# Geometry of the Dual Grassmannian

BY

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THESIS

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To my mother,

Rita Marie Abdelkerim,

my biggest advocate and ally.

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# SUMMARY

Linear sections of Grassmannians provide important examples of varieties. The geometry of these linear sections is closely tied to the spaces of Schubert varieties contained in them. In this monograph, we describe the spaces of Schubert varieties contained in hyperplane sections of $G(2,n)$. The group $\mathbb{P}GL(n)$ acts with finitely many orbits on the dual of the Plücker space $\mathbb{P}^*(\bigwedge^2 V)$. The orbits are determined by the singular locus of $H \cap G(2,n)$. For $H$ in each orbit, we describe the spaces of Schubert varieties contained in $H \cap G(2,n)$. We also discuss some generalizations to $G(k,n)$.

# CHAPTER 1

# INTRODUCTION

Armand Borel produced seminal work on the actions of linear algebraic groups which helped place classical algebraic geometry on a more solid foundation (Borel, 2000). In particular, he characterized a type of subgroup $B$ of a Lie group $G$ such that the set of cosets $G/B$ has the structure of a projective algebraic variety. When specializing to the group $GL(n, \mathbb{C})$, the rich combinatorial structure of the geometry of a variety constructed in this way can be expressed concretely in terms of matrices and vector spaces.

Hermann Grassmann wrote one of the first treatises on linear algebra (Grassmann, 1844). He was the first to exhibit the notion of the exterior algebra of a vector space $V$, providing a geometric interpretation. An element of the exterior algebra of $V$ is a formal sum of vector subspaces of $V$, where a *decomposable* element, that is, a homogeneous element of degree $k$ that can be expressed as a single term, corresponds to a $k$-plane.

Let $V$ be a vector space of dimension $n$. In this thesis we examine the orbits of the action of the projective linear group $\mathbb{P}GL(n)$ on $\mathbb{P}^*(\bigwedge^k V)$, the projectivization of the $k$-th exterior power of the dual vector space $V^*$. The points of $\mathbb{P}^*(\bigwedge^k V)$ correspond to hyperplanes of the projective space generated by $k$-vectors $v_{i_1} \wedge \cdots \wedge v_{i_k}$ where $v_{i_j}$ are vectors in $V$.

Julius Plücker gave an embedding of the first Grassmannian that is not a projective space, $G(2, 4)$, into a projective space of dimension 5 and showed that it is a quadric hypersurface. *Plücker coordinates*, also known as *Grassmann coordinates*, are determinants of $k \times k$ minors of a $k \times n$ matrix whose row vectors form a basis for a $k$-plane $\Lambda$ in $V$, *i.e.*, a point $[\Lambda] \in G(k,n)$. It only makes sense for an element $\alpha$ of $\mathbb{P}(\bigwedge^k V)$ to represent a single vector subspace $\Lambda$ if we can write $\alpha$ with only one summand, that is, if $\alpha = v_1 \wedge \cdots \wedge v_k$ for some $v_1, \ldots, v_k \in V$. The minimal number of summands with which you can write an element $\Lambda$ of the $k$-th exterior power of $V$ is $\binom{r}{k}$, where $r$ is the dimension of the subspace $W = \mathrm{Ann}(\Lambda^\perp)$, with $\Lambda^\perp = \{v^* \in V^* \mid i(v^*)(\Lambda) = 0\}$ and $i(v^*) : \bigwedge^k V \to \bigwedge^{k-1} V$ is the contraction operator (Griffiths and Harris, 1978, p. 210). So $\alpha$ is a decomposable element if and only if $\dim W = k$. This condition induces the *Plücker relations* in the Plücker coordinates. It turns out (Griffiths and Harris, 1978; Hodge and Pedoe, 1994; Kleiman and Laksov, 1972; Donagi, 1977) that the ideal of $G(k,n)$ is generated by the $\binom{n}{k+1}$ Plücker relations, which are all quadratic. This in particular shows that the Grassmannian is nondegenerate, meaning that it is not contained in any hyperplane in its Plücker embedding.

As a result, one may ask about the hyperplane sections of $G(k,n)$, in particular which ones are singular and the nature of their singular loci. But since $G(k,n)$ is a smooth variety (it is homogeneous for the action of $\mathbb{P}GL(n)$), the collection of singular hyperplane sections of $G(k,n)$ is the dual variety $G(k,n)^*$ (see (Harris, 1992)). We show that for most values of $k$ and $n$ with $k \leq n/2$, $G(k,n)^*$ is a hypersurface of $\mathbb{P}^*(\bigwedge^k V)$.

Griffiths and Harris (Griffiths and Harris, 1978) outline the algebraic topology of the complex Grassmannian via a decomposition into *Schubert cells.* The closure of a Schubert cell is a *Schubert variety.* The equivalence classes of Schubert varieties generate additively the cohomology groups of the Grassmannian. The multiplicative structure of the cohomology ring is determined by special Schubert classes $\sigma_{\lambda,0,\ldots,0}$, a representative of which is (Coskun, 2010) a Schubert variety of $k$-planes meeting a fixed vector space $F_{n-k-1}$ in dimension at least 1. Kleiman and Laksov (Kleiman and Laksov, 1972), Hodge and Pedoe (Hodge and Pedoe, 1994), and Griffiths and Harris (Griffiths and Harris, 1978) show that the Grassmannian is a smooth rational variety and give combinatorial nomenclature for Schubert varieties. Great care must be taken when interpreting integers as dimensions of vector spaces or of projective spaces. Here we will be as clear as possible in this matter.

We fix a flag $F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n = V$ of vector spaces with $\dim F_i = i$. In (Griffiths and Harris, 1978), a general Schubert variety is defined as follows:

$$\Sigma_{\lambda_1,\ldots,\lambda_k} = \{\Lambda \mid \dim(\Lambda \cap F_{n-k+i-\lambda_i}) \geq i\}.$$

More specifically, we view a Schubert variety as depending on the *partial flag* $F_{n-k+1-\lambda_1} \subset F_{n-k+2-\lambda_2} \subset \cdots \subset F_{n-\lambda_k}$. We will often either write Schubert varieties as $\Sigma(F_{a_1} \subset \cdots \subset F_{a_k})$ or translate to the previous notation via $\Sigma_{n-k+1-a_1,\ldots,n-a_k}$.

We also see in (Kleiman and Laksov, 1972; Hodge and Pedoe, 1994; Griffiths and Harris, 1978) formulas for the degree of the Grassmannian and any of its Schubert varieties; they show that Schubert varieties are irreducible; they provide rigor for the correspondence between multiplication of classes in the cohomology ring $H^*(G(k,n))$ and intersection of representative varieties.

The combinatorics of multiplication in the cohomology ring of the Grassmannian (and more generally, flag varieties and homogeneous spaces) is a rich area of study. Littlewood and Richardson (Littlewood and Richardson, ) first gave the structure constants of the cohomology ring of the Grassmannian from the viewpoint of symmetric functions. Fulton (Fulton, 1997) and Fulton and Harris (Fulton and Harris, 1991) provide an overview of the connection between the geometry and the representation theory viewpoints.

Given a smooth variety $Y \subset \mathbb{P}^r$, we denote by $Y^*$ the locus of hyperplanes $H$ containing the tangent space to a point of $Y$. This is called the *dual variety* to $Y$ and is a subvariety of $\mathbb{P}^{r*}$, the space of hyperplanes in $r$-dimensional projective space (Shafarevich, 1994). Because a point $y$ of a hyperplane section $H \cap Y$ is singular if and only if $H$ contains the tangent space to $Y$ at $y$, the dual variety parameterizes singular hyperplane sections of $Y$. In this thesis we exhibit the geometry of the dual variety $G(k,n)^*$ to the Grassmannian, focusing almost exclusively on the case $k = 2$.

Donagi (Donagi, 1977) shows that $G(2,n)^*$ is a hypersurface of $\mathbb{P}^*(\bigwedge^2 V)$ if $n$ is even and is of codimension 3 otherwise. He uses classical techniques such as group actions, geometric interpretations of linear algebra calculations, and to a small extent automorphism groups. He notes that a hyperplane $H$ corresponds to a skew-symmetric bilinear form $Q_H$ on $V$, which always has even rank, and stratifies $\mathbb{P}^*(\bigwedge^2 V)$ by subsets of $H$ such that rank $H \leq 2j$ since the only projective invariant of a skew-symmetric bilinear form is its rank. Donagi also examines pencils and nets of hyperplane sections of $G(2,n)$. Finally, he states and offers a proof of a result of Segre that says there are four orbits of the action of $\mathbb{P}GL(6)$ on $\mathbb{P}^*(\bigwedge^3 V)$, $\dim V = 6$.

