Cohomological Insights for Complex Surface Automorphisms with Positive Entropy

BY

PAUL RESCHKE
B.A., Amherst College, 2004
M.S., University of Illinois at Chicago, 2010

THESIS

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Chicago, 2013

Chicago, Illinois

Defense Committee:

Laura DeMarco, Chair and Advisor
Izzet Coskun
Majid Hadian-Jazi
Sarah Koch, Harvard University
Ramin Takloo-Bighash

To Jenna, who does not waver.

ACKNOWLEDGMENTS

I thank Laura DeMarco, my advisor, for the many hours she has spent enthusiastically discussing my work (and much other math as well) with me. She is, unfailingly, a patient, thoughtful, and dedicated mentor. I especially thank Laura for encouraging me to seek out my own avenues of research and to develop a personal research agenda for both my graduate studies and the future.

I thank Izzet Coskun for answering my many (and often repetitive) questions with precision and thoroughness. I thank Curt McMullen for his eagerness to view my work and to offer insightful questions for further study. I thank Serge Cantat, Igor Dolgachev, Chong Gyu Lee, and Christian Schnell for helpful discussions about the connections and applications of my work to a variety of related mathematical topics.

I thank Mattias Jonsson, Roland Roeder, and Dan Thompson for giving me valuable opportunities to speak about my work.

I thank Holly Krieger for challenging me, inspiring me, and being a great friend. I thank Karen Zaya, Natalie McGathey, and Richard Abdelkerim for being exemplary officemates. I thank Dimitris Diochnos, Emily Cilli-Turner, Andy Brasile, and Jim Freitag for making time to enjoy Chicago with me. I thank Jonah Gaster, Cesar Lozano Huerta, Luigi Lombardi, Hexi Ye, Matt Wechter, Nick Gardner, and Michael Siler for camaraderie and illuminating conversations.

I thank my parents for providing assurance and optimism. I thank Jenna for uncountably many kindnesses. And I thank Aivry for taking a few naps from time to time.

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SUMMARY

We investigate automorphisms of compact Kähler surfaces, focusing on automorphisms with positive entropy. Such maps lend themselves to applications of tools from both complex analysis and complex algebraic geometry. Indeed, Cantat [(1);(2);(3)] used techniques from both arenas to develop some of the key initial results in this area of research–namely, that any connected compact Kähler surface admitting a positive-entropy automorphism is necessarily bimeromorphic to a torus, a K3 surface, an Enriques surface, or the projective plane, and that any such automorphism necessarily has a unique measure of maximal entropy. Here, we focus on using tools from Hodge theory and algebraic geometry to more fully explore the interactions between the dynamical properties of positive-entropy automorphisms and their cohomological behaviors.

The first connection between cohomological actions and dynamics of surface automorphisms is the result due collectively to Gromov (4), Yomdin (5), and Friedland (6) that the entropy of an automorphism σ of a connected compact Kähler surface X is equal to the logarithm of the spectral radius of σ^* on $H^{1,1}(X)$. We develop a refined cohomological interpretation of entropy that precisely describes the action on line bundles induced by any positive-entropy automorphism of a complex projective surface. This interpretation leads to a cohomological characterization of positive-entropy automorphisms with no periodic curves and ultimately to a distinguished means of constructing the measures of maximal entropy for a large class of positive-entropy automorphisms.

SUMMARY (Continued)

We also develop a variety of examples of compact Kähler surface automorphisms–specifically, torus and Kummer surface automorphisms–with positive entropy. The examples we construct are all synthetic, meaning that we infer the existence of each surface automorphism from the existence of an isometry of a corresponding cohomological structure. McMullen [(7);(8);(9);(10)] introduced synthetic constructions of automorphisms in part to begin a classification of the possible values that can arise as entropies of surface automorphisms. Here, we completely characterize the possible values of entropy for two-dimensional complex torus automorphisms. Furthermore, we use our refined cohomological interpretation of entropy to further differentiate the values in terms of those that occur on projective tori and those that occur on non-projective tori.

CHAPTER 1

INTRODUCTION

For a continuous self-map f of a compact Hausdorff space X, we define the topological entropy of f in the manner of Adler, Kronheim, and McAndrew (11): for any finite open cover $\mathcal{U}$ of X, set

$$h(\mathcal{U}, f) = \lim_{n\to\infty} \frac{1}{n} \log H(f^{-1}\mathcal{U} \vee \cdots \vee f^{-n}\mathcal{U}),$$

where $H(f^{-1}\mathcal{U} \vee \cdots \vee f^{-n}\mathcal{U})$ is the minimum number of open sets needed to cover X from the minimal common refinement of the covers $f^{-j}\mathcal{U}$; then the topological entropy of f is

$$h(f) = \sup_{\mathcal{U}} h(\mathcal{U}, f),$$

where the supremum is taken over all finite open covers of X. When X is a compact Kähler manifold and f is holomorphic, results due to Gromov (4), Yomdin (5), and Friedland (6) show that

$$h(f) = \max_{0\leq j\leq \dim(X)} \log \rho(f^* : H^{j,j}(X) \to H^{j,j}(X)),$$

where ρ denotes the spectral radius. (See also (12), §2.)

Since we will focus on entropies of (holomorphic) automorphisms of compact Kähler surfaces, it would suffice for us to take entropy to be defined only in terms of cohomological actions. However, we will make occasional use of the original definition. Also, the original definition suggests a dynamical interpretation of the meaning of entropy: the entropy of a map is some measure of how complicated the orbits of subsets become under iteration of the map.

A connected compact Kähler curve (i.e., a Riemann surface) X cannot admit an automorphism with positive entropy; this result follows from the observations that $H^{0,0}(X)$ and $H^{1,1}(X)$ are one-dimensional and that any automorphism must induce an invertible transformation of $H^{0,0}(X)$ and $H^{1,1}(X)$. In fact, every automorphism in genus zero (i.e., where $X = \mathbb{P}^1$) is a Möbius transformation, every automorphism in genus one (i.e., where X is an elliptic curve) has some iterate which is a translation, and, by Hurwitz's theorem, every automorphism in genus greater than one (i.e., where X is hyperbolic) has some iterate which is the identity map. (See (13) for details.)

A connected compact Kähler surface X may admit automorphisms with positive entropy. In this setting, the entropy of an automorphism is given by the spectral radius of its action on $H^{1,1}(X)$ (which may have arbitrarily large dimension); thus the intersection theory for compact Kähler surfaces plays an essential role in the study of positive-entropy surface automorphisms. A crucial first result is that the entropy of any such automorphism must be the logarithm of a Salem number (i.e., a real algebraic integer greater than one whose minimal polynomial is reciprocal and has only one root with magnitude greater than one). (See §2.4 below.)

Explicit examples show that positive-entropy surface automorphisms can exhibit complicated dynamical phenomena such as Siegel disks, infinite sets of saddle periodic points, and non-trivial partitions by Fatou and Julia sets. (See, e.g., (14), (15), (3), (7), and (8).) On the other hand, with the exception of a class of torus automorphisms characterized by Gizatullin

(16), any zero-entropy automorphism of a connected compact Kähler surface has some iterate that is isotopic to the identity map. (See (2) and (3), §2.4.) Thus two important goals for the field of complex surface dynamics are to fully understand the distinguishing characteristics of positive-entropy automorphisms and to complete the catalogue of examples of such automorphisms. In this thesis, we pursue these goals by investigating the cohomological actions induced by surface automorphisms. We primarily use intersection theory and other tools from Hodge theory.

The following result by Cantat (1) shows that examples of positive-entropy automorphisms can be found only on four types of surfaces: if σ is a dynamically minimal positive-entropy automorphism of a connected compact Kähler surface X, then X can only be a complex torus, a K3 surface, an Enriques surface, or a rational surface. A surface automorphism is dynamically minimal if no exceptional curve (of the first kind–i.e., whose contraction yields again a smooth surface) on the surface is periodic for the automorphism. If a surface automorphism has a periodic curve that is exceptional, then the automorphism descends to an automorphism of the surface obtained by contraction of the orbit of the exceptional curve; conversely, if a surface automorphism has a periodic point, then the automorphism ascends to an automorphism of the blow-up of the surface along the orbit of the point (for which the exceptional curves coming from the blow-up are periodic). The entropy of any surface automorphism is equal to the entropy of its dynamically minimal version. (See §2.4 below.) A two-dimensional complex torus is a quotient of $\mathbb{C}^2$ by a rank-four $\mathbb{Z}$-lattice, a K3 surface is a simply connected compact Kähler surface whose canonical bundle is trivial, an Enriques surface is a quotient of a K3 surface by an involution with no fixed points, and a rational surface is a Kähler surface that is bimeromorphic to $\mathbb{P}^2$; K3 surfaces and two-dimensional complex tori can be projective or non-projective, while Enriques surfaces and rational surfaces are always projective. (See (17) for details.) It follows from work by Nagata (18) that any rational surface admitting a positive-entropy automorphism is necessarily a blow-up of $\mathbb{P}^2$ at ten or more points. (See also (3), §10.3.)

A special class of K3 surfaces consists of those that arise from quotients of tori: let X be a two-dimensional complex torus, and let i be the involution on X coming from multiplication by -1 on $\mathbb{C}^2$; then the Kummer surface associated to X is the blow-up of X/i at its sixteen singular points. Every Kummer surface is a K3 surface, and a Kummer surface is projective if and only if it arises from the quotient of a projective torus; moreover, every two-dimensional complex torus automorphism descends without a change in entropy to an automorphism of the Kummer surface associated to the torus. (See (7), §4, and §5 below.) Thus a result concerning two-dimensional complex torus automorphisms will typically also apply to Kummer surface automorphisms that are induced by torus automorphisms. However, a Kummer surface may admit additional automorphisms beyond those which are induced by torus automorphisms. (See, e.g., (19).)

1.1 Distinguished Line Bundles

For automorphisms of smooth complex projective surfaces, we develop a refined cohomological interpretation of entropy:

Theorem 1.1 *Let X be a connected compact Kähler surface, and let σ be an automorphism of X. Let λ be a Salem number of degree s, let $S(t)$ be the minimal polynomial for λ, and let*

$W_S(\sigma)$ be the kernel of the action of $S(\sigma^*)$ on $\mathrm{Pic}(X)$. *Then the following four statements are equivalent:*

1) *$W_S(\sigma)$ contains a line bundle with a non-trivial Chern class;*

2) *$W_S(\sigma)$ contains an s-dimensional sublattice of line bundles with non-trivial Chern classes;*

3) *X is projective and the entropy of σ is* $\log(\lambda) > 0$*; and*

4) *$W_S(\sigma)$ contains a nef and big line bundle.*

(See §3 below for the proof.) The degree of a Salem number is the degree of its minimal polynomial. Given any $\mathbb{Z}$-module endomorphism ϕ and any polynomial

$$Q(t) = q_0 + q_1 t + \cdots + q_n t^n$$

with integer coefficients, there is a naturally defined $\mathbb{Z}$-module endomorphism

$$Q(\phi) = q_0 \mathbf{1} + q_1 \phi + \cdots + q_n \phi^n,$$

where $\mathbf{1}$ is the identity map and $\phi^j = \phi^{\circ j}$ is the j-fold iteration $\phi \circ \cdots \circ \phi$ for any $j \in \mathbb{N}$; the map $S(\sigma^*)$ in Theorem 1.1 is defined in this way. Since the entropy of any automorphism of a connected compact Kähler surface must be zero or the logarithm of a Salem number, Theorem 1.1 accounts for all smooth complex projective surface automorphisms with positive entropy. Also, any monic irreducible polynomial satisfying case 1 in Theorem 1.1 must be either a Salem polynomial or a cyclotomic polynomial. (See §2.4 below.)

Suppose that σ is an automorphism of a smooth complex projective surface X with entropy $\log(\lambda) > 0$, and let $S(t)$ be the minimal polynomial for λ. We will say that a line bundle $L \in \mathrm{Pic}(X)$ is *distinguished* (by σ) if $S(\sigma^*)L$ is trivial.

In the special case where λ is a degree-two Salem number, Theorem 1.1 shows that any automorphism σ of a smooth complex projective surface X with entropy $\log(\lambda)$ must satisfy

$$\sigma^* L + (\sigma^{-1})^* L = (\lambda + \lambda^{-1}) L$$

for any distinguished line bundle L; in particular, $(X; \sigma, \sigma^{-1})$ is a dynamical system of two morphisms associated to L, as defined by Kawaguchi (20), for any such X, σ, and L.

We show that case 4 in Theorem 1.1 cannot be improved to state in general that there is an ample line bundle in $W_S(\sigma)$:

Theorem 1.2 *Let X be a smooth complex projective surface, and let σ be an automorphism of X with entropy $\log(\lambda) > 0$. Then the following two statements are equivalent:*

1) *There is an ample distinguished line bundle on X; and*

2) *No curve on X is periodic for σ.*

(See §3.4 and §3.5 below for the proof.) Theorem 1.2 shows that, in general, the dichotomy between automorphisms with periodic curves and those without periodic curves can be interpreted as a dichotomy of the sets of distinguished line bundles associated to these automorphisms. A projective surface automorphism that admits an ample distinguished line bundle must be dynamically minimal; however an automorphism for which no exceptional curve is periodic may still have a periodic curve. Bedford and Kim (21) showed that the set of dynamically minimal positive-entropy automorphisms of rational surfaces includes automorphisms with periodic curves and automorphisms without periodic curves. A positive-entropy automorphism of a two-dimensional complex torus cannot have a periodic curve, while any Kummer surface automorphism that is induced by a two-dimensional complex torus automorphism must have a periodic curve. (See §4.4 below.) It follows from work by McMullen [(9),(10)] that the set of positive-entropy automorphisms of non-Kummer projective K3 surfaces includes automorphisms with periodic curves and automorphisms without periodic curves.

We also give a cohomological characterization of entropy for automorphisms of non-projective surfaces:

Theorem 1.3 *Let X be a connected compact Kähler surface, and let σ be an automorphism of X. Let λ be a Salem number, and let $S(t)$ be the minimal polynomial for λ. Then the following two statements are equivalent:*

1) *There is a non-trivial class $w \in H^{2,0}(X)$ such that $S(\sigma^*)w = 0$; and*

2) *X is non-projective and the entropy of σ is $\log(\lambda) > 0$.*

(See §3.5 below for the proof.) As in Theorem 1.1, any monic irreducible polynomial satisfying case 1 in Theorem 1.3 must be either a Salem polynomial or a cyclotomic polynomial; in fact, any eigenvalue for σ^* on $H^{2,0}(X)$ in Theorem 1.3 must have magnitude one. (See §2.4 below.)

Theorems 1.1 and 1.3 show that the distinction between a positive-entropy automorphism of a projective surface and that of a non-projective surface can be seen in the positioning of the cohomological eigenspaces corresponding to the Galois conjugates of the Salem numbers giving the entropies. In particular, we have the following corollary of Theorems 1.1 and 1.3:

Corollary 1.4 *Let X be a connected compact Kähler surface, and let σ be an automorphism of X. Suppose that the entropy of σ is the logarithm of a degree-two Salem number. Then X is projective.*

Except on one small point, our proofs of Theorems 1.1, 1.2, and 1.3 (and Corollary 1.4) do not use the fact that only certain types of Kähler surfaces admit positive-entropy automorphisms. Indeed, only when converting information about actions on Néron-Severi groups to information about actions on Picard groups do we restrict the types of surfaces considered. (See §3.3 below.) So the proof of Theorem 1.3 makes no assumption about the type of X, while the proofs of Theorems 1.1 and 1.2 make no assumptions about the type of X when we replace $\mathrm{Pic}(X)$ with $\mathrm{NS}(X)$ and line bundles with classes in $\mathrm{NS}(X)$. If they are restricted to a given fixed type of surface, the proofs of the theorems can be made shorter (or in some cases obviated); however, the shorter proofs are not uniform across all types of surfaces.

