Upper Bounds on the Density of Two Radius Packings of Disks in the Plane

by

Ali Mohajer
B.A. (Valparaiso University) 1995
B.S. (Valparaiso University) 1995
M.S. (University of Illinois at Chicago) 2012

Thesis submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics (Pure Mathematics)
in the Graduate College of the
University of Illinois at Chicago, 2018

Chicago, Illinois

Defense Committee:
Peter B Shalen, Chair and Advisor
Marc Culler
Daniel Groves
David Dumas
Ramin Takloo-Bighash
Thomas Kennedy (University of Arizona)

To Celestia, who waited.

ACKNOWLEDGMENTS

Thanks, first, to my parents, Batul and Morteza, whose love and support are limitless; to my spouse, Ana, whose love, patience, and confidence have been my source of strength; and to my brother, Kiyarash, who possesses rare grace, and for whom I have boundless admiration. Thanks to my family, who have taught me who I am, and who help me remember when I forget.

Thanks to my advisor, Peter Shalen, who has set an example for kindness, thoughtfulness, generosity, and expansiveness of spirit which I will forever strive toward.

Thanks to Samad Hedayat, who has been a constant source of inspiration since childhood, and who encouraged me to return to academia. I filled his blackboard with the most esoteric markings I could conjure when I was 9, and asked him to assign meaning to them.

Thanks to my friends, old and new, whom I treasure and hope to deserve.

Thanks to my many teachers, who have gifted me with their vision.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

SUMMARY

In this work we consider how well a packing consisting of disks of two distinct radii can cover the plane. The main result, which sharpens the best known upper bound on two-radius packings where the ratio of the smaller radius to the larger is 0.7, is presented in Theorem 3.3.1.

The density of a planar packing of disks with unequal radii cannot exceed that of a packing of congruent disks if the radii of the disks are sufficiently close. Between 1950 and 1970, "sufficiently close" was sharpened from "nearly equal" to ratios of radii within the interval $[0.742\ldots, 1]$ (1) (2) (3) (4). It is not yet known whether this interval can be extended.

We call the infimum of the ratio of the radii of the disks in a packing the **homogeneity** of that packing. A universal upper bound on the density of a packing of unequal disks as a function of homogeneity has been found (5), but this bound is usually not sharp. For two-radius packings there are less than a dozen discrete values in the homogeneity interval (0,1) where a sharp upper bound on the density of the packing has been found, usually where special regularity features exist (6), (7), (8), (9). In this paper we work at a homogeneity where such regularity conditions cannot be found.

The general heuristic when searching for an upper bound on density has been to partition the underlying space (in this case the Euclidean plane) in a manner which allows global inferences to be made from local bounds on density which depend on special characteristics of the partition. Perhaps the most familiar way of partitioning the underlying space is by considering a lattice, wherein each cell of the partition is congruent to the fundamental domain. Here we partition the plane by considering a Delaunay triangulation on the centers of the disks in a saturated packing. The regularity conditions present in a Delaunay triangulation lead to geometric constraints on dense triangles in the packing. These constraints, taken in conjunction with certain topological properties of the triangulation, allow us to arrive at a global density bound.

CHAPTER 1

STATEMENT OF PROBLEM AND HISTORY

A **packing** of disks in a convex subset of the plane is a non-overlapping arrangement of disks in that subset. We define the **density of a packing in a bounded set** as the ratio of the area of the intersection of the packing with the set to the area of the set. This definition extends naturally to unbounded convex sets by considering a lim sup over the family of all bounded convex subsets of the given convex set. The **homogeneity** of a packing of unequal disks is the infimum of the ratio of the radii of any two disks in the packing. An upper bound on the density of a planar packing of disks of a given homogeneity is sought.

In 1773 Joseph-Louis Lagrange proved in his study of quadratic forms (10) (11) that among lattice packings of disks of equal size, the hexagonal lattice arrangement has maximal density. This density is $\frac{\pi}{\sqrt{12}}$.

In 1910, Axel Thue proved the general result that $\frac{\pi}{\sqrt{12}}$ is the greatest density attainable by any arrangement of equal disks in the plane (12). Thue's method was to partition the plane into three classes of region (sectors concentric with a disk in the packing, triangular regions which arise when disks in the packing are close enough to each other, and empty space) and reason that in each of these types of region the density is bounded above by $\frac{\pi}{\sqrt{12}}$.

In 1953, Laszlo Fejes Toth examined the question of finding the largest possible packing density of incongruent disks (1), and showed that if the homogeneity of a packing is sufficiently

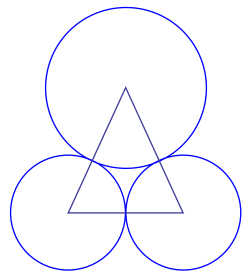

Figure 1: Florian's triangle

close to 1, the density of the packing cannot exceed $\frac{\pi}{\sqrt{12}}$. His methods involved the following elements: density is defined using a hexagonal region which grows to fill the plane; the hexagonal region is decomposed into polygons associated with the disks in the packing; an appropriately defined convex function is used to bound the density of each disk in its associated polygon; and the fact that the average number of sides of the partitioning polygons is 6 is used in conjunction with properties of convex functions to obtain the density bound.

In 1960, August Florian proved (5) that the density of a packing of unequal disks with homogeneity h cannot exceed the density of an arrangement of three mutually tangent disks of radii 1, h, and h in the triangle having the centers of these disks as its vertices (see Figure 1). This work provides an upper bound over the entire interval of homogeneity; however, the bound is usually not sharp.

In 1963, by examining local geometry and refining the methods of Fejes Toth, Florian (2) extended the interval of homogeneity where the density of a packing cannot exceed $\frac{\pi}{\sqrt{12}}$ to $[0.906\ldots, 1]$.

This interval was further extended to [0.742. . . , 1] by Gerd Blind in 1969 (3) and independently by Gabor Fejes Toth in 1972 (4) following ideas of Karoly Boroczky mentioned in Laszlo Fejes Toth's *Lagerungen. . .* (13) and in private communication. The methods involve the use of certain isoperimetric inequalities and convexity. The number 0.742. . . is constructed using the function describing the area of a regular polygon inscribing a unit disk.

In 2002, Gerd Blind and Roswitha Blind generalized the method used in Laszlo Fejes Toth's proof from 1953, which required the use of a hexagonal region in the definition of density, to one that only requires convexity (14). Where L. Fejes Toth used the Voronoi cells of centers of the disks in the packing to decompose a hexagonal region of the plane, the decomposition of the convex body in the Blinds' work is done instead using the power lines of the disks within it. This decomposition is better suited to packings of disks with different radii, since disks in an unequal packing may not be properly contained in the Voronoi cells of their centers.

In a **compact packing**, each disk is tangent to a ring of disks, each of which is tangent to its two cyclic neighbors. Thomas Kennedy showed in (9) that if we restrict our attention to packings of disks with two distinct radii, there are only 9 homogeneities for which it is possible to construct compact packings. All of these homogeneities lie outside the interval $[0.742\ldots, 1]$.

In 2003, Aladar Heppes showed in (7) that the density realized in six compact packings are maximal for their respective homogeneities, and asserted that the methods in his paper can be used to show that the density of these compact packings are locally maximal with respect to homogeneity, and furthermore, that at homogeneities 0.6 and 0.67, density is bounded above by $\frac{\pi}{\sqrt{12}}$. Heppes' method involves fixing a packing of a given homogeneity, obtaining a triangulation of the plane, and then further decomposing the triangles of the triangulation in order to obtain a decomposition of the plane into cells whose density does not exceed the target density (that of the compact packing). Some empirical data supporting the assertions made in (7) is available at a site maintained by Kennedy (15).

Given a discrete set of points $\mathcal{V}$ in the plane, the **Voronoi cell** (also called the **Dirichlet domain**) of a point $v \in \mathcal{V}$ is the set of all points of the plane which are at least as close to v as to any other point of $\mathcal{V}$. In 1992, Wu-Yi Hsiang gave a simple proof of Thue's theorem for disks of equal radius (16) by appealing to the Voronoi decomposition of the plane associated with the centers of the disks in the packing. In a packing of homogeneity 1, each disk is contained in the Voronoi cell of its center. This fact, which is used in Hsiang's paper, does not generally hold for packings of unequal disks.

A **Delaunay triangulation** of the plane is a triangulation of the plane using a discrete set of points as its vertices, with the additional property that the circumcircle of each triangle contains no points of the vertex set in its interior. Figure 2 illustrates this property.

A Delaunay triangulation can be shown to contain the dual graph of the Voronoi decomposition of its vertices, and so, it is perhaps not surprising to see techniques involving Delaunay decompositions or a hybrid of Voronoi and Delaunay decompositions in packing research. See, for instance, the work of Hales and Ferguson (17).

The size of the minimal angle in triangulations on a set of points $\mathcal{V}$ is maximized by a Delaunay triangulation. Furthermore, it is possible to obtain a universal lower bound on the

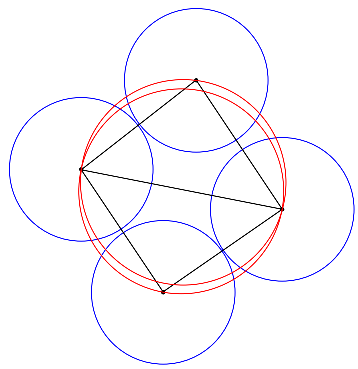

Figure 2: Delaunay triangles with their associated circumcircles in a packing of unit disks

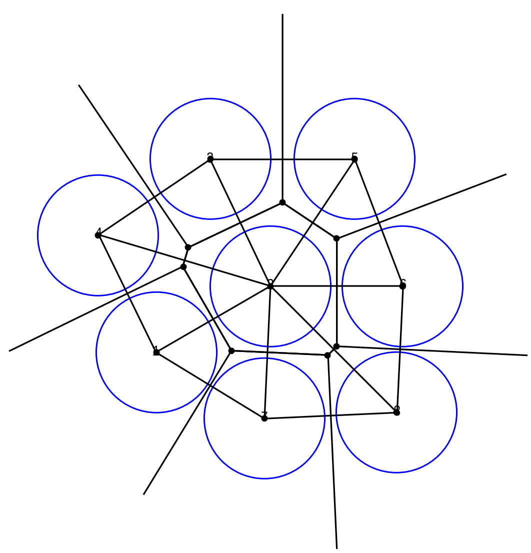

Figure 3: The superimposed Voronoi and Delaunay graphs of an irregular packing

size of the smallest angle in Delaunay triangulations on the centers of disks in **saturated** (see Definition 2.1.3) packings of a fixed homogeneity h, as we shall see in Section 2.2.

In 2010, Hai-Chau Chang and Lih-Chung Wang posted a simple proof of Thue's theorem (18) in which they consider a Delaunay triangulation on the centers of the disks in the packing and bound the density of the packing in each triangle by appealing to the regularity features of the triangulation. The technical details which are a prominent feature in all the other papers mentioned in this introduction are entirely subsumed in the Delaunay triangulation, and the proof is otherwise truly elementary. The argument in (18) makes tacit use of the fact that no triangle in a Delaunay triangulation on the centers of disks in a saturated packing of homogeneity 1 intersects a disk whose center is not a vertex of that triangle, except possibly in a set of measure zero. This is not generally true in packings of unequal disks.

The methods of L. Fejes Toth(19), Florian(2), Blind(3), and G. Fejes Toth (4) make use of a convex function taking as input the area of a regular polygon inscribed in a unit disk. In particular, Blind makes use of such a function to arrive at the leftmost endpoint of the homogeneity interval $[0.742\ldots, 1]$ in which the density of a packing cannot exceed $\frac{\pi}{\sqrt{12}}$. The quantized nature of the inputs of such functions, as well as the shape of the plot of empirical data on known packing densities for various homogeneities, suggests that the interval $[0.742\ldots, 1]$ can be extended for the target density $\frac{\pi}{\sqrt{12}}$, and that sharper upper density bounds may be attainable throughout the interval (0,1), except at homogeneities which admit compact packings.

CHAPTER 2

PRELIMINARIES

2.1 Definitions and Preliminary Notions

Definition 2.1.1. *A **packing** of disks in a convex subset of the plane is a non-overlapping arrangement of disks contained in that subset.*

Definition 2.1.2. *Let $\mathcal{P}$ denote the set of all disks in a packing. Let B_r be the disk of radius r centered at the origin, and let $|X|$ denote the area measure of the set X. We define the **density of $\mathcal{P}$ in a bounded set** B:*

$$\rho(\mathcal{P}, B) := \frac{|\mathcal{P} \cap B|}{|B|},$$

*and the **density of $\mathcal{P}$ in the plane**:*

$$\rho := \limsup_{r \to \infty} \frac{|\mathcal{P} \cap B_r|}{|B_r|}.$$

Definition 2.1.3. *A packing is said to be **saturated** if there is no room to add an additional disk.*

Definition 2.1.4. *The **homogeneity** of a packing is the infimum of the ratio of radii of disks in the packing.*

Whenever we refer to a packing of homogeneity h, we assume that the radius of the largest disk in the packing is 1. A saturated disk packing of homogeneity h in the plane, therefore, satisfies the following two conditions:

- The distance between any two centers of disks in the packing is at least $2h$.
- Any point of the plane is at most distance $1 + h$ from the center of some disk in the packing.

Definition 2.1.5. *A* ***triangulation*** *of a discrete set of points $\mathcal{V}$ in the plane is a decomposition of the convex hull of $\mathcal{V}$ into pairwise-disjoint open simplices such that the faces of each simplex are also in the collection and the vertices are the points of $\mathcal{V}$.*

The closure of each 2-simplex in a triangulation is referred to as a triangle of the triangulation.

Observation 2.1.6. *The length of every edge of a triangle in a triangulation on the centers of disks in a packing of homogeneity h is at least $2h$.*

Definition 2.1.7. *A trangulation on a discrete set of points is called a* ***Delaunay triangulation*** *if no vertex is contained inside the circumcircle of any triangle of the triangulation.*

A discrete set of points $\mathcal{V}$ in the plane admits a Delaunay triangulation if, for two positive constants r and R it satisfies two properties:

- Every two distinct points of $\mathcal{V}$ are at least distance r apart.
- The distance of no point of the plane is more than R from some point of V.

A proof of this fact due to P. Gruber appears in (20), following the work of Delone (*aka* Delaunay) in (21) and Delone and Ryshkov in (22).

Notice that a saturated packing of disks in the plane satisfies both conditions above, and so admits a Delaunay triangulation.

Lemma 2.1.8. *The circumradius of every triangle in a Delaunay triangulation on a saturated disk packing of homogeneity h in the plane is strictly less than $1 + h$.*

Proof. In a Delaunay triangulation, no vertex of the triangulation is properly contained in the circumcircle of any triangle, and in a saturated packing of homogeneity h, every point of the plane, in particular the circumcenter of any triangle, is at least distance $1 + h$ from the center of some disk in the packing. □

Notation 2.1.9. *When we refer to an arbitrary triangle T, we assume its angles are labelled α, β, and γ, at respective vertices A, B, and C, so that $\alpha \geq \beta \geq \gamma$. The letters a, b, and c refer to the lengths of the edges opposite α, β, and γ.*

Definition 2.1.10. *To each triangle T in a triangulation on the centers of disks in a packing we assign a* ***combinatorial type*** *$[r_A, r_B, r_C]$, where r_A, r_B, and r_C are radii of the disks at A, B, and C respectively.*

Observation 2.1.11. *Triangles in a two-radius packing of homogeneity h belong to one of the eight combinatorial types $[1, 1, 1], [1, 1, h], [1, h, 1], [h, 1, 1], [1, h, h], [h, 1, h], [h, h, 1]$, and $[h, h, h]$.*

The angles of a triangulation on the centers of disks in a packing cut each disk into sectors. In a Delaunay triangulation on the centers of disks in a saturated packing of homogeneity h it is sometimes the case that a triangle intersects, in a set of positive measure, a sector belonging to a disk whose center is not a vertex of that triangle. Informally, one may think of such an intersection as the "overhanging lip" of a sector cut out by an adjacent triangle. To facilitate computations and to associate each sector to a unique triangle, we make the following definition:

Definition 2.1.12. *To each triangle T of combinatorial type $[r_A, r_B, r_C]$ we associate a **sector area**:*

$$sa(T) := \frac{1}{2}\left(\alpha r_A^2 + \beta r_B^2 + \gamma r_C^2\right),$$

which is the sum of the areas of the sectors cut from the disks at the vertices of the triangle by the angle at each vertex of the triangle.