Piontkowski and Van de Ven in (Piontkowski and de Ven, 1999) examine $G(2,n)$ principally from the perspective of automorphism groups. They also show the odd/even result mentioned above. In addition they explore the homogeneity of the automorphism groups of sections by higher codimension linear spaces.

Reinterpreting the results of Donagi, we can classify the singular loci of hyperplane sections of $G(k,n)$ by examining the orbits of the action of $\mathbb{P}GL(n)$ on $\mathbb{P}^*(\bigwedge^k V)$. This is because two hyperplane sections are projectively equivalent iff their singular loci are isomorphic, and the singular locus of a hyperplane section completely depends on the rank of the corresponding skew-symmetric bilinear form.

Linear sections of Grassmannians provide examples that play an important role in many branches of algebraic geometry, including the classification of varieties, derived equivalences and mirror symmetry. For example, general codimension four linear sections of $G(2,5)$ are Del Pezzo surfaces of degree five (see (Coskun, 2006)) and general codimension seven linear sections of $G(2,7)$ are Calabi-Yau threefolds (see (Borisov and Căldăraru, 2009), (Rødland, 2000)). The geometry of a linear section $X$ of a Grassmannian is closely tied to the spaces of Schubert varieties contained in $X$, which provide crucial information about the cohomology and Hodge structure of $X$ (see (Donagi, 1977) and Chapter 6 of (Griffiths and Harris, 1978)). In this work we will describe the spaces of Schubert varieties contained in a hyperplane section of a Grassmannian.

Let $G(k,n)$ denote the Grassmannian parameterizing $k$-dimensional subspaces of a fixed $n$-dimensional vector space $V$. Let $\lambda$ denote a partition whose parts satisfy

$$n-k \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0.$$

When writing a partition, the parts that are equal to zero are often omitted. For many purposes, it is more convenient to group together the parts of $\lambda$ that are equal. We will write $\lambda$ also as $\lambda = (\mu_1^{i_1}, \cdots, \mu_t^{i_t})$ and set $k_s = \sum_{j=1}^{s} i_j$, where $\mu_1 > \mu_2 > \cdots > \mu_t$ and

$$\mu_1 = \lambda_1 = \cdots = \lambda_{k_1}, \mu_2 = \lambda_{k_1+1} = \cdots = \lambda_{k_2}, \ldots, \mu_t = \lambda_{k_{t-1}+1} = \cdots = \lambda_k.$$

Given a partition $\lambda$ and a flag $F_\bullet : F_1 \subset F_2 \subset \cdots \subset F_n = V$, the Schubert variety $\Sigma_\lambda(F_\bullet)$ is defined as

$$\Sigma_\lambda(F_\bullet) = \{[W] \in G(k,n) \mid \dim(W \cap F_{n-k+i-\lambda_i}) \geq i\}. \tag{1.1}$$

We will often abuse notation by dropping the reference to the flag. When we would like to emphasize the flag elements $F_{n-k+i-\lambda_i}$ imposing rank conditions, we will write $\Sigma_\lambda(F_{n-k+1-\lambda_1} \subset \cdots \subset F_{n-\lambda_k})$. The cohomology class $\sigma_\lambda$ of the Schubert variety depends only on the partition $\lambda$ and not on the choice of flag. The Schubert classes $\sigma_\lambda$, as $\lambda$ varies over all allowed partitions, form a $\mathbb{Z}$-basis for the cohomology of $G(k,n)$ (Griffiths and Harris, 1978, §1.5).

The Plücker map embeds the Grassmannian $G(k,n)$ in $\mathbb{P}(\bigwedge^k V)$. Let $H$ be a hyperplane in $\mathbb{P}(\bigwedge^k V)$. Let

$$X(\lambda, H) = \{\Sigma_\lambda(F_\bullet) \mid \Sigma_\lambda(F_\bullet) \subset G(k,n) \cap H\}$$

denote the space of Schubert varieties with class $\sigma_\lambda$ contained in $G(k,n) \cap H$. In the next section, we will see that $X(\lambda, H)$ is a closed algebraic subset of a suitable partial flag variety ($X(\lambda, H)$ may be reducible). The purpose of this thesis is to describe $X(\lambda, H)$ in detail when $k = 2$ and $H$ is arbitrary. We will also discuss some generalizations to larger $k$.

There is a natural incidence correspondence

$$\mathcal{I}(\lambda) = \{(\Sigma_\lambda(F_\bullet), H) \mid \Sigma_\lambda(F_\bullet) \subset H\}$$

parameterizing pairs of a Schubert variety $\Sigma_\lambda(F_\bullet)$ and a hyperplane $H$ in the Plücker space containing $\Sigma_\lambda(F_\bullet)$. Let $\pi_2$ denote the natural projection to $\mathbb{P}^*(\bigwedge^k V)$. The first problem we address is characterizing the image of $\pi_2$. Before stating our theorems, we recall the case of $G(2,4)$.

*Example* 1.0.1 (Spaces of Schubert varieties in $G(2,4)$). The Plücker map embeds $G(2,4)$ in $\mathbb{P}^5$ as a quadric hypersurface $Q$. The image of a Schubert variety $\Sigma_{2,1}$ is a line on $Q$. Conversely, every line on $Q$ is a Schubert variety with class $\sigma_{2,1}$. Therefore, the Fano variety $\mathcal{F}_1(Q)$ parameterizing lines on $Q$ is isomorphic to the flag variety $F(1,3;4)$ (Harris, 1992, §6).

Let $X = G(2,4) \cap H$ be a smooth hyperplane section of $G(2,4)$. Then $X$ is a smooth quadric threefold. The Fano variety $\mathcal{F}_1(X)$ parameterizing lines on $X$ is the orthogonal Grassmannian $OG(2,5)$, which is isomorphic to $\mathbb{P}^3$.

On the other hand, let $Y = G(2,4) \cap \Sigma_1(V_2 \subset V_4)$ be a singular hyperplane section of $G(2,4)$. Then $Y$ is a cone over a smooth quadric surface. The Fano variety $\mathcal{F}_1(Y)$ parameterizing lines on $Y$ has two irreducible components $Z_1$ and $Z_2$. Both $Z_1$ and $Z_2$ are isomorphic to the blow-up of $\mathbb{P}^3$ along a line. The two components $Z_1$ and $Z_2$ intersect exactly along the exceptional divisor of the two blow-ups. The components $Z_1$ and $Z_2$ can be geometrically described as follows. Let $l = \Sigma_{2,1}(F_1 \subset F_3)$ be a line on $G(2,4)$. The line $l$ is contained in $Y$ if all the two-dimensional subspaces parameterized by $l$ intersect $V_2$ defining $\Sigma_1(V_2 \subset V_4)$ non-trivially. There are two possibilities. Either $V_2 \subset F_3$ and $F_1$ is an arbitrary one-dimensional subspace of $F_3$; or $F_3$ is arbitrary and $F_1 = F_3 \cap V_2$. These two possibilities correspond to the two components $Z_1$ and $Z_2$.

The image of a Schubert variety $\Sigma_{1,1}$ or $\Sigma_2$ under the Plücker map is a plane on the quadric hypersurface $Q$. Conversely, every plane on $Q$ is a Schubert variety of the form $\Sigma_{1,1}$ or $\Sigma_2$. These varieties are parameterized by $\mathbb{P}^{3*}$ and $\mathbb{P}^3$, respectively. By the Lefschetz Hyperplane Theorem (Griffiths and Harris, 1978, §1.2), a smooth quadric threefold does not contain any planes. For otherwise the degree of the plane, which is one, would be divisible by the degree of $Q \cap H$, which is two. Therefore, the smooth hyperplane section $X$ of $G(2,4)$ does not contain any Schubert varieties $\Sigma_{(1,1)}$ or $\Sigma_2$. On the other hand, $Y$ is a cone over a quadric surface. Such a threefold has two one-dimensional families of planes both parameterized by $\mathbb{P}^1$. The two components are distinguished by the cohomology class of the planes they parameterize. Hence, the space of Schubert varieties of the type $\Sigma_{1,1}$ or $\Sigma_2$ on $Y$ are both parameterized by $\mathbb{P}^1$. Notice that in these two cases the incidence correspondences $\mathcal{I}(1,1)$ and $\mathcal{I}(2)$ both have dimension $5 = \dim(\mathbb{P}^*(\bigwedge^2 V))$; however, the second projection is not surjective (Harris, 1992, Example 12.5).