1.2 Measures of Maximal Entropy on Projective Surfaces

Given an automorphism σ of a compact Kähler surface X, we can define a measure-theoretic entropy for any Borel probability measure μ on X satisfying $\sigma_*\mu = \mu$: for any finite measurable partition $\mathcal{P}$ of X, set

$$h_\mu(\mathcal{P}, \sigma) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\sigma^{-1}\mathcal{P} \vee \cdots \vee \sigma^{-n}\mathcal{P}),$$

where

$$H_\mu(\sigma^{-1}\mathcal{P} \vee \cdots \vee \sigma^{-n}\mathcal{P}) = \sum_P \mu(P)\log(\mu(P))$$

is summed over all elements P in the minimal common refinement of the partitions $\sigma^{-j}\mathcal{P}$; then the measure-theoretic entropy of σ with respect to μ is

$$h_\mu(\sigma) = \sup_{\mathcal{P}} h_\mu(\mathcal{P}, \sigma),$$

where the supremum is taken over all finite measurable partitions of X. (See, e.g., (22).) The Variational Principle gives

$$h(\sigma) = \sup h_\mu(\sigma),$$

where the supremum is taken over all σ-invariant Borel probability measures on X. (See (23).) A measure of maximal entropy for σ is any measure that realizes the supremum as a maximum.

Cantat [(2),(3)] showed that σ has a unique measure of maximal entropy μ_σ whenever $h(\sigma) > 0$, and that this measure can be expressed as a wedge product $T_+ \wedge T_-$ of positive closed $(1,1)$-currents T_+ and T_- on X that are, respectively, dilated and contracted under the action on currents induced by σ. (See §4.1 below.) We use Theorems 1.1 and 1.2 to develop a distinguished means of obtaining the measures of maximal entropy for certain projective surface automorphisms:

Theorem 1.5 *Let X be a smooth complex projective surface, and let σ be an automorphism of X with entropy $\log(\lambda) > 0$. Suppose that either X is not a rational surface or no curve on X*

is periodic for σ. Then there is a distinguished nef and big line bundle on X whose Chern class contains a semi-positive curvature form; moreover, if ω_0 is such a semi-positive form, then the inductively defined sequence

$$\{\omega_n = (\lambda + \lambda^{-1})^{-1}(\sigma^*\omega_{n-1} + (\sigma^{-1})^*\omega_{n-1})\}_{n\in\mathbb{N}}$$

converges weakly to a positive current T with the property that the measure $T \wedge T$ is $c\mu_\sigma$ for some positive real number c.

(See §4.3 below for the proof.) If σ has no periodic curves in Theorem 1.5, then ω_0 can be taken to be a Kähler form; otherwise, the assumption that the surface is a torus, a K3 surface, or an Enriques surface allows an application of a theorem due to Kawamata to give the existence of ω_0. Thus Theorem 1.5 sets apart rational surface automorphisms with periodic curves among all positive-entropy projective surface automorphisms; it is not known if there even can be a form ω_0 as in Theorem 1.5 for a rational surface automorphism with a periodic curve.

Cantat [(2),(3)] showed also (using work by Bedford, Lyubich, and Smillie [(24),(25)]) that the isolated periodic points for a positive-entropy projective surface automorphism σ are equidistributed with respect to μ_σ. (See §4.2 below.) Via this result, work by Kawaguchi (20) and Lee (26) leads to an alternative proof of Theorem 1.5 in the special case where λ is a degree-two Salem number and σ has no periodic curves; so Theorem 1.5 generalizes this special case.

Since Supp(μ_σ) is contained in the Julia set for any positive-entropy surface automorphism σ (where the Julia set is defined to be the locus on which the set of forward and backward iterates of σ fails to be a normal family), a complete understanding of these supports may aid in the characterization of positive-entropy automorphisms that exhibit Siegel disks or other non-trivial Fatou components. (See (3), §7.) Every two-dimensional complex torus automorphism with positive entropy has a full Julia set. (See §4.4 below.) On the other hand, there are examples of positive-entropy non-projective K3 surface automorphisms and positive-entropy rational surface automorphisms with periodic curves for which the Julia sets are zero-dimensional (and the automorphisms have Siegel disks). (See (14), (15), (7), and (8).) Beyond these cases, descriptions of the supports of the measures of maximal entropy or the Julia sets are not known for positive-entropy surface automorphisms in general; the examples of positive-entropy surface automorphisms where the Julia sets are not known include projective K3 surface automorphisms and rational surface automorphisms without periodic curves. (See, e.g., (21) and (7).) Thus the dichotomy of automorphisms given by Theorem 1.5 could in fact coincide with the differentiation of automorphisms according to whether or not they have full Julia sets. Given these observations, we speculate that there is never a form ω_0 as in Theorem 1.5 for a rational surface automorphism with a periodic curve–and further that this type of projective surface automorphism is the only one for which the Julia sets can be strictly proper. The Julia set is at least (analytically) Zariski dense for any compact Kähler surface automorphism with positive entropy. (See §4.2 below.)

1.3 Synthetic Constructions of Torus Automorphisms

Since any automorphism of a two-dimensional complex torus X preserves the Hodge decomposition of $H^1(X, \mathbb{C})$, the roots of the characteristic polynomial for the action of the automorphism on $H^1(X,\mathbb{C})$ necessarily consist of two complex conjugate pairs (so any real such root is a double root). In fact, any degree-four polynomial in $\mathbb{Z}[t]$ whose solution set has the form

$$\{\gamma_1, \gamma_2, \overline{\gamma_1}, \overline{\gamma_2}\}$$

with $|\gamma_1\gamma_2| = 1$ is the characteristic polynomial for the action on the first cohomology group induced by some two-dimensional complex torus automorphism; synthesis is the process of constructing torus automorphisms that realize such characteristic polynomials. (See §5.1 below.)

We use synthesis to characterize all of the values of entropy that occur for automorphisms of two-dimensional complex tori:

Theorem 1.6 *Let $S(t)$ be the minimal polynomial for a Salem number λ of degree d.*

1) *If $d = 6$, then $\log(\lambda)$ is the entropy of some two-dimensional complex torus automorphism if and only if $S(1) = -m^2$ for some integer m and $S(-1) = n^2$ for some integer n.*

2) *If $d = 4$, then $\log(\lambda)$ is the entropy of some two-dimensional complex torus automorphism if and only if one of the following three cases holds: (a) $S(1) = -m^2$ for some integer m; (b) $S(-1) = n^2$ for some integer n; or (c) $S(1) = -(1/2)m^2$ for some integer m and $S(-1) = (1/2)n^2$ for some integer n.*

3) *If $d = 2$, then $\log(\lambda)$ is the entropy of some two-dimensional complex torus.*

These cases constitute all possible positive values of entropy for two-dimensional complex torus automorphisms.

(See §5.2 below for the proof.) Theorem 1.6 also gives all possible positive values of entropy for Kummer surface automorphisms that are induced by torus automorphisms. Additionally, since the intersection form makes $H^2(\mathbb{C}^2/\Lambda, \mathbb{Z})$ (for any two-dimensional complex torus $\mathbb{C}^2/\Lambda$) isomorphic to the even unimodular lattice of signature $(3, 3)$, Theorem 1.6 has an application to the study of lattice isometries: let $L_{3,3}$ be the unique even unimodular lattice of signature $(3, 3)$; work by Gross and McMullen (27) shows that a degree-six Salem polynomial $S(t)$ is the characteristic polynomial for some isometry of $L_{3,3}$ if $S(1) = -1$ and $S(-1) = 1$, and that any degree-six Salem polynomial $S(t)$ that is the characteristic polynomial for some isometry of $L_{3,3}$ must satisfy $S(1) = -m^2$ for some integer m and $S(-1) = n^2$ for some integer n; case 1 in Theorem 1.6 completes the picture in this special case by showing that the necessary condition on $S(t)$ is in fact sufficient as well.

Ghys and Verjovsky (28) describe the set of two-dimensional complex tori with infinite automorphism groups in terms of the lattices in $\mathbb{C}^2$ giving the tori. Corollary 1.4 shows that every torus from case 3 in Theorem 1.6 is necessarily an abelian surface. Since the dimensions of $H^{2,0}(X)$ and $H^{1,1}(X)$ are, respectively, one and four for any two-dimensional complex torus X, we have in this setting the following complementary corollary of Theorems 1.1 and 1.3:

Corollary 1.7 *Let X be a two-dimensional complex torus, and let σ be an automorphism of X. Suppose that the entropy of σ is the logarithm of a degree-six Salem number. Then X is non-projective.*

We show also that tori from case 2 in Theorem 1.6 can be projective or non-projective:

Theorem 1.8 *Let λ be a degree-four Salem number such that* $\log(\lambda)$ *is the entropy of some two-dimensional complex torus automorphism. Then* $\log(\lambda)$ *is both the entropy of some abelian surface automorphism and the entropy of some non-projective two-dimensional complex torus automorphism.*

(See §5.3 below for the proof.) If X is an abelian surface that admits an automorphism whose entropy is the logarithm of a degree-four Salem number, then the Picard rank of X must be four; on the other hand, if X is a non-projective two-dimensional complex torus that admits such an automorphism, then the Picard rank of X must be two–so that X is in fact an example of a compact Kähler manifold with a non-trivial Picard group but no divisors. (See §5.4 below.)

Tori of the form $E \times E$, where E is an elliptic curve, admit many straightforward explicit examples of positive-entropy automorphisms, and the automorphism groups of such tori can be complicated. (See §5.4 below.) On the other hand, the set of entropies exhibited by the automorphism group of a non-projective two-dimensional complex torus is either trivial or equal to

$$\{k \log(\lambda) | k \in \mathbb{N}_0\}$$

for some Salem number λ. (See §5.5 below.) In a related vein, we show that an entropy that occurs for a two-dimensional complex torus automorphism will generally only occur on finitely many different two-dimensional complex tori:

Theorem 1.9 *Let λ be a Salem number such that* $\log(\lambda)$ *is the entropy of some two-dimensional complex torus automorphism. Then either:*

1) *One of* $\lambda+\lambda^{-1}+2$ *or* $\lambda+\lambda^{-1}-2$ *is the square of an integer; or*

2) *The set of two-dimensional complex tori that admit automorphisms whose entropies are* $\log(\lambda)$ *is finite.*

(See §5.5 below for the proof.) Every Salem number from case 1 in Theorem 1.9 has degree two, and for each such Salem number there is a positive-dimensional set of parameters defining abelian surfaces–including surfaces of the form $E \times E$ as well as simple abelian surfaces–that admit automorphisms whose entropies are logarithms of the Salem number. (See §5.4 below.)

Characterizations of the possible values of positive entropy for automorphisms of K3 surfaces, Enriques surfaces and rational surfaces remain open, although partial progress has been made on several fronts. (See, e.g., (9), (10), (29), and (30).) We expect that the tools we use to address entropies on two-dimensional complex tori will prove to be useful in the study of entropies on other surfaces as well.

1.4 Further Questions

1) *What are the possible supports of measures of maximal entropy for positive-entropy automorphisms of compact Kähler surfaces?*

As indicated in §1.2, a key step in answering this question will be determining whether or not any projective surface that is not rational can admit a positive-entropy automorphism whose measure of maximal entropy does not have full support. Additionally, understanding the nature of the supports when they are not full will be an important advancement.

2) *Given a Salem number* λ*, which compact Kähler surfaces admit automorphisms with entropy* $\log(\lambda)$*?*

The results in §1.3 answer this question for two-dimensional complex tori. While significant partial answers to this question have been achieved for other compact Kähler surfaces, much work remains for a complete characterization.

3) *Are there cohomological means for understanding measures of maximal entropy for strictly birational maps on the projectice plane?*

Diller and Favre (31) showed that there is a large class of birational maps on $\mathbb{P}^2$ with positive (cohomologically defined) entropy that do not admit bimeromorphic conjugacies to automorphisms (in contrast to the fact that any positive-entropy bimeromorphic map on a non-rational compact Kähler surface is necessarily conjugate to a positive-entropy automorphism). Towards an understanding of cohomological dynamics and measures of maximal entropy in this setting, the development of theorems analogous to Theorems 1.1, 1.2, 1.3, and 1.5 will be a useful goal.

4) *How are cohomological actions related to dynamics of endomorphisms and rational self-maps in general?*

Positive-entropy surface automorphisms form only a small subset of the universe of dynamical systems on compact Kähler manifolds. Given the importance of cohomological structures in the study of compact Kähler manifolds in general, the study of dynamics on such manifolds will benefit greatly from an expanded understanding of the connections between cohomological actions and dynamical behaviors. Examples and case-specific results will be important initial progress in this direction. One important aspect of this line of inquiry will be the pursuit of characterizations of the possible (logarithmic) dynamical degrees (analogues of entropies for surface automorphisms) for maps on compact Kähler manifolds in general.

CHAPTER 2

REVIEW OF COHOMOLOGICAL STRUCTURES

Before proceeding to the proofs of the theorems in §1, we recall a variety of facts about cohomology groups (§2.1), line bundles (§2.2), and differential forms (§2.3) on compact Kähler surfaces. In §2.4, we present some previously known results about the implications of the existence of an automorphism with positive entropy on a compact Kähler surface.

Let X be a connected compact Kähler surface, and let σ be an automorphism of X.

2.1 Cohomology Groups

The Hodge decomposition gives, for $k \in \mathbb{N}_0$,

$$H^k(X, \mathbb{C}) = H^k(X, \mathbb{Z}) \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}(X),$$

where each $H^{p,q}(X)$ is isomorphic to the corresponding Dolbeault cohomology group and also each $H^{p,0}(X)$ is isomorphic to $H^p(X, \mathcal{O}_X)$. (For details, see (17).) Moreover,

$$\overline{H^{p,q}(X)} = H^{q,p}(X)$$

for any p and q; thus, in particular, there is another decomposition

$$H^2(X, \mathbb{R}) = H^2(X, \mathbb{Z}) \otimes \mathbb{R} = H^2(X, \mathbb{C})_{\mathbb{R}} = H^{1,1}(X)_{\mathbb{R}} \oplus (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}.$$

The Picard group of X is denoted $\mathrm{Pic}(X)$, and the Néron-Severi group of X is denoted $\mathrm{NS}(X)$; the first Chern map from $\mathrm{Pic}(X)$ to $H^2(X,\mathbb{Z})$ is denoted c_1. (For details, see (17) and (32).) When torsion is factored out, the image of c_1 is $\mathrm{NS}(X)$; moreover, the Lefschetz theorem on (1,1) classes gives

$$\mathrm{NS}(X) = H^2(X,\mathbb{Z}) \cap H^{1,1}(X).$$

The rank of $\mathrm{NS}(X)$ (as a finitely generated abelian group) is called the Picard rank of X. The kernel of c_1 is called the Picard variety of X and is denoted $\mathrm{Pic}^0(X)$; it is a complex torus of dimension equal to the rank of $H^1(X,\mathcal{O}_X)$ as a complex vector space. Finally, the cup product defines a symmetric bilinear form on $H^2(X,\mathbb{Z})$ and a compatible quadratic form on $H^2(X,\mathbb{R})$; the form on $H^2(X,\mathbb{Z})$ coincides with the image of the bilinear form on $\mathrm{Pic}(X)$ coming from intersections of curves. The pull-back map σ^* on each of these spaces is an automorphism that preserves the corresponding pairing; moreover, these pull-back maps commute with the Chern map. The intersection of two elements o_1 and o_2 is denoted $o_1.o_2$, while the self-intersection of an element o is denoted o^2.

2.2 Line Bundles

The following consequence of Grauert's criterion provides a means of determining whether or not X is projective.

Theorem 2.1 ((17), Theorem IV.6.2) *A connected compact complex surface X is projective if and only if there is a line bundle $L \in \mathrm{Pic}(X)$ with $L^2 > 0$.*

The following further consequence of Grauert's criterion determines when a line bundle with positive self-intersection is in fact ample itself.