Definition 2.1.13. *Fix a target density ρ_0. Let a triangle T have area $|T|$ and sector area $sa(T)$. We define the **surfeit of** T **relative to** ρ_0 to be the quantity:*

$$sf(T) := sa(T) - \rho_0|T|.$$

Informally, one may think of the surfeit of a triangle as a measure of the excess weight of its sector area compared to a congruent triangle in which the packing has density ρ_0.

Definition 2.1.14. *Let W be a finite set of triangles which are pairwise disjoint or meet in at most a set of measure zero. We define the surfeit of W to be:*

$$sf(W) := \sum_{T \subset W} sf(T)$$

Informally, we may think of the surfeit of the entire triangulation of an infinite packing as involving the upper sum (the lim sup of a sequence of partial sums) of a series. Showing that the density of a packing is less than or equal to a target density is tantamount to showing that this upper sum is non-positive. This is discussed in detail in the proof of Theorem 3.3.1.

In this paper it is often useful to consider families of similar triangles. A triangle T is determined, up to similarity, by two angles, so we make the following definition:

Definition 2.1.15. *When T is labeled as in 2.1.9, we call the tuple (β, γ) the **similarity type** of T.*

We may think of the similarity type of a triangle T as a point in the $\beta\gamma$-plane. Such a point lies in the region bounded by the lines $\beta = \gamma$, $\beta = \pi/2 - \gamma/2$, and the vertical line $\gamma = 0$. We will see in Section 2.2 that a Delaunay triangulation imposes restrictions on the size of the smallest angle of a triangle. As a result, when a Delaunay triangulation is in hand, this region may be further restricted. It is useful to classify triangles by combinatorial type when restricting similarity types in this way.

Definition 2.1.16. *The **permissible region** for triangles of combinatorial type $[r_A, r_B, r_C]$ belonging to a Delaunay triangulation on the centers of disks in a two-radius packing of homogeneity h is the set of points in the $\beta\gamma$-plane corresponding to similarity types which do not violate angle restrictions imposed by the Delaunay triangulation.*

Definition 2.1.17. *Let v be a vertex of a triangle in a triangulation. We define the **star of** **v** to be the set of all triangles which have v as a vertex.*

Definition 2.1.18. *The number of triangles in the star of a vertex v is called the* ***valence*** *of the star of v, or, sometimes, the valence of v.*

Naturally, if all triangles in a triangulation on a packing carry non-positive surfeit, the density of that packing is bounded above by the target density. Usually this is too much to hope for, so an alternative approach is to determine what types of triangles can carry positive surfeit, and group them together with triangles of sufficiently negative surfeit so that the sum of surfeits in each group is non-positive.

Informally, a **heavy** triangle is one which has positive or nearly positive surfeit. The precise meaning of "heavy" is described in each chapter in which the word is used. Also informally, we call a triangle **thin** when the length of one edge is significantly greater than that of the other two. As we shall see in detail in Section 3.7, when a heavy Delaunay triangle is thin, its circumradius is nearly maximal. As a result, the properties of the Delaunay triangulation allow us to determine a lower bound on the area of the heavy-thin triangle's long-edge neighbor. This lower bound on the neighbor's area gives an upper bound on neighbor's surfeit. We then group the heavy triangle with its lighter neighbor, and sometimes also with other nearby triangles, and reallocate surfeit within this group. This gives rise to the notion of **adjusted surfeit**, which we will make precise in Definition 3.2.8.

We will find that triangles of positive adjusted surfeit belong to only a few combinatorial types, and within each of these combinatorial types, the similarity types of triangles of positive adjusted surfeit are constrained to small regions of the $\beta\gamma$-plane. These observations are used to make deductions about the adjusted surfeit of stars in the triangulation, which are then used to arrive at an upper bound on the density of the packing.

2.2 Permissible Triangles

We seek upper bounds on density, so we may assume that the packings we consider are saturated, and that a Delaunay triangulation on the centers of the disks in the packing is in hand. It is then possible to obtain a lower bound on the size of the smallest angle in a triangle of a given combinatorial type, and in a few cases it is also possible to obtain a sharper lower bound on the second smallest angle as well. For instance, in a $[1, 1, 1]$ triangle, the length of each side, in particular the side of length c opposite the smallest angle γ is bounded below by 2. Letting r represent the circumradius of this triangle, we obtain the relation

$$\frac{\sin \gamma}{c} = \frac{1}{2r}$$

which allows us to conclude that $\gamma > \sin^{-1}(\frac{1}{1+h})$. The corresponding angle bounds for all combinatorial types are summarized in Table I.

$[1, 1, 1]$	$\gamma \leq \beta \leq \frac{\pi}{2} - \frac{\gamma}{2}$	$\sin^{-1}(\frac{1}{1+h}) < \gamma \leq \frac{\pi}{3}$
$[1, 1, h]$	$\gamma \leq \beta \leq \frac{\pi}{2} - \frac{\gamma}{2}$	$\sin^{-1}(\frac{1}{1+h}) < \gamma \leq \frac{\pi}{3}$
$[1, h, 1]$	$\sin^{-1}(\frac{1}{1+h}) < \beta \leq \frac{\pi}{2} - \frac{\gamma}{2}$	$\frac{\pi}{6} < \gamma \leq \frac{\pi}{3}$
$[h, 1, 1]$	$\gamma \leq \beta \leq \frac{\pi}{2} - \frac{\gamma}{2}$	$\frac{\pi}{6} < \gamma \leq \frac{\pi}{3}$
$[1, h, h]$	$\gamma \leq \beta \leq \frac{\pi}{2} - \frac{\gamma}{2}$	$\frac{\pi}{6} < \gamma \leq \frac{\pi}{3}$
$[h, 1, h]$	$\gamma \leq \beta \leq \frac{\pi}{2} - \frac{\gamma}{2}$	$\frac{\pi}{6} < \gamma \leq \frac{\pi}{3}$
$[h, h, 1]$	$\frac{\pi}{6} < \beta \leq \frac{\pi}{2} - \frac{\gamma}{2}$	$\sin^{-1}(\frac{h}{1+h}) < \gamma \leq \frac{\pi}{3}$
$[h, h, h]$	$\gamma \leq \beta \leq \frac{\pi}{2} - \frac{\gamma}{2}$	$\sin^{-1}(\frac{h}{1+h}) < \gamma \leq \frac{\pi}{3}$

TABLE I: Angle constraints by combinatorial type.
The bounds apply to saturated packings of homogeneity h.

2.2.1 A Surfeit Bounding Function

The area of any triangle T can be computed using an edge length and the sines of its angles using the formula $|T| = \frac{1}{2}a^2\dfrac{\sin\beta\sin\gamma}{\sin\alpha}$, or the obvious modification of this formula where a different edge length is used. Edge lengths are bounded below by the sum of the radii of the disks at the endpoints, and $\alpha = \pi - (\beta+\gamma)$, so the function:

$$f_{[r_A,r_B,r_C]}(\beta,\gamma) := \frac{1}{2}\left(\pi r_A^2 - \beta(r_A^2 - r_B^2) - \gamma(r_A^2 - r_C^2) - \rho_0 \max\left\{\begin{array}{l} (r_B+r_C)^2\dfrac{\sin\beta\sin\gamma}{\sin(\beta+\gamma)} \\ (r_A+r_C)^2\dfrac{\sin(\beta+\gamma)\sin\gamma}{\sin\beta} \\ (r_A+r_B)^2\dfrac{\sin(\beta+\gamma)\sin\beta}{\sin\gamma} \end{array}\right\}\right)$$

gives an upper bound on the surfeit of a triangle of combinatorial type $[r_A, r_B, r_C]$ relative to the target density ρ_0.

The first terms in $f_{[r_A,r_B,r_C]}$ are simply the sector area associated with a triangle of combinatorial type $[r_A, r_B, r_C]$ with similarity type (β, γ). The **max** function gives twice the area of the smallest triangle of similarity type (β, γ) which admits disks of radii r_A, r_B, and r_C at its vertices so that the disks do not overlap.

The domain of $f_{[r_A,r_B,r_C]}$ is the region of the $\beta\gamma-$plane corresponding to permissible triangles of combinatorial type $[r_A, r_B, r_C]$ as shown in Table I.

2.3 Positive Surfeit Regions

In the works of Thue, L. Fejes Toth, Blind, Chang and Wang and others it was possible to arrive at a decomposition of the plane in which no component had density greater than the target density. Here, when ρ_0 is chosen to be less than the Florian bound at a given homogeneity, the surfeit of certain triangles will be positive; however, assigning the domain of the $f_{[r_A,r_B,r_C]}$ according to the information in Table I significantly constrains the geometry of these positive-surfeit triangles.

For example, Figure 4 shows that a $[1, 1, 0.7]$ triangle in a saturated two-radius packing of homogeneity 0.7 cannot carry positive surfeit (the lower surface is the graph of the bounding function and the upper plane represents surfeit 0). Figure 5 shows that a $[0.7, 0.7, 0.7]$ triangle carries positive surfeit only in a small region of the $\beta\gamma$-plane, which appears as a darker sliver in the bottom portion of the graph.

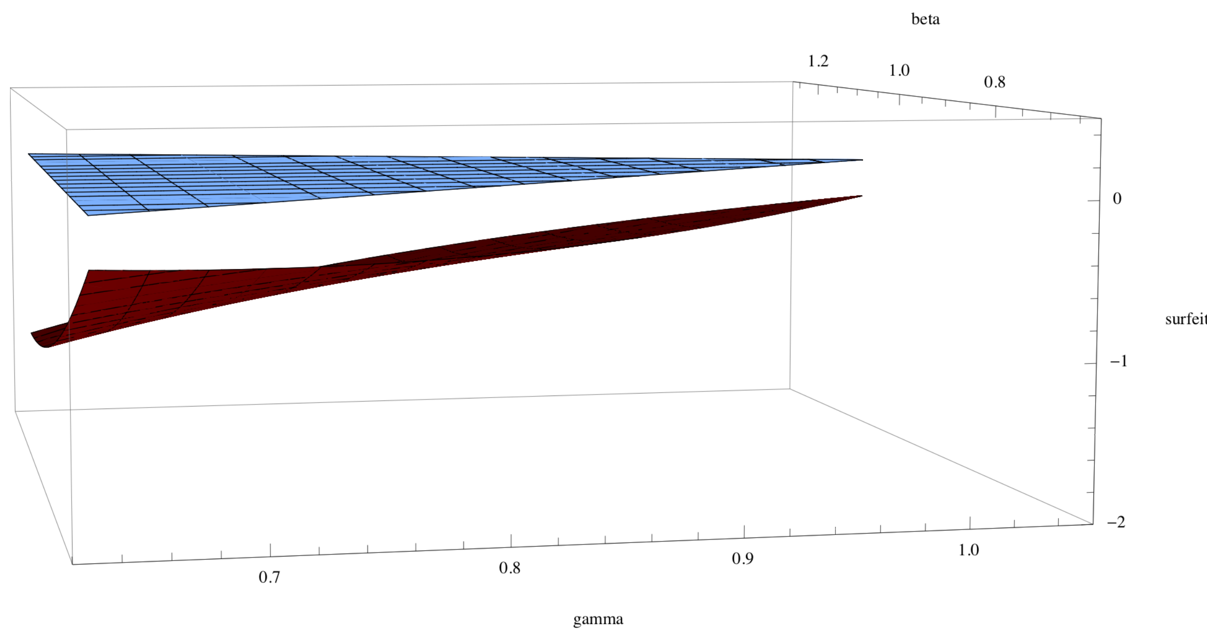

Figure 4: Surfeit bound for [1, 1, 0.7] triangles

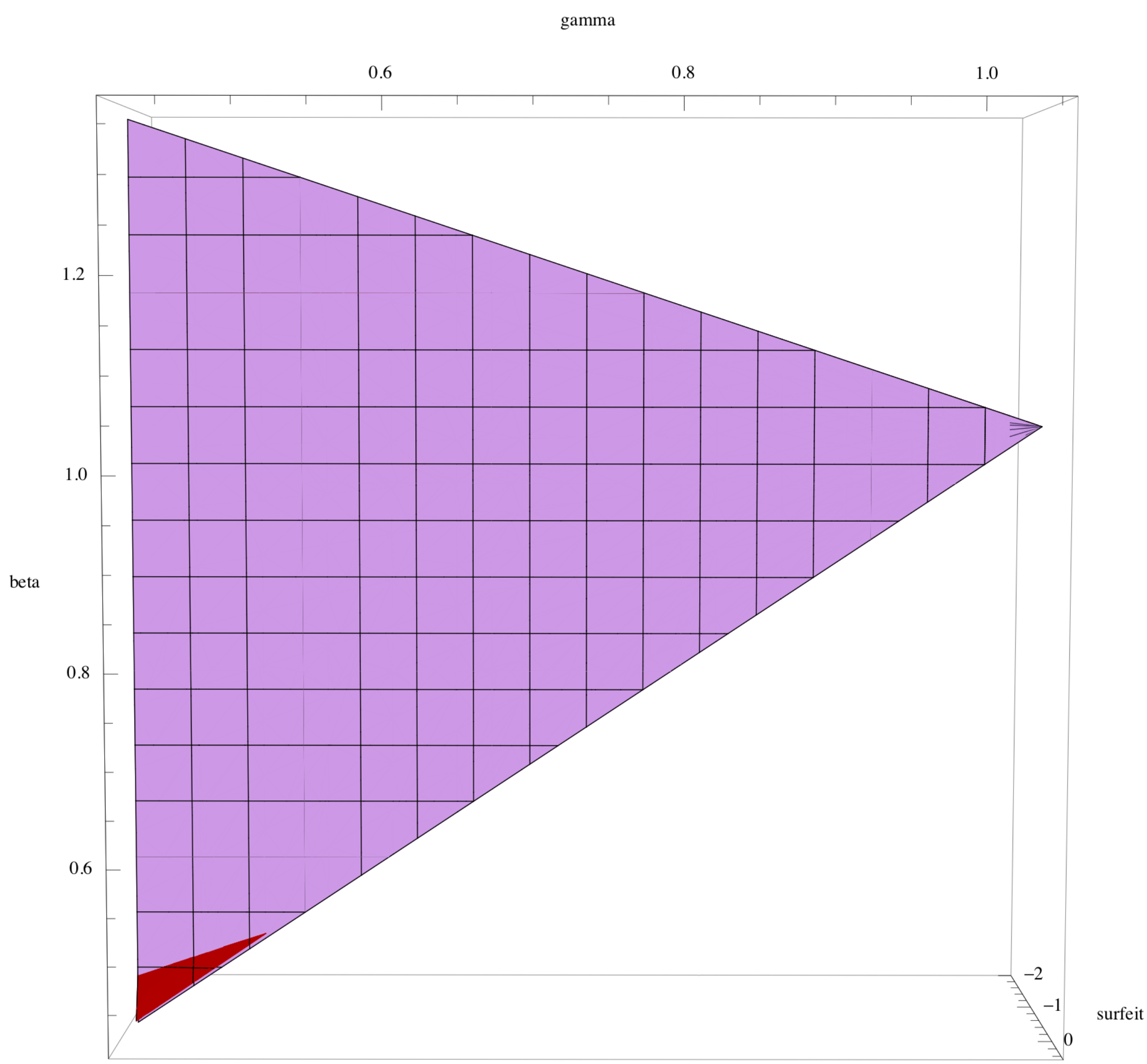

Figure 5: Surfeit bound for [0.7, 0.7, 0.7] triangles

CHAPTER 3

A NEW UPPER DENSITY BOUND FOR TWO-RADIUS PACKINGS OF HOMOGENEITY 0.7

In this chapter we establish an upper bound of 0.909346 on the density of two-radius packings of homogeneity 0.7, which is sharper than Florian's bound of approximately 0.909347. We assume each packing is saturated, and that a Delaunay triangulation on the centers of its disks is in hand. When we say "triangle," we mean a triangle of such a triangulation.