In general, $\mathbb{P}GL(n)$ acts with finitely many orbits on $\mathbb{P}^*(\bigwedge^2 V)$ (Donagi, 1977, §2). A hyperplane $H$ in $\mathbb{P}(\bigwedge^2 V)$ may be viewed as a skew-symmetric matrix $Q_H$. The dimension of the kernel of $Q_H$ is the invariant that determines the orbits of $\mathbb{P}GL(n)$ on $\mathbb{P}^*(\bigwedge^2 V)$ (Donagi, 1977, §2). The dense open orbit corresponds to hyperplanes $H$ such that $G(2,n) \cap H$ is smooth. The dual variety $G(2,n)^*$ parameterizing hyperplanes tangent to $G(2,n)$ decomposes into finitely many orbits depending on the singular locus of $H \cap G(2,n)$. For $H \in G(2,n)^*$, the singular locus of $G(2,n) \cap H$ is a Schubert variety of the form $\Sigma_{2r,2r}$ for some $1 \leq r \leq \lfloor \frac{n-2}{2} \rfloor$ (Donagi, 1977, §2). Let $S_r$ denote the locus in $\mathbb{P}^*(\bigwedge^2 V)$ parameterizing hyperplanes $H$ such that the singular locus of $G(2,n) \cap H$ contains a Schubert variety of the form $\Sigma_{2r,2r}$. By convention, we set $S_{\lceil \frac{n-1}{2} \rceil}$ to be $\mathbb{P}^*(\bigwedge^2 V)$. We thus have

$$S_1 \subset S_2 \subset \cdots \subset S_{\lfloor \frac{n-2}{2} \rfloor} \subset S_{\lceil \frac{n-1}{2} \rceil}$$

and the $\mathbb{P}GL(n)$ orbits on $\mathbb{P}^*(\bigwedge^2 V)$ are the locally closed subsets $S_r \setminus S_{r-1}$.

Our first theorem characterizes the image of $\pi_2(\mathcal{I}(\lambda))$ when $k = 2$.

**Theorem 1.0.2.** *Let $\lambda = (a, b)$ be a partition for $G(2, n)$. The image of the map*

$$\pi_2 : \mathcal{I}(a, b) \to \mathbb{P}^*(\bigwedge^2 V)$$

*contains $S_r$ if and only if $\lceil \frac{a+b}{2} \rceil \geq r$. In particular, the map $\pi_2$ is surjective if and only if $\lceil \frac{a+b}{2} \rceil > \frac{n-2}{2}$.*

In particular, if $H \in S_r \setminus S_{r-1}$, then $X((a, b), H)$ is not empty if and only if $\lceil \frac{a+b}{2} \rceil \geq r$. This raises the question of describing $X((a, b), H)$ in cases it is not empty. Our second theorem addresses this question.

Let $Q$ be a skew-symmetric form on an $n$-dimensional vector space. If $Q$ is non-degenerate, then $n = 2r$ has to be even. A linear space $W$ is isotropic with respect to $Q$ if the restriction of $Q$ to $W$ is identically zero. Given a vector space $W$, let $W^{\perp}$ denote the set of vectors $v \in V$ such that $v^T Q w = 0$ for every $w \in W$. If $Q$ is non-degenerate, the variety parameterizing the $k$-dimensional isotropic subspaces of $F_{2r}$ is called *the isotropic Grassmannian $SG(k, 2r)$*. An isotropic subspace of a non-degenerate skew-symmetric form has at most half the dimension, hence $k \leq r$.

**Theorem 1.0.3.** *Let $H$ be a hyperplane in $\mathbb{P}(\bigwedge^2 V)$ such that $[H] \in \mathbb{P}^*(\bigwedge^2 V)$ is contained in the $\mathbb{P}GL(n)$ orbit $S_r \setminus S_{r-1}$. Let $F_{n-2r}$ be the kernel of the corresponding skew-symmetric form $Q_H$. Let $(a, b)$ be a partition for $G(2, n)$ such that $\lceil \frac{a+b}{2} \rceil \geq r$. Let*

$$M = \max(0, n - 1 - a - \min(r, b)) \quad \textit{and} \quad N = \min\left(n - a - 1, n - r - \frac{a + b + 1}{2}\right).$$

1. *Assume that* $a \neq b$. *Then the irreducible components* $Z_j$ *of* $X((a,b),H)$ *are in one-to-one correspondence with integers* $M \leq j \leq N$. *The irreducible component* $Z_j$ *parameterizes pairs* $(V_{n-a-1} \subset V_{n-b})$ *in* $F(n-a-1, n-b; n)$ *such that* $V_{n-a-1}$ *is a* $Q_H$*-isotropic subspace with* $\dim(V_{n-a-1} \cap F_{n-2r}) \geq j$ *and* $V_{n-b}$ *is a linear space* $V_{n-a-1} \subset V_{n-b} \subset V_{n-a-1}^{\perp}$ *with* $\dim(V_{n-b} \cap F_{n-2r}) \geq 2n - 2r - a - b - 1 - j$. *The dimension of* $Z_j$ *is given by*

$$\dim(Z_j) = (a+1-b)(a+b+j-n+1) - j\frac{(4r+3a+3j-3n+4)}{2} + \frac{(n-a-1)(3a+j-n+4)}{2}.$$

2. *Assume that* $a = b$. *Then* $X((a,a),H)$ *parameterizes* $Q_H$*-isotropic subspaces of dimension* $n-a$. *In particular,* $X((a,a),H)$ *is irreducible and*

$$\dim(X((a,a),H)) = \begin{cases} \frac{r^2+r}{2} + (n-a)(a-r) & \text{if } n \geq a+r \\ \frac{(n-a)(3a-n+1)}{2} & \text{if } n < a+r \end{cases}$$

Some special cases of the theorem are worth highlighting for the beauty of the geometry. For example, when $H$ corresponds to a skew-symmetric form of rank exactly $2r$, then $X((r,r),H)$ is isomorphic to the Lagrangian Grassmannian $SG(r,2r)$. If $a+b+1 = 2r$, then $X((a,b),H)$ is isomorphic to the isotropic Grassmannian $SG(b,2r)$. This is the content of Corollary (1.0.5). Finally, if $a+1 \geq 2r$, then the space of Schubert varieties of the form $\Sigma_{a,0}$ contained in

$H \cap G(2,n)$ is isomorphic to the Grassmannian $G(n-a-1, n-2r)$ (Cor 1.0.6). In all of these situations the spaces of Schubert varieties contained in the specified type of hyperplane are irreducible.

Though we focus mainly on hyperplane sections, we show in Corollary 1.0.7 that the largest linear space that can be contained in a general codimension two linear sections of $G(2,n)$ is of dimension $n-3$.

The simplest case to see geometrically is the case where $H$ belongs to $S_1$, the set of hyperplane sections of $G(2,n)$ that are themselves Schubert varieties of the form $\Sigma_1(F_{n-2} \subset F_n)$. Given $a > b > 0$, we demonstrate in Corollary 1.0.8 that $X((a,b),H)$ consists of two irreducible components, each of which is a Schubert variety in the flag variety $F(n-a-1, n-b; n)$.

When $n-2 > k > 2$, $\mathbb{P}GL(n)$ no longer acts with finitely many orbits on $\mathbb{P}^*(\bigwedge^k V)$ except when $k = 3$ and $n = 6$, 7, or 8 (Donagi, 1977, §2). It is, therefore, unrealistic to hope for as complete a classification of the spaces $X(\lambda, H)$. However, $X(\lambda, H)$ can be easily described for $H$ in certain orbits of $\mathbb{P}GL(n)$.

For example, $\mathcal{I}(\lambda)$ surjects onto $\mathbb{P}^*(\bigwedge^k V)$ if either $\lambda = (n-k, \ldots, n-k, i)$ with $i > 0$ or $\lambda_1 = n-k$ and $\lambda_k = n-k-1$. Note that this first type of partition corresponds to a linear Schubert variety of codimension at least $(k-1)(n-k)+1$, and the second type corresponds to a linear Schubert variety when $\lambda_1 = \cdots = \lambda_{k-1} = n-k$. This in particular means that every smooth hyperplane section contains a Schubert variety $\Sigma_\lambda$ for the partitions $\lambda$ described above.

On the other hand, when $n-k > \lambda_1, \ldots, \lambda_{k-1}$ and $\lambda_k = 0$, the second projection of the incidence variety $\mathcal{I}(\lambda)$ is contained in $G(k,n)^*$. Hence no smooth hyperplane section of $G(k,n)$

can contain such a Schubert variety. This and the previous paragraph are the content of Prop 1.0.9.