Theorem 2.2 ((17), Theorem IV.6.4) *Let X be a connected compact complex surface. Then a line bundle $L \in \mathrm{Pic}(X)$ is ample if and only if $L^2 > 0$ and $L.[D] > 0$ for any effective divisor D on X.*

If X is projective, then every line bundle on X is a non-empty class of linearly equivalent divisors; that is, $\mathrm{Pic}(X)$ is precisely the group of divisors on X modulo linear equivalence. (See (32), §II.4 and §II.6.) Since the pull-back of an effective divisor is again an effective divisor, the set of effective divisor classes

$$\mathcal{E}(X) = \{[D] \in \mathrm{Pic}(X) | D \text{ is an effective divisor}\}$$

is preserved by σ^*. (See (17), §I.6, and (32), §II.6.) Thus, since σ^* also preserves the intersection pairing on $\mathrm{Pic}(X)$, the set of ample line bundles on X is preserved by σ^* as well. The following property of ample line bundles is a consequence of the Hodge index theorem: if $L \in \mathrm{Pic}(X)$ is ample, then $L.L' \neq 0$ for any $L' \in \mathrm{Pic}(X)$ with $(L')^2 > 0$. (See (32), §V.1.) A line bundle $L \in \mathrm{Pic}(X)$ is called nef if $L.[D] \geq 0$ for any effective divisor D on X; if X is projective, a nef line bundle on X is called big if it has positive self-intersection. (See (17), §I.6, §IV.7, and §IV.12.) Thus any ample line bundle on X is necessarily nef and big. The following property of line bundles with positive self-intersection is a consequence of the Riemann-Roch theorem: if $L \in \mathrm{Pic}(X)$ has $L^2 > 0$ and $L.H > 0$ for some ample $H \in \mathrm{Pic}(X)$, then $L^{\otimes m} \in \mathcal{E}(X)$ for some $m \in \mathbb{N}$. (See (32), §V.1.) If $\hat{X}$ is the blow-up of X at a point and E is the exceptional curve for the blow-up, then the Picard group of $\hat{X}$ is given by

$$\mathrm{Pic}(\hat{X}) \cong \mathrm{Pic}(X) \times <[E]>;$$

moreover, the intersection form on $\mathrm{Pic}(\hat{X})$ is compatible with the intersection form on $\mathrm{Pic}(X)$ (which is orthogonal to $<[E]>$). (See (17), §I.9.) Finally, the canonical line bundle on X is denoted K_X.

2.3 Differential Forms

The space of complex differential forms on X is

$$\Omega(X) = \bigoplus_{0\le r\le 4} \Omega^r(X) = \bigoplus_{0\le r\le 4} \left(\bigoplus_{p+q=r} \Omega^{p,q}(X)\right),$$

where each $\Omega^{p,q}(X)$ is the space of complex differential (p,q)-forms on X; $\Omega^{p,q}(X)$ is trivial unless $0 \le p, q \le 2$. (For details, see (17) and (33).) A complex differential form is real if it is equal to its own complex conjugate. In terms of differential forms, the intersection pairing on $H^2(X,\mathbb{R})$ is given by, for real d-closed 2-forms ω_a and ω_b,

$$[\omega_a].[\omega_b] = \int_X \omega_a \wedge \omega_b.$$

For an irreducible curve $Y \subseteq X$ and a real d-closed 2-form ω on X, the intersection pairing is given by

$$[\omega].\,\mathrm{c}_1([Y]) = \int_Y \omega;$$

this formula extends linearly to give the intersection of ω with $\mathrm{c}_1([D])$ for any divisor D on X. Since the wedge product of any form in $\Omega^1(X)$ with itself is zero, the decomposition

$$H^2(X,\mathbb{R}) = H^{1,1}(X)_{\mathbb{R}} \oplus (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$$

is necessarily an orthogonal decomposition with respect to the intersection pairing. Thus the pairing restricts to a non-degenerate quadratic form on $H^{1,1}(X)_{\mathbb{R}}$.

Theorem 2.3 ((17), Theorem IV.2.14) *Let X be a connected compact Kähler surface, and let $h^{1,1}(X)$ be the dimension of $H^{1,1}(X)_{\mathbb{R}}$. Then $H^{1,1}(X)_{\mathbb{R}}$ has signature $(1, h^{1,1}(X)-1)$ under the quadratic form induced by the cup product.*

This result leads to an analogue of the Hodge index theorem that applies to real closed $(1,1)$-forms: if v_1 and v_2 are linearly independent in $H^{1,1}(X)_{\mathbb{R}}$ with $v_1^2 > 0$ and $v_2^2 \geq 0$, then $v_1.v_2 \neq 0$.

If ω is a real $(1,1)$-form on X, then ω can be written locally as

$$\omega|_U = i\alpha_{11}dz_1 \wedge d\overline{z_1} + i\alpha_{12}dz_1 \wedge d\overline{z_2} + i\alpha_{21}dz_2 \wedge d\overline{z_1} + i\alpha_{22}dz_2 \wedge d\overline{z_2},$$

where $(\alpha_{ij}(x))$ is a Hermitian matrix at every point $x \in U$; w is (semi-)positive if $(\alpha_{ij}(x))$ is positive (semi-)definite at all points in X. A Kähler form is a real $(1,1)$-form that is positive and d-closed. The positive-definiteness forces a Kähler form to have positive intersection with any effective divisor on X. The assumption that X is Kähler means that X admits at least one Kähler form. If κ_1 and κ_2 are two Kähler forms on X, then $a\kappa_1 + b\kappa_2$ is also a Kähler form on X for any positive real numbers a and b. Thus the set of Kähler classes forms a convex cone $C_K(X) \subseteq H^{1,1}(X)_{\mathbb{R}}$, called the Kähler cone of X; it is contained in the convex cone

$$C_+(X) = \{v \in H^{1,1}(X)_{\mathbb{R}} | v^2 > 0, v.[\kappa] > 0 \, \forall \, \text{Kähler form} \, \kappa\},$$

called the positive cone of X. The Kähler cone is open in $H^{1,1}(X)_{\mathbb{R}}$, and, moreover, it is given explicitly by

$$C_K(X) = \{v \in C_+(X) | v.\, \mathrm{c}_1([Y]) > 0 \, \forall \, \text{curve} \, Y \subseteq X \, \text{with} \, Y^2 < 0\}.$$

(See (34), (35), (36), and (37), §1.) Thus the Chern class of a line bundle $L \in \mathrm{Pic}(X)$ is represented by a Kähler form if and only if L is ample. (This is the content of the Kodaira embedding thoerem; see (38), §9, and (37), §1.) So X is projective if and only if $C_K(X) \cap \mathrm{NS}(X) \neq \emptyset$.

If κ is a Kähler form on X given locally by

$$\kappa|_U = i\alpha_{11} dz_1 \wedge d\overline{z_1} + i\alpha_{12} dz_1 \wedge d\overline{z_2} + i\alpha_{21} dz_2 \wedge d\overline{z_1} + i\alpha_{22} dz_2 \wedge d\overline{z_2},$$

then $\sigma^*\kappa$ is given locally by

$$(\sigma^*\kappa)|_{\sigma^{-1}(U)} = i\beta_{11} d\sigma_1 \wedge d\overline{\sigma_1} + i\beta_{12} d\sigma_1 \wedge d\overline{\sigma_2} + i\beta_{21} d\sigma_2 \wedge d\overline{\sigma_1} + i\beta_{22} d\sigma_2 \wedge d\overline{\sigma_2},$$

where $(\beta_{ij}(y) = \alpha_{ij}(\sigma(y)))$ is positive definite at every $y \in \sigma^{-1}(U)$. Since σ_1 and σ_2 give local holomorphic coordinates at $\sigma^{-1}(x)$ for any pair of local holomorphic coordinates z_1 and z_2 at any point $x \in X$, it follows that $\sigma^*\kappa$ is Kähler. Thus the Kähler cone of X is preserved by σ^*.

2.4 Positive Entropy

A real algebraic integer is called a Salem number if it is greater than one and it has a reciprocal minimal polynomial with exactly two roots off the unit circle; the minimal polynomial for a Salem number is called a Salem polynomial. The proof of the following theorem is based on an argument by McMullen [(7), §3].

Theorem 2.4 *Let σ be an automorphism of a connected compact Kähler surface. Then the entropy of σ is either zero or the logarithm of a Salem number.*

Proof: Let A be the set of eigenvalues (not counting multiplicity) for σ^* on $H^{1,1}(X)_{\mathbb{R}}$, and let A' be the set of eigenvalues for σ^* on $(H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$. Since the characteristic polynomial for the action of σ^* on $H^2(X, \mathbb{R}) = H^2(X, \mathbb{Z}) \otimes \mathbb{R}$ has integer coefficients, every element of A or A' is an algebraic integer; moreover, A and A' are each invariant under complex conjugation.

Since any element of $(H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$ can be represented by a 2-form with local expression

$$\beta dz_1 \wedge dz_2 + \overline{\beta} d\overline{z_1} \wedge d\overline{z_2},$$

$(H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$ must be positive definite (or trivial). Also, since the signature of the quadratic form on $H^{1,1}(X)_{\mathbb{R}}$ is $(1, h^{1,1}(X) - 1)$, $H^{1,1}(X)_{\mathbb{R}}$ cannot contain a totally isotropic subspace of dimension greater than one.

For each eigenvalue $\alpha \in A$, let $E(\alpha) \subseteq H^{1,1}(X)$ be the generalized eigenspace for σ^* corresponding to α; so each $E(\alpha)$ is a complex vector subspace, and

$$H^{1,1}(X) = \bigoplus_{\alpha} E(\alpha).$$

For each eigenvalue $\alpha \in A$, set

$$E'(\alpha) = E(\alpha) \oplus E(\overline{\alpha}) = E(\alpha) \oplus \overline{E(\alpha)}$$

(so $E'(\alpha) = E(\alpha)$ if α is real); then

$$H^{1,1}(X)_{\mathbb{R}} = \bigoplus_{\alpha} E'(\alpha)_{\mathbb{R}},$$

where the sum omits one element from each conjugate pair of non-real eigenvalues. If e_1 and e_2 are eigenvectors corresponding, respectively, to eigenvalues α_1 and α_2 in A, then $e_1.e_2 = (\sigma^* e_1).(\sigma^* e_2) = 0$ unless $\alpha_1 \overline{\alpha_2} = 1$ (where the intersection form on $H^{1,1}(X) = H^{1,1}(X)_{\mathbb{R}} \otimes \mathbb{C}$ is the indefinite hermitian inner product induced by the quadratic form on $H^{1,1}(X)_{\mathbb{R}}$). So, if it were the case that $\alpha^{-1} \notin A$ for some $\alpha \in A$, then every element in $E'(\alpha)_{\mathbb{R}}$ would be perpendicular to $H^{1,1}(X)_{\mathbb{R}}$–which cannot happen, since the quadratic form on $H^{1,1}(X)_{\mathbb{R}}$ is non-degenerate; thus A is invariant under inversion.

If there were an eigenvalue $\beta \in A'$ with $|\beta| \neq 1$, then $(H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$ would necessarily contain a non-trivial isotropic subspace (namely, $(E(\beta) \oplus E(\overline{\beta}))_{\mathbb{R}}$, where $E(\beta)$ and $E(\overline{\beta})$ are the generalized eigenspaces for σ^* on $H^{2,0}(X) \oplus H^{0,2}(X)$ corresponding to β and $\overline{\beta}$), which cannot be the case. So the magnitude of every element of A' is one.

Now suppose that the entropy of σ is positive, so that A contains an eigenvalue λ with $|\lambda| > 1$. If λ were not real, then $E'(\lambda)_{\mathbb{R}}$ would be a totally isotropic subspace of $H^{1,1}(X)_{\mathbb{R}}$ with dimension greater than one, which cannot exist; so λ is real. Similarly, if λ' were another eigenvalue in A with magnitude greater than one, then $E'(\lambda)_{\mathbb{R}} \oplus E'(\lambda')_{\mathbb{R}}$ would be a totally isotropic subspace of $H^{1,1}(X)_{\mathbb{R}}$ with dimension greater than one, which cannot exist; so λ is the only element of A with magnitude greater than one, and it is an eigenvalue with multiplicity one. Now let κ be a Kähler form on X, and let $e_+ \in E(\lambda)$ and $e_- \in E(\lambda^{-1})$ be real eigenvectors satisfying

$$(e_+ + e_-)^2 > 0 \text{ and } (e_+ + e_-).[\kappa] > 0,$$

which are guaranteed to exist by the non-degeneracy of the quadratic form on $H^{1,1}(X)_{\mathbb{R}}$. Then it follows that

$$(ae_+ + be_-).[\kappa'] > 0 \,\forall\, a, b > 0, \,\forall\, \kappa' \in C_K.$$

So, in particular, from

$$(\sigma^*(e_+ + e_-)).[\sigma^*\kappa] = (\mathrm{sgn}(\lambda)(|\lambda|e_+ + |\lambda^{-1}|e_-)).[\sigma^*\kappa] > 0,$$

it follows that λ is positive.

All together, the eigenvalues of σ^* on $H^2(X, \mathbb{Z})$ (counting multiplicity) are precisely λ, λ^{-1}, and $h^2(X) - 2$ algebraic integers with magnitude one. If λ were not a Galois conjugate of λ^{-1}, then λ^{-1} would be an algebraic integer with Mahler measure equal to one that is not a root of unity–which, by a theorem of Kronecker, cannot exist. (See (39), §1.) Thus λ is a Salem number. □

The proof of Theorem 2.4 shows also that the irreducible factors of the characteristic polynomial for the action of σ^* on $H^2(X, \mathbb{Z})$ can only be cyclotomic polynomials and at most one Salem polynomial. The degree of a Salem number is the degree of its minimal polynomial; it is necessarily even. If the entropy of σ is $\log(\lambda)$, then the entropy of σ^{-1} is also $\log(\lambda)$ and (for all $k \in \mathbb{N}$) the entropy of σ^k is $\log(\lambda^k)$. If λ is a Salem number of degree s, then λ^k is a Salem number of degree s as well; indeed, the set of Galois conjugates of λ^k is precisely the set of all k-th powers of Galois conjugates of λ.

If there is an exceptional curve E on X that is periodic for σ, then σ descends to an automorphism σ' of the surface X' obtained from X by contraction of the orbit of E; moreover, since the orbit of $\mathrm{c}_1([E])$ in $\mathrm{NS}(X) \subseteq H^{1,1}(X)_{\mathbb{R}}$ is preserved by σ^* and only contributes a cyclotomic factor to the characteristic polynomial for σ^*, σ' and σ must have the same entropy.

(See also (40), §2, and (3), §4.1.) Conversely, if $x \in X$ is a periodic point for σ, then σ extends (without a change in entropy) to an automorphism of the blow-up of X at the points in the orbit of x. Complex tori (of dimension two), K3 surfaces, and Enriques surfaces are all minimal, in the sense that they have no exceptional curves; thus any automorphism with positive entropy of a surface that is not rational necessarily descends to an automorphism of a surface with no exceptional curves.

CHAPTER 3

DISTINGUISHED LINE BUNDLES

In this section, we prove Theorems 1.1, 1.2, and 1.3. In §3.1, §3.2, and §3.3, we address case 1, case 2, and case 3 in Theorem 1.1. In §3.4, we address Theorem 1.2 and case 4 in Theorem 1.1. We complete the proofs of Theorems 1.1, 1.2, and 1.3 in §3.5.

