is also inherently classical. For an example, see proposition 4.19.

Theorem 5. Let $(\mathcal{L}, \mathcal{A})$ be an equational M-system such that $\mathcal{A} \vdash$ E3$'$, and also such that there exists an $\mathcal{L}$-term u with no variables and an $\mathcal{L}$-term $t(x, y)$ such that

$$\mathcal{A} \vdash x \approx y \leftrightarrow t(x, y) \approx u.$$

Then CL is derivable from $\mathcal{A}$.

Proof. As in the previous theorem, it suffices to show that any two atomic $\mathcal{L}$-wffs are compatible with respect to $\mathcal{A}$. Let $u_1 \approx v_1$ and $u_2 \approx v_2$ be arbitrary atomic $\mathcal{L}$-wffs. By lemma 4.10,

$$\mathcal{A} \vdash t(u_1, v_1) \approx u \ \widetilde{C}\ \ t(u_2, v_2) \approx u.$$

But by assumption and R5, we have that $u_1 \approx v_1$ is logically equivalent to $t(u_1, v_1) \approx u$ (w.r.t. $\mathcal{A}$) and $u_2 \approx v_2$ is logically equivalent to $t(u_2, v_2) \approx u$ (w.r.t. $\mathcal{A}$). Hence replacement yields that

$$\mathcal{A} \vdash u_1 \approx v_1 \ \widetilde{C}\ \ u_2 \approx v_2.$$

$\square$

From the two theorems just proven, we see that strong transitivity of equality seems to be a very *classical* property. Of course, algebraic systems with cancellative terms are extremely common and occur whenever there is a group structure (such as the additive structure in arithmetic, on vector spaces, or on operators), and the orthomodular lattice axioms are amongst those for which an appropriate constant term (namely 1) exists. Hence, for any such algebraic system, if one wishes to find any non-standard models, one *cannot* impose strong transitivity.

4.2 Quantum Monoids and Quantum Groups

We now investigate certain axiomatizations of monoids and group, as well as non-standard models of these axiomatizations. We begin by showing that the usual group axioms are strong enough to prove that groups have cancellative terms in quantum logic, so that by theorem 4 above, the usual group axioms (along with strong transitivity) are inherently classical. We then move on to considering examples, the first of which is a non-standard (conservative) model of a group which satisfies only weak transitivity (not strong), vindicating our claim that E3$'$ is strictly stronger than E3 in quantum logic. We then present a non-standard (conservative) model of a monoid which *does* satisfy E3$'$ — this shows that we cannot prove an analog of the above theorems which would show that strong transitivity and substitution *by themselves* are inherently classical. Our final example of the section is a non-conservative model of a monoid, showing that non-conservative models do, indeed, exist.

4.2.1 Groups Axioms with Strong Transitivity are Inherently Classical

We define $\mathcal{L}_G := \{e, *, \cdot^{-1}\}$ where the symbol 'e' represents the identity element, so $\alpha(e) := 0$, and '$\cdot^{-1}$' represents the operation of taking an element to its inverse, so $\alpha(\cdot^{-1}) := 1$, and finally '$*$' represents the multiplication, so that $\alpha(*) := 2$. We then let the group axioms $\mathcal{A}_G$ be $\mathcal{E}$ along with the wffs G1-G5 (with $x, y, z \in \mathcal{B}_V$).

(G1) $(\forall x)(\forall y)(\forall z)\ [(x * y) * z \approx x * (y * z)]$

(G2) $(\forall x)(e * x \approx x)$ and $(\forall x)(x * e \approx x)$

(G3) $(\forall x)(x * x^{-1} \approx e)$ and $(\forall x)(x^{-1} * x \approx e)$

(G4) $(\forall x)(\forall y)(x \approx y \to x^{-1} \approx y^{-1})$

(G5) $(\forall x)(\forall y)(\forall z)(x \approx y \to x * z \approx y * z) \wedge (\forall x)(\forall y)(\forall z)(x \approx y \to z * x \approx z * y)$

Note that (G4) and (G5) are just substitution for $\cdot^{-1}$ and $*$, respectively.

Obviously, groups under classical logic have cancellative terms — namely (for group multiplication '$*$' and group elements a, b, c) we have that $a * c = b * c$ implies $a = b$. This follows simply by multiplying on the right with c^{-1}. We now show that the usual group axioms still have this property under quantum logic.

Proposition 4.12. Let $x, y, z \in \mathcal{B}_V$. Then $\mathcal{A}_G \vdash (x * z \approx y * z) \to (x \approx y)$.

Proof. We wish to find an appropriate chain of equalities to use lemma 4.3. We will represent '$*$' by simple concatenation to make this proof a little easier to read. First, G2 yields $x \approx xe$ (by Q6 and R5), then G3 gives $e \approx zz^{-1}$ (again using Q6 and R5), and so this along with G5 (using R5) gives $xe \approx x(zz^{-1})$. Then, by G1 (using Q6 and R5) we have $x(zz^{-1}) \approx (xz)z^{-1}$. Now, G5 (by Q6 and R5) gives $(xz \approx yz) \to ((xz)z^{-1} \approx (yz)z^{-1})$. We then have the 'reverse chain' using 'y' instead of 'x'. Specifically, G1 (using Q6 and R5) gives $y(zz^{-1}) \approx (yz)z^{-1}$, and recalling $e \approx zz^{-1}$, we have that G5 (using R5) gives $ye \approx y(zz^{-1})$, finally G2 yields $y \approx ye$ (by Q6 and R5). This establishes the necessary chain, so that lemma 4.3 yields the desired results. □

This proposition then gives that the group axioms $\mathcal{A}_G$ along with strong transitivity are inherently classical.

Proposition 4.13. In the language $\mathcal{L}_G$, the set $\mathcal{A}_G \cup \{\text{E3}'\}$ is inherently classical.

Proof. The above proposition shows that $t(x, y) := x * y$ is a cancellative term. Also, $t(x, y)$ satisfies substitution by G5. Hence, the result follows by theorem 4. $\square$

We now study a model of the group axioms $\mathcal{A}_G$ which does *not* satisfy strong transitivity.

4.2.2 Non-standard Model of a Group

Example 4.14. For $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{00, 01, 10, 11\}$, we define the interpretation $\mathfrak{F}_G$ of $\mathcal{L}_G^{\mathcal{F}}$ by $e \mapsto 00$, $\cdot^{-1} \mapsto -$, and $* \mapsto +$ where the operations '+' and '−' are given their usual meaning in $\mathbb{Z}_2 \times \mathbb{Z}_2$, and we write ij as shorthand for (i, j). Then[1] $\nu : (\mathbb{Z}_2 \times \mathbb{Z}_2)^2 \to \mathrm{MO}_3$ is defined to be the unique pre-truth map satisfying

$$\nu(00, 01) = \nu(10, 11) = v_1, \quad \nu(00, 10) = \nu(01, 11) = v_2, \quad \nu(00, 11) = \nu(01, 10) = v_3.$$

Then let $[\![\cdot]\!]$ be the unique truth valuation w.r.t. $(\mathcal{L}_G, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathfrak{F}_G, \mathrm{MO}_3)$ given by lemma 4.6, so that by the same lemma the $\mathcal{L}_G$-structure $\mathfrak{M}_K := (\mathbb{Z}_2 \times \mathbb{Z}_2, \mathrm{MO}_3, [\![\cdot]\!], \mathfrak{F}_G)$ is easily seen to model $\mathcal{E}$. Since $\mathfrak{M}_K$ is built from the Klein four-group, G1-G3 holds in $\mathfrak{M}_K$ by lemma 4.7. It remains to check substitution for $\cdot^{-1}$ (G4) and $*$ (G5), which can be done straightforwardly by brute force computation. Hence $\mathfrak{M}_K \vDash \mathcal{A}_G$.

---

[1]For the definition of MO_3 see example 4.4.

Given our theorem for the inherent classicality of algebras with cancellative operations satisfying strong transitivity (theorem 4), E3$'$ cannot hold in the model $\mathfrak{M}_K$. To see that it fails, consider the extended $\mathcal{L}_G$-wff

$$(01 \approx 10) \to [(10 \approx 11) \to (01 \approx 11)].$$

We see that $[\![01 \approx 10]\!] = \nu(01, 10) = v_3$ while

$$[\![(10 \approx 11) \to (01 \approx 11)]\!] = \neg v_1 \vee (v_1 \wedge v_2) = \neg v_1,$$

and $v_3 \nleq \neg v_1$ in MO_3, so we have that $v_3 \to \neg v_1 \neq 1$ by lemma C.14, which shows that axiom E3$'$ does not hold in $\mathfrak{M}_K$. In order to find a non-standard model which satisfies E3$'$ we will need to consider an axiom system with no cancellative terms.

4.2.3 Non-standard Model of a Monoid satisfying Strong Transitivity

We define $\mathcal{L}_{Mon} := \{e, *\}$ (recall example 2.1) with $\alpha(e) := 0$ and $\alpha(*) := 2$, and let the monoid axioms $\mathcal{A}_{Mon}$ be $\mathcal{E}$ along with the following set of axioms (with $x, y, z \in \mathcal{B}_V$).

(M1) $(\forall x)(\forall y)(\forall z)\ [(x * y) * z \approx x * (y * z)]$

(M2) $(\forall x)(e * x \approx x)$ and $(\forall x)(x * e \approx x)$

(M3) $(\forall x)(\forall y)(\forall z)(x \approx y \to x * z \approx y * z) \wedge (\forall x)(\forall y)(\forall z)(x \approx y \to z * x \approx z * y)$

Note that these are just the group axioms $\mathcal{A}_G$ which do not contain the symbol '$\cdot^{-1}$'.

Example 4.15. We define an entire class of conservative models for $\mathcal{A}_{Mon}$. Let $n \in \{1, 2, \ldots\}$ and define $\bar{n} := \{1, 2, \ldots, n\}$. Then define the interpretation $\mathfrak{F}_M$ by $e \mapsto 1$, and with '$*$' interpreted by $i * j := \max(i, j)$ for $i, j \in \bar{n}$. Then let ν be defined by (for all $i, j \in \bar{n}$)

$$\nu(i,j) := \begin{cases} v_1 & \text{if } \{i,j\} = \{1,2\} \\ v_2 & \text{if } \{i,j\} = \{n, n-1\} \\ \delta_{ij} & \text{otherwise} \end{cases}$$

where δ_{ij} is the Kronecker delta. Clearly ν is a pre-truth map, which gives a unique truth valuation (for $n \geq 3$) $[\![\cdot]\!]$ by lemma 4.6. Then (for $n \geq 3$) define the $\mathcal{L}_{Mon}$-structure $\mathfrak{n} := (\bar{n}, \mathrm{MO}_2, [\![\cdot]\!], \mathfrak{F}_M)$.

It is straightforward to verify (using lemma 4.7) that $\mathfrak{n}$ is indeed a model for $\mathcal{A}_{Mon}$ whenever $n \geq 3$. Also, it is easy to see that strong transitivity is satisfied in this model if

$$\nu(a,b) \leq \neg\nu(a,c) \vee \big(\nu(a,c) \wedge \nu(b,c)\big) \tag{4.4}$$

holds in MO_2 for all $a, b, c \in \bar{n}$. The reader will find it straightforward to check that, for $n \geq 4$, the inequality in equation 4.4 is indeed satisfied, so that E3$'$ holds in $\mathfrak{n}$ for all $n \geq 4$.

This class of examples demonstrates that without cancellativity, one may find non-standard models which satisfy E3$'$, so that strong transitivity and substitution by themselves cannot be sufficient to guarantee that an M-system has no non-standard models.

4.2.4 Non-conservative Model of a Monoid

We define the language $\mathcal{L}'_{Mon} \coloneqq \{*\}$ with $\alpha(\cdot) \coloneqq 2$, and then define a set $\mathcal{A}'_{Mon}$ of alternate monoid axioms. Let $\mathcal{A}'_{Mon}$ be the following axiom (with $x, y \in \mathcal{B}_V$)

$$\text{(M2}'\text{)} \quad (\exists x)\Big((\forall y)[(x \cdot y \approx y) \wedge (y \cdot x \approx y)]\Big)$$

along with the axioms $\mathcal{E}$ as well as M1 and M3 from the previous section. In this presentation of the monoid axioms, we have incorporated the identity element by a 'there exists' statement, rather than treating it as a constant (0-ary) operation. In the presence of the schema CL, these two presentations of the monoid axioms have exactly the same models; however without this schema we will exhibit a model of $\mathcal{A}'_{Mon}$ that has an underlying set with a binary operation which can never be a model of $\mathcal{A}_{Mon}$, and as such, this model is non-conservative.

Example 4.16. Let $A = \{a, b, c\}$, define $\nu : A^2 \to \mathrm{MO}_2$ to be the unique pre-truth map satisfying $\nu(a, b) = v_1$, $\nu(a, c) = v_2$, and $\nu(b, c) = 0$, and define an interpretation of '$*$' on A by the following multiplication table.

$*$	a	b	c
a	a	a	a
b	a	b	a
c	a	a	c

Then consider the $\mathcal{L}'_{Mon}$-structure $\mathfrak{M}_M \coloneqq (A, \mathrm{MO}_2, [\![\cdot]\!], \mathfrak{F}'_M)$, where $[\![\cdot]\!]$ is the unique truth valuation given by lemma 4.6 from ν. This lemma gives that E1-E2 hold in $\mathfrak{M}_M$, and since

$(A, *)$ forms a semi-group, lemma 4.7 gives us immediately that M1 holds in $\mathfrak{M}_M$. That the remainder of $\mathcal{A}'_{Mon}$ does indeed hold in $\mathfrak{M}_M$ is straightforward, but mildly tedious, to check.

However, by inspection of the above multiplication table, one can see that there is no (classical) identity element, which immediately shows that this cannot be a conservative model.

The example above also provides a nice illustration of the odd behavior of formal statements involving '$\exists$' when the truth values are not the classical $\{0, 1\}$. (Recall the discussion of section 1.1.4.)

4.3 Quantum Orthomodular Lattices and Boolean Algebras

In this section, we look at treating axiomatizations of orthomodular lattices within the framework of quantum mathematics.[1] First, we prove that a commonly used axiom system for orthomodular lattices is inherently classical under the additional assumption of strong transitivity, using theorem 5. We then present a natural class of structures in which all of these orthomodular lattice axioms hold *except* for one of the substitution axioms, and we discuss two possible reductions for substitution. We then consider the theory of Boolean algebras within quantum mathematics — one may expect that any of the usual axiomatizations of these algebraic systems, so closely tied to classical logic, would only allow for standard models, even when

[1]Of course, this is a necessary first step in conducting a "second-order perturbation" of logic away from classical toward the quantum — we would eventually like to give a *quantum* treatment of the semantics for the first-order quantum logic presented in this work where, when constructing models in this new semantics, we would use the *quantum OMLs* we now begin to study as the underlying truth value algebra, instead of the *classical OMLs* which now underlie our semantics.

only requiring weak transitivity. We show that this is not the case by presenting a non-standard model of one such axiomatization of Boolean algebras.

4.3.1 Orthomodular Lattice Axioms with Strong Transitivity are Inherently Classical

As we have mentioned, this discussion will be a little 'meta' compared to the previous algebraic structures considered. We begin by discussing an M-system $(\mathcal{L}_{OL}, \mathcal{A}_{OML})$ associated with orthomodular lattices. We define the language[1] $\mathcal{L}_{OL} := \{0, \neg, \wedge\}$, where $\alpha(0) := 0$, $\alpha(\neg) := 1$, and $\alpha(\wedge) := 2$. Also, we denote the set of axioms OL1-OL8 and OM below along with $\mathcal{E}$ by $\mathcal{A}_{OML}$ (where $x, y, z \in \mathcal{B}_V$, and $1 := \neg 0$ as well as $t \vee u := \neg(\neg t \wedge \neg u)$, $t \to u := \neg t \vee (t \wedge u)$, for any terms t, u)[2]

(OL1) $(\forall x)(\forall y)\big[(x \wedge y) \approx (y \wedge x)\big]$

(OL2) $(\forall x)(\forall y)(\forall z)\big([(x \wedge y) \wedge z] \approx [x \wedge (y \wedge z)]\big)$

(OL3) $(\forall x)(\forall y)\big(x \approx [x \wedge (x \vee y)]\big)$

(OL4) $(\forall x)[x \approx \neg(\neg x)]$

(OL5) $(\forall x)\Big(\big[(x \wedge 0) \approx 0\big] \wedge \big[(x \vee 0) \approx x\big]\Big)$

---

[1]We have now triple-loaded the symbols '$\wedge$' and '$\neg$' with meaning — they refer to three *distinct* things, (1) logical connectives which combine wffs, (2) operations in the truth value algebra for any given model, and now (3) operation symbols in our language $\mathcal{L}_{OL}$. We trust that this will not cause any confusion since context will always distinguish the desired meaning — the axiom OL5 is about as confusing as this triple meaning will get.

[2]While we will use the defined notions of '$\vee$' and '1' as function symbols, we will reserve '$\to$' and '$\leftrightarrow$' to only be used as logical symbols and operations in the truth value algebra — they will not be used as defined function symbols.

(OL6) $(\forall x)[(x \vee \neg x) \approx 1]$

(OL7) $(\forall x)(\forall y)(x \approx y \to \neg x \approx \neg y)$

(OL8) $(\forall x)(\forall y)(\forall z)[x \approx y \to (x \wedge z) \approx (y \wedge z)]$

(OM) $(\forall x)(\forall y)\big([x \wedge y] \approx [x \wedge (\neg x \vee (x \wedge y))]\big)$

Note that OL7 above is just substitution for $\neg$, that OL8 is just substitution for $\wedge$ (in the presence of OL1), and that removing OM from $\mathcal{A}_{OML}$ yields an axiomatization of ortholattices.[1] Since the natural $\mathcal{L}_{OL}$-structures will not actually model *all* the above axioms (in particular, OL8 fails), we will define $\mathcal{A}^{-}_{OML}$ to be $\mathcal{E}$ along with OM and OL1-OL7. Before presenting these natural structures, we will prove that $\mathcal{A}_{OML}$ along with strong transitivity is inherently classical. We will need a couple lemmas to establish this fact.

Lemma 4.17. Let u be any $\mathcal{L}_{OL}$-term. Then $\mathcal{A}_{OML} \vdash u \approx (u \wedge u)$.

Proof. By Q6 and R5 we have $\big[(u \wedge 0) \approx 0\big] \wedge \big[(u \vee 0) \approx u\big]$, so Q3 and R5 yield $(u \vee 0) \approx u$. The OL8 (using Q6 and R5) gives

$$\big[(u \vee 0) \approx u\big] \to \big[u \wedge (u \vee 0) \approx u \wedge u\big],$$

so that R5 then gives $u \wedge (u \vee 0) \approx u \wedge u$. Of course, OL3 gives (using Q6 and R5) that $u \wedge (u \vee 0) \approx u$. Then, using lemma 4.3 (with $A = \top$) yields that $\top \to u \approx (u \wedge u)$, and so (2) in prop. 2.4 yields the desired result. □

[1]The commutativity of '$\wedge$' (OL1) saves us from needing to have substitution for both slots as an axiom.

Lemma 4.18. Let $x, y \in \mathcal{B}_V$, and let $t(x,y) \coloneqq [\neg x \vee (x \wedge y)] \wedge [\neg y \vee (x \wedge y)]$. Then $\mathcal{A}_{OML} \vdash (x \approx y) \leftrightarrow (1 \approx t(x,y))$.

Proof. By (5) in proposition 2.4, it suffices to prove both $(x \approx y) \to [1 \approx t(x,y)]$ as well as $[1 \approx t(x,y)] \to (x \approx y)$. First we will prove $(x \approx y) \to (1 \approx t(x,y))$: repeated uses of OL7-OL8 (using R1) give $(x \approx y) \to [t(x,x) \approx t(x,y)]$. Also, lemma 4.17 gives that $[\neg x \vee (x \wedge x)] \approx t(x,x)$. That same lemma also gives that $x \approx (x \wedge x)$, so that OL7-OL8 along with OL1 yield $(\neg x \vee x) \approx \neg x \vee (x \wedge x)$ (with multiple uses of R5). Now OL6 gives $x \vee \neg x \approx 1$, so that lemma 4.3 gives the desired $(x \approx y) \to (1 \approx t(x,y))$.

We will now prove $[1 \approx t(x,y)] \to (x \approx y)$: OL8 yields (using Q6 and R5)

$$[1 \approx t(x,y)] \to [1 \wedge x \approx t(x,y) \wedge x].$$

Now, one can proceed to use OL1,OL2, and OM, along with OL5 and OL2 and finally OL3 to construct a chain which allows use of lemma 4.3 to establish that $[1 \approx t(x,y)] \to [x \approx (x \wedge y)]$. Similarly, using OL8 with 'y' and following the same procedure yields $[1 \approx t(x,y)] \to [y \approx (x \wedge y)]$, which gives using the aforementioned lemma, the desired result. □

This lemma then yields the desired proposition.

Proposition 4.19. The set $\mathcal{A}_{OML} \cup \{\mathrm{E3}'\}$ is inherently classical.

Proof. $\mathcal{A} \vdash (x \approx y) \leftrightarrow (1 \approx t(x,y))$ by lemma 4.18. Then take $u \coloneqq 1$ and $t(x,y) \coloneqq x \leftrightarrow y$ in theorem 5. □

4.3.2 A Natural Class of Non-standard Orthomodular Lattices

Example 4.20. We now describe a class of models for the M-system $(\mathcal{L}_{OL}, \mathcal{A}^{-}_{OML})$.[1] Let L be a complete orthomodular lattice (with operations $0, \neg, \wedge$), and define $\nu : L^2 \to L$ to be

$$\nu(a,b) := (a \wedge b) \vee (\neg a \wedge \neg b) = a \leftrightarrow b,$$

where we use the symbol '$\leftrightarrow$' from the definition C.8.[2] ν is manifestly a pre-truth map. Also, ran $\nu = L$ since $\nu(a,1) = a$ for any $a \in L$ by lemma C.14. Let $[\![\cdot]\!]$ be the unique truth valuation extending ν by lemma 4.6. Then define the $\mathcal{L}_{OL}$-structure $\mathfrak{M}_L := (L, L, [\![\cdot]\!], \mathfrak{F})$ (with $\mathfrak{F}$ the interpretation sending each symbol to the operation of the same symbol[3]). By lemma C.14 we know that $a \leftrightarrow b$ (for any a, b elements of an OML) satisfies the requisite condition in lemma 4.6, so that $\mathfrak{M}_L \vDash \mathcal{E}$. By lemma 4.7, $\mathfrak{M}_L$ satisfies OL1-OL6, and OM. That $\mathfrak{M} \vDash$ OL7 follows directly from lemma C.14.

However, this class of models for $(\mathcal{L}_{OL}, \mathcal{A}^{-}_{OML})$ does not, in general, satisfy either strong transitivity or substitution for '$\wedge$' (OL8). To see that this is so, consider the model $\mathfrak{M}_L$ using

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[1]As mentioned previously, these structures do not generically model OL8, and hence they cannot model the full OML axioms $\mathcal{A}_{OML}$.

[2]Lemma C.14 demonstrates the above equality.

[3]This is possible due to our aforementioned "triple-loading" of the symbols '$\wedge$', and '$\neg$' with meaning, so that these function symbols in $\mathcal{L}^{\mathcal{F}}$ are interpreted as the meet and join in L.

the orthomodular lattice $L = \mathrm{MO}_2 \times B_2$ (see example 4.5 from earlier). First, examining strong transitivity, we easily compute that $\nu(a,c) = (a \leftrightarrow c) = b$, $\nu(c,e) = d$ and $\nu(a,e) = c$, so that

$$[\![a \approx c]\!] = \nu(a,c) = b \nleq \neg d = \big(\nu(c,e) \to \nu(a,e)\big) = [\![c \approx e \to e \approx a]\!],$$

which shows that axiom E3′ is not satisfied (using lemma C.14) in $\mathfrak{M}_L$. Using the same model we see that

$$[\![\neg a \approx \neg c]\!] = (\neg a \leftrightarrow \neg c) = b \nleq \neg e = (0 \leftrightarrow e) = \big([\![\neg a \wedge e]\!] \leftrightarrow [\![\neg c \wedge e]\!]\big) = [\![\neg a \wedge e \approx \neg c \wedge e]\!],$$

showing that OL8 does not generically hold in $\mathfrak{M}_L$.

As per our discussion in section 4.1.1, we could replace OL8 with the reduced substitution RSub[$\wedge$]. Now, if we restrict ourselves to OMLs L that are irreducible and satisfy the relative center property, it is easy (using lemma 3.11 and proposition 3.9) to see that this reduced substitution holds in any such $\mathfrak{M}_L$ (as defined in example 4.20). Moreover, these examples satisfy the axiom schema $\mathbf{T}(\psi) \to \psi$,[1] so that by proposition 4.2 these examples model the set of axioms $\mathcal{A}^-_{OML} \cup \{\mathrm{RSub}[\wedge]\}$ which is classically equivalent to the usual axioms $\mathcal{A}_{OML}$.

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[1]The wff $\mathbf{T}(\psi) \to \psi$ is a tautology in any equational M-system by E1 and lemma 3.10.

There is the potential to do somewhat better, and obtain a reduction of $\mathcal{A}_{OML}$ which holds in a broader class of these natural models. It is relatively straightforward to see that if $a, b, c \in L$ all commute, then

$$[\![a \approx b]\!] \leq [\![a \wedge c \approx b \wedge c]\!],$$

in these models, so that OL8 holds in the context of commuting elements. Unfortunately, we have not been able to use this fact to find a reduction of OL8 which holds in all our natural models defined above.

The fact that unreduced substitution for the operation $\wedge$ (OL8) does not generically hold for the above natural and intuitive models of the axioms $\mathcal{A}^{-}_{OML}$ suggests that the usual statement of substitution in classical mathematics is not the most "suitable" reduction for use in quantum mathematics. That is, the usual statement of substitution for $\wedge$ — as intuitive as it is — is a classical statement of that property, and not what a hypothetical subatomic philosopher of mathematics would consider the "appropriate" notion of substitution.

4.3.3 Non-standard Model of the usual Boolean Algebra Axioms

In what follows, we discuss an M-system $(\mathcal{L}_{BA}, \mathcal{A}_{BA})$ associated with Boolean algebras. Now, the usual axiomatization of Boolean algebras use the same language as OMLs, namely $\mathcal{L}_{OL}$.[1] We let $\mathcal{A}_{BA}$ denote the set of axioms $\mathcal{A}_{OML}$ along with BA below[2]

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[1] One can also axiomatize Boolean algebras as a special class of *rings* — see (21) for details.

[2] The axioms so defined are not classically independent, for example OM can be derived from the other axioms contained in $\mathcal{A}_{BA}$. Also, note that here we define Boolean algebras as being OMLs with all their elements commuting, rather than by distributivity as in section 1.1.3.

(BA) $(\forall x)(\forall y)[x \approx (x \wedge y) \vee (x \wedge \neg y)]$.

Since $\mathcal{A}_{OML} \subseteq \mathcal{A}_{BA}$, by proposition 4.19 we cannot enforce strong transitivity if we expect to find a non-standard model of the Boolean algebra axioms. However, if we do not enforce strong transitivity and only consider $\mathcal{A}_{BA}$, then there do exist non-standard models, as the following example shows.

Example 4.21. Let $B := \{1, 0, a, \neg a\}$ be the free Boolean algebra on one generator (i.e. the Boolean diamond), and recall $\mathrm{MO}_2 := \{1, 0, v_1, v_2, \neg v_1, \neg v_2\}$. Let $\mathfrak{F}$ be the natural interpretation of the function symbols on B, and define ν to be the unique pre-truth map on B satisfying $\nu(i, \neg i) = 0$ for all $i \in B$ as well as $\nu(a, 0) = \nu(\neg a, 1) = v_1$ and $\nu(a, 1) = \nu(\neg a, 0) = v_2$. Then let $[\![\cdot]\!]$ be the truth valuation given by lemma 4.6, so that the structure $\mathfrak{M}_B := (B, \mathrm{MO}_2, [\![\cdot]\!], \mathfrak{F})$ is a model of $\mathcal{A}_{BA}$ by lemma[1] 4.6 and lemma 4.7, as well as a straightforward confirmation of OL7 and OL8.

From this we see that the axioms $\mathcal{A}_{BA}$, though closely tied with classical logic, still admit non-standard models in our first-order quantum logic.

4.4 Quantum Linear Algebra

It is fitting that the final piece of mathematics we develop with quantum logic is the very area of mathematics responsible for its inception (5), namely linear algebra. Of course, since linear algebra is the branch of mathematics used in formulating the quantum theory, there is reason to hope that applying quantum logic to mathematics may be uniquely powerful in this

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[1] That the hypotheses of these lemmas are satisfied by ν is easy to verify.

context. In this section, we examine both linear algebra as well as the algebra of operators on Hilbert spaces. In order to keep the discussion simple, we will not attempt a quantum mathematical treatment of the topological aspects of these spaces, but rather focus on their algebraic structure.

For both vector spaces and operator algebras, we will see that a natural class of conservative models present themselves. Moreover, the vector space models are related to those of operator algebras in a very natural way. For the case of vector spaces, these natural models satisfy substitution for all their operations. This is not the case, however, for the operation of multiplication in the operator algebras. Furthermore, we show that the operator algebra models yield precisely the OML models discussed above (see example 4.20) when they are restricted to the lattice of projection operators. Finally, we discuss how the Schrödinger and von Neumann equations may be modified within the framework of quantum linear algebra to allow the possibility of evolutions beyond what is allowed in standard quantum mechanics formulated on classical linear algebra.

4.4.1 A Natural Class of Non-standard Models for the Vector Space Axioms

Since complex Hilbert spaces are those relevant for quantum mechanics, we restrict our attention to vector spaces over the complex numbers $\mathbb{C}$. We define the language

$$\mathcal{L}_{VS}^{\mathcal{F}} := \{|0\rangle, -, +\} \cup \{\lambda \;:\; \lambda \in \mathbb{C}\},$$

with $\alpha(|0\rangle) = 0$, $\alpha(-) = 1$, $\alpha(+) = 2$, and $\alpha(\lambda) = 1$ for all $\lambda \in \mathbb{C}$. For any term t, we denote $\lambda(t) := \lambda t$ for any $\lambda \in \mathbb{C}$. We then define the following vector space axioms (for $x, y, z \in \mathcal{B}_V$)

(VS1) $(\forall x)(\forall y)(\forall z)[x + (y + z) \approx (x + y) + z]$

(VS2) $(\forall x)(x + |0\rangle \approx x)$

(VS3) $(\forall x)[x + (-x) \approx |0\rangle]$

(VS4) $(\forall x)(\forall y)(x + y \approx y + x)$

(VS5) $(\forall x)[(\lambda + \mu)x \approx \lambda x + \mu y$ for each $\lambda, \mu \in \mathbb{C}$

(VS6) $(\forall x)(\forall y)[\lambda(x + y) \approx \lambda x + \lambda y]$ for each $\lambda \in \mathbb{C}$

(VS7) $(\forall x)(1x \approx x)$

(VS8) $(\forall x)[(\lambda\mu)x \approx \lambda(\mu x)]$ for each $\lambda, \mu \in \mathbb{C}$

(VS9) $(\forall x)(\forall y)(\forall z)[x \approx y \to (x + z \approx y + z)] \wedge (\forall x)(\forall y)(\forall z)[x \approx y \to (z + x \approx z + y)]$

(VS10) $(\forall x)(\forall y)(x \approx y \to \lambda x \approx \lambda y)$ for each $\lambda \in \mathbb{C}$.

Note that VS9 and VS10 are just substitution for addition and scalar multiplication, respectively. We then let $\mathcal{A}_{VS}$ refer to the set[1] of axioms VS1-VS10 along with $\mathcal{E}$.

For any Hilbert space $\mathcal{H}$ we will form a conservative model of the above axioms naturally utilizing the linear structure of the space. We will retain the notational conventions established in section 1.2.2, and also, for any $|\psi\rangle \in \mathcal{H}$, we define $|\psi\rangle^{\perp} := \{|\psi\rangle\}^{\perp}$.

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[1]Unlike our other axiom systems considered so far, this set is infinite since (some of) the axioms consist of one instance for each $\lambda, \mu \in \mathbb{C}$.

Example 4.22. Let $\mathcal{H}$ be a Hilbert space, and let $L_{\mathcal{H}}$ be the lattice[1] of closed linear subspaces of $\mathcal{H}$ (with operations $0, \cdot^{\perp}, \cap$), and define the interpretation $\mathfrak{F}_{\mathcal{H}}$ of the function symbols in the usual way. We define $\nu : \mathcal{H}^2 \to L_{\mathcal{H}}$ by $\nu(|\psi\rangle, |\phi\rangle) \coloneqq (|\psi\rangle - |\phi\rangle)^{\perp}$, so that ν is manifestly a pre-truth map. We wish to show that

$$\big(|\psi\rangle - |\phi\rangle\big)^{\perp} \cap \big(|\phi\rangle - |\chi\rangle\big)^{\perp} \subseteq \big(|\psi\rangle - |\chi\rangle\big)^{\perp}. \tag{4.5}$$

To see that this containment is satisfied, suppose that $|\eta\rangle \in \big(|\psi\rangle - |\phi\rangle\big)^{\perp} \cap [|\phi\rangle - |\chi\rangle]^{\perp}$; this gives that $\langle\eta|\psi\rangle - \langle\eta|\phi\rangle = 0$ and $\langle\eta|\phi\rangle - \langle\eta|\chi\rangle = 0$, from which it follows that $\langle\eta|\psi\rangle = \langle\eta|\chi\rangle$. However, this shows that $|\eta\rangle \in \big(|\psi\rangle - |\chi\rangle\big)^{\perp}$, and since $|\eta\rangle$ was generic, equation 4.5 holds. Then, we let $\mathfrak{M}_{\mathcal{H}} \coloneqq (\mathcal{H}, L_{\mathcal{H}}, [\![\cdot]\!], \mathfrak{F}_{\mathcal{H}})$ be the $\mathcal{L}_{VS}$-structure existing by virtue of lemma 4.6 which also gives that (since we have established equation equation 4.5) $\mathfrak{M}_{\mathcal{H}} \vDash \mathcal{E}$. Lemma 4.7 then gives that VS1-VS8 hold in $\mathfrak{M}_{\mathcal{H}}$.

Furthermore, it is easy to see that

$$[\![|\psi\rangle \approx |\phi\rangle]\!] = \ker(|\psi\rangle - |\phi\rangle)^{\perp} = \ker(\lambda|\psi\rangle - \lambda|\phi\rangle)^{\perp} = [\![\lambda|\psi\rangle \approx \lambda|\phi\rangle]\!] \quad \text{for all } \lambda \in \mathbb{C},$$

as well as that

$$[\![|\psi\rangle \approx |\phi\rangle]\!] = [\![|\psi\rangle + |\chi\rangle \approx |\phi\rangle + |\chi\rangle]\!] \quad \text{for all } |\chi\rangle \in \mathcal{H}.$$

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[1] This lattice is complete and orthomodular in general, and is actually a modular ortholattice in the finite dimensional case — see section 1.2.2.

So that not only do VS9 and VS10 hold in $\mathfrak{M}_{\mathcal{H}}$, but also, letting $t(x, y) \coloneqq x + y$, we see that $\mathfrak{M}_{\mathcal{H}}$ satisfies LC[t] and RC[t]. Hence, since this model $\mathfrak{M}_{\mathcal{H}}$ is clearly non-standard, we cannot have strong transitivity holding in this model by theorem 4.

Now that we have a class of non-standard models for the mathematics underlying quantum mechanics, it's natural to examine the Schrödinger equation (using the usual classical notion of a derivative) in this light. We write the Schrödinger equation as (where H is the Hamiltonian operator, which we take to be bounded for simplicity and we set $\hbar = 1$)

$$i\frac{d}{dt}|\psi\rangle \approx H|\psi\rangle. \tag{4.6}$$

Now, the most straightforward way to utilize the Schödinger equation within the realm of quantum mathematics is simple to take the wff in equation 4.6 as an axiom. However, if we take this approach, it is easy to see that the usual unitary quantum dynamics is completely retained, and no new dynamics is allowed. This is because for any evolving state $|\psi(t)\rangle$, equation 4.6 holds iff

$$[\![i\frac{d}{dt}|\psi(t)\rangle \approx H|\psi(t)\rangle]\!] = \mathcal{H} = |0\rangle^{\perp},$$

since $[\![\cdot]\!]$ is a truth valuation and $\mathcal{H}$ is the top element of $L_{\mathcal{H}}$, which is to say that we must have

$$i\frac{d}{dt}|\psi(t)\rangle - H|\psi(t)\rangle = |0\rangle,$$

which means that the Schrödinger equation must be classically satisfied. Hence, attempting to do quantum mechanics on these non-standard models of the vector space axioms leads to the exact same dynamical evolution that we know and love from the use of classical linear algebra!

However, we have seen that sometimes the usual formulation of any given axiom in classical mathematics (for example, substitution) is not necessarily the most "suitable" form of that axiom in the quantum mathematical context, and perhaps some reduction of that axiom is more "appropriate". The most wild hope would be that one could find a reduction of the Schrödinger equation (or, more generally, the von Neumann equation) which would, under quantum mathematics, yield not only the unitary evolutions, *but also the measurement evolutions* as solutions to that reduced equation. We will next examine operator algebras so that we can begin such an attempt at reducing the von Neumann equation.

4.4.2 A Natural Class of Non-standard Models of Operator Algebra Axioms

There are a variety of operator algebra languages and axioms (for example, those of B^*-algebras, C^*-algebras, von Neumann algebras, etc.) — as in the previous section, we are only interested in algebraic properties, and so we will use only algebraic axioms and ignore any topological considerations. The (equational) language we utilize is just $\mathcal{L}_{OA} := \mathcal{L}_{VS} \cup \{*, \cdot^\dagger\}$ where $\{*, \cdot^\dagger\}$ are function symbols with $\alpha(*) = 2$ (representing the operator product) and $\alpha(\cdot^\dagger) = 1$ (representing the adjoint). We then define the following axioms (with $x, y, z \in \mathcal{B}_V$, and where $\bar{\lambda}$ denotes the complex conjugate of λ for any $\lambda \in \mathbb{C}$)

(OA1) $(\forall x)(\forall y)(\forall z)[x * (y * z) \approx (x * y) * z]$

(OA2) $(\forall x)(\forall y)(\forall z)\big([x * (y + z) \approx (x * y) + (x * z)] \wedge [(x + y) * z \approx (x * z) + (y * z)]\big)$

(OA3) $(\forall x)(\forall y)\big([\lambda(x * y) \approx (\lambda x) * y] \wedge [\lambda(x * y) \approx x * (\lambda y)]\big)$ for each $\lambda \in \mathbb{C}$

(OA4) $(\forall x)[(x^\dagger)^\dagger \approx x]$

(OA5) $(\forall x)[(\lambda x)^\dagger \approx \bar{\lambda} x^\dagger$ for each $\lambda \in \mathbb{C}$

(OA6) $(\forall x)(\forall y)\big([(x + y)^\dagger \approx x^\dagger + y^\dagger] \wedge [(x \cdot y)^\dagger \approx y^\dagger \cdot x^\dagger]$

(OA7L) $(\forall x)(\forall y)(\forall z)\big[(x \approx y) \to (x * z \approx y * z)\big]$

(OA7R) $(\forall x)(\forall y)(\forall z)\big[(x \approx y) \to (z * x \approx z * y)\big]$

Where we have broken up substitution for the multiplication '$*$' into a 'left' and 'right' substitution (OA7L and OA7R respectively) since only OA7L will be satisfied by our natural models constructed shortly hereafter. Let $\mathcal{A}_{OA}^-$ consist of VS1-VS10 along with the above axioms OA1-OA6, OA7L and $\mathcal{E}$. The full operator algebra axioms are then $\mathcal{A}_{OA} \coloneqq \mathcal{A}_{OA}^- \cup \{\text{OA7R}\}$.