A Schubert variety of the form $\Sigma_{n-k-1,\ldots,n-k-1,0}(F_k \subset V)$ is special because it depends on the flag element $F_k$ and consists of $k$-dimensional vector subspaces intersecting $F_k$ in dimension at least $k-1$. It follows that the linear span of this particular Schubert variety is precisely the tangent space to $G(k,n)$ at the point $[F_k]$. It turns out that, when considering families of such Schubert varieties in a hyperplane section, we obtain that $\pi_2(\mathcal{I}(\lambda))$ is precisely $G(k,n)^*$, meaning that a general singular hyperplane section contains a Schubert variety whose linear span is the tangent space to a point of $G(k,n)$, whereas no smooth hyperplane section can contain this type of Schubert variety (Cor. 1.0.10).

It is very rare to have an explicit, concrete resolution of singularities of a variety. We obtain such a resolution for the dual of the Grassmannian in its Plücker embedding by considering $\Sigma_\lambda$ such that $\lambda_1 = \cdots = \lambda_{k-1} = n-k-1$ and $\lambda_k = 0$. Let $N = \binom{n}{k} - k(n-k) - 2$, the dimension of the projective space $\mathbb{P}(\bigwedge^k V)$ after conditions have been imposed on it by the projective tangent space to a point of $G(k,n)$. Then the incidence correspondence $\mathcal{I}(\lambda)$ is a $\mathbb{P}^N$-bundle over $G(k,n)$, and $\pi_2$ is a birational map onto $G(k,n)^*$ that gives a resolution of singularities of $G(k,n)^*$.

# CHAPTER 2

# PRELIMINARIES: THE GEOMETRY OF GRASSMANNIANS

In this chapter, we recall some basic facts about Grassmannians and their Schubert varieties in the Plücker embedding that we did not cover in the Introduction. We outline connections between points of intersection of projective tangent spaces and their corresponding vector spaces. We also classify all linear spaces that can be contained in the Grassmannian. For the reader's convenience, we sketch the proofs of some classical facts about $G(2,n)^*$. We refer the reader to (Griffiths and Harris, 1978) and (Harris, 1992) for facts about Grassmannians and Schubert varieties, to (Donagi, 1977) and (Piontkowski and de Ven, 1999) for facts about the dual variety $G(2,n)^*$, and to (Billey and Lakshmibai, 2000), (Lakshmibai and Seshadri, 1984), and (Coskun, 2010) for facts about singularities of Schubert varieties.

## 2.1 Parameter spaces of Schubert varieties.

Although it is standard in the literature to define a Schubert variety by (Equation 1.1), the Schubert variety does not determine the flag. In fact, the Schubert variety does not even determine the elements of the flag $F_{n-k+i-\lambda_i}$ that impose the rank conditions defining the Schubert variety.

For example, $\Sigma_{1,1}(F_2 \subset F_3)$ and $\Sigma_{1,1}(F_2' \subset F_3)$ define the same Schubert variety in $G(2,4)$ for any two $F_2$ and $F_2'$, two-dimensional subspaces contained in $F_3$. Once a two-dimensional subspace $W$ is contained in $F_3$, then $W$ automatically intersects any two-dimensional subspace of $F_3$ non-trivially.

In order to characterize the flags that define the same Schubert variety, it is more convenient to group the repeated parts in the partition $\lambda$. Often in the literature we express $\lambda$ as $\lambda = (\mu_1^{i_1}, \ldots, \mu_r^{i_t})$, where

$$\lambda_1 = \cdots = \lambda_{i_1} = \mu_1, \lambda_{i_1+1} = \cdots \lambda_{i_1+i_2} = \mu_2, \cdots, \lambda_{i_1+\cdots+i_{t-1}+1} = \cdots = \lambda_k = \mu_t$$

and

$$n - k \geq \mu_1 > \mu_2 > \cdots > \mu_t \geq 0.$$

For simplicity, set $k_s = \sum_{j=1}^{s} i_j$. In particular, $k_t = k$. The Schubert variety $\Sigma_\lambda(F_\bullet)$ can equivalently be defined as

$$\Sigma_\lambda(F_\bullet) = \{[W] \in G(k,n) \mid \dim(W \cap F_{n-k+k_j-\mu_j}) \geq k_j \text{ for } 1 \leq j \leq t\}. \tag{2.1}$$

Once $W$ intersects $F_{n-k+k_s-\mu_s}$ in a $k_s$-dimensional subspace, it intersects $F_{n-k+k_s-\mu_s-j}$ in a subspace of dimension at least $k_s - j$. Consequently, the rank conditions in (Equation 2.1) imply all the rank conditions in (Equation 1.1). Conversely, the Schubert variety determines the linear spaces $F_{n-k+k_s-\mu_s}$ for $1 \leq s \leq t$ because only the last entry of a consecutive string of equal entries imposes new rank conditions. Thus we can use the partial flag variety $F(n - k + k_1 -$

$\mu_1, \ldots, n-\mu_t; n)$ as a parameter space for Schubert varieties in $G(k,n)$ with cohomology class $\sigma_\lambda$. The space $X(\lambda, H)$ is then naturally a closed algebraic subset of $F(n-k+k_1-\mu_1, \ldots, n-\mu_t; n)$.

We have a natural incidence correspondence $\mathcal{I}(\lambda)$

$$\begin{array}{ccc} & \mathcal{I}(\lambda) = \{(\Sigma_\lambda(F_\bullet), H) \mid \Sigma_\lambda(F_\bullet) \subset H\} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ F(n-k+k_1-\mu_1, \ldots, n-\mu_t; n) & & \mathbb{P}^*(\bigwedge^k V) \end{array}$$

consisting of pairs of a Schubert variety $\Sigma_\lambda(F_\bullet)$ and a hyperplane containing it. We prove some facts about this incidence correspondence below.

**Proposition 2.1.1.** *The first projection $\pi_1$ realizes $\mathcal{I}(\lambda)$ as a projective bundle over the partial flag variety $F(n-k+k_1-\mu_1, \ldots, n-\mu_t; n)$. The fibers are isomorphic to $\mathbb{P}H^0(I_{\Sigma_\lambda}(1))$, where $I_{\Sigma_\lambda}$ denotes the ideal sheaf of $\Sigma_\lambda$, and are all projective spaces of the same dimension.*

*Proof.* The $\pi_1$-preimage of a point in $F(n-k+k_1-\mu_1, \ldots, n-\mu_t; n)$ is the set of hyperplanes $H$ containing a fixed Schubert variety of the form $\Sigma_\lambda$. Because of the inclusion-reversing correspondence between varieties and ideals, the fiber is precisely the projectivization of the vector space of homogeneous linear polynomials generated by the Plücker coordinates vanishing on $\Sigma_\lambda$. The space of global sections $H^0(\mathcal{I}_{\Sigma_\lambda}(1))$ of the first twist of the ideal sheaf of $\Sigma_\lambda$ has exactly this characterization, so the fiber over $\pi_1$ is $\mathbb{P}(H^0(\mathcal{I}_{\Sigma_\lambda}(1)))$. □

Consequently, $\mathcal{I}(\lambda)$ is irreducible and smooth (Shafarevich, 1994, Theorem I.6.8). Note, however, that the second projection $\pi_2$ is rarely flat and much harder to understand.

## 2.2 The Plücker embedding of the Grassmannian.

The Grassmannian $G(k,n)$ is a smooth, projective variety of dimension $k(n-k)$. The Plücker map embeds $G(k,n)$ into $\mathbb{P}(\bigwedge^k V)$. The image of the Grassmannian under this embedding is the space of totally decomposable wedges.

Let $\lambda$ be an admissible partition for $G(k,n)$ and define $r_j = n-k+j-\lambda_j$. Suppose a Schubert variety $\Sigma_\lambda$ is given by the partial flag $F_{r_1} \subset \cdots \subset F_{r_k}$ and we choose a basis $\{e_i\}$ so that $F_i$ is generated by $e_1, \ldots, e_i$. Then we can determine the equations in Plücker coordinates of $\Sigma_\lambda$ as follows (Kleiman and Laksov, 1972; Hodge and Pedoe, 1994). In Figure 1 if we

$$\begin{array}{llllll} e_1 & \ldots & e_{r_1} & & & \\ e_1 & \ldots & \ldots & e_{r_2} & & \\ \vdots & & & & \ddots & \\ e_1 & \ldots & \ldots & \ldots & \ldots & e_{r_k} \end{array}$$

Figure 1. Rows of basis elements of flag spaces.

choose one vector from each row with no repetitions and take their wedge product, we know that this multivector is contained in the Plücker image of $\Sigma_\lambda$. On the other hand, any multivector $e_{i_1} \wedge \cdots \wedge e_{i_k}$ where any $i_j$ is larger than $r_j$ will not be contained in $\Sigma_\lambda$, hence the Plücker coordinate corresponding to such a multivector vanishes on $\Sigma_\lambda$. We have proved the following important fact.