3.1 From Distinguished Chern Classes to Positive Entropy

Proposition 3.1 *Let X be a connected compact Kähler surface, and let σ be an automorphism of X. Let λ be a Salem number, and let $S(t)$ be the minimal polynomial for λ. Suppose that there is a non-trivial Chern class $\mathrm{c}_1(L) \in \mathrm{NS}(X)$ such that $S(\sigma^*)\,\mathrm{c}_1(L) = 0$. Then X is projective and σ has entropy $\log(\lambda)$.*

Proof: The set of eigenvalues of the linear action of $S(\sigma^*)$ on $\mathrm{NS}(X) \otimes \mathbb{R}$ is precisely the set of all $S(\alpha)$ where α is an eigenvalue of the action of σ^*. Since zero is an eigenvalue of $S(\sigma^*)$, the eigenvalues of σ^* must include a root of $S(t)$; moreover, since the characteristic polynomial for the action of σ^* on $\mathrm{NS}(X) \otimes \mathbb{R}$ has integer coefficients, it must have $S(t)$ as a factor. So the entropy of σ must be $\log(\lambda)$. Let D_+ be the eigenspace for σ^* corresponding to the eigenvalue λ, let D_- be the eigenspace for σ^* corresponding to the eigenvalue λ^{-1}, and let $E = D_+ \oplus D_-$. Then the dimension of E is two, and, since any vector in D_+ or D_- must have zero self-intersection, the signature of $E \subseteq \mathrm{NS}(X) \otimes \mathbb{R}$ is $(1, 1)$. If the self-intersection of every element in $\mathrm{NS}(X)$ were non-positive, then the same would be true for every element in

$\mathrm{NS}(X)\otimes\mathbb{Q}$, which is dense in $\mathrm{NS}(X)\otimes\mathbb{R}$; but then, by continuity, the self-intersection of every element in $\mathrm{NS}(X)\otimes\mathbb{R}$ would be non-positive, which is not the case. Thus $\mathrm{NS}(X)$ must contain some Chern class $\mathrm{c}_1(L_0)$ with

$$\mathrm{c}_1(L_0)^2 = L_0^2 > 0,$$

and X must be projective. □

3.2 From Positive Entropy to Distinguished Chern Classes

Proposition 3.2 *Let X be a connected compact Kähler surface, and let σ be an automorphism of X. Let λ be a Salem number, let $S(t)$ be the minimal polynomial for λ, and let s be the degree of $S(t)$. Suppose that X is projective and that the entropy of σ is $\log(\lambda)$. Then there is an s-dimensional sublattice of $\mathrm{NS}(X)$ that is annihilated by $S(\sigma^*)$.*

Proof: The eigenvalues of σ^* acting on $H^{1,1}(X)_{\mathbb{R}}$ are λ, λ^{-1}, and $h^{1,1}-2$ algebraic integers with magnitude one (counting multiplicity). Let D_+ be the eigenspace corresponding to λ, let D_- be the eigenspace corresponding to λ^{-1}, and let $E = D_+ \oplus D_-$. So E has signature $(1,1)$ and $E^\perp$ has signature $(0, h^{1,1}-2)$. Let v_1 be a non-trivial element in D_+, let v_2 be a non-trivial element in D_-, and let $\{v_3, \ldots, v_{h^{1,1}}\}$ be a basis for $E^\perp$. Then, with respect to the basis $\{v_1, \ldots, v_{h^{1,1}}\}$, σ^* is given in matrix form as

$$\sigma^* = \begin{pmatrix} \begin{matrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{matrix} & 0 \\ 0 & J \end{pmatrix},$$

for some $(h^{1,1}-2) \times (h^{1,1}-2)$ matrix J. For each $k \in \mathbb{N}$, define $||J^k||$ by

$$||J^k|| = \max\{||J^k \vec{x}||/||\vec{x}|| \,|\, \vec{x} \in E^{\perp} - \{0\}\},$$

where the norm on $E^{\perp} \cong \mathbb{R}^{h^{1,1}-2}$ is the standard Euclidean norm. Since the spectral radius of J is one, J must satisfy, by a result of Gelfand,

$$\lim_{k\to\infty} ||J^k|| = 1.$$

(See (41), §4.2.) So, for any $\vec{x} = (x_1, \ldots, x_{h^{1,1}})$ in $H^{1,1}(X)_{\mathbb{R}}$,

$$\lim_{k\to\infty} \lambda^{-k}(\sigma^*)^k \vec{x} = (x_1, 0, \ldots, 0)$$

and

$$\lim_{k\to\infty} \lambda^{-k}((\sigma^{-1})^*)^k \vec{x} = (0, x_2, 0, \ldots, 0).$$

If $\mathrm{NS}(X) \otimes \mathbb{R}$ were contained in $E^{\perp}$, then $\mathrm{NS}(X) \otimes \mathbb{R}$ would be negative definite and X could not be projective; so $\mathrm{NS}(X) \otimes \mathbb{R}$ must contain some element $\vec{x} = (x_1, \ldots, x_{h^{1,1}})$ with either $x_1 \neq 0$ or $x_2 \neq 0$. Thus, since it is invariant under σ^* and closed, $\mathrm{NS}(X) \otimes \mathbb{R}$ must contain either D_+ or D_-. It follows that $S(t)$ must be a factor in the characteristic polynomial for σ^* acting on $\mathrm{NS}(X) \otimes \mathbb{R}$. So there is a subspace $E' \subseteq \mathrm{NS}(X) \otimes \mathbb{Q}$ such that E' is invariant under σ^* and the characteristic polynomial for σ^* acting on E' is $S(t)$. Thus $E' \cap \mathrm{NS}(X)$ is an s-dimensional sublattice that is annihilated by $S(\sigma^*)$. $\square$

3.3 From Chern Classes to Line Bundles

Let X be a connected compact Kähler surface, and let σ be an automorphism of X. If X is a finite blow-up of a K3 surface, an Enriques surface, or a rational surface, then $b_1(X)$ is zero; so the dimension of $H^1(X, \mathcal{O}_X)$ is zero, and

$$\mathrm{c}_1 : \mathrm{Pic}(X) \to \mathrm{NS}(X)$$

is an isomorphism of non-torsion elements. (See (17), §I.9, §III.4, §V.1, §VIII.2, and §VIII.15.) If X is not one of these three types of surfaces and σ has positive entropy, then X must be a finite blow-up of a torus; so the dimension of $H^1(X, \mathcal{O}_X)$ is two. In any case, for any polynomial $S(t) \in \mathbb{Z}[t]$, if a line bundle $L \in \mathrm{Pic}(X)$ satisfies $S(\sigma^*)L = 0$, then the Chern class $\mathrm{c}_1(L)$ also satisfies $S(\sigma^*)\,\mathrm{c}_1(L) = 0$.

Proposition 3.3 *Let X be a connected compact Kähler surface, and let σ be an automorphism of X. Let λ be a Salem number, and let $S(t)$ be its minimal polynomial. Suppose that the entropy of σ is $\log(\lambda)$ and that $\mathrm{c}_1(L_2) \in \mathrm{NS}(X)$ is a Chern class satisfying $S(\sigma^*)\,\mathrm{c}_1(L_2) = 0$. Then there is a line bundle $L_1 \in \mathrm{Pic}(X)$ that satisfies $S(\sigma^*)L_1 = 0$ and $\mathrm{c}_1(L_1) = \mathrm{c}_1(L_2)$.*

Proof: If X is not bimeromorphic to a torus, then the statement is evident, and, moreover, $L_1 = L_2$.

If X is a torus, then $H^*(X, \mathbb{Z})$ is generated by $H^1(X, \mathbb{Z})$ via the cup product; so, in particular, the six eigenvalues for the action of σ^* on $H^2(X, \mathbb{Z})$ are precisely the products of all pairs among the four eigenvalues for the action of σ^* on $H^1(X, \mathbb{Z})$. Since the eigenvalues for the action of σ^* on $H^2(X,\mathbb{Z})$ are λ, λ^{-1}, and four algebraic integers with magnitude one, the eigenvalues for the action of σ^* on $H^1(X,\mathbb{Z})$ must be two algebraic integers with magnitude $\sqrt{\lambda}$ and two algebraic integers with magnitude $\sqrt{\lambda}^{-1}$; so, since no root of $S(t)$ has magnitude $\sqrt{\lambda}$ or $\sqrt{\lambda}^{-1}$, the action of $S(\sigma^*)$ on $H^1(X,\mathbb{R})$ must be surjective. Moreover, since $\mathrm{Pic}^0(X)$ is the quotient of $H^1(X,\mathcal{O}_X)$ by an embedding of $H^1(X,\mathbb{Z})$ (that commutes with σ^*), the action of $S(\sigma^*)$ must in fact be surjective on $\mathrm{Pic}^0(X)$. Thus there is some $L_0 \in \mathrm{Pic}^0(X)$ such that

$$S(\sigma^*)L_0 = S(\sigma^*)L_2,$$

and the line bundle

$$L_1 = L_2 - L_0$$

satisfies $S(\sigma^*)L_1 = 0$ with $\mathrm{c}_1(L_1) = \mathrm{c}_1(L_2)$.

If X is a blow-up of a torus $\mathbb{T}$ at r points, then the Picard group of X is given by

$$\mathrm{Pic}(X) \cong \mathrm{Pic}(\mathbb{T}) \times \mathbb{Z}^r,$$

the Néron-Severi group of X is given by

$$\mathrm{NS}(X) \cong \mathrm{NS}(\mathbb{T}) \times \mathbb{Z}^r,$$

and the first Chern map is given by, for $L \in \mathrm{Pic}(\mathbb{T})$ and $(l_1, \ldots, l_r) \in \mathbb{Z}^r$,

$$\mathrm{c}_1((L, (l_1, \ldots, l_r))) = (\mathrm{c}_1(L), (l_1, \ldots, l_r)).$$

Thus, since the action of σ^* on $\mathrm{NS}(X)$ respects the product structure, the existence of a line bundle $L_1 \in \mathrm{Pic}(X)$ such that $\mathrm{c}_1(L_1) = \mathrm{c}_1(L_2)$ and $S(\sigma^*)L_1 = 0$ follows from the proof for the case when X is a torus. $\square$

3.4 Nef and Big Line Bundles

Let X be a smooth complex projective surface, and suppose that σ is an automorphism of X with entropy $\log(\lambda) > 0$. Let $S(t)$ be the minimal polynomial for λ, and let s be the degree of $S(t)$. For the action of σ^* on $H^{1,1}(X)_{\mathbb{R}}$, let D_+ be the eigenspace corresponding to λ, let D_- be the eigenspace corresponding to λ^{-1}, and let $E = D_+ \oplus D_-$. Let $\{v_1, \ldots, v_{h^{1,1}}\}$ be a basis for $H^{1,1}(X)_{\mathbb{R}}$ such that $v_1 \in D_+$, $v_2 \in D_-$, and $v_j \in E^\perp$ for all other j. Since $C_K(X)$ is open in $H^{1,1}(X)_{\mathbb{R}}$, it must contain some element v whose first two coordinates are some non-trivial elements $e_+ \in D_+$ and $e_- \in D_-$. Then

$$\lim_{k\to\infty} \lambda^{-k}(\sigma^*)^k v = e_+$$

and

$$\lim_{k\to\infty} \lambda^{-k}((\sigma^{-1})^*)^k v = e_-.$$

Thus, since σ^* preserves the Kähler cone, $\overline{C_K(X)}$ must contain e_+ and e_-, as well as $ae_+ + be_-$ for any two non-negative real numbers a and b.

Let $u = ae_+ + be_-$, with a and b any two positive real numbers. If u^2 were zero, then E would be a two-dimensional totally isotropic subspace of $H^{1,1}(X)_{\mathbb{R}}$, which cannot be the case; so u^2 is positive. Also, $u.\,\mathrm{c}_1([D])$ is non-negative for any effective divisor D on X. (See also (40), §2.)

Proposition 3.4 ((40), Proposition 3.1) *Assume the hypotheses and notation of the preceding text in this section, and let C be an irreducible curve on X. Then C is periodic for σ if and only if $u.\,\mathrm{c}_1([C])$ is zero. There are only finitely many such curves on X.*

Thus, in particular, u is a Kähler class if and only if no curve on X is periodic for σ. If C is a curve on X with $u.\,\mathrm{c}_1([C]) = 0$, then, by the analogue of the Hodge index theorem, C must have negative self-intersection.

Let $E' \subseteq \mathrm{NS}(X) \otimes \mathbb{Q}$ be the s-dimensional subspace that is annihilated by $S(\sigma^*)$; so E is contained in $E' \otimes \mathbb{R}$. Let $\mathrm{NS}'(X)$ be the s-dimensional sublattice $E' \cap \mathrm{NS}(X)$; so $\mathrm{NS}'(X)$ is precisely the set of all Chern classes in $\mathrm{NS}(X)$ that are annihilated by $S(\sigma^*)$. Since $E^\perp$ is negative definite, $\mathrm{NS}'(X) \cap C_+(X)$ must contain some non-trivial Chern class $\mathrm{c}_1(L_+)$.

Suppose that C is an irreducible curve on X such that $\sigma^k(C) = C$ for some $k \in \mathbb{N}$, and let $S_k(t)$ be the minimal polynomial for λ^k; so $S_k(t)$ is again a Salem polynomial of degree s. Since the entropy of σ^k is $\log(\lambda^k)$ and every invariant space for σ^* is also invariant for $(\sigma^*)^k = (\sigma^k)^*$, it follows that $\mathrm{NS}'(X)$ is annihilated by $S_k((\sigma^*)^k)$. Thus, for any $\mathrm{c}_1(L) \in \mathrm{NS}'(X)$,

$$0 = (S_k((\sigma^*)^k)[C]).(S_k((\sigma^*)^k)L) = (\Sigma^2)([C].L),$$

where Σ is the sum of the coefficients of $S_k(t)$; but since one is not a root of $S_k(t)$, Σ cannot be zero–which forces $[C].L = 0$. So, in particular, $\mathrm{NS}'(X) \cap C_K(X)$ must be empty if σ has any periodic curves.

Proposition 3.5 *Assume the hypotheses and notation of the preceding text in this section, and suppose that no curve on X is periodic for σ. Then there is an ample line bundle $L \in \mathrm{Pic}(X)$ satisfying $S(\sigma^*)L = 0$.*

Proof: Let $\{w_3, \ldots, w_s\}$ be a basis for $E^\perp$ in $E' \otimes \mathbb{R}$; so $\{v_1, v_2, w_3, \ldots, w_s\}$ is a basis for $E' \otimes \mathbb{R}$. Since $(L_+)^2$ is positive, the first two coordinates of $\mathrm{c}_1(L_+)$ must be non-zero; moreover, since $\mathrm{c}_1(L_+)$ has non-negative intersection with both e_+ and e_-, the first two coordinates of $\mathrm{c}_1(L_+)$ must be ae_+ and be_- for some positive numbers a and b. So

$$\lim_{k\to\infty} \lambda^{-k}((\sigma^*)^k + ((\sigma^{-1})^*)^k)\,\mathrm{c}_1(L_+) = ae_+ + be_-,$$

which is a Kähler class. Since $C_K(X)$ is open, there is some positive integer k' such that the k'-th iterate of the sequence is also a Kähler class. So $((\sigma^*)^{k'} + ((\sigma^{-1})^*)^{k'})\,\mathrm{c}_1(L_+)$ is a Kähler class and an element of $\mathrm{NS}'(X)$, and $((\sigma^*)^{k'} + ((\sigma^{-1})^*)^{k'})L_+$ is an ample line bundle that is mapped into $\mathrm{Pic}^0(X)$ by $S(\sigma^*)$. $\square$

If no curve on X is periodic for σ and the degree of $S(t)$ is two, then $\mathrm{c}_1(L_+)$ is an element of $E \cap C_K(X)$ and L_+ itself is ample. In general (whether or not σ has periodic curves), if the degree of $S(t)$ is two, then $\mathrm{c}_1(L_+)$ is an element of $E \cap \overline{C_K(X)}$ and L_+ is nef and big. (See also (40), §3.)

Suppose that C is an irreducible curve on X with non-negative self-intersection; so $\mathrm{c}_1([C])$ is an element of $\overline{C_+(X)}$. Then, since the intersection of any two elements in $C_+(X)$ is positive, the intersection of $\mathrm{c}_1([C])$ with any element of $C_+(X)$ must be non-negative. Thus the only barrier to the existence of a nef and big line bundle whose Chern class is contained in $\mathrm{NS}'(X)$ is the set of irreducible curves on X with negative self-intersection.