We use the bounded linear operators on any Hilbert space to construct a model for $\mathcal{A}_{OA}$. As in the previous section, let $L_{\mathcal{H}}$ be the lattice of closed linear subspaces of any Hilbert space $\mathcal{H}$, and let $\mathcal{B}(\mathcal{H})$ denote the set of bounded linear operators on $\mathcal{H}$.

As before, we search for a natural choice for a pre-truth map — two obvious candidate maps $\sigma, \tau : \mathcal{B}(\mathcal{H})^2 \to L_{\mathcal{H}}$ present themselves:

1. $\sigma(A, B) \coloneqq \ker(A - B)$
2. $\tau(A, B) \coloneqq \ker(A^\dagger - B^\dagger)$,

While σ may seem to be the more natural choice at first, it will be enlightening at this point to connect back to the models discussed in the previous section on vector spaces.

In classical mathematics, for $A, B \in \mathcal{B}(\mathcal{H})$, the equation $A \approx B$ simply means $A|\psi\rangle \approx B|\psi\rangle$ (with '$\approx$' equality in the *vector space*) for all $|\psi\rangle \in \mathcal{H}$. Now for a Hilbert space $\mathcal{H}$, consider the non-standard model $\mathfrak{M}_{\mathcal{H}}$ defined in the previous section. Then

$$\begin{aligned}
[\![(\forall|\psi\rangle)(A|\psi\rangle \approx B|\psi\rangle)]\!] &= \bigwedge_{|\psi\rangle\in\mathcal{H}} \nu(A|\psi\rangle, B|\psi\rangle) = \bigcap_{|\psi\rangle\in\mathcal{H}} ((A-B)|\psi\rangle)^{\perp} \\
&= \{|\phi\rangle \in \mathcal{H} \mid \langle\phi|(A-B)|\psi\rangle = 0 \text{ for all } |\psi\rangle \in \mathcal{H}\ \} \\
&= \{|\phi\rangle \in \mathcal{H} \mid (A-B)^{\dagger}|\phi\rangle = 0\ \} = \ker(A^{\dagger} - B^{\dagger}) \\
&= \tau(A, B).
\end{aligned} \tag{4.7}$$

Given the naturalness of τ with regards to our previous vector space models, this will be the choice of pre-truth map we use to construct our non-standard operator algebra model. Of course, σ would also give a natural $\mathcal{L}_{OA}$-structure with properties similar to the one we now construct.

Example 4.23. Let $\mathcal{H}$ be a Hilbert space, $L_{\mathcal{H}}$ the closed lattice of subspaces of $\mathcal{H}$, and $\mathcal{B}(\mathcal{H})$ the bounded linear operators on $\mathcal{H}$. Let $\mathfrak{F}_{\mathcal{B}(\mathcal{H})}$ be the usual interpretation of the function symbols in $\mathcal{L}_{OA}$ as operations in $\mathcal{B}(\mathcal{H})$ (where we interpret '$*$' as operator multiplication, which we represent by simple concatenation, i.e. $A * B$ is interpreted as AB), and let τ be the pre-truth map given by (for $A, B \in \mathcal{B}(\mathcal{H})$)

$$\tau(A, B) := \ker(A^{\dagger} - B^{\dagger})$$

as defined above. It is easy to see that ran $\tau = L_{\mathcal{H}}$. We also have (for any $A, B, C \in \mathcal{B}(\mathcal{H})$)

$$\begin{aligned}
\tau(A,B) \wedge \tau(B,C) &= \ker(A^\dagger - B^\dagger) \cap \ker(B^\dagger - C^\dagger) \\
&= \{|\psi\rangle \in \mathcal{H} \ : \ A^\dagger|\psi\rangle = B^\dagger|\psi\rangle \text{ and } B^\dagger|\psi\rangle = C^\dagger|\psi\rangle\} \\
&\subseteq \{|\psi\rangle \in \mathcal{H} \ : \ A^\dagger|\psi\rangle = C^\dagger|\psi\rangle\} \\
&= \ker(A^\dagger - C^\dagger) = \tau(A,C).
\end{aligned}$$

Using this pre-truth map, we define $\mathfrak{B}_{\mathcal{H}} := (\mathcal{B}(\mathcal{H}), L_{\mathcal{H}}, [\![\cdot]\!], \mathfrak{F}_{\mathcal{B}_{\mathcal{H}}})$ to be the $\mathcal{L}_{OA}$ structure given by lemma 4.6, so that $\mathfrak{B}_{\mathcal{H}} \vDash \mathcal{E}$. Lemma 4.7 then gives that VS1-VS8 and OA1-OA6 are satisfied in $\mathfrak{B}_{\mathcal{H}}$. Similar considerations to example 4.22 show that VS9 and VS10 are satisfied. Considering OA7L, we have that (for $A, B, C \in \mathcal{B}(\mathcal{H})$)

$$[\![A \approx B]\!] = \ker(A^\dagger - B^\dagger) \subseteq \ker\left(C^\dagger(A^\dagger - B^\dagger)\right) = \ker\left((AC)^\dagger - (BC)^\dagger\right) = [\![AC \approx BC]\!],$$

so that OA7L holds in $\mathfrak{M}_{\mathcal{H}}$. To see that OA7R generically fails, consider the Pauli matrices σ_1, σ_2 and σ_3 as well as the zero operator 0. We easily see that

$$[\![\sigma_2 \approx 0]\!] = \ker(\sigma_2) \nsubseteq \ker\left(\sigma_1\right) = \ker\left(\sigma_2\sigma_3\right) = [\![\sigma_2\sigma_3 \approx 0\sigma_3]\!].$$

Here we are in a better position to reduce OA7R, since the reduction RSub$[*, 2]$ discussed in section 4.1.1 will do the trick. This can be easily seen from lemma 3.11 and propositions 4.2 and 3.9, since the projection lattice of a Hilbert space is irreducible and satisfies the relative center property (lemma C.21 and proposition C.24). From this we see that $\mathfrak{B}_{\mathcal{H}}$ is a model of a natural reduction of the axioms $\mathcal{A}_{OA}$, namely $\mathcal{A}^{-}_{OA} \cup \{\mathrm{RSub}[*, 2]\}$.

Now, as discussed in section 1.2.2, the lattice of closed subspaces of $\mathcal{H}$ can naturally be identified with the lattice of projection operators, and so we can use $\mathfrak{B}_{\mathcal{H}}$ to construct a $\mathcal{L}_{OL}$ structure in the following natural way.

Example 4.24. Let $\mathfrak{F}$ be the natural interpretation ($\neg \mapsto \cdot^{\perp}$, $\wedge \mapsto \cap$), and define the truth valuation $[\![\cdot]\!]'$ by (for any $V, W \subseteq \mathcal{H}$ closed linear subspaces)

$$[\![V \approx W]\!]' := \tau(P_V, P_W),$$

where P_V, P_W are the projection operators onto V, W respectively, and extend to the remaining wffs by proposition 3.12. Then define the $\mathcal{L}_{OL}$-structure $\mathfrak{L}_{\mathcal{H}} := (L_{\mathcal{H}}, L_{\mathcal{H}}, [\![\cdot]\!]', \mathfrak{F})$.

We will now see that the structure $\mathfrak{L}_{\mathcal{H}}$ is precisely the structure $\mathfrak{M}_{L_{\mathcal{H}}}$ of example 4.20.[1] Clearly, the only thing we need to check is that the truth valuations of the two models are the same. Now, for $V, W \in L_{\mathcal{H}}$ with associated (Hermitian) projectors P_V and P_W, we have $[\![V \approx W]\!]' = \ker(P_V - P_W)$. Now $|\psi\rangle \in \ker(P_V - P_W)$ iff $P_V|\psi\rangle = P_W|\psi\rangle$. First, assume

$$|\psi\rangle \in [\![V \approx W]\!] = (V \cap W) \vee (V^{\perp} \cap W^{\perp}),$$

[1]Recall that $\mathfrak{M}_{L_{\mathcal{H}}}$ has truth valuation satisfying $[\![V \approx W]\!] = (V \cap W) \vee (V^{\perp} \cap W^{\perp})$ where '$P \vee Q$' is the closed linear span of P and Q.

so that $|\psi\rangle = |\phi\rangle + |\chi\rangle$ with $|\psi\rangle \in (V \cap W)$ and $|\chi\rangle \in (V^\perp \cap W^\perp)$. Then

$$P_V|\psi\rangle = P_V|\phi\rangle + P_V|\chi\rangle = |\phi\rangle = P_W|\phi\rangle + P_W|\chi\rangle = P_W|\psi\rangle, \tag{4.8}$$

and so we have $[\![V \approx W]\!] \subseteq [\![V \approx W]\!]'$. Now consider $|\psi\rangle \in [\![V \approx W]\!]'$, so that $P_V|\psi\rangle = P_W|\psi\rangle \in V \cap W$. Then, for I the identity operator in $\mathcal{B}(\mathcal{H})$, since $P_{V^\perp} = I - P_V$ and $P_{W^\perp} = I - P_W$, we have $P_{V^\perp}|\psi\rangle = P_{W^\perp}|\psi\rangle \in (V^\perp \cap W^\perp)$ and

$$|\psi\rangle = P_V|\psi\rangle + P_{V^\perp}|\psi\rangle \in (V \cap W) \vee (V^\perp \cap W^\perp) = [\![V \approx W]\!], \tag{4.9}$$

so $[\![V \approx W]\!]' \subseteq [\![V \approx W]\!]$. Hence, we have both inclusions, which gives that $[\![V \approx W]\!] = [\![V \approx W]\!]'$ for all $V, W \in L_{\mathcal{H}}$, showing that the model $\mathfrak{L}_{\mathcal{H}}$ just described is indeed the structure $\mathfrak{M}_{L_{\mathcal{H}}}$. Of course, in this case, $\mathfrak{M}_{L_{\mathcal{H}}}$ is a model of the natural reduction of the OML axioms, namely $\mathcal{A}^-_{OML} \cup \{\mathrm{RSub}[\wedge]\}$.

We cannot help but believe that, for any given Hilbert space $\mathcal{H}$, the natural interplay between the model of the (reduced) OML axioms $\mathfrak{M}_{L_{\mathcal{H}}}$, the model of the vector space axioms $\mathfrak{M}_{\mathcal{H}}$, and the model $\mathfrak{B}_{\mathcal{H}}$ of the (reduced) operator algebra axioms hints at a profound significance of these models with respect to quantum theory.

4.4.3 The von Neumann Equation in Quantum Mathematics

One can express quantum states as time-evolving density operators, and the analog of the Schrödinger equation is, of course, the von Neumann equation $\frac{d}{dt}\rho = -i[H, \rho]$. A similar argument to the one given above for the Schrödinger equation tells us that if we take (vN) $\frac{d}{dt}\rho \approx -i[H,\rho]$

as an axiom of quantum mechanics, we again have only the standard unitary dynamics of quantum mechanics.

Now, however, we will go a little farther than we did in analyzing the Schrödinger equation and attempt a simple reduction of the von Neumann equation. First, define

$$B\rho := \frac{d}{dt}\rho + i[H,\rho],$$

where we note that since our operator algebra models $\mathfrak{B}_{\mathcal{H}}$ satisfy substitution for '+', the $\mathcal{L}_{OA}$-wffs $B\rho \approx 0$ and $\frac{d}{dt}\rho \approx -i[H,\rho]$ have the exact same truth value in these models. It seems fairly natural to attempt a reduction utilizing the commutator (as discussed in section 3.2.1). Since the only other operator in sight (other than B and ρ and the fixed H) is 0, a natural reduction to consider is

(vN′) $c\,(B\rho \approx 0, \rho \approx 0) \to B\rho.$

Since (using that $\rho^\dagger = \rho$ and $(B\rho)^\dagger = B\rho)$) we have (using the truth valuation $[\![\cdot]\!]$ in our natural model $\mathfrak{B}_{\mathcal{H}}$) $[\![\rho \approx 0]\!] = \ker(\rho)$ and $[\![B\rho \approx 0]\!] = \ker(B\rho)$, there is no reason to suspect *a priori* that we have only standard unitary dynamics for all states ρ. Surprisingly, as the following proposition show, (vN′) allows for exactly the same dynamical evolutions as (vN) in our natural models $\mathfrak{B}_{\mathcal{H}}$.

Before establishing the proposition, we prove the following useful lemmas. We define $\hat{0} := \{|0\rangle\}$, so that $\hat{0}$ is the subspace consisting of only the zero vector. Also recall that '$\vee$' is interpreted as the closure of the linear span in $L_{\mathcal{H}}$.

Lemma 4.25. In the model $\mathfrak{B}_{\mathcal{H}}$, for any two operators $\rho, A \in \mathcal{B}(\mathcal{H})$, the following two statements are equivalent

1. $[\![c(\rho \approx 0, A \approx 0)]\!] \leq [\![A \approx 0]\!]$
2. $\ker(\rho) \cap \ker(A)^{\perp} = \hat{0} = \ker(\rho)^{\perp} \cap \ker(A)^{\perp}$

Proof. First define $V := \ker(\rho)$ and $W := \ker(A)$. Statement (1) above is then $c(V, W) \subseteq W$ (with c interpreted in the lattice $L_{\mathcal{H}}$) — explicitly, statement (1) is simply

$$(V \cap W) \vee (V \cap W^{\perp}) \vee (V^{\perp} \cap W) \vee (V^{\perp} \cap W^{\perp}) \subseteq W. \tag{4.10}$$

Now statement (2) above is simply that $V \cap W^{\perp} = \hat{0} = V^{\perp} \cap W^{\perp}$. Of course, (equation 4.10) implies (2) because $W \cap W^{\perp} = \hat{0}$, and the converse implication is evident as well. □

We will also need the following simple lemmas involving only (classical) linear algebra.

Lemma 4.26. Let $A, \rho \in \mathcal{B}(\mathcal{H})$ be such that $\rho \neq 0$, $\ker(\rho)^{\perp} \cap \ker(A)^{\perp} = \hat{0}$ and also let A satisfy, in the direct sum decomposition $\mathcal{H} = \ker(\rho)^{\perp} \oplus \ker(\rho)$,

$$A = \begin{pmatrix} X & Y \\ Y^{\dagger} & 0 \end{pmatrix}. \tag{4.11}$$

Then $A = 0$.

Proof. The condition $\ker(\rho)^{\perp} \cap \ker(A)^{\perp} = \hat{0}$ is equivalent to $\ker(\rho) \vee \ker(A) = \mathcal{H}$. Then, for any $|\psi\rangle \in \mathcal{H}$, we have that $|\psi\rangle = |\phi\rangle + |\chi\rangle$, where $|\phi\rangle \in \ker(\rho)$ and $|\chi\rangle \in \ker(A)$. Of course, $|\chi\rangle = |\eta\rangle + |\omega\rangle$, with $|\eta\rangle \in \ker(\rho)^{\perp}$ and $|\omega\rangle \in \ker(\rho)$. Then, using matrix notation for the direct sum decomposition as above, we have

$$|\psi\rangle = \begin{pmatrix} |\eta\rangle \\ |\phi\rangle + |\omega\rangle \end{pmatrix}, \tag{4.12}$$

and in particular as $|\psi\rangle$ ranges over $\mathcal{H}$, we must have η ranging over all of $\ker\rho$.

Now $A|\psi\rangle = A|\phi\rangle + A|\chi\rangle = A|\phi\rangle$, and this means

$$\begin{aligned} A|\psi\rangle &= \begin{pmatrix} X & Y \\ Y^{\dagger} & 0 \end{pmatrix} \begin{pmatrix} |\eta\rangle \\ |\phi\rangle + |\omega\rangle \end{pmatrix} = \begin{pmatrix} X|\eta\rangle + Y(|\phi\rangle + |\omega\rangle) \\ Y^{\dagger}|\eta\rangle \end{pmatrix} \\ &= A|\phi\rangle = \begin{pmatrix} Y|\phi\rangle \\ |0\rangle \end{pmatrix}, \end{aligned} \tag{4.13}$$

And so we must have $Y^{\dagger}|\eta\rangle = |0\rangle$, and since $|\eta\rangle$ was generic, this means that $Y^{\dagger} = 0 = Y$. But then this immediately gives $\ker(\rho) \subseteq \ker(A)$, and so $\ker(\rho) \vee \ker(A) = \mathcal{H}$ means that $\ker(A) = \mathcal{H}$, i.e. $A = 0$. □

Lemma 4.27. Assume $\rho(t)$ is a time-dependent density operator (with t the time parameter) on $\mathcal{H}$. Then for each time t, in the direct sum decomposition $\mathcal{H} = \ker(\rho(t))^{\perp} \oplus \ker(\rho(t))$, $B\rho(t)$ must be of the form

$$B\rho(t) = \begin{pmatrix} X & Y \\ Y^{\dagger} & 0 \end{pmatrix}. \tag{4.14}$$

for some linear transformations X, Y.

Proof. Let $N = \dim \mathcal{H}$ (which may be infinite). Consider a time-dependent orthonormal basis $\{|\psi_i(t)\rangle\}_i^N$ for $\mathcal{H}$ in which $\rho(t)$ is diagonal for all t, so that

$$\rho(t) = \sum_{i=1}^{N} \alpha_i(t)|\psi_i(t)\rangle\langle\psi_i(t)|. \tag{4.15}$$

Then $\sum_{i=1}^{N} \alpha_i(t) = 1$ with $\alpha_i(t) \geq 0$ for all i since $\rho(t)$ is a density operator. Using equation 4.15, we compute

$$\frac{d\rho}{dt} = \sum_{i=1}^{N} \frac{d\alpha_i(t)}{dt}|\psi_i(t)\rangle\langle\psi_i(t)| + \sum_{i=1}^{N} \alpha_i(t)\Big(|\frac{d\psi_i(t)}{dt}\rangle\langle\psi_i(t)| + |\psi_i(t)\rangle\langle\frac{d\psi_i(t)}{dt}|\Big), \tag{4.16}$$

from which we can then obtain the matrix elements for $\frac{d}{dt}\rho$, namely

$$\Big(\frac{d\rho}{dt}\Big)_{jk} = \frac{d\alpha_j(t)}{dt}\delta_{jk} + \alpha_k\langle\psi_j(t)|\frac{d\psi_k(t)}{dt}\rangle + \alpha_j\langle\frac{d\psi_j(t)}{dt}|\psi_k(t)\rangle. \tag{4.17}$$

Let $f_{jk} \coloneqq \langle\psi_j(t)|\frac{d\psi_k(t)}{dt}\rangle$, and note that $f_{jk} + \bar{f}_{kj} = 0$. (This follows easily from $\frac{d}{dt}\langle\psi_j(t)|\psi_k(t)\rangle = \frac{d}{dt}\delta_{ij} = 0$.) Given this, a simple computation yields

$$\left(\frac{d\rho}{dt}\right)_{jk} = \frac{d\alpha_j(t)}{dt}\delta_{jk} + (\alpha_k - \alpha_j)f_{jk}. \tag{4.18}$$

where the f_{jk}'s and α_i's all depend on t. Now, $(i[H,\rho])_{jk} = i(\alpha_k - \alpha_j)H_{jk}$, and so for all t we have

$$[(B\rho(t)]_{jk} = \frac{d\alpha_j(t)}{dt}\delta_{jk} + (\alpha_k - \alpha_j)(f_{jk} + iH_{jk}). \tag{4.19}$$

Then, we see that for any fixed vector $|\phi\rangle \in \ker\rho(t)$ we can expand $|\phi\rangle = \sum_{i=1}^N \beta_i|\psi_i(t)\rangle$ where $\beta_i = 0$ for all i such that $\alpha_i \neq 0$. Using this, for any fixed t one may compute $\langle\phi|B\rho(t)|\chi\rangle$ for any $|\phi\rangle, |\chi\rangle \in \ker\rho(t)$ and verify the conclusion of the above lemma. $\square$

With the aid of the previous lemmas, we are now ready to show that our reduced von Neumann equation is equivalent to the usual von Neumann equation,

Proposition 4.28. Let $\rho(t)$ be a time dependent density matrix. Then in the model $\mathfrak{B}_{\mathcal{H}}$, ρ satisfies (vN) iff it satisfies (vN′).

Proof. Since (vN′) is clearly weaker than (vN), only one direction is non-trivial, and so assume $\rho(t)$ is a time-dependent density operator on $\mathcal{H}$ satisfying (vN′). Fix t. By lemma 4.25, because (vN′) is satisfied, we must have that $\hat{0} = \ker(\rho(t))^\perp \cap \ker(B\rho(t))^\perp$. Then by lemma 4.27 and lemma 4.26, we must have that $B\rho(t) = 0$ for any t, i.e. that (vN) is satisfied. $\square$

While one may find this result somewhat discouraging for the prospect of using a reduction of the von Neumann equation in order to arrive at a generalized notion of evolution in a quantum mechanics based on quantum linear algebra, we actually consider these preliminary results to be rather promising. Ultimately, we want any generalized evolutions to not be *too* general, ideally they would contain *only* unitary and measurement evolutions — and certainly we would not want to obtain evolutions that would take density operators to other non-density operators *which did not represent states at all*! What the above proposition suggests is that even reduced versions of the von Neumann equation "want" to take states to states, and this gives us reason to have hope that the project of obtaining measurement evolutions from the (appropriate reduction of the) von Neumann equation by developing quantum mechanics with quantum mathematics may very well succeed.

This, however, is a project we will leave for the future.

CHAPTER 5

QUANTUM SET THEORY

Classical axiomatic set theory, using either the Zermelo-Fraenkel axioms with choice (ZFC), or the von Neumann-Bernays-Gödel axioms (NBG), can be used to provide a foundation for virtually all of classical mathematics.[1] Ideally, we would like to find a foundation for *quantum mathematics*, and the most natural hope is that we could accomplish this with a *quantum set theory.* As may be expected, developing a quantum set theory which is up to the task is by no means a trivial exercise. One renowned logician of the 20th century, Gaisi Takeuti, has made some impressive first strides[2] in this direction (42), but unfortunately the technical machinery he develops is a bit unwieldy — in his own words

> "A development of mathematics with quantum logic is not impossible. However I now feel that it is not very worthwhile because of its extreme difficulty." (43)

Given such an ominous admonition, what else can we say but *challenge accepted!*

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[1]For a nice, but not too technical discussion, see Chapter XI in MacLane (32).

[2]See also (37; 38; 47; 48) for further developments of Takeuti's work.

In this chapter, we will present a reduction of the classical ZFC axioms.[1] This version differs from Takeuti's reduction (42) in that the form of our reduced axioms is much simpler,[2] and this simplicity allows us to construct models of these axioms based on the projection lattices of (separable complex) Hilbert spaces which are more amenable to mathematical investigation than Takeuti's quantum sets.

Before moving on to the ZFC axioms, we make a few general comments on axiomatic set theory, and also discuss the goals of a *quantum* axiomatic set theory. In section 5.1, we review the classical statement of the ZFC axioms,[3] as well as the classical set theoretic universe, before proceeding to give our reduced version of the axioms. In section 5.2, we proceed to construct models of the reduced axioms presented.[4] In section 5.3 we give a qualitative comparison of our quantum set theory to that constructed by Takeuti.

Axiomatic Set Theory

Classical axiomatic set theory, in contrast to naive set theory (see appendix A), is the development of set theory by means of (classical) predicate logic from a list of axioms, typically

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[1]See section 5.1 for a standard presentation of the ZFC axioms, and section 5.1.4 for our reduced version.

[2]Of course this is a subjective statement. We provide a more detailed explanation of this assertion in section 5.3.

[3]For those interested in classical NBG set theory we recommend Chapter 4 of Mendelson's book (35).

[4]The models we construct for our quantum set theory (as well as Takeuti's models) are dependent upon the existence of a model of *classical set theory* — such models are discussed in further detail in section 5.1.2.

either ZFC or NBG.[1] Perhaps the largest difference between axiomatic set theory as usually developed and in naive set theory is that axiomatic set theory (typically) everything under consideration is either a set or a class, and there are no "primitive" elements.[2] This is a little easier to understand after one has seen a model of such an axiomatic set theory, so we postpone a full discussion of this matter until section 5.1.2.

At this point, it is worth mentioning that it is not known if the ZFC axioms (or any other axioms axiomatizing set theory for that matter) are consistent within classical logic[3] — this is essentially a consequence of Gödel's second incompleteness theorem along with the fact that set theory provides the foundation for all of mathematics. This will not concern us, however, and we make no claim that quantum logic can resolve this problem. In fact, we will utilize the full power of classical ZFC in constructing our "quantum set theory" within quantum logic.

As for any axiomatic theory, one can then proceed to make formal deductions,[4] as well as construct models of the axioms.[5] In our development of quantum set theory, we will focus primarily on the construction of a model of (a reduction of) the ZFC axioms. Before moving on

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[1] We will only concern ourselves with the ZFC axioms.

[2] Although see (39) for a development of axiomatic set theory where this is not the case.

[3] See (18) for an introductory discussion.

[4] For example, we could provide formal deductions for the various propositions stated in appendix A.

[5] In section 5.1.2 we provide a "model" of classical set theory.

to a discussion the usual ZFC axioms, we will now set some modest goals we wish to accomplish in developing quantum set theory.

Goals of Quantum Axiomatic Set Theory

Just as the major goal of classical set theory is to provide a foundation for all of classical mathematics, the main goal of a *quantum* set theory is to provide a foundation for *quantum* mathematics. However, since quantum mathematics has not been developed in any detail, it would be difficult to apply this criteria. Hence, we put forth two modest goals which, we believe, are a respectable minimal criteria that any attempt at quantum set theory should satisfy.

First, since quantum logic is subclassical (and therefore includes all of classical logic and mathematics as a special case), quantum set theory must be a generalization of classical set theory. Specifically, the models of quantum set theory with the standard $\{0,1\}$ truth values should be exactly the models of classical set theory.

Second, quantum set theory should *at least* be powerful enough to develop a "quantum version" of the natural numbers suitable for constructing a "quantum arithmetic", and of course, these must reduce to classical arithmetic over the usual natural numbers $\mathbb{N}$ when the truth values are the standard $B_2 = \{0,1\}$.

The quantum set theory we develop fulfills both of these criteria.

5.1 The Zermelo-Fraenkel with Choice Axioms

In this section, we first present and briefly discuss the classical ZFC axioms — those interested in a more detailed discussion are invited to read Enderton's magnificent book *Elements* *of Set Theory* (19). We then present our reduced version of these axioms, and proceed to prove that our reduced axioms are indeed classically equivalent to the usual ZFC axioms.

Throughout our discussion of set theory, we work with the language $\mathcal{L}_{set} := \{\in\}$, where $\in$ is a binary predicate. It will be useful to establish some notational shorthand. We define

$$(x = y) := (\forall u)(u \in x \leftrightarrow u \in y), \tag{5.1}$$

as well as

$$(x \neq y) := \neg(x = y) \quad \text{and} \quad x \notin y := \neg(x \in y), \tag{5.2}$$

and

$$x \subseteq y := (\forall u)(u \in x \rightarrow u \in y). \tag{5.3}$$

5.1.1 Classical Zermelo Fraenkel with Choice

In order to state the ZFC axioms concisely, we need to establish a little more notation. Since in this section we restrict ourselves to classical logic, for a given wff $\psi(x)$, if the statement $(\exists x)\psi(x)$ holds in a given model, then there exists some set A in that model such that $\psi(A)$ holds.[1] Hence, a statement in set theory of the form

$$(\exists x)(\forall u)(u \in x \leftrightarrow \psi(u))$$

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[1]This is because the truth value algebra of such a model is B_2.

which holds in a given model yields a set A in that model whose elements are precisely those u which satisfy $\psi(u)$, and any set B in this model which also satisfies this property will satisfy the formal statement '$A = B$'. (given the definition of equality above, equation 5.1). Hence we can use a symbol to refer to that set in any given model,[1] and we do this for the following cases:[2]

Empty Set: We define $\varnothing$ (using ZFC3) to be the set satisfying (for any choice of set x)

$(\forall u)(u \in \varnothing \leftrightarrow u \in x \wedge u \neq u)$.

Pairs and Singletons: For any two sets x and y, we define $\{x, y\}$ to be the set satisfying

$(\forall u)(u \in \{x, y\} \leftrightarrow u = x \vee u = y)$, and define $\{x\} \coloneqq \{x, x\}$. These exist by ZFC2 below.

Intersection: For any two sets x and y, we define (using ZFC3) $x \cap y$ to be the set satisfying

$(\forall u)(u \in x \cap y \leftrightarrow u \in x \wedge u \in y)$.

Union: For any set x, we define $\bigcup x$ to be the set which satisfies the statement

$(\forall u)(u \in \bigcup x \leftrightarrow (\exists z)(u \in z \wedge z \in x))$. This set exists by ZFC4 below. Then for any two sets x and y we define $x \cup y \coloneqq \bigcup\{x, y\}$.

Power Set: For any set x, we define (using ZFC5) $\mathcal{P}(x)$ to be the set satisfying

$(\forall u)(u \in \mathcal{P}(x) \leftrightarrow u \subseteq x)$.

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[1] Strictly speaking, when writing $\mathcal{L}_{set}$-sentences, we should *not* be referring to objects in models at all, but we follow the common practice. We will shortly note an alternative way of stating the axioms that does not involve defining symbols which refer to sets in models.

[2] The reader will note that the defined symbols are only used in axioms which are numbered higher than the axiom which allows for the definition — this ensures that we avoid any circular reasoning.

Set Builder Notation: For any sets x and y and any wff ψ, we define $\{u \in x : \psi(u, y)\}$ to be that set containing exactly those elements of u for which $\psi(u, y)$ is true. This set exists by ZFC3 below.

If we were not using the above defined symbols, our axioms would be considerably lengthened, since we would need to include in each axiom which utilizes a defined symbol an additional clause that would substitute for said symbol. For example, the axiom of regularity (ZFC8) below would read as follows without the use of $\varnothing$:

$$(\forall z)[(\forall u)((u \notin z)) \rightarrow (\forall x)(x \neq z \rightarrow (\exists y)(y \in x \wedge y \cap x = z))],$$

rather than the significantly simpler

$$(\forall x)(x \neq \varnothing \rightarrow (\exists y)(y \in x \wedge y \cap x = \varnothing)).$$

Our other axioms could be similarly rewritten without the use of the symbols defined above. This procedure would yield an equivalent formulation of the axioms, since ZFC1 suffices to prove all of the equality and substitution axioms in classical logic.[1]

We are now ready to state the Zermelo-Fraenkel axioms for set theory.

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[1]If one is working beyond the context of irreducible truth value algebras (i.e. beyond B_2 since we are considering classical ZFC here), one *must* think of the defined symbols as a shorthand for abbreviating the axioms (as is done for ZFC8 above), since in this case statements involving '$\exists$' do *not* necessarily give a corresponding set in any given model. See section 3.2.2.

ZFC1 *Extensionality:* $(\forall x)(\forall y)[x = y \to (\forall z)(x \in z \leftrightarrow y \in z)]$.

ZFC2 *Pairing:* $(\forall x)(\forall y)(\exists z)(\forall u)(u \in z \leftrightarrow u = x \vee u = y)$.

ZFC3 *Separation Schema:* For ψ any wff,

$$(\forall x)(\forall y)(\exists z)(\forall u)(u \in z \leftrightarrow u \in x \wedge \psi(u, y)).$$

ZFC4 *Union:* $(\forall x)(\exists y)(\forall u)(u \in y \leftrightarrow (\exists z)(u \in z \wedge z \in x))$.

ZFC5 *Power Set:* $(\forall x)(\exists y)(\forall u)(u \in y \leftrightarrow u \subseteq x)$.

ZFC6 *Infinity:* $(\exists x)(\varnothing \in x \wedge (\forall y)(y \in x \to y \cup \{y\} \in x))$.

ZFC7 *Replacement Schema:* For ψ any wff,

$$[(\forall x)(\forall y)(\forall z)(\psi(x, y) \wedge \psi(x, z) \to y = z)]$$

$$\to (\forall x)(\exists z)(\forall u)[u \in z \leftrightarrow (\exists y)(y \in x \wedge \psi(y, u)).$$

ZFC8 *Regularity:* $(\forall x)[x \neq \varnothing \to (\exists y)(y \in x \wedge y \cap x = \varnothing)]$.

ZFC9 *Choice:*

$$(\forall z)\big([(\forall x)(\forall y)(x \in z \to x \neq \varnothing) \wedge (x \in z \wedge y \in z \wedge x \neq y \to x \cap y = \varnothing)]$$

$$\to (\exists s)(\forall t)[t \in z \to (\exists u)(s \cap t = \{u\})]\big).$$

For those unfamiliar with axiomatic set theory, we will give a brief description of the above axioms. First, we should point out that we are working with equality '=' as a defined predicate, rather than a part of our language $\mathcal{L}_{set}$. If we had taken the equality symbol '=' as part of our language, we would instead have

$$(\forall x)(\forall y)[x = y \rightarrow (\forall u)(u \in x \leftrightarrow u \in y)]$$

as our extensionality axiom. In this case, extensionality would simply say that *sets are determined by their members*, and we would further require equality to satisfy substitution axioms (see section 4.1.1). Since we have *defined* equality to satisfy the above requirement, our extensionality axiom instead provides a basis upon which to *prove* that equality so defined satisfies substitution. Since this fact will be useful in the sequel, we mark it off as a proposition (5.1) at the end of this section.

As for the axiom schema of separation, this is a weakening of the (perhaps more intuitive) axiom schema of abstraction which states that for any wff ψ,

$$(\exists z)(\forall u)[(u \in z) \leftrightarrow \psi(u)],$$

i.e. there exists a set z such that u is a member of z iff $\psi(u)$ is true. This axiom was used in the original naive formulation of set theory, however this axiom leads to Russel's paradox[1],

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[1]See appendix A.

and so is replaced in ZFC by the weaker axiom of separation, which is basically the axiom of abstraction restricted to some given set.

The axioms of pairing, union, power set, and infinity are straightforward. Pairing states that, given any two sets, one can form a new set whose members are exactly the two original sets. Union states that given a collection of sets, one can obtain a new set whose members are exactly the members of the original collection. Power set says that for any set, the set of all subsets of that set forms a set itself. And infinity states that an infinite set exists (and, in particular, there is at least one set).

The replacement schema and regularity are more of interest for set theory in itself as opposed to set theory as a framework for mathematics. Replacement was not included in Zermelo's original axioms, and was added by Fraenkel in 1922. It has important proof-theoretic consequences and is necessary for the construction of the higher cardinals (see (22)). Regularity prevents some counter-intuitive behavior such as sets being elements of themselves (see proposition 5.2).

Finally, the axiom of choice is one of the most well-known axioms in all of mathematics, and is certainly the most controversial axiom in ZFC. This axiom states that for any collection of sets, one can form a new set with exactly one element from each set in the original collection. For a brief discussion of the axiom of choice, see appendix A.4, and for a lengthy and in-depth investigation of this axiom, see (27).

Two Simple Facts of Classical ZFC

The first proposition below we prove in quantum logic since the proof is the same as for classical logic.

Proposition 5.1. In the language $\mathcal{L}_{set}$, ZFC1 implies that equality ('=') as defined in equation 5.1 above satisfies the substitution property, i.e. we have both

$$\begin{aligned} \text{ZFC1} &\vdash (\forall x)(\forall y)[x = y \rightarrow (\forall z)(x \in z \leftrightarrow y \in z)] \\ &\vdash (\forall x)(\forall y)[x = y \rightarrow (\forall z)(z \in x \leftrightarrow z \in y)] \end{aligned}$$

Proof. The second statement follows from the definition of '=' using Q3 and R5, and the first is simply a restatement of axiom ZFC1. □

Proposition 5.2. In the language $\mathcal{L}_{set}$, the sentence $(\forall x)(x \notin x)$ is classically derivable from the ZFC axioms.

Proof. We know that $\{x\} \neq \varnothing$ since $x \in \{x\}$. Then by ZFC8, we have that there is some $y \in \{x\}$ such that $y \cap \{x\} = \varnothing$. But by definition of the singleton, we know that $y \in \{x\}$ means that $y = x$, and hence $x \cap \{x\} = \varnothing$, so that for any $z \in x$, we know that $z \notin \{x\}$, i.e. that $z \neq x$. Hence $x \notin x$. □

These two propositions will be useful in the sequel.

5.1.2 The Classical Universe of Sets

Now that we have seen the ZFC axioms, we will explore the prototypical "model" of these axioms,[1] namely the classical universe of sets. However, before diving in to the classical universe,

[1] As we will see, this "model" actually uses a fair amount of (classical) set theory in its construction, and so is not useful in proving the consistency of ZFC, as per section 1.1.4.

we will first discuss the most naive approach to constructing an "intuitive" model of the ZFC axioms of set theory. This approach defines a "set" to be any conceivable collection — anything one can imagine goes. The collection of all cows would be a set, as would the collection of all collections, and the collection of all collections which are not members of themselves. Ah! Here we are in trouble, since we have just run face first into Russel's paradox.[1] If we want a candidate for a model of ZFC, we need to do better than this.

One safe way is to proceed constructively.[2] We begin with one set, say the empty set $\varnothing$, and define $V_0 = \varnothing$ and then proceed inductively, at each stage taking the power set of the previous stage. Explicitly, $V_{i+1} = \mathcal{P}(V_i)$ for every $i \in \mathbb{N}$. But clearly this is not quite going to work, since at each stage all of our sets will be finite, and we need (by ZFC6) infinite sets! Hence, in order to construct a set theoretic universe we will use transfinite induction, where instead of running over all the natural numbers, we run over all the ordinals.[3]

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[1] See the appendix A.

[2] For the reader encountering unfamiliar notions, knowledge of the elementary set theory covered in appendix A suffices for the discussion here.

[3] The reader unfamiliar with ordinal numbers and transfinite induction is referred to section A.3 of the appendix.

Explicitly, let **Ord** be the class of ordinal numbers. Define

$$\begin{aligned} V_{\mathbf{0}} &:= \varnothing \\ V_{\beta+1} &:= \mathcal{P}(V_\beta) \\ V_\beta &:= \bigcup_{\gamma\in\beta} V_\gamma \quad \text{if } \beta \text{ is a limit ordinal.} \\ \mathfrak{V} &:= \bigcup_{\beta\in\mathbf{Ord}} V_\beta \quad \text{where the union is understood to give rise to a proper class.} \end{aligned} \tag{5.4}$$

That V_β is indeed defined for every $\beta \in \mathbf{Ord}$ follows by the principle of transfinite induction (see theorem A.5), along with the fact (see def. A.23) that every ordinal is either $\varnothing$, a successor ordinal, or a limit ordinal, as well as (see proposition A.8) that every ordinal is well-ordered by the membership relation. We then define the truth value of the atomic sentences by whether or not a given set is "actually" an element of another set via the above construction.

Of course, here we have done nothing but rob Peter to pay Paul. The entire apparatus of transfinite recursion is *built* on set theory, and so we cannot hope to prove the existence of the model above described without first *assuming* the consistency of the axioms of set theory. Only if we take the (classical) consistency of ZFC on faith can we proceed with our development of quantum set theory.

Finally, we should comment on one major difference between the classical universe of sets constructed above, and the more intuitive notion of "all possible collections". In the classical universe, everything in sight is a set — *including every element of every set.* This is certainly a more elegant, if less intuitive, approach, allowing us to dispense with tracking whether any particular object in our model is a set or not, but runs counter to the way set theory is naively used in developing mathematics — for example, when doing arithmetic one considers sets of numbers (such as the prime numbers, the even numbers, etc.), but unless one is a set theorist, one does not typically think of the *numbers themselves* as being sets. Of course, when building branches of mathematics from set theory, such as arithmetic, treating particular objects of study (such as numbers) as *not* being sets simply amounts to ignoring their internal structure *as sets*, and so in treating *everything* as a set one loses nothing. In chapter 6 we develop basic arithmetic from set theory, and if one is uncomfortable working in a mathematical world where everything is a set, that discussion should suffice to convince oneself that nothing *untoward* is being done.