3.1 Constants and Definitions Used Only in This Chapter

For this chapter, we define:

- $h := 0.7$,
- $\rho_0 := 0.909346$,
- $s := 2.016 \times 10^{-6}$.

Definition 3.1.1. ***Surfeit*** *means surfeit relative to* ρ_0.

Definition 3.1.2. *We say a triangle is* ***heavy*** *if its surfeit is at least* $-6s$.

As an aid to visualization, note the approximate numeric values some of the angle bounds from Table I as they apply to triangles in this chapter: $\sin^{-1}(\frac{1}{1+h}) \approx 0.6289$ rad $\approx 36.03°$ and $\sin^{-1}(\frac{h}{1+h}) \approx 0.4244$ rad $\approx 24.32°$.

3.2 Summary of the Argument

The overall structure of the argument leading to the main result of this chapter is presented below. The technical results asserted as claims are proven later in this chapter.

Claim 3.2.1. *A triangle of type* $[h, h, 1]$ *has surfeit at most* s.

Claim 3.2.2. *A heavy triangle must be of type* $[h, h, 1], [1, h, h]$, *or* $[h, h, h]$.

Claim 3.2.3. *If a triangle of type* $[1, h, h]$ *or* $[h, h, h]$ *is heavy, then it is obtuse, and we can compute, by type, a lower bound on circumradius and upper bounds on the sizes of each acute angle. (Saturation gives an upper bound of* $1 + h$ *on the circumradius of any triangle.)*

Corollary 3.2.4. *As a result of Claim 3.2.3, we can also compute, by type, upper bounds on the lengths of the two smallest edges and a lower bound on the length of the long edge of any heavy* $[1, h, h]$ *or* $[h, h, h]$ *triangle.*

Claim 3.2.5. *The upper and lower bounds on edge lengths from Corollary 3.2.4 allow us to conclude that if two heavy triangles of type* $[1, h, h]$ *or* $[h, h, h]$ *share an edge, that shared edge must either be the long edge of both or a short edge of both.*

Claim 3.2.6. *If a triangle of type* $[1, h, h]$ *or* $[h, h, h]$ *is heavy, then the surfeit of its long-edge adjacent neighbor is negative enough to compensate. More precisely, if* y *is the surfeit of the neighbor* D *and if* D *has a total of* n *heavy* $[1, h, h]$ *or* $[h, h, h]$ *neighbors of surfeits* $x_1, \ldots, x_n$, *then* $y + \sum_{i=1}^{n} (x_i + 7s) < -6s$.

Corollary 3.2.7. *If two heavy triangles of type* $[1, h, h]$ *or* $[h, h, h]$ *share an edge, that shared edge cannot be the long edge of either.*

Definition 3.2.8. *Let D be any triangle in a Delaunay triangulation on a saturated two-radius packing of homogeneity h. Let n_D be the number of triangles which share an edge with D and also:*

a) are of type $[1, h, h]$ or $[h, h, h]$,

b) are heavy, and

c) share their longest edge with D.

We define ***the adjusted surfeit of D*** *as follows:*

Case 1*: If $n_D = 0$ and D is not itself a heavy triangle of type $[1, h, h]$ or $[h, h, h]$, then we define the adjusted surfeit of D to be equal to the surfeit of D.*

Case 2*: If D is a heavy triangle of type $[1, h, h]$ or $[h, h, h]$, then we define the adjusted surfeit of D to be $-7s$.*

Case 3*: If $n_D > 0$, name the triangles which share an edge with D and satisfy all three conditions above $T_1, \ldots, T_{n_D}$, call their surfeits $x_1, \ldots, x_{n_D}$ respectively, and let y be the surfeit of D. The adjusted surfeit of D is defined to be $y + \sum_{i=1}^{n_D}(x_i + 7s)$.*

If D belongs to ***Case 3****, we define* ***the cluster of D*** *to be the set $\{D, T_1, \ldots, T_{n_D}\}$.*

Proposition 3.2.9. *Adjusted surfeit is well-defined.*

Proof. In order to show that the cases above comprise a partition of the triangulation, it suffices to show that a triangle cannot simultaneously belong to Case 2 and Case 3. Let D be a heavy triangle of type $[1, h, h]$ or $[h, h, h]$. By Corollary 3.2.7, no two heavy triangles of type $[1, h, h]$ or $[h, h, h]$ can share a long edge, so $n_D = 0$. □

It is now possible to conclude the following:

Proposition 3.2.10. *Any triangle of adjusted surfeit greater than* $-6s$ *is of type* $[h, h, 1]$.

Proof. Let D be a triangle with adjusted surfeit greater than $-6s$, and consider the cases in Definition 3.2.8. In Case 2 the assigned adjusted surfeit is $-7s$, and in Case 3 the assigned adjusted surfeit is less than $-6s$ by Claim 3.2.6, so the adjusted surfeit of D has been assigned according to Case 1. This means that D is heavy, and is not of type $[1, h, h]$ or $[h, h, h]$. By Claim 3.2.2, D is of type $[h, h, 1]$. □

We extend the definition of adjusted surfeit to finite unions of triangles:

Definition 3.2.11. *The* ***adjusted surfeit of a finite union of triangles*** *is the sum of the adjusted surfeits of the triangles in the union.*

Proposition 3.2.12. *No triangle belongs to more than one cluster.*

Proof. By construction, every cluster is identified by a unique triangle D for which $n_D > 0$, and heavy triangles $T_1, \ldots, T_{n_D}$ which each share their longest edge with D. By Claim 3.2.6, D is not heavy, and so cannot be a member of another cluster. By Claim 3.2.3, each of $T_1, \ldots, T_{n_D}$ is obtuse, and so has a unique longest edge and cannot be a heavy triangle in another cluster. □

Lemma 3.2.13. *The adjusted surfeit of a cluster is the same as its surfeit. The adjusted surfeit of a triangle not belonging to a cluster is also the same as its surfeit.*

Proof. Clusters arise only when the adjusted surfeit of some triangle D is assigned according to Case 3 of Definition 3.2.8, and consist of at most 4 triangles $D, T_1, \ldots, T_{n_D}$ with respective surfeits $y, x_1, \ldots, x_D$. By Proposition 3.2.12, clusters are disjoint. Notice that the adjusted surfeit of D is $y + \sum_{i=1}^{n_D}(x_i + 7s)$ and the surfeit of each of the triangles $T_1, \ldots, T_{n_D}$ is $-7s$. $\square$

We say that a collection of triangles intersects a set if any element of that collection intersects the set.

Corollary 3.2.14. *If X is a bounded set and W is defined as follows:*

$$W := \{\textit{clusters intersecting } X\} \cup \{\textit{triangles contained in } X\},$$

then, as a result of Observation 3.2.13 and Proposition 3.2.12, the surfeit of W is the same as the adjusted surfeit of W.

Claim 3.2.15. *No more than six $[h, h, 1]$ triangles of adjusted surfeit greater than $-6s$ may occur in any star, and no star may be composed entirely of triangles of this type. (See Proposition 3.8.1).*

Corollary 3.2.16. *No star in the packing has positive adjusted surfeit.*

Proof. Since s is a positive quantity, it follows from Proposition 3.2.10 that only triangles of type $[h, h, 1]$ may make a positive contribution to the adjusted surfeit of a star. By Claim 3.2.15 no star may have more than 6 such triangles. The adjusted surfeit of a triangle of type $[h, h, 1]$ is assigned according to Case 1 of Definition 3.2.8 and is equal to its surfeit, which, by Claim 3.2.1, is at most s, so the contribution of triangles of positive adjusted surfeit to the surfeit of any star is at most $6s$. By Proposition 3.2.10, the adjusted surfeit of any triangle not of type

$[h, h, 1]$ is less that $-6s$, so, by Claim 3.2.15, every star must have at least one triangle with adjusted surfeit less than $-6s$. □

Observation 3.2.17. *The diameter of every triangle is bounded above by $2(1+h)$, so the diameter of every cluster is bounded above by a constant depending only on h, as is the diameter of every star.*

Proposition 3.2.18. *The area of every triangle is bounded below by a constant depending only on h.*

Proof. Choose a triangle T at will, let r be its circumradius, and label T as described in 3.1. Then, $|T| = \frac{1}{2}ab\sin\gamma$. The length of each edge is bounded below by $2h$, and

$$\sin\gamma = \frac{c}{2r} > \frac{2h}{2(1+h)}.$$

□

Proposition 3.2.19. *The set of absolute values of surfeits of triangles is bounded.*

Proof. Let T be a triangle with associated sector area $\mathrm{sa}(T)$. Then the surfeit of T is $\mathrm{sa}(T) - \rho_0|T|$. The sector area of any triangle is bounded above by $\pi/2$, so we may take $\pi/2$ as a coarse upper bound for the surfeit of T, and every triangle is inscribed in a circumcircle of radius less than $1+h$, so we may take $-\pi\rho_0(1+h)^2$ as a coarse lower bound on the surfeit of T. □

Theorem 3.3.1, which states that the density of a two-radius packing of homogeneity h is at most ρ_0, follows:

3.3 Proof of the Main Theorem

Theorem 3.3.1. *The density of a two-radius packing of homogeneity h is at most ρ_0.*

Proof. Let $\mathcal{P}$ denote the set of all disks in the packing, and let B_r be the disk of radius r centered at the origin. Then the density ρ of the packing is:

$$\rho := \limsup_{r\to\infty} \frac{|\mathcal{P} \cap B_r|}{|B_r|}.$$

All the disks meeting the boundary of B_r are contained in the annulus $B_{r+1} \backslash B_{r-1}$, so the sum of their areas is at most linear in r. On the other hand, $|B_r|$ grows quadratically with r. This means we may replace the numerator in the expression above with the sum of the areas of disks meeting B_r and write:

$$\rho = \limsup_{r\to\infty} \frac{\sum\limits_{D\cap B_r \neq \emptyset} |D|}{|B_r|}.$$

Let $\mathcal{T}$ denote the set of all triangles in a Delaunay triangulation on this packing. The diameter of each triangle of $\mathcal{T}$ is bounded above as noted in Observation 3.2.17, so all triangles carrying sector area associated with a disk meeting ∂B_r are contained in an annulus of bounded width. The area of each triangle is bounded below as shown in Proposition 3.2.18, so the number of these triangles is linear in r. This means we can replace the numerator of the expression above with the sum of sector areas associated with triangles contained in B_r, and the denominator with the total area of triangles contained in B_r. If we let $\mathrm{sa}(T)$ denote the sector area associated with a triangle T, we may write:

$$\rho = \limsup_{r\to\infty} \frac{\sum\limits_{T\subset B_r} \mathrm{sa}(T)}{\sum\limits_{T\subset B_r} |T|},$$

so

$$\rho - \rho_0 = \limsup_{r\to\infty} \frac{\sum\limits_{T\subset B_r} \mathrm{sa}(T) - \rho_0 \sum\limits_{T\subset B_r} |T|}{\sum\limits_{T\subset B_r} |T|}.$$

Let $\mathrm{sf}(T)$ represent the surfeit of T. Since $\mathrm{sf}(T) := \mathrm{sa}(T) - \rho_0|T|$, we may write:

$$\rho - \rho_0 = \limsup_{r\to\infty} \frac{\sum\limits_{T\subset B_r} \mathrm{sf}(T)}{\sum\limits_{T\subset B_r} |T|}.$$

Now define:

$$W_r := \{\text{triangles in clusters intersecting } B_r\} \cup \{\text{triangles contained in } B_r\}.$$

Clusters are bounded in diameter (Observation 3.2.17), so triangles in W_r that are not contained in B_r are contained in an annulus of bounded width. The area of each triangle is bounded below (Proposition 3.2.18), so the number of triangles in this annulus is at most linear in r. Since the area of each triangle is also bounded above, the quantity

$$\sum_{T\in W_r} |T| - \sum_{T\subset B_r} |T|$$

is at most linear in r. Additionally, since the surfeit of each triangle is bounded in absolute value (Proposition 3.2.19), the quantity

$$\mathrm{sf}(W_r) - \sum_{T \subset B_r} \mathrm{sf}(T)$$

is also at most linear in r. As a result, we may write:

$$\rho - \rho_0 = \limsup_{r\to\infty} \frac{\mathrm{sf}(W_r)}{\sum\limits_{T\in W_r} |T|}.$$

Let $\mathrm{ajsf}(W_r)$ denote the adjusted surfeit of W_r. By Corollary 3.2.14, $\mathrm{ajsf}(W_r) = \mathrm{sf}(W_r)$, so we may write:

$$\rho - \rho_0 = \limsup_{r\to\infty} \frac{\mathrm{ajsf}(W_r)}{\sum\limits_{T\in W_r} |T|}.$$

Let V_r be the disjoint union of all stars intersecting B_r. Stars are bounded in diameter as noted in Observation 3.2.17 and the area of each triangle is bounded below as shown in Proposition 3.2.18, so V_r is a triple cover of B_r except in a bounded neighborhood of ∂B_r, and the number of stars in this bounded neighborhood of ∂B_r is linear in r. This means the quantity

$$\mathrm{ajsf}(W_r) - \frac{1}{3} \sum_{S\in V_r} \mathrm{ajsf}(S)$$

is at most linear in r. We may write:

$$\rho - \rho_0 = \limsup_{r\to\infty} \frac{\frac{1}{3}\sum\limits_{S\in V_r} \mathrm{ajsf}(S)}{\sum\limits_{T\in W_r} |T|}.$$

By Corollary 3.2.16, every term in the numerator is non-positive, so the result follows. $\square$

3.4 The Surfeit Bounding Function for Two Radius Packings

For two-radius packings it is possible to use a simpler form of the surfeit bounding function described in 2.2.1. This is because of two facts: first, there are only two choices of radius; second, since $\gamma \leq \beta \leq \pi - (\beta + \gamma) < \pi - \beta$, we have:

$$\sin\gamma \leq \sin\beta \leq \sin(\beta + \gamma).$$

As a result, at least one branch of $f_{[r_A, r_B, r_C]}$ is never selected. For instance, when the combinatorial type is $[1, 1, 1]$, the third branch of the max function is always selected and the first two may be ignored, so

$$f_{[1,1,1]}(\beta, \gamma) = \frac{\pi}{2} - 2\rho_0 \frac{\sin(\beta + \gamma)\sin\beta}{\sin\gamma}.$$

The boundary of the domain of $f_{[r_A, r_B, r_C]}$ is piecewise smooth for every choice of r_A, r_B, and r_C. Where applicable, the curve dividing the branches of $f_{[r_A, r_B, r_C]}$ in the $\beta\gamma-$plane is also smooth, so it is possible to use calculus.

3.5 Proof of Claim 3.2.1

Proposition 3.5.1. *The surfeit of a triangle of type* $[h, h, 1]$ *is at most* s.