**Proposition 2.2.1.** *The Plücker coordinates vanishing on the Schubert variety $\Sigma_\lambda$ are precisely those with the multi-indices $(i_1, \ldots, i_k)$ where for at least one $j$, we have that $i_j > r_j$, where $r_j = n - k + j - \lambda_j$. In particular, this means that every Schubert variety is cut out of $G(k, n)$ by (very special) hyperplanes.* □

Specializing to the case $k = 2$, we obtain the following lemma that we will repeatedly use in the sequel.

**Lemma 2.2.2.** *The dimension of the vector space of hyperplanes containing a Schubert variety $\Sigma_{a,b}$ in $G(2, n)$ is given by*

$$h^0(I_{\Sigma_{a,b}}(1)) = \binom{n}{2} - \binom{n-b}{2} + \binom{a-b+1}{2}.$$

*Proof.* There are $\binom{n}{2}$ total elements in a basis of $\mathbb{P}(\bigwedge^2 V)$. We subtract from this the number of Plücker coordinates that do not vanish on $\Sigma_\lambda$, counting this number as follows. Looking at (Figure 2) we see that all of the nonvanishing Plücker coordinates are among those that correspond to the multivectors obtained by choosing two elements in the second row. Within these, the ones that do vanish have multi-indices $(\ell, m)$ where $\ell$ and $m$ are chosen from the last $a - b + 1$ indices appearing in the second row of (Figure 2). Thus there are $\binom{n-b}{2} - \binom{a-b+1}{2}$ Plücker coordinates that do not vanish on $\Sigma_\lambda$, so we obtain the result. □

*Remark* 2.2.3. Note that we can instead simply count the number of vanishing Plücker coordinates as follows. Again looking at (Figure 2), we see that if a vanishing Plücker coordinate involves a vector from the first row, the choice of second index must come from $n-b+1, \ldots, n$.

$$\begin{array}{llllll} e_1 & \dots & e_{n-a-1} & & & \\ e_1 & \dots & e_{n-a-1} & e_{n-a} & \dots & e_{n-b} \end{array}$$

Figure 2. The case $k = 2$ for a Schubert variety of the form $\Sigma_{a,b}$.

In other words, there are $b(n - a - 1)$ such Plücker coordinates. On the other hand, if a coordinate does not involve a choice of index from the first row, it necessarily involves choosing both indices from the $a + 1$ indices $n - a, \dots, n$. Hence the number of vanishing Plücker coordinates on $\Sigma_\lambda$ is $b(n - a - 1) + \binom{a+1}{2}$, which the reader may verify is equal to $\binom{n}{2} - \binom{n-b}{2} + \binom{a-b+1}{2}$

Applying the Theorem on the Dimension of Fibers (Shafarevich, 1994, Theorem I.6.7) to the first projection $\pi_1 : \mathcal{I}(a, b) \to F(n - a - 1, n - b; n)$, we obtain the following corollary.

**Corollary 2.2.4.** *If $a = b$, then the first projection*

$$\pi_1 : \mathcal{I}(a, a) \to F(n - a; n) = G(n - a, n)$$

*exhibits $\mathcal{I}(a, a)$ as a projective space bundle over $G(n - a, n)$ with fibers of dimension*

$$\binom{n}{2} - \binom{n-a}{2} - 1.$$

*In particular, $\mathcal{I}(a, a)$ is irreducible and*

$$\dim(\mathcal{I}(a, a)) = \frac{a(4n - 3a - 1)}{2} - 1.$$

*If $a > b$, then the first projection*

$$\pi_1 : \mathcal{I}(a,b) \to F(n-a-1, n-b; n)$$

*exhibits $\mathcal{I}(a,b)$ as a projective space bundle over $F(n-a-1, n-b; n)$ with fibers of dimension*

$$\binom{n}{2} - \binom{n-b}{2} + \binom{a-b+1}{2} - 1.$$

*In particular, $\mathcal{I}(a,b)$ is irreducible and*

$$\dim(\mathcal{I}(a,b)) = n(a+b+1) - \frac{a^2+3a}{2} - b^2 - 2.$$

□

In the Plücker embedding, the linear subspaces of $G(k,n)$ have a concrete description.

**Lemma 2.2.5.** *A line on $G(k,n)$ corresponds to a family of $k$-dimensional subspaces of $V$ that contain a fixed $(k-1)$-dimensional subspace and are contained in a fixed $(k+1)$-dimensional subspace.*

*Proof.* Fix a basis $\{e_r\}$ of $V$ so that the linear embedding of $\mathbb{P}^1$ into $\mathbb{P}(\bigwedge^k V)$ is given by $[x:y] \mapsto [x:y:0:\cdots:0]$. This choice can be made by projective equivalence: if $[x:y]$ maps to a point with $x$ in the $i$th position, $y$ in the $j$th position, and zeros elsewhere, we can transform the basis of $V$ so that $x$ is in the first position and $y$ is in the second. Then $p_{1,2,\ldots,k-1,k} = x$ and $p_{1,2,\ldots,k-1,k+1} = y$. If $x = 0$ or $y = 0$, then the image is a single wedge product

$e_1 \wedge \cdots \wedge e_{k-1} \wedge e_k$ or $e_1 \wedge \cdots \wedge e_{k-1} \wedge e_{k+1}$. If both $x$ and $y$ are nonzero, then the image of $[x : y]$ corresponds to

$$x(e_1 \wedge \cdots \wedge e_{k-1} \wedge e_k) + y(e_1 \wedge \cdots \wedge e_{k-1} \wedge e_{k+1}) = e_1 \wedge \cdots \wedge e_{k-1} \wedge (xe_k + ye_{k+1})$$

so that every point in the image of $\mathbb{P}^1$ is actually an element of $G(k, n)$. In particular, if all Plücker coordinates except $p_{1,2,\ldots,k-1,k}$ and $p_{1,2,\ldots,k-1,k+1}$ vanish, then the image of $\mathbb{P}^1$ is equal to the Schubert variety given by the partial flag $F_1 \subset F_2 \subset \cdots \subset F_{k-1} \subset F_{k+1}$ where $F_r$ is generated by $e_1, \ldots, e_r$. Conversely, if we begin with $\Sigma_\lambda$ given by this type of partial flag, we can construct a linear embedding of $\mathbb{P}^1$ into $G(k, n)$ whose image is $\Sigma_\lambda$. □

More generally, a linear space of dimension $s$ on $G(k, n)$ corresponds to either (1) a family of $k$-dimensional subspaces that contain a fixed $(k-1)$-dimensional space $F_{k-1}$ and are contained in a fixed $(k+s)$-dimensional subspace $F_{k+s}$; or (2) a family of $k$-dimensional subspaces that are contained in a fixed $(k+1)$-dimensional subspace $F_{k+1}$ and contain a fixed $(k-s)$-dimensional subspace $F_{k-s}$ (Harris, 1992, §6). Case (1) only exists if we have $k + s \leq n$, and this space is linear because given any $(k+1)$-dimensional subspace $G_{k+1}$ contained in $F_{k+s}$ that contains $F_{k-1}$, the Schubert variety of $k$-dimensional subspaces contained in $G_{k+1}$ and containing $F_{k-1}$ lies completely in this family. In other words, every line generated by two points of $\Sigma(F_1 \subset \cdots \subset F_{k-1} \subset F_{k+s})$ is contained in that Schubert variety. Similarly, case (2) only exists if $s \leq k$, and given any $G_{k-1}$ containing $F_{k-s}$ and contained in $F_{k+1}$, the corresponding line is contained in $\Sigma(F_1 \subset \cdots \subset F_{k-s} \subset F_{k-s+2} \subset \cdots \subset F_{k+1})$. We have proved the following, which will be indispensable in the sequel.

**Proposition 2.2.6** (Linear Spaces in the Grassmannian)**.** *A subvariety of $G(k,n)$ is isomorphic to $\mathbb{P}^s$ if and only if it is a Schubert variety of the form $\Sigma_\lambda$, where either $\lambda = (n-k,\dots,n-k, n-k-s)$ or $\lambda = ((n-k)^{k-s}, (n-k-1)^s)$.* □

It is worthwhile to restate this for the case $k = 2$:

**Proposition 2.2.7.** *The linear spaces in $G(2,n)$ are precisely the Schubert varieties of the form $\Sigma_{n-2,i}$ or $\Sigma_{n-3,n-3}$.* □

## 2.3 Singularities of Schubert varieties.

In order to minimize confusion we will denote the point in the Grassmannian $G(k,n)$ corresponding to a $k$-dimensional subspace $W$ by $[W]$.