5.1.3 Generalizing the Classical Universe

This section is only intended to provide some background for those who wish to obtain a slightly better idea of Takeuti's set theory, and may be omitted without loss of continuity. Also, as in Takeuti, we treat '=' as a binary predicate rather than a defined notion. For a more detailed account, see either (2) or (34).

Recall proposition A.4, which states that for any given set A, there is an isomorphism between the power set $\mathcal{P}(A)$ and the set of maps B_2^A from A to the standard truth value algebra $B_2 = \{0, 1\}$. Since B_2 is a Boolean algebra, it is natural to generalize the classical universe of sets by taking some Boolean algebra B, and replacing $\mathcal{P}(V_\beta)$ with B^{V_β} in equation 5.4 to obtain a *Boolean-valued* model of set theory, i.e. a model of set theory with Boolean truth values. That is, for B any complete Boolean algebra, we define[1]

$$
\begin{aligned}
&V_0^{(B)} := \varnothing \\
&V_{\beta+1}^{(B)} := B^{V_\beta^{(B)}} \\
&V_\beta^{(B)} := \bigcup_{\gamma\in\beta} V_\gamma^{(B)} \quad \text{if } \beta \text{ is a limit ordinal.} \\
&\mathfrak{V}^{(B)} := \bigcup_{\beta\in\mathbf{Ord}} V_\beta^{(B)} \quad \text{where the union is understood to give rise to a proper class.}
\end{aligned} \tag{5.5}
$$

Then, for any complete Boolean algebra B, $\mathfrak{V}^{(B)}$ will serve as the underlying class of a model of the ZFC axioms. Below, we define a truth valuation $[\![\cdot]\!]^{(B)}$ for these models by specifying the valuation on the evaluated atomic sentences, and then extending this valuation to all wffs (as per proposition 3.12). First, for any $f \in \mathfrak{V}^{(B)}$, define the (classical) set $\mathcal{D}(f)$ to be the domain of f. Then we further define by transfinite induction (for any $f, g \in \mathfrak{V}^{(B)}$)

$$[\![f \in g]\!]^{(B)} := \bigvee_{h\in\mathcal{D}(g)} \left(g(h) \wedge [\![f = y]\!]^{(B)}\right) \tag{5.6}$$

$$[\![f = g]\!]^{(B)} := \bigwedge_{h\in\mathcal{D}(f)} \left(f(h) \to [\![h \in g]\!]^{(B)}\right) \wedge \bigwedge_{j\in\mathcal{D}(g)} \left(g(j) \to [\![j \in f]\!]^{(B)}\right). \tag{5.7}$$

---

[1] Our construction of Boolean-valued sets follows Bell (2).

Boolean-valued models such as these are useful as a different window on the method of *forcing* developed by Cohen. Forcing has been useful in constructing independence proofs in axiomatic set theory, for example of the continuum hypothesis.[1] See (12) for a detailed treatment of the use of forcing in independence proofs, and (2; 34) for the connection between Boolean-valued models and independence proofs.

5.1.4 Reduced Zermelo Fraenkel with Choice

Just as in the statement of the standard ZFC axioms, it will behoove us to make a few definitions before defining our reduced versions of these axioms. Just as we referred to any element of the underlying class of a model of ZFC as a 'set', we will refer to any element of the underlying class of a model of the reduced ZFC axioms below as a 'quantum set'.[2]

First, we continue to use the definitions of '$=$','$\neq$', '$\notin$',and '$\subseteq$' from the previous section (equation 5.1, equation 5.2, equation 5.3, respectively), and we further define

$$(x \doteq y) := (\forall z)(x \in z \leftrightarrow y \in z). \tag{5.8}$$

---

[1]The most basic version of the continuum hypothesis states that there is no set whose cardinality is strictly larger than the natural numbers and strictly smaller than the real numbers.

[2]We actually refer to the objects in a slightly more general class of $\mathcal{L}_{set}$-structures as 'quantum sets' — see def. 5.2 below.

Further, for a given variable y, we will abuse notation slightly, defining

$$(x \in y^*) := (\exists z)(z \doteq y \wedge x \in z)$$

$$(y^* \in z) := (y \in z) \tag{5.9}$$

which will allow us to think of $\cdot^*$ (by '$\cdot^*$', we mean the "assignment" $f \mapsto f^*$ as per the note following def. A.15) as if it were a unary function symbol.

Next, we recall the discussion of section 3.2.2, where we introduced, for any given wff ψ, the formula schema $\mathbf{C}(\psi)$ and $\mathbf{T}(\psi)$. Specifically, for the language $\mathcal{L}_{set}$, these become

$$\mathbf{C}(\psi) = (\forall s)(\forall t)(\varphi_{s \in t}(\psi) \rightarrow \psi) \tag{5.10}$$

and

$$\mathbf{T}(\psi) = (\forall s)(\forall t)(s \in t \rightarrow \psi), \tag{5.11}$$

where we recall that $\varphi_x(y) = x \vee (\neg x \wedge y)$ is the Sasaki projection.

In the aforementioned discussion, we noted that by reducing a statement of the form '$(\exists x)\psi(x)$' into the statement '$(\exists x)\mathbf{T}(\psi(x))$', we could still guarantee the existence[1] of some object a satisfying $\psi(a)$ in any model for which the reduced statement holds, *even when the*

[1]Due to the axiom RZFC12 below.

truth values are non-standard.[1] Given this fact, we will define the symbols for the notions of (quantum) power set, union, etc.[2] in a fashion similar to the classical case for use in the axioms below. We define:[3]

Empty Set: We define (using RZFC3) $\varnothing$ to be the quantum set satisfying (for any choice of x)

$(\forall u)(u \in \varnothing \leftrightarrow u \neq u \wedge u \in x)$

Singletons: For any set x, we define $\{x\}$ to be the quantum set satisfying

$(\forall u)(u \in \{x\} \leftrightarrow u^* = x^*)$. This exists by axiom RZFC2.

Intersection: For any two sets x and y, we define $x \cap y$ to be the quantum set satisfying

$(\forall u)(u \in x \cap y \leftrightarrow u \in x \wedge u^* \in y)$, which exists by RZFC3.

Pairwise Union: For any two sets x and y, we define $x \cup y$ to be the quantum set which satisfies

$(\forall u)(u \in z \leftrightarrow u \in x \vee u \in y)$, which exists by RZFC10.

Set Union: For any set x, we define $\bigcup x$ to be the quantum set which satisfies the statement

$(\forall u)(u \in \bigcup x \leftrightarrow (\exists z)(u \in z \wedge z \in x))$, This set exists by axiom RZFC4.

---

[1]Recall that, as per the discussion in section 3.2.2, this requires also that the truth value algebra be irreducible. By our completeness theorem 3 we can restrict our attention to models with irreducible truth values insofar as we wish to determine what wffs are derivable in our axiomatic set theory.

[2]Of course, just as for classical ZFC, when using reducible truth values we could simply think of these symbols as a shorthand for abbreviating the axioms, keeping in mind that the symbol would no longer refer to any particular set in a given model. See section 5.1.1.

[3]Since the notation is so standard, and because we will virtually always be working with the reduced ZFC axioms, we use the same notation as for the corresponding notions in the case of classical ZFC. No confusion should arise — the only place both notions are used near each other is in the proof of proposition 5.4, where we make explicit note of which is being used.

Power Set: For any set x, we define $\mathcal{P}(x)$ to be the quantum set satisfying

$(\forall u)(u \in \mathcal{P}(x) \leftrightarrow u^* \subseteq x)$, which exists by RZFC5.

Set Builder Notation: For any sets x and y and any wff ψ, we define $\{u \in x \ : \ \psi(u^*, y)\}$ to be that quantum set satisfying $(\forall u)(u \in z \leftrightarrow u \in x \wedge \psi(u^*, y))$, which exists by RZFC3.

With the above symbols and operations defined, we are ready to state our reduced version of the ZFC axioms (which collectively we refer to as RZFC).

(RZFC1) *Extensionality:* $(\forall x)(\forall y)[\mathbf{T}(x = y) \rightarrow (\forall z)(x \in z \leftrightarrow y \in z)]$

(RZFC2) *Singleton:* $(\forall x)(\exists z)\mathbf{T}\big((\forall u)(u \in z \leftrightarrow u^* = x^*)\big)$

(RZFC3) *Separation Schema:* For ψ any wff,

$$(\forall x)(\forall y)(\exists z)\mathbf{T}\big[(\forall u)(u \in z \leftrightarrow u \in x \wedge \psi(u^*, y))\big]$$

(RZFC4) *Union:* $(\forall x)(\exists y)\mathbf{T}\big[(\forall u)(u \in y \leftrightarrow (\exists z)(u \in z \wedge z \in x))\big]$

(RZFC5) *Power Set:* $(\forall x)(\exists y)\mathbf{T}\big[(\forall u)(u \in y \leftrightarrow u^* \subseteq x)\big]$.

(RZFC6) *Infinity:* $(\exists x)\mathbf{T}\big(\varnothing \in x \wedge (\forall y)(y \in x \rightarrow y \cup \{y\} \in x)\big)$

(RZFC7) *Replacement Schema:* For ψ any wff,

$$[(\forall x)(\forall y)(\forall z)(\psi(x, y) \wedge \psi(x, z) \rightarrow y = z)]$$
$$\rightarrow (\forall x)(\exists z)\mathbf{T}\big[(\forall u)[u \in z \leftrightarrow (\exists y)(y \in x \wedge \psi(y^*, u^*))]\big].$$

(RZFC8) *Regularity:* $(\forall x)(x \neq \varnothing \to (\exists y)(y \in x \wedge y \cap x = \varnothing))$

(RZFC9) *Choice:*

$$(\forall z)\big([(\forall x)(\forall y)(x \in z \to x \neq \varnothing) \wedge (x \in z \wedge y \in z \wedge x \neq y \to x \cap y = \varnothing)]$$
$$\to (\exists s)\mathbf{T}\big[(\forall t)[t \in z \to (\exists u)(s \cap t = \{u\})]\big]\big)$$

(RZFC10) *Pairwise Union:* $(\forall x)(\forall y)(\exists z)\mathbf{T}[(\forall u)(u \in z \leftrightarrow u \in x \vee u \in y)]$

(RZFC11) $*$*-classicality:* $(\forall x)[(\forall u)\ \mathbf{C}(u \in x) \to x = x^*]$

(RZFC12) $\mathbf{T}$-*normality:* For ψ any wff, $\mathbf{C}(\mathbf{T}(\psi)) \wedge [\mathbf{T}(\psi) \to \psi]$

For technical reasons that will become apparent in the sequel, we have replaced the classical pairing axiom with the singleton and pairwise union axiom. The astute reader will be wondering if, using the notation defined above, that $\bigcup(\{x\} \cup \{y\}) = x \cup y$. As we will see in section 5.2, this is *not* generically the case.

Before showing that the RZFC axioms are a reduction of the ZFC axioms, we will first demonstrate that our defined notion of equality ('=') has the properties we would expect.

Proposition 5.3. In the language $\mathcal{L}_{set}$, we have that

1. $\vdash (\forall x)(x = x)$
2. $\vdash (\forall x)(\forall y)(x = y \to y = x)$

3. $\vdash (\forall x)(\forall y)(\forall z)(x = y \wedge y = z \rightarrow x = z)$

4. $\vdash (\forall x)(\forall y)[x = y \rightarrow (\forall z)(z \in x \leftrightarrow z \in y)]$

5. RZFC1 $\vdash (\forall x)(\forall y)[\mathbf{T}(x = y) \rightarrow (\forall z)(x \in z \leftrightarrow y \in z)]$

Proof. Number (4) above is already established in proposition 5.1, and (5) follows trivially, since the statement to be proved *is* RZFC1. To utilize the full power of orthomodular lattice theory, we will prove the remaining items above using the completeness theorem, so that we only need to examine each of the above sentences in an arbitrary $\mathcal{L}_{set}$-structure — to this end let $\mathfrak{M}$ be an $\mathcal{L}_{set}$-structure with truth valuation $[\![\cdot]\!]$ and underlying class $\mathfrak{A}$.[1]

First, considering (1), for any $a \in \mathfrak{A}$, we have

$$[\![a = a]\!] = [\![(\forall u)(u \in a \leftrightarrow u \in a)]\!] = \bigwedge_{a \in \mathfrak{A}} \Big([\![u \in a]\!] \leftrightarrow [\![u \in a]\!]\Big) = 1,$$

where the last equality is by lemma C.14. Then (1) holds by lemma 2.12 (and the completeness theorem).

Considering (2), for any $a, b \in \mathfrak{A}$, we have

$$[\![a = b]\!] = \bigwedge_{c \in \mathfrak{A}} \Big([\![c \in a]\!] \leftrightarrow [\![c \in b]\!]\Big) = [\![b = a]\!],$$

---

[1]We urge the reader to pay careful attention to the '=' which is a defined notion in $\mathcal{L}_{set}$ and the '=' which represents equality in the truth value algebra of the model.

where the final equality holds by lemma C.14, so that we have

$$[\![a = b \to b = a]\!] = \left([\![a = b]\!] \to [\![b = a]\!]\right) = 1$$

by the same lemma, and so (2) holds by lemma 2.12 and completeness as well.

Moving on to (3), for any $a, b, c \in \mathfrak{A}$, we have

$$\begin{aligned}
\left([\![a = b]\!] \wedge [\![b = c]\!]\right) &= \Big[\bigwedge_{d \in \mathfrak{A}} \left([\![d \in a]\!] \leftrightarrow [\![d \in b]\!]\right) \wedge \bigwedge_{e \in \mathfrak{A}} \left([\![e \in b]\!] \leftrightarrow [\![e \in c]\!]\right)\Big] \\
&= \bigwedge_{f \in \mathfrak{A}} \Big[\left([\![f \in a]\!] \leftrightarrow [\![f \in b]\!]\right) \wedge \left([\![f \in b]\!] \leftrightarrow [\![f \in c]\!]\right)\Big] \\
&\leq \bigwedge_{f \in \mathfrak{A}} \left([\![f \in a]\!] \leftrightarrow [\![f \in c]\!]\right) = [\![a = c]\!],
\end{aligned}$$

where the second line is obtained by definition of the greatest lower bound, and the inequality follows from lemma C.14. Then (3) holds by lemma 2.12 and completeness. □

Note: Note that (1-3) above are just the equality axioms of definition 4.2. Also (4-5) above are simply substitution axioms (with (5) reduced) for the predicate '$\in$' (see section 4.1.1). Compare with proposition 5.1 concerning equality in (unreduced) ZFC.

We will now demonstrate that the RZFC axioms are, indeed, a reduction of the ZFC axioms from the previous section.

Proposition 5.4. The RZFC axioms (RZFC1-RZFC12) presented above are a reduction of the ZFC axioms (ZFC1-ZFC9) presented in section 5.1.1.

Proof. We need to show that the classical and reduced ZFC axioms are equivalent in the presence of the schema CL (i.e. using classical logic). RZFC12 is a tautology of classical logic (so is automatically implied by ZFC). RZFC8 is unchanged from the classical axiomatization (so the reduced versions imply the classical versions and vice versa). RZFC1, RZFC3-RZFC7 and RZFC9 now include the '**T** operator', but this is simply the identity (up to logical equivalence) in classical logic,[1] and so we can ignore this operator for the purposes of proving equivalence.

Axioms RZFC3, RZFC5, and RZFC7 also replace some instances of $x, y, \ldots$ with $x^*, y^*, \ldots$. Clearly, if RZFC11 holds in a model with the standard $\{0,1\}$ truth values, then, since $\mathbf{C}(\psi)$ is satisfied in any such model, we have $x = x^*$, and hence the classical versions of these axioms are logically equivalent to the reduced ones by the substitution property of '=' (see proposition 5.1). Hence, the RZFC axioms automatically implies the classical axioms ZFC1, ZFC3, ZFC5, and ZFC7 — moreover, if we demonstrate that RZFC11 is implied by the classical ZFC axioms, we obtain the reduced axioms RZFC3, RZFC5, and RZFC7 for free. Hence we only need to show that (in classical logic) ZFC2 follows from the RZFC axioms, as well as that the reduced axioms RZFC2, RZFC10 and RZFC11 follow from the classical ZFC axioms (and CL).

The reduced axioms RZFC2 and RZFC10 follow trivially from the classical pairing and union, since we have the classical $\{x\}$ as well as the classical union $x \cup y$. Now, we show that

---

[1]For proving the ZFC axioms are derivable from the RZFC axioms, the '**T** operator' is the 'identity operator' by RZFC12. For proving the RZFC axioms from ZFC, we always have $\varnothing \in \{\varnothing\}$ in ZFC, so that '**T**' is the 'identity operator' by lemma 3.10.

the ZFC axioms of section 5.1.1 imply RZFC11 ($*$-classicality). By definition of '$=$', '$u \in x^*$', and '$\doteq$', we have

$$\begin{aligned}(x = x^*) &\leftrightarrow (\forall u)(u \in x \leftrightarrow u \in x^*)\\ &\leftrightarrow (\forall u)[u \in x \leftrightarrow (\exists z)(z \doteq x \wedge u \in z)]\\ &\leftrightarrow (\forall u)[u \in x \leftrightarrow (\exists z)((\forall s)(z \in s \leftrightarrow x \in s) \wedge u \in z)]. \qquad (5.12)\end{aligned}$$

Hence, we only need show the double implication in the final line. If $u \in x$, then taking $z = x$, we see that the RHS of the implication in the bottom line holds. Conversely, if the RHS of said implication holds, then by the (classical) pairing axiom, there is some z with[1] $z \in \{x\}$ and $u \in z$, so since $z = x$ this implies $u \in x$. This establishes the double implication, so we see that $x = x^*$ so that RZFC11 holds.

Next, we assume the RZFC axioms, and wish to show that ZFC2 follows. But for arbitrary sets x, y we have the singleton sets[2] $\{x\}$, $\{y\}$ by RZFC2, and hence we have the set $\{x\} \cup \{y\}$ by axiom RZFC10, which shows that ZFC2 is indeed implied by the RZFC axioms combined with CL. $\square$

---

[1]Here $\{x\}$ is the singleton existing by virtue of the *classical* axioms.

[2]Here $\{x\}$ and $\{y\}$ are the singletons existing by virtue of the *reduced* axioms.

The above proposition shows that we have attained our first goal of quantum set theory, namely that when we restrict ourselves to the standard $B_2 = \{0,1\}$ truth values, we recapture exactly classical ZFC set theory.

5.2 Quantum Models of Set Theory

In this section we construct models of the reduced ZFC axioms presented in section 5.1.4. As we will see explicitly, this construction relies on a number of properties of projection lattices which are not general properties of orthomodular lattices.

5.2.1 Universes of Quantum Sets

Given a complete orthomodular lattice L, we will now define an associated universe of quantum sets. We utilize the definition of the classical universe $\mathfrak{V}$ from section 5.1.2, as well as the notion of a class function from appendix A.

Definition 5.1. Let L be an orthomodular lattice, let $\mathfrak{K}$ be a class, and let $f \in L^{\mathfrak{K}}$. Then the *support of* f (denoted $\sup f$) is given by

$$\sup f := \{k \in \mathfrak{K} : f(k) \neq 0\}. \tag{5.13}$$

Let L be an OML. For any $A \in \mathfrak{V}$, and for any $\hat{f} : A \to L\backslash\{0\}$, we can naturally identify $\hat{f}$ with the class function f from $\mathfrak{V}$ into L given by $f(B) = f(A)$ if $B \in A$ and 0 if $B \notin A$, so that $\sup f = A$. Similarly, for any class function $f \in L^{\mathfrak{V}}$ such that $\sup f$ is a set, we can naturally think of f as a map $\hat{f} : \sup f \to L\backslash\{0\}$ by taking the restriction. Given this natural correspondence, we will identify such objects in the sequel.[1]

Definition 5.2. Let L be a complete orthomodular lattice, and $\mathfrak{V}$ the classical universe of sets. The *L-valued universe (of sets)*, which we denote $\mathbb{Q}_L$, is the proper class

$$\mathbb{Q}_L := \{ f \in L^{\mathfrak{V}} \ : \ \sup f \text{ is a (classical) set } \} \tag{5.14}$$

where the "set" builder notation is understood in the sense of classes,[2] and where each $f \in \mathbb{Q}_L$ is called a *L-valued set*, or *quantum set*,[3] if L is clear from the context. We then define the truth valuation $[\![\cdot]\!]$ (for L-valued sets f, g) by

$$[\![f \in g]\!] := g(\sup f),$$

and extend this to all wffs (as per proposition 3.12). Then the $\mathcal{L}_{set}$-structure $(\mathbb{Q}_L, L, [\![\cdot]\!], \varnothing)$ is denoted $\mathcal{Q}_L$, and called the *L-valued set structure.*

---

[1]This is done solely because it will serve to greatly streamline our presentation and keep our focus away from notational headaches, essentially because it allows us to not have to constantly keep track of the domain of any given function.

[2]As mentioned above, we identify a given class function $f \in L^{\mathfrak{V}}$ such that $\sup f$ is a set with the set function $\hat{f} : \sup f \to L\backslash\{0\}$, so that we can form the class of all such f.

[3]We will use this terminology even though the models we construct satisfy the RZFC axioms only for OMLs with certain properties; the reason for this is that these L-valued sets still satisfy almost all of the RZFC axioms, and are, in any case, a straightforward generalization of the sets of the classical universe.

Of course, when $L = B_2$, this reproduces the classical universe (as per the discussion in section 5.1.3). Before proceeding any further, we should point out that there is an odd property of the models just constructed which suggests they may not suffice as a final and authoritative model of quantum set theory. The problem is essentially that, for a given quantum set f, and for any two other quantum sets g and h with the same support, we have that $g \in f \leftrightarrow h \in f$ holds in any model $\mathcal{Q}_L$ defined above, so that our quantum sets don't "separate points". One could think of our quantum sets as being somehow a subclass of all "true" quantum sets (yet to be discovered) satisfying a certain "symmetry condition", namely that the elements of our quantum sets are invariant under any permutation of quantum sets which leaves the support of every quantum set fixed. Under this interpretation, it's worth speculating that this symmetry is what makes it possible for our quantum sets to satisfy such a nice reduction of the ZFC axioms but yet still be rich enough to provide an interesting model of set theory which goes well beyond the classical universe.

5.2.2 The Quantum Sets satisfy the (reduced) Zermelo-Fraenkel with Choice Axioms

In this section we shall show that for L a projection lattice of a separable complex Hilbert space, the model $\mathcal{Q}_L$ of definition 5.2 is a model[1] for the RZFC axioms. However, many of the axioms hold for any complete orthomodular lattice L, and others hold more generally than in projection lattices. For each axiom, we will state the largest class of OMLs for which we know

[1]We reiterate that this requires the assumption that the classical universe is indeed a well-defined notion.

it holds. That the projection lattices then model all the axioms will be a corollary of these results. Along the way we will prove some lemmas and make some definitions which are set aside since they are computationally useful. The reader not interested in the technical details of proving that our quantum sets satisfy the RZFC axioms can skip ahead directly to theorem 6 and its corollary 5.17.

Definition 5.3. Let L be a complete orthomodular lattice, and let $A, B \in \mathfrak{V}$. Then define the L-valued set $\delta_A : \mathfrak{V} \to L$ by

$$\delta_A(B) := \begin{cases} 1 & \text{if } A = B \\ \\ 0 & \text{if } A \neq B, \end{cases} \tag{5.15}$$

and define the L-valued set $\chi_A : \mathfrak{V} \to L$ by

$$\chi_A(B) := \begin{cases} 1 & \text{if } B \in A \\ \\ 0 & \text{if } B \notin A, \end{cases} \tag{5.16}$$

Note: For any set A in the classical universe, we have that $\sup \delta_A = \{A\}$, and that $\sup \chi_A = A$.

Lemma 5.5. Let L be a complete orthomodular lattice, let $\mathcal{Q}_L$ be as in definition 5.2, and let f, g be L-valued sets. Then

$$[\![f \doteq g]\!] = \begin{cases} 1 & \text{if } \sup f = \sup g \\ \\ 0 & \text{if } \sup f \neq \sup g \end{cases} \tag{5.17}$$

and

$$[\![g \in f^*]\!] = \begin{cases} 1 & \text{if } \sup g \in \sup f \\ \\ 0 & \text{if } \sup g \notin \sup f \end{cases} \tag{5.18}$$

and also

$$[\![f^* = g^*]\!] = [\![f \doteq g]\!]. \tag{5.19}$$

Proof. First, using the definition of $f \doteq g$, we have

$$\begin{aligned} [\![f \doteq g]\!] &= [\![(\forall z)(f \in z \leftrightarrow g \in z)]\!] \\ &= \bigwedge_{h \in \mathbb{Q}_L} \big([\![f \in h]\!] \leftrightarrow [\![g \in h]\!] \big) \\ &= \bigwedge_{h \in \mathbb{Q}_L} \big(h(\sup f) \leftrightarrow h(\sup g) \big), \end{aligned}$$

and so we immediately see that if $\sup f = \sup g$, then $h(\sup f) = h(\sup g)$, so that $(h(\sup f) \leftrightarrow h(\sup g)) = 1$ for any $h \in \mathbb{Q}_L$ by lemma C.14, and so $[\![f \doteq g]\!] = 1$. On the other hand, if $\sup f \neq \sup g$, we have (from definition 5.3) that

$$[\![f \doteq g]\!] \leq \big(\delta_{\sup f}(\sup f) \leftrightarrow \delta_{\sup f}(\sup g) \big) = (1 \leftrightarrow 0) = 0,$$

which establishes equation 5.17.

Next, using the definition of $g \in f^*$, we have

$$[\![g \in f^*]\!] = [\![(\exists z)(z \doteq f \wedge g \in z)]\!] = \bigvee_{h \in \mathbb{Q}_L} \big([\![h \doteq f]\!] \wedge [\![g \in h]\!] \big),$$

but from equation 5.17, we have that $[\![h \doteq f]\!]$ is non-zero only when $\sup h = \sup f$, and so only these h's contribute to the join. Hence, we have

$$[\![g \in f^*]\!] \;=\; \bigvee_{\substack{h\in\mathbb{Q}_L \\ \sup h = \sup f}} \big(1 \wedge h(\sup g)\big) \;\geq\; \chi_{\sup f}(\sup g),$$

so by definition 5.3, we see that if $\sup g \in \sup f$ then $[\![g \in f^*]\!] = 1$. On the other hand, if $\sup g \notin \sup f$, then for any h with $\sup h = \sup f$, we must have $h(\sup g) = 0$, and so $[\![g \in f^*]\!] = 0$, which establishes equation 5.18.

Finally, by definition of '=', and equation 5.18, we have

$$\begin{aligned} [\![f^* = g^*]\!] &= [\![(\forall x)(x \in f^* \leftrightarrow x \in g^*]\!] \\ &= \bigwedge_{h\in\mathbb{Q}_L} [\![h \in f^*]\!] \leftrightarrow [\![h \in g^*]\!]. \end{aligned}$$

Now if $\sup f = \sup g$, by equation 5.18, we have that $([\![h \in f^*]\!] \leftrightarrow [\![h \in g^*]\!]) = 1$ for any $h \in \mathbb{Q}_L$, so that $[\![f^* = g^*]\!] = 1$. If, on the other hand, $\sup f \neq \sup g$, then there exists some classical set A such that $A \in \sup f$ but $A \notin \sup g$ (or vice versa). Either way, for this A, we have (by definition 5.3, equation 5.18, and lemma C.14) that

$$([\![\chi_A \in f^*]\!] \leftrightarrow [\![\chi_A \in g^*]\!]) = 0,$$

and so $[\![f^* = g^*]\!] = 0$. Hence, by equation 5.17, this establishes equation 5.19. $\square$

Lemma 5.6. Let L be a complete orthomodular lattice, let $\mathcal{Q}_L$ be as in definition 5.2, let $\psi(x)$ be an (extended) $\mathcal{L}_{set}$-wff, and let f be an L-valued set. Then

$$[\![\psi(f^*)]\!] = [\![\psi(\chi_{\sup f})]\!].$$

Proof. We prove this as a consequence of lemma 3.14, so we only need show that both $[\![f^* \in g]\!] = [\![\chi_{\sup f} \in g]\!]$ and $[\![g \in f^*]\!] = [\![g \in \chi_{\sup f}]\!]$ for any two quantum sets f and g. First, for any quantum set g, we have

$$[\![f^* \in g]\!] = [\![f \in g]\!] = g(\sup f) = g(\sup \chi_{\sup f}) = [\![\chi_{\sup f} \in g]\!]$$

by the definition of the expression $f^* \in g$ and $\chi_{\sup f}$. For the other case, we have that $[\![g \in f^*]\!] = \chi_{\sup f}(\sup g) = [\![g \in \chi_{\sup f}]\!]$ by lemma 5.5. □

Proposition 5.7. Let L be a complete orthomodular lattice. Then $\mathcal{Q}_L$ satisfies RZFC2-RZFC5 as well as RZFC10.

Proof. For RZFC2, we need

$$\mathcal{Q}_L \vDash (\forall x)\mathbf{T}\big((\exists z)(\forall u)(u \in z \leftrightarrow u^* = x^*\big),$$

i.e. (using lemma 3.8) it suffices to show that for any quantum set f, there exists some other quantum set g such that

$$[\![(\forall u)(u \in g \leftrightarrow u^* = f^*)]\!] = 1.$$

Taking $g = \delta_{\sup f}$, we see that

$$[\![(\forall u)(u \in \delta_{\sup f} \leftrightarrow u^* = f^*)]\!] = \bigwedge_{h \in \mathfrak{Q}_L} \big([\![\delta_{\sup f}(\sup h)]\!] \leftrightarrow [\![h^* = f^*]\!]\big) = 1,$$

where the last equality follows from equation 5.15 in definition 5.3 as well as equation 5.19 and equation 5.17 from lemma 5.5.

To show the RZFC3 holds, we need (for any wff ψ) that

$$\mathcal{Q}_L \vDash (\forall x)(\forall y)(\exists z)\mathbf{T}\big[(\forall u)(u \in z \leftrightarrow u \in x \wedge \psi(u^*, y))\big],$$

and so (again using lemma 3.8) it suffices to show for any quantum sets f, g that there exists some quantum set h such that

$$[\![(\forall u)(u \in h \leftrightarrow u \in f \wedge \psi(u^*, g))]\!] = 1,$$

i.e. that, for any $j \in \mathfrak{Q}_L$, that $h(\sup j) = f(\sup j) \wedge [\![\psi(j^*, g)]\!]$. But we can simply define, for any $A \in \mathfrak{V}$, that

$$h(A) \coloneqq f(A) \wedge [\![\psi(\chi_A, g)]\!].$$

By the (classical) schema of separation, $\sup h$ is indeed a set (contained in $\sup f$). The result the follows immediately from lemma 5.6.

Considering RZFC4, we must show

$$\mathcal{Q}_L \models (\forall x)(\exists y)\mathbf{T}\big[(\forall u)(u \in y \leftrightarrow (\exists z)(u \in z \wedge z \in x))\big],$$

and so it is sufficient to find, given an arbitrary quantum sets f, another quantum set g such that

$$\bigwedge_{h \in \mathfrak{Q}_L} \Big([\![h \in g]\!] \leftrightarrow \bigvee_{j \in \mathfrak{Q}_L} \big([\![h \in j]\!] \wedge [\![j \in f]\!] \big) \Big) = 1.$$

Equivalently, we need to satisfy, for every $h \in \mathfrak{Q}_L$, that

$$g(\sup h) = \bigvee_{j \in \mathfrak{Q}_L} \big(j(\sup h) \wedge f(\sup j) \big).$$

But clearly to satisfy the previous equation we can simply *define*, for any $A \in \mathfrak{V}$,

$$g(A) := \bigvee_{j \in \mathfrak{Q}_L} \big(j(A) \wedge f(\sup j) \big),$$

and since $g(A) \neq 0$ implies that $A \in \sup j \in \sup f$ for some $j \in \mathfrak{Q}_L$, we have that $\sup g$ is indeed a set, so that RZFC4 is satisfied.

Moving on to RZFC5, we need to show

$$\mathcal{Q}_L \models (\forall x)(\exists y)\mathbf{T}\big[(\forall u)(u \in y \leftrightarrow u^* \subseteq x)\big],$$

and so it suffices to demonstrate for any quantum set f, that there is another quantum set g such that

$$[\![(\forall u)(u \in g \leftrightarrow u^* \subseteq f)]\!] = 1.$$

Define g by (for all $A \in \mathfrak{V}$)

$$g(A) := [\![\chi_A \subseteq f]\!],$$

and so if $A \in \sup g$, then since

$$[\![\chi_A \subseteq f]\!] = \bigwedge_{h \in \mathfrak{Q}_L} \big(\chi_A(\sup h) \to f(\sup h)\big) = \bigwedge_{B \in A} f(B)$$

(using lemma C.14), we must have $A \subseteq \sup f$, so that $\sup g \subseteq \mathcal{P}(\sup f)$, which shows that $\sup g$ is a set. Then we have

$$\begin{aligned}
[\![(\forall u)(u \in g \leftrightarrow u^* \subseteq f)]\!] &= \bigwedge_{j \in \mathfrak{Q}_L} \big([\![j \in g]\!] \leftrightarrow [\![j^* \subseteq f]\!]\big) \\
&= \bigwedge_{j \in \mathfrak{Q}_L} \Big(g(\sup j) \leftrightarrow [\![\chi_{\sup j} \subseteq f]\!]\Big) = 1,
\end{aligned}$$

where we have used the definition of g and lemmas 5.6 and C.14 to obtain the final equality.

Considering now RZFC10, we must show that

$$\mathcal{Q}_L \vDash (\forall x)(\forall y)(\exists z)\mathbf{T}[(\forall u)(u \in z \leftrightarrow u \in x \vee u \in y)],$$

and so it suffices to show that for quantum sets f, g that there exists some quantum set h such that

$$[\![(\forall u)(u \in h \leftrightarrow u \in f \vee u \in g)]\!] = 1.$$

Define h by (for any $A \in \mathfrak{V}$)

$$h(A) := f(A) \vee g(A),$$

and since clearly $\sup h = \sup f \cup \sup g$, we see that $\sup h$ is indeed a set. Then we have

$$\begin{aligned}[\![(\forall u)(u \in z \leftrightarrow u \in x \vee u \in y)]\!] &= \bigwedge_{j \in \mathbb{Q}_L} \Big([\![j \in h]\!] \leftrightarrow \big([\![j \in f]\!] \vee [\![j \in g]\!]\big)\Big) \\ &= \bigwedge_{j \in \mathbb{Q}_L} \Big(h(\sup j) \leftrightarrow \big(f(\sup j) \vee g(\sup j)\big)\Big) = 1\end{aligned}$$

by the definition of h and lemma C.14. □

Since the other RZFC axioms use symbols defined under the assumption of RZFC1 and RZFC12, we must prove these axioms before continuing, which we only know to be true for certain OMLs, namely those satisfying the *relative center property* (see def. C.13).

Proposition 5.8. Let L be a complete irreducible orthomodular lattice satisfying the relative center property. Then $\mathcal{Q}_L$ satisfies RZFC1 and RZFC12.

Proof. First we consider RZFC12. Since L has the relative center property, we have that $[\![\mathbf{C}(\mathbf{T}(\psi))]\!] = 1$ by lemma 3.11 (the other hypothesis of that lemma is obviously true by definition of $\mathcal{Q}_L$). Also (using that $[\![\varnothing \in \{\varnothing\}]\!] = 1$, for instance), we have that $\mathbf{T}([\![\psi]\!]) \to [\![\psi]\!] = 1$ by lemma 3.10. Hence

$$\mathcal{Q}_L \vDash \mathbf{C}(\mathbf{T}(\psi)) \wedge [\mathbf{T}(\psi) \to \psi].$$

Next, considering RZFC1, we must show

$$\mathcal{Q}_L \vDash (\forall x)(\forall y)[\mathbf{T}(x = y) \to x \doteq y],$$

i.e. we must show (by lemma C.14) for any quantum sets f, g that

$$[\![\mathbf{T}(f = g)]\!] \leq [\![f \doteq g]\!].$$

By the above, $\mathcal{Q}_L \vDash$ RZFC12, so that $[\![\mathbf{T}(f = g)]\!] \in \{0, 1\}$ by proposition 3.9. The statement trivially holds whenever $[\![f = g]\!] \neq 1$ (since then $[\![\mathbf{T}(f = g)]\!] = 0$ by the aforementioned proposition), so consider the case where $[\![f = g]\!] = 1$. By the definition of '=' this means that $f(A) = g(A)$ for all $A \in \mathfrak{V}$, and in particular $\sup f = \sup g$. A simple computation then yields $[\![f \doteq g]\!] = 1$, so that RZFC1 indeed holds in $\mathcal{Q}_L$.

This follows trivially from equation 5.19 in lemma 5.5.

□

Now that we have proven the above axioms, so that we know the defined symbols do represent unique quantum sets, we can establish the following.

Lemma 5.9. Let L be a complete irreducible orthomodular lattice satisfying the relative center property, and let $\mathcal{Q}_L$ be as in definition 5.2. Then, for f, g any quantum sets, and any $A \in \mathfrak{V}$

1. $\varnothing(A) = 0$
2. $\{f\}(A) = \delta_{\sup f}(A)$
3. $(f \cap g)(A) = f(A) \wedge g(A)$
4. $(f \cup g)(A) = f(A) \vee g(A)$
5. $f(\sup f) = 0$
6. $(f \cup \{f\})(A) = \begin{cases} 1 & \text{if } A = \sup f \\ f(A) & \text{if } A \in \sup f \\ 0 & \text{otherwise.} \end{cases}$
7. $\mathcal{P}(f)(A) = \bigwedge_{B \in A} f(B)$
8. $\bigcup f(A) = \bigvee_{\substack{B \in \sup f \\ A \in B}} f(B)$

Proof. For number (1), the empty set is defined to satisfy $f \in \varnothing \leftrightarrow (f \neq f)$, but

$$[\![f = f]\!] = \bigwedge_{h \in \mathfrak{Q}_L} ([\![h \in f]\!] \leftrightarrow [\![h \in f]\!]) = 1,$$

so $[\![f \in \varnothing]\!] = \varnothing(\sup f) = 0$ for all $f \in \mathfrak{Q}_L$, hence $\varnothing(A) = 0$ for all $A \in \mathfrak{V}$.

Numbers (2-4) above follow directly from the proof of proposition 5.7 and the fact that any quantum set f is determined by the truth values (for every $g \in \mathfrak{Q}_L$) $[\![g \in f]\!]$.

For number (5), we note that by the classical axiom of regularity no set may be a member of itself (proposition 5.2), and so (5) follows from the fact that $\sup f \notin \sup f$.

For number (6) we note that the cases are mutually exclusive by (5). We then use (4) and (2) to compute

$$f \cup \{f\}(A) = f(A) \vee \{f\}(A) = f(A) \vee \delta_{\sup f}(A),$$

which gives the above result.

For number (7) and (8), since the sets which are given by RZFC4 and RZFC5 are unique, from the proof of proposition 5.2, we know that

$$\mathcal{P}(f)(A) = [\![\chi_A \subseteq f]\!] = \bigwedge_{h \in \mathfrak{Q}_L} \big(\chi_A(\sup h) \to f(\sup h)\big) = \bigwedge_{B \in A} f(B),$$

as well as that

$$\bigcup f(A) = \bigvee_{j \in \mathfrak{Q}_L} \big(j(A) \wedge f(\sup j)\big) = \bigvee_{B \in \mathfrak{V}} \big(\chi_B(A) \wedge f(B)\big) = \bigvee_{\substack{B \in \sup f \\ A \in B}} f(B).$$

$\square$

<u>Note</u>: From this can immediately see that at least one intuitive property of the classical universe is preserved by our quantum sets, namely that for any quantum set f, we have that $[\![f \notin f]\!] = 1$ by (5) above.