Proof. For a triangle of type $[h, h, 1]$ the surfeit bounding function simplifies to:

$$f_{[h,h,1]}(\beta,\gamma) := \frac{1}{2}\left(h^2\pi + (1-h^2)\gamma - \rho_0 \max \left\{ \begin{array}{l} (1+h)^2 \left(\dfrac{\sin(\beta+\gamma)\sin\gamma}{\sin\beta} \right) \\ (2h)^2 \left(\dfrac{\sin(\beta+\gamma)\sin\beta}{\sin\gamma} \right) \end{array} \right\} \right).$$

Let f_1 and f_2 represent the branches of $f_{[h,h,1]}$. Then:

$$\frac{\partial f_1}{\partial \beta} = \frac{1}{2}\frac{\rho_0(1+h)^2\sin^2\gamma}{\sin^2\beta}$$

and

$$\frac{\partial f_2}{\partial \gamma} = \frac{1}{2}\left((1-h^2) + \frac{\rho_0(2h)^2\sin^2\beta}{\sin^2\gamma} \right),$$

so both f_1 and $_2$ have non-zero gradient for permissible $[h, h, 1]$ triangles. The extrema of $f_{[h,h,1]}$ therefore occur either on the curve $(1+h)\sin\gamma = 2h\sin\beta$ where $f_1 = f_2$, or on the boundary of its domain. Along the curve where the branches of $f_{[h,h,1]}$ agree, the surfeit bounding function may be parametrized by γ and expressed as follows:

$$f_{[h,h,1]}\Big|_{(1+h)\sin\gamma = 2h\sin\beta} = \frac{h^2\pi}{2} + \frac{(1-h^2)\gamma}{2} - \rho_0 h(1+h)\sin\left(\sin^{-1}\left(\frac{1+h}{2h}\sin\gamma \right) + \gamma \right).$$

This restriction of $f_{[h,h,1]}$ has positive second derivative for all permissible values of γ and attains its maximum value, which is less than 2.016×10^{-6}, at $\gamma = 2\sin^{-1}(h/(1+h))$.

The smooth components of the boundary of the domain of $f_{[h,h,1]}$, counterclockwise from the point $(\sin^{-1}(\frac{h}{1+h}), \frac{\pi}{6})$, are described below. The behavior of the bounding function on each boundary component is also described.

- The vertical line $\gamma = \sin^{-1}\frac{h}{1+h}$ restricted to $\frac{\pi}{6} < \beta \leq \frac{\pi}{2} - \frac{\gamma}{2}$. Here the bounding function evaluates to the branch f_2, is parametrizable in β, and is decreasing in β under this parametrization.

- The line $\beta = \pi/2 - \gamma/2$ restricted to $\sin^{-1}(\frac{h}{1+h}) < \gamma \leq 2\sin^{-1}(h/(1+h))$. Here the bounding function evaluates to the branch f_2, is parametrizable in γ, and is increasing in γ under this parametrization.

- The line $\beta = \pi/2 - \gamma/2$ restricted to $2\sin^{-1}(h/(1+h)) \leq \gamma \leq \pi/3$. Here the bounding function evaluates to the branch f_1, is parametrizable in γ, and is decreasing in γ under this parametrization.

- The line $\beta = \gamma$ restricted to $\pi/6 \leq \gamma \leq \pi/3$. Here the bounding function evaluates to the branch f_1, is parametrizable in γ, and has positive second derivative in γ under this parametrization.

- The horizontal line $\beta = \pi/6$ restricted to $\sin^{-1}(\frac{h}{1+h}) < \gamma \leq \sin^{-1}(\frac{h}{1+h})$. Here the bounding function evaluates to the branch f_1, is parametrizable in γ, and is decreasing in γ under this parametrization.

Since $f_{[h,h,1]}(\sin^{-1}(\frac{h}{1+h}), \frac{\pi}{6})$ is less than -0.001, the result follows.

$\square$

3.6 Proof of Claim 3.2.2

In this section we show that types $[1,1,1], [1,1,h], [1,h,1], [h,1,1]$, and $[h,1,h]$ admit no heavy triangles. This establishes Claim 3.2.2 of section 3.2.

Proposition 3.6.1. *The surfeit of a triangle of type* $[1,1,1]$ *is bounded above by -0.0042.*

Proof. The surfeit bounding function for triangles of type $[1,1,1]$ may be simplified as described in 3.4 and written:

$$f_{[1,1,1]}(\beta,\gamma) = \frac{\pi}{2} - 2\rho_0 \frac{\sin(\beta+\gamma)\sin\beta}{\sin\gamma}.$$

Since

$$\frac{\partial f_{[1,1,1]}}{\partial \gamma} = 2\rho_0 \frac{\sin^2\beta}{\sin^2\gamma},$$

which is non-zero of the domain of $f_{[1,1,1]}$, this function does not attain its maximum on the interior of its domain.

The permissible domain is enclosed by the curves

- $\beta = \gamma$ for γ between $\sin^{-1}(1/(1+h))$ and $\pi/3$,
- $\beta = \dfrac{\pi}{2} - \dfrac{\gamma}{2}$ for γ between $\sin^{-1}(1/(1+h))$ and $\frac{\pi}{3}$,
- $\gamma = \sin^{-1}(1/(1+h))$ for β between $\sin^{-1}(1/(1+h))$ and $\pi/2 - (\sin^{-1}(1/(1+h)))/2$.

Straightforward analysis of the sort shown in Proposition 3.5.1 shows that $f_{[1,1,1]}$ attains its maximum at $(\pi/3, \pi/3)$ and is bounded above by -0.0042. $\square$

Proposition 3.6.2. *The functions* $f_{[1,1,1]}$ *and* $f_{[1,1,h]}$*, which have the same domain, satisfy* $f_{[1,1,1]} > f_{[1,1,h]}$*, so the surfeit of a triangle of type* $[1,1,h]$ *is also bounded above by -0.0042.*

Proof. The bounding function $f_{[1,1,h]}$ simplifies to:

$$f_{[1,1,h]}(\beta,\gamma)=\frac{\pi}{2}-\frac{1-h^2}{2}\gamma-2\rho_0\frac{\sin(\beta+\gamma)\sin\beta}{\sin\gamma},$$

which is strictly less than $f_{[1,1,1]}(\beta,\gamma)$.

$\square$

Proposition 3.6.3. *The surfeit of a triangle of type* $[1,h,1]$ *is bounded above by* -0.0021.

Proof. The surfeit bounding function for triangles of type $[1,h,1]$ may be simplified as described in 3.4 and written:

$$f_{[1,h,1]}(\beta,\gamma):=\frac{1}{2}\left(\pi-(1-h^2)\beta-\rho_0\max\begin{cases}(2)^2\left(\dfrac{\sin(\beta+\gamma)\sin\gamma}{\sin\beta}\right)\\(1+h)^2\left(\dfrac{\sin(\beta+\gamma)\sin\beta}{\sin\gamma}\right)\end{cases}\right).$$

For computation, define

$$f_1(\beta,\gamma):=\frac{1}{2}\left(\pi-(1-h^2)\beta-(2)^2\rho_0\left(\frac{\sin(\beta+\gamma)\sin\gamma}{\sin\beta}\right)\right)$$

and

$$f_2(\beta,\gamma)=\frac{1}{2}\left(\pi-(1-h^2)\beta-(1+h)^2\rho_0\left(\frac{\sin(\beta+\gamma)\sin\beta}{\sin\gamma}\right)\right).$$

On the permissible domain for $[1,h,1]$ triangles $\frac{\partial f_1}{\partial\gamma}=0$ only when $\beta=\gamma=\pi/3$; however, $\frac{\partial f_1}{\partial\beta}$ is non-zero here. Also, $\frac{\partial f_2}{\partial\beta}$ is non-zero for permissible $[1,h,1]$ triangles. So f_1 and f_2 have non-zero gradient on the domain of $f_{[1,h,1]}$.

The restriction of $f_{[1,h,1]}$ to the curve $2\sin\gamma = (1+h)\sin\beta$, where $f_1 = f_2$, is bounded above by its value at $(\frac{\pi}{6}, \sin^{-1}(\frac{1}{1+h}))$, which is less than -0.0021.

The smooth components of the boundary of the domain of $f_{[1,h,1]}$, counterclockwise from the point $(\frac{\pi}{6}, \sin^{-1}(\frac{1}{1+h}))$, are described below. The behavior of the bounding function on each boundary component is also described.

- The vertical line $\gamma = \pi/6$ restricted to $\sin^{-1}(\frac{1}{1+h}) < \beta \leq \frac{\pi}{2} - \frac{\gamma}{2}$. Here the bounding function is decreasing in β and is bounded above by -0.0021.
- The line $\beta = \pi/2 - \gamma/2$ restricted to $\pi/6 < \gamma \leq 2\sin^{-1}((1+h)/4)$. Here the bounding function evaluates to the branch f_2, is parametrizable in γ, is increasing in γ under this parametrization and is bounded above by -0.117.
- The line $\beta = \pi/2 - \gamma/2$ restricted to $2\sin^{-1}((1+h)/4) \leq \gamma \leq \pi/3$. Here the bounding function evaluates to the branch f_1, is parametrizable in γ, and is decreasing in γ under this parametrization.
- The line $\beta = \gamma$ restricted to $\sin -1(1/(1+h)) \leq \gamma \leq \pi/3$. Here the bounding function evaluates to the branch f_1, is parametrizable in γ, and has positive second derivative in γ under this parametrization.
- The horizontal line $\beta = \sin -1(1/(1+h))$ restricted to $\pi/6 < \gamma \leq \sin^{-1}(\frac{1}{1+h})$. Here the bounding function evaluates to the branch f_1, is parametrizable in γ, and is decreasing in γ under this parametrization.

So, on the boundary of the permissible domain for $[1, h, 1]$ triangles, $f_{[1,h,1]}$ attains its maximum at $(\sin^{-1}(\frac{1}{(1+h)}), \frac{\pi}{6})$, and is bounded above by -0.0021. $\square$

Proposition 3.6.4. *The surfeit of a triangle of type* $[h, 1, 1]$ *is bounded above by* -0.000073.

Proof. The function below gives an upper bound for the surfeit of $[h, 1, 1]$ triangles:

$$f_{[h,1,1]}(\beta, \gamma) := \frac{1}{2}\left(h^2\pi + (1-h^2)(\beta+\gamma) - \rho_0 \max \begin{cases} (2)^2 \left(\dfrac{\sin\beta \sin\gamma}{\sin(\beta+\gamma)} \right) \\ (1+h)^2 \left(\dfrac{\sin(\beta+\gamma)\sin\beta}{\sin\gamma} \right) \end{cases} \right).$$

An examination of the behavior of this function on the interior of its domain, on the bounding curve separating its branches, and on its boundary shows it is bounded above by its value at $(\cos^{-1}(1/(1+h)), \cos^{-1}(1/(1+h)))$, which is less than -0.000073. $\square$

Proposition 3.6.5. *The functions* $f_{[h,1,h]}$ *and* $f_{[h,1,1]}$, *which have the same domain, satisfy* $f_{[h,1,1]} > f_{[h,1,h]}$, *so the surfeit of a triangle of type* $[h, 1, h]$ *is also bounded above by* -0.000073.

Proof. The bounding function $f_{[h,1,h]}$ simplifies to:

$$f_{[h,1,h]}(\beta, \gamma) := \frac{1}{2}\left(h^2\pi + (1-h^2)(\beta+\gamma) - \rho_0(1+h)^2 \left(\frac{\sin(\beta+\gamma)\sin\beta}{\sin\gamma} \right) \right).$$

which is strictly less than $f_{[h,1,1]}$. $\square$

3.7 Pair Analysis for heavy $[1, h, h]$ and $[h, h, h]$ triangles

In this section:

- Propositions 3.7.1 and 3.7.2 establish Claims 3.2.3 and 3.2.5.
- Propositions 3.7.6, 3.7.8, and 3.7.3 establish Claim 3.2.6 for heavy $[1, h, h]$ triangles.
- Propositions 3.7.7, 3.7.9, and 3.7.4 establish the validity of Claim 3.2.6 for heavy $[h, h, h]$ triangles.

Proposition 3.7.1. *In any permissible triangle of type* $[1, h, h]$ *of surfeit greater than* $-6s$*, each of the two smallest angles* γ *and* β *of are no larger than* $0.628821 \approx 36.0287°$*, so such a triangle is obtuse; each of the two shortest edges of such a triangle is no longer than* 2*; the long edge of such a triangle is no shorter than* 2.749*; and the circumradius is at least* 1.445.

Proof. The function below gives an upper bound for the surfeit of a $[1, h, h]$ triangle:

$$f_{[1,h,h]}(\beta, \gamma) := \frac{1}{2}\left(\pi - (1 - h^2)(\beta + \gamma) - \rho_0(1 + h)^2 \left(\frac{\sin(\beta + \gamma)\sin\beta}{\sin\gamma}\right)\right).$$

It is not hard to show using methods similar to those in the previous sections that if $\beta + \gamma \geq \pi/2$ then $f_{[1,h,h]}(\beta, \gamma) \leq -6s$. Suppose now that $f_{[1,h,h]}(\beta, \gamma) > -6s$. Then $\beta + \gamma < \pi/2$, which means that $\sin(\beta + \gamma) > \sin 2\gamma$, so

$$(1 - h^2)\gamma + \frac{\rho_0}{2}(1 + h)^2 \sin 2\gamma < \frac{\pi}{2} + 6s$$

which means $\gamma < 0.628821 \approx 36.0287°$.

To establish the upper bound for β, notice that the curve implicitly defined by $f_{[1,h,h]} = -6s$ meets the permissible domain in a single connected component, is monotone increasing with respect to γ, intersects the line $\gamma = \pi/6$ on the left and the line $\beta = \gamma$ on the right. Since $f_{[1,h,h]}(\pi/6, \pi/6) > -6s$, the region for which $f_{[1,h,h]} > -6s$ lies beneath this curve, so the upper bound obtained for γ in the preceding paragraph may also be used as an upper bound for β.

Now let a, b and c represent respectively the lengths of the edges of such a triangle in descending order of length. The circumradius r of any triangle is bounded above by $1 + h$, so the upper bound on the lengths of b are obtained from $b = 2r \sin \beta$ and $c = 2r \sin \gamma$.

We also have:

$$a = \frac{\sin(\beta + \gamma)}{\sin \gamma} c.$$

It is not hard to show, by the now-familiar method of searching for critical points on the interior and boundary of a domain, that when β and γ are within the bounds established above, $\sin(\beta + \gamma)/\sin \gamma$ is at least 1.6174. Since c is at least $1 + h$, a is at least 2.749.

The lower bound on circumradius follows from $r = c/2 \sin \gamma$. $\square$

Proposition 3.7.2. *In any permissible triangle of type* $[h, h, h]$ *of surfeit greater than* $-6s$*, each of the two smallest angles* γ *and* β *of are no larger than* $0.521281 \approx 29.8672°$*, so such a triangle is obtuse; each of the two shortest edges of such a triangle is no longer than* 1.6932*; the long edge of such a triangle is no shorter than* 2.4281*; and the circumradius is at least* 1.405.

Proof. The argument is very similar to that given in the previous proposition, except that the length of the short edge of a triangle of type $[h, h, h]$ is bounded below by $2h$.

The function below gives an upper bound for the surfeit of a $[h, h, h]$ triangle:

$$f_{[h,h,h]}(\beta, \gamma) := \frac{1}{2}\left(h^2\pi - \rho_0(2h)^2\left(\frac{\sin(\beta+\gamma)\sin\beta}{\sin\gamma}\right)\right).$$

Once again, it's not hard to show that if $\beta + \gamma \geq \pi/2$ then $f_{[h,h,h]}(\beta, \gamma) \leq -6s$. Suppose now that $f_{[h,h,h]}(\beta, \gamma) > -6s$. Then $\beta + \gamma < \pi/2$, which means that $\sin(\beta + \gamma) > \sin 2\gamma$, so

$$\frac{\rho_0}{2}(2h)^2 \sin 2\gamma < \frac{h^2\pi}{2} + 6s$$

which means $\gamma < 0.521281 \approx 29.8672°$.

Also as in the previous proposition, we establish the upper bound for β by noting that the curve implicitly defined by $f_{[1,h,h]} = -6s$ meets the permissible domain in a single connected component, is monotone increasing with respect to γ for permissible values of β and γ, intersects the line $\gamma = \sin^{-1}\frac{h}{1+h}$ on the left and the line $\beta = \gamma$ on the right, and that $f_{[1,h,h]} > -6s$ beneath this curve, so the upper bound obtained for γ in the preceding paragraph may also be used as an upper bound for β.