The tangent space $T_{[W]}G(k,n)$ is naturally isomorphic to $\mathrm{Hom}(W, V/W)$ (Harris, 1992, §16). We denote by $\mathbb{T}_{[W]}G(k,n)$ the projective closure of the tangent space and call it the *projective tangent space* to $G(k,n)$ at the point $[W]$. We will often abbreviate this simply as $\mathbb{T}_{[W]}$. We can explicitly describe the projective tangent space to $G(k,n)$. Choose a basis $e_1,\dots,e_n$ for $V$ so that $W$ is given as the span of the vectors $e_1,\dots,e_k$. Then under the Plücker embedding, the image of $[W]$ is $e_1 \wedge e_2 \wedge \cdots \wedge e_k$. Let $i_1,\dots,i_k$ be a set such that the cardinality of the set $\{i_1,\dots,i_k\} - \{1,2,\dots,k\}$ is at most one. Since we can replace any of the elements $1 \le i \le k$ by one of the elements $k < j \le n$, there are $k(n-k)+1$ such sets. The projective tangent space to $G(k,n)$ at $W$ is spanned by the $k(n-k)+1$ points in $\mathbb{P}(\bigwedge^k V)$ defined by setting all the Plücker coordinates but $p_{i_1,\dots i_k}$ equal to zero (Donagi, 1977, §1.3). To prove this description of the tangent space, observe that the line spanned by $p_{i_1,\dots,i_k}$ and $p_{1,2,\dots,k}$ is contained in the Grassmannian $G(k,n)$. Since the tangent space at $[W]$ contains every line passing through $[W]$, we conclude that the projective tangent space contains the projective space generated by these linearly independent lines. Since they both have dimension $k(n-k)$, we conclude that they are equal. Note that what we are doing here is starting with $e_1 \wedge \cdots e_k$ and assigning to each of $e_1, \dots, e_k$ a choice of $e_{k+1}, \dots, e_n$, which precisely determines an element of $\mathrm{Hom}(W, V/W)$ since the vector space generated by $e_{k+1}, \dots, e_n$ is isomorphic to $V/W$. This is one indication of why it is important that we are working over a field.

Given a partition $\lambda$, a singular partition $\lambda^s$ associated to $\lambda$ is obtained by adding a hook to the partition $\lambda$ (see Figure 3). More explicitly, if $\lambda = (\mu_1^{i_1}, \dots, \mu_t^{i_t})$, then $\lambda^s$ is any of the partitions

$$(\mu_1^{i_1}, \dots, \mu_{u-2}^{i_{u-2}}, (\mu_{u-1}+1)^{i_{u-1}+1}, \mu_u^{i_u-1}, \mu_{u+1}^{i_{u+1}}, \dots, \mu_t^{i_t})$$

provided that they are admissible for $G(k,n)$, where it is understood that if $\mu_{u-1}+1 = \mu_{u-2}$ those parts have to be grouped together. For example, if $(5,3,2,2,1)$ is a partition for $G(5,11)$, then the singular partitions are $(6,6,2,2,1), (5,4,4,2,1)$ and $(5,3,3,3,3)$.

The singular locus of the Schubert variety $\Sigma_\lambda(F_\bullet)$ is the union of $\Sigma_{\lambda^s}(F_\bullet)$ as $\lambda^s$ varies over all allowable singular partitions associated to $\lambda$. In particular, $\Sigma_{a,b}$ in $G(2,n)$ is smooth if and only if $a = n-2$ or $a = b$. Otherwise, the singular locus of $\Sigma_{a,b}(F_{n-1-a} \subset F_{n-b})$ is $\Sigma_{a+1,a+1}(F_{n-2-a} \subset F_{n-1-a})$ (Coskun, 2010).

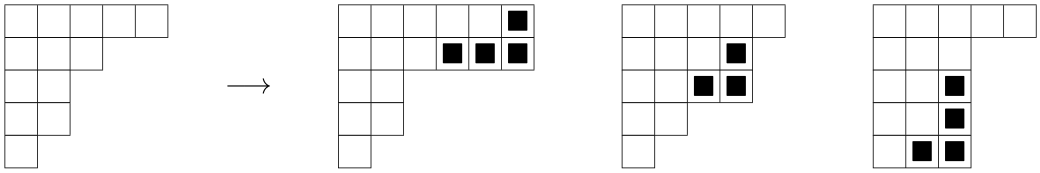

Figure 3. Examples of adding a hook to the Young tableau corresponding to $\Sigma_{5,3,2,2,1}$ in $G(5,11)$.

**Lemma 2.3.1.** *Let $H$ be a hyperplane in $\mathbb{P}(\bigwedge^k V)$. Let $V_1$ be a linear space with* $\dim(V_1) \geq k$ *such that $H \cap G(k,n)$ is singular at every $[W] \in G(k,n)$ such that $W \subset V_1$. Then for any linear space $U$ such that* $\dim(U \cap V_1) \geq k-1$*, $[U] \in G(k,n) \cap H$.*

*Proof.* First, observe that if a line $l$ on $G(k,n)$ intersects the singular locus of $H \cap G(k,n)$, then by Bezout's Theorem (Hartshorne, 1977, I.7.7), $l$ is contained in $H \cap G(k,n)$. For suppose that the intersection $l \cap (H \cap G(k,n))$ is proper. Then there will be precisely $(\deg l)(\deg H)$=1 points in the intersection. Call this point $p$. However, since $l$ meets the singular locus of the hyperplane section, $l \subset \mathbb{T}_p \subset H$. Since we assumed $l \subset G(k,n)$, the intersection cannot be proper and the claim is proved.

Hence, for any $k$-dimensional subspace $U$ that intersects $V_1$ in a subspace of dimension $k-1$, we have $[U] \in H \cap G(k,n)$. This is immediate by assumption if $U \subset V_1$. We may assume that $U \not\subset V_1$. Let $F_{k-1} = U \cap V_1$ and let $W$ be a $k$-dimensional subspace of $V_1$ containing $F_{k-1}$. Then the $k$-dimensional subspaces contained in $Span(U, W)$ and containing $F_{k-1}$ are parameterized by a line $l$ in $G(k,n)$. The line $l$ contains $[W]$ which is a singular point of

$H \cap G(k,n)$ by assumption. Hence $l \subset H \cap G(k,n)$. Since $[U]$ is also a point on $l$, we conclude that $[U] \in H \cap G(k,n)$. This concludes the proof of the lemma. □

**Lemma 2.3.2.** *Let $H$ be a hyperplane in $\mathbb{P}(\bigwedge^2 V)$. Let $V_1, V_2$ be two linear subspaces of $V$ such that $\dim(V_i) \geq 2$. Assume that $H \cap G(2,n)$ is singular along every two-dimensional subspace contained in $V_i$, $1 \leq i \leq 2$. Then $H \cap G(2,n)$ contains every two-dimensional subspace that intersects $Span(V_1, V_2)$ non-trivially and is singular along every two-dimensional subspace that is contained in $Span(V_1, V_2)$.*

*Proof.* Note that in order to prove the lemma, we may replace $V_2$ with a linear space complementary to $V_1 \cap V_2$. This is because the lemma will be all the more true if $V_1$ and $V_2$ intersect. We may, therefore, assume the most general situation, namely $V_1 \cap V_2 = 0$. Next, let $W$ be a two-dimensional subspace that intersects $Span(V_1, V_2)$ in a one-dimensional subspace $F_1$. Then there exists a two-dimensional subspace containing $F_1$ and intersecting both $V_1$ and $V_2$ non-trivially. To construct this two-dimensional subspace $W'$ take the span of the two one-dimensional subspaces $G_1 = V_1 \cap Span(F_1, V_2)$ and $G_1' = V_2 \cap Span(F_1, G_1)$. Let $F_3$ be the three-dimensional subspace spanned by $W$ and $W'$. The two-dimensional subspaces contained in $F_3$ are parameterized by a plane $P$ in $G(2,n)$ (see Proposition 2.2.7). There are two special lines $l_1$ and $l_1'$ on this plane, parameterizing two-dimensional subspaces containing $G_1$, respectively, $G_1'$ and contained in $F_3$. Since each of these two-dimensional spaces intersect $V_1$ or $V_2$ non-trivially, $l$ and $l'$ are contained in $H \cap G(2,n)$. By Bezout's Theorem, we conclude that $P \subset H \cap G(2,n)$, for if $P \cap H$ were a proper intersection in $\mathbb{P}(\bigwedge^2 V)$, then there would only be one line contained in both, but we have just demonstrated that there are two, so the intersection cannot be proper. Therefore, $[W] \in H \cap G(2,n)$. Since $H \cap G(2,n)$ is projective and contains the dense open subset of the Schubert variety of $[W]$ such that $\dim(W \cap Span(V_1, V_2)) = 1$, we conclude that $H \cap G(2,n)$ contains every $[W]$ such that $W \cap Span(V_1, V_2) \neq 0$. This proves the first part of the lemma.