We will now deliver on a promise to show that there are quantum sets f and g such that $\bigcup\big(\{f\} \cup \{g\}\big) \neq f \cup g$.

Example 5.10. From the above we know that for any $A \in \mathfrak{V}$ that $(f \cup g)(A) = f(A) \vee g(A)$. However

$$\bigcup\big(\{f\} \cup \{g\}\big)(A) = \bigvee_{\substack{B \in \sup(\{f\} \cup \{g\}) \\ A \in B}} \big(\{f\} \cup \{g\}\big)(B)$$

$$= \bigvee_{\substack{B \in \{\sup f, \sup g\} \\ A \in B}} \big(\delta_{\sup f}(B) \vee \delta_{\sup g}(B)\big)$$

$$= \begin{cases} 1 & \text{if } A \in \sup f \cup \sup g \\ 0 & \text{otherwise.} \end{cases}$$

where we have used that

$$\big(\{f\} \cup \{g\}\big)(A) = \delta_{\sup f}(A) \vee \delta_{\sup g}(A),$$

so that $\sup(\{f\} \cup \{g\}) = \{\sup f, \sup g\}$. Hence, we clearly have $\bigcup\big(\{f\} \cup \{g\}\big) \neq f \cup g$ for any quantum sets f, g such that there is some $A \in \mathfrak{V}$ with $f(A) \vee g(A) \neq 1$.

There is an another possible approach to reducing the ZFC axioms in which some of the remaining axioms can be satisfied for more general OMLs — we discuss this in more detail at the end of this section.

Lemma 5.11. Let L be a complete OML, let $\mathcal{Q}_L$ be as in definition 5.2, and let ω be the first infinite ordinal. Also, for any $f \in \mathfrak{Q}_L$ define (for any $A \in \mathfrak{V}$)

$$f^+(A) := \begin{cases} 1 & \text{if } A = \sup f \\ f(A) & \text{if } A \in \sup f \\ 0 & \text{otherwise,} \end{cases} \tag{5.20}$$

and define $g_\varnothing \in \mathfrak{Q}_L$ by $g_\varnothing(A) := 0$ for every $A \in \mathfrak{V}$. Further, for any quantum sets f, g assume[1] that $f \cap g$ is the quantum set in $\mathfrak{Q}_L$ which satisfies (3) in lemma 5.9.

Then we have

1. $[\![g_\varnothing \in \chi_\omega]\!] \wedge \bigwedge_{h \in \mathfrak{Q}_L} [\![(h \in \chi_\omega \to h^+ \in \chi_\omega)]\!] = 1$
2. If $L \neq \{0, 1\}$, then

$$\bigwedge_{h \in \mathfrak{Q}_L} \Big([\![h \neq g_\varnothing]\!] \to \bigvee_{j \in \mathfrak{Q}_L} \big[[\![j \in h]\!] \wedge [\![j \cap h = g_\varnothing]\!] \big] \Big) = 1$$

3. If $L \neq \{0, 1\}$, then for any quantum set f and any extended $\mathcal{L}_{set}$-wff $\psi(s, t)$ (with $s, t \in \mathcal{B}_V$)

$$[\![(\forall x)(x \in f \to x \neq g_\varnothing) \to (\exists s)(\forall t)(t \in f \to \psi(s, t))]\!] = 1.$$

---

[1] If $f \cap g$ is defined as a consequence of the RZFC axioms (such as whenever L satisfies the relative center property), then the assumption is not necessary. Otherwise we can simply take this assumption as the *definition* of $f \cap g$ for purposes of this lemma.

Proof. First we consider (1). Note that $\varnothing \in \omega$ and for any $\beta \in \omega$, we have $\beta + 1 = \beta \cup \{\beta\} \in \omega$. Also, for any quantum set h, we have that $\sup h^{+} = \sup h \cup \{\sup h\}$. Then, using that $\sup g_{\varnothing} = \varnothing$, we compute

$$\llbracket g_{\varnothing} \in \chi_{\omega} \rrbracket \wedge \bigwedge_{h \in \mathfrak{Q}_L} \llbracket (h \in \chi_{\omega} \to h^{+} \in \chi_{\omega}) \rrbracket = \chi_{\omega}(\varnothing) \wedge \bigwedge_{h \in \mathfrak{Q}_L} [\chi_{\omega}(\sup h) \to \chi_{\omega}(\sup h^{+})]$$

$$= 1 \wedge \bigwedge_{A \in \mathfrak{V}} [\chi_{\omega}(A) \to \chi_{\omega}(A \cup \{A\})] = 1,$$

which establishes (1).

Considering (2), for any quantum sets j and h we have (using lemma C.14)

$$\llbracket h \neq g_{\varnothing} \rrbracket = \neg \bigwedge_{k \in \mathfrak{Q}_L} \big(h(\sup k) \leftrightarrow g_{\varnothing}(\sup k) \big) = \bigvee_{A \in \mathfrak{V}} \neg (h(A) \leftrightarrow 0) = \bigvee_{A \in \mathfrak{V}} h(A),$$

as well as

$$\llbracket j \cap h = g_{\varnothing} \rrbracket = \bigwedge_{k \in \mathfrak{Q}_L} \big([j(\sup k) \wedge h(\sup k)] \leftrightarrow g_{\varnothing}(\sup k) \big) = \bigwedge_{B \in \mathfrak{V}} \big(\neg j(B) \vee \neg h(B) \big).$$

Using these then yields

$$\bigwedge_{h \in \mathfrak{Q}_L} \Big(\llbracket h \neq g_{\varnothing} \rrbracket \to \bigvee_{j \in \mathfrak{Q}_L} \big[\llbracket j \in h \rrbracket \wedge \llbracket j \cap h = g_{\varnothing} \rrbracket \big] \Big)$$

$$= \bigwedge_{h \in \mathfrak{Q}_L} \Big(\big[\bigvee_{A \in \mathfrak{V}} h(A) \big] \to \bigvee_{j \in \mathfrak{Q}_L} \big[h(\sup j) \wedge \bigwedge_{B \in \mathfrak{V}} \big(\neg j(B) \vee \neg h(B) \big) \big] \Big).$$

By lemma C.14, it suffices to show that for any quantum set h, we have

$$\bigvee_{A\in\mathfrak{V}} h(A) \leq \bigvee_{j\in\mathbf{Q}_L} \left[h(\sup j) \wedge \bigwedge_{B\in\mathfrak{V}} \big(\neg j(B) \vee \neg h(B)\big)\right],$$

and hence it is sufficient to show that, for any $A \in \mathfrak{V}$, that

$$h(A) \leq \bigvee_{\substack{j\in\mathbf{Q}_L \\ \sup j = A}} \left[h(\sup j) \wedge \bigwedge_{B\in\sup h\cap\sup j} \big(\neg j(B) \vee \neg h(B)\big)\right]. \tag{5.21}$$

Now we proceed by cases. First, the case $h(A) = 0$ is trivial, 0 is the bottom element of L (this is always the case if L is trivial). Next, if $A = \varnothing$, then the inequality (equation 5.21) is satisfied since any quantum set j such that $\sup j = \varnothing$ also satisfies $\sup j \cap \sup h = \varnothing$, so that

$$\bigvee_{\substack{j\in\mathbf{Q}_L \\ \sup j = \varnothing}} \left[h(\sup j) \wedge \bigwedge_{B\in\sup h\cap\sup j} \big(\neg j(B) \vee \neg h(B)\big)\right] = h(\varnothing).$$

For the third case, assume $A \neq \varnothing$ and also that $h(A) \neq 0$ and $h(A) \neq 1$. Then define a quantum set j_0 by $j_0(B) := \neg h(A)$ for every $B \in \sup h \cap A$, $j_0(B) := 1$ for every $B \in A \setminus \sup h$, and $j_0(B) := 0$ otherwise. Then $\sup j_0 = A$ since $h(A) \neq 1$. Then the inequality (equation 5.21) is satisfied, since we have

$$\bigwedge_{B\in\sup h\cap\sup j_0} \big(\neg j(B) \vee \neg h(B)\big) = \bigwedge_{B\in\sup h\cap A} \big(h(A) \vee \neg h(B)\big) \geq h(A).$$

For the final case, we consider $A \neq \varnothing$, and $h(A) = 1$. Since $L \neq \{0,1\}$ (and L non-trivial), there exists some $a \in L$ with $a \notin \{0,1\}$. Define $j_1(B) := a$, and $j_2(B) := \neg a$ for all $B \in \sup h \cap A$, and $j_1(B) := j_2(B) := 1$ for all $B \in A \backslash \sup h$, and $j_1(B) := j_2(B) = 0$ otherwise, so that $\sup j_1 = \sup j_2 = A$. Then the inequality (equation 5.21) is satisfied, since

$$\bigwedge_{B \in \sup h \cap \sup j_1} \big(\neg j_1(B) \vee \neg h(B)\big) = \bigwedge_{B \in \sup h \cap A} \big(\neg a \vee h(B)\big),$$

and

$$\bigwedge_{B \in \sup h \cap \sup j_2} \big(\neg j_2(B) \vee \neg h(B)\big) = \bigwedge_{B \in \sup h \cap A} \big(a \vee h(B)\big),$$

so that

$$\begin{aligned}\bigvee_{\substack{j \in \mathbb{Q}_L \\ \sup j = A}} & \Big[h(\sup j) \wedge \bigwedge_{B \in \sup h \cap \sup j} \big(\neg j(B) \vee \neg h(B)\big)\Big] \\ & \geq \Big[\bigwedge_{B \in \sup h \cap A} \big(\neg a \vee h(B)\big)\Big] \vee \Big[\bigwedge_{B \in \sup h \cap A} \big(a \vee h(B)\big)\Big] \geq \neg a \vee a = 1\end{aligned}$$

which establishes (2).

Finally, we consider (3) above. First, by lemma C.14, it suffices to show

$$[\![(\forall x)(x \in f \to x \neq g_\varnothing)]\!] \leq [\![(\exists s)(\forall t)(t \in f \to \psi(s,t))]\!]. \tag{5.22}$$

Computing the LHS of the inequality (equation 5.22) yields

$$\begin{aligned}
[\![(\forall x)(x \in f \to x \neq g_\varnothing)]\!] &= \bigwedge_{h \in \mathfrak{Q}_L} \big([\![h \in f]\!] \to [\![h \neq g_\varnothing]\!]\big) \\
&= \bigwedge_{h \in \mathfrak{Q}_L} \big(f(\sup h) \to \big[\bigvee_{A \in \mathfrak{V}} \neg h(A)\big]\big) \\
&= \bigwedge_{B \in \mathfrak{V}} \bigwedge_{\substack{h \in \mathfrak{Q}_L \\ \sup h = B}} \big(f(B) \to \big[\bigvee_{A \in B} \neg h(A)\big]\big).
\end{aligned}$$

Then considering the RHS of inequality (equation 5.22) we have

$$\begin{aligned}
[\![(\exists s)(\forall t)(t \in f \to \psi(s,t))]\!] &= \bigvee_{h \in \mathfrak{Q}_L} \Big[\bigwedge_{j \in \mathfrak{Q}_L} \big([\![j \in f]\!] \to [\![\psi(h,j)]\!]\big)\Big] \\
&= \bigvee_{h \in \mathfrak{Q}_L} \Big[\bigwedge_{j \in \mathfrak{Q}_L} \Big(\neg f(\sup j) \vee \big(f(\sup j) \wedge [\![\psi(h,j)]\!]\big)\Big)\Big] \\
&\geq \bigwedge_{j \in \mathfrak{Q}_L} \neg f(\sup j) = \bigwedge_{B \in \mathfrak{V}} \neg f(B),
\end{aligned}$$

and so it suffices to show that, for any $B \in \mathfrak{V}$, that

$$\bigwedge_{\substack{h \in \mathfrak{Q}_L \\ \sup h = B}} \big(f(B) \to \big[\bigvee_{A \in B} \neg h(A)\big]\big) \leq \neg f(B). \tag{5.23}$$

If $f(B)=0$, then $\neg f(B)=1$, and so the inequality (equation 5.23) is automatically satisfied. If $f(B)\neq 0$, define $h_0(A) := f(B)$ if $A \in B$, and $h_0(A)=0$ otherwise, so $\sup h_0 = B$. Then

$$\begin{aligned}\bigwedge_{\substack{h\in\mathfrak{Q}_L\\ \sup h=B}} \Big(f(B) \to \Big[\bigvee_{A\in B} \neg h(A)\Big]\Big) &\leq \Big(f(B) \to \Big[\bigvee_{A\in B} \neg h_0(A)\Big]\Big)\\ &= \big(f(B) \to \neg f(B)\big) = \neg f(B)\end{aligned}$$

by lemma C.14, and so (3) is established.

$\square$

From this technical lemma, we obtain as a direct corollary

Proposition 5.12. Let L be a complete irreducible OML which satisfies the relative center property and let $\mathcal{Q}_L$ be as in definition 5.2. Then $\mathcal{Q}_L$ satisfies RZFC6, RZFC8, and RZFC9.

Proof. First, we see that $\mathcal{Q}_L \vDash$ RZFC6 iff

$$[\![(\exists x)\mathbf{T}(\varnothing \in x \wedge (\forall y)(y \in x \to y \cup \{y\} \in x))]\!] = 1,$$

but

$$\begin{aligned}[\![(\exists x)(\varnothing \in x \wedge (\forall y)(y \in x \to y \cup \{y\} \in x))]\!] &= \bigvee_{j\in\mathfrak{Q}_L} \mathbf{T}\Big([\![g_\varnothing \in j]\!] \wedge \bigwedge_{h\in\mathfrak{Q}_L} [\![(h \in j \to h' \in j)\Big)\\ &\geq [\![g_\varnothing \in \chi_\omega]\!] \wedge \bigwedge_{h\in\mathfrak{Q}_L} [\![(h \in \chi_\omega \to h' \in \chi_\omega)]\!]\\ &= 1\end{aligned}$$

where in the first equality we have used that $g_\varnothing = \varnothing$ (as quantum sets) as well as that $h' = h \cup \{h\}$ (by lemma 5.9), and the final equality follows immediately from (1) in lemma 5.11.

Now, if $L = \{0, 1\}$, then our model is just the classical universe, and so RZFC8 and RZFC9 follow by proposition 5.4, and so we need only consider the case in which $L \neq \{0, 1\}$. Also, the axioms trivially hold if L is trivial, so we assume L non-trivial as well.

To show that $\mathcal{Q}_L \vDash$ RZFC8, we need to show

$$\llbracket (\forall x)(x \neq \varnothing \to (\exists y)(y \in x \wedge y \cap x = \varnothing)) \rrbracket = 1,$$

and so we compute (using that $g_\varnothing = \varnothing$)

$$\begin{aligned} &\llbracket (\forall x)(x \neq \varnothing \to (\exists y)(y \in x \wedge y \cap x = \varnothing)) \rrbracket \\ &\qquad = \bigwedge_{h \in \mathbb{Q}_L} \Big(\llbracket h \neq g_\varnothing \rrbracket \to \bigvee_{j \in \mathbb{Q}_L} \big[\llbracket j \in h \rrbracket \wedge \llbracket j \cap h = g_\varnothing \rrbracket \big] \Big) = 1 \end{aligned}$$

where the last equality follows from (2) in lemma 5.11.

Finally, we see that $\mathcal{Q}_L \vDash$ RZFC9 iff

$$\begin{aligned} &\llbracket (\forall z)\big([(\forall x)(\forall y)(x \in z \to x \neq \varnothing) \wedge (x \in z \wedge y \in z \wedge x \neq y \to x \cap y = \varnothing)] \\ &\quad \to (\exists s)\mathbf{T}\big[(\forall t)[t \in z \to (\exists u)(s \cap t = \{u\})]\big]\big) \rrbracket = 1, \end{aligned}$$

so it suffices to show (by lemmas C.14 and 3.8, and since $\varnothing = g_\varnothing$), for any quantum set f, that

$$[\![(\forall x)(\forall y)(x \in f \to x \neq g_\varnothing) \wedge (x \in f \wedge y \in f \wedge x \neq y \to x \cap y = g_\varnothing)]\!]$$
$$\leq [\![(\exists s)(\forall t)[t \in f \to (\exists u)(s \cap t = \{u\})]]\!]. \tag{5.24}$$

However, trivially we have that

$$[\![(\forall x)(\forall y)(x \in f \to x \neq g_\varnothing) \wedge (x \in f \wedge y \in f \wedge x \neq y \to x \cap y = \varnothing)]\!]$$
$$\leq [\![(\forall x)(\forall y)(x \in f \to x \neq g_\varnothing)]\!]. \tag{5.25}$$

Define the wff $\psi(s,t) := (\exists u)(s \cap t = \{u\})$, and then by lemma 5.11, we have

$$[\![(\forall x)(x \in f \to x \neq g_\varnothing) \to (\exists s)(\forall t)(t \in f \to \psi(s,t))]\!] = 1,$$

which by lemma C.14 is true iff

$$[\![(\forall x)(x \in f \to x \neq g_\varnothing)]\!] \leq [\![(\exists s)(\forall t)(t \in f \to \psi(s,t))]\!]. \tag{5.26}$$

Combining the inequalities (equation 5.25) and (equation 5.26) yields the inequality (equation 5.24), which establishes RZFC9. □

<u>Note</u>: Only in the "classical case" ($L = \{0,1\}$) do we need to use that the classical axiom of regularity (resp. choice) holds in order to establish the reduced regularity (resp. choice)

axiom. When $L \neq \{0,1\}$, the proof given actually doesn't use the fact that the classical universe satisfies the regularity (resp. choice) axiom at all.

The remaining two reduced ZFC axioms are slightly more subtle, and we will need different criteria than the relative center property to prove them.

Lemma 5.13. Let L be a complete OML, and let $\mathcal{Q}_L$ be as in definition 5.2, and let G be the group of continuous (ortholattice) automorphisms of L. Further let $\psi(x_1,\ldots,x_n)$ be an $\mathcal{L}_{set}$-wff and let $f, f_1, \ldots, f_n$ be quantum sets. Then for any $\mu \in G$ we have

1. $\mu(0) = 0$ and $\mu(1) = 1$.
2. $\sup \mu \circ f = \sup f$
3. $\mathfrak{Q}_L = \{\mu \circ g \ : \ g \in \mathfrak{Q}_L\}$
4. $\mu\big([\![\psi(f_1,\ldots,f_n)]\!]\big) = [\![\psi(\mu \circ f_1,\ldots,\mu \circ f_n)]\!]$.

Proof. First, since μ is an ortholattice automorphism on L, we have that $\mu(0) = 0$ and $\mu(1) = 1$ by definition.

Next, since $\mu(0) = 0$ by (1), we know that for any $A \in \mathfrak{V}$, $\mu \circ f(A) = 0$ iff $f(A) = 0$, and hence $\sup \mu \circ f = \sup f$.

For (3), note that if g is a quantum set, then so is $\mu \circ g$. Also, this trivially means that $\mu^{-1} \circ g$ is a quantum set. Then $g = \mu \circ (\mu^{-1} \circ g)$, establishing the desired equality.

We then prove (4) above by induction on the construction of evaluated wffs. For the base case, we see that (for any quantum sets g, h)

$$\mu\big([\![g \in h]\!]\big) = \mu(h(\sup g)) = \mu \circ h(\sup \mu \circ g) = [\![\mu \circ g \in \mu \circ h]\!],$$

where we have used (2) above. For the inductive steps, consider evaluated wffs $\psi(g_1, \ldots, g_m)$ and $\xi(g_1, \ldots, g_m)$ with quantum sets $g_1, \ldots, g_m$ for which (4) holds. Then we have

$$\mu\big([\![\neg\psi(g_1, \ldots, g_m)]\!]\big) = \neg\mu\big([\![\psi(g_1, \ldots, g_m)]\!]\big) = [\![\neg\psi(\mu \circ g_1, \ldots, \mu \circ g_m)]\!],$$

as well as

$$\begin{aligned} \mu\big([\![\psi(g_1, \ldots, g_n)]\!] \wedge [\![\xi(g_1, \ldots, g_m)\big) &= [\![\psi(\mu \circ g_1, \ldots, \mu \circ g_m) \wedge \xi(\mu \circ g_1, \ldots, \mu \circ g_m)]\!] \\ &= [\![(\psi \wedge \chi)(\mu \circ g_1, \ldots, \mu \circ g_m)]\!], \end{aligned}$$

and finally

$$\begin{aligned} \mu\big([\![(\forall x)\psi(x, g_2, \ldots, g_m)]\!]\big) &= \bigwedge_{h \in \mathbb{Q}_L} \mu\big([\![\psi(h, g_2, \ldots, g_m)]\!]\big) \\ &= \bigwedge_{h \in \mathbb{Q}_L} [\![\psi(\mu \circ h, \mu \circ g_2, \ldots, \mu \circ g_m)]\!] \\ &= \bigwedge_{h \in \mathbb{Q}_L} [\![\psi(h, \mu \circ g_2, \ldots, \mu \circ g_m)]\!] \\ &= [\![(\forall x)\psi(x, \mu \circ g_2, \ldots, \mu \circ g_m)]\!], \end{aligned}$$

where the second to last equality follows by (3) above. □

Lemma 5.14. Let L be a complete OML which is rotatable, and let $\mathcal{Q}_L$ be as in definition 5.2. Then for any $A, B \in \mathfrak{V}$, and any $\mathcal{L}_{set}$-wff $\psi(x, y)$, we have $[\![\psi(\chi_A, \chi_B)]\!] \in \{0, 1\}$.

Proof. Since 0 and 1 are the only fixed points of the group of continuous automorphisms on L (denote this group G), it suffices to show that $[\![\psi(\chi_A, \chi_B)]\!]$ is a fixed point of G. Define $a := [\![\psi(\chi_A, \chi_B)]\!]$, so by lemma 5.13 we have

$$\mu(a) = \mu\big([\![\psi(\chi_A, \chi_B)]\!]\big) = [\![\psi(\mu \circ \chi_A, \mu \circ \chi_B)]\!] = [\![\psi(\chi_A, \chi_B)]\!] = a,$$

where we have used the following fact: for any $Y \in \mathfrak{V}$ and $\mu \in G$, we have $\mu \circ \chi_Y = \chi_Y$. To see this, note that for any $Y, Z \in \mathfrak{V}$, we have that $\chi_Y(Z) \in \{0, 1\}$ by definition, and so $\mu \circ \chi_Y(Z) = \chi_Y(Z)$ for any $Z \in \mathfrak{V}$ by lemma 5.13. Hence $\mu \circ \chi_Y = \chi_Y$.. □

Proposition 5.15. Let L be a complete OML which is rotatable, and let $\mathcal{Q}_L$ be as in definition 5.2. Then $\mathcal{Q}_L$ satisfies RZFC7.

Proof. We need to show

$$\begin{aligned}\mathcal{Q}_L \vDash &[(\forall x)(\forall y)(\forall z)(\psi(x, y) \wedge \psi(x, z) \to y = z)]\\ &\qquad \to (\forall x)(\exists z)\mathbf{T}\big[(\forall u)[u \in z \leftrightarrow (\exists y)(y \in x \wedge \psi(y^*, u^*))\big].\end{aligned}$$

It suffices to assume that

$$[\![(\forall x)(\forall y)(\forall z)(\psi(x,y) \wedge \psi(x,z) \rightarrow y = z)]\!] \neq 0,$$

and then prove that, for any quantum set f, there is some other quantum set g such that

$$[\![(\forall u)\big(u \in g \leftrightarrow (\exists y)[y \in f \wedge \psi(y,u^*)]\big)]\!] = 1,$$

i.e. that for any quantum set h

$$g(\sup h) = \bigvee_{j \in \mathfrak{Q}_L} \big(f(\sup j) \wedge [\![\psi(j^*, h^*)]\!]\big).$$

But this will be automatically be satisfied (by lemma 5.6) if we define, for any $A \in \mathfrak{V}$,

$$g(A) := \bigvee_{B \in \mathfrak{V}} \big(f(B) \wedge [\![\psi(\chi_B, \chi_A)]\!]\big) = \bigvee_{j \in \mathfrak{Q}_L} \big(f(\sup j) \wedge [\![\psi(\chi_{\sup j}, \chi_A)]\!]\big),$$

and so we only need show that $\sup g$ is a set.

Now, we see immediately that, for any $A \in \mathfrak{V}$, that $g(A) \neq 0$ iff there is some $B \in \mathfrak{V}$ such that $f(B) \wedge [\![\psi(\chi_B, \chi_A)]\!] \neq 0$. But by lemma 5.14, $[\![\psi(\chi_B, \chi_A)]\!] \in \{0, 1\}$, and so in order for $g(A) \neq 0$ there must exist some $B \in \mathfrak{V}$ such that $B \in \sup f$ and $[\![\psi(\chi_B, \chi_A)]\!] = 1$.

We will show that, for any given A, the class of all sets B satisfying both $B \in \sup f$ as well as $[\![\psi(\chi_B, \chi_A)]\!] = 1$ is indeed a set using the *classical* replacement axiom with regard to the statement $\Psi(B, A)$ which states[1] that '$[\![\psi(\chi_B, \chi_A)]\!] = 1$'.

To show that Ψ satisfies the hypothesis of *classical* replacement, we assume that, for generic $X, Y, Z \in \mathfrak{V}$, that $\Psi(X, Y)$ and $\Psi(X, Z)$ are true, i.e. that both $[\![\psi(\chi_X, \chi_Y)]\!] = 1$ and $[\![\psi(\chi_X, \chi_Z)]\!] = 1$. Now recall our assumption that

$$[\![(\forall x)(\forall y)(\forall z)(\psi(x, y) \wedge \psi(x, z) \to y = z)]\!] \neq 0,$$

which implies that,

$$[\![(\psi(\chi_X, \chi_Y)]\!] \wedge [\![\psi(\chi_X, \chi_Z)]\!] \to [\![Y = Z]\!] \neq 0,$$

but since $\Psi(X, Y)$ and $\Psi(X, Z)$ are true, this means (using lemma C.14 that

$$\big(1 \to [\![\chi_Y = \chi_Z]\!]\big) = [\![\chi_Y = \chi_Z]\!] \neq 0,$$

but clearly $[\![\chi_Y = \chi_Z]\!] \in \{0, 1\}$, and so we must have $[\![\chi_Y = \chi_Z]\!] = 1$, so that $\chi_Y = \chi_Z$ as quantum sets, which means that $Y = Z$ as classical sets.

---

[1] We are being rather informal here, since we have not used a formal language when using the classical set theory by which we defined our model $\mathcal{Q}_L$. However, it would be simple (although tedious) to write every statement in dealing with the construction of $\mathcal{Q}_L$ formally, in which case we would arrive at the statement '$[\![\psi(\chi_B, \chi_A)]\!] = 1$' as a *formal wff* in classical set theory, and it is this formal wff to which $\Psi(B, A)$ refers.

Hence $\Psi(X,Y)$ satisfies the hypothesis of classical replacement, and so for any $Z \in \mathfrak{V}$, there exists some $S \in \mathfrak{V}$ such that for any $T \in \mathfrak{V}$, $T \in S$ iff there exists some $U \in \mathfrak{V}$ such that $U \in Z$ and $\Psi(U,T)$. Taking $Z = \sup f$, we have that $T \in S$ iff there exists a $U \in \mathfrak{V}$ with $U \in \sup f$ and $[\![\psi(\chi_U, \chi_T)]\!] = 1$, which is to say that $T \in S$ iff $T \in \sup g$, so that $\sup g = S$, which is indeed a classical set.

□

Proposition 5.16. Let L be a complete atomic OML which satisfies the exchange axiom (EA), and let $\mathcal{Q}_L$ be as in definition 5.2. Then $\mathcal{Q}_L$ satisfies RZFC11.

Proof. If $L = \{0,1\}$, then RZFC11 holds by proposition 5.4, so we can assume that $L \neq \{0,1\}$, and in particular the height of L is greater than or equal to 3. We need to show

$$\mathcal{Q} \vDash (\forall x)[(\forall u)\ \mathbf{C}(u \in x) \to x = x^*].$$

By lemma C.14, it suffices to show, for any quantum set f, that

$$\bigwedge_{h \in \mathfrak{Q}_L} [\![\mathbf{C}(h \in f)]\!] \leq [\![f = f^*]\!]. \tag{5.27}$$

Now, by definition of '**C**', we have

$$\bigwedge_{h\in\mathfrak{Q}_L} [\![\mathbf{C}(h\in f)]\!] = \bigwedge_{h\in\mathfrak{Q}_L}\ \bigwedge_{j,k\in\mathfrak{Q}_L} \big(\varphi_{[\![j\in k]\!]}([\![h\in f]\!]) \to [\![h\in f]\!]\big)$$

$$= \bigwedge_{A\in\mathfrak{V}} \Big[\bigwedge_{a\in L} \big(\varphi_a(f(A)) \to f(A)\big)\Big]$$

$$= \bigwedge_{A\in\sup f} \Big[\bigwedge_{a\in L} \big(\varphi_a(f(A)) \to f(A)\big)\Big],$$

where for the last equality we have used that $\varphi_x(0) = 0$ by lemma C.13, while we also have

$$[\![f = f^*]\!] = \bigwedge_{h\in\mathfrak{Q}_L} [\![h\in f \leftrightarrow h\in f^*]\!] = \bigwedge_{A\in\sup f} f(A).$$

by lemma 5.5. Since for $f = \varnothing$ this gives $[\![f = f^*]\!] = 1$, in order to establish ZFC11 it will suffice to show, for any $b\in L$ with $b\neq 0$, that

$$\Big[\bigwedge_{a\in L} \varphi_a(b) \to b\Big] \leq b.$$

But this follows directly from lemma C.20, and so RZFC11 is established. □

Putting together everything above establishes

Theorem 6. Let L be a complete, irreducible, atomic, rotatable OML which satisfies EA and the relative center property. Then $\mathcal{Q}_L$ is a model of RZFC.

Proof. This simply collects the results of propositions 5.7, 5.8, 5.12, 5.16, and 5.15. □

Corollary 5.17. Let L be the projection lattice of a separable complex Hilbert space $\mathcal{H}$. Then $\mathcal{Q}_L$ is a model of RZFC1-12.

Proof. This follows immediately from the previous theorem along with propositions C.23 and C.24 as well as lemma C.21. □

An Alternate Approach to Reducing ZFC

As we mentioned in the previous section, some of the ZFC axioms can be made to hold more generally than in the context of projection lattices of Hilbert space, however we would need to change our reduction of these axioms. The strategy for achieving this more general reduction is to eliminate the '**T** operator' from our RZFC formulas, and introduce new function symbols for the empty set, singleton, etc., since statements involving '$\exists$' would no longer guarantee the existence of any particular set. By making this change, we can satisfy all of the modified RZFC axioms[1] (with the exception of RZFC3 and RZFC7), but one technical hurdle remains.

This hurdle involves the axioms of separation (RZFC3) and replacement (RZFC11). For separation, we need to add a function symbol *for every wff* ψ in order to utilize set-builder notation in all its generality. However, every time we add another function symbol to our language, we then have more wffs in our language, which requires the addition of yet more function symbols, and it is not clear that this process would ever stabilize giving us a well-defined language.

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[1]We also would need a replacement for RZFC1, a natural candidate which is easily shown to do the job is $(\forall x)(\forall y)[x^* = y^* \rightarrow (\forall z)(x \in z \leftrightarrow y \in z)]$.

For the axiom of replacement, the situation is even worse. In this case, we need to add a function symbol not for *every* wff ψ, but only for ψ satisfying the hypothesis of the implication in the axiom. Since in general the truth value of a given wff ψ may be hard to determine, this leaves us in a position where in order to even know what symbols are in our language, we need to solve a rather difficult problem.

These technical hurdles are the reason for the approach we have taken to reducing the ZFC axioms. However, if the above hurdles were resolved, then we may be able to say that $\mathcal{Q}_L$ is a model of quantum set theory for a much broader class of complete OMLs L than what has been proven above.

5.3 Comparison to Takeuti's Quantum Set Theory

Takeuti's reduction of ZFC, along with his model of these reduced axioms, is presented in (42). Unfortunately, due to slight differences in his framework, as well as the many technical details involved in his reduction of the ZFC axioms, we will not be able to give a full description of the quantum set theory he develops, and will content ourselves with giving a qualitative description of the main differences between our construction in the previous section and his quantum set theory.

First, Takeuti's reduction of the ZFC axioms is more drastic in a number of ways. Most obviously, he expands his language by introducing not only a new unary function,[1] similar to our $\cdot^*$, but also new predicates[2] (related to the commutation of certain truth values).[3]

Also, most of his reduced axioms take the form $\psi(x_1, x_2, \ldots) \to \chi(x_1, x_2, \ldots)$ (for χ some ZFC axiom, and ψ a wff), where for many "quantum sets" $f, g, \ldots$ in Takeuti's quantum model we frequently have $[\![\psi(f, g, \ldots)]\!] = 0$. Hence, for many tuples of Takeuti's quantum sets, his reduced axioms *do not actually force the relevant ZFC axiom to hold!* This is in marked difference to our quantum sets, in which we place no particular constraints on which tuples of quantum sets a given axiom holds.[4]

The model Takeuti constructs to satisfy his reduced axioms is also markedly different than the models constructed in the previous section in other ways as well. Speaking intuitively, Takeuti's model gives a much "bigger" quantum set theory. As seen above, in our models a quantum set is just a map from some set in the classical universe of sets $\mathfrak{V}$ into a given projection lattice L. Takeuti's construction of the quantum sets, on the other hand, proceeds

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[1]For those referring to his paper, the symbol he uses is '$\check{\cdot}$', i.e. applying this function to a variable x would yield $\check{x}$.

[2]Again for those referring to his original paper, the symbols for these predicates are $\underline{\vee}$ and $\perp\!\!\!\perp$. Note that Takeuti uses the symbol $\underline{\vee}$ to represent multiple predicates of different arity.

[3]Note that unlike our **C** and **T** operators as well as our "function" '$\cdot^*$' which are merely notational shorthand, these predicates and this function symbol must be added by hand to the language $\mathcal{L}_{set}$ in Takeuti's framework.

[4]That is, no constraints beyond the conditions on L which are needed to make $\mathcal{Q}_L$ a model of RZFC.

via transfinite recursion starting with the set of all maps from the empty set into L, but then at each successive ordinal allows maps from *any quantum set already constructed* into L. Since his sets contain all of the usual "classical sets" as a subclass, we see that our class of quantum sets $\mathfrak{Q}_L$ forms a "small" subclass of Takeuti's sets,[1] although the truth valuations of our quantum set structures are entirely different from and dramatically simpler than Takeuti's.

Finally, for the *cognoscente*,[2] we note that Takeuti's models are a straightforward generalization of the construction of Boolean-valued models of set theory, where one takes the fixed complete Boolean algebra B which serves as the truth values for a particular Boolean-valued model, and replaces this B with a particular complete orthomodular lattice — namely, the projection lattice of a (complex separable) Hilbert space. It is perhaps not surprising that Takeuti's quantum sets are beset from the beginning with technical difficulties — as mentioned before, the Boolean-valued models which form the starting point for Takeuti's generalization are used to prove independence results in set theory, i.e. they are *built* to behave unintuitively compared to the classical universe! That our quantum sets are an attempt to generalize directly from the intuitive classical universe is perhaps another reason accounting for the technical simplicity of our models compared to Takeuti's.

We will now proceed to the next chapter, where we use our quantum sets to construct "quantum natural numbers" which will form the basis for a "quantum arithmetic".

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[1]This is to say the underlying class of our models $\mathcal{Q}_L$ form a subclass of the underlying class of Takeuti's universe.

[2]And also for those patient readers who trudged their way through section 5.1.3.

CHAPTER 6

QUANTUM ARITHMETIC

In this chapter we use the quantum set theory developed in chapter 5 to construct a quantum version of the natural numbers (similar to the construction of the usual natural numbers as finite ordinals), and then use these quantum natural numbers to build a natural quantum model of (one particular) axiomatization[1] of arithmetic.

In section 6.1, we begin by presenting the basics of axiomatic arithmetic. We first discuss the successor fragment, which is essentially an axiomatic treatment of *counting*, and then present an axiomatization of arithmetic (including addition and multiplication). We also briefly discuss the connection of our axioms to first-order Peano arithmetic, and present Dunn's theorem on the inherent classicality of the Peano axioms. We finish the section by showing how one uses classical set theory to construct the usual natural numbers $\mathbb{N}$.

In the following section (6.2), we proceed with the construction of our quantum model of arithmetic. We first discuss the concept of a 'successor' in the framework of quantum logic, and then proceed to construct a (classical) set of 'quantum natural numbers' (for any given complete OML L). We then define a sum and product on these numbers which we use to construct an $\mathcal{L}_A$-structure (for the language $\mathcal{L}_A$ of arithmetic), and show that this structure

[1]By Dunn's theorem (15), any model of the usual Peano axioms would be be a standard model with a Boolean truth value algebra, as mentioned in 4.1.3.

satisfies the axioms of the successor fragment (for any such L). However, these structures satisfy the full arithmetic axioms *if and only if* L is modular. This gives our promised arithmetical characterization of the lattice-theoretic property of modularity.

We next investigate (in section 6.3) some of the simple properties of the quantum models of arithmetic we have developed. We first show that while the usual substitution axioms do not hold in our models, our natural reduction of substitution presented in chapter 4 does. We then show that any two-variable identity from the usual arithmetic on $\mathbb{N}$ is still an identity in the framework of quantum arithmetic, but that once one considers expressions in more than two variables this is not the case. In particular, we show that both associativity and distribution of multiplication over addition no longer hold in our quantum arithmetic.

In the final section (6.4), we restrict ourselves to a particular case of the 'quantum natural numbers' we have constructed — namely those which are built from our $\mathcal{L}_{set}$-structures $\mathcal{Q}_L$ where L is the projection lattice of a complex separable Hilbert space $\mathcal{H}$. We find that the quantum natural numbers in this context correspond precisely to the (bounded) quantum observables on $\mathcal{H}$ with whole number eigenvalues, and as such our arithmetical sum and product give a "new" sum and product of these observables. While this new sum and product reduce to the standard sum and product on commuting observables, this is no longer the case when one considers non-commuting observables. We demonstrate some remarkable properties of this new sum and product, including the fact that any given eigenvalue of the new sum of two observables A and B is equal to the sum of some eigenvalue of A with some eigenvalue of B, and similarly for the product. We conclude by showing that our new sum and product are the unique binary operations on observables satisfying some physically motivated criteria.

6.1 Arithmetical Basics

The theory of arithmetic (at least as far as the usual first-order treatment is concerned) is represented by an equational M-system (see chapter 4).[1] Also, like other equational M-systems we have considered, (many presentations of) the arithmetic axioms contain $\mathcal{E}$ (i.e. the equality axioms E1-E3) as well as substitution axioms. We present the prototypical first-order arithmetical axioms in this section, namely those of Peano arithmetic. As mentioned previously (section 4.1.3), these axioms are inherently classical. As such, we develop an alternate axiomatization which encodes many of the usual properties of arithmetic.

Arithmetic, as we consider it, is taken to have four operations — the first is a constant representing 'zero', and another is a unary 'successor operation', which takes a single number and yields the "next number". The other two operations represent addition and multiplication. One can also consider the theory containing only the successor and zero operations (known as the successor fragment) in isolation from the full arithmetical theory. It is this fragment we first consider.