Now let a, b and c represent respectively the lengths of the edges of such a triangle in descending order of length. The circumradius r of any triangle is bounded above by $1 + h$, so the upper bound on the lengths of b are obtained from $b = 2r\sin\beta$ and $c = 2r\sin\gamma$.

We also have:

$$a = \frac{\sin(\beta+\gamma)}{\sin\gamma} c.$$

When β and γ are within the bounds established above, an examination of critical points in the interior and the boundary of this domain shows that $\sin(\beta+\gamma)/\sin\gamma$ is at least 1.7343. Since c is at least $2h$, a is at least 2.4281.

The lower bound on circumradius follows from $r = c/2\sin\gamma$. □

Notice that the upper bounds on the lengths of short edges established above is less than the lower bound on the lengths of the long edges. Claims 3.2.3 and 3.2.5 are now established.

The pair of propositions below establish upper bounds for the surfeit of $[1,h,h]$ and $[h,h,h]$ triangles:

Proposition 3.7.3. *The surfeit of a triangle of type* $[1,h,h]$ *is bounded above by* 0.1658.

Proof. The surfeit bounding function $f_{[1,h,h]}$ simplifies to:

$$f_{[1,h,h]}(\beta,\gamma) := \frac{1}{2}\left(\pi - (1-h^2)(\beta+\gamma) - \rho_0(1+h)^2\left(\frac{\sin(\beta+\gamma)\sin\beta}{\sin\gamma}\right)\right).$$

This function has non-zero gradient on its domain since

$$\frac{\partial f_{[1,h,h]}}{\partial\beta} = -(1-h^2) - \rho_0(1+h)^2\left(\frac{\sin(2\beta+\gamma)}{\sin\gamma}\right) < 0.$$

We note that $f_{[1,h,h]}(\pi/6,\pi/6)$ is bounded above by 0.1658 and examine $f_{[1,h,h]}$ on its boundary components, clockwise from $(\pi/6,\pi/6)$:

- The restriction of $f_{[1,h,h]}$ to $\gamma = \pi/6$ is decreasing in β.
- The restriction of $f_{[1,h,h]}$ to $\beta = \pi/2 - \gamma/2$ is increasing in γ.
- The restriction of $f_{[1,h,h]}$ to $\beta = \gamma$ has positive second derivative with respect to γ.

Since $f_{[1,h,h]}(\pi/6, \pi/6) > f_{[1,h,h]}(\pi/3, \pi/3)$, the result follows.

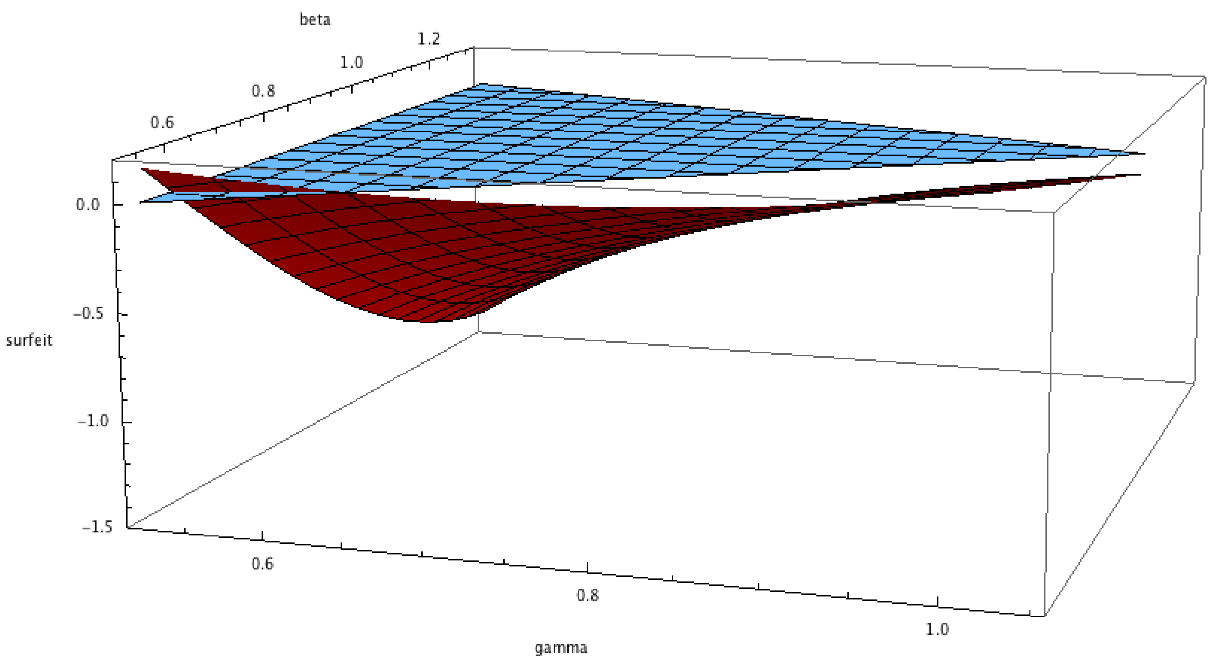

Figure 6: Surfeit bound for [1, 0.7, 0.7] triangles
The graph of $f_{[1,h,h]}$ (the curved surface) lies mostly below 0 and attains its maximum at $(\frac{\pi}{6}, \frac{\pi}{6})$.

$\square$

Proposition 3.7.4. *The surfeit of a triangle of type $[h,h,h]$ is bounded above by* 0.1009*.*

Proof. The surfeit bounding function $f_{[h,h,h]}$ simplifies to:

$$f_{[h,h,h]}(\beta,\gamma) := \frac{1}{2}\left(h^2\pi - \rho_0(2h)^2\left(\frac{\sin(\beta+\gamma)\sin\beta}{\sin\gamma}\right)\right).$$

The reset of the argument follows exactly the same form as the proof of Proposotion 3.7.3:

This function has non-zero gradient on its domain since

$$\frac{\partial f_{[1,h,h]}}{\partial\beta} = -\rho_0(2h)^2\left(\frac{\sin(2\beta+\gamma)}{\sin\gamma}\right) < 0.$$

We note that $f_{[1,h,h]}(\sin^{-1}(\frac{h}{1+h}),\sin^{-1}(\frac{h}{1+h}))$ is bounded above by 0.1009 and examine $f_{[1,h,h]}$ on its boundary components, clockwise from $(\pi/6,\pi/6)$:

- The restriction of $f_{[1,h,h]}$ to $\gamma = \sin^{-1}(\frac{h}{1+h})$ is decreasing in β.
- The restriction of $f_{[1,h,h]}$ to $\beta = \pi/2 - \gamma/2$ is increasing in γ.
- The restriction of $f_{[1,h,h]}$ to $\beta = \gamma$ has positive second derivative with respect to γ.

Since $f_{[1,h,h]}(\sin^{-1}(\frac{h}{1+h}),\sin^{-1}(\frac{h}{1+h})) > f_{[1,h,h]}(\pi/3,\pi/3)$, the result follows. □

To establish Claim 3.2.6 we consider four separate cases in which a heavy $[1,h,h]$ or $[h,h,h]$ is adjacent along its long edge to a triangle with a disk of radius h or 1 at the opposing vertex.

Lemma 3.7.5. *Suppose a point D lies outside the circumcircle of a triangle ABC. If ABC is obtuse at A, and if D lies on the side of BC opposite to A, then the radius of the circumcircle of DBC is greater than that of ABC.*

Proof. Let ABC and D be given as in the statement of the lemma. Let $\mathcal{L}$ be the perpendicular bisector of BC, and let H be that intersection of $\mathcal{L}$ with the circumcircle of ABC which lies in the same half-plane as D relative to BC. Now consider the circumcircle of DBC. Since D lies outside the circumcircle of ABC, so does the entire arc $\widehat{DBC}$, and in particular the intersection of this arc with $\mathcal{L}$, which we name H'. So H' lies above H on $\mathcal{L}$ relative to BC. Let $\eta := \angle BHC$ and $\eta' := \angle BH'C$. Since H' lies above H, $\eta' < \eta$, and since α is obtuse, η is acute, so $\sin \eta' < \sin \eta$. So, since the triangles BHC and $BH'C$ share the edge BC, the circumradius of $BH'C$ is greater than the circumradius of BHC. The result follows from noting that H lies on the circumcircle of ABC, and H' lies on the circumcircle of BDC. □

Proposition 3.7.6. *Let ABC be a $[1, h, h]$ triangle of surfeit greater than $-6s$. Under our labeling convention, BC is the longest edge of ABC. Let $D'BC$ be the triangle sharing this edge with ABC. If the disk centered at D' is of radius h, then the surfeit of $D'BC$ is no more than -0.6004.*

Proof. Consider a $[1, h, h]$ triangle ABC of similarity type (β, γ) of surfeit greater than $-6s$. Suppose that its long-edge adjacent neighbor $D'BC$ has a disk of radius h at D'. Then D' is in the half-plane opposite to A relative to BC and satisfies the following two properties:

- $|D'B| \geq 2h$,
- D' lies on or outside the circumcircle of ABC.

We seek an upper bound on the surfeit of $D'BC$. Since the sector area of $D'BC$ is equal to 0.245π, this amounts to finding a lower bound on the area of $D'BC$, which in turn amounts to minimizing the height of D' relative to BC. We now construct a new point D as follows:

Consider the perpendicular bisector $\mathcal{L}$ of BC. If D' and B are in opposite half-planes with respect to $\mathcal{L}$, let D_0 be the symmetric point with respect to $\mathcal{L}$, otherwise let $D_0 := D'$. Now construct the perpendicular from D_0 to $\mathcal{L}$ and let D_1 be the projection of D_0 onto $\mathcal{L}$. There are three cases:

Case 1: The line segment D_0D_1 intersects neither the circumcircle of ABC nor the circle of radius $2h$ centered at B. In this case, let D be that intersection of the circle of radius $2h$ centered at B and the circumcircle of ABC which lies in the half-plane of D' relative to BC. Notice that D is no higher than D' relative to BC, so $|DBC| \leq |D'BC|$.

Case 2: The line segment D_0D_1 intersects the circumcircle of ABC, but not the circle of radius $2h$ centered at B. In this case, again, let D be the intersection of the circle of radius $2h$ centered at B and the circumcircle of ABC which lies in the half-plane of D' relative to BC. Once again, $|DBC| \leq |D'BC|$.

Case 3: The line segment D_0D_1 intersects both the circumcircle of ABC and the circle of radius $2h$ centered at B. In this case, let D be that intersection of D_0D_1 and the disk of radius $2h$ centered at B which lies outside the circumcircle of ABC. If two such points exist, pick the one closer to $\mathcal{L}$. In this case $|DBC| = |D'BC|$.

Now notice that D satisfies the following three properties:

- $|BD| = 2h$,

- D lies on or outside the circumcircle of ABC,

- $|DBC| \leq |D'BC|$

Let δ be the angle of DBC at D.

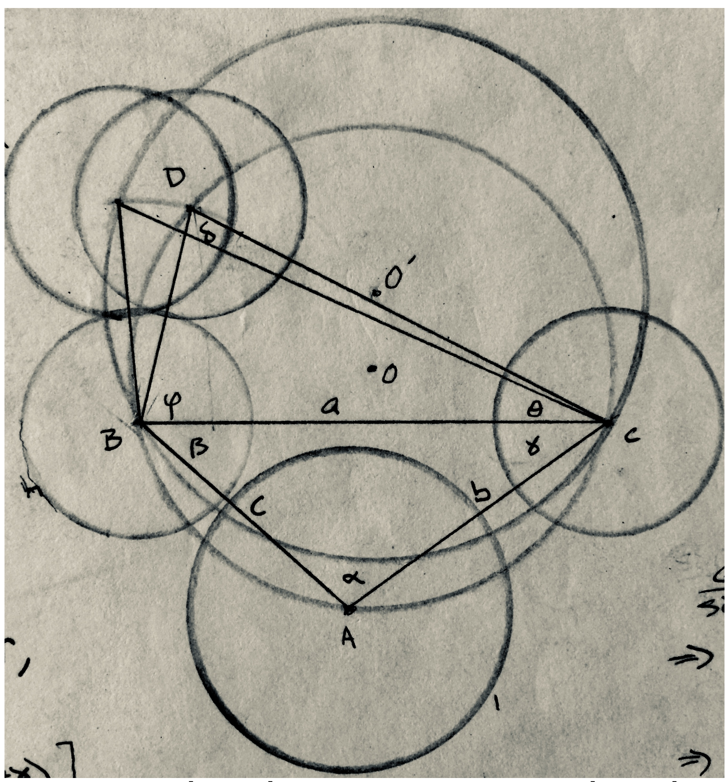

Figure 7: A heavy $[1, h, h]$ triangle paired with an $[h, h, h]$ triangle.
In this representation, D has been constructed following either Case 1 or Case 2. The unlabelled vertex corresponds to a construction of D following Case 3.

Since $|BD| = 2h$, the area of DBC is determined by the angle δ as shown below:

$$\begin{aligned}
|BDC| &= \frac{1}{2}|DB||DC|\sin\delta \\
&= h\,(2h\cos\delta + |BC|\cos\theta)\sin\delta \\
&= h\left(2h\cos\delta + |BC|\cos\left(\sin^{-1}\left(\frac{2h}{|BC|}\sin\delta\right)\right)\right)\sin\delta \\
&= h\left(2h\cos\delta + |BC|\sqrt{1-\left(\frac{2h}{|BC|}\sin\delta\right)^2}\right)\sin\delta \\
&= h\left(2h\cos\delta + \sqrt{|BC|^2-(2h\sin\delta)^2}\right)\sin\delta.
\end{aligned}$$

Now for β and γ bounded so that the surfeit of ABC is at least $-6s$, that is, subject to the bounds established in Proposition 3.7.1, we have:

$$|BC| = |AB|\frac{\sin(\beta+\gamma)}{\sin\gamma} \geq (1+h)\frac{\sin(\beta+\gamma)}{\sin\gamma} > 2h\frac{\sin(\beta+\gamma)}{\sin\gamma} > 2.264$$

so

$$|BCD| > h^2\sin 2\delta + h\sin\delta\sqrt{5.127 - 1.96\sin^2\delta}.$$

In order to find a lower bound for the expression on the right hand side of the inequality above, we seek upper and lower bounds on δ.

Let r and r' denote the circumradii of ABC and DBC respectively. Then, since BC is common to both triangles,

$$\delta = \sin^{-1}\left(\frac{r}{r'}\sin(\beta+\gamma)\right).$$

To find a lower bound for δ we note: $\sin(\beta+\gamma)$ is monotone increasing in both β and γ when the surfeit of ABC is at least $-6s$; the circumradius of every triangle is bounded above by $1+h$, so $r' < 1+h$; and

$$r = \frac{|AB|}{2\sin\gamma} > \frac{1+h}{2\sin\gamma},$$

so

$$\delta > \sin^{-1}\left(\frac{\sin(\beta+\gamma)}{2\sin\gamma}\right) > \sin^{-1}\left(\frac{\sin(1.25762)}{2\sin(0.628810)}\right) > 0.94199 \approx 53.97°.$$

To establish an upper bound for δ we note that since D and ABC satisfy the conditions of Lemma 3.7.5, $r' \geq r$. So δ is at most $\beta+\gamma$, which, by Proposition 3.7.1 is at most 1.25762 when the surfeit of ABC is at least $-6s$.

Analysis of the right hand side of the inequality

$$|BCD| > h^2 \sin 2\delta + h\sin\delta\sqrt{5.127 - 1.96\sin^2\delta}$$

on the interval $0.94199 < \delta < 1.25762$ shows that $|BCD| > 1.5066$, which establishes the result. □

Proposition 3.7.7. *Let ABC be an $[h,h,h]$ triangle of surfeit greater than $-6s$ having BC as its longest edge. Let $D'BC$ be the triangle sharing this edge with ABC. If the disk centered at D' is of radius h, then the surfeit of $D'BC$ is no more than -0.6816.*

Proof. The argument follows the structure of the proof of Proposition 3.7.6, with the difference that the disk at A is of radius h.