Proposition 3.6 *Assume the hypotheses and notation of the preceding text in this section. Then there is a nef and big line bundle $L \in \mathrm{Pic}(X)$ satisfying $S(\sigma^*)L = 0$.*

Proof: For some $q \in \mathbb{N}$, $(L_+)^{\otimes q}$ is an effective divisor class; let Y_+ be an effective divisor in this class. Let $B \in \mathbb{Z}$ be the minimum value of the self-intersection of an irreducible curve in the support of Y_+; so, for any $k \in \mathbb{N}$, B is a lower bound for the self-intersection of an irreducible curve in the support of

$$((\sigma^*)^k + ((\sigma^{-1})^*)^k)Y_+.$$

If an irreducible curve has negative intersection with an effective divisor, then the curve must be in the support of the divisor; thus, for any $k \in \mathbb{N}$,

$$(((\sigma^*)^k + ((\sigma^{-1})^*)^k)((L_+)^{\otimes q})$$

has non-negative intersection with every irreducible curve on X with either non-negative self-intersection or self-intersection less than B. Let $\mathcal{C}$ be the set of all irreducible curves on X with self-intersection at least B and at most -1. Let w be the limit of the sequence

$$\{\lambda^{-k}((\sigma^*)^k + ((\sigma^{-1})^*)^k)\,\mathrm{c}_1((L_+)^{\otimes q})\}_{k\in\mathbb{N}};$$

so $w = ae_+ + be_-$ for some positive numbers a and b, and hence has positive self-intersection and non-negative intersection with every curve on X. Let $w' = ae_+ - be_-$, let $m \in \mathbb{N}$ be the dimension of $\mathrm{NS}(X) \otimes \mathbb{R}$, and let $\{w_3, \ldots, w_m\}$ be a basis for $E^\perp$ in $\mathrm{NS}(X) \otimes \mathbb{R}$; so $\{w, w'\}$ is a basis for E, $\{w, w', w_3, \ldots, w_m\}$ is a basis for $\mathrm{NS}(X) \otimes \mathbb{R}$, and $\{w', w_3, \ldots, w_m\}$ is a basis for $< w >^\perp$ in $\mathrm{NS}(X) \otimes \mathbb{R}$. Thus, since the signature of $\mathrm{NS}(X) \otimes \mathbb{R}$ is $(1, m-1)$, the set

$$\{v \in \mathrm{NS}(X) \otimes \mathbb{R} | v^2 = K, 0 \leq v.w \leq 1\}$$

is homeomorphic to the set

$$\{\vec{x} \in \mathbb{R}^{m-1} | -K \leq ||\vec{x}||^2 \leq 1 - K\},$$

and hence is compact, for each $K \in \{B, \ldots, -1\}$. So, in particular, the set of all curves in $\mathcal{C}$ intersecting w with value one or less is finite. Let $\mathcal{C}_0$ be the set of curves in $\mathcal{C}$ that have intersection zero with w; then every curve in $\mathcal{C}_0$ is periodic, and hence must have intersection zero with every element of $\mathrm{NS}'(X)$ as well. Let $\epsilon > 0$ be the minimum value of the intersection of a curve in $\mathcal{C} - \mathcal{C}_0$ with w. Suppose that $\{C_l\}_{l\in\mathbb{N}}$ is a sequence of curves in $\mathcal{C} - \mathcal{C}_0$; then the sequence

$$\{y_l = (1/(\mathrm{c}_1([C_l]).w))\,\mathrm{c}_1([C_l])\}_{l\in\mathbb{N}}$$

is contained in the compact set

$$\{v \in \mathrm{NS}(X) \otimes \mathbb{R} | v.w = 1, B/\epsilon^2 \leq v^2 \leq 0\},$$

and hence must have a subsequence converging to some element $y \in \mathrm{NS}(X) \otimes \mathbb{R}$. If there were also a sequence $\{x_l\}_{l\in\mathbb{N}}$ of elements in $\mathrm{NS}(X) \otimes \mathbb{R}$ converging to w such that $\mathrm{c}_1([C_l]).x_l \leq \epsilon/2$ for every l, then it would follow that $y.w = 1$ while $y_l.x_l \leq 1/2$ for every l–which cannot happen. So there is an open neighborhood U of w in $\mathrm{NS}(X) \otimes \mathbb{R}$ such that every element of U intersects every curve in $\mathcal{C} - \mathcal{C}_0$ with value greater than $\epsilon/2$, and thus there is some positive integer k' such that

$$(((\sigma^*)^{k'} + ((\sigma^{-1})^*)^{k'})L_+).[C] > \frac{\lambda^{k'}\epsilon}{2}\,\forall\, C \in \mathcal{C} - \mathcal{C}_0.$$

So $((\sigma^*)^{k'} + ((\sigma^{-1})^*)^{k'})L_+$ is a nef and big line bundle that is mapped into $\mathrm{Pic}^0(X)$ by $S(\sigma^*)$. $\square$

3.5 Proofs of Theorems 1.1, 1.2, and 1.3

Proof of Theorem 1.1: By Proposition 3.1, case 1 implies case 3; by Propositions 3.2 and 3.3, case 3 implies case 2; and it is evident that case 2 implies case 1. Proposition 3.6 then shows that any of case 1, case 2, or case 3 imply case 4; and it is evident that case 4 implies case 1. $\square$

Proof of Theorem 1.2: Proposition 3.5 is one direction; the other direction follows immediately from the discussion preceding Proposition 3.5. □

Proof of Theorem 1.3: Let A be the set of eigenvalues (not counting multiplicity) for σ^* on $H^{1,1}(X)_{\mathbb{R}}$, and let A' be the set of eigenvalues for σ^* on $(H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$. There is a non-trivial class $w \in H^{2,0}(X)$ with $S(\sigma^*)w = 0$ if and only if A' contains a root of $S(t)$; in this case, w is an eigenvector corresponding to a root of $S(t)$.

Suppose that X is non-projective, and that the entropy of σ is $\log(\lambda)$. By the proof of Theorem 2.4, $S(t)$ is the only non-cyclotomic irreducible factor in the characteristic polynomial for σ^* on $H^2(X, \mathbb{Z})$. So there is a subspace $W \subseteq H^2(X, \mathbb{Q})$ such that W is invariant under σ^* and the characteristic polynomial for the action of σ^* on W is $S(t)$–and $W \otimes \mathbb{C}$ is precisely the set of classes in $H^2(X, \mathbb{C})$ that are annihilated by $S(\sigma^*)$. If every root of $S(t)$ were contained in A, then W would be contained in $H^{1,1}(X)_{\mathbb{R}}$ and it would follow from Theorem 3.1 that X is projective–which is a contradiction. So case 2 implies case 1.

Suppose that A' contains a root of $S(t)$. By the proof of Theorem 2.4, the entropy of σ is $\log(\lambda)$ and each Galois conjugate of λ has multiplicity one as an eigenvalue of σ^* on $H^2(X, \mathbb{C})$. If X were projective, then it would follow from Proposition 3.2 that every eigenspace corresponding to a Galois conjugate of λ is contained in $H^{1,1}(X)$–which is a contradiction. So case 1 implies case 2. □

CHAPTER 4

MEASURES OF MAXIMAL ENTROPY

In this chapter, we discuss various means of characterizing the unique measure of maximal entropy for a compact Kähler surface automorphism with positive entropy. We present background material on currents and measures in §4.1, and we discuss some facts about the supports of measures of maximal entropy for positive-entropy automorphisms in §4.2. We prove Theorem 1.5 in §4.3, and we present some results specific to two-dimensional complex torus automorphisms in §4.4.

We will make implicit use of the following fact (which follows from the definition of measure-theoretic entropy): if X is a Borel space, σ is an automorphism of X, and $A \subseteq X$ is a Borel measurable set that is invariant under σ, then

$$h_\mu(\sigma) = \begin{cases} h_{(\mu|_{X-A})}(\sigma|_{X-A}) & (\mu(A) = 0) \\ \mu(A) h_{[(1/\mu(A))\cdot \mu|_A]}(\sigma|_A) + \mu(X-A) h_{[(1/\mu(X-A))\cdot \mu|_{X-A}]}(\sigma|_{X-A}) & (0 < \mu(A) < 1) \\ h_{(\mu|_A)}(\sigma|_A) & (\mu(A) = 1) \end{cases}$$

for any σ-invariant Borel probability measure μ on X.

Let X be a connected compact Kähler surface, and let σ be an automorphism of X.

4.1 Positive Currents

An r-current on X is a real-valued linear functional on $\Omega^{4-r}(X)_{\mathbb{R}}$ that is continuous with respect to the topology of uniform convergence. The space of currents on X is

$$W(X) = \bigoplus_{0 \leq r \leq 4} W^r(X),$$

where each $W^r(X)$ is the space of r-currents on X. (See also (42), §I.1, and (17), §I.11.) The exterior derivative d on differential forms induces a map on currents: if T is an r-current, then dT is the $r+1$ current given by, for $\eta \in \Omega^{3-r}(X)$,

$$dT(\eta) = (-1)^{r+1} T(d\eta).$$

Since $d^2 = 0$ as a map on currents, d defines a cochain complex of spaces of currents, which gives rise to the cohomology groups $H^r_W(X)$. For any $\omega \in \Omega^r(X)_{\mathbb{R}}$, let T_ω be the r-current given by, for $\eta \in \Omega^{4-r}(X)_{\mathbb{R}}$,

$$T_\omega(\eta) = \int_X \omega \wedge \eta;$$

then the map from $\Omega(X)_{\mathbb{R}}$ to $W(X)$ given by $\omega \mapsto T_\omega$ commutes with d and induces isomorphisms from the De Rham cohomology groups $H^r(X, \mathbb{R})$ to the groups $H^r_W(X)$. Any d-closed r-current on X descends to a linear functional on $H^{4-r}(X, \mathbb{R})$ (since it is trivial on d-exact forms in $\Omega^{4-r}(X)_{\mathbb{R}}$); two d-closed r-currents represent the same cohomology class in $H^r_W(X)$ if and only if they give the same linear functional on $H^{4-r}(X, \mathbb{R})$. For a d-closed current

$T \in W^2(X)$ and a d-closed form $\omega \in \Omega^2(X)_{\mathbb{R}}$, the intersection pairing on $H^2(X,\mathbb{R})$ is given by $[T].[\omega] = T(\omega)$. The push-forward by σ of an r-current T is the r-current $\sigma_* T$ given by, for $\eta \in \Omega^{4-r}(X)_{\mathbb{R}}$, $\sigma_* T(\eta) = T(\sigma^* \eta)$; the pull-back of an r-current T by σ is the r-current $\sigma^* T = (\sigma^{-1})_* T$. The pull-back map σ^* commutes with the map from $\Omega(X)_{\mathbb{R}}$ to $W(X)$ given by $\omega \mapsto T_\omega$.

A $(1,1)$-current on X is a 2-current which takes non-zero values only on elements of $\Omega^{1,1}(X)_{\mathbb{R}}$; since every element of $\Omega^2(X)_{\mathbb{R}}$ decomposes uniquely as the sum of a real $(1,1)$-form and a form in $(\Omega^{2,0}(X) \oplus \Omega^{0,2}(X))_{\mathbb{R}}$, every 2-current on X decomposes uniquely as the sum of a $(1,1)$-current and a 2-current whose restriction to $\Omega^{1,1}(X)_{\mathbb{R}}$ is trivial. (See also (3), §5.1, and (20), §3.2.) From the formula for the intersection pairing, it follows that the cohomology class of any d-closed $(1,1)$-current must be a class in $H^{1,1}(X)_{\mathbb{R}}$. Also, any current T_ω associated to a real $(1,1)$-form ω on X is necessarily a $(1,1)$-current; thus every class in $H^{1,1}(X)_{\mathbb{R}}$ is represented by some d-closed $(1,1)$-current. A $(1,1)$-current T is positive if it satisfies $T(\eta) \geq 0$ for any semi-positive $\eta \in \Omega^{1,1}(X)_{\mathbb{R}}$.

Proposition 4.1 *Let X be a connected compact Kähler surface. Then the trivial current is the unique positive current on X representing the trivial class in $H^{1,1}(X)_{\mathbb{R}}$.*

Proof: Suppose that T is a positive current representing the trivial class in $H^{1,1}(X)_{\mathbb{R}}$. If T were non-trivial, then there would be some (non-closed) form $\eta \in \Omega^{1,1}(X)_{\mathbb{R}}$ such that $T(\eta) < 0$; but then there would be some Kähler form $\kappa \in \Omega^{1,1}(X)_{\mathbb{R}}$ such that

$$T(\eta + \kappa) = T(\eta) < 0$$

with $\eta + \kappa$ a positive form, which cannot happen. □

Any 4-current on X extends to a linear functional on the space of continuous real-valued functions on X, and hence also to a linear functional on the space of non-negative (and non-infinite) Borel measurable functions on X; thus, in particular, a 4-current on X that is positive on non-negative functions is the same thing as a finite Borel measure on X (and vice versa). The wedge product of a positive $(1,1)$-current T with a semi-positive $(1,1)$-form ω is the measure $T \wedge T_\omega$ given by, for a smooth function $\phi \in \Omega^0(X)_{\mathbb{R}}$,

$$(T \wedge T_\omega)(\phi) = T(\phi\omega);$$

if T' is a positive $(1,1)$-current that is a weak limit of semi-positive $(1,1)$-forms, then the wedge product $T \wedge T'$ is the weak limit of the wedge products of T with the semi-positive forms. The following result by Cantat shows how the action of σ^* on $W(X)$ gives rise to a unique measure of maximal entropy for σ.

Theorem 4.2 ((2) and (3)) *Let X be a connected compact Kähler surface, and suppose that σ is an automorphism of X with entropy $\log(\lambda) > 0$. Then there are positive closed $(1,1)$-currents T_+ and T_- on X with the following properties:*

1) *$\sigma^* T_+ = \lambda T_+$ and $\sigma_* T_- = \lambda T_-$;*

2) *T_+ and T_- are the unique positive $(1,1)$-currents representing their respective cohomology classes;*

3) If T is another positive d-closed $(1,1)$-current on X, then the sequences

$$\{\lambda^{-k}(\sigma^k)^*T\}_{k\in\mathbb{N}} \; and \; \{\lambda^{-k}(\sigma^k)_*T\}_{k\in\mathbb{N}}$$

converge weakly to $([T_-].[T])T_+$ and $([T_+].[T])T_-$, respectively; and

4) $T_+ \wedge T_-$ is the unique measure of maximal entropy for σ.

Since the currents T_+ and T_- in Theorem 4.1 are, respectively, dilated and contracted by σ^*, their wedge self-products must both be trivial. Thus, for any positive real numbers a and b, the current $aT_+ + bT_-$ has wedge self-product

$$a^2(T_+ \wedge T_+) + 2ab(T_+ \wedge T_-) + b^2(T_- \wedge T_-) = 2ab(T_+ \wedge T_-),$$

which is a positive multiple of the measure of maximal entropy for σ. Moreover,

$$\sigma^*(aT_+ + bT_-) + (\sigma^{-1})^*(aT_+ + bT_-) = (\lambda + \lambda^{-1})(aT_+ + bT_-)$$

for any a and b. The cohomology classes $[T_+]$ and $[T_-]$ are necessarily eigenvectors for the action of σ^* on $H^{1,1}(X)_{\mathbb{R}}$ (corresponding to the eigenvalues λ and λ^{-1}, respectively) and are necessarily contained in $\overline{C_K(X)}$; also, $[T_+].[T_-] = 1$ (since $T_+ \wedge T_-$ is a probability measure). The measures of maximal entropy for σ, σ^{-1}, and all iterates of σ or σ^{-1} are necessarily the same.

Proposition 4.3 *Let X be a smooth complex projective surface, and suppose that σ is an automorphism of X with entropy $\log(\lambda) > 0$. Let L be a nef and big line bundle on X, and suppose that $\mathrm{c}_1(L)$ contains some semi-positive form ω_0. Then the inductively defined sequence*

$$\{\omega_n = (\lambda + \lambda^{-1})^{-1}(\sigma^*\omega_{n-1} + (\sigma^{-1})^*\omega_{n-1})\}_{n\in\mathbb{N}}$$

converges weakly to a current T with the property that $T \wedge T$ is some positive scaling of the measure of maximal entropy for σ.