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[1] We use the symbol '=' for the equality in this chapter instead of the symbol '≈' used in chapter 4.

6.1.1 Successor Fragment

First we define our equational language[1] $\mathcal{L}_S$ by specifying the function symbols $\mathcal{L}_S^{\mathcal{F}} := \{0, \cdot'\}$, with $\alpha(0) = 0$ and $\alpha(\cdot') = 1$. Of course, '0' represents the number 'zero' and ' $\cdot'$ ' represents the 'successor' operation. We then define the following wffs (where we define $t \neq u := \neg(t = u)$ for terms t and u):

(S1) $(\forall x)(x' \neq 0)$

(S2) $(\forall x)(x \neq x'), (\forall x)(x \neq x''), \ldots$

(S3) $(\forall x)(\forall y)(x = y \rightarrow x' = y')$

(S4) $(\forall x)(\forall y)(x' = y' \rightarrow x = y)$

(S5) $(\forall x)[(x \neq 0) \rightarrow [(\exists y)(x = y')]$.

Note that S2 above represents an infinite set of axioms, and that S3 is simply substitution for the successor operation. We then define the successor axioms $\mathcal{A}_S$ to be S1-S5 above along with the equality axioms $\mathcal{E}$. The axioms listed above are classically equivalent to the usual axiomatization of the successor fragment of first-order Peano arithmetic[2] — see (18) for details.

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[1]Since the equality predicate for this theory will be the equality defined in the previous chapter for sets (equation 5.1) suitably restricted, we will use the usual symbol '=' to denote this predicate as opposed to the symbol '≈' that was used in chapter 4.

[2]The induction schema from the standard Peano successor axioms has been replaced by S5 and the infinite sequence of axioms S2.

6.1.2 Full Arithmetic

Having established the language and axioms for the successor theory, we can now consider the full arithmetic. We define the language $\mathcal{L}_A \coloneqq \mathcal{L}_S \cup \{\dot{+}, \dot{\times}\}$ with arity $\alpha(\dot{+}) = 2$ and $\alpha(\dot{\times}) = 2$. We then define the following wffs.

(A1) $(\forall x)(x \dot{+} 0 = x)$

(A2) $(\forall x)(\forall y)[x \dot{+} y' = (x \dot{+} y)']$

(A3) $(\forall x)(x \dot{\times} 1 = x)$

(A4) $(\forall x)(\forall y)(x \dot{\times} y' = (x \dot{\times} y) \dot{+} x)$.

We then take our arithmetical axioms $\mathcal{A}_A$ to be A1-A4 above along with $\mathcal{A}_S$. For purposes of discussion, we also define the induction schema (which is used in first-order Peano arithmetic):

(Ind) $\Big(\psi(0) \wedge (\forall x)[\psi(x) \to \psi(x')]\Big) \to (\forall y)\psi(y)$ for any $\mathcal{L}_A$-wff $\psi(z)$ (with $z \in \mathcal{B}_V$).

This will allow us to define the Peano axioms[1] to be $\mathcal{A}_{PA} \coloneqq \mathcal{A}_A \cup \text{Ind}$. The axioms $\mathcal{A}_A$ are not classically equivalent to $\mathcal{A}_{PA}$, and in fact are strictly weaker (in both classical as well as quantum logic). This will not be a great concern to us — they are already stronger than one simple theory of arithmetic,[2] and as we will see, they have an extremely natural non-standard model which justifies their use.

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[1]We have a few extra axioms over the usual treatment, but as these are all theorems of the usual Peano axioms this will not bother us.

[2]That is, Robinson arithmetic.

With the above definitions, we are now in a position to state the theorem of Dunn which has elicited so much commentary from us. (For a proof, see (15).)

Proposition 6.1. *Dunn's Theorem:* In the language $\mathcal{L}_A$, the set $\mathcal{A}_{PA}$ is inherently classical.

From this theorem we see that any non-standard $\mathcal{L}_A$-structures *cannot* satisfy $\mathcal{A}_{PA}$.

6.1.3 Classical Arithmetic

The original inspiration for axiomatic arithmetic is just the ordinary arithmetic we all learned in grade school based on the natural numbers $\mathbb{N}$. This arithmetic on $\mathbb{N}$ forms what we will call the *usual model* of the M-system $(\mathcal{L}_A, \mathcal{A}_A)$, where the function symbols are interpreted in the obvious way. Now, as mentioned in the previous chapter, we can "build" this arithmetic within the framework of (classical) axiomatic set theory. Since we will perform a similar construction in *quantum* axiomatic set theory to obtain the *quantum* natural numbers, it will be useful to now briefly discuss the typical method by which the natural numbers are constructed in classical set theory.

The Natural Numbers from Classical Set Theory

The reader familiar with ordinal numbers (see section A.3 in the appendix) already knows where we are headed — namely, the finite ordinals represent the natural numbers, and the successor operation is interpreted in the ordinal numbers by $\mathbf{n} \mapsto \mathbf{n} \cup \{\mathbf{n}\}$ (for any ordinal $\mathbf{n}$). For the reader unfamiliar with ordinal numbers, we will now discuss the relevant details. First, we need a few notions in *classical* set theory.

Definition 6.1. Let A be a set. Then $A \cup \{A\}$ (denoted A') is called the *classical successor to* A. A set S is called *inductive* if both $\varnothing \in S$, and also $A \in S$ implies that $A' \in S$ for any set A. A set S is called *transitive* if, for any sets A and B with $A \in B$ and $B \in S$, we also have $A \in S$.

If we think of $\varnothing$ as being a natural choice to represent 'zero', then the classical successor as defined above gives a natural way to construct all the remaining natural numbers. To wit, we define

$$\begin{aligned}
&\mathbf{0} := \varnothing \\
&\mathbf{1} := \mathbf{0}' = \varnothing' = \{\varnothing\} = \{\mathbf{0}\} \\
&\mathbf{2} := \mathbf{1}' = \{\varnothing\}' = \{\varnothing, \{\varnothing\}\} = \{\mathbf{0}, \mathbf{1}\} \\
&\mathbf{3} := \mathbf{2}' = \{\varnothing, \{\varnothing\}\}' = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}
\end{aligned} \tag{6.1}$$

and so on, namely for $n > 3$ take $\mathbf{n} := \mathbf{n} \cup \{\mathbf{n}\} = \{\mathbf{0}, \mathbf{1}, \ldots, \mathbf{n} - \mathbf{1}\}$.

We then denote the collection of all such natural numbers by ω. Clearly, ω is an inductive set. Moreover, it is not hard to see that ω is characterized by the property that it is contained in every inductive set — i.e. ω is the smallest inductive set. We can immediately see that every natural number as defined above is a transitive set, as is ω itself. Indeed, for any set A (in classical set theory), there is a (slightly complicated) method of characterizing the elements of ω in terms of the property of being a transitive set, but these details won't concern us (23). Both the properties just discussed (namely that of being a transitive set and an inductive set) will prove useful in the construction of our natural quantum model of arithmetic.

Now, the above considerations show that the natural numbers ω (as defined above) are identical to the usual natural numbers $\mathbb{N}$ as far as the successor fragment of arithmetic is concerned. As such, we will identify $\mathbb{N}$ and ω, and treat n and $\mathbf{n}$ as being the "same" object (for any $n \in \mathbb{N}$). Now, in principle we could describe the usual addition and multiplication on $\mathbb{N}$ in terms of the sets in ω, but this turns out to be a little complicated and doesn't do much to help build one's intuition. Also, we will construct our addition and multiplication of our 'quantum natural numbers' directly, without interpreting these operations on sets. Hence we will now leave our discussion of classical sets and arithmetic, and move on to the quantum case.

6.2 Quantum Arithmetic from Quantum Sets

We begin by generalizing the notion of the classical successor, as well as the notions of transitive and inductive sets. We then form, for any given complete OML L, the (classical) set of quantum natural numbers, and define a natural interpretation of the successor on these quantum natural numbers, as well as addition and multiplication operators. We then proceed to show that the structures we have created do, indeed, always model $\mathcal{A}_S$, and also that these structures model $\mathcal{A}_A$ iff L is modular.

6.2.1 Quantum Set Theoretic Preliminaries

In the previous section, we defined the classical successor of a set A to be the set $A \cup \{A\}$. In this way, starting from $\varnothing$, we constructed the natural numbers. The intuitive idea here is that to a set A we add a single new "last" element — namely A itself. Once we move to quantum sets, this is not necessarily the most natural way to define a successor.

As we stated, the successor function is supposed to represent "counting". Also, we mentioned that we could think of numbers as transitive sets. Now, if we use the classical successor in quantum set theory, it is easy to see that this operation will not generically preserve the transitivity of sets. For example, take any quantum set f with $\sup f = \{\varnothing\}$, and $[\![\varnothing \in f]\!] \neq 1$.[1] Then using the classical successor on this quantum set gives

$$[\![\varnothing \in f^*]\!] \wedge [\![f^* \in f \cup \{f\}]\!] = 1 \quad \text{but} \quad [\![\varnothing \in f \cup \{f\}]\!] = [\![\varnothing \in f]\!] \neq 1,$$

showing that transitivity is ruined! A little thought shows that the way to preserve transitivity would be to create a new set from the original by adding the empty set to the "beginning", and then "pushing the other sets in f down the line" (preserving truth values).[2] This idea motivates the following definition, which is (admittedly) a little unwieldy.[3]

Definition 6.2. In the language $\mathcal{L}_{set}$ define (for $x \in \mathcal{B}_V$)[4]

$$S(f) := \mathcal{P}\big(f^* \cup \bigcup f^*\big) \quad \text{and} \quad \psi(x,y) := (\exists z)(x = z^* \cup \{z^*\} \wedge z \in y) \vee x = \varnothing$$

---

[1]Such a set must trivially be transitive because no set is in $\varnothing$.

[2]Of course, while the empty set is at the "beginning" of the natural numbers (thought of as sets), this is not the case for sets more generically. However, since our only use for the successor is the construction of a quantum version of such numbers, this will not concern us.

[3]If we did keep the classical definition for the successor f' of any quantum set f — that is, if we took $f' = f \cup \{f\}$ — we *would* obtain the exact same quantum natural numbers, even though the successor fragment axioms would not be satisfied.

[4]We use the notation established in sections 5.1.4 and 5.2.2.

and the define the *successor to* x, denoted x', by

$$x' := \{u \in S(x) : \psi(u^*, x)\}.$$

<u>Note</u>: This set exists by the separation axiom (RZFC3). Now, while a generic L does not give a $\mathcal{Q}_L$ which satisfies the RZFC axioms, in this case it is still possible to define a successor for the 'quantum sets' in $\mathfrak{Q}_L$ for any complete OML L. For such an L and $f \in \mathfrak{Q}_L$, *define* f' by equation 6.2 in the following proposition.

The following proposition shows the (quantum) successor behaves as in our intuitive description, and also agrees with the classical successor in the model $\mathcal{Q}_{B_2}$ when we restrict this quantum successor to the natural numbers (elements of ω).[1]

Proposition 6.2. Let L be any OML. In the $\mathcal{L}_{set}$-structure $\mathcal{Q}_L$, the successor f' of a quantum set $f \in \mathfrak{Q}_L$ is given by (for $A \in \mathfrak{V}$)[2]

$$f'(A) = \begin{cases} 1 & \text{if} \quad A = \varnothing \\ f(B) & \text{if} \quad A = B \cup \{B\} \text{ for some } B \in \mathfrak{V} \\ 0 & \text{otherwise.} \end{cases} \tag{6.2}$$

---

[1] It does not, in general (i.e. for an arbitrary classical set), reduce to the classical successor function in this model. As the successor is designed for the purposes of constructing the natural numbers, this fact won't cause us any concern.

[2] As per the above note, we take this as the *definition* when $\mathcal{Q}_L$ does not model the RZFC axioms, so in this case there is nothing to prove.

Proof. We use lemmas 5.9 and 5.6 frequently without comment in this proof. First, we compute (for any $A \in \mathfrak{V}$)

$$\begin{aligned} \bigcup f^*(A) &= [\![\chi_A \in \bigcup f^*]\!] = \left(\bigcup \chi_{\sup f}\right)(A) = \bigvee_{A \in B \in \mathfrak{V}} \chi_{\sup f}(B) \\ &= \begin{cases} 1 & \text{if there exists a } B \in \mathfrak{V} \text{ with } A \in B \in \sup f \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The aforementioned lemmas then also yield that $\left(f^* \cup \bigcup f^*\right)(A) = 1$ if either (i) $A \in \sup f$, or (ii) there exists a $B \in \mathfrak{V}$ such that $A \in B \in \sup f$. We then easily see that $\mathcal{P}\left(f^* \cup \bigcup f^*\right)(\varnothing) = 1$, and also for any $B \in \sup f$ that $\mathcal{P}\left(f^* \cup \bigcup f^*\right)(B \cup \{B\}) = 1$, since, for any $C \in B$, we have that $C \in B \in \sup f$, and also $B \in \sup f$, so that $\left(f^* \cup \bigcup f^*\right)(C) = 1$ for all $C \in B \cup \{B\}$. From this we see that, if $g \in \mathfrak{Q}_L$ such that $\sup g = B \cup \{B\}$ for some $B \in \mathfrak{V}$, then

$$[\![\varnothing \in S(f)]\!] = S(f)(\varnothing) = 1 \quad \text{and} \quad [\![g \in S(f)]\!] = S(f)(B \cup \{B\}) = 1.$$

Then we consider

$$[\![\psi(\varnothing^*, f)]\!] = [\![(\exists z)(\varnothing^* = z^* \cup \{z^*\} \wedge z \in f)]\!] \vee [\![\varnothing^* = \varnothing]\!] \geq [\![\varnothing^* = \varnothing]\!] \geq 1,$$

which shows that $f'(\varnothing) = [\![\varnothing \in f']\!] = [\![\varnothing \in S(f)]\!] \wedge [\![\psi(\varnothing^*, f)]\!] = 1.$

Next, let $g \in \mathfrak{Q}_L$ be a quantum set with $\sup g = B \cup \{B\}$ for some $B \in \mathfrak{V}$. Then

$$\llbracket \psi(g^*, f) \rrbracket = \llbracket (\exists z)(g^* = z^* \cup \{z^*\} \wedge z \in f) \rrbracket \vee \llbracket g^* = \varnothing \rrbracket$$

$$= \llbracket g^* = \chi_B \cup \{\chi_B\} \rrbracket \wedge \llbracket \chi_B \in f \rrbracket = f(B),$$

where we have used lemma 5.14, showing that for any A with $A = B \cup \{B\}$, we have that

$$f'(A) = f'(\sup g) = \llbracket g \in f' \rrbracket = \llbracket g \in S(f) \rrbracket \wedge \llbracket \psi(g^*, f) \rrbracket = f(B).$$

Finally, we assume that $A \neq \varnothing$, and $A \neq B \cup \{B\}$ for any $B \in \mathfrak{V}$. We will show that $f'(A) = 0$ by contradiction, so assume not, and let $h \in \mathfrak{Q}_L$ be any quantum set with $\sup h = A$. Then we have

$$0 \neq f'(A) = \llbracket h \in f' \rrbracket \leq \llbracket \psi(h^*, f) \rrbracket,$$

so that $\llbracket \psi(h^*, f) \rrbracket \neq 0$. Hence, we have either (a) $\llbracket h^* = \varnothing \rrbracket \neq 0$ or (b) $\llbracket (\exists z)(h^* = z^* \cup \{z^*\} \wedge z \in y) \rrbracket \neq 0$. First considering case (a), we have $\llbracket h^* = \varnothing \rrbracket = \llbracket \chi_A = \varnothing \rrbracket$ so that if $\llbracket h^* = \varnothing \rrbracket \neq 0$ we must have $A = \varnothing$ which is a contradiction. Next considering case (b), we note that (for any quantum set $g \in \mathfrak{Q}_L$) $\llbracket h^* = g^* \cup \{g^*\} \rrbracket$ takes on only the values 0 and 1 (by lemma 5.14), so hence if $\llbracket (\exists z)(h^* = z^* \cup \{z^*\} \wedge z \in y) \rrbracket \neq 0$, there must be some quantum set g with $A = \sup g \cup \{\sup g\}$ which is also a contradiction. These two contradictions show that we must have $f'(A) = 0$ if $A \neq \varnothing$ and $A \neq B \cup \{B\}$, completing the proof. $\square$

Transitive and Inductive Quantum Sets

Now that we know how to compute the successor in our natural model of quantum set theory, we can generalize the notions of transitive and inductive sets to these models (in fact, even to structures $\mathcal{Q}_L$ with L any complete OML).

Definition 6.3. Let L be a complete OML, let f be an L-valued set, and let $[\![\cdot]\!]$ be the truth valuation of $\mathcal{Q}_L$. Then f is said to be *inductive* if $[\![\varnothing \in f]\!] = 1$ and $[\![(\forall x)(x \in f \to x' \in f)]\!] = 1$. f is said to be *a transitive set* if

$$[\![(\forall x)(\forall y)\big([(y \in f) \wedge (x \in y)] \to (x \in f)\big)]\!] = 1.$$

In the framework of classical set theory, the property of being inductive was sufficient to define the natural numbers in terms of sets. In our quantum set theory, however, we will also need the property of being a transitive set to get a handle on the quantum natural numbers.

6.2.2 The Quantum Natural Numbers

The first step in constructing a model of the arithmetical axioms $\mathcal{A}_A$ is to form a candidate (classical) set to use as the underlying set for an $\mathcal{L}_A$-structure. We will obtain such a set for any complete OML L (which we will call ω_L). As any particular ω_L will be the analog of the set $\mathbb{N}$ in the usual model of arithmetic, we will call elements of these sets *quantum natural numbers*.

We define a little notation to make the definition of a quantum natural number easier to parse. Let $\psi_I(x)$ be given by (for $x, y, z \in \mathcal{B}_V$)[1]

$$\psi_I(x) := (\forall z)[[(\varnothing \in z) \wedge (\forall y)(y \in z \rightarrow y' \in z)] \rightarrow x \in z],$$

(meaning "x is an element of every inductive set"), and let $\psi_T(x)$ be defined by

$$\psi_T(x) := (\forall y)(\forall z)\Big([(y \in x) \wedge (z \in y)] \rightarrow (z \in x)\Big)$$

(meaning "x is a transitive set"). Then we can then define our quantum natural numbers.

Definition 6.4. Let L be a complete OML, and let $\mathcal{Q}_L$ be the structure of def. 5.2. Then $f \in \mathbb{Q}_L$ is said to be a *quantum natural number* (or simply *quantum number*)[2] if

$$[\![\psi_I(f) \wedge \psi_T(f)]\!] = 1.$$

We then define the (classical) set $\omega_L := \{f \in \mathbb{Q}_L \ : \ [\![\psi_I(f) \wedge \psi_T(f)]\!] = 1\}$.

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[1]Technically speaking, the y' which occurs in the definition of ψ_I is only defined when $\mathcal{Q}_L \models$ RZFC. For the case of more generic L, to keep everything truly kosher we would need to go through the tedious process of adding a unary function symbol to $\mathcal{L}_{set}$ and then interpreting this function symbol as the successor we have defined. We will leave the details to those readers who have nothing better to do with their time.

[2]Not to be confused with the quantum numbers describing a quantum state!

<u>Note</u>: Intuitively speaking, we take the quantum natural numbers to be those quantum sets f for which the statements 'f is a member of every inductive set' and 'f is a transitive set' are both true.

The following proposition shows that we can give a nice characterization of the quantum natural numbers.

Proposition 6.3. L be a complete OML, let $\mathcal{Q}_L$ be the structure of def. 5.2, and let $f \in \mathfrak{Q}_L$ be a quantum set. Then (recalling that ω is the set of natural numbers thought of as classical sets)

1. $[\![\psi_I(f)]\!] = 1$ iff $\sup f \in \omega$,
2. $[\![\psi_T(f)]\!] = 1$ iff $f(A) \le f(B)$ whenever $B \in A$ (for any $A, B \in \mathfrak{V}$).

Proof. We we first will prove (1) above. First we compute

$$\begin{aligned}[\![\psi_I(f)]\!] &= [\![(\forall z)[[(\varnothing \in z) \wedge (\forall y)(y \in z \to y' \in z)] \to x \in z]]\!] \\ &= \bigwedge_{h \in \mathfrak{Q}_L} \Big(\big[h(\varnothing) \wedge \bigwedge_{g \in \mathfrak{Q}_L} (h(\sup g) \to h(\sup g')) \big] \to h(\sup f) \Big),\end{aligned}$$

so by lemma C.14, we see that $[\![\psi_I(f)]\!] = 1$ iff for all $h \in \mathfrak{Q}_L$ we have that

$$h(\varnothing) \wedge \bigwedge_{g \in \mathfrak{Q}_L} (h(\sup g) \to h(\sup g')) \le h(\sup f). \tag{6.3}$$

First we assume that $[\![\psi_I(f)]\!] = 1$. Now if $\sup g \notin \omega$, then $\chi_\omega(\sup g) = 0$, so $\chi_\omega(\sup g) \to \chi_\omega(\sup g') = 1$ by lemma C.14. On the other hand, if $\sup g \in \omega$, then $\sup g' \in \omega$ as well, as can be seen from prop. 6.2. This means that $\chi_\omega(\sup g') = 1$ and so, $\chi_\omega(\sup g) \to \chi_\omega(\sup g') = 1$ again by lemma C.14. Then we take $h = \chi_\omega$ in equation equation 6.3, which gives (since $\varnothing \in \omega$ so that $\chi_\omega(\varnothing) = 1$)

$$\chi_\omega(\varnothing) \wedge \bigwedge_{g \in \mathbb{Q}_L} (\chi_\omega(\sup g) \to \chi_\omega(\sup g')) = 1 \le \chi_\omega(\sup f),$$

so that $\sup f \in \omega$.

Next, we assume that $\sup f \in \omega$. We will show that equation 6.3 is satisfied (for any $h \in \mathbb{Q}_L$) for any $\sup f \in \omega$ by induction on ω. The base case $0 = \varnothing \in \omega$ is trivial, so assume that, for $n \in \omega$,

$$h(\varnothing) \wedge \bigwedge_{g \in \mathbb{Q}_L} [h(\sup g) \to h(\sup g')] \le h(n).$$

Then we trivially have, for any $n \in \omega$ and $h \in \mathbb{Q}_L$, that

$$h(\varnothing) \wedge \bigwedge_{g \in \mathbb{Q}_L} [h(\sup g) \to h(\sup g')] \le h(\sup \chi_n) \to h(\sup \chi_n') = h(n) \to h(n+1).$$

The previous two inequalities then give

$$h(\varnothing) \wedge \bigwedge_{g \in \mathbb{Q}_L} [h(\sup g) \to h(\sup g')] \le h(n) \wedge [h(n) \to h(n+1)] = h(n) \wedge h(n+1) \le h(n+1)$$

where the equality holds by orthomodularity, which completes the induction.

Next, we will demonstrate (2). First, note that $[\![\psi_T(f)]\!] = 1$ iff for any $g \in \mathfrak{Q}_L$ and $B \in \mathfrak{V}$ (using lemma C.14) we have

$$f(\sup g) \wedge g(B) \leq f(B). \tag{6.4}$$

First, assume $[\![\psi_T(f)]\!] = 1$. Then for any $A \in \mathfrak{V}$ and $B \in A$, equation 6.4 yields that (taking $g = \chi_A$) $f(A) \leq f(B)$. Next, assume $f(A) \leq f(B)$ for any $B \in A \in \mathfrak{V}$. For any $B \in \mathfrak{V}$ and $g \in \mathfrak{Q}_L$, if $g(B) \wedge f(\sup g) \neq 0$, then $B \in \sup g \in \sup f$, so that $f(\sup g) \leq f(B)$ by assumption, which yields that equation 6.4 is satisfied. □

By the above proposition, the fact that a quantum natural number f satisfies $[\![\psi_I(f)]\!] = 1$ forces $\sup f$ to be an (ordinary) natural number,[1] and the requirement that $[\![\psi_T(f)]\!] = 1$ forces (for each $i, j \in \omega$) that $f(j) \leq f(i)$ whenever $i \leq j$, so that the $f(i)$'s form a decreasing sequence of elements in the truth value algebra L associated with the model. Moreover, these criteria characterize the quantum natural numbers for any given L. Of course, when $L = B_2$ one recovers the ordinary natural numbers, with the number natural number 'n' corresponding to that $f \in \omega_{B_2}$ with $f(i) = 1$ iff $i < n$.

Notational Conventions for Quantum Natural Numbers

We will now present some conventions for dealing with quantum natural numbers that we will utilize extensively throughout the remainder of this chapter. First, we will frequently denote a quantum natural number by an upper case Latin letter $A, B, C, \ldots$. Then, for $n \in \omega \backslash \{0\}$ and

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[1]Hence, also that $\sup f$ is a transitive set.

$A \in \omega_L$,[1] we define $A_n := A(n-1)$ to reduce notational clutter, and $A_0 = 1$ for technical reasons. It is also easy to see in this notation that

$$(A')_n = \begin{cases} 1 & \text{if } n = 0 \\ A_{n-1} & \text{if } n \geq 1, \end{cases}$$

and we will frequently leave off parentheses, writing A'_n for $(A')_n$.

Finally, we will frequently identify $A \in \omega_L$ with its associated decreasing sequence, which we will present as (where $n \geq \sup A$)

$$\big(A(0), A(1), \ldots, A(n-1)\big) = \big(A_1, \ldots, A_n\big).$$

In this notation, the successor of $A \in \omega_L$ produces a new tuple by simply adding a '1' to the left side of the above tuple, so that

$$(A_1, \ldots A_n)' = (1, A_1, \ldots, A_n).$$

Also, by our definition the representation is only unique up to trailing zeroes. As an example, we have (for any $a \in L$)

$$(1, a, a, 0, 0) = (1, a, a, 0) = (1, a, a).$$

---

[1]Note that we can think of A as a map from ω to L, since $supA \in \omega$.

As mentioned earlier (section 6.1.3), we will identify $\mathbb{N}$ and ω in the sequel.

Addition and Multiplication of Quantum Numbers

In this section we define an interpretation of the addition '$\dot{+}$' and multiplication '$\dot{\times}$' symbols on quantum natural numbers. We then proceed to prove some technical results concerning the properties of these operations which we will use in proving that the axioms $\mathcal{A}_A$ hold.

Definition 6.5. Let L be a complete OML, and let $A, B \in \omega_L$. Then we define $A \dot{+} B$ and $A \dot{\times} B$ by (for $n \in \mathbb{N}$, where the indices k and j are understood to run only over elements of $\mathbb{N}$)

$$(A \dot{+} B)_n := \bigvee_{k+j \geq n} \big(A_k \wedge B_j\big) \quad \text{and} \quad (A \dot{\times} B)_n := \bigvee_{k \cdot j \geq n} \big(A_k \wedge B_j\big). \tag{6.5}$$

Of course, we need to show that $A \dot{+} B$ and $A \dot{\times} B$ so defined are actually quantum natural numbers. This is done in proposition 6.7, the proof of which will require a number of technical lemmas which we will present after the following simple definition.

Definition 6.6. Let L be a complete OML, and let $A \in \omega_L$ be a quantum natural number. Then $a \in \mathbb{N}$ is called an *eigenvalue of* A if $A_a \neq A_{a+1}$. The set of all the eigenvalues of A is called the *spectrum of* A, which we denote $\sigma(A)$.

As may be expected, this notion is a generalization of the linear algebraic notion of an eigenvalue. The connection will become clear in section 6.4.1 — in particular see proposition 6.20.

Lemma 6.4. Let L be a complete OML, and let $A, B \in \omega_L$ be quantum natural numbers. Then

$$(A \dot{+} B)_n = \bigvee_{k=0}^{n} A_k \wedge B_{n-k} = \bigvee_{\substack{k+j \geq n \\ k \in \sigma(A) \\ j \in \sigma(B)}} A_k \wedge B_j.$$

Proof. By definition we trivially have

$$\bigvee_{k=0}^{n} A_k \wedge B_{n-k} \leq \bigvee_{k+j \geq n} A_k \wedge B_j = (A \dot{+} B)_n.$$

Consider $A_k \wedge B_j$ with $k + j \geq n$, and $k \leq n$. It follows that $j \geq n - k$, and since the B_i's form a decreasing sequence, we have that $B_j \leq B_{n-k}$, and so also $A_k \wedge B_j \leq A_k \wedge B_{n-k}$, and so we also see that every non-zero term in the join on the RHS of the above inequality is also in the join on the LHS, establishing the other inequality, giving equality.

The second equality is easy to see, for any $l \in \mathbb{N}$, let l_A be the smallest eigenvalue of A greater than or equal to l, and similarly let l_B be the smallest eigenvalue of B greater than or equal to l. It is easy to see that $A_k = A_{k_A}$ and $B_j = B_{j_B}$ for any $j, k \in \mathbb{N}$. The result follows trivially from this fact. □

Lemma 6.5. Let L be a complete OML, and let $A, B \in \omega_L$ be quantum natural numbers. Then

$$(A \dot{\times} B)_n = \bigvee_{s=0}^{n} A_s \wedge B_{[\frac{n}{s}]} = \bigvee_{\substack{k \cdot j \geq n \\ k \in \sigma(A) \\ j \in \sigma(B)}} A_k \wedge B_j.$$

where $[\frac{n}{s}]$ denotes the smallest integer greater than $\frac{n}{s}$.

Proof. By definition we trivially have

$$\bigvee_{s=0}^{n} A_s \wedge B_{[\frac{n}{s}]} \leq \bigvee_{k\cdot j\geq n} A_k \wedge B_j = (A \dot{\times} B)_n.$$

If $n = 0$ the result is trivial, so consider $n \neq 0$, and also consider $A_k \wedge B_j$ with $k \cdot j \geq n$ and $k \leq n$. It follows that $j \geq \frac{n}{k}$, so that $j \geq [\frac{n}{k}]$. We then have that $A_k \wedge B_{[\frac{n}{k}]} \geq A_k \wedge B_j$ since the B_i's form a decreasing sequence. We then see that every term in the join on the RHS of the above inequality is less than some term in the join on the LHS, yielding the other inequality, which means the two expressions are equal.

The second equality follows trivially by the same considerations in the previous lemma. □

Lemma 6.6. Let L be a complete OML, and let $A, B \in \omega_L$ be quantum natural numbers, and let $n \in \mathbb{N}$. Then $(A \dot{+} B)_{n+1} \leq (A \dot{+} B)_n$, and $(A \dot{\times} B)_{n+1} \leq (A \dot{\times} B)_n$.

Proof. By definition we have

$$(A \dot{+} B)_n := \bigvee_{k+j\geq n} \big(A_k \wedge B_j\big) \quad \text{and} \quad (A \dot{+} B)_{n+1} := \bigvee_{k\cdot j\geq n+1} \big(A_k \wedge B_j\big),$$

and so the result follows trivially (since $k + j \geq n + 1$ implies $k + j \geq n$). For the multiplication we similarly have

$$(A \dot{\times} B)_n := \bigvee_{kj\geq n} \big(A_k \wedge B_j\big) \quad \text{and} \quad (A \dot{\times} B)_{n+1} := \bigvee_{kj\geq n+1} \big(A_k \wedge B_j\big),$$

and so the inequality holds for the similar reasons. □

Proposition 6.7. Let L be a complete OML, and let $A, B \in \omega_L$. Then $A \dot{+} B \in \omega_L$ and $A \dot{\times} B \in \omega_L$.

Proof. This follows trivially from lemmas 6.6 and proposition 6.3. □

6.2.3 Quantum Models of Arithmetic

We now have all our pieces in play to construct the $\mathcal{L}_A$-structures which will model our arithmetical axioms $\mathcal{A}_A$. First, we will use the truth valuation from our quantum set theory to construct a truth valuation for the quantum natural numbers in the obvious way.

Definition 6.7. Let L be a complete OML, let $\mathcal{Q}_L$ be the structure of def. 5.2 with truth valuation $[\![\cdot]\!]_S$, and let ω_L be the quantum natural numbers. Define ν to be the map from the atomic evaluated ω_L-extended $\mathcal{L}_A$-sentences into L given by $\nu(A = B) := [\![A = B]\!]_S$ for all $A, B \in \omega_L$. Then the truth valuation $[\![\cdot]\!]$ obtained by extending ν to all $\mathcal{L}_A$-wffs (given by proposition 3.12) is called the *set-inspired truth valuation w.r.t. L.*

Note: We can easily see from this definition that (for $A, B \in \omega_L$ and $N := \max(\sup A, \sup B)$)

$$[\![A = B]\!] = \bigwedge_{i \in \mathbb{N}} (A_i \leftrightarrow B_i) = \bigwedge_{i=1}^{N} (A_i \leftrightarrow B_i).$$

We can now define the structures which will serve as models for both $\mathcal{A}_S$ and $\mathcal{A}_A$.

Definition 6.8. Let L be a complete OML, let $\mathfrak{F}_L$ be the interpretation of $\mathcal{L}_A$ in ω_L obtained by the following assignments: let $0 \mapsto \varnothing$, interpret the symbol $\cdot'$ as the successor of definition

6.2, and interpret $\dot{+}$ and $\dot{\times}$ as in definition 6.5. Further let $[\![\cdot]\!]$ be the set-inspired truth valuation w.r.t. L. Then we define the *quantum L-number structure* (denoted $\mathfrak{N}_L$) by

$$\mathfrak{N}_L := (\omega_L, L, [\![\cdot]\!], \mathfrak{F}_L).$$

We begin by showing that (for L any complete OML) $\mathfrak{N}_L$ (thought of as an $\mathcal{L}_S$ structure[1]) is a model for the successor axioms $\mathcal{A}_S$. We will then proceed to prove a number of technical results which will ultimately allow us to show that $\mathfrak{N}_L \vDash \mathcal{A}_A$ iff L is modular. We actually find that most of the arithmetical axioms hold in $\mathfrak{N}_L$ even if L is not modular — we will make this apparent in the upcoming lemmas.

Theorem 7. Let L be a complete OML. Then $\mathfrak{N}_L \vDash \mathcal{A}_S$.

Proof. The equality axioms E1-E3 follow directly from proposition 5.3 due to the definition of the truth valuation in $\mathfrak{N}_L$ (def. 6.8).

Examining S1, we see that we need to show, for any $A \in \omega_L$, that $[\![A' = 0]\!] = 0$. But we have

$$[\![A' = 0]\!] = \bigwedge_{i \geq 1} A_i' \leftrightarrow 0 \leq A_1' \leftrightarrow 0 = 0$$

by proposition 6.2 and lemma C.14, so that S1 holds in $\mathfrak{N}_L$.

Next we consider S2. For any $A \in \omega_L$, and $n \in \{1, 2, \ldots\}$, we let $A^{(n)}$ represent the successor operation applied n times to A. Then, in order to establish S2, we need to show (for any A and

[1]Namely, by simply forgetting about the multiplication and the addition.

n as above) that $[\![A^{(n)} = A]\!] = 0$. It is easy to see that $A^{(n)}_m = A_{m-n}$ if $m > n$, and 1 otherwise. Using this gives

$$\begin{aligned}
[\![A^{(n)} = A]\!] &= \bigwedge_{i\in\mathbb{N}} (A^{(n)}_i \leftrightarrow A_i) = \bigwedge_{i\le n} A_i \wedge \bigwedge_{i>n} (A_{i-n} \leftrightarrow A_i) \\
&= A_n \wedge \bigwedge_{i>n} (A_{i-n} \to A_i) \\
&= A_n \wedge A_1 \wedge \bigwedge_{i\ge 1} (A_i \to A_{i+n}).
\end{aligned}$$

where we have used that the A_i's form a decreasing sequence, and lemma C.14. By induction on j (for any $n \ge 1$), we will show that $A_1 \wedge \bigwedge_{i\ge 1}(A_i \to A_{i+n}) \le A_j$ for any $j \ge 1$. The base case $j = 1$ is trivial, so assume the result holds for some given j. Then

$$A_1 \wedge \bigwedge_{i\ge 1}(A_i \to A_{i+n}) = A_j \wedge \bigwedge_{i\ge 1}(A_i \to A_{i+n}) \le A_j \wedge (A_j \to A_{j+n}) = A_{j+n} \le A_{j+1},$$

where the first equality follows from our inductive assumption, we have used orthomodularity to obtain the last equality, and we have also used that the A_i's form a decreasing sequence. Since there is some j such that $A_j = 0$, this gives that $[\![A^{(n)} = A]\!] = 0$ as required.

We next consider S3 and S4 simultaneously, since they will both be satisfied if we can show (for any $A, B \in \omega_L$) that $[\![A = B]\!] = [\![A' = B']\!]$. Computing gives

$$[\![A' = B']\!] = \bigwedge_{i\in\mathbb{N}}(A'_i \leftrightarrow B'_i) = \bigwedge_{i\ge 1}(A_{i-1} \leftrightarrow B_{i-1}) = \bigwedge_{i\in\mathbb{N}}(A_i \leftrightarrow B_i) = [\![A = B]\!],$$

so that both these axioms hold in $\mathfrak{N}_L$.

Finally, we consider S5. To establish that this axiom holds in $\mathfrak{N}_L$, it suffices to show (for any $A \in \omega_L$), that

$$[\![A \neq 0]\!] \leq [\![(\exists y)(A = y')]\!] = \bigvee_{B \in \omega_L} [\![A = B']\!],$$

by lemma C.14. Also by that lemma, we have $[\![A \neq 0]\!] = \neg \bigwedge_{i \geq 1} \neg A_i = A_1$. Now, defining $C_0 = 1$ and $C_i = A_{i-1}$ for $i \geq 1$, we have that $C_i' = A_i$ for $i \neq 1$, so

$$\bigvee_{B \in \omega_L} [\![A = B']\!] \geq [\![A = C']\!] = \bigwedge_{i \in \mathbb{N}} (A_i \leftrightarrow C_i') = A_1 \leftrightarrow 1 = A_1,$$

establishing S5. □

Now that we have established that (for L any complete OML) our quantum L-number structure models the successor axioms, we will show that these structures also model A1-A3.

Lemma 6.8. Let L be a complete OML. Then $\mathfrak{N}_L \vDash$A1-A3.

Proof. We begin by considering A1. To show that this axiom holds in $\mathfrak{N}_L$, we need to show that $[\![A \dotplus 0 = A]\!] = 1$. But (recalling that $0_b = 1$ iff $b = 0$, and 0 otherwise)

$$(A \dotplus 0)_n = \bigvee_{a+b \geq n} (A_a \wedge 0_b) = \bigvee_{a \geq n} A_a = A_n,$$

showing that A1 holds in $\mathfrak{N}_L$.

Next we consider A2, for which it suffices to show that (for any $A, B \in \omega_L$ and $n \in \mathbb{N}$) that $(A \dotplus B')_n = (A \dotplus B)'_n$. We compute (for $n \geq 1$, since the $n = 0$ case is trivial)

$$(A \dotplus B')_n = \bigvee_{a+b\geq n} (A_a \dotplus B'_b) = A_n \vee \bigvee_{\substack{a+b\geq n \\ b\neq 0}} (A_a \dotplus B_{b-1}) = A_n \vee \bigvee_{a+b+1\geq n} (A_a + B_b)$$
$$= \bigvee_{a+b\geq n-1} (A_a \dotplus B_b) = (A \dotplus B)_n,$$

showing that A2 indeed holds.

Finally, we consider A3. For any $A \in \omega_L$ and $n \geq 1$, a simple computation yields (using that $1_b = 1$ for $b \leq 1$, and 0 otherwise)

$$(A \dot{\times} 1)_n = \bigvee_{ab\geq n} (A_a \wedge 1_b) = \bigvee_{a\geq n} A_a = A_n,$$

so that A3 does indeed hold. □

Lemma 6.9. Let L be a complete OML. Then $\mathfrak{N}_L \vDash$A4 iff L is modular.