Construct the point D exactly as in Proposition 3.7.6. As before, we have:

$$|BDC| = h\left(2h\cos\delta + \sqrt{|BC|^2 - (2h\sin\delta)^2}\right)\sin\delta;$$

with

$$|BC| = |AB|\frac{\sin(\beta+\gamma)}{\sin\gamma} \geq 2h\frac{\sin(\beta+\gamma)}{\sin\gamma}.$$

Here β and γ are subject to the bounds established in Proposition 3.7.2. As a result,

$$\frac{\sin(\beta+\gamma)}{\sin\gamma} > 1.73436.$$

This means

$$|BCD| > h^2\sin 2\delta + h\sin\delta\sqrt{5.8957 - 1.96\sin^2\delta}.$$

Now, as before, we find bounds on δ in order to analyze the behavior of the right hand side of the inequality above. Following the form of Proposition 3.7.6, we let r and r' denote the circumradii of ABC and DBC respectively and compute:

$$\delta = \sin^{-1}\left(\frac{r}{r'}\sin(\beta+\gamma)\right).$$

Now, since the disk at A is of radius h

$$r = \frac{|AB|}{2\sin\gamma} > \frac{h}{\sin\gamma},$$

so

$$\delta > \sin^{-1}\left(\frac{h}{1+h}\frac{\sin(\beta+\gamma)}{\sin\gamma}\right) > \sin^{-1}\left(\frac{h}{1+h}1.73436\right) > 0.7954 \approx 45.57^\circ.$$

Now to establish an upper bound for δ we note that since D and ABC satisfy the conditions of Lemma 3.7.5, $r' \geq r$. So δ is at most $\beta+\gamma$, which, by Proposition 3.7.2 is at most 1.0426 when the surfeit of ABC is at least $-6s$.

Analysis of the right hand side of the inequality

$$|BCD| > h^2 \sin 2\delta + h \sin\delta\sqrt{5.8957 - 1.96\sin^2\delta}$$

on the interval $0.7954 < \delta < 1.0426$ shows that $|BCD| > 1.5960$. So,

$$\mathrm{sf}(D'BC) = 0.245\pi - \rho_0|D'BC| \leq 0.245\pi - \rho_0|DBC| < -0.6816.$$

$\square$

Proposition 3.7.8. *Let ABC be a $[1,h,h]$ triangle of surfeit greater than $-6s$ having BC as its longest edge. Let $D'BC$ be the triangle sharing this edge with ABC. If the disk centered at D' is of radius 1, then the surfeit of $D'BC$ is no more than -0.9279.*

Proof. Suppose that $D'BC$ is labelled so that the angle at D' is called δ'. Notice now that the sector area of $D'BC$ is dependent on δ'. In particular,

$$\mathrm{sa}(D'BC) = \frac{1}{2}(h^2\pi + (1-h^2)\delta')$$

which is an increasing function of δ'.

We construct D so that $|DBC| \leq |D'BC|$, and the angle δ at D in the triangle DBC satisfies $\delta \geq \delta'$. Doing so will allow us to bound the surfeit of $D'BC$ above by the quantity:

$$\frac{1}{2}(h^2\pi + (1-h^2)\delta) - \rho_0|DBC|.$$

The construction of D is similar to the previous proposition, with the difference that here we consider a circle of radius $1+h$ centered at B.

Notice that since D' is the center of a disk of radius 1, $|D'B| \geq 1+h$. Also, by the Delaunay condition, D' lies on or outside the circumcircle of ABC.

Construct the perpendicular bisector $\mathcal{L}$ of BC. If D' and B are in opposite half-planes with respect to $\mathcal{L}$, let D_0 be the reflection of D' with respect to $\mathcal{L}$; otherwise, let $D_0 := D'$. Now construct a perpendicular from D_0 to $\mathcal{L}$, let D_1 be the projection of D_0 onto $\mathcal{L}$, and consider the line segment D_0D_1 There are three cases:

Case 1: D_0D_1 intersects neither the circumcircle of ABC nor the circle of radius $1+h$ centered at B. In this case, let D be the intersection of the circle of radius $1+h$ centered at B and the circumcircle of ABC which lies in the half-plane of D' relative to BC. Notice that

D is no higher than D' relative to BC, so $|DBC| \leq |D'BC|$. It is not hard to show, using Proposition 3.7.1, that D and B lie on the same side of $\mathcal{L}$.

Case 2: D_0D_1 intersects the circumcircle of ABC, but not the circle of radius $1+h$ centered at B. In this case, again, let D be the intersection of the circle of radius $1+h$ centered at B and the circumcircle of ABC which lies in the half-plane of D' relative to BC. Once again, $|DBC| \leq |D'BC|$.

Case 3: D_0D_1 intersects both the circumcircle of ABC, and the circle of radius $1+h$ centered at B. In this case, let D be that intersection of the perpendicular and the disk of radius $1+h$ centered at B which lies outside the circumcircle of ABC. If two such points exist, pick the one closer to $\mathcal{L}$. In this case $|DBC| = |D'BC|$.

In DBC let δ be the angle at D.

In Case 1 and Case 2, since D' is on or outside the circumcircle of ABC and D is on the circumcircle of ABC and both are in the same half-plane relative to BC, and both δ' and δ are opposite BC, by Lemma 3.7.5 and the Law of Sines, $\delta \geq \delta'$.

In Case 3, to show $\delta \geq \delta'$, first note that both these angles are acute. This is because, by reasoning similar to that of the previous paragraph, both are smaller in measure than an angle δ_1 with its vertex on the circumcircle of ABC, subtending the chord BC and in the half-plane opposite to A. The angle at A is obtuse, so δ_1 is acute. It's enough, therefore, to show that $\sin\delta \geq \sin\delta'$.

By construction, $|D_0BC| = |DBC|$. Now $|D_0C| \geq |DC|$ since C lies in the opposite half-plane to both D_0 and D relative to $\mathcal{L}$. Furthermore, $|DB| = 1 + h$, also by construction, so $|D_0B| \geq |DB|$. It follows that $\sin\delta \geq \sin\delta'$.

The following properties are now satisfied:

- $|BD| = 1 + h$,
- D lies on or outside the circumcircle of ABC,
- $|DBC| \leq |D'BC|$
- $\delta \geq \delta'$

The rest of the argument runs similarly to the proof of Proposition 3.7.6:

Since $|BD| = 1 + h$, the area of DBC is determined by the angle δ as shown below:

$$\begin{aligned}
|BDC| &= \frac{1}{2}|DB||DC|\sin\delta \\
&= \frac{1}{2}(1+h)\left((1+h)\cos\delta + |BC|\cos\theta\right)\sin\delta \\
&= \frac{1}{2}(1+h)\left((1+h)\cos\delta + |BC|\cos\left(\sin^{-1}\left(\frac{1+h}{|BC|}\sin\delta\right)\right)\right)\sin\delta \\
&= \frac{1}{2}(1+h)\left((1+h)\cos\delta + |BC|\sqrt{1-\left(\frac{1+h}{|BC|}\sin\delta\right)^2}\right)\sin\delta \\
&= \frac{1}{2}(1+h)\left((1+h)\cos\delta + \sqrt{|BC|^2-((1+h)\sin\delta)^2}\right)\sin\delta.
\end{aligned}$$

Now,

$$|BC| = |AB|\frac{\sin(\beta+\gamma)}{\sin\gamma} \geq (1+h)\frac{\sin(\beta+\gamma)}{\sin\gamma}.$$

Now, β and γ are subject to the bounds of Proposition 3.7.1, since the surfeit of ABC is at least $-6s$, so:

$$\frac{\sin(\beta+\gamma)}{\sin\gamma} > 1.617,$$

and

$$|BCD| > \frac{1}{2}(1+h)^2\left(\cos\delta + \sqrt{1.617^2 - \sin^2\delta}\right)\sin\delta.$$

In order to find a lower bound for the expression on the right hand side of the inequality above, we seek upper and lower bounds on δ.

As in Proposition 3.7.6, let r and r' denote the circumradii of ABC and DBC respectively. Then, since BC is common to both triangles,

$$\delta = \sin^{-1}\left(\frac{r}{r'}\sin(\beta+\gamma)\right).$$

To find a lower bound for δ we note: $\sin(\beta+\gamma)$ is monotone increasing in both β and γ when the surfeit of ABC is at least $-6s$; the circumradius of every triangle is bounded above by $1+h$, so $r' < 1+h$; and

$$r = \frac{|AB|}{2\sin\gamma} > \frac{1+h}{2\sin\gamma},$$

so

$$\delta > \sin^{-1}\left(\frac{\sin(\beta+\gamma)}{2\sin\gamma}\right) > \sin^{-1}\left(\frac{1.617}{2}\right) > 0.94160 \approx 53.95^\circ.$$

To establish an upper bound for δ we note that since D and ABC satisfy the conditions of Lemma 3.7.5, $r' \geq r$. So δ is at most $\beta + \gamma$, which, by Proposition 3.7.1 is at most 1.25762 when the surfeit of ABC is at least $-6s$.

Analysis of the right hand side of the inequality:

$$|BCD| > \frac{1}{2}(1+h)^2 \left(\cos\delta + \sqrt{1.617^2 - \sin^2\delta}\right)\sin\delta$$

on the interval $0.94199 < \delta < 1.25762$ shows that $|BCD| > 2.219$. Since δ is bounded above by 1.25762, and since the expression:

$$\frac{1}{2}(h^2\pi + (1-h^2)\delta) - \rho_0|DBC|$$

gives an upper bound on the surfeit of $D'BC$, the result follows. $\square$

Proposition 3.7.9. *Let ABC be an $[h,h,h]$ triangle of surfeit greater than $-6s$ having BC as its longest edge. Let $D'BC$ be the triangle sharing this edge with ABC. If the disk centered at D' is of radius 1, then the surfeit of $D'BC$ is no more than -0.7820.*

Proof. The argument is similar in form to the proof of Proposition 3.7.8. The only difference is that the disk centered at A has radius h, so $|AB|$ is bounded below by $2h$ instead of $1+h$. Also, bounds on β and γ are obtained from Proposition 3.7.2 rather than Proposition 3.7.1.

$\square$

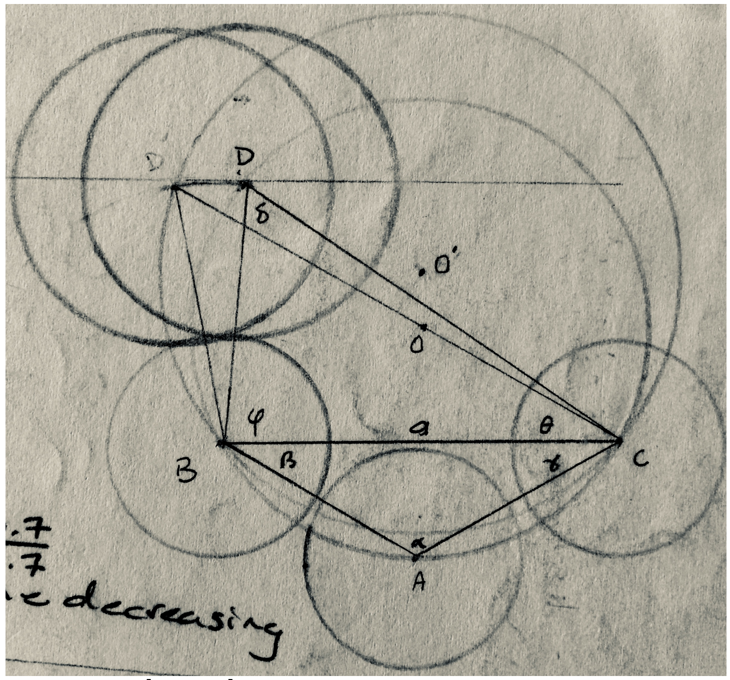

Figure 8: A heavy $[h, h, h]$ triangle paired with a triangle with a 1 vertex.
Two possible constructions of D are depicted in this representation.

3.8 Proof of Claim 3.2.15

Proposition 3.8.1. *No more than six heavy triangles of type* $[h, h, 1]$ *may appear in any star, and no star may consist only of such triangles.*

Proof. In Proposition 3.5.1 it was established that $f_{[h,h,1]}$ attains its maximum value on the boundary component $\beta = \pi/2 - \gamma/2$ when $\gamma = 2\sin^{-1}(\frac{h}{1+h}) \approx 0.848779$.

Numeric analysis shows that $f_{[h,h,1]}$ is strictly less than $-6s$ everywhere on its domain except in the region bounded by $\gamma = 0.848769$, $\beta = \pi/2 - \gamma/2$, $\gamma = 0.848803$ and $\beta = 1.14639$, as shown in the figure below.

As a result, we may conclude that the angles of a heavy triangle of type $[h, h, 1]$ are subject to the following bounds:

- $0.848769 < \gamma < 0.848803$
- $1.146394 < \beta < 1.146411$
- $1.146378 < \alpha < 1.146429$.

Now a triangle of type $[h, h, 1]$ belonging to a star with a 1 disk at the central vertex (a $1-$star) must have its γ angle at the center of the star. In this case, the result follows from noting that $\pi - 7 * 0.848769 < \sin^{-1}(h/(1+h))$, and that no permissible triangle may have an angle less than $\sin^{-1}(h/(1+h))$.

If a triangle of type $[h, h, 1]$ belongs to a star with an h disk at the central vertex, then it must have its β or α angle at the center of the star. It is easy to see that no combination of values in the ranges established above for α and β add to 2π, and that no more than two such triangles may belong to an $h-$star. $\square$

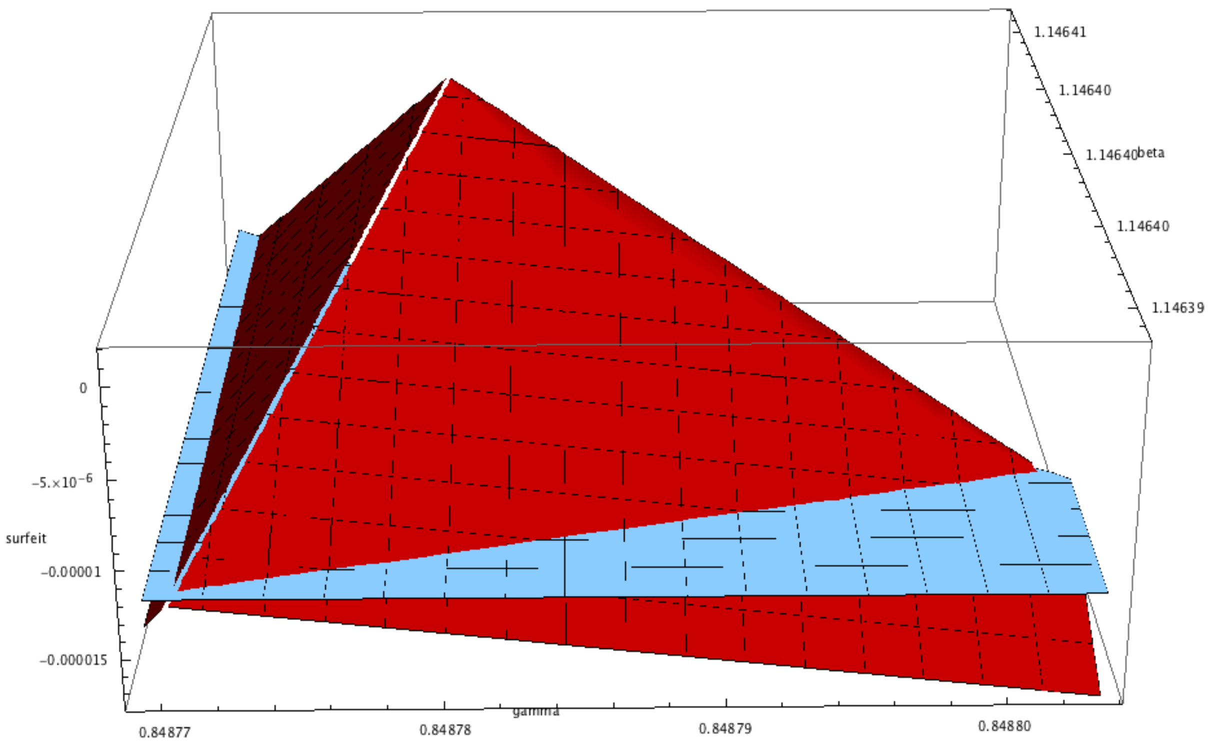

Figure 9: The permissible region for heavy $[h, h, 1]$ triangles.
The portion of the graph of $f_{[h,h,1]}$ depicted here is restricted to the domain established in the proof of Proposition 3.8.1. The horizontal plane is at height $-6s$.