Proof: Since ω_0 is semi-positive and d-closed, the current T_{ω_0} is positive and d-closed; also, $\mathrm{c}_1(L)$ and $[T_{\omega_0}]$ are the same cohomology class. Since $[T_+]$ and $[T_-]$ span the eigenspaces corresponding, respectively, to the eigenvalues λ and λ^{-1}, the Chern class of any nef and big line bundle on X must have positive intersection with both $[T_+]$ and $[T_-]$. So, in particular, the sequence

$$\{T_n = \lambda^{-n}((\sigma^n)^*T_{\omega_0} + ((\sigma^{-1})^n)^*T_{\omega_0})\}_{n\in\mathbb{N}}$$

converges weakly to $aT_+ + bT_-$ for some positive real numbers a and b. For any $k \in \mathbb{N}$,

$$T_{\omega_{2k}} = \left(\frac{1}{\lambda+\lambda^{-1}}\right)^{2k} \sum_0^k \binom{2k}{j} \lambda^{2k-2j} T_{2k-2j}$$

(where $T_0 = T_{\omega_0}$) and

$$T_{\omega_{2k-1}} = \left(\frac{1}{\lambda+\lambda^{-1}}\right)^{2k-1} \sum_0^{k-1} \binom{2k-1}{j} \lambda^{2k-1-2j} T_{2k-1-2j};$$

also, for any $m \in \mathbb{N}$,

$$\sum_0^m \binom{m}{j} \lambda^{m-2j} = (\lambda+\lambda^{-1})^m > 2^m = \sum_0^m \binom{m}{j}.$$

(Compare (20), §3.3.2.) Moreover,

$$\lim_{k\to\infty}\left(\frac{1}{\lambda+\lambda^{-1}}\right)^{2k}\binom{2k}{k-l}\lambda^{2l} = \lim_{k\to\infty}\left(\frac{1}{\lambda+\lambda^{-1}}\right)^{2k-1}\binom{2k-1}{k-l}\lambda^{2l-1} = 0$$

for any $l \in \mathbb{N}_0$, and

$$\lim_{m\to\infty}\left(\frac{1}{\lambda+\lambda^{-1}}\right)^{m}\sum_{m/2<j\leq m}\binom{m}{j}\lambda^{m-2j} = 0.$$

Thus it follows that, for any real $(1,1)$-form η, the sequences $\{T_{\omega_{2k}}(\eta)\}_{k\in\mathbb{N}}$ and $\{T_{\omega_{2k-1}}(\eta)\}_{k\in\mathbb{N}}$ both converge to $aT_+(\eta) + bT_-(\eta)$. $\square$

4.2 Periodic Analytic Subsets

Suppose that σ has positive entropy. By definition, the entropy of σ restricted to the orbit of any periodic point must be zero. Suppose that C is an irreducible curve on X that is periodic for σ; so C is a fixed curve for σ^k for some $k \in \mathbb{N}$. A finite sequence of blow-ups of singular points of C yields a surface X' in which C', the strict transform of C, is non-singular. (See (17), §II.7.) Since σ^k must preserve the set of non-singular points on C, each blown-up point in the construction of X' must be periodic for σ^k; so σ^k extends to an automorphism of X', which in turn restricts to an automorphism (with entropy zero) of the non-singular curve C'. Thus the restriction of σ^k to C must have entropy zero. Since the entropy of any iterate of σ restricted to some σ-invariant set is at least the entropy of σ on the set, the entropy of σ restricted to the orbit of any periodic curve must be zero.

Proposition 4.4 *Let X be a compact Kähler surface, and let σ be an automorphism of X. Suppose that σ has positive entropy, and that C is a proper irreducible analytic subset of X that is periodic for σ. Then $\mu_\sigma(C) = 0$.*

Proof: Let $k \in \mathbb{N}$ be the period of C under σ; so the entropy of σ^k restricted to C is zero. Let ν be the measure on X given by, for a Borel set A,

$$\nu(A) = (1 - \mu_\sigma(C))^{-1}\mu_\sigma(A - (A \cap C)).$$

If $\mu_\sigma(C)$ were positive, then ν would be a σ^k-invariant probability measure on X such that the entropy of σ^k with respect to ν is strictly greater than the entropy of σ^k with respect to μ_σ–which cannot exist. □

If the support of μ_σ were contained in some analytic subset of X, then it would be contained in some periodic analytic subset–that is, some finite union of points and irreducible curves on X, each periodic for σ; but then the support of μ_σ would have measure zero, which cannot be the case. Thus the support of μ_σ is Zariski dense in X. If $\mu_\sigma(\{x\})$ were positive for some point $x \in X$, then x would necessarily be a periodic point (since $\mu_\sigma(X) = 1$)–and hence could not have positive measure; thus μ_σ has no atoms on X. The same argument shows that (since μ_σ has no atoms) the measure of any irreducible curve on X must in fact be zero.

For any $k \in \mathbb{N}$, the set of points in X of exact period k for σ is an analytic subset $Z_k \subseteq X$; the set $\mathrm{Per}(\sigma, k)$ of isolated points of exact period k is the complement in Z_k of the union of all of the curves contained in Z_k. (See (3), §4.2.2.) Since there can only be finitely many curves on X that are periodic for σ, there is an upper bound on the values of k for which $\mathrm{Per}(\sigma, k)$ is a proper subset of Z_k. The set of all isolated periodic points for σ is Zariski dense in X. (See (3), §4.4.3.) The following result by Cantat shows that μ_σ is determined by the sets $\mathrm{Per}(\sigma, k)$ if X is projective; $\mathrm{Per}^*(\sigma, n)$ denotes the union of all $\mathrm{Per}(\sigma, k)$ with $1 \leq k \leq n$.

Theorem 4.5 ((2) and (3)) *Let X be a smooth complex projective surface, and suppose that σ is an automorphism of X with positive entropy. Then the isolated periodic points for σ are equidistributed with respect to the measure of maximal entropy for σ, in the sense that the sequence*

$$\left\{ (|\,\mathrm{Per}^*(\sigma, n)|)^{-1} \sum\nolimits_{x \in \mathrm{Per}^*(\sigma, n)} \delta_x \right\}_{n \in \mathbb{N}}$$

converges weakly to μ_σ.

4.3 Semi-Positive Forms and Proof of Theorem 1.5

Suppose that X is projective and that E is an effective divisor on X. The effective divisor class $[E] \in \mathcal{E}(X)$ is base-point free if for any $x \in X$ there is an effective divisor in $[E]$ that does not contain x. A line bundle in $\mathcal{E}(X)$ is globally generated if it is base-point free as an effective divisor class. (See (32), §II.7.)

Theorem 4.6 ((33), Corollary 4.3.19) *Every globally generated line bundle on a smooth complex projective surface has a semi-positive form in its Chern class.*

If a line bundle $L \in \mathrm{Pic}(X)$ has a semi-positive form ω in its Chern class, then the intersection properties of ω guarantee that L is nef. Combined with Theorem 4.6, the following theorem (due to Kawamata) provides a means of finding line bundles whose Chern classes contain semi-positive forms.

Theorem 4.7 ((43), Theorem 2) *Let A be a nef and big divisor on a smooth complex projective surface X, and suppose that $K_X \otimes A$ is nef. Then $(K_X \otimes A)^{\otimes q}$ is globally generated for some $q \in \mathbb{N}$.*

Theorem 4.7 is immediately applicable to surfaces whose canonical bundles are numerically trivial.

Proposition 4.8 *Let X be a smooth complex projective surface, and let σ be an automorphism of X. Suppose that X is not a rational surface, that σ has positive entropy, and that L is a distinguished nef and big line bundle on X. Then there is some $q \in \mathbb{N}$ such that $L^{\otimes q}$ has a semi-positive form in its Chern class.*

Proof: Since X is not a rational surface, it must be birational to a K3 surface, an Enriques surface, or an abelian surface. If X is a K3 surface or an abelian surface, then K_X is trivial; so

$$K_X \otimes L = L$$

is nef (and big), and $L^{\otimes q}$ is globally generated for some $q \in \mathbb{N}$. (See (17), §VI.1.) If X is an Enriques surface, then $K_X \otimes K_X$ is trivial and K_X must have zero intersection with every line bundle on X; so $K_X \otimes L$ is nef (and big), and

$$(K_X \otimes L)^{\otimes q} = K_X^{\otimes (q \bmod 2)} \otimes L^{\otimes q}$$

is globally generated for some $q \in \mathbb{N}$–from which it follows that $L^{\otimes 2q}$ is globally generated. (See (17), §VIII.15.)

More generally (if X is not necessarily minimal), let $\mathcal{F}$ be the set of exceptional curves on X; so $\mathcal{F}$ is a finite set, and every $E \in \mathcal{F}$ is periodic for σ. Let X' be the surface obtained from X by contraction of all curves in $\mathcal{F}$; so the Picard group of X is given by

$$\mathrm{Pic}(X) \cong \mathrm{Pic}(X') \times \left(\bigoplus_{E \in \mathcal{F}} <[E]> \right).$$

Since every $E \in \mathcal{F}$ must have zero intersection with L, L can be expressed as

$$L = (L', 0) \in \mathrm{Pic}(X),$$

for some nef and big $L' \in \mathrm{Pic}(X')$. Since X' is a K3 surface, an Enriques surface, or an abelian surface, $(L')^{\otimes q}$ is base-point free (as an effector divisor class) for some $q \in \mathbb{N}$. For any $x \in X$, let x' be the image of x in X'; then there is some effective divisor D_x representing $(L')^{\otimes q}$ that does not contain x'. Thus, for any $x \in X$, the proper transform of D_x in X is an effective divisor representing $L^{\otimes q}$ that does not contain x; so $L^{\otimes q}$ is globally generated. □

Proof of Theorem 1.5: By Proposition 3.6, there is a nef and big distinguished line bundle $L \in \mathrm{Pic}(X)$; by Proposition 4.8, some multiple $L^{\otimes q}$ is a distinguished nef and big line bundle whose Chern class contains a semi-positive form. The weak convergence of the sequence and the properties of the limit are given by Proposition 4.3. □

4.4 Periodic Points on Tori

Suppose that X is a two-dimensional complex torus, and that σ has positive entropy. Since $H^*(X, \mathbb{Z})$ is generated by $H^1(X, \mathbb{Z})$, the Lefschetz number for σ is

$$\sum (-1)^j \operatorname{Tr}(\sigma^* : H^j(X, \mathbb{Z}) \to H^j(X, \mathbb{Z})) = (1 - \gamma_1)(1 - \gamma_2)(1 - \overline{\gamma_1})(1 - \overline{\gamma_2}) \neq 0,$$

where γ_1 and γ_2 are the eigenvalues for σ^* on $H^{1,0}(X)$; so the Lefschetz fixed point theorem guarantees that there is some $x_0 \in X$ such that $\sigma(x_0) = x_0$. (See also (44), §2.1.) Thus σ is a self-isogeny (i.e., a surjective endomorphism that preserves a group structure) of X if X is a given a group structure in which x_0 is the identity element.

Since K_X is trivial, it follows from the adjunction formula that any irreducible curve C on X with $C^2 < 0$ would necessarily be a smooth rational curve; however, a complex torus cannot contain a smooth rational curve (since the embedding of the projective line into the torus would lift to a non-trivial map from the projective line to affine space, which cannot exist). (See (17), II.11, and (40), §3.) Thus no curve on X can be periodic for σ, and every periodic point for σ must be isolated.

Proposition 4.9 *Let X be a two-dimensional complex torus, and suppose that σ is an invertible self-isogeny of X with positive entropy. Then a point $x \in X$ is periodic for σ if and only if x is a torsion point.*

Proof: For any $m \in \mathbb{N}$,

$$\{x \in X | mx = 0\}$$

is finite and preserved by σ; so every torsion point on X is periodic for σ.

Suppose now that there is a non-torsion point $x \in X$ such that $\sigma^n(x) = x$ for some $n \in \mathbb{N}$. Then

$$\{mx | m \in \mathbb{Z}\}$$

is an infinite set of periodic points for σ with period at most n. However, since no curve on X is periodic for σ, $Z_n^* = \mathrm{Per}^*(\sigma, n)$ contains only finitely many points, which is a contradiction.

□

Since σ has positive entropy, both γ_1 and γ_2 have magnitude different from one (so σ is a hyperbolic automorphism when X is viewed as a real four-dimensional torus). It follows that μ_σ is the Haar measure on X, which has full support. (See (45), §III, and (46).) Theorem 4.5 and Proposition 4.9 give an alternate proof of this fact when X is an abelian surface: if $x \in X$ is a torsion point, then there is some $k \in \mathbb{N}$ such that $\sigma^k(x) = x$; so $\mathrm{Per}^*(\sigma^k, n)$ is invariant under translation by x for any $n \in \mathbb{N}$, and therefore $\mu_{\sigma^k} = \mu_\sigma$ is invariant under translation by x as well; thus $y + x$ is contained in $\mathrm{Supp}(\mu_\sigma)$ for any $y \in \mathrm{Supp}(\mu_\sigma)$ and any torsion point $x \in X$, so that $\mathrm{Supp}(\mu_\sigma)$ contains a dense subset of (and is therefore equal to) X.

CHAPTER 5

SYNTHESIS OF TORUS AUTOMORPHISMS

In this chapter, we characterize the possible values of positive entropy for two-dimensional complex torus automorphisms, and we describe the tori on which the possible values of positive entropy occur. We present the process of synthesis in §5.1, and we prove Theorem 1.6 in §5.2. In §5.3, we introduce the notion of a reorientation of a two-dimensional complex torus by a positive-entropy automorphism, and we use this notion to prove Theorem 1.8. In §5.4, we present several detailed examples of two-dimensional complex torus automorphisms with positive entropy. Finally, in §5.5, we briefly describe the full automorphism groups for certain two-dimensional complex tori and we prove Theorem 1.9.

Let X be a two-dimensional complex torus, and suppose that σ is an automorphism of X with positive entropy; so X is a quotient $\mathbb{C}^2/\Lambda$ such that σ is the quotient of a map $F \in \mathrm{GL}_2(\mathbb{C})$ satisfying $F(\Lambda) = \Lambda$. Since $H^{1,0}(X)$ is spanned by dz_1 and dz_2 (for a choice of coordinates z_1 and z_2 on $\mathbb{C}^2$), it follows that $\sigma^* = F^T$ on $H^{1,0}(X)$.

Now let $\tilde{X}$ be the Kummer surface associated to X. For any translation τ on X, the quotient of X by $\tau \circ i \circ \tau^{-1}$ (where i is the involution coming from multiplication by -1 on $\mathbb{C}^2$) is isomorphic to X/i; so the construction of $\tilde{X}$ does not depend on the choice of the group structure on X. In particular, σ descends to an automorphism $\tilde{\sigma}$ of $\tilde{X}$ (since it commutes with

i when it is taken to be a group homomorphism). The blow-up of X/i at each of its sixteen singular points gives a rational curve on $\tilde{X}$ with self-intersection -2; so

$$H^{1,1}(\tilde{X}) = H^{1,1}(X) \oplus \left(\bigoplus_1^{16} <[C_j]>\right).$$

Moreover, since each C_j is periodic for $\tilde{\sigma}$, the entropy of σ and $\tilde{\sigma}$ are the same. (See also (7), §4.) If $x \in X$ is periodic for σ and not fixed by i, then the image of x in $\tilde{X}$ is periodic for $\tilde{\sigma}$. Conversely, if $\tilde{x} \in \tilde{X}$ is periodic for $\tilde{\sigma}$ and not contained in any C_j, then $\tilde{x}$ is the image of a periodic point for σ.

5.1 Synthetic Constructions

Let γ_1 and γ_2 be the eigenvalues of F. Then the eigenvalues of σ^* on $H^1(X, \mathbb{Z})$ are γ_1, γ_2, $\overline{\gamma_1}$, and $\overline{\gamma_2}$; since σ^* is invertible, it follows that $|\gamma_1\gamma_2| = 1$. The proof of the following theorem is adapted from an argument by McMullen [(7), §4].