Proof. First, we assume that L is modular, and let $A, B \in \omega_L$. Since the image of A and B are each chains in L, by Jónsson's theorem C.7, we can use distributivity in all the following expressions (recalling that all of our joins are finite), and we do so. First we compute (for $n \geq 1$, since the $n = 0$ case is trivial)

$$(A \dot{\times} B')_n = \bigvee_{ab\geq n} (A_a \wedge B'_b) = \bigvee_{ab\geq n} (A_a \wedge B_{b-1}) = \bigvee_{a(b+1)\geq n} (A_a \wedge B_b).$$

Then we also have

$$[(A \dot{\times} B) \dot{+} A]_n = \bigvee_{ij \geq n} [(A \dot{\times} B)_i \wedge A_j] = \bigvee_{ij \geq n} \Big(\big[\bigvee_{k+l \geq i} (A_k \wedge B_l) \big] \wedge A_j \Big)$$
$$= \bigvee_{ij \geq n} \Big(\bigvee_{k+l \geq i} (A_j \wedge A_k \wedge B_l) \Big) = \bigvee_{ij \geq n} \Big(\bigvee_{k+l \geq i} (A_{\max(j,k)} \wedge B_l) \Big).$$

It is then easy to see that $(A \dot{\times} B')_n \leq [(A \dot{\times} B) \dot{+} A]_n$, since for a generic $A_a \wedge B_b$ with $a(b+1) \geq n$, take $k = 1$, $j = a$ and $l = b$, along with $i = b+1$ and $j = a$ so that $A_a \wedge B_b$ occurs in the join defining $[(A \dot{\times} B) \dot{+} A]_n$. Similarly, we see that any term $A_{\max(j,k)} \wedge B_l$ in the join defining $[(A \dot{\times} B) \dot{+} A]_n$ also occurs as a term in the join defining $(A \dot{\times} B')_n$, since we can take $a = \max(j,k)$ and $b = l$. The two inequalities show that in fact these two expressions are equal for each $n \geq 1$, so that A4 is satisfied.

Next, we assume that L is not modular, so by proposition C.5 L has a sublattice isomorphic (as a lattice) to the lattice N_5 of example C.4. Using the same labeling of elements in that example, define $A, B \in \omega_L$ by

$$A := (t, y, x) \quad \text{and} \quad B := (t, z, z).$$

As mentioned earlier, $B' = (1, t, z, z)$, and it is easy to compute

$$A \dot{\times} B = (t, t, t, b, b) \quad \text{and so} \quad [(A \dot{\times} B) \dot{+} A]_5 = y.$$

Meanwhile another computation yields

$$(A \dot{\times} B')_5 = x.$$

Then, by lemma C.14, A4 cannot be satisfied. $\square$

We then have (as a trivial corollary of the previous results) the following promised theorem characterizing modularity in terms of models of arithmetic.

Theorem 8. Let L be a complete OML. Then $\mathfrak{N}_L \vDash \mathcal{A}_A$ iff L is modular.

Proof. This simply combines the results of lemmas 6.8 and 6.9 as well as theorem 7. $\square$

6.3 Properties of Quantum Arithmetic

In this section we investigate some of the properties of the models $\mathfrak{N}_L$ of arithmetic developed above (where L is a complete modular ortholattice). First, we examine the substitution axioms, and provide examples showing that they are not satisfied in general. We also discuss reductions of these axioms. Next we examine expressions which are polynomials in two variables, and show that amongst this class of expressions, *all* identities which hold in the usual arithmetic of the natural numbers also hold in our models $\mathfrak{N}_L$.

Moving beyond this class of examples, we consider associativity and the distributivity of multiplication over addition, and show that generically neither of these are satisfied. We do, however, provide a simple criteria for showing that distributivity and associativity do hold.

6.3.1 Substitution

The following examples (6.10 and 6.11) show that we do not generically have substitution for addition and multiplication in our models $\mathfrak{N}_L$, even if L is the projection lattice of a Hilbert space. For the first example, we consider our multiplication $\dot{\times}$.

Example 6.10. In order to see that substitution for $\dot{\times}$ does not hold generally, let $\mathcal{H}$ be a complex two dimensional Hilbert space and let L be the associated projection lattice (see section 1.2.2). Take

$$A = (I, I, I, P, P, P), \quad B = (I, I, I, I, P, P), \quad \text{and} \quad C = (I, I, I, I, Q, Q),$$

where $P \neq Q$ and $P, Q \neq I, 0$. Now, we easily compute (using lemma C.14)

$$[\![A = B]\!] = \bigwedge_{i \in \mathbb{N}} (A_i \leftrightarrow B_i) = (P \leftrightarrow I) \wedge (P \leftrightarrow P) = P.$$

Then, an explicit calculation gives that

$$A \dot{\times} C = (I, I, \ldots I, P, P, \ldots P),$$

where there are $18 I$'s and $6 P$'s; also, we obtain $B \dot{\times} C = (I, I, \ldots I)$, where there are $24 I$'s.

Finally, we have that $[\![A \dot{\times} C = B \dot{\times} C]\!] = 0$ since $\bigwedge_{i \in I} (A \dot{\times} C)_i \wedge (B \dot{\times} C)_i$ has at least one term that is $P \leftrightarrow Q = 0$. Note that $P \leftrightarrow Q = 0$ follows from the fact that $P \rightarrow Q = \neg P \vee (P \wedge Q) = \neg P$

and $Q \to P = \neg Q \vee (P \wedge Q) = \neg Q$, so that $P \leftrightarrow Q = \neg P \wedge \neg Q = 0$ since we consider a two-dimensional example and we take $P \neq Q$ and $P, Q \neq I, 0$.

And so, since $P \nleq 0$, we see that $[\![A = B]\!] \nleq [\![A \dot{\times} C = B \dot{\times} C]\!]$, which shows that substitution for $\dot{\times}$ does not hold even generically in our models.

We now consider substitution for the operation $\dot{+}$.

Example 6.11. In order to see that substitution for $\dot{\times}$ does not hold generally, again take $\mathcal{H}$ to be a complex two dimensional Hilbert space, and let L be the associated projection lattice. Take

$$A = (I, I, I), \quad B = (Q, Q, Q), \quad \text{and} \quad C = (I, P, P),$$

where $P \neq Q$ and $P, Q \neq I, 0$. Now, note that $[\![A = B]\!] = Q$. That this is what $A = B$ valuates to follows from the fact that in $\bigwedge_{i \in I} A_i \leftrightarrow B_i$, the only contributions come from $I \leftrightarrow Q = Q$ (since all other terms are $0 \leftrightarrow 0 = 1$); thus, we have that $[\![A = B]\!] = Q$. An explicit calculation gives

$$A \dot{+} C = (I, I, I, I, P, P) \quad \text{and} \quad B \dot{+} C = (I, I, I, Q, Q),$$

from which we see that

$$[\![A \dot{\times} C = B \dot{\times} C]\!] = \bigwedge_{i \in I} (A \dot{\times} C)_i \wedge (B \dot{\times} C)_i$$

$$= (I \leftrightarrow I) \wedge (I \leftrightarrow I) \wedge (I \leftrightarrow I) \wedge (I \leftrightarrow Q) \wedge (P \leftrightarrow Q) \wedge (P \leftrightarrow 0) = Q \wedge 0 \wedge P^{\perp} = 0,$$

where we have used the fact that $P \leftrightarrow Q = 0$. As such, we see that $[\![A \dot{+} C = B \dot{+} C]\!] = 0$, and so, since $Q \nleq 0$, we see that $[\![A = B]\!] \nleq [\![A \dot{+} C = B \dot{+} C]\!]$. This shows that substitution for $\dot{+}$ does not hold in general.

Since substitution for $\dot{+}$ and $\dot{\times}$ does not hold in general, it is natural to try the reduced versions of substitution discussed in section 4.1.1. Demonstrating the reduced axioms RSub[$\dot{+}$] and RSub[$\dot{\times}$] for a particular $\mathfrak{N}_L$ amounts to showing (for all $f, g, h \in \omega_L$)

$$[\![\mathbf{T}(f = g) \to (g \dot{+} h = g \dot{+} h)]\!] = 1 \quad \text{and} \quad [\![\mathbf{T}(f = g) \to (f \dot{\times} h = g \dot{\times} h)]\!] = 1,$$

since our operations are clearly commutative (recalling the definition of the '**T** operator' — def. 3.2.2)

The above statements can easily be verified in $\mathfrak{N}_L$ (for L modular) using proposition 3.9, since by lemma C.21 we have $\mathfrak{N}_L \models \mathbf{C}(\mathbf{T}(\psi)) \wedge [\mathbf{T}(\psi) \to \psi]$ for any wff ψ (from lemmas 3.10 and 3.11). Hence, we can add the reduced substitution axioms to $\mathcal{A}_A$ along with the axiom $\mathbf{T}(\psi) \to \psi$, to arrive at a presentation of the arithmetic axioms which are classically equivalent to $\mathcal{A}_A \cup \{\text{Sub}[\dot{\times}], \text{Sub}[\dot{+}]\}$ (by proposition 4.2).

6.3.2 Two-variable Expressions in Quantum Arithmetic

We will now show that all two-variable identities of standard arithmetic are satisfied in our quantum arithmetic. (The reader will recall that the informal notion of a "polynomial" is represented in formal logic by the notion of a 'term'.) This will follow simply from the following proposition.

Proposition 6.12. Let L be a complete modular ortholattice, and let $p(x,y)$ be an $\mathcal{L}_A$-term. Then for any $A, B \in \omega_L$, we have that (for any $n \in \mathbb{N}$)

$$\hat{p}(A,B)_n = \bigvee_{p(j,k) \geq n} (A_j \wedge B_k), \tag{6.6}$$

where $\hat{p}(A,B)$ is the evaluation of p in $\mathfrak{N}_L$, and $p(j,k)$ represents p evaluated in $\mathbb{N}$ (the usual model of arithmetic).[1]

Proof. For notational ease we make everyone take their hats off, and use the same symbol to represent a term and its evaluation. We then prove this proposition by induction on the construction of extended $\mathcal{L}_A$-terms. Since there is only one constant symbol in this language, there are two base cases. For the case of the constant 0, we recall that 0 is interpreted as $\varnothing$, i.e. the map which sends every $n \in \omega$ to 0 in L. Hence, by definition $0_0 = 1$ and $0_n = 0$ for all $n \geq 1$. We see by inspection that this agrees with

$$0_n = \bigvee_{0 \geq n} (A_n \wedge B_k)$$

[1]Recall that the evaluation $\hat{p}(A,B)$ is simply the quantum number obtained by taking the polynomial expression $p(x,y)$, plugging in the quantum numbers A for x and B for y, and then computing.

from the fact that $A_0 = B_0 = 1$ and the join over an empty set is the bottom element. The other base case for a term is $p(x, y) = x$, and we easily see that for this p,

$$\bigvee_{p(j,k)\geq n} (A_j \wedge B_k) = \bigvee_{j\geq n} (A_j \wedge B_k) = \bigvee_{j\geq n} (A_j) = A_n = \hat{p}(A, B).$$

We now consider the inductive steps. The first operation to consider is the successor. Assume that $f(x, y)$ is a term satisfying equation 6.6. We need to show that $[f(x, y)]'$ also satisfies this equation. It is easy to see that for any quantum natural number C, we have that $(C')_n = C_{n-1}$ for all $n \geq 1$ (and, of course, $(C')_0 = 1$. From this, we compute (for $n \geq 1$ — the $n = 0$ case is trivial).

$$[f(A, B)']_n = [f(A, B)]_{n-1} = \bigvee_{f(j,k)\geq n-1} (A_j \wedge B_k) = \bigvee_{f(j,k)'\geq n} (A_j \wedge B_k)$$

for all $n \geq 1$ (since the successor is just '+1' in $\mathbb{N}$).

For the next inductive step, we consider the sum and product simultaneously — to this end, let $* \in \{+, \times\}$ and assume that equation 6.6 holds for the two terms $f(x, y)$ and $g(x, y)$. First, using lemma C.7, we see that the set of all A_i's and B_j's generate a distributive sublattice of L. Hence, we can use this (lattice) distributivity with abandon.[1]

$$
\begin{aligned}
[f(A,B) \dot{*} g(A,B)]_n &= \bigvee_{k*l\geq n} \Big(\Big[\bigvee_{f(a,b)\geq k} (A_a \wedge B_b) \Big] \wedge \Big[\bigvee_{g(c,d)\geq l} (A_c \wedge B_d) \Big] \Big) \\
&= \bigvee_{k*l\geq n} \Big[\bigvee_{\substack{f(a,b)\geq k \\ g(c,d)\geq l}} \big(A_a \wedge B_b \wedge A_c \wedge B_d \big) \Big] \\
&= \bigvee_{k*l\geq n} \Big[\bigvee_{\substack{f(a,b)\geq k \\ g(c,d)\geq l}} \big(A_{\max(a,c)} \wedge B_{\max(b,d)} \big) \Big],
\end{aligned}
$$

where we have used that the A_i's and B_j's form a decreasing sequence. Now, first note that we have

$$
\bigvee_{k*l\geq n} \Big[\bigvee_{\substack{f(a,b)\geq k \\ g(c,d)\geq l}} \big(A_{\max(a,c)} \wedge B_{\max(b,d)} \big) \Big] \geq \bigvee_{f(r,s)*g(r,s)\geq n} (A_r \wedge B_s),
$$

as we can see that (for any r and s satisfying $f(r,s) * g(r,s) \geq n$) the term $A_r \wedge B_s$ is in the join on the LHS of the above expression by taking $a = c = r$ and $b = d = s$ along with $k = f(r,s)$ and $l = g(r,s)$. To get the other inequality, note that for any $k, l, a, b, c, d \in \mathbb{N}$ such that $k * l \geq n$, $f(a,b) \geq k$ and $g(c,d) \geq l$, we can take $r = \max(a,c)$ and $s = \max(b,d)$. Then $f(r,s) \geq f(a,b)$ and $g(r,s) \geq g(c,d)$ (due to the monotonicity of polynomials in arithmetic on

[1] Although we appear to have infinite joins, since both A and B are quantum natural numbers, in fact the joins only run over a finite number of terms.

natural numbers), and this means that $f(r,s) * g(r,s) \geq k * l \geq n$, so that any term in the LHS of the above expression is also in the right, yielding that in fact we have

$$[f(A,B) \dot{*} g(A,B)]_n = \bigvee_{k*l \geq n} \Big[\bigvee_{\substack{f(a,b)\geq k \\ g(c,d)\geq l}} \big(A_{\max(a,c)} \wedge B_{\max(b,d)} \big) \Big] = \bigvee_{f(r,s)*g(r,s)\geq n} (A_r \wedge B_s)$$

which completes the induction. □

From this we can easily see that any two-variable identity in the addition of the usual natural number (e.g. $[(x+y)x] + 3 = (x^2+1) + (xy+2)$) is trivially satisfied in our quantum arithmetic, since for two two-variable polynomials such that $p(x,y) = q(x,y)$ in the ordinary natural numbers we have (for any $n \in \mathbb{N}$, retaining the notation of the previous proposition)

$$[\hat{p}(A,B)]_n = \bigvee_{p(a,b)\geq n} (A_a \wedge B_b) = \bigvee_{q(a,b)\geq n} (A_a \wedge B_b) = [\hat{q}(A,B)]_n.$$

6.3.3 Failure of Associativity and Distributivity

We now provide examples showing that our addition and multiplication are generically non-associative. First we consider $\dot{+}$.

Example 6.13. In order to see that associativity of $\dot{+}$ does not hold in general, let L be the projection lattice of a complex two dimensional Hilbert space and let $A_1 = P$, $B_1 = Q$, and $C_! = R$ for distinct one-dimensional projectors P, Q, R, and then $A_n = B_n = C_n = 0$ for all $n \geq 2$. We have that

$$A \dot{+} B = (P \vee Q, P \wedge Q) \quad \text{and} \quad B \dot{+} C = (Q \vee R, Q \wedge R),$$

while

$$(A \dot{+} B) \dot{+} C = (P \vee Q \vee R, (P \wedge Q) \vee (R \wedge (P \vee Q)), P \wedge Q \wedge R).$$

We compute

$$(A \dot{+} B) \dot{+} C = (I, R).$$

On the other hand, we have that

$$A \dot{+} (B \dot{+} C) = (P \vee Q \vee R, (Q \wedge R) \vee (P \wedge (Q \vee R)), P \wedge Q \wedge R),$$

and so

$$A \dot{+} (B \dot{+} C) = (I, P).$$

Thus, we see that

$$A \dot{+} (B \dot{+} C) \neq (A \dot{+} B) \dot{+} C,$$

and as such, $\dot{+}$ is not an associative operation.

Now, we provide an example showing $\dot{\times}$ is not associative in general.

Example 6.14. In order to see that associativity of $\dot{\times}$ does not hold in general, let L be the projection lattice of a complex two dimensional Hilbert space, and let $A = (I, P)$, $B = (I, Q)$, and $C = R$, where P, Q, R are distinct one-dimensional projectors. We have that

$$A \dot{\times} B = (I, P \vee Q, P \wedge Q, P \wedge Q) = (I, I),$$

while

$$B \dot{\times} C = R.$$

Now,

$$(A \dot{\times} B) \dot{\times} C = (R, R),$$

while

$$A \dot{\times} (B \dot{\times} C) = R.$$

Thus, we see that

$$A \dot{\times} (B \dot{\times} C) \neq (A \dot{\times} B) \dot{\times} C,$$

and as such, $\dot{\times}$ is not an associative operation in general.

We now show that multiplication does not generically distribute over addition in our quantum arithmetic.

Example 6.15. Let L be the projection lattice of a complex two dimensional Hilbert space, and further let

$$A = (I, I, P, P), \quad B = (I, I, Q, Q, Q, Q), \quad \text{and} \quad C = (I, R, R, R),$$

where $P \neq Q$, $P \neq R$ and $Q \neq R$; also $P, Q, R \neq I, 0$. We have that

$$(B \dot{+} C) = (I, I, I, I, I, I, Q),$$

and an explicit calculation shows that $A \dot{\times} (B \dot{+} C)$ is given by $14I$'s and $10P$'s. Now, we also have that

$$(A \dot{\times} B) = (I, I, I, I, I, I, I, I, Q, Q, Q, Q),$$

and

$$(A \dot{\times} C) = (I, I, I, I, R, R, R, R).$$

From these, one can then show that $(A \dot{\times} B) \dot{+} (A \dot{\times} C)$ is given by $16I$'s. Thus, we see that

$$A \dot{\times} (B \dot{+} C) \neq (A \dot{\times} B) \dot{+} (A \dot{\times} C),$$

showing that multiplication does not generically distribute over multiplication.

6.3.4 Nice Behavior of Arithmetic for Certain Numbers

We now prove that multiplication distributes over addition for quantum numbers A, B, C when their images are contained in a distributive sublattice (which does not need to be a subortholattice).[1] More generally, one can also demonstrate a generalization of proposition 6.12, showing that all polynomial identities in $\mathbb{N}$ still hold for any set of quantum natural numbers whose images all live in a distributive sublattice — the proof is almost identical to the previous proposition, the notation is just a little unwieldy.

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[1] Jónsson's theorem gives a nice characterization of how chains (recall that the images of our quantum numbers form chains) in modular lattices generate distributive sublattices — see proposition C.6.

Proposition 6.16. Let L be a modular ortholattice, let $A, B, C \in \omega_L$ be quantum natural numbers, and furthermore assume that the images of A, B, C (considered as maps from ω to L) generate a distributive sublattice. Then

$$A \dot{\times} (B \dot{+} C) = (A \dot{\times} B) \dot{+} (A \dot{\times} C).$$

Proof. First we compute

$$\begin{aligned}[A \dot{\times} (B \dot{+} C)]_n &= \bigvee_{jk \geq n} (A_j \wedge (B \dot{+} C)_k) = \bigvee_{jk \geq n} \Big(A_j \wedge \bigvee_{l+m \geq k} (B_l \wedge C_m) \Big) \\ &= \bigvee_{jk \geq n} \Big(\bigvee_{l+m \geq k} (A_j \wedge B_l \wedge C_m) \Big), \end{aligned} \tag{6.7}$$

where we have used distributivity to obtain the final equality. Then, we also have

$$\begin{aligned}[(A \dot{\times} B) \dot{+} (A \dot{\times} C)]_n &= \bigvee_{a+b \geq n} \big[(A \dot{\times} B)_a \wedge (A \dot{\times} C)_b \big] = \bigvee_{a+b \geq n} \Big(\bigvee_{cd \geq a} (A_c \wedge B_d) \wedge \bigvee_{ef \geq b} (A_e \wedge C_f) \Big) \\ &= \bigvee_{a+b \geq n} \Big(\bigvee_{\substack{cd \geq a \\ ef \geq b}} (A_c \wedge B_d \wedge A_e \wedge C_f) \Big) \\ &= \bigvee_{a+b \geq n} \Big(\bigvee_{\substack{cd \geq a \\ ef \geq b}} (A_{\max(c,e)} \wedge B_d \wedge C_f) \Big). \end{aligned} \tag{6.8}$$

First consider a generic element $A_j \wedge B_l \wedge C_m$ in the final join in equation 6.7. Taking $c = e = j$, $d = l$, and $f = m$, along with $a = jl$ and $b = jm$, we easily see that $a + b = j(l + m) \geq jk \geq n$, so that this term occurs in the join of equation 6.8. Similarly, for a generic $A_{\max(c,e)} \wedge B_d \wedge C_f$ occurring in the join in equation 6.8, if we take $j = \max(c, e)$, $l = d$, $m = f$, and also $k = d + f$, then we see that $jk = j(d+f) \geq cd + ef \geq a + b \geq n$, so that this term also occurs in the join of equation 6.7. Hence, these joins are over the exact same sets, and so must be equal.

$\square$

The above results show that modularity is important not only for satisfying the axiom A4, but also for ensuring that our arithmetic is sufficiently "well-behaved". We now move on to consider the most most interesting quantum number structures.

6.4 Arithmetic over Projection Lattices

For the remainder of the section, we let $P(\mathcal{H})$ denote the projection lattice associated to any given Hilbert space $\mathcal{H}$. Recalling the discussion of section 1.2.2, as well as propositions 1.16 and C.24, a projection lattice $P(\mathcal{H})$ over a complex separable Hilbert space $\mathcal{H}$ is a complete OML, and is modular iff $\mathcal{H}$ is finite dimensional. We will see that the quantum numbers $\omega_{P(\mathcal{H})}$ are in 1-1 correspondence with the bounded observables (i.e. bounded Hermitian operators)[1] on $\mathcal{H}$ with (only) whole number eigenvalues on $\mathcal{H}$ — as such we identify a quantum natural number with its corresponding observable.[2]

Using this identification, the arithmetical addition and multiplication have some have some intriguing properties which we state for the sum ($\dot{+}$) — the corresponding statements also hold for the product ($\dot{\times}$). First, for quantum natural numbers A and B which commute (as

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[1]As is customary, we blur the distinction between an observable and its corresponding Hermitian operator.

[2]This correspondence will be made precise shortly.

observables), we have that $A \dot{+} B$ is simply the usual (linear algebraic) operator sum. In fact, something stronger is true regardless of whether A and B commute — for any simultaneous eigenstate $|\psi\rangle$ of A and B (with eigenvalues a and b respectively), we have that $|\psi\rangle$ is an eigenstate of $A \dot{+} B$ with eigenvalue $a + b$ (just as for the usual linear algebraic sum). Finally, very much *unlike* the usual sum from linear algebra, every eigenvalue of $A \dot{+} B$ is equal to the sum of some eigenvalue of A with some eigenvalue of B. And, as mentioned before, the product $(\dot{\times})$ has similar properties.

Finally, we conclude this section with a uniqueness theorem which may hopefully be useful in determining the appropriate physical interpretation of this new sum and product of observables.

6.4.1 Quantum Numbers as Observables

For any separable[1] Hilbert space $\mathcal{H}$, consider a quantum natural number $A \in \omega_{P(\mathcal{H})}$. As described above, A is an order reversing map from ω to $P(\mathcal{H})$ that is eventually 0 (see section 6.2.2). As per the discussion of that section, for $n = \sup A$, we have that $A_1, \ldots, A_n$ are a sequence of decreasing non-zero projection operators in $\mathcal{H}$, and $A_{n+1} = 0$. There is then an obvious bounded observable with (only) whole number eigenvalues which we construct from A, namely

$$\sum_{i=1}^{n} A_n.$$

[1] Keep in mind that such quantum natural numbers will not model the arithmetical axioms $\mathcal{A}_A$ unless the Hilbert space is finite dimensional.

As such, we will identify A with this observable. Moreover, it is straightforward to see that any bounded observable with only whole number eigenvalues corresponds to some quantum natural number in this fashion (in particular, *all* the observables with only whole number eigenvalues when $\mathcal{H}$ is finite dimensional).

Example 6.17. Consider an operator $\hat{A}$ on a three dimensional Hilbert space $\mathcal{H}$ with orthonormal basis $\{|\psi_1\rangle, |\psi_2\rangle, |\psi_4\rangle\}$, such that $\hat{A}|\psi_i\rangle = i|\psi\rangle$ for $i \in \{1,2,4\}$, so that $\hat{A}$ corresponds to the matrix (in this basis)

$$\hat{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Letting I be the identity operator, $P = |\psi_4\rangle\langle\psi_4|$, and $Q = |\psi_4\rangle\langle\psi_4| + |\psi_2\rangle\langle\psi_2|$, we have that $\hat{A}$ corresponds to the quantum natural number

$$A := \big(I, Q, P, P\big),$$

i.e. $A_1 = I$, $A_2 = Q$, and $A_3 = A_4 = P$, as the reader will easily verify.

Before examining our arithmetical sum ($\dot{+}$) and product ($\dot{\times}$) interpreted as a new sum and product on *quantum observables*, we will prove a few technical lemmas, as well as a proposition showing that the definition of 'eigenvalue' of a quantum natural number agrees with the standard notion of an eigenvalue of an observable.

Lemma 6.18. Let $\mathcal{H}$ be a complex separable Hilbert space, and let $A \in \omega_{P(\mathcal{H})}$ be a quantum natural number. Then the spectral decomposition of the observable corresponding to A is given by

$$A = \sum_{k=1}^{\sup A} k\,(A_k - A_{k-1}).$$

Proof. Spectral decompositions are unique, so we merely need show that the above equality holds. For this, we simply compute

$$\sum_{k=1}^{\sup A} k(A_k - A_{k-1}) = \sum_{k=1}^{\sup A} kA_k - \sum_{k=2}^{\sup A+1} (k-1)A_k = A_1 + \sum_{k=2}^{\sup A} A_k = A$$

where we have used that $A_l = 0$ for all $l > \sup A$. □

Lemma 6.19. Let $\mathcal{H}$ be a complex separable Hilbert space, and let $A \in \omega_{P(\mathcal{H})}$ be a quantum natural number. Then $a \in \mathbb{N}$ is an eigenvalue of the observable corresponding to A with eigenvector $|\psi\rangle$ iff

$$A_a|\psi\rangle = |\psi\rangle \quad \text{and} \quad A_{a+1}|\psi\rangle = |0\rangle,$$

recalling that $|0\rangle$ is the zero vector.

Proof. First assume that $A_a|\psi\rangle = |\psi\rangle$ and $A_{a+1}|\psi\rangle = |0\rangle$. Then

$$A|\psi\rangle = \sum_{i=1}^{\sup A} A_i|\psi\rangle = \sum_{i=1}^{a} A_i|\psi\rangle + \sum_{i=a+1}^{\sup A} A_i|\psi\rangle = \sum_{i=1}^{a} |\psi\rangle = a|\psi\rangle.$$

where we have used that the A_i's are a decreasing sequence of projection operators, showing that $|\psi\rangle$ is an eigenvector of A with eigenvalue a.

Next, assume that $|\psi\rangle$ is an eigenvector of A with eigenvalue a. Using the spectral decomposition of A from lemma 6.18, we see that $|\psi\rangle$ is an eigenvector of $A_{i-1} - A_i$ for all $i \in \mathbb{N}$, and since (with I the identity operator) $A_i = I - \sum_{j=1}^{i}(A_{j-1} - A_j)$, $|\psi\rangle$ is an eigenvector of each of the A_i's, with decreasing sequence of eigenvalues a_i (each of which is 0 or 1). Then, since

$$a|\psi\rangle = A|\psi\rangle = \sum_{i=1}^{\sup A} A_i|\psi\rangle = \sum_{i=1}^{\sup A} a_i|\psi\rangle$$

we immediately see that $a_i = 1$ for all $i \le a$, and $a_i = 0$ for all $i > a$. □

The above lemma makes it easy to see that our notion of an eigenvalue of a quantum natural number (see def. 6.6) is a generalization of the notion of an eigenvalue of an observable. We make this precise in the following proposition.

Proposition 6.20. Let $\mathcal{H}$ be a complex separable Hilbert space, and let $A \in \omega_{P(\mathcal{H})}$ be a quantum natural number with corresponding observable $\hat{A}$. Then a is an eigenvalue of A (as a quantum natural number, def. 6.6) iff a is an eigenvalue of $\hat{A}$ (by the usual linear algebraic definition of eigenvalue).

Proof. By the lemma 6.19, it is obvious that if a is an eigenvalue of $\hat{A}$ (according to the linear algebra definition), it is an eigenvalue of A by our definition (6.6). Conversely, if a is an eigenvalue of A by definition 6.6, then $A_a - A_{a+1}$ is not the zero operator, so that (by lemma 6.18), a is an eigenvalue of $\hat{A}$ according to the linear algebra definition. □

Having established these properties of our interpretation of quantum natural numbers as quantum mechanical observables, we can begin to investigate our arithmetical sum and product in this context.

6.4.2 Arithmetical Addition and Multiplication as Viewed on Observables

As mentioned above, one interesting feature of these arithmetical operations when interpreted as a new sum and product on quantum mechanical observables is that $\dot{\times}$ and $\dot{+}$ correspond to ordinary multiplication and addition, respectively, when we restrict ourselves to commuting observables. This is a simple corollary of the more powerful result mentioned earlier, namely that this new sum and product "respect eigenvectors", which we prove after a simple technical lemma.

Lemma 6.21. Let $\mathcal{H}$ be a complex separable Hilbert space, let $|\psi\rangle \in \mathcal{H}$, and let P and Q be projection operators on $\mathcal{H}$ such that $P|\psi\rangle = Q|\psi\rangle = |\psi\rangle$. Then $(P \wedge Q)|\psi\rangle = |\psi\rangle$.

Proof. We use the correspondence of the projection lattice with the subspace lattice. By assumption we have $|\psi\rangle \in \ker(P)^{\perp}$ and $\ker Q^{\perp}$, so that $|\psi\rangle \in \ker(P)^{\perp} \cap \ker(Q)^{\perp}$ which is the meet of these subspaces — again using the correspondence this gives $|\psi\rangle \in \ker(P \wedge Q)^{\perp}$ which is the desired result. □

Proposition 6.22. Let $\mathcal{H}$ be a complex separable Hilbert space, and let $A, B \in \omega_{P(\mathcal{H})}$ be quantum natural numbers. Further let $|\psi\rangle \in \mathcal{H}$ satisfy $A|\psi\rangle = a|\psi\rangle$ and $B|\psi\rangle = b|\psi\rangle$. Then

$$(A \dot{+} B)|\psi\rangle = (a+b)|\psi\rangle \quad \text{and} \quad (A \dot{\times} B)|\psi\rangle = ab|\psi\rangle.$$

Proof. Let $f(x,y)$ be either $x+y$ or xy, and $\hat{f}(A,B)$ either $A \dot{+} B$ or $A \dot{\times} B$, respectively. By lemma 6.19, it suffices to show that

$$[\hat{f}(A,B)]_{f(a,b)}|\psi\rangle = |\psi\rangle \quad \text{and} \quad [\hat{f}(A,B)]_{f(a,b)+1}|\psi\rangle = |0\rangle.$$

Now

$$[\hat{f}(A,B)]_{f(a,b)}|\psi\rangle = \bigvee_{f(k,l)\geq f(a,b)} \big(A_k \wedge B_l\big)|\psi\rangle = \bigvee_{\substack{f(k,l)\geq f(a,b)\\ k\leq a,\ l\leq b}} \big(A_k \wedge B_l\big)|\psi\rangle,$$

where the final equality holds by definition of the meet. Now, if $k \leq a$ and $l \leq b$, but $k+l \geq a+b$, then we must have $k = a$ and $b = l$ (and similarly if $kl \geq ab$). Hence we have that

$$[\hat{f}(A,B)]_{f(a,b)}|\psi\rangle = A_a \wedge B_b|\psi\rangle = |\psi\rangle$$

by lemma 6.21. Similarly, we have that

$$[\hat{f}(A,B)]_{f(a,b)+1}|\psi\rangle = \bigvee_{\substack{f(k,l)\geq f(a,b)+1\\ k\leq a,\ l\leq b}} \big(A_k \wedge B_l\big)|\psi\rangle,$$

but in this case $k \leq a$, $l \leq b$ and $f(k,l) \geq f(a,b)+1$ is impossible to satisfy! A careful reading of the definition shows that the join over an empty set is the bottom element, yielding the desired result. □

We then have the following promised corollary showing that our sum and product reduce to the ordinary (linear algebraic) sum and product on commuting observables.

Corollary 6.23. Let $\mathcal{H}$ be a complex finite dimensional Hilbert space, and let $A, B \in \omega_{P(\mathcal{H})}$ be quantum natural numbers such that $AB = BA$. Then $A \dot{+} B = A + B$, and $A \dot{\times} B = AB$.[1]

Proof. Let n be the dimension of $\mathcal{H}$. $AB = BA$ implies that there exists an orthonormal basis for $\mathcal{H}$ consisting of common eigenvectors of A and B — let $\{|\psi_i\rangle \ : \ i = 1, \ldots, n\}$ be a such a basis where $A|\psi_i\rangle = a_i|\psi_i\rangle$ and $B|\psi_i\rangle = b_i|\psi_i\rangle$. By proposition 6.22, we have that

$$(A \dot{+} B)|\psi_i\rangle = (a_i + b_i)|\psi_i\rangle \quad \text{and} \quad (A \dot{\times} B)|\psi_i\rangle = a_i b_i|\psi_i\rangle,$$

showing that $A \dot{+} B$ and $A \dot{\times} B$ agree with $A + B$ and AB, respectively, on a basis. Hence they must be the same linear operators. □

Of course, the results above can be appropriately extended to arbitrary polynomials involving $\dot{\times}$ and $\dot{+}$ by induction. We now demonstrate a property of this new sum and product which is even more intriguing than those demonstrated above — namely that for *any* observables A and B (not necessarily commuting), every eigenvalue of $A \dot{+} B$ is of the form $a + b$, where a and b are eigenvalues of A and B respectively. Similarly, every eigenvalue of $A \dot{\times} B$ is of the form ab with a and b eigenvalues of A and B, respectively.[2]

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[1] We take the symbol '+' to denote ordinary addition of linear operators, and concatenation to represent operator multiplication.

[2] As the reader will note from the statement of the proposition, this result applies beyond the context of projection lattices, and in fact holds for our generalization of the notion of 'eigenvalue' to any quantum natural number.

Proposition 6.24. Let L be a complete OML, and let $A, B \in \omega_L$ be quantum natural numbers. Then

1. If $c \in \sigma(A \dot{+} B)$, then $c = a + b$ with $a \in \sigma(A)$ and $b \in \sigma(B)$.
2. If $c \in \sigma(A \dot{\times} B)$, then $c = ab$ with $a \in \sigma(A)$ and $b \in \sigma(B)$.

Proof. Let $\hat{f}(A, B)$ be either $A \dot{+} B$ or $A \dot{\times} B$, and let $f(k, j)$ be $k + j$ or kj, respectively. Then if c is an eigenvalue of $\hat{f}(A, B)$, we have that (by lemmas 6.4 and 6.5)

$$\bigvee_{\substack{f(k,j) \geq c \\ k \in \sigma(A) \\ j \in \sigma(B)}} A_k \wedge B_j \neq \bigvee_{\substack{f(k,j) \geq c+1 \\ k \in \sigma(A) \\ j \in \sigma(B)}} A_k \wedge B_j .$$

Of course, every $A_k \wedge B_j$ occurring in the join on the RHS of the above expression is also in the LHS, so the only way the two expressions are not equal is if there is some $k, j \in \mathbb{N}$ such that (i) $k \in \sigma(A)$, (ii), $j \in \sigma(B)$, and (iii) $f(k, j) \geq c$ but $f(k, j) \ngeq c + 1$. Of course, (iii) is obviously equivalent to $f(k, j) = c$, yielding the desired result.

□

As with the previous result, this proposition can be extended by induction to any polynomial in the symbols '$\dot{+}$' and '$\dot{\times}$'.

6.4.3 Towards an Interpretation in Projection Lattices

We have shown above that our new arithmetical sum ($\dot{+}$) and product ($\dot{\times}$) on observables with whole number eigenvalues satisfy some remarkable properties. It would be even more remarkable if they were the *unique* sum and product satisfying these properties. Unfortunately, as of yet we do not know if this is the case.

However, we can find some criteria to put on candidates for a new sum/product which single out our new arithmetical operations. While these criteria do include one item which is of a technical mathematical nature, they are otherwise physically motivated.

Our first criteria is that the candidate sum (respectively, product) must 'respect eigenvectors'. By this, we mean that for two operators A and B and a common eigenvector $|\psi\rangle$ with eigenvalues a and b respectively, the candidate sum (resp. product) of A and B must have $|\psi\rangle$ as an eigenvector with eigenvalue the sum (resp. product) of a and b. We have proved that $\dot{+}$ and $\dot{\times}$ satisfy this criteria in proposition 6.22.

To state the second criteria, we need to discuss our interpretation of quantum natural numbers as observables with whole number eigenvalues. We recall that any such observable A is simply a sum of decreasing projection operators $A = \sum_{i=1}^{\sup A} A_i$ (so $i \leq j$ implies $A_j \leq A_i$). Such observables have a natural interpretation as a "sequence of filters". Of course, all the projectors A_i in the sequence commute, so we can think of a measurement of A as a measurement of all the A_i's successively, starting with A_1, then measuring A_2, and so on up through $A_{\sup A}$. Since these projectors are decreasing, once one measures a '0' outcome from one of these projectors, all the remaining projectors must also yield a '0'. Our second criteria is that the candidate sum (resp. product) operator respect this "filter interpretation" of the observables. More precisely, we require of our candidate sum (resp. product) that if $|\psi\rangle$ is both an eigenstate of the n^{th} filter of A with eigenvalue 1, and an eigenstate of the m^{th} filter of B with eigenvalue 1, then $|\psi\rangle$ must be an eigenstate of the $(n+m)^{\text{th}}$ filter in the case of the sum, and an eigenstate of the $(nm)^{\text{th}}$ filter in the case of the product, with eigenvalue 1 in both cases. Of course, we should show that our arithmetical sum and product satisfy this second criteria.

Proposition 6.25. Let $\mathcal{H}$ be a separable Hilbert space, let $A, B \in \omega_{P(\mathcal{H})}$ be quantum natural numbers, and let $|\psi\rangle \in \mathcal{H}$ be such that $A_n|\psi\rangle = B_m|\psi\rangle = |\psi\rangle$. Then $(A \dot{+} B)_{n+m}|\psi\rangle = |\psi\rangle$ and $(A \dot{\times} B)_{nm}|\psi\rangle = |\psi\rangle$.

Proof. Since $A_n|\psi\rangle = B_m|\psi\rangle = |\psi\rangle$, we have $(A_n \wedge B_m)|\psi\rangle = |\psi\rangle$ by lemma 6.21. Since clearly $n+m \geq n+m$ and $nm \geq nm$, the result then follows trivially from the definition of $\dot{+}$ and $\dot{\times}$. □

We now state and prove our uniqueness results for $\dot{\times}$ and $\dot{+}$ in the following two propositions.