CHAPTER 4

FUTURE DIRECTION

A reasonable goal in extending the techniques of this thesis would be to establish $\frac{\pi}{\sqrt{12}}$ as an upper bound for the density of two-radius packings for a single value of h outside the interval $[0.742\ldots, 1]$ (concretely, I consider $h = 0.7$), and to then show that the reasoning applies to a neighborhood of the point. This chapter contains conjectures, partial results, and a description of techniques used to this end. Additional work is needed on positive surfeit stars of valence 7 in order to establish the result.

An eventual goal of the research program is to extend the interval $[0.742\ldots, 1]$ to its leftmost limit. Further extensions to the research program could include establishing sharp bounds for the density of two-radius planar packings on the entire interval of homogeneity, extending the work to packings which comprise disks of arbitrary radius subject to a given homogeneity bound, establishing density bounds in two-radius packings where the relative number of disks of each radius is constrained, and extending the work to higher dimensions and spaces of nonzero curvature.

4.1 Dense Stars

We classify a star by the radius of the disk centered at and the valence of its central vertex; for instance, we may refer to an h star of valence 5. Since every triangle belongs to three distinct stars, in order to show that the surfeit of the triangulation is non-positive, it is enough to show that the upper sum of the surfeits of all the stars in the triangulation is non-positive.

For a fixed homogeneity we define the quantity σ_n to mean the maximum possible surfeit of a star of valence n. For a two-species packing of homogeneity 0.7, combinatorial considerations in conjunction with analysis of the sort mentioned in the previous chapters lead us to conclude that the only stars in the triangulation with positive or near positive surfeit have either valence 5 or 7 and belong to only a handful of combinatorial types. In fact, we show that $\sigma_6 = 0$, $\sigma_5 > 0 > \sigma_n$ for $n < 5$, and $\sigma_7 > \sigma_n$ for $n > 7$. Thus, since the average valence of a star in the triangulation is 6, in order to show that the surfeit of the triangulation is non-positive, it is enough to show that $\sigma_5 + \sigma_7 \leq 0$. The bulk of the remaining work consists of finding upper bounds for σ_5 and σ_7.

For a two-species packing of homogeneity 0.7, the 5-star in Figure 10 carries a positive surfeit which is approximately, but not less than, 0.01056, so this quantity serves as a lower bound on σ_5. Similarly, the 7-star in Figure 11 witnesses a lower bound of $-0.0183\ldots$ for σ_7.

Consider now the process for determining σ_5 in a packing of homogeneity 0.7:

Numerical analysis of the surfeit bounding function f in the neighborhood of local maxima in the $\beta\gamma$-plane, pair-analysis for thin triangles, and computer-assisted matching of candidate triangles allow us to conclude that the upper bound for σ_5 is realized by a star comprising the same triangle types as the ones shown in Figure 10.

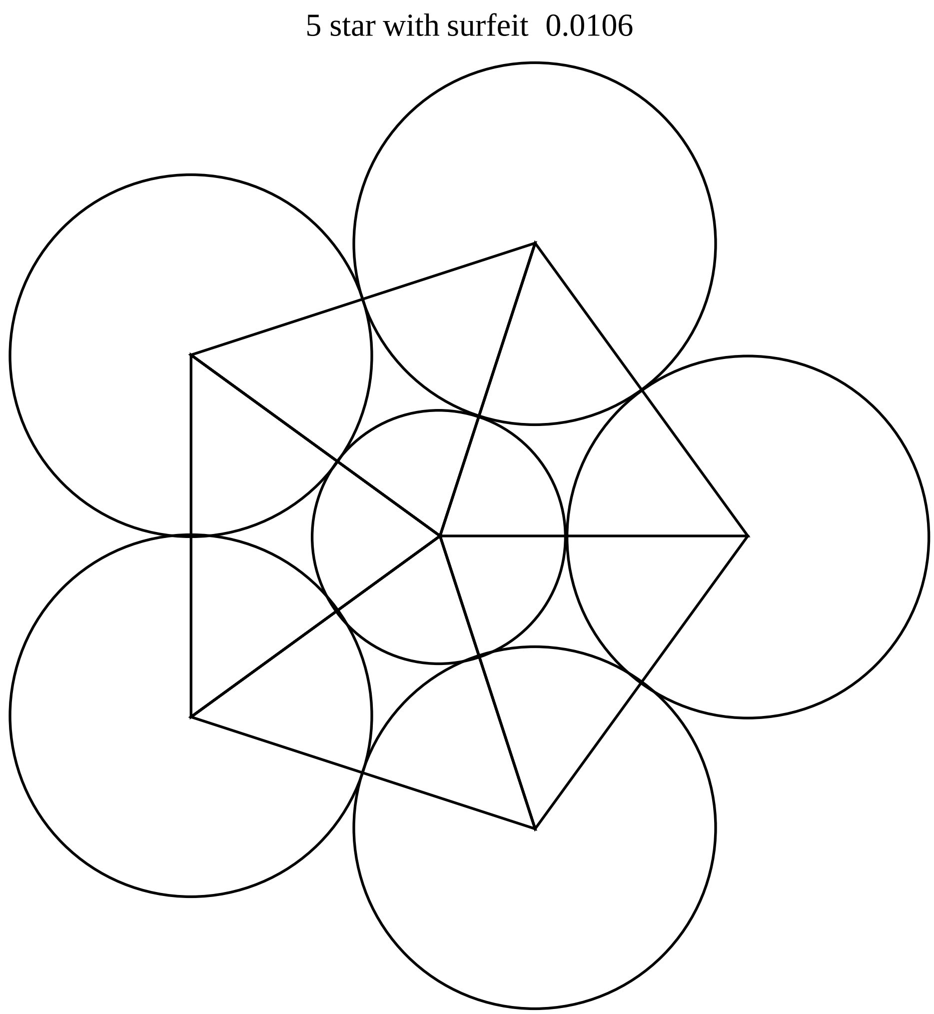

Figure 10: A dense 0.7-star of valence 5.

The central disk appears to be tangent to the five large disks dirrounding it, but, in fact, is not. A simple calculation shows that the radius of a disk inscribing a regular pentagon of side length 2 is approximately 1.701, slightly more than the sum of the radii at the endpoints of a radial edge.

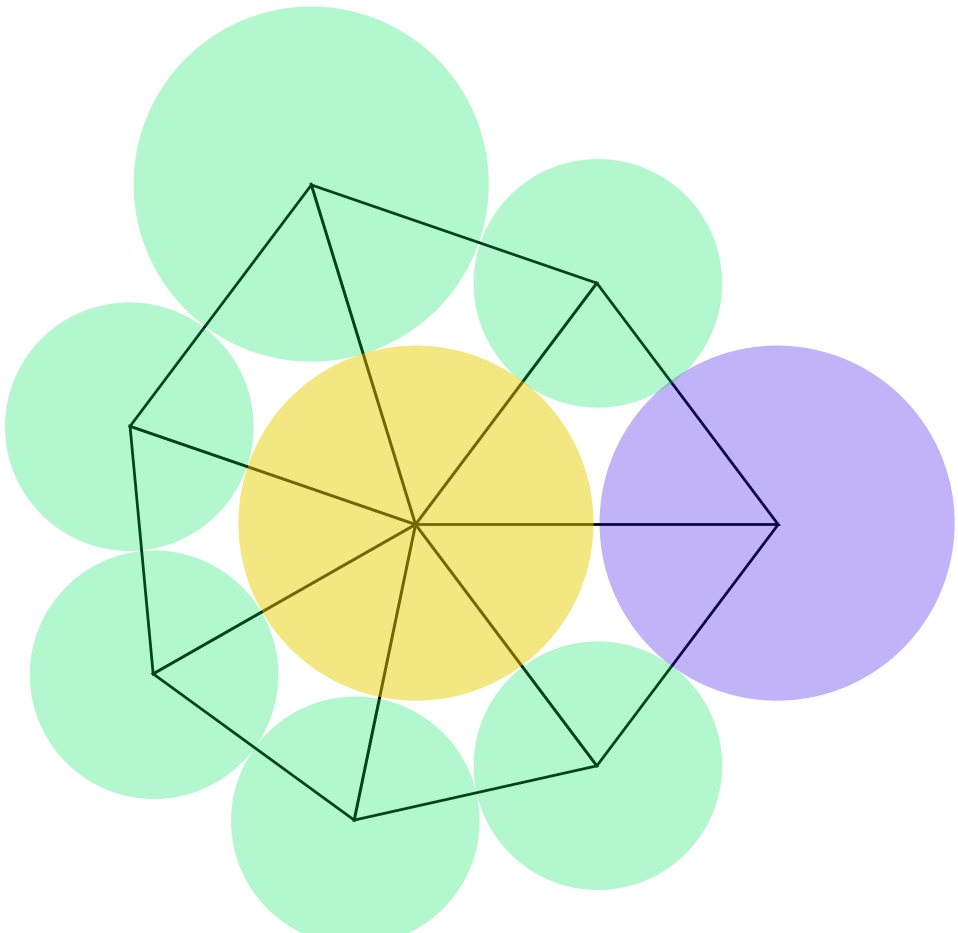

Figure 11: A heavy 1-star of valence 7

This 1-star of valence 7 with 13 tangencies has an approximate surfeit of -0.01831. It is the densest one known. The large disk on the right is tangent to its cyclic neighbors, but not to the central disk.

We represent the similarity type of a 5-star by a point in $\mathbb{R}^{10}$ (the coordinates represent the interior angles of the triangulated polygon) and consider a perturbation of the star within a neighborhood defined by appropriate bounds on the angles of the constituent triangles.

The following function of a vector $\vec{\theta}$ in $\mathbb{R}^{10}$ gives an upper bound on the surfeit of a 5-star with a central disk of radius 0.7 and radial disks of radius 1:

$$s(\vec{\theta}) := 0.49\pi + \frac{1}{2}\sum_{i=0}^{4}\left[\theta_{2i} + \theta_{2i+1} - \frac{2\pi}{\sqrt{3}}\left(\frac{\sin\theta_{2i+1}\sin\theta_{2i+2}}{\sin(\theta_{2i+1}+\theta_{2i+2})}\right)\right]$$

where θ_i is an interior angle of the triangulated pentagon and the indices are taken modulo 10.

Let $\mathbb{1}_{10} := (1, 1, \ldots, 1) \in \mathbb{R}^{10}$. Then $\vec{\theta^0} := \frac{3\pi}{10}\mathbb{1}_{10}$ represents the interior angles of a triangulated regular 5-star.

A deformation of the star is represented by $\vec{\theta^0} + \vec{x}$ for some appropriately chosen vector $\vec{x}$. We are interested in a perturbation of the star which results in an increase in surfeit from that of the regular 5-star, so we may restrict $\vec{x}$ to a ball of some appropriately chosen radius r about the origin. The value for r will be determined from constraints on the geometry of the constituent triangles of the star.

By the Mean Value Theorem, for every $x \in B_r(0)$ there is a constant $c_x \in (0, 1)$ such that

$$s(\vec{\theta^0} + \vec{x}) - s(\vec{\theta^0}) \quad = \quad < \nabla s(\vec{\theta^0} + c_x\vec{x}),\ \vec{x} >,$$

and so

$$s(\vec{\theta^0} + \vec{x}) \quad \leq \quad s(\vec{\theta^0}) + \left|< \nabla s(\vec{\theta^0} + c_x\vec{x}),\ \vec{x} >\right|.$$

Each coordinate of $\nabla s(\vec{\theta})$ is of the form $k_1 + k_2 \frac{sin^2\theta_j}{sin^2(\theta_i+\theta_j)}$ for fixed constants k_1 and k_2. This means that $\nabla s(\vec{\theta^0})$ has all coordinates equal. Furthermore, since a vector $\vec{\theta} := \vec{\theta^0} + \vec{x}$ represents the similarity type of a triangulated 5-gon, we have $\sum_{i=0}^{9} \theta_i = 3\pi$ and $\sum_{i=0}^{9} x_i = 0$. This means that the perturbation vector $\vec{x}$ is orthogonal to $\nabla s(\vec{\theta^0})$ and so $< \nabla s(\vec{\theta^0}), \vec{x} > = 0$. So we may write:

$$s(\vec{\theta^0} + \vec{x}) \quad \leq \quad s(\vec{\theta^0}) + \left| < \nabla s(\vec{\theta^0} + c_x\vec{x}) - \nabla s(\vec{\theta^0}),\ \vec{x} > \right|;$$

furthermore, by the Cauchy-Schwarz inequality,

$$\left| < \nabla s(\vec{\theta^0} + c_x\vec{x}) - \nabla s(\vec{\theta^0}),\ \vec{x} > \right| \quad \leq \quad |x| \left| \nabla s(\vec{\theta^0} + c_x\vec{x}) - \nabla s(\vec{\theta^0}) \right|,$$

so, since $s(\vec{\theta^0}) = 0.01056\ldots$ and $x \in B_r(0)$, we may conclude:

$$s(\vec{\theta^0} + \vec{x}) \quad < \quad 0.0106 + r \left| \nabla s(\vec{\theta^0} + c_x\vec{x}) - \nabla s(\vec{\theta^0}) \right|$$

for some $c_x \in (0, 1)$.

To choose r, we note that the surfeit of a triangle of type $[0.7, 1, 1]$ is no more than 0.00329. Thus, if any triangle in the perturbed 5-star has a surfeit less than -0.0026, the surfeit of the star will be less than 0.01056, which is a lower bound for σ_5. So we need only consider triangles with surfeit greater than -0.0026. Examination of the bounding function f allows us to conclude that the angles θ_i in the perturbation are subject to the bounds $\frac{3\pi}{10} - 0.0076 < \theta_i < \frac{3\pi}{10} + 0.0076$, so it is sufficient to choose $r = 0.0241$.

Every coordinate of $\nabla s(\vec{\theta^0} + c_x\vec{x}) - \nabla s(\vec{\theta^0})$ is of the form:

$$\frac{2\pi}{\sqrt{3}}\left[\left(\frac{\sin\left(\frac{3\pi}{10}\right)}{\sin\left(\frac{3\pi}{5}\right)}\right)^2 - \left(\frac{\sin\left(\frac{3\pi}{10} + c_x x_j\right)}{\sin\left(\frac{3\pi}{5} + c_x(x_i + x_j)\right)}\right)^2\right],$$

so, after some analysis, we are able to conclude that

$$r\left|\nabla s(\vec{\theta^0} + c_x\vec{x}) - \nabla s(\vec{\theta^0})\right| < 0.0043,$$

and we obtain an upper bound on σ_5 of 0.0149. This is sharper than the upper bound of 0.01645 which is obtained by multiplying the maximum surfeit of a $[0.7, 1, 1]$ triangle by 5. (We conjecture that this upper bound can be improved, and that in fact the surfeit of the regular 5-star from the previous paragraphs is maximal.)

The main challenge that remains is to find an upper bound for σ_7 that is at most $-\sigma_5$, since showing that $\sigma_5 + \sigma_7 \leq 0$ will prove that every two-radius packing of homogeneity 0.7 has density at most $\frac{\pi}{\sqrt{12}}$.