Next, we prove that a hyperplane section of $G(2,n)$ that contains a Schubert variety of the form $\Sigma_{a,0}(F_{n-1+a} \subset F_n)$ is singular along a Schubert variety of the form $\Sigma_{a+1,a+1}(F_{n-2+a} \subset F_{n-1+a})$. This will conclude the proof of the second part of the lemma. Let $v \wedge w$ represent the Plücker point of a two-dimensional subspace contained in $F_{n-1+a}$. Choose coordinates for $V$ so that $F_{n-1+a}$ is spanned by $e_1, \dots, e_{n-1+a}$ with $e_1 = v$ and $e_2 = w$. Then the defining polynomial of a hyperplane containing $\Sigma_{a,0}$ is a linear combination of the Plücker coordinates $p_{i,j}$ with $n-1+a < i < j \leq n$. The tangent space to $G(2,n)$ in its Plücker embedding at the point $e_1 \wedge e_2$ is given by the span of the points $e_1 \wedge e_i$ and $e_2 \wedge e_j$ with $2 \leq i \leq n$ and $3 \leq j \leq n$. All the Plücker coordinates containing $\Sigma_{a,0}$ vanish at all these points spanning the tangent space to the Grassmannian. Hence, all these hyperplanes contain the tangent space at all the points of $\Sigma_{a+1,a+1}$. We conclude that the linear section $H \cap G(2,n)$ is singular along $\Sigma_{a+1,a+1}$. This concludes the proof of the lemma. □

*Remark* 2.3.3. We chose to give this proof because similar arguments can be used for $G(k,n)$. For $G(2,n)$, one can prove the previous lemma using the correspondence between hyperplanes and skew-symmetric forms. By assumption, $V_1$ and $V_2$ are in the kernel of the skew-symmetric form $Q_H$. Therefore, the span of $V_1$ and $V_2$ is also in the kernel. The lemma then follows by observing that $H \cap G(2,n)$ is singular along $[W]$, where $W$ is in the kernel of $Q_H$.

It follows from Lemma 2.3.2 that the singular locus of a hyperplane section $H \cap G(2,n)$ is either empty or a Schubert variety of the form $\Sigma_{a,a}$ parameterizing two-dimensional subspaces contained in a vector space $F_{n-a}$. Simply let $F_{n-a}$ be the span of all the two-dimensional subspaces $W$ where $[W]$ is a singular point of $G(2,n) \cap H$. Furthermore, $a$ has to be even. To see this use the correspondence between the hyperplane $H$ and the skew-symmetric form $Q_H$. The codimension of the kernel of a skew-symmetric form is even since the restriction of the skew-symmetric form to a complementary linear space is non-degenerate. Hence, $a$ has to be even. Conversely, every $\Sigma_{2r,2r}$ occurs as the singular locus of some hyperplane section of $G(2,n)$. This can be seen by explicitly writing the skew-symmetric form $e_1 \wedge e_2 + e_3 \wedge e_4 + \cdots + e_{2r-1} \wedge e_{2r}$, whose kernel has codimension $2r$. Finally, Darboux's Theorem (McDuff and Salamon, 1998, §2) guarantees that the hyperplanes corresponding to the skew-symmetric forms with the same dimensional kernel form one orbit under $\mathbb{P}GL(n)$. This concludes the proof of the following well-known statement alluded to in the Introduction.

**Proposition 2.3.4.** *((Donagi, 1977, §2)) The group $\mathbb{P}GL(n)$ acts with finitely many orbits on $\mathbb{P}^*(\bigwedge^2 V)$. The orbits are indexed by an integer $1 \leq r \leq \lceil \frac{n-1}{2} \rceil$. The orbit corresponding to $r < \lceil \frac{n-1}{2} \rceil$ consists of hyperplanes $H$ such that the singular locus of $H \cap G(2,n)$ is a Schubert variety of the form $\Sigma_{2r,2r}$. The open orbit corresponding to $r = \lceil \frac{n-1}{2} \rceil$ is the complement of the dual variety $G(2,n)^*$ parameterizing hyperplanes $H$ such that $H \cap G(2,n)$ is smooth.* □

Let $r \leq \frac{n-2}{2}$. A hyperplane $[H] \in S_r \setminus S_{r-1}$ is singular along $\Sigma_{2r,2r}$, which parameterizes linear spaces contained in $F_{n-2r}$. By Lemma 2.3.1, $H \cap G(2,n)$ contains the Schubert variety $\Sigma_{2r-1,0}$ parameterizing linear spaces intersecting $F_{n-2r}$. Conversely, we saw in the proof of Lemma 2.3.2 that a hyperplane containing $\Sigma_{2r-1,0}(F_{n-2r} \subset F_n)$ is singular along the Schubert variety $\Sigma_{2r,2r}$ parameterizing linear spaces that are contained in $F_{n-2r}$. We conclude that $H$ contains a unique $\Sigma_{2r-1,0}$. In particular, the map $\pi_2 : \mathcal{I}(2r-1,0) \to S_r$ is birational and a resolution of singularities of $S_r$. Furthermore, the Theorem on the Dimension of Fibers and Corollary 2.2.4 then imply the following corollary.

**Corollary 2.3.5.** *([§2](Donagi, 1977)) The codimension of $S_r$ in $\mathbb{P}^*(\bigwedge^k V)$ is $\binom{n-2r}{2}$.* □

In particular, we have the following well-known corollary.

**Corollary 2.3.6.** *([§2](Donagi, 1977) or (Piontkowski and de Ven, 1999)) When $n$ is even, then the dual $G(2,n)^*$ is a hypersurface. When $n$ is odd $G(2,n)^*$ has codimension three.* □

Finally, if $n-2 > k > 2$, then the dual of $G(k,n)$ in its Plücker embedding is a hypersurface, and at a general point $[H] \in G(k,n)^*$, the singular locus of $H \cap G(k,n)$ consists of one singular point. For the convenience of the reader, we provide an elementary proof. Since $G(k,n)$ is isomorphic to $G(n-k,n)$, we may further assume that $2k \leq n$. To discuss properties of $G(k,n)^*$, we need to examine how pairs of projective tangent spaces intersect. This question is answered by the following lemma.

**Lemma 2.3.7.** *Let $[W_1]$ and $[W_2]$ be distinct points of $G(k,n)$, and let $s = \dim(W_1 \cap W_2)$. Then*

$$\mathbb{T}_{[W_1]}G(k,n) \cap \mathbb{T}_{[W_2]}G(k,n) = \begin{cases} \varnothing, & \textit{if } s < k-2 \\ \mathbb{P}^3, & \textit{if } s = k-2 \\ \mathbb{P}^{n-1}, & \textit{if } s = k-1. \end{cases}$$

*Proof.* Let $\mathcal{E} = \{e_1, \ldots, e_n\}$ be a basis for $V$ such that $W_1 \cap W_2 = \langle e_1, \ldots, e_s \rangle$, $W_1 = \langle e_1, \ldots, e_s, e_{s+1}, \ldots, e_k \rangle$ and $W_2 = \langle e_1, \ldots, e_s, e_{k+1}, \ldots, e_{2k-s} \rangle$. Via the Plücker embedding, we may represent $[W_1]$ as $[e_1 \wedge \cdots \wedge e_s \wedge e_{s+1} \wedge \cdots \wedge e_k]$ and $[W_2]$ as $[e_1 \wedge \cdots \wedge e_s \wedge e_{k+1} \wedge \cdots \wedge e_{2k-s}]$.

Let $\mathcal{E}_1 := \{e_1, \ldots, e_k\}$ and $\mathcal{E}_2 := \{e_1, \ldots, e_s, e_{k+1}, \ldots, e_{2k-s}\}$. The basis of the tangent space to $[W_1]$ consists of $[e_1 \wedge \cdots \wedge e_k]$ and all elements of the form $[e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_k \wedge e_j]$, where $e_j$ comes from $\mathcal{E} - \mathcal{E}_1$. Similarly, the basis for $\mathbb{T}_{[W_2]}$ consists of $[e_1 \wedge \cdots \wedge e_s \wedge e_{k+1} \wedge \cdots \wedge e_{2k-s}]$ and all elements of the form $[e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_s \wedge e_{k+1} \wedge \cdots \wedge e_{2k-s} \wedge e_j]$, where $e_j \in \mathcal{E} - \mathcal{E}_2$.

The set of basis elements of (the affine cone over) $\mathbb{T}_{[W_1]}$ and the set of basis elements of (the affine cone over) $\mathbb{T}_{[W_2]}$ are in one-to-one correspondence, respectively, with the following sets:

$$\mathcal{B}_1 := \{S \subset \mathcal{E} \mid \#S = k, \#(S - \mathcal{E}_1) \leq 1\}$$

$$\mathcal{B}_2 := \{T \subset \mathcal{E} \mid \#T = k, \#(T - \mathcal{E}_2) \leq 1\}.$$

So, we want to explore the number of elements of $\mathcal{B}_1 \cap \mathcal{B}_2$ for different choices of $s$.