Proposition 6.26. Let $\mathcal{H}$ be a complex finite dimensional Hilbert space, and let $* : \omega_L^2 \to \omega_L$ be a map satisfying (for every $A, B \in \omega_L$).

1. For every $n \in \mathbb{N}$, $(A * B)_n$ is a lattice polynomial[1] in the A_i's and B_j's *which is independent of A and B.*

2. For $|\psi\rangle \in \mathcal{H}$ such that $A_n|\psi\rangle = B_m|\psi\rangle = |\psi\rangle$ with $n, m \in \mathbb{N}$, we have that $(A * B)_{nm}|\psi\rangle = |\psi\rangle$.

3. If $|\psi\rangle$ is a simultaneous eigenstate of A and B with eigenvalues a and b, respectively, then $|\psi\rangle$ is an eigenstate of $A * B$ with eigenvalue ab.

Then $A * B = A \dot{\times} B$ for every $A, B \in \omega_L$.

[1] That is, a polynomial in $\wedge$ and $\vee$, but not $\neg$.

Proof. We first show that

$$(A * B)_n \geq (A \dot{\times} B)_n \quad \text{for all } n \in \mathbb{N}. \tag{6.9}$$

First, recall that $(A \dot{\times} B)_n = \bigvee_{jk \geq n} (A_j \wedge B_k)$, and so it suffices to show that $(A * B)_n \geq A_j \wedge B_k$ for all $j, k \in \mathbb{N}$ with $jk \geq n$. In order to show this, we need to show that for any $|\psi\rangle \in \mathcal{H}$ with $A_j \wedge B_k|\psi\rangle = |\psi\rangle$, we have $(A * B)_n|\psi\rangle = |\psi\rangle$. But since $A_j \geq A_j \wedge B_k$ (and similarly for B_k, we have that $A_j|\psi\rangle = B_k|\psi\rangle = |\psi\rangle$, and so the desired inequality (equation 6.9 follows directly by assumption (2) above (and the fact that $A * B \in \omega_L$ so that the $(A * B)_i$'s form a decreasing sequence).

Now, using assumption (1) and lemma C.3 (and recalling that the A_i's and B_j's form decreasing sequences), we have that (for any $n \in \mathbb{N}$)

$$(A * B)_n = \bigvee_{i,j \in \mathbb{N}} (A_{a_i} \wedge B_{b_j}) \tag{6.10}$$

for some set of a_i's and b_j's in $\mathbb{N}$ (and also note that wlog we can assume the join is actually finite).

We will finally show the desired equality by contradiction — to this end assume that there are some $A, B \in \omega_L$ and $n \in \mathbb{N}$ for which the inequality in equation 6.9 is strict. If this is so, we immediately see that there must be $a, b \in \mathbb{N}$ with $ab < n$ that appear in the join in equation 6.10, and since we are assuming, by (1), that the lattice polynomials defining $A * B$ are independent of A and B, this must be true for that n *regardless* of the particular A and B under consideration.

In particular, we can consider an A and B such that there is a $|\psi\rangle \in \mathcal{H}$ such that $A_a|\psi\rangle = B_b|\psi\rangle = |\psi\rangle$ and also $A_{a+1}|\psi\rangle = B_{b+1}|\psi\rangle = |0\rangle$, i.e. $|\psi\rangle$ is an eigenvector of A with eigenvalue a and an eigenvector of B with eigenvalue b. Hence, by assumption (3), we must have that $|\psi\rangle$ is an eigenvector of $A * B$ with eigenvalue ab, i.e. $(A * B)_{ab}|\psi\rangle = |\psi\rangle$ while $(A * B)_{ab+1}|\psi\rangle = |0\rangle$. But since $ab < n$, we have $ab + 1 \leq n$, so we must have $(A * B)_n|\psi\rangle = |0\rangle$ as well. However, since $A_a \wedge B_b \leq (A * B)_n$ by lemma 6.21, we must have $(A * B)_n|\psi\rangle = |\psi\rangle$ which is a contradiction, completing the proof. □

Proposition 6.27. Let $\mathcal{H}$ be a complex finite dimensional Hilbert space, and let $* : \omega_L^2 \to \omega_L$ be a map satisfying (for every $A, B \in \omega_L$).

1. For every $n \in \mathbb{N}$, $(A*B)_n$ is a lattice polynomial in the A_i's and B_j's *which is independent of A and B.*
2. For $|\psi\rangle \in \mathcal{H}$ such that $A_n|\psi\rangle = B_m|\psi\rangle = |\psi\rangle$ with $n, m \in \mathbb{N}$, we have that $(A*B)_{n+m}|\psi\rangle = |\psi\rangle$.
3. If $|\psi\rangle$ is a simultaneous eigenstate of A and B with eigenvalues a and b, respectively, then $|\psi\rangle$ is an eigenstate of $A * B$ with eigenvalue $a + b$.

Then $A * B = A \dot{+} B$ for every $A, B \in \omega_L$.

Proof. The proof of this proposition is almost identical to the proof of proposition 6.26, except that all the multiplications are switched to additions. □

From the above propositions, we see that our new arithmetical sum and product of observables are the unique operations satisfying the physically motivated criteria spoken of earlier (conditions (2) and (3) in the above propositions), along with a technical mathematical requirement (condition (1)). Hopefully, we will be able to go beyond the above results and come up with entirely physical criteria that single out our new sum and product operations on observables. Such criteria would be helpful in arriving at an interpretation of this new arithmetical sum and product, opening the door to possible applications of these operations in quantum computation and quantum information. As of yet, this is still a work in progress.

CHAPTER 7

CONCLUSION

We have taken quite a journey. Starting from the simple premise that the reasoning we use to develop mathematics should be based on the natural logical structure encoded in the statements one can make concerning the measurement of quantum observables, we have managed to construct a rich and powerful first-order quantum logic, and using this formal logic, we have discovered a plethora of new and intriguing quantum mathematical structures.

In this final chapter, we summarize the major results of our investigation of quantum logic and quantum mathematics (section 7.1). Following this summary, we then discuss some of the myriad open problems and new questions our investigation has unearthed in section 7.2. We then conclude this work with a few brief comments in section 7.3.

7.1 Summary of Major Results

The first task accomplished was the construction of a first-order quantum logic powerful enough to develop quantum mathematics. This was done in terms of both a formal deduction system as well as a semantics, which we proved were both sound and complete with respect to one another. We also proved a more powerful completeness result, namely that our first-order quantum logic is sound and complete with respect to a restricted semantics where one only allows models with irreducible truth value algebras.

We then saw that this more powerful completeness result allowed us to use statements with '$\exists$' in the usual way — namely to guarantee the existence of certain objects in any given model satisfying certain formal sentences. Prior to this work, there was no known way to accomplish this task in first-order quantum logic.

After demonstrating these fundamental results concerning first-order quantum logic in and of itself, we proceeded to investigate quantum mathematics by first examining languages where the sole predicate represented 'equality'. After discussing two different formal sentences asserting that this equality predicate is transitive (a weak and a strong version), we proceeded to show that a number of algebraic axiom systems were inherently classical[1] when these axioms included the strong transitivity of equality, suggesting the "classicalness" of this property.

We then presented a number of illustrative algebraic systems in quantum logic, including quantum versions of monoids, groups, orthomodular lattices, vector spaces, and operator algebras. Of course, (classical) orthomodular lattices, as well as Hilbert spaces and the set of bounded operators on a Hilbert space, all have natural associated orthomodular lattices.[2] Using this association, we constructed natural models of (a choice of) the OML axioms, the vector space axioms, and the operator algebra axioms. What is more, we found that these three natu-

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[1]Recall that an axiom system is inherently classical if all the theorems that can be proved from that system are the same under either classical and quantum logic.

[2]In the case of an OML L, the associated OML is just L. For a Hilbert space $\mathcal{H}$, or for the bounded linear operators on $\mathcal{H}$, the associated OML is the projection lattice of $\mathcal{H}$.

ral classes of quantum mathematical structures all have a beautiful interplay and compatibility with one another.

To conclude our examination of algebraic systems in quantum mathematics, we made some brief preliminary investigations into interpreting the Schrödinger and von Neumann equations in the framework of our aforementioned natural models. While we noted that in principle such an interpretation could yield not only unitary but also measurement evolutions from these equations, we fell short of achieving this practically, and instead only managed to show that a simple formal statement classically equivalent to the von Neumann equation yielded only the usual unitary dynamics in our natural models of operator algebras. However, since it would easily have been possible for a quantum mathematical treatment of the von Neumann equation to allow evolutions which did not even take states to states (but rather to linear operators not representing states at all), we considered this to be a not entirely unpromising result.[1]

We then began an investigation of axiomatic set theory in the framework of quantum mathematics. After presenting a natural reduction of the usual ZFC axioms, we demonstrated an intuitive quantum model of these axioms which combines a rich and powerful structure generalizing the notion of ordinary sets with a mathematical elegance making it possible to investigate quantum set theory to a degree not easily achievable with the quantum set theory developed previously by Takeuti.

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[1] We have examined various other reductions of the von Neumann equation which do allow for some measurement evolutions, but this is very much a work in progress.

We then used the quantum set theory we developed to construct a quantum version of the natural numbers (in fact, one for every complete OML L) in a fashion analogous to the construction of the usual natural numbers in the context of classical set theory, and then presented a natural addition and multiplication on these quantum natural numbers. We discovered that these quantum natural numbers satisfied an axiomatization of the successor fragment of Peano arithmetic for any such L. As noted previously, there can be no non-standard model of the usual axioms of full Peano arithmetic (by a theorem of Dunn), and as such we examined our quantum natural numbers in the context of a weaker axiomatization of arithmetic still stronger than a commonly studied axiomatization — namely that of Robinson arithmetic. Surprisingly, we found that the quantum natural numbers we constructed satisfied these arithmetical axioms *if and only* if the L used in their construction was modular, yielding an arithmetical characterization of this important lattice-theoretic property.

We then considered our quantum natural numbers for a particular case of utmost interest for us — namely those for which the OML L used in their construction was the projection lattice of a complex separable Hilbert space $\mathcal{H}$. For this case, we found that the quantum natural numbers are entirely equivalent to the set of usual quantum mechanical observables on $\mathcal{H}$ which are bounded and possess only whole number eigenvalues. In particular, when $\mathcal{H}$ is finite dimensional (so that the projection lattice is modular, whereby the associated quantum natural numbers satisfy the arithmetical axioms), the quantum natural numbers are in *1-1 correspondence* with all the observables which have solely whole number eigenvalues. In this context, we can interpret our arithmetical sum and product as a "new" sum and product on such quantum observables, and we find that this new sum and product have some remarkable properties.

Like the usual operator sum and product from (classical) linear algebra, our new arithmetical sum and product "respect eigenvectors", which is to say that for two observables A and B with a common eigenvector $|\psi\rangle$ (having eigenvalues a and b, respectively), the arithmetical sum of A and B also has $|\psi\rangle$ as an eigenvector with eigenvalue equal to the sum of a and b (and similarly for the product). However, in complete contrast to the operator sum and product, our new sum and product also "respect *eigenvalues*", in that when A and B do *not* commute, the new sum (resp. product) of A and B has eigenvalues which are each the sum (resp. product) of an eigenvalue of A with an eigenvalue of B. Since eigenvalues represent the possible outcomes of the measurement of observables, this strongly suggests that our new sum and product have an extremely natural physical interpretation related to measurement. We have made some preliminary investigations into this interpretation, but so far a natural and purely physical elucidation of this arithmetical sum and product has eluded us.

7.2 Looking to the Future

This work has left us with a number of fascinating open questions, the first of which are related to the first-order logic we have formulated. Just as we have proved a more powerful completeness result allowing the restriction of our semantics to irreducible models, we would like to know if there are other subclasses of models for which an analogous completeness result holds. In the classical case, we have completeness with respect to the single Boolean algebra B_2, and this algebra has other interesting properties beyond irreducibility, such as being subdirectly irreducible[1] and being complete as a lattice. We would like to know if our first-order quantum logic is complete with respect to models whose truth value algebras are either all subdirectly irreducible or all complete. Also, we have treated reduction of axioms in a bit of an ad hoc fashion, trying to find "suitable" reductions as needed. It would be incredibly useful to have a general "theory" of reduction which, in some way, "explains" why certain reductions are the "correct" reductions for our first-order quantum logic. Finally, as mentioned before, our approach to quantum logic can be thought of as a "first-order perturbation" away from classical logic toward a "true" quantum logic, and so the natural next step would be to create a "second-order perturbation". The most obvious way to do this would be to use *quantum* set theory in the formal construction of $\mathcal{L}$-structures (for any language $\mathcal{L}$),[2] and this is definitely an important future area to investigate.

Moving beyond questions concerning only the form and properties of the logic we have developed, we would also like a better understanding of the properties of specific mathematical systems. For example, we would like to know which M-systems are inherently classical — ideally, this understanding would tie in with the aforementioned desired theory of reduction, and help to explain why certain sets of wffs are more "suitable" for use as axioms in quantum mathematics. Also, at a more concrete level, there are vast new vistas of mathematical possibilities opened up for investigation which include not only the reinvestigation of every system of axioms heretofore

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[1]For a statement of this important universal algebraic property, see Burris (9).

[2]Recall that, even for our quantum set theory, all of our models are built with classical underlying sets (or classes).

considered in classical mathematics (such as groups, rings, fields, topological spaces, manifolds, category theory, etc.), but also (since quantum logic is subclassical) there is the possibility of investigating systems of axioms which are *classically inconsistent*.

Of course, one of the most intriguing areas to investigate in this general context is a quantum mathematical treatment of the theory of quantum mechanics itself. Without a doubt, (and as mentioned earlier) one of the most interesting questions raised is whether one can obtain exactly the unitary and measurement evolutions as solutions to some reduced version of the von Neumann equation as interpreted in quantum linear algebra.

Extending our investigations into set theory may be extremely helpful in addressing virtually all of the problems just posed. Of course, the ultimate goal is to show that some quantum set theory is indeed rich enough to provide a foundation for all of quantum mathematics, and for this we will almost certainly need to develop a yet richer non-standard model of quantum set theory than the one presented here. Interpreting our sets as a "symmetric subclass" of these "true" quantum sets[1] will hopefully prove a useful starting point in the development of this richer quantum set theory. Of course, the quantum set theory we have developed is already quite powerful, and it will no doubt prove highly worthwhile to investigate how much of quantum mathematics can be founded on these simple non-standard models — one natural place to start this investigation is with a study of "quantum ordinals" as well as "quantum cardinalities"

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[1]See the discussion in section 5.2.

in our quantum set theory. Also, another interesting question in regards to these models is whether one can find a "better" reduced version of the ZFC axioms which they satisfy.

One area of quantum mathematics that we have already shown can be developed within our quantum set theory is quantum arithmetic. Given the intriguing developments which have occurred in our preliminary investigation of this arithmetic, it is of paramount importance to pursue this further. One fascinating question is whether one can obtain a recursively enumerable[1] axiomatization of our natural models of quantum arithmetic which is strong enough to prove *every* first-order arithmetical statement which holds in these models.[2]

Also of paramount importance is to go beyond the quantum natural numbers to "quantum real numbers". Although the work is still in progress, we have managed such a generalization, and one immediate goal of this work is to be able to construct this generalization using our quantum set theory. Once the "quantum real numbers" can be thoroughly understood, this opens up the possibility of developing a "quantum calculus", which brings up the most wildly optimistic hope we will dare to utter. This hope is that if one begins with (a proper choice of) the usual equations of classical physics, and then interprets these equations as "quantum differential equations" over the "quantum real numbers", the development of this theory within quantum mathematics would yield *exactly* either quantum mechanics or quantum field the-

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[1] Essentially, an axiomatization is recursively enumerable if an iPhone (given an infinite amount of time) could write it down.

[2] Of course, the content of Gödel's (first) incompleteness theorem is that one cannot do this for the ordinary natural numbers in classical logic.

ory (depending on the exact classical equations used). If this hope turned out to be true, it would yield a brave new interpretation of why the physical world behaves in a "quantum" fashion. Namely, we would see that the basic *equations* of classical physics are indeed true, but the physics heretofore deduced from these equations (i.e. the entire content of classical physical theory) doesn't describe the world at a fundamental level because we have derived the *wrong* consequences of these classical equations by using the *wrong* logic to reason about the mathematics describing the physical world. While, as we have said, hoping for such developments would be wildly optimistic, if this project were achieved it would be nothing short of revolutionary.

7.3 Final Remarks

As the preliminary investigations of this work have revealed, quantum logic opens up the doors to a brave new world of mathematical inquiry. It is beyond doubt that this new quantum mathematics contains astounding and marvelous mathematical vistas which are as of yet unexplored. What is more, this exploration of quantum mathematics has the potential to both revolutionize the foundations of quantum mechanics as well as provide new insights into the nature of logic and thought itself. This work has documented but a few miniscule steps into the unimagined vastness of intellectual investigation that quantum mathematics promises, and we look forward with delight to what wonders may yet be uncovered in this fascinating field.

APPENDICES

Appendix A

ELEMENTARY SET THEORY

In this appendix we provide enough elementary (classical) set theory to obviate the need for outside references. We proceed from the perspective of naive set theory, but the interested reader is referred to (19) to see this material developed axiomatically.

A.1 Basic Notions

There are three primitive notions in naive set theory, that of *object*, *set*, and *being an element of*.[1] Everything in the universe of discourse is an object, but only certain objects are sets. 'Being an element of' is a binary relation between an object and a set, and for a given object a and set A we write $a \in A$ to mean that a is an element of A, and $a \notin A$ to mean that a is not an element of A. Note that since sets are (special cases of) objects, we can form sets of sets, and sets of sets of sets, and so on. We will also use the primitive logical notion of *being identically equal to*.[2] Two objects a and b are identically equal if they are the same object, in which case we write $a = b$. From these primitive notions we will proceed to develop all of the set theory needed for this document.

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[1]In contrast to ZFC which has no distinct notion of object, or, equivalently, in which the only objects are sets. However, for a development of axiomatic set theory in which not all objects are sets, see (39).

[2]This differs from our treatment axiomatic set theory in chapter 5 in which we treat equality as a defined notion.

Appendix A (Continued)

First, sets are determined entirely by their members, so that two sets are identically equal if and only if they have the same members. This is known as the property of *extensionality*.

Proposition A.1. Let A and B be sets. Then $A = B$ if for every object a, $a \in A$ iff $a \in B$.

We can then define the notion of subset.

Definition A.1. Let A and B be sets. Then A is a subset of B (denoted $A \subseteq B$, or $B \supseteq A$) if, for every object $a \in A$, we have that $a \in B$.

Note: For any set A, we have $A \subseteq A$. Also, by extensionality, for any two sets A and B, we have that $A = B$ if and only if both $A \subseteq B$ and $B \subseteq A$.

Using extensionality, we can define sets by precisely specifying their members.

Definition A.2. Let A, B be sets. Define

Empty Set: The set containing no elements (denoted $\varnothing$).

A union B: The set whose elements are exactly those objects a which belong to either A or B (denote $A \cup B$).

Finite lists: Given any finite list of objects $a_1, \ldots, a_n$, we can form the set whose elements are exactly those objects in this list. Denote this set by $\{a_1, \ldots, a_n\}$.

A intersect B: The set whose elements are exactly those objects a such that a belongs to both A and B (denoted $A \cap B$).

The power set of A: The set whose elements are exactly the subsets of A (including $\varnothing$ and A itself). Denote this set by $\mathcal{P}(A)$.

Appendix A (Continued)

Set builder notation: For any set A, and any property[1] P, we can form the set whose elements are exactly those elements $a \in A$ such that a satisfies P, which we denote $\{a \in A : a \text{ satisfies } P\}$.

Note: First, for any set A, we have that $\varnothing \subseteq A$, since every element of $\varnothing$ is an element of A vacuously. Also, for any two sets A and B we have $A \cap B \subseteq A \subseteq A \cup B$. Third, note that when we use set builder notation, we must begin with some set A (rather than just taking the collection of all objects satisfying some property P) in order to avoid Russel's paradox.[2] However, we will occasionally use set-builder notation without this restriction, but doing so will (in general) only result in a class,[3] not necessarily a set.

Set builder notation allows us a large amount of flexibility in construction sets. For example consider the following:

Definition A.3. Let A and B be sets. We define A *subtract* B (denoted $A \backslash B$) by

$$A \backslash B := \{a \in A : a \notin B\}.$$

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[1]As discussed here, the notion of a 'property' is a little nebulous, although the intuitive notion should suffice. In the framework of predicate logic we could replace this with a more precise treatment — instead of a 'property P', we would utilize a wff $\psi(x)$ (where x is a free variable), and we would then replace the phrase 'a satisfies P', with the formal assertion '$\psi(a)$'.

[2]We take the property P to be the property of not being an element of oneself. Then if we consider the set S of all sets which are not members of themselves (i.e. $S := \{x : \psi(x)\}$, where $\psi(x) := x \notin x$). If $S \notin S$, then $S \in S$ by definition of $\psi(x)$, but if $S \in S$, then by the definition of $\psi(x)$, we have$S \notin S$. Either possibility leads to a contradiction, and so S cannot be a set.

[3]See section A.2.5.

Finally, we will also consider intersections and unions of arbitrary (not necessarily finite) collections of sets. They are defined as follows — let $\mathcal{A}$ be a set all of whose elements are sets. Then we define the sets[1]

$$\bigcap \mathcal{A} := \{a \; : \; a \in A \text{ for every } A \in \mathcal{A}\}$$
$$\bigcup \mathcal{A} := \{a \; : \; a \in A \text{ for every } A \in \mathcal{A}\}. \tag{A.1}$$

We will sometimes use the alternate notation

$$\bigcup_{A \in \mathcal{A}} A := \bigcup \mathcal{A}, \qquad \bigcap_{A \in \mathcal{A}} A := \bigcap \mathcal{A}.$$

A.2 Relations and Maps

A.2.1 Relations on Sets

In order to define what we mean by a relation on a set, we first need a notion of ordered pair. The basic idea is that, for two objects a and b, we want a new object (a, b) such that $(a, b) = (c, d)$ if and only if $a = b$ and $c = d$. While we could simply introduce such a thing as a primitive notion, it is instructive to see how we can use the basic notions above to *build* ordered pairs from scratch.

[1] The reader may be worried that, as per the discussion following def. A.2, that these intersections and unions may not actually be sets. This is not the case — the interested reader is referred to Enderton (19).

Definition A.4. Let a and b be objects. Define

$$(a, b) \coloneqq \{\{a\}, \{a, b\}\},$$

and then any such (a, b) is called an *ordered pair.* Then, for any objects $a_1, \ldots, a_n$ define inductively

$$(a_1, \ldots, a_n) \coloneqq ((a_1, \ldots, a_{n-1}), a_n),$$

and call $(a_1, \ldots, a_n)$ an *ordered n-tuple* (or simply a *tuple*).

It is straightforward to see that ordered pairs so defined have the desired property — we include the proof since it is instructive to see such things worked out in gory detail at least once.

Proposition A.2. Let a, b, c, d be objects. Then $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Proof. First, assume that $a = c$ and $b = d$. Then $\{a, b\} = \{c, d\}$ by extensionality. Hence, also by extensionality, we have that

$$(a, b) = \{\{a\}, \{a, b\}\} = \{\{a\}, \{c, d\}\} = \{\{c\}, \{c, d\}\} = (c, d).$$

For the converse, assume $(a, b) = (c, d)$. Then $\{a\} \in (c, d)$, so either $a = c$ or $\{a\} = \{c, d\}$. If $a = c$, then by extensionality we must have $\{a, b\} = \{c, d\}$, but this means that we must have $b = d$, again by extensionality. If, on the other hand, we have $\{a\} = \{c, d\}$ then we must also have $a = c = d$, and since we must also have $\{a, b\} = \{c\}$, this means that $a = b = c = d$. $\square$

Appendix A (Continued)

Now that we have ordered pairs at our disposal, we can construct *sets* of ordered pairs. Begin with any two objects a, b and two sets A, B such that $a \in A$ and $b \in B$. Note that $\{a\} \in \mathcal{P}(A) \subseteq \mathcal{P}(A \cup B)$. Also, $\{a, b\} \subseteq A \cup B$, so $\{a, b\} \in \mathcal{P}(A \cup B)$. Hence $(a, b) \subseteq P(A \cup B)$, i.e. $(a, b) \in \mathcal{P}(\mathcal{P}(A \cup B))$. This leads us to the definition

Definition A.5. Let A and B be sets, and define the *product of* A *and* B (denote $A \times B$) by

$$A \times B := \{(a, b) \in \mathcal{P}(\mathcal{P}(A \cup B)) : a \in A \text{ and } b \in B\}.$$

Let $A_1, \ldots, A_n$ be a finite list of sets. Then define the *product of the* A_i's (denote $A_1 \times \cdots \times A_n$, or $\prod_{i=1}^{n} A_i$) inductively by

$$\prod_{i=1}^{n} A_i := (A_1 \times \cdots \times A_{n-1}) \times A_n.$$

If $A_1 = \cdots = A_n$ we define $A^n := A_1 \times \cdots \times A_n$, and also define $A^0 := \{\varnothing\}$.

We are now in a position to define relations on sets. Let us first consider an example of a binary relation, namely the relation 'strictly less than' ($<$) on the integers $\mathbb{Z}$. It is customary to think of the statement $n < m$ as specifying something *intrinsic* about the pair of integers n and m, namely that $m - n$ is positive. However, we could just as easily think of the relation $<$ extrinsically — $<$ simply subdivides pairs of integers (i.e. elements of $\mathbb{Z} \times \mathbb{Z}$) into two sets, those pairs (n, m) for which $n < m$ is true, and those for which $n < m$ is false. In fact, keeping track of both of these sets is redundant — for each pair $(n, m) \in \mathbb{Z} \times \mathbb{Z}$, we know that $n < m$ is true or false but not both. Hence, we may think of the relation $<$ as being defined by the set of pairs (n, m) for which $n < m$ is true. This motivates the general definition of relations on sets.

Appendix A (Continued)

Definition A.6. Let $n \in \{1, 2, 3, \ldots\}$, and let $A_1, \ldots A_n$ be sets. Then an *n-ary relation* between $A_1, \ldots,$ and A_n. is a subset $R \subseteq A_1 \times \cdots \times A_n$. If $A_1 = \cdots = A_n = A$, then we say that R is an *n-ary relation on A*. If $n = 1$ we call R *unary*, and if $n = 2$ we call the relation *binary*. For R a binary relation we define $aRb := (a, b) \in R$.

Note: Unary relations are essentially just predicates, for example if we define $P \subseteq \mathbb{Z}$ by $P := \{n \in \mathbb{Z} : z > 0\}$ then P is just the predicate of positivity. For the example discussed above, we would define $< := \ \{(n, m) \in \mathbb{Z} \times \mathbb{Z} : m - n \in P\}$, which would be a binary relation on $\mathbb{Z}$.

A.2.2 Equivalence Relations and Partitions

Some of the most important relations are equivalence relations.

Definition A.7. Let A be a set, and $\sim$ be a binary relation on A. Then $\sim$ is called an *equivalence relation* on A if $\sim$ satisfies the following three properties.

Reflexivity: $a \sim a$ for all $a \in A$

Symmetry: If $a \sim b$, then $b \sim a$ for all $a, b \in A$

Transitivity: If $a \sim b$ and $b \sim c$, then $a \sim c$ for all $a, b, c \in A$

If $\sim$ is an equivalence relation, then for any $a \in A$, the set $\{b \in A \ : \ b \sim a\}$ is called the *equivalence class of a modulo* $\sim$, and we denote this equivalence class $[a]_\sim$.

There is a natural relationship between equivalence relations and partitions.

Definition A.8. Let A be a set. Then a set $\mathcal{A} \subseteq \mathcal{P}(A)$ is called a *partition of A* if both

1. $X \cap Y = \varnothing$ for all $X, Y \in \mathcal{A}$,

2. $\bigcup \mathcal{A} = A$.

The natural relationship is given in the following proposition (see Enderton (19) for a proof).

Proposition A.3. Let A be a set, let $\mathcal{A}$ be a partition of A, and let $\equiv$ be an equivalence relation on A. Define the binary relation $\sim$ on A by (for all $a, b \in A$)

$$a \sim b \quad \text{iff there exists some } X \in \mathcal{A} \text{ with both } a, b \in X,$$

and define

$$\mathfrak{X} := \{X \in \mathcal{P}(A) \ : \ X = [a]_{\equiv} \text{ for some } a \in A.\}.$$

Then $\sim$ is an equivalence relation, $\mathfrak{X}$ is a partition of A, and $\sim\, = \,\equiv$ iff $\mathcal{A} = \mathfrak{X}$.

A.2.3 Maps Between Sets

Now that we have defined relations, we can define maps between sets. The usual way of thinking of a map f from some set A to another B, is that it is a rule such that for every $a \in A$, we assign some object $f(a)$ which must be in B. We can make this all very explicit in terms of relations, without the need of any additional machinery.

Definition A.9. Let A and B be sets. A binary relation f between A and B is said to be a *map* (or equivalently a *function*) *from A to B* (denoted $f : A \to B$) if for every $a \in A$, there is a unique $b \in B$ such that $(a, b) \in f$. For a given a, we denote this unique b by $f(a)$. If f is a function from A to B, we say that A is the *domain of* f (denoted dom f), and define the *range of* f (denote ran f) to be the set $\{b \in B \ : \ b = f(a) \text{ for some } a \in A\}$.

Ex: Consider the function on the integers which assigns to any number its square, i.e. $n \mapsto n^2$. The relation f which corresponds to this function is $\{(n, n^2) \in \mathbb{Z} \times \mathbb{Z} \ : \ n \in \mathbb{Z}\}$.

Note: By the above definition, the codomain is not a well-defined notion for any given map. For example, the set f defined above may have codomain of $\mathbb{Z}$, or only the non-negative integers depending on context, and in general $f : A \to B$ implies that $f : A \to \operatorname{ran} f$ (*but not conversely*). This will not be relevant for us, but is worth mentioning since it can be a point of confusion for those more accustomed to thinking of functions from a category-theoretic perspective.

Every function defines a number of sets which it will be useful to designate.

Definition A.10. Let A, B, and C, be sets, and $f : A \to B$, and $g : B \to C$. Further let $A_0 \subseteq A$ and $B_0 \subseteq B$.

1. The *image of* A_0 *under* f (denoted $f(A_0)$) is defined by

$$f(A_0) := \{b \in B \ : \ b = f(a) \text{ for some } a \in A_0\}.$$

2. The *preimage of* B_0 *under* f (denoted $f^{-1}(B_0)$) is defined by

$$f^{-1}(B_0) := \{a \in A \ : \ f(a) \in B_0\}.$$

3. The *composition of f with g* (denoted $g \circ f$) is the set

$$g \circ f := \{(a, c) \in A \times C \ : \ (a, b) \in f \text{ and } (b, c) \in g \text{ for some } b \in B\}.$$

4. The *restriction of f to A_0* (denoted $f|_{A_0}$) is given by

$$f|_{A_0} := \{(a, b) \in f \ : \ a \in A_0\}.$$

Note: The image of A under f is just the range of f, i.e. $f(A) = \text{ran } f$. Also, the composition $g \circ f$ is clearly a function from A to C, and the restriction $f|_{A_0}$ is a map from A_0 to B.

Given two sets A and B, it will be useful to consider the set of all maps between them.

Definition A.11. Let A and B be sets. Define

$$B^A := \{f \in A \times B \ : \ f \text{ is a map from } A \text{ to } B\}. \tag{A.2}$$

A.2.4 Isomorphisms of Sets

Having functions at our disposal, we can now define a notion of isomorphism of sets.

Definition A.12. Let A and B be sets, and $f : A \to B$. If for any $a, b \in A$, we have that $f(a) = f(b)$ implies that $a = b$, we say that f is *1-1*, or equivalently that f is *injective.* If, for any $b \in B$ there exists some $a \in A$ such that $f(a) = b$, then we say f is *onto*, or equivalently *surjective.*

If f is both injective and surjective, we say that f is an *set isomorphism*, or equivalently that f is *bijective*, and we then say that A and B are *isomorphic as sets*, or *in 1-1 correspondence.*

Note: Finite sets are isomorphic iff they contain the same number of elements.

At this point we would like to claim that '*being isomorphic*' is an equivalence relation, but we face a technical hurdle, namely that the collection of all sets is not a set, and equivalence relations are defined on sets. If not for this, we could take equivalence classes of sets under the relation of being isomorphic, and use this to define a notion of number (cardinality) generalizing the finite numbers. As it stands, we will have to take another approach.

Before moving on, we would like to note an interesting isomorphism between the subsets of a given set A and the maps from A into the standard truth values $\{0,1\}$.

Proposition A.4. Let A be a set, and define $2 := \{0,1\}$. Then $\mathcal{P}(A)$ is isomorphic to 2^A.

Proof. Define $f : 2^A \to \mathcal{P}(A)$ by, for every $t : A \to 2$,

$$f(t) := \{a \in A \ : \ t(a) = 1\}.$$

f so defined is clearly a map from 2^A to $\mathcal{P}(A)$. This map is 1-1, since if $f(t) = f(s)$ for any two $t, s : A \to 2$, then for any $a \in A$ either $t(a) = 1$ or $t(a) = 0$. If $t(a) = 1$, then $a \in f(t) = f(s)$ so $s(a) = 1 = t(a)$, and if $t(a) = 0$ then $a \notin f(t) = f(s)$ so that $s(a) \neq 1$, and so $s(a) = 0 = t(a)$. Hence $s(a) = t(a)$ for every $a \in A$, and so $s = t$, showing that f is 1-1.

To see that f is onto, consider $X \subseteq \mathcal{P}(A)$. Define $u : A \to 2$ by (for every $a \in A$)

$$u(a) := \begin{cases} 1 & \text{if } a \in X \\ \\ 0 & \text{if } a \notin X. \end{cases}$$

Clearly $u \in 2^A$, and also $f(u) = X$, so that f is onto. Hence, f is an isomorphism, establishing that $\mathcal{P}(A)$ is isomorphic to 2^A. □

Note: An intuitive way to understand this isomorphism is to think of a subset of A as being those elements a such that the statement '$a \in A$' is true.

A.2.5 Classes of Sets and Class Functions

We have already noted that the collection of all sets satisfying some given property may not be a set. However, for some given property P, we would like to be able to refer to the collection of all sets satisfying this property as *something*, and this motivates the definition of a *class*.

Definition A.13. A *class* is any collection of objects. If a class $\mathfrak{K}$ is not a set, we say that $\mathfrak{K}$ is a *proper class*.

Ex: The collection of all sets is a proper class, while the collection of all sets contained in a given set A is a class, but not a proper class (indeed, this class is just $\mathcal{P}(A)$, the power set of A).

For a given property P, we then use the following notation (similar to set-builder) to designate classes. Namely,

$$\{a \ : \ a \text{ satsifies property } P\} \tag{A.3}$$

Appendix A (Continued)

is defined to be the class of all objects satisfying property P. We then abuse the '$\in$' notation, and so for a given class $\mathfrak{K}$ and object a, we write '$a \in \mathfrak{K}$' to mean that 'the object a is in the class $\mathfrak{K}$'. Also, for two given class $\mathfrak{K}_1$ and $\mathfrak{K}_2$, we define

$$\mathfrak{K}_1 \cup \mathfrak{K}_2 := \{a \ : \ a \in \mathfrak{K}_1 \text{ or } a \in \mathfrak{K}_2\},$$

and write $\mathfrak{K}_1 \subseteq \mathfrak{K}_2$ iff for every $a \in \mathfrak{K}_1$, we also have $a \in \mathfrak{K}_2$.

Definition A.14. If, for two given classes $\mathfrak{K}_0$ and $\mathfrak{K}$, every object $a \in \mathfrak{K}_0$ also satisfies $a \in \mathfrak{K}$, then we say that $\mathfrak{K}_0$ is a *subclass of* $\mathfrak{K}$ (denoted $\mathfrak{K}_0 \subseteq \mathfrak{K}$).

This motivates the definition of a class function. Recall that a function is a subset of a product set satisfying the "function property", namely if A is the domain of a function f, then for every $a \in A$ such that both $(a,b) \in f$ and $(a,c) \in f$, we have $b = c$. This leads us to define a *class function*.

Definition A.15. Let $\mathfrak{F}$ be a class such that every element of $\mathfrak{F}$ is an ordered pair, and for any objects a, b and c such that $(a,b) \in \mathfrak{F}$ and $(a,c) \in \mathfrak{F}$, we have $b = c$. Then $\mathfrak{F}$ is said to be a *class function*, and we define $\mathfrak{F}(a)$ to be the unique object b such that $(a,b) \in \mathfrak{F}$. We also say that, for classes $\mathfrak{L}$ and $\mathfrak{K}$, a class function $\mathfrak{F} \in \mathfrak{L}^{\mathfrak{K}}$ if for all $(a,b) \in \mathfrak{F}$, we have $a \in \mathfrak{F}$ and $b \in \mathfrak{L}$.

<u>Note</u>: If a class function is a set, then it is a (regular set) function, so that functions are a special case of class functions. Also, we will take this opportunity to define a convention for dealing with class functions. Rather than using the usual '$\mathfrak{F}(a)$' for the image of an object under a given class function $\mathfrak{F}$, sometimes we may instead use some expression such as a^*. In

Appendix A (Continued)

such a case, we will use the symbol ' $\cdot$ ' to indicate where the symbol representing the object(s) should go. For example, for $a \mapsto a^*$ we would use ' $\cdot^*$ ' to represent the associated class function, and for $(a, b) \mapsto \varphi_a(b)$ we would use $\varphi.(\cdot)$ to represent the associated class function.

We will also define the domain and range of a class function $\mathfrak{F}$ so that the concepts agree with those already defined when $\mathfrak{F}$ is a set. We will actually need the following definitions for *any* class when constructing the ordinals.

Definition A.16. Let $\mathfrak{F}$ be a class. Then define the *domain of* $\mathfrak{F}$ (denoted dom $\mathfrak{F}$) by

$$\text{dom } \mathfrak{F} := \{a \ : \ \text{for some object } b,\ (a, b) \in \mathfrak{F}\},$$

and define the *range of* $\mathfrak{F}$ (denoted ran $\mathfrak{F}$) by

$$\text{ran } \mathfrak{F} := \{b \ : \ \text{for some object } a,\ (a, b) \in \mathfrak{F}\},$$

and for any subclass $\mathfrak{K} \subseteq \text{dom } \mathfrak{F}$, we define

$$\mathfrak{F}(\mathfrak{K}) := \{b \ : \ a \in \mathfrak{K} \text{ and } (a, b) \in \mathfrak{F}\}.$$

Also, we will write $\mathfrak{F} : \mathfrak{A} \to \mathfrak{B}$ to mean that dom $\mathfrak{F} = \mathfrak{A}$ and ran $\mathfrak{F} \subseteq \mathfrak{B}$.

Finally, we will need the following notion in discussions of model theory.

Definition A.17. Let $\mathfrak{K}$ be a class, and $n \in \{1, 2, \ldots\}$. Define

$$\mathfrak{K}^n := \{(k_1, \ldots, k_n) \ : \ k_1, \ldots, k_n \in \mathfrak{K}\},$$

and $\mathfrak{K}^0 := \{\varnothing\}$. Then an *$n$-ary operation on* $\mathfrak{K}$ is a class function $\mathfrak{F}$ such that for any pair $(a, b) \in \mathfrak{F}$, we have that both $a \in \mathfrak{K}^n$ and $b \in \mathfrak{K}$.