The method described above for bounding σ_5 benefits strongly from the symmetry present in the regular 5-star. In the case of σ_7, computer-aided combinatorial analysis of allowable triangle types yields three combinatorial types of candidate star, none of which possess the strong symmetry of a regular 5-star, and so computations become significantly more complicated. For instance, it is not possible to appeal to the orthogonality of the perturbation vector, so in order to use the methods employed in bounding σ_5, cumbersome modifications and correction terms must be employed.

I have developed some numerical analysis and visualization software to inform and facilitate the computations necessary for determining σ_7. This software makes it possible to see the effects of simple perturbations in real-time and to plot more complex perturbations. These visualizations have so far led to computational efficiencies by dimension reduction through imposing tangency constraints, and point to the likelihood that convexity may be used to simplify computations further. See Figure 12 for an example of a perturbation slider-toy, and Figure 13 for a plot of surfeit for a 7-star undergoing a two-dimensional perturbation.

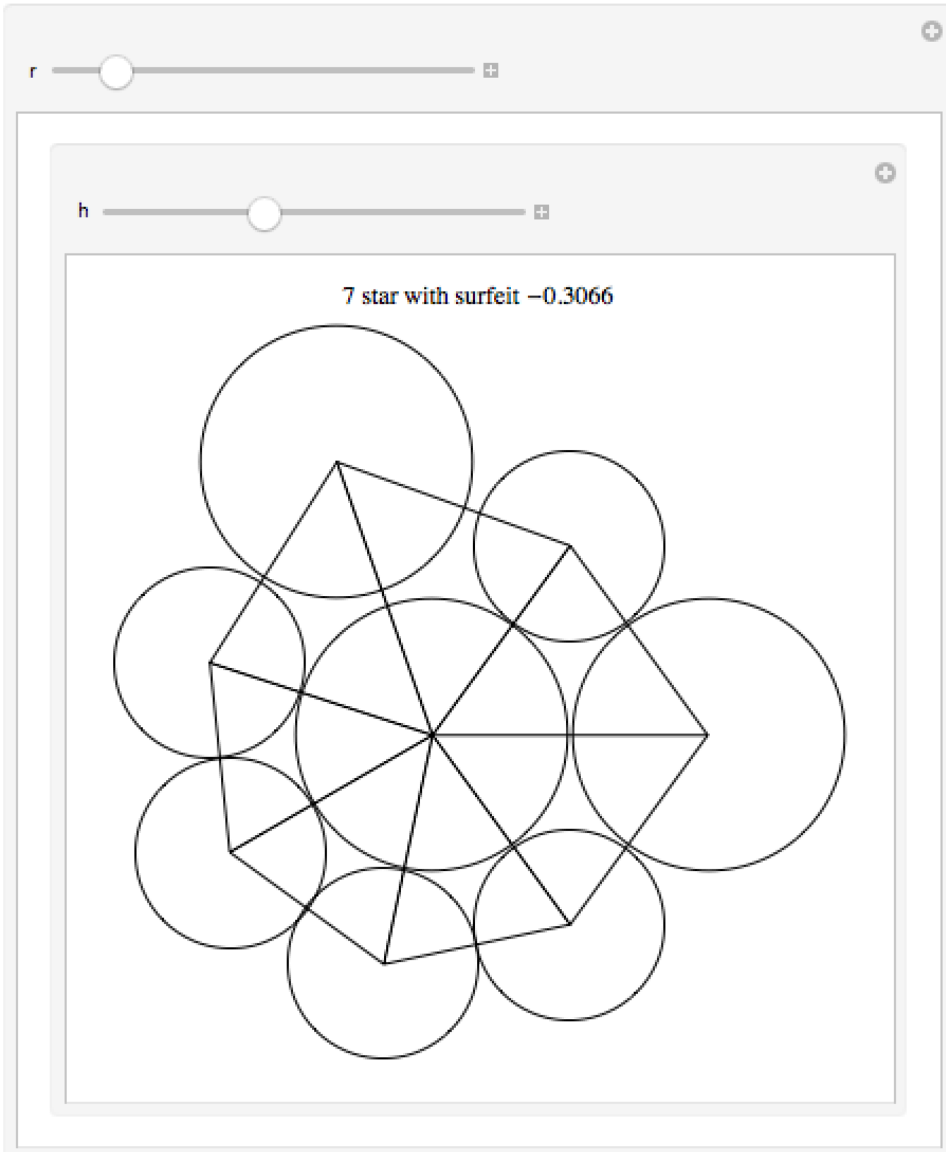

Figure 12: Perturbation slider-toy.

Manual perturbation is possible using the slider tools. The slider marked "r" adjusts the radial distance of the large disk at top-left from the origin, which coincides with the position of the central disk's center when both sliders are in their leftmost position. The slider marked "h" perturbs the top left disk by a rotation of its center relative to its starting position. The toy maintains as many tangencies as possible under the perturbation, shifting the positions of the other disks where necessary. Surfeit is calculated in real-time.

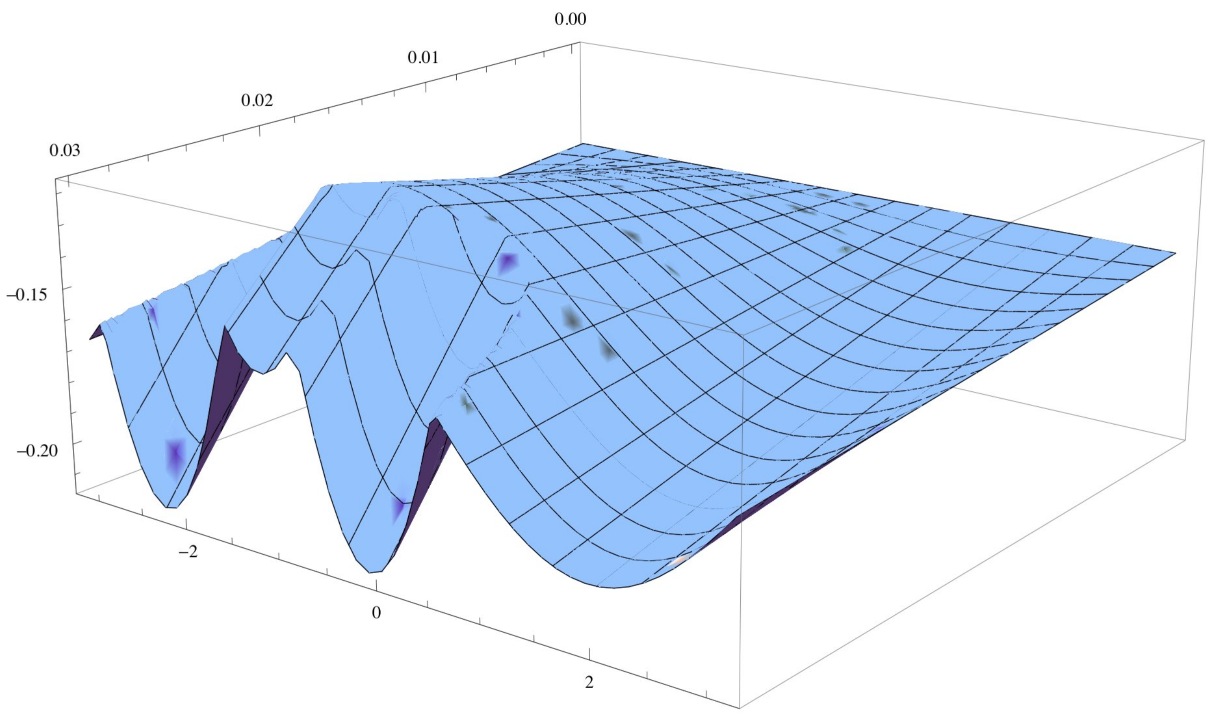

Figure 13: Surfeit plot for a 7-star under a perturbation.
This figure shows a plot of surfeit for a 7-star under perturbation. The vertical height of the surface is the surfeit of the 7-star. The domain is a polar neighborhood of the configuration space of one of the outer disks of the 7-star as described in the caption of Figure 12.

CITED LITERATURE

1. Fejes Toth, L.: Lagerungen in der Ebene, auf der Kugel und im Raum. Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Band LXV. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1953.

2. Florian, A.: Dichteste Packung inkongruenter Kreise. Monatsh. Math., 67:229–242, 1963.

3. Blind, G.: Uber unterdeckungen der ebene durch kreise. J. Reine Angew. Math., 236:145–173, 1969.

4. Fejes Toth, G.: Covering the plane by convex discs. Acta Mathematica Academiae Scientiarum Hungaricae, 23:263–270, 1972.

5. Florian, A.: Ausfullung der ebene durch kreise. Rend. Circ. Mat. Palermo (2), 9:300–312, 1960.

6. Heppes, A.: On the densest packing of discs of radius 1 and $\sqrt{2}-1$. Studia Sci. Math. Hungar., 36(3-4):433–454, 2000.

7. Heppes, A.: Some densest two-size disc packings in the plane. Discrete Comput. Geom., 30(2):241–262, 2003. U.S.-Hungarian Workshops on Discrete Geometry and Convexity (Budapest, 1999/Auburn, AL, 2000).

8. Kennedy, T.: A densest compact planar packing with two sizes of discs. ArXiv Mathematics e-prints, December 2004.

9. Kennedy, T.: Compact packings of the plane with two sizes of discs. Discrete Comput. Geom., 35(2):255–267, 2006.

10. Lagrange, J.: Recherches d'arithmetique. Nouveaux Memoires de L'Academie royal des Sciences et Belles-Lettres de Berlin, pages 265–312, 1773.

11. Zong, C.: Sphere Packings. Universitext. Springer-Verlag, New York, 1999.

12. Thue, A.: Uber die dichteste zusammenstellung von kongruenten kreisen in einer ebene. Christiania Vid. Selsk. Skr., 1:3–9, 1910.

13. Fejes Toth, L.: Lagerungen in der Ebene auf der Kugel und im Raum. Springer-Verlag, Berlin-New York, 1972. Zweite verbesserte und erweiterte Auflage, Die Grundlehren der mathematischen Wissenschaften, Band 65.

14. Blind, G. and Blind, R.: Packings of unequal circles in a convex set. Discrete Comput. Geom., 28(1):115–119, 2002.

15. Kennedy, T.: `http://math.arizona.edu/~tgk/pack_two_discs`.

16. Hsiang, W.-Y.: A simple proof of a theorem of Thue on the maximal density of circle packings in E^2. Enseign. Math. (2), 38(1-2):125–131, 1992.

17. Hales, T. and Ferguson, S.: The Kepler Conjecture. Springer, New York, 2011.

18. Chang, H.-C. and Wang, L.-C.: A Simple Proof of Thue's Theorem on Circle Packing. ArXiv e-prints, September 2010.

19. Fejes Toth, L.: Regular figures. A Pergamon Press Book. The Macmillan Co., New York, 1964.

20. Gruber, P. M.: Convex and Discrete Geometry, volume 336 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Berlin, 2007.

21. Delone, B. N.: Sur la sphere vide. Proc. Internat. Congr. Math. (Toronto 1924), Univ. Toronto Press, (1):695–700, 1928.

22. Delone, B. N. and Ryvskov, S. S.: Extremal problems of the theory of positive quadratic forms. Trudy Mat. Inst. Steklov., 112:203–223, 387, 1971. Collection of articles dedicated to Academician Ivan Matveeviv c Vinogradov on his eightieth birthday, I.

VITA

Ali Mohajer

Curriculum Vitae

1835 Soniat Street
New Orleans, LA 70115
+1 (708) 446 5337
ali.mohajer@gmail.com

Education

2018 (anticipated) **PhD in Pure Mathematics**, *University of Illinois at Chicago*, Chicago, IL.
Thesis advisor: Peter B. Shalen

2012 **MS in Pure Mathematics**, *University of Illinois at Chicago*, Chicago, IL.

1995 **BS, BA**, *Valparaiso University*, Valparaiso, IN.
BS double major: Chemistry; Individualized (with a concentration in Physics)
BA major/minor: English/English Writing

Experience

Teaching

2018 - present **Adjunct Professor**, *Tulane University Department of Mathematics*, New Orleans, LA.
- Fall 2018 – MATH-1230: Statistics for Scientists (40 students)

2018 - present **Instructor**, *Loyola University New Orleans Department of Mathematics and Computer Science*, New Orleans, LA.
- Fall 2018 – MATH-A118: Precalculus Mathematics (88 students)

2018 - present **Visiting Scholar**, *University of Illinois at Chicago*, Chicago, IL.
- Fall 2018 – Math 496: Independent Study (MCL Project - Two-radius Planar Disk Packing) (1 student)

2017 - 2018 **Lecturer**, *University of Illinois at Chicago*, Chicago, IL.
- Fall 2017 – MATH 310: Applied Linear Algebra (100 students); MATH 121: Precalculus Mathematics (260 students).
- Spring 2018 – MATH 310: Applied Linear Algebra (80 students); MATH 220: Introduction to Differential Equations (200 students); MATH 496: Independent Study - Efficiency of Planar Disk Packings (3 students).

2014 - 2018 **Instructor, Number Theory**, *Johns Hopkins University Center for Talented Youth*, Lancaster, PA.

2015 - 2017 **Instructor**, *Math Circles of Chicago, Pilsen grades 10-12*, Chicago, IL.

2011 - 2015 **Instructor – Emerging Scholars Program**, *University of Illinois at Chicago*, Chicago, IL.
ESP workshops at UIC are collaborative learning environments for students seeking additional challenge. Students work in small groups on advanced materials associated with standard course offerings.
- Fall 2013 through Spring 2015 – MATH 294: Special Topics (Intro to Advanced Math);
- Fall 2012 and Spring 2013 – MATH 211: ESP Workshop for Calculus III;
- Spring 2012 – MATH 179: ESP Workshop for Calculus I;
- Fall 2011 – MATH 181: ESP Workshop for Calculus II.

2011 - 2013 **Instructor (Summer Programs)**, *University of Illinois at Chicago*, Chicago, IL.
- Summer 2013 – MATH 090: Intermediate Algebra (Summer College);
- Summer 2011/12 – MATH 090: Intermediate Algebra (Summer Enrichment Workshop).

2010, '11, '16, '17 **Teaching Assistant**, *University of Illinois at Chicago*, Chicago, IL.
- Spring 2017 – MATH 121: Precalculus
- Spring 2016 – MCS 275: Programming Tools and File Management
- Fall 2011 – MATH 181: Calculus II (Discussion);
- Spring 2011 – MATH 210: Calculus III (Lab);
- Fall 2010 – MATH 165: Calculus for Business (Discussion).

Industry

1992–2009 **Operations Support / Technology Consultant / IT Manager**.
Technical and management roles, including:
- Technical Operations Manager (consultant) – Chicago Transit Authority (2005);
- Senior IT Manager, Networks and Infrastructure – Kana Software (2002-2003);
- Program Manager, Windows Printing and Imaging – Microsoft Corporation (1999-2000);
- Technical Support – Valparaiso University (1992-1995).

Awards

- UIC MSCS Graduate Teaching Award – Spring 2014

Service

- Co-President, UIC Mathematics Graduate Students Association, 2013/14
- Organizer of the MSCS Graduate Student Colloquium (2015-16).
- President, 3150 N Lake Shore Drive Condominium Association. (Board Member since 2011; Treasurer 2012 – 2017.)

Languages

English	**native speaker**	*Minor in English Writing at Valparaiso University*
Farsi	**native speaker**	*College level reading and writing skills.*
Arabic	**reading proficient; basic communication skills**	*Modern Standard Arabic*

Computer skills

math software	Mathematica, Maple	publishing	LaTeX
programming	C++, Fortran, Python	office	Excel

Conferences and seminars

University of Michigan, Ann Arbor, Invited Speaker at Geometry Seminar, April 2017.

Temple University, Philadelphia, Speaker at Graduate Student Conference in Algebra, Geometry, and Topology, June, 2018

Research interest

Density bounds on infinite packings of unequal spheres.