Theorem 5.1 *Let $P(t) \in \mathbb{Z}[t]$ be monic of degree four with roots γ_1, γ_2, $\overline{\gamma_1}$, and $\overline{\gamma_2}$ such that $|\gamma_1\gamma_2| = 1$. Then there are a two-dimensional complex torus X and an automorphism σ of X such that $P(t)$ is the characteristic polynomial for σ^* on $H^1(X, \mathbb{Z})$.*

Proof: Let $M \in M_{4\times 4}(\mathbb{Z})$ be a block diagonal matrix such that each block is the companion matrix of a distinct factor of $P(t)$ (counting multiplicity); so the characteristic polynomial for M is $P(t)$, and M is an element of $\mathrm{GL}_4(\mathbb{Z})$ that is diagonalizable over $\mathbb{C}$. Let v_1 and v_2 be eigenvectors in $\mathbb{C}^4$ for M corresponding, respectively, to the eigenvalues γ_1 and γ_2; then $\overline{v_1}$ and $\overline{v_2}$ are eigenvectors for M corresponding, respectively, to $\overline{\gamma_1}$ and $\overline{\gamma_2}$. So

$$M = \begin{pmatrix} \Re\gamma_1 & -\Im\gamma_1 & 0 & 0 \\ \Im\gamma_1 & \Re\gamma_1 & 0 & 0 \\ 0 & 0 & \Re\gamma_2 & -\Im\gamma_2 \\ 0 & 0 & \Im\gamma_2 & \Re\gamma_2 \end{pmatrix}$$

with respect to the real basis

$$\{v_1 + \overline{v_1}, \imath(v_1 - \overline{v_1}), v_2 + \overline{v_2}, \imath(v_2 - \overline{v_2})\}.$$

It follows that

$$\begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}$$

preserves some rank-four $\mathbb{Z}$-lattice in $\mathbb{C}^2$. $\square$

5.2 Positive Values of Entropy and Proof of Theorem 1.6

Since σ has positive entropy, γ_1 and γ_2 can be ordered so that

$$|\gamma_1| > 1 > |\gamma_2|,$$

in which case the entropy of σ is $\log(|\gamma_1|^2)$. Also, the eigenvalues of σ^* on $H^2(X, \mathbb{Z})$ are $|\gamma_1|^2$, $|\gamma_2|^2$, $\gamma_1\gamma_2$, $\gamma_1\overline{\gamma_2}$, $\overline{\gamma_1}\gamma_2$, and $\overline{\gamma_1\gamma_2}$. So, as in the proof of Theorem 2.4, the characteristic polynomial for σ^* on $H^2(X, \mathbb{Z})$ is monic and reciprocal with four roots on the unit circle and two positive real roots off the unit circle.

Proposition 5.2 *Let $Q(t) \in \mathbb{Z}[t]$ be a monic and reciprocal of degree six with four roots on the unit circle and two positive real roots off the unit circle. Then the following two statements are equivalent:*

1) *There are a two-dimensional complex torus X and an automorphism σ of X such that the characteristic polynomial for σ^* on $H^2(X, \mathbb{Z})$ is $Q(t)$; and*

2) *$Q(1) = -m^2$ and $Q(-1) = n^2$ for some integers m and n.*

Proof: Write

$$Q(t) = t^6 + at^5 + bt^4 + ct^3 + bt^2 + at + 1$$

(so a, b, and c are integers).

Suppose that

$$Q(1) = 2 + 2a + b + c = -m^2$$

and

$$Q(-1) = 2 - 2a + b - c = n^2$$

for some integers m and n. Then m and n are either both odd or both even, so that $j = (1/2)(n+m)$ and $k = (1/2)(n-m)$ are both integers. Since $jk = b+1$ and $j^2 + k^2 = -c - 2a$, the roots of $Q(t)$ must be the pairwise products of the distinct roots of

$$P(t) = t^4 + jt^3 - at^2 + kt + 1.$$

The hypotheses on the roots of $Q(t)$ force $P(t)$ to have roots occuring in conjugate pairs. So Theorem 5.1 shows that case 2 implies case 1.

Suppose that σ is an automorphism of a two-dimensional complex torus X such that the characteristic polynomial for σ^* on $H^2(X,\mathbb{Z})$ is $Q(t)$. Then the entropy of σ is the logarithm of the unique real root of $Q(t)$ with magnitude greater than one (which is necessarily a Salem number). Let

$$P(t) = t^4 + jt^3 - at^2 + kt + 1 \in \mathbb{Z}[t]$$

be the characteristic polynomial for σ^* on $H^1(X,\mathbb{Z})$. Then the relationship between the roots of $Q(t)$ and the roots of $P(t)$ gives

$$Q(t) = t^6 + at^5 + (jk-1)t^4 - (j^2+k^2+2a)t^3 + (jk-1)t^2 + at + 1.$$

So, in particular,

$$Q(1) = -(j-k)^2$$

and

$$Q(-1) = (j+k)^2.$$

Thus case 1 implies case 2. $\square$

Proof of Theorem 1.6: Since $h^2(X) = 6$ for any two-dimensional complex torus X, any positive-entropy two-dimensional complex torus automorphism must have as its entropy the logarithm of a Salem number of degree two, four, or six.

Suppose $d = 6$. Then any automorphism σ of a two-dimensional complex torus X whose entropy is $\log(\lambda)$ must have $S(t)$ as the characteristic polynomial for the action of σ^* on $H^2(X, \mathbb{Z})$. So Proposition 5.2 gives case 1.

Suppose $d = 4$. Then any automorphism σ of a two-dimensional complex torus X whose entropy is $\log(\lambda)$ must have $S(t)$ as a factor in the characteristic polynomial for the action of σ^* on $H^2(X, \mathbb{Z})$–so the characteristic polynomial must have the form

$$Q(t) = S(t)(t^2 + at + 1),$$

with $a \in \{-2, -1, 0, 1, 2\}$. If $a = -2$ (resp., $a = 2$), then $Q(t)$ satisfies the conditions of Proposition 5.2 if and only if $S(-1)$ (resp., $-S(1)$) is the square of an integer; if $a = -1$ (resp., $a = 1$), then $Q(t)$ may only satisfy the conditions of Proposition 5.2 if $-S(1)$ (resp., $S(-1)$) is the square of an integer; if $a = 0$, then $Q(t)$ satisfies the conditions of Proposition 5.2 if and only if $S(t)$ satisfies condition 2(c). So Proposition 5.2 gives case 2.

Suppose $d = 2$. Then

$$Q(t) = S(t)(t-1)^2(t+1)^2$$

is monic and reciprocal of degree six with four roots on the unit circle and two positive real roots off the unit circle, and satisfies

$$Q(1) = Q(-1) = 0.$$

So Proposition 5.2 gives case 3. □

5.3 Reorientations and Proof of Theorem 1.8

Choose a basis for $\mathbb{C}^2$ with respect to which

$$F = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}.$$

In this basis, set

$$\Lambda' = \{(z_1, z_2) | (z_1, \overline{z_2}) \in \Lambda\}$$

and

$$F' = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \overline{\gamma_2} \end{pmatrix}.$$

Then $F'(\Lambda') = \Lambda'$, so that F' induces an automorphism σ' of $X' = \mathbb{C}^2/\Lambda'$ with the same entropy as σ; X' is called the reorientation of X by σ, and σ' is called the reorientation of σ. The constructions of X' and σ' are independent of the choice of basis diagonalizing F.

Proposition 5.3 *Let σ be an automorphism of a two-dimensional complex torus X whose entropy is the logarithm of a degree-four Salem number, and let X' be the reorientation of X by σ. Then exactly one of X or X' is projective.*

Proof: Let λ be the Salem number such that $\log(\lambda)$ is the entropy of σ, and let $S(t)$ be minimal polynomial for λ; so the characteristic polynomial for σ^* on $H^2(X, \mathbb{Z})$ has the form

$$Q(t) = S(t)(t^2 + at + 1),$$

with $a \in \{-2, -1, 0, 1, 2\}$. Let γ_1 and γ_2 be the eigenvalues for σ^* on $H^{1,0}(X)$; then exactly one of $\gamma_1\gamma_2$ or $\gamma_1\overline{\gamma_2}$ is a root of unity. Let σ' be the reorientation of σ; so the eigenvalue for σ^* on $H^{2,0}(X)$ is $\gamma_1\gamma_2$, while the eigenvalue for $(\sigma')^*$ on $H^{2,0}(X')$ is $\gamma_1\overline{\gamma_2}$. Thus the statement follows from Theorem 1.3. □

Proof of Theorem 1.8: The statement is a direct consequence of Proposition 5.3 and the fact that a positive-entropy automorphism of a two-dimensional complex torus has the same entropy as its reorientation. □

5.4 Explicit Examples

Example 5.4 *Let λ be a degree-two Salem number; so*

$$q = \lambda + \lambda^{-1}$$

is an integer greater than 2, and the minimal polynomial for λ is

$$S(t) = t^2 - qt + 1.$$

Suppose that $q+2=r^2$ *(resp.,* $q-2=r^2$*) for some integer* r*, and let* $A=(a_{ij})\in \mathrm{GL}_2(\mathbb{Z})$ *be a matrix with determinant* 1 *(resp.,* -1*) and trace* r*; so the eigenvalues of* A *are* $\mathrm{sgn}(r)\sqrt{\lambda}$ *and* $\mathrm{sgn}(r)\sqrt{\lambda}^{-1}$ *(resp.,* $\mathrm{sgn}(r)\sqrt{\lambda}$ *and* $-\mathrm{sgn}(r)\sqrt{\lambda}^{-1}$*). Then any (necessarily projective) two-dimensional complex torus of the form* $E\times E$*, where* E *is an elliptic curve, admits an automorphism* σ *given by*

$$\sigma(e_1,e_2)=(a_{11}e_1+a_{12}e_2,a_{21}e_1+a_{22}e_2)$$

whose entropy is $\log(\lambda)$*. (See also (7), §4, and (40), §3.)*

By the Poincaré reducibility theorem, every abelian surface is either simple or isogenous to $E_1\times E_2$ *for some elliptic curves* E_1 *and* E_2*; moreover, if an abelian surface is isogenous to* $E_1\times E_2$ *for some non-isogenous elliptic curves* E_1 *and* E_2*, then the surface cannot admit an automorphism with positive entropy. (See also (47), §IV.19.) So it follows from Corollary 1.4 that every two-dimensional complex torus that admits an automorphism with entropy* $\log(\lambda)$ *is an abelian surface that is either simple or isogenous to some* $E\times E$ *(with* E *an elliptic curve).*

Let γ_1 *and* γ_2 *be the eigenvalues of* A *(so* $r=\gamma_1+\gamma_2$ *and* $\det(A)=\gamma_1\gamma_2$*), and suppose that* Λ *is a lattice in* $\mathbb{C}^2$ *with a basis of the form*

$$\{(1,1),(\gamma_1,\gamma_2),(z_1,z_2),(\gamma_1 z_1,\gamma_2 z_2)\}$$

(for some complex numbers z_1 *and* z_2*). Since* γ_1 *and* γ_2 *are both roots of* $t^2-rt+\det(A)$*,*

$$\begin{pmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{pmatrix}$$

restricts to

$$\begin{pmatrix} 0 & -\det(A) & 0 & 0 \\ 1 & r & 0 & 0 \\ 0 & 0 & 0 & -\det(A) \\ 0 & 0 & 1 & r \end{pmatrix}$$

on Λ*; so* $\mathbb{C}^2/\Lambda$ *admits an automorphism whose eigenvalues on* $H^{1,0}(\mathbb{C}^2/\Lambda)$ *are* γ_1 *and* γ_2 *(and whose entropy is therefore* $\log(\lambda)$*). Suppose further that* $z_1 = \imath$ *and* $z_2 = \delta\imath$ *for some non-zero* $\delta \in \mathbb{R}$*. The abelian surface* $\mathbb{C}^2/\Lambda$ *contains an elliptic curve if and only if it contains two distinct isogenous elliptic curves, in which case there are elements*

$$(\zeta_1, \zeta_2) = (k_1, k_1) + (k_2\gamma_1, k_2\gamma_2) + (k_3\imath, k_3\delta\imath) \in \Lambda$$

(so each k_j *is an integer) and* $c + d\imath \in \mathbb{C}$ *with* $d \neq 0$ *such that* $(c + d\imath)(\zeta_1, \zeta_2) \in \Lambda$*. If the equations*

$$ck_1 + ck_2\gamma_1 - dk_3 + dk_1\imath + dk_2\gamma_1\imath + ck_3\imath = l_1 + l_2\gamma_1 + l_3\imath + l_4\gamma_1\imath$$

and

$$ck_1 + ck_2\gamma_2 - dk_3\delta + dk_1\imath + dk_2\gamma_2\imath + ck_3\delta\imath = l_1 + l_2\gamma_2 + l_3\delta\imath + l_4\gamma_2\delta\imath$$

have a simultaneous non-trivial solution with $k_1, \ldots, k_3, l_1, \ldots, l_4 \in \mathbb{Z}$ *and* $c, d \in \mathbb{R}$ *(with* $d \neq 0$*), then*

$$dk_3(\delta k_1 + \delta k_2\gamma_1 - k_1 - k_2\gamma_2) = (l_2k_1 - l_1k_2)(\gamma_1 - \gamma_2) = l_4k_3\delta(\gamma_1 - \gamma_2)$$

implies $\delta \in \mathbb{Q}$ *or*

$$\delta(k_1 + k_2\gamma_1) = k_1 + k_2\gamma_2$$

or

$$\delta(l_3 + l_4\gamma_2)(k_1 + k_2\gamma_1) = (l_3 + l_4\gamma_1)(k_1 + k_2\gamma_2).$$

So $\mathbb{C}^2/\Lambda$ *is simple for a generic choice of* δ *(including, for example, any* δ *not contained in* $\mathbb{Q}(\gamma_1)$*).*

Example 5.5 *Let* λ *be a degree-two Salem number that does not satisfy the hypothesis of Example 5.4. Then testing all possible degree-four monic reciprocal polynomials* $C(t)$ *with no roots off the unit circle shows that* $Q(t) = S(t)C(t)$ *can only satisfy the conditions of Proposition 5.2 if* $C(t)$ *has the form*

$$(t^2 + a_1t + 1)(t^2 + a_2t + 1)$$

with $a_1 \neq a_2 \in \{-2, -1, 0, 1, 2\}$*. As in the proof of Theorem 1.6,* $Q(t)$ *satisfies the conditions of Proposition 5.2 for any degree-two Salem polynomial* $S(t)$ *if* $a_1 = -2$ *and* $a_2 = 2$ *(or vice versa).*

Let σ *be an automorphism of an abelian surface* X *whose entropy is* $\log(\lambda)$*; so the eigenvalues of* σ^* *on* $(H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$ *are some* β *and* $\overline{\beta}$ *with* $|\beta| = 1$*. Then* a_1 *and* a_2 *can be ordered so that* $t^2 + a_1t + 1$ *is the characteristic polynomial for* σ^* *on* $(H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$ *and*

$$Q_0(t) = S(t)(t^2 + a_2t + 1)$$

is the characteristic polynomial for σ^* *on* $H^{1,1}(X)_{\mathbb{R}}$*. Since no root of* $Q_0(t)$ *is an eigenvalue for* σ^* *on* $(H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$*, the subspace of* $H^2(X, \mathbb{R})$ *that is annihilated by* $Q_0(\sigma^*)$ *is precisely* $H^{1,1}(X)_{\mathbb{R}}$*. Since* $Q_0(\sigma^*)$ *annihilates a four-dimensional subspace of* $H^2(X, \mathbb{Q})$*, the Picard rank of* X *must be four. It follows that* X *is isogenous to a torus of the form* $E \times E$*, where* E *is an elliptic curve with complex multiplication. (See (48).)*

Let $E = \mathbb{C}/\mathbb{Z}[\sqrt{2-q}]$ *(where* $q = \lambda + \lambda^{-1}$*, as in Example 5.4). Then*

$$\begin{pmatrix} 0 & -1 \\ 1 & (\sqrt{q-2})\imath \end{pmatrix}$$

gives an automorphism of $E \times E$*, as in Example 5.4, whose entropy is* $\log(\lambda)$*.*

Example 5.6 *If* X *is an abelian surface that admits an automorphism whose entropy is the logarithm of a degree-four Salem number, then Theorem 1.1 shows that the Picard rank of* X *is four–so that* X *is isogenous to a torus of the form* $E \times E$ *with complex multiplication, as in Example 5.5. If* X *is a non-projective two-dimensional complex torus that admits an automorphism* σ *whose entropy is the logarithm of a degree-four Salem number* λ*, so that the characteristic polynomial for* σ^* *on* $H^2(X, \mathbb{Z})$ *has the form*

$$S(t)(t^2 + at + 1)$$

with $S(\lambda) = 0$ *and* $a \in \{-2, -1, 0, 1, 2\}$*, then Theorem 1.3 shows that* $t^2 + at + 1$ *is a factor in the characteristic polynomial for* σ^* *on* $H^{1,1}(X)_{\mathbb{R}}$*; it follows, in a fashion similar to Example 5.6, that the Picard rank of* X *is two.*

Let $E = \mathbb{C}/\mathbb{Z}[\sqrt{-D}]$ with $D \in \mathbb{N}$. Then any matrix of the form

$$\begin{pmatrix} 0 & -1 \\ 1 & b_1 + (b_2\sqrt{D})\imath \end{pmatrix},$$

where b_1 and b_2 are integers, gives an automorphism of $E \times E$, as in Examples 5.4 and 5.5; the characteristic polynomial for the action of the automorphism on $\mathrm{NS}(E \times E)$ is

$$t^4 - (b_1^2 + b_2^2 D)t^3 + (2b_1^2 - 2b_2^2 D - 2)t^2 - (b_1^2 + b_2^2 D)t + 1,$$

which is a degree-four Salem polynomial whenever it does not have the form

$$t^4 - pt^3 - (2 \pm 2p)t^2 - pt + 1,\ t^4 - pt^3 + (1 \pm p)t^2 - pt + 1,\ \text{or } t^4 - pt^3 + 2t^2 - pt + 1$$

(for $p \in \mathbb{N}_0$). (See also (49), §5.2.) Let A_{b_1,b_2} be such a matrix, and let σ_{b_1,b_2} be the corresponding automorphism of $E \times E$. Then the eigenvectors of A_{b_1,b_2} are

$$\left(1, \frac{-b_1 - b_2\imath \pm \sqrt{b_1^2 - b_2^2 + 2b_1b_2\imath - 4}}{2}\right),$$

and the reorientation of $E \times E$ by σ_{b_1,b_2} can be given concretely via an explicit change of basis for A_{b_1,b_2}; since any lattice giving $E \times E$ as a quotient of $\mathbb{C}^2$ is invariant under the map that sends (z_1, z_2) to $(\sqrt{D}\imath z_1, \sqrt{D}\imath z_2)$, the lattice giving the reorientation must be invariant under the map that sends (z_1, z_2) to $(\sqrt{D}\imath z_1, -\sqrt{D}\imath z_2)$.