A.3 Ordinal Numbers and Transfinite Induction

In this section we define the ordinal numbers, and in order to do this we will need to discuss proofs by transfinite induction and definitions by transfinite recursion — these are also used in the construction of the classical universe of sets as well as our models of quantum set theory. We finish with a brief discussion of cardinalities of sets. This section follows Chapter 7 in Enderton (19) closely, and the reader is referred there for details and proofs.

A.3.1 Transfinite Induction and Recursion

We begin with the basic notions of a strict linear ordering,[1] a well-ordering, and a segment.

Definition A.18. Let A be a set, and let $<$ be a binary relation on A. $<$ is called a *strict linear ordering* on A if both

1. Transitivity: For every $a, b, c \in A$ with $a < b$ and $b < c$ we have $a < c$.

---

[1] We could develop the theory in this section using chains rather than strict linear orderings, but we follow Enderton (19) for ease of reference.

Appendix A (Continued)

2. Trichotomy: For every $a, b \in A$, exactly one of the following

$$a < b, \qquad b < a, \qquad a = b$$

holds.

Definition A.19. Let A be a set and $<$ a strict linear ordering on A. We say that $<$ is a *well-ordering* if every nonempty subset $X \subseteq A$ has a least element under $<$. The pair $(A, <)$ is then called a *well-ordered structure*.

Ex: The quintessential example of a well-ordered set is the natural numbers $\mathbb{N}$ with the usual strict ordering.

Definition A.20. Let A be a set with $a \in A$, and $<$ a strict linear ordering. Define the *segment of A before a* (denoted seg a) by

$$\text{seg } a := \{b \in A \ : \ b < a\}.$$

With these definitions we proceed with the principle of transfinite induction.

Proposition A.5. Let A be a set and $<$ a well-ordering on A. Further let $B \subseteq A$, and assume that seg $a \subseteq B$ implies that $a \in B$ for every $a \in A$. Then $B = A$.

Note: This principle reduces to the standard principle of induction when we take $A = \mathbb{N}$ and $<$ the standard ordering.

The above theorem allows us to do proofs by transfinite induction on any well-ordered set. Suppose we wish to prove that (for some well-ordered set A) every $a \in A$ satisfies some property P. If we can show that whenever every element of seg a satisfies property P then a does as well, the above theorem shows us that all of A satisfies P (just by taking B to be the set of elements satisfying P).

We are now ready to discuss definition by transfinite recursion,[1] which we will need to define the ordinals.

Proposition A.6. Let $(A, <)$ be a well-ordered structure, and let $\mathfrak{F}$ be a class function such that for every set x there exists an object y with $(x, y) \in \mathfrak{F}$. Then there exists a unique function f with domain A such that

$$(f|_{\text{seg } a}, f(a)) \in \mathfrak{F}$$

for all $a \in A$.

The connection of the above theorem to definitions by recursion is perhaps best illustrated by an example. Take the natural numbers[2] $\mathbb{N}$ with $<$ the standard ordering. Let $\{ \ (\{(n,n) : n < m\}, m) \ : \ m \in \mathbb{N}\} \subseteq \mathfrak{F}$ and also $(\varnothing, 0) \in \mathfrak{F}$ (and fill out $\mathfrak{F}$ so that it satisfies the hypothesis of the above theorem). Then the unique function f with domain $\mathbb{N}$ given by proposition A.6

[1]The theorem below requires the axiom of replacement (see section 5.1) to prove.

[2]At this point we do *not* consider the natural numbers to be sets — rather they are objects which are not sets. In particular $0 \neq \varnothing$ here.

is just the identity. To see this, since seg $0 = \varnothing$, we have $f(0) = 0$, but then $f(1) = 1$ since $f_{\text{seg }1} = \{(0,0)\}$, and $(\{(0,0)\},1) \in \mathfrak{F}$, etc.

A.3.2 Ordinal Numbers

We can now use definition by transfinite recursion to define the ordinal numbers.

Definition A.21. Let $(A,<)$ be a well-ordered structure, and take $\mathfrak{K} := \{(x, \operatorname{ran} x) \,:\, x \text{ is a set}\}$. Then let f be the unique function with domain A given by proposition A.6 such that

$$f(a) = \operatorname{ran} f|_{\text{seg } a} = f(\text{seg } a) = \{f(x) \,:\, x < a\}.$$

Then ran f is called the $\in$-image of $(A,<)$. If, for a given set α, there exists a well-ordered structure $(A,<)$ such that α is the $\in$-image of $(A,<)$, then α is called the *ordinal number of* $(A,<)$, or just an *ordinal number*. We denote the class of all ordinals by **Ord**.

Note: **Ord** is a proper class — this fact is known as the Burali-Forti theorem.

Example A.7. Take $A = \{0,1,2\}$ with the usual (strict linear) ordering. Then we compute the function f to be

$$f(0) = \{f(x) \,:\, x < 0\} = \varnothing$$

$$f(1) = \{f(x) \,:\, x < 1\} = \{f(0)\} = \{\varnothing\}$$

$$f(2) = \{f(x) \,:\, x < 2\} = \{f(0), f(1)\} = \{\varnothing, \{\varnothing\}\}.$$

Hence the ordinal number of $(A,<)$ is just ran$f = \{\varnothing, \{\varnothing\}\}$.

Appendix A (Continued)

From the above example we can see the beginning of a pattern. In fact this allows us to consider the natural numbers as a special case of the ordinal numbers.

Definition A.22. Let $n \in \mathbb{N} := \{0, 1, \ldots\}$, with $<$ the standard ordering on $\mathbb{N}$. Then define *the n^{th} ordinal* (denoted $\mathbf{n}$) to be the ordinal number of $(\{m \in \mathbb{N} : m \leq n\}, <)$. We define the first infinite ordinal ω to be the ordinal number of $(\mathbb{N}, <)$.

<u>Note</u>: Hence we have $\mathbf{0} = \varnothing$, $\mathbf{1} = \{\varnothing\}$, $\mathbf{2} = \{\varnothing, \{\varnothing\}\}$, etc.

In order to construct the classical universe of sets we will need the following definition and theorem.

Definition A.23. Let α be an ordinal number. Define $\alpha + 1 := \alpha \cup \{\alpha\}$. If there is some ordinal number β such that $\alpha = \beta + 1$ then α is said to be a *successor ordinal.* If there is no such β, and $\alpha \neq \varnothing$, then α is said to be a *limit ordinal.*

Proposition A.8. Let α be an ordinal number, and $\mathcal{A}$ a set of ordinal numbers. Define $\epsilon_\alpha := \{(A, B) \in \alpha \times \alpha \ : \ A \in B\}$. Then

1. $(\alpha, \epsilon_\alpha)$ is a well-ordered structure.
2. $\varnothing = \alpha$ or $\varnothing \in \alpha$.
3. $\alpha + 1$ is an ordinal number.
4. $\bigcup \mathcal{A}$ is an ordinal number.

A.4 The Axiom of Choice and Zorn's Lemma

We finish this appendix on set theory with a discussion of the Axiom of Choice and Zorn's lemma. The Axiom of Choice, in plain English,[1] goes as follows.

> For any set $\mathcal{A}$ consisting of disjoint, non-empty sets, there is a set B such that every $b \in B$ is a member of exactly one set $A \in \mathcal{A}$.

This axiom of set theory is certainly intuitive, and can in fact be proven from the other axioms if we restrict $\mathcal{A}$ above to be a finite collection, essentially by making a choice of an element from each $A \in \mathcal{A}$ and forming B from those chosen elements. The Axiom of Choice then allows us follow this procedure for for arbitrary infinite collections as well.

While the statement of the Axiom of Choice is rather intuitive, certain other statement which are equivalent[2] are certainly less so. For example, it has been shown that the statement 'every vector space has a basis' is equivalent to the Axiom of Choice (7). Zorn's lemma is another statement which is classically equivalent to the Axiom of Choice.

Lemma A.9. *Zorn's Lemma:* Let A be a set partially ordered under $\leq$, such that for every chain $C \subseteq A$, there exists some $a \in A$ such that a is an upper bound for C. Then A has a maximal element under $\leq$.

---

[1]See section 5.1 for a formal treatment.

[2]At least in the presence of the other axioms of set theory under classical logic.

Appendix A (Continued)

Zorn's lemma makes frequent appearance in classical mathematics, where it is used to prove that every vector space has a basis, that every ring has a maximal ideal, etc. (See Aluffi (1) for example). We use it in the proof of theorem 3.

Appendix B

UNIVERSAL ALGEBRA

We will need only a few rudiments of universal algebra, for a thorough introduction to the subject see (9) (although we do not follow his notation precisely). Universal algebra can be developed in terms of a (classical) equational logic, but as we need the results of this section before we discuss equational logic in the main text of this document, we develop what we need only in terms of set theory.

B.1 Algebras and Homomorphisms

Definition B.1. Let A and F be nonempty sets such that F, and every $f \in F$ is a map from A^n to A for some $n \in \mathbb{N}$. Then the pair (A, F) (as well as just the set A) is said to be an *algebra with operations* F. If $f : A^n \to A$, then f is said to be an *n-ary operation on* A, and f is said to be of *arity* n. The *type of* (A, F) is the map $\alpha : F \to \mathbb{N}$ assigning to each $f \in F$ its arity. If F is finite, (A, F) is said to have a finite number of operations. If F has m elements, with $m \in \mathbb{N}$, then (A, F) is said to be a $(n_1, n_2, \ldots, n_m)$-algebra, where $n_1, \ldots, n_m$ are the arities of the elements of F in non-increasing order.

Example B.1. For any group G with identity 'e', multiplication '$*$', and inverse operation $\cdot^{-1}$, we can make a few different algebras out of G. First, $(G, \{e, *, \cdot^{-1}\})$ is a $(2, 1, 0)$-algebra (with $\alpha(e) = 0$, $\alpha(*) = 2$, and $\alpha(\cdot^{-1}) = 1$). Or we could forget some of the structure, and take $(G, \{*\})$ as a (2)-algebra, and so on.

Appendix B (Continued)

An example more relevant for this document is that of a lattice.

Example B.2. Let L be a lattice with meet '$\wedge$' and join '$\vee$'. Then L forms a $(2,2)$-algebra $(L, \{\wedge, \vee\})$.

Ortholattices form another relevant example.

Example B.3. Let L be an OL with meet '$\wedge$', join '$\vee$', involution '$\neg$', top element '1' and bottom element '0'. Then L forms a $(2,2,1,0,0)$-algebra $(L, \{\wedge, \vee, \neg, 1, 0\})$, or just a $(2,1,0)$-algebra $(L, \{\wedge, \neg, 0\})$.

We now need to define what it means for two algebras to be of the *same type* — essentially this will mean that they have the 'same' functions.

Definition B.2. Let (A_1, F_1) be an algebra of type α and (A_2, F_2) an algebra of type β. If there exists an isomorphism (of sets) $\iota : F_1 \to F_2$ such that $\beta \circ \iota = \alpha$ then A_1 and A_2 are said to be of the *same type*, and the isomorphism ι is called a *type identification*.

We can then define structure-preserving maps between algebras of the same type.

Definition B.3. Let (A_1, F_1) (with type α) and (A_2, F_2) be algebras of the same type with type identifier ι. Then a *F_1-homomorphism from A_1 to A_2 w.r.t. ι* is a map $h : A_1 \to A_2$ such that

$$\iota(f)(h(a_1), \ldots, h(a_{\alpha(f)}) = h(f(a_1, \ldots, a_{\alpha(f)})).$$

for every $f \in F_1$ and $a_1, \ldots, a_{\alpha(f)} \in A_1$.

Note: Usually the type identifier will be implicit, as the algebras under considerations will use the same symbols to denote identified operations. We will then refer to h as simply a homomorphism.

Definition B.4. Let (A_1, F_1) and (A_2, F_2) be algebras of the same type, and let $h : A_1 \to A_2$ be a homomorphism. If h is also a set isomorphism, then it is called an *algebra isomorphism.* If there exists an algebra isomorphism from (A_1, F_1) to (A_2, F_2), then (A_1, F_1) and (A_2, F_2) are said to be *isomorphic.*

B.2 Subalgebras

Definition B.5. Let (A, F) be an algebra of type α, let $B \subseteq A$. We say that B is a *subalgebra of* (A, F) if, for every $f \in F$ and $b_1, \ldots, b_{\alpha(f)} \in B$ we have that $f(b_1, \ldots, b_{\alpha(f)}) \in B$.

Note: When B is a subalgebra of (A, F), we will frequently say that B is a subalgebra of A.

Proposition B.4. Let (A, F) be an algebra, let I be an index set, and let B_i be a subalgebra of (A, F) for each $i \in I$. Then $\bigcap_{i \in I} B_i$ is a subalgebra of A.

Definition B.6. Let (A, F) be an algebra, and let $A_0 \subseteq A$. The *subalgebra generated by* A_0 is defined to be

$$\bigcap \{B \subseteq A \; : \; A_0 \subseteq B \text{ and } B \text{ is a subalgebra of } A\}.$$

B.3 Products and Irreducibility

We will only need products of two algebras for this work.

Definition B.7. Let (A_1, F_1) and (A_2, F_2) be algebras of the same type with type identifier ι. For each $f \in F_1$, define the $\alpha(f)$-ary operation $\hat{f}$ on $A_1 \times A_2$ by (for every $a_1 \in A_1$ and $a_2 \in A_2$)

$$\hat{f}(a_1, a_2) := (f(a), \iota(f)(a)),$$

and let $\hat{F} := \{\hat{f} \ : \ f \in F_1\}$. Then *the product of* (A_1, F_1) *and* (A_2, F_2) is the algebra given by $(A_1 \times A_2, \hat{F})$.

Note: If A is the product of A_1 and A_2, then clearly A is of the same type as A_1 and A_2.

We then have the following definition.

Definition B.8. Let (A, F) be an algebra. If there exists two algebras (A_1, F_1) and (A_2, F_2) both of the same type as (A, F), and (A, F) is isomorphic to the product of (A_1, F_1) and (A_2, F_2), then (A, F) is said to be *reducible*. Otherwise, (A, F) is said to be *irreducible*.

B.4 Free Algebras

We now consider free algebras.

Definition B.9. Let F be a set with a map $\alpha : F \to \mathbb{N}$, and let A be a nonempty set. Then the *free algebra with operations* F *on* A (denote $\mathcal{F}(A)$) is the unique algebra (up to isomorphism) with operations F of type α such that for any algebra (B, F') (also of type α) and every set map $f : A \to B$, there exists a unique homomorphism $\hat{f} : \mathcal{F}(A) \to B$ such that $\hat{f}(a) = f(a)$ for every $a \in A$. Also, for any $a \in A$, a is called a *basic element of* $\mathcal{F}(A)$.

That the above algebra always exists and is well-defined is proved in Burris (9).

Appendix B (Continued)

B.5 Congruences on Algebras and Quotients

Just as one can take a partition of a set, and form a new set (the set of equivalence classes), for a given algebra (A, F), it is possible to take a partition of A (with the right properties), and form an algebra on the equivalence classes of A of the same type.

Definition B.10. Let (A, F) be an algebra of type α, let $f \in F$, and let '$\sim$' be an equivalence relation on A. Then '$\sim$' is called a *congruence w.r.t. f* if, for any $a_1, \ldots, a_{\alpha(f)}, b_1, \ldots, b_{\alpha(f)} \in A$ such that $a_i \sim b_i$ for $i \in \{1, \ldots, \alpha(f)\}$, we have

$$f(a_1, \ldots, a_n) \sim f(b_1, \ldots, b_n).$$

If '$\sim$' is a congruence w.r.t. every $f \in F$, then '$\sim$' is simply called a *congruence on A*.

We then can define the following quotient algebra.

Proposition B.5. Let (A, F) be an algebra of type α, and let '$\sim$' be a congruence on A. Then for each $f \in F$, define[1] $\hat{f} : (A/\sim)^{\alpha(f)} \to (A/\sim)$ by $\hat{f}([a]_\sim) := [f(a)]_\sim$. Let $\hat{F} := \{\hat{f} \ : \ f \in F\}$ and define $\beta : \hat{F} \to \mathbb{N}$ by $\beta(\hat{f}) := \alpha(f)$. Then

1. $(A/\sim, \hat{F})$ is an algebra of type β.
2. (A, F) and $(A/\sim, \hat{F})$ are of the same type, with type identifier $f \mapsto \hat{f}$.
3. The map given by $a \mapsto [a]_\sim$ (for every $a \in A$) is a surjective algebra homomorphism.

---

[1] That this is a well-defined notion requires proof — again the reader is referred to Burris (9).

This proposition motivates the following definition.

Definition B.11. Let (A, F) be an algebra of type α, and let '$\sim$' be a congruence on A. Then the algebra $(A/\sim, \hat{F})$ from proposition B.5 is called the *quotient algebra of A by '$\sim$'*.

Just as quotients give rise to homomorphisms, homomorphisms give rise to quotients.

Proposition B.6. Let (A_1, F_1) and (A_2, F_2) be algebras of the same type, and $h : A_1 \to A_2$ a homomorphism. Then the binary relation '$\sim$' on A_1 given by

$$a \sim b \quad \text{iff} \quad h(a) = h(b)$$

is a congruence on A.

Appendix C

LATTICE THEORY

This appendix contains a hodge-podge of concepts and results from lattice theory that we will use in various proofs in the main text. The reader interested in a nice introduction to lattices is referred to (13).

C.1 Lattices and Posets

We begin by discussing complete lattices and continuous homomorphisms.

Definition C.1. Let L be a lattice. L is said to be *complete* if for every $A \subseteq L$, both $\bigwedge A$ and $\bigvee A$ exist.

Lemma C.1. Let L be a lattice, with $a, b, c \in L$. Then

1. If $a \leq b$ and $a \leq c$, then $a \leq b \wedge c$.
2. If L is complete, with $B \subseteq L$, and $a \leq b$ for every $b \in B$, then $a \leq \bigwedge_{b \in B} b = \bigwedge B$.
3. $a \leq b$ iff $a \wedge b = a$ iff $a \vee b = b$.

Definition C.2. Let L_1 and L_2 lattice, with $h : L_1 \to L_2$ a lattice homomorphism. h is called *continuous* if, for every $A, B \subseteq L_1$ we have that

$$h(\bigvee A) = \bigvee h(A) \quad \text{and} \quad h(\bigwedge B) = \bigwedge h(B)$$

whenever $\bigvee A$ and $\bigwedge B$ exist.

Appendix C (Continued)

Lemma C.2. For lattices L_1 and L_2, the natural projection map $p_1 : L_1 \times L_2 \to L_1$ is a continuous homomorphism.

Proof. That the projection map is homomorphism is a straightforward fact of universal algebra (see (9)). In order to show that the homomorphism is continuous, we consider only meets — the argument works similarly for joins. To that end, consider $A \subseteq L_1 \times L_2$. First, since we know that $\bigwedge A \leq a$ for all $a \in A$, we also know that $p_1(\bigwedge A) \leq p_1(a)$ for all $a \in A$. Next, if we assume that there is a $z \in L_1$ such that $z \leq q$ for all $q \in p_1(A)$, then (since projections are surjective) we have $p_1(b) \leq p_1(a)$ for some $b \in L_1 \times L_2$ with $z = p_1(b)$ and all $a \in A$. But then we clearly have $(p_1(b), 0) \leq (p_1(a), p_2(a)) = a$ for all $a \in A$, so that $(p_1(b), 0) \leq \bigwedge A$, and hence $z = p_1(b) \leq p_1(\bigwedge A)$, showing that $p_1(\bigwedge A) = \bigwedge p_1(A)$. □

The following lemma shows that any element of a distributive lattice can be put in *disjunctive normal form.*

Lemma C.3. Let L be a distributive lattice, and let $A \subseteq L$ be a finite set which generates L as a lattice (via $\wedge$ and $\vee$). Then for any $y \in L$, there exists $g_1, \ldots, g_n \in A$ such that

$$y = a_1 \vee a_2 \vee \ldots \vee a_m,$$

where $a_i = g_{i_1} \wedge g_{i_2} \wedge \ldots \wedge g_{i_n}$

Proof. This is a simple induction on the generation of L by elements of A. □

We finish this section by discussing chains and the concept of height.

Appendix C (Continued)

Definition C.3. Let P be a set partially ordered under $\leq$. P is called a *chain* if for every $a, b \in P$, either $a \leq b$ or $b \leq a$. If P is finite, the *length of* P (denoted $l(P)$) is the number of elements of P.

Note: Another convention common in the literature is that the length is defined to be the number of elements *minus 1*.

Definition C.4. Let P be any poset partially ordered under $\leq$. The *height of* P (denote Ht P is defined to be the supremum of the set

$$\{l(C) \ : \ C \subseteq P \text{ and } C \text{ is a finite chain under } \leq\}$$

if the supremum exists, otherwise Ht P is defined to be ∞. If the height of P is finite, then P is said to be of *finite height*.

C.1.1 Modular Lattices

We present a few results having to do specifically with modular lattices, but begin by presenting a lattice which is *not* modular.

Example C.4. Let $N_5 = \{b, t, x, y, z\}$ with the smallest partial order on L satisfying $b \leq x \leq y \leq t$, and $b \leq z \leq t$.

For a proof of the following proposition see Davey (13).

Proposition C.5. Let L be a lattice. Then L is modular iff it does not contain a sublattice isomorphic to N_5.

Appendix C (Continued)

For a proof of the following see (30).

Proposition C.6. *Jónsson's theorem*: Let L be a modular lattice with partial order $\leq$, and let $C_1, \ldots, C_n$ be nonempty subsets of L which are chains under $\leq$ (suitably restricted). Then the sublattice of L generated by $\bigcup_{i=1}^{n} C_i$ is distributive iff for any $x_1, \ldots, x_n$ with $x_i \in C_i$, the sublattice generated by $\{x_1, \ldots, x_n\}$ is distributive.

Lemma C.7. Let L be a modular lattice with partial order $\leq$, and let C_1 and C_2 be chains under $\leq$ (suitably restricted). Then $C_1 \cup C_2$ generate a distributive sublattice.

Proof. Let $x_1 \in C_1$ and $x_2 \in C_2$. Then one can easily see that the sublattice generated by $\{x_1, x_2\} = \{x_1, x_2, x_1 \wedge x_2, x_1 \vee x_2\}$ which is easily seen to be distributive. □

C.2 Orthomodular Lattices

We first consider some useful facts concerning ortholattices more generally before considering orthomodular lattices in particular.

C.2.1 Ortholattices

We will need the following basic fact about ortholattices.

Lemma C.8. Let L be a non-trivial ortholattice. Then

1. Ht $L \geq 2$
2. Ht $L = 2$ iff $L = B_2$.

Proof. Since L is non-trivial, we have that $0 \neq 1$, and since $0 < 1$ we have that Ht $L \geq 2$, establishing (1) above.

If $L \neq B_2$, then there is some $a \in L$ with $a \notin \{0,1\}$, and hence $0 < a$ and $a < 1$, so that $\{0,a,1\}$ is a chain with three elements so if $L \neq \{0,1\}$, then Ht $L \geq 3$. For the other implication, if Ht $L \geq 3$, then there is some chain $C \subseteq L$ with at least 3 elements, so that $L \neq \{0,1\}$. □

The following definition will also prove useful.

Definition C.5. Let L be an ortholattice, and let $a,b \in L$. If $a \leq \neg b$, then we say that a *is orthogonal to* b (write $a \perp b$).

Note: Orthogonality is easily seen to be symmetric and irreflexive. Also, for any $a,b \in L$ with $a \perp b$, we have $a \wedge b = 0$.

C.2.2 Commuting Elements and the Center of an Orthomodular Lattice

The following subset of on OML will frequently prove of interest.

Definition C.6. Let L be an OML. Then the *center of* L is defined to be the set

$$\{c \in L \;:\; c\ C\ a \text{ for all } a \in L\},$$

and is denote $Z(L)$.

The following are both proved in Kalmbach (31), p. 24.

Proposition C.9. Let L be an OML. Then $Z(L)$ is a subalgebra of L which is a Boolean algebra. If L is also complete, then so is $Z(L)$.

Proposition C.10. Let L be an OML with $a \in L$. Then the set of elements of L which commute with a form a subalgebra closed under infinite meets and joins (whenever they exist).

We will use the following in our discussion of reducing axioms.

Definition C.7. Let L be an OML. We define, for any $a, b \in L$, the *commutator of* a *and* b (denoted $c(a,b)$ to be)

$$c(a,b) := [(a \wedge b) \vee (a \wedge \neg b)] \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)]$$

The following are proved in Kalmbach (31), p. 25-26.

Proposition C.11. Let L be an OML, with $a, b \in L$. Then $c(a,b) = 1$ iff $a \; C \; b$.

Proposition C.12. *Foulis-Holland Theorem:* Let L be an orthomodular lattice, with $a, b, d \in L$ such that $a \; C \; d$ and $b \; C \; d$. Then the (ortholattice) subalgebra generated by $\{a, b, d\}$ is distributive.

C.2.3 The Sasaki Projection and the Sasaki Hook

We will need the following operations when dealing with OMLs frequently.

Definition C.8. Let L be an orthomodular lattice, and let $a, b \in L$. We define the *Sasaki Hook* (denoted '$\cdot \to \cdot$') to be the binary operation on L given by

$$a \to b := \neg a \vee (a \wedge b) \tag{C.1}$$

and the *Sasaki projection* (denoted '$\varphi_{\cdot}(\cdot)$') to be the binary operation defined by

$$\varphi_a(b) := a \vee (\neg a \wedge b). \tag{C.2}$$

Appendix C (Continued)

Finally, define

$$a \leftrightarrow b := (a \to b) \wedge (b \to a).$$

Note: For any $a, b \in L$, the Sasaki hook and projection are related by $a \to b = \neg\varphi_a(\neg b)$.

Lemma C.13. Let L be an orthomodular lattice, and φ the Sasaki projection. Then for any $a, b \in L$,

1. $a \; C \; b \quad \text{iff} \quad \varphi_a(b) \le b$
2. $\varphi_a(0) = 0$

Proof.

1. This is 12.(iii) in Kalmbach (31), p. 156.
2. $\varphi_a(0) = a \wedge (\neg a \vee 0) = a \wedge \neg a = 0$.

□

Lemma C.14. Let L be an orthomodular lattice and '$\to$' the Sasaki hook (of def. C.8), and let $a, b, c \in L$.

1. $a \leftrightarrow b = b \leftrightarrow a$
2. $a \to b = 1$ iff $a \le b$
3. $1 \to a = a$
4. $a \leftrightarrow b = 1$ iff $a = b$
5. $a \leftrightarrow 1 = a$

Appendix C (Continued)

6. $a \leftrightarrow 0 = a \rightarrow 0 = \neg a$

7. $a \rightarrow \neg a = \neg a$

8. $a \leftrightarrow b = (a \wedge b) \vee (\neg a \wedge \neg b) = \neg a \leftrightarrow \neg b$.

9. $(a \leftrightarrow b) \wedge (b \leftrightarrow c) \leq a \leftrightarrow c$

Proof.

1. Follows trivially from the definition.

2. If $a \rightarrow b = \neg a \vee (a \wedge b) = 1$ then meeting both sides with a yields (using orthomodularity) that

$$a = a \wedge (\neg a \vee (a \wedge b)) = a \wedge b,$$

i.e. that $a \leq b$. Conversely, if $a \leq b$, then $a \wedge b = a$ so $a \rightarrow b = \neg a \vee a = 1$.

3. $1 \rightarrow a = \neg 1 \vee (1 \wedge a) = 0 \vee a = a$

4. $a \leftrightarrow b = 1$ iff $a \rightarrow b = b \rightarrow a = 1$, by (2) this is true iff $a \leq b$ and $b \leq a$, i.e. iff $a = b$.

5. Trivially by (2) and (3) above.

6. By (2) above $0 \rightarrow a = 1$, so $a \leftrightarrow 0 = a \rightarrow 0 = \neg a \vee (a \wedge 0) = \neg a$.

7. $a \rightarrow \neg a = \neg a \vee (a \wedge \neg a) = \neg a \vee 0 = \neg a$.

8. We can easily use the Foulis-Holland theorem to compute this. It is easy to see that $a \wedge b \ C \ \neg a$ and $a \wedge b \ C \ \neg b$, so we simply undistribute

$$a \leftrightarrow b = [\neg a \vee (a \wedge b)] \wedge [\neg b \vee (b \wedge a)] = (\neg a \wedge \neg b) \vee (b \wedge a).$$

9. This can be found in (14) (where he has $a\Delta b := \neg[a \leftrightarrow b]$). See p. 437.

□

C.2.4 Atoms and the Exchange Property

Definition C.9. Let L be a lattice with bottom element 0, and let $a \in L$. a is called an *atom* if a covers 0. L is called *atomic* if for every $x \in L$ with $x \neq 0$, there exists an atom $p \in L$ such that $x \geq p$. L is called *atomistic* if every $a \in L$ satisfies

$$a = \bigvee\{p \in L \ : \ p \leq a, \text{ and } p \text{ is an atom}\}.$$

The following two propositions are proved in Kalmbach (31), p. 140.

Proposition C.15. Let L be an atomic OML. Then L is atomistic.

Proposition C.16. Let L be an OML. Then the following three conditions are equivalent.

1. If (for any $a, b \in L$) a covers $a \wedge b$, then $a \vee b$ covers b.
2. If (for any $a \in L$ and atom $p \in L$) $p \nleq a$, then $p \vee a$ covers a.
3. If (for any $a \in L$ and atoms $p, q \in L$) the conditions $p \leq q \vee a$ and $p \wedge a = 0$ imply $q \leq p \vee a$.

The following are all important properties possessed by the projection lattices.

Definition C.10. Let L be an atomic OML. If L satisfies any of the three equivalent conditions of proposition C.16, then L is said to satisfy the *exchange axiom* (EA).

Appendix C (Continued)

Definition C.11. Let L be an atomic OML. If, for every pair of atoms $p, q \in L$ such that $p \perp q$, we have that there exists an atom $r \in L$ such that $r \leq p \vee q$ and also $r \not C p$ and $r \not C q$, then L is said to satisfy the *atomic bisection property* (or *ABP*).

Definition C.12. Let L be an atomic OML. If L satisfies ABP and also, for every triple of distinct atoms $p, q, r \in L$ we have that $p \leq q \vee r$ imply both $q \leq p \vee r$ and $r \leq p \vee q$, then L is said to satisfy the *superposition principle* (or *SP*).

The following lemma is theorem 14.8.9 in (4).

Lemma C.17. Let L be an atomic OML satisfying the exchange axiom. Then L is irreducible iff L satisfies SP.

Lemma C.18. L atomic OML, with $a, s \in L$, s an atom, and s and a commute. Then $s \leq a$ or $s \perp a$.

Proof. To see this, note that we have

$$s \; C \; a \leftrightarrow s = (s \wedge a) \vee (s \wedge \neg a),$$

and also $s \in \Omega(L)$ implies that $s \wedge x = s$ or $s \wedge x = 0$, and if $s \wedge a = 0$, then $s \wedge \neg a = s$ (or vice versa) — thus, $s \leq a$ or $s \perp a$. □

The following is proved in Kalmbach, p. 143.

Lemma C.19. Let L be an atomic OML satisfying EA, and let $a, b \in L$ with $a \perp b$, and let $p \in L$ be an atom with $p \leq a \vee b$ but $p \nleq b$. Then $(p \vee b) \wedge a$ is an atom.

Appendix C (Continued)

We will need the following when proving the RZFC axioms hold in certain OMLs.

Lemma C.20. Let L be a complete, atomic, irreducible OML satisfying EA. Then for any $y \in L$ with $y \notin \{0,1\}$, we have

$$\bigwedge_{x \in L} \big(\varphi_x(y) \to y\big) = 0.$$

Proof. If $L = B_2$ the above holds vacuously. Otherwise, since $L \neq \{0,1\}$ in this case, we must have that L is height greater than 2 (lemma C.8). Then since L is an atomic, irreducible, OML such that EA holds and the height of L is greater than 2, both superposition (SP) and the atomic bisection property (ABP) hold (lemma C.17). Let $\Omega(L)$ denote the set of all atoms of L. Since L is an atomic OML, L is atomistic (proposition C.15), so for any $y \in L$, we have that both

$$y = \bigvee_{\substack{p \leq y \\ p \in \Omega(L)}} p \qquad \text{and} \qquad \neg y = \bigvee_{\substack{q \leq \neg y \\ q \in \Omega(L)}} q,$$

and each of the joins above are non-empty (since $y \notin \{0,1\}$) so

$$1 = y \vee \neg y = \Big[\bigvee_{\substack{p \leq y \\ p \in \Omega(L)}} p\Big] \vee \Big[\bigvee_{\substack{q \leq \neg y \\ q \in \Omega(L)}} q\Big] = \bigvee_{\substack{p \leq y, q \leq \neg y \\ p, q \in \Omega(L)}} (p \vee q).$$

Now, for each $p, q \in \Omega(L)$ with $p \leq y$, $q \leq \neg y$, we have $p \perp q$ (since $p \leq y \leq \neg q$), and so since L satisfies ABP, we choose some $r_{pq} \in \Omega(L)$ which is such that $r_{pq} \leq p \vee q$ and r_{pq} does not commute with either p or q, and furthermore, by SP, we have that $p \vee q = q \vee r_{pq} = p \vee r_{pq}$. We further claim that r_{pq} does not commute with y. By lemma C.18, r_{pq} commutes with y iff either $r_{pq} \leq y$ or $r_{pq} \leq \neg y$. But if $r_{pq} \leq y$, then $r_{pq} \vee p = p \vee q \leq y$, so that $q \leq y \wedge \neg y = 0$ which

Appendix C (Continued)

is a contradiction since $q \in \Omega(L)$. A similar contradiction results if $r_{pq} \leq \neg y$, and so r_{pq} cannot commute with y.

Now, $r_{pq} \in \Omega(L)$ implies that $\neg r_{pq}$ is a coatom, and $r_{pq} \leq p \vee q$ implies that $p \vee q \nleq \neg r_{pq}$ (by transitivity of $\leq$ since $r_{pq} \neq 0$). Then, $\neg r_{pq} \wedge (p \vee q)$ is an atom by EA.

But then, by lemma C.19 (using that $p \nleq r_{pq}$ since both $p, r_{pq} \in \Omega(L)$, and also $r_{pq} \perp \neg r_{pq}$) we see that $z_{pq} \coloneqq \neg r_{pq} \wedge (p \vee r_{pq})$ is an atom. Since $z_{pq} \leq \neg r_{pq}$, clearly $z_{pq} \neq r_{pq}$, and then by SP we have

$$z_{pq} \vee r_{pq} = p \vee z_{pq} = p \vee q.$$

Additionally, we must have that z_{pq} does not commute with y, by the same argument that r_{pq} does not commute with y. Hence, we have that

$$1 = \bigvee_{\substack{p \leq y, q \leq \neg y \\ p,q \in \Omega_L}} (p \vee q) = \bigvee_{\substack{p \leq y, q \leq \neg y \\ p,q \in \Omega_L}} (r_{pq} \vee z_{pq}) \leq \bigvee_{\substack{r \in \Omega(L) \\ r \not\mathcal{C}\, y}} r,$$

and taking the negation yields,

$$0 = \bigwedge_{\substack{r \in \Omega(L) \\ r \not\mathcal{C}\, y}} \neg r. \tag{C.3}$$

But for r any atom that doesn't commute with y, we have

$$\varphi_r(y) = r \wedge (\neg r \vee y) = r \wedge 1 = r,$$

since $y \nleq \neg r$ and $\neg r$ is a coatom, so that

$$\varphi_r(y) \to y = \neg r \vee (r \wedge y) = \neg r \vee 0 = \neg r.$$

Plugging this back into equation C.3 yields

$$0 = \bigwedge_{\substack{r \in \Omega(L) \\ r \not\subset y}} \neg r = \bigwedge_{\substack{r \in \Omega(L) \\ r \not\subset y}} \big(\varphi_r(y) \to y\big) \geq \bigwedge_{\substack{x \in L \\ x \not\subset y}} \big(\varphi_x(y) \to y\big).$$

□

The following property will be useful in many ways.

Definition C.13. Let L be an orthomodular lattice. L is said to satisfy the *relative center property* if for any $a \in L$, the center of any interval $[0, a]$ is exactly the set $\{a \wedge b \ : \ b \text{ is in the center of } L\}$.

Lemma C.21. The following OMLs satisfy the relative center property.

1. Any complete modular ortholattice.
2. Projection lattices of any Hilbert space.
3. Projection lattices of any von Neumann algebra.

Proof. For (1) and (3) see Theorem 14 in Kalmbach (31), pages 110-111. (2) is a special case of (3). □

Lemma C.22. Let L be a complete orthomodular lattice which satisfies the relative center property, and let $a \in L$. Then $\bigwedge_{b \in L} b \to a$ is in the center of L.

Proof. Note that an element is in the center of L iff its negation is, and

$$\neg \bigwedge_{b \in L} b \to a = \bigvee_{b \in L} \neg(\neg b \vee (b \wedge a)) = \bigvee_{b \in L} \varphi_b(\neg a),$$

(with $\varphi_b(a)$ the Sasaki projection), and so it suffices to show that $\bigvee_{b \in L} \varphi_b(a)$ is in the center of L if L satisfies the relative center property. But this then follows directly from Theorem 14 in Kalmbach (31) (page 110) and Theorem 7 (page 108) along with proposition 10 in Chevalier (10) (specifically the equivalence of (b) and (d)). □

Definition C.14. Let L be a complete OML, and let G be the group of continuous ortholattice automorphisms on L. L is said to be *rotatable* if the only fixed points of L under the action of G are 0 and 1.

Proposition C.23. The projection lattice of a separable complex Hilbert space is rotatable.

Proof. It is easily verified that for any unitary transformation U, the map which takes any projection operator P to $U^\dagger P U$ induces an automorphism on the projection lattice. Of course, the only projectors fixed by all such automorphisms are I and 0. □

All of the following facts are established in Kalmbach (31).

Proposition C.24. The projection lattice of a separable complex Hilbert space is an atomic, irreducible complete OML satisfying the exchange axiom.

GLOSSARY OF NOTATION

$\{ a : P \}$	class of all a which satisfy property P	defs. A.2, A.13
$\varnothing$	empty set	def. A.2, secs. 5.1.1, 5.1.4
$\dot{+}$, $\dot{\times}$	sum and product on quantum natural numbers	def. 6.5
$A \backslash B$	$\{a \in A : a \notin B\}$ for sets A, B	def. A.3
α	arity function for a language or algebra	defs. 2.2, B.1
$\mathcal{B}$	set of basic symbols for formal syntax	def. 2.1
$\mathcal{B}_S$	set of basic logical symbols $\{\wedge, \neg, \forall\}$	def. 2.1
$\mathcal{B}_V$	set of variables $\{\upsilon_1, \upsilon_2, \ldots\}$	def. 2.1
$\mathfrak{B}^{\mathfrak{A}}$	class of maps from $\mathfrak{A}$ to $\mathfrak{B}$ for classes $\mathfrak{A}, \mathfrak{B}$	defs. A.11, A.15
$\mathbb{C}$	the complex numbers	chapter 1
$f \circ g$	composition of functions f, g	def. A.10
$\mathbb{N}$	the natural numbers $\{0, 1, 2, \ldots\}$	chapter 1
$\mathfrak{N}_L$	quantum L-number structure for complete OML L	def. 6.8
Ord	class of ordinal numbers	def. A.21
$\mathcal{Q}_L$	L-valued set structure for a complete OML L	def. 5.2
$\mathfrak{Q}_L$	the class of all quantum sets for a complete OML L	def. 5.2
$\mathbb{R}$	the real numbers	chapter 1
$\sigma(A)$	spectrum of a quantum natural number A	def. 6.6
ω	first infinite ordinal / natural numbers as sets	def. A.22, sec. 6.1.3
ω_L	L-valued quantum natural numbers for complete OML L	def. 6.4

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