We consider three cases: (i) $s < k-2$, (ii) $s = k-2$, and (iii) $s = k-1$. Observe that, if $s = k$, then there is nothing to prove, as in that case $W_1 = W_2$.

(i) Assume $s < k-2$. Let $S \in \mathcal{B}_1$ and $T \in \mathcal{B}_2$, and assume $S = T$. So $S$ differs from $\mathcal{E}_1$ by at most one element. Let $S' := S \cap (\mathcal{E}_1 - \mathcal{E}_2)$. We know that this is nonempty because $\mathcal{E}_1 - \mathcal{E}_2$ consists of $k - s > 2$ elements, and since $S$ differs from $\mathcal{E}_1$ by at most one element, $\#S' \geq 2$. Since $S'$ differs from $\mathcal{E}_2$ by at least two elements, any set of cardinality $k$ containing $S'$ cannot belong to $\mathcal{B}_2$, hence $S' \not\subset T$. This is a contradiction, so when $s < k-2$, $\mathcal{B}_1 \cap \mathcal{B}_2 = \varnothing$. Thus $\langle \mathcal{B}_1 \cap \mathcal{B}_2 \rangle = \{0\}$, so $\mathbb{T}_{W_1} \cap \mathbb{T}_{W_2} = \mathbb{P}(\{0\}) = \varnothing$.

(ii) Assume $s = k-2$. If $U \in \mathcal{B}_1 \cap \mathcal{B}_2$, then

(a) $\#(U - \{e_1, \ldots, e_{k-2}, e_{k-1}, e_k\}) \leq 1$,

(b) $\#(U - \{e_1, \ldots, e_{k-2}, e_{k+1}, e_{k+2}\}) \leq 1$.

In other words, in order for $U$ to satisfy both conditions (a) and (b), $U$ must contain exactly one of $e_{k-1}$ or $e_k$ *and* exactly one of $e_{k+1}$ or $e_{k+2}$. This results in precisely $\#(\mathcal{B}_1 \cap \mathcal{B}_2) = 4$ elements, which means $\mathbb{T}_{W_1} \cap \mathbb{T}_{W_2} = \mathbb{P}\left(\langle \mathcal{B}_1 \cap \mathcal{B}_2 \rangle\right) \cong \mathbb{P}^3$.

(iii) Suppose $s = k-1$ and that $U \in \mathcal{B}_1 \cap \mathcal{B}_2$ in this case. Explicitly, this means that $U$ differs from $\{e_1, \ldots, e_k\}$ by at most one element and from $\{e_1, \ldots, e_{k-1}, e_{k+1}\}$ by at most one element. If $U$ contains $\{e_1 \ldots, e_{k-1}\}$, then $U$ must also contain one of $e_k, e_{k+1}, \ldots$, or $e_n$. This results in $n - k + 1$ such $U$'s.

Say $U$ contains $\{e_1, \ldots, e_{k-2}\}$. In order to satisfy $U \in \mathcal{B}_1 \cap \mathcal{B}_2$, $U$ must contain *both* $e_k$ and $e_{k+1}$. Similarly, if $U$ contains $\{e_1, \ldots, e_{k-3}, e_{k-1}\}$, $U$ must contain both $e_k$ and $e_{k+1}$; if $U$ contains $\{e_1, \ldots, e_{k-4}, e_{k-2}, e_{k-1}\}$, it must contain $e_k$ and $e_{k+1}$; and so on, up to the case where $U$ contains $\{e_2, \ldots, e_{k-1}\}$, again meaning that $U$ contains both $e_k$ and $e_{k+1}$.

In other words, to contain a proper subset of $\{e_1, \ldots, e_{k-1}\}$ (the intersection of $\mathcal{E}_1$ and $\mathcal{E}_2$ for this case) *forces* containment of *both* $e_k$ and $e_{k+1}$ and exclusion of $e_{k+2}, \ldots, e_n$.

Hence the number of elements in $\mathcal{B}_1 \cap \mathcal{B}_2$ is $(n-k+1)+(k-1) = n$, so if $s = k-1$, $\mathbb{T}_{W_1} \cap \mathbb{T}_{W_2} \cong \mathbb{P}^{n-1}$.

□

Let $U = G(k,n) \times G(k,n) - \Delta$ be the complement of the diagonal $\Delta$ in $G(k,n) \times G(k,n)$. Consider the incidence correspondence

$$J = \{([W_1],[W_2],H) \mid \mathbb{T}_{[W_1]}, \mathbb{T}_{[W_2]} \subset H\}$$

consisting of a point $([W_1],[W_2])$ in $U$ and a hyperplane $H$ containing the projective tangent spaces to $G(k,n)$ at both points. Let $\pi_1$ and $\pi_2$ denote the projection to $U$ and $\mathbb{P}^*(\bigwedge^k V)$, respectively. Note that for every $[H]$ in $\pi_2(J)$, the hyperplane section $H \cap G(2,n)$ contains at least two singular points.

Let $U_1$ be the locus in $U$ parameterizing $\{([W_1],[W_2]) \mid \dim(W_1 \cap W_2) < k-2\}$. Then by Lemma 2.3.7 the fibers of $\pi_1$ over $U_1$ are projective spaces of dimension $\binom{n}{k} - 2k(n-k) - 3$. Observe that $U_1$ has dimension $2k(n-k)$ by the Theorem on the Dimension of Fibers: if we view $U_1$ as an incidence correspondence itself and project to either $G(k,n)$, we see that the fiber will be the complement of a Schubert variety. The Theorem on the Dimension of Fibers applied to $\pi_1$ implies that $\dim(\pi_1^{-1}(U_1)) = \binom{n}{k} - 3$, hence $\pi_2(\pi_1^{-1}(U_1))$ has codimension at least two in $\mathbb{P}^*(\bigwedge^k V)$.

Let $U_2$ be the locus in $U$ parameterizing $\{([W_1], [W_2]) \mid \dim(W_1 \cap W_2) = k-2\}$. Lemma 2.3.7 tells us that the $\pi_1$-fibers over $U_2$ are projective spaces of dimension $\binom{n}{k} - 2k(n-k) + 1$. As with $U_1$ if we project $U_2$ onto $G(k,n)$, we have that the fiber over a point $[W']$ is an open subset of a Schubert variety of the form

$$\{[W] \mid \dim(W \cap W') \leq k-2\} = \Sigma_{n-2-k,\dots,n-2-k,0,0}$$

so that $U_2$ has dimension $2(n-2) + k(n-k)$. From this we obtain that $\dim(\pi_1^{-1}(U_2)) = \binom{n}{k} - 1 - (k(n-k) - 2(n-2) - 2)$. We want to show, then, that $k(n-k) - 2(n-2) - 2 \geq 2$. Notice that since $k > 2$, this inequality is equivalent to $n \geq \frac{k^2}{k-2}$. Now since $n \geq 2k$ by assumption, to show $2k \geq \frac{k^2}{k-2}$ would imply the above inequality. A simple calculation shows that this inequality holds if $k \geq 4$ or $k = 3$ and $n \geq 9$. If $k = 3$ and $n = 6, 7$, or $8$, we observe that the general fiber dimension of $\pi_2$ on $\pi_1^{-1}(U_2)$ is 6,4 and 2, respectively. Let $W_1 = Span(e_1, e_2, e_3)$ and let $W_2 = Span(e_1, e_4, e_5)$. A hyperplane $H$ containing $\mathbb{T}_{[W_1]}$ and $\mathbb{T}_{[W_2]}$ can be expressed as $\sum_{i=6}^{n}(a_i p_{24i} + b_i p_{34i} + c_i p_{25i} + d_i p_{35i}) = 0$ in Plücker coordinates. Consider two-dimensional subspaces $Y$ in $Span(e_2, e_3, e_4, e_5)$ that satisfy $a_i e_2 \wedge e_4 + \cdots + d_i e_3 \wedge e_5 = 0$ for $6 \leq i \leq n$. Then $H$ contains the tangent space to the three-dimensional subspace $Span(e_1, Y)$. The claim about the fiber dimension of $\pi_2$ follows. Hence, $\pi_2(\pi_1^{-1}(U_2))$ has codimension at least two in $\mathbb{P}^*(\bigwedge^k V)$ in these cases as well.

Let $U_3$ be the locus in $U$ parameterizing $\{([W_1], [W_2]) \mid \dim(W_1 \cap W_2) = k-1\}$. Then the fibers of $\pi_1