If $\Lambda \subseteq \mathbb{C}^2$ is a lattice that is invariant under the map that sends (z_1, z_2) to $(\sqrt{D}\imath z_1, -\sqrt{D}\imath z_2)$ (with $D \in \mathbb{N}$), so that Λ has a basis of the form

$$\{(u_1, u_2), (v_1, v_2), (\sqrt{D}\imath u_1, -\sqrt{D}\imath u_2), (\sqrt{D}\imath v_1, -\sqrt{D}\imath v_2)\},$$

then there is a form ω on $\mathbb{C}^2/\Lambda$ such that $[\Re\omega]$ and $[\sqrt{D}\Im\omega]$ span a two-dimensional lattice in $\mathrm{NS}(X)$*; in terms of the chosen basis for $\mathbb{C}^2$,*

$$\omega = (u_1\overline{v_2} - v_1\overline{u_2})^{-1} dz_1 \wedge d\overline{z_2}$$

has this property. A torus that can be expressed as the quotient of $\mathbb{C}^2$ by a lattice that is invariant under the map that sends (z_1, z_2) to $(\sqrt{D}\imath z_1, -\sqrt{D}\imath z_2)$ is called a J_D-torus; the fact that (for any $D \in \mathbb{N}$) certain J_D-tori admit automorphisms whose entropies are logarithms of degree-four Salem numbers gives an alternative proof of the previously known result that a generic J_D torus has Picard rank two. (See (50), Appendix B.) The intersection form is negative definite on $\mathrm{NS}(X)$ *for any J_D-torus X with Picard rank two; so, since a two-dimensional complex torus cannot contain a curve with negative self-intersection, a generic J_D-torus (including any J_D-torus that admits an automorphism with positive entropy) has no divisors.*

5.5 Finiteness Results and Proof of Theorem 1.9

Suppose now that X is non-projective. Fujiki [(51), §5] asserts that in this case the invertible self-isogeny group of X is isomorphic to

$$\mathbb{Z} \times (\mathbb{Z}/m\mathbb{Z})$$

for some $m \in \{2, 4, 6\}$–so that, in particular, there is some (infinite-order) invertible self-isogeny σ_0 of X such that the invertible self-isogeny group of X is generated by σ_0 and finitely many finite-order self-isogenies; since any two invertible self-isogenies of X commute with one another and any translation on X induces the identity map on $H^*(X, \mathbb{Z})$, it follows that the set of entropies exhibited by the automorphism group of X is

$$\{k \log(\lambda_0) | k \in \mathbb{N}_0\},$$

where $\log(\lambda_0)$ is the entropy of σ_0. Oguiso (52) proves a related result for K3 surfaces: if $\tilde{X}$ is a non-projective K3 surface that admits a positive-entropy automorphism, then the automorphism group of $\tilde{X}$ maps onto $\mathbb{Z}$ with a finite kernel. Indeed, taking $\tilde{X}$ to be the Kummer surface associated to X and noting that the invertible self-isogeny group of X maps into the automorphism group of $\tilde{X}$ with a kernel of order two leads to an alternative proof of the characterization of the set of entropies exhibited by the automorphism group of X; in this argument, the fact that any finite-order automorphism of X must commute with σ_0 comes from direct testing of the possible finite-order actions on $H^*(X, \mathbb{Z})$. In contrast to the case where X is non-projective, the automorphism group of an abelian surface can be quite complicated–as indicated in Examples 5.4, 5.5, and 5.6; the invertible self-isogeny group of a torus of the form $E \times E$ (where E is an elliptic curve), for example, is isomorphic to $\mathrm{GL}_2(K)$, where K is the self-isogeny ring of E, and the entropy of any invertible self-isogeny of $E \times E$ is the logarithm of the square of the spectral radius of its image in $\mathrm{GL}_2(K)$.

Suppose that σ_1 and σ_2 are positive-entropy automorphisms of, respectively, two-dimensional complex tori X_1 and X_2 such that γ_1 and γ_2 are the eigenvalues of both σ_1^* on $H^{1,0}(X_1)$ and σ_2^* on $H^{1,0}(X_2)$ (so that σ_1 and σ_2 both have entropy $\log(|\gamma_1|^2)$, and σ_1^* and σ_2^* are conjugate in $\mathrm{GL}_2(\mathbb{C})$), and set

$$G = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}.$$

So there are lattices Λ_1 and Λ_2 in $\mathbb{C}^2$ such that $G(\Lambda_1) = \Lambda_1$, $G(\Lambda_2) = \Lambda_2$, $X_1 = \mathbb{C}^2/\Lambda_1$, $X_2 = \mathbb{C}^2/\Lambda_2$, and σ_1 and σ_2 are the quotients of G by, respectively, Λ_1 and Λ_2. If there is an (algebraic) isomorphism $\phi : X_1 \to X_2$ such that $\sigma_2 = \phi \circ \sigma_1 \circ \phi^{-1}$, then there is a matrix $\Phi \in \mathrm{GL}_2(\mathbb{C})$ such that $G \circ \Phi = \Phi \circ G$, $\Phi(\Lambda_1) = \Lambda_2$, and ϕ is the quotient of Φ; so

$$G|_{\Lambda_2} = \Phi|_{\Lambda_1} \circ G|_{\Lambda_1} \circ \Phi^{-1}|_{\Lambda_2}.$$

Suppose now that γ_1 and γ_2 are not real. If there is a matrix $B \in \mathrm{GL}_4(\mathbb{Z})$ such that $G|_{\Lambda_2} = BG|_{\Lambda_1}B^{-1}$, then there are matrices C and D in $\mathrm{GL}_4(\mathbb{R})$ such that both $C^{-1} \circ B \circ G|_{\Lambda_1} \circ B^{-1} \circ C$ and $D^{-1} \circ G|_{\Lambda_1} \circ D$ are equal to

$$\begin{pmatrix} \Re\gamma_1 & -\Im\gamma_1 & 0 & 0 \\ \Im\gamma_1 & \Re\gamma_1 & 0 & 0 \\ 0 & 0 & \Re\gamma_2 & -\Im\gamma_2 \\ 0 & 0 & \Im\gamma_2 & \Re\gamma_2 \end{pmatrix};$$

it follows that $C^{-1} \circ B \circ D$ defines a matrix $\Phi \in \mathrm{GL}_2(\mathbb{C})$ that commutes with G and satisfies $\Phi(\Lambda_1) = \Lambda_2$. So, since each σ_j^* on $H^1(X_j, \mathbb{Z})$ is given by $(G|_{\Lambda_j})^T$, σ_1 and σ_2 are the same automorphism (of the same torus) if and only if σ_1^* on $H^1(X_1, \mathbb{Z})$ and σ_2^* on $H^1(X_2, \mathbb{Z})$ are conjugate in $\mathrm{GL}_4(\mathbb{Z})$; Example 5.4 shows that this statement does not hold when γ_1 and γ_2 are real. The following result by Latimer and MacDuffee shows that the set of $\mathrm{GL}_4(\mathbb{Z})$-conjugacy classes of matrices all having some fixed set of eigenvalues can be quite complicated.

Theorem 5.7 ((53)) *Let $P(t) \in \mathbb{Z}[t]$ be a monic irreducible polynomial of degree r. Then the set of $\mathrm{GL}_r(\mathbb{Z})$-conjugacy classes of matrices with characteristic polynomial $P(t)$ is in bijective correspondence with the ideal class group for $\mathbb{Z}[t]/P(t)$ in $\mathbb{Q}(t)/P(t)$.*

Since the ideal class group is finite for any order in a number field, there are only finitely many $\mathrm{GL}_r(\mathbb{Z})$-conjugacy classes of matrices with characteristic polynomial $P(t)$ in Theorem 5.7. (See, e.g., (54), §6.2.)

Proof of Theorem 1.9: Let $S(t)$ be the minimal polynomial for λ. Since there are only finitely many monic reciprocal polynomials in $\mathbb{Z}[t]$ of degree at most four with no roots off the unit circle, there are only finitely many degree-six monic reciprocal polynomials in $\mathbb{Z}[t]$ with $S(t)$ as a factor and four roots on the unit circle. Let

$$Q(t) = t^6 + at^5 + bt^4 + ct^3 + bt^2 + at + 1$$

be a polynomial in $\mathbb{Z}[t]$ with $S(t)$ as a factor and four roots on the unit circle such that $Q(1) = -m^2$ and $Q(-1) = n^2$ for some integers m and n; then any polynomial of the form $t^4 + \cdots + 1 \in \mathbb{Z}[t]$ with the property that the pairwise products of its distinct roots are the roots of $Q(t)$ must be one of

$$t^4 + jt^3 - at^2 + kt + 1,\ t^4 - jt^3 - at^2 - kt + 1,$$

$$t^4 + kt^3 - at^2 - jt + 1, \text{ or } t^4 - kt^3 - at^2 + jt + 1,$$

where $j = (1/2)(n + m)$ and $k = (1/2)(n - m)$.

Let $P(t)$ be a polynomial of the form $t^4 + \cdots + 1 \in \mathbb{Z}[t]$ such that the roots of $Q(t)$ are the pairwise products of the distinct roots of $P(t)$. Then $P(t)$ is reducible if and only if it has a real root, in which case the multiset of roots of $P(t)$ must be one of

$$\{\sqrt{\lambda}, \sqrt{\lambda}, \sqrt{\lambda}^{-1}, \sqrt{\lambda}^{-1}\},\ \{\sqrt{\lambda}, \sqrt{\lambda}, -\sqrt{\lambda}^{-1}, -\sqrt{\lambda}^{-1}\},$$

$$\{-\sqrt{\lambda}, -\sqrt{\lambda}, \sqrt{\lambda}^{-1}, \sqrt{\lambda}^{-1}\}, \text{ or } \{-\sqrt{\lambda}, -\sqrt{\lambda}, -\sqrt{\lambda}^{-1}, -\sqrt{\lambda}^{-1}\}$$

–so that either $\sqrt{\lambda} + \sqrt{\lambda}^{-1}$ or $\sqrt{\lambda} - \sqrt{\lambda}^{-1}$ is an integer and therefore either $\lambda + 2 + \lambda^{-1}$ or $\lambda - 2 + \lambda^{-1}$ is the square of an integer. So, if case 1 does not hold, then it follows from Theorem 5.7 that there are only finitely many $\mathrm{GL}_4(\mathbb{Z})$-conjugacy classes of matrices in $\mathrm{GL}_4(\mathbb{Z})$ with characteristic polynomial $P(t)$; moreover, given such a conjugacy class and a choice of two roots γ_1 and γ_2 of $P(t)$ with $|\gamma_1\gamma_2| = 1$, there is exactly one two-dimensional complex torus X that admits an automorphism σ such that σ^* on $H^1(X, \mathbb{Z})$ is in the conjugacy class and the eigenvalues for σ^* on $H^{1,0}(X)$ are γ_1 and γ_2.

If σ is an automorphism of a two-dimensional complex torus X with entropy $\log(\lambda)$, then the characteristic polynomial for σ^* on $H^2(X, \mathbb{Z})$ must be some $Q(t)$ as above, and the characteristic polynomial for σ^* on $H^1(X, \mathbb{Z})$ must be some corresponding $P(t)$ as above. $\square$

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VITA

Education

University of Illinois - Chicago, Chicago, IL

- Ph.D in Pure Mathematics, expected 2013
  Advisor: Laura G. DeMarco
    - M.S. in Pure Mathematics, 2010
    - UIC University Fellowship, 2008
    - Amherst College Fellowships, 2009, 2010, 2011

Amherst College, Amherst, MA

- B.A. in Mathematics and Physics, *magna cum laude*, 2004
    - Phi Beta Kappa, 2003
    - The Bassett Physics Prize, 2002
    - Hughes Fellowship, 2001

Papers

2013. Reschke, Paul. *Salem Numbers and Automorphisms of Abelian Surfaces*, in preparation.

2013. Krieger, Holly and Reschke, Paul. *Cohomological Conditions on Endomorphisms of Projective Varieties*, in preparation.

2012. Reschke, Paul. *Distinguished Line Bundles for Complex Surface Automorphisms*, preprint, submitted. arXiv:1205.1074

2012. Reschke, Paul. *Salem Numbers and Automorphisms of Complex Surfaces.* Mathematical Research Letters, 19(2):475–482.

2004. Reschke, Paul. *Rational Periodic and Preperiodic Points of Polynomial Functions.* Submitted to the Department of Mathematics and Computer Science of Amherst College in partial fulfillment of the requirements for the degree of Bachelor of Arts with Honors. Faculty Advisor: Professor Rob L. Benedetto. April 12.

Employment

Lecturer, Teaching Assistant, Grader, Tutor
University of Illinois - Chicago, Chicago, IL
August 2009 - Present

- ◇ Lecturer for Math 210, Calculus III, Fall 2012
- ◇ Lecturer for Math 090, Intermediate Algebra, Summer 2012
- ◇ TA, Spring 2011, Fall 2010, Summer 2010, Fall 2009

Senior Analyst, Analyst, Associate Analyst, Research Associate
NERA Economic Consulting, Boston, MA
Environmental Economics Practice
July 2004 - February 2008

Seminars

UIC Graduate Student Colloquium

- ◇ *Co-Founder, Co-Organizer*, Spring 2012, Fall 2012

Presentations

2013. "Salem Numbers and Abelian Surface Automorphisms." Special Session on Dynamical Systems: Thermodynamic Formalism and Connections with Geometry, AMS Spring Western Sectional Meeting, Boulder, CO. April 14.

2013. "Salem Numbers and Complex Surface Automorphisms." AMS Special Session on Complex Dynamics, AMA/MAA Joint Math Meetings, San Diego, CA. January 9.

2012. "Salem Numbers and Complex Surface Automorphisms." Seminar on Ergodic Theory and Probability, Ohio State University. November 29.

2012. "Salem Numbers and Projective Surface Automorphisms." Seminar on Dynamical Systems, Indiana University-Purdue University Indianapolis. November 16.

2012. "Distinguished Line Bundles for Complex Surface Automorphisms." Seminar on Several Complex Variables, University of Michigan. September 24.

2012. "Distinguished Line Bundles for Complex Surface Automorphisms." Semester Program on Complex and Arithmetic Dynamics, ICERM. April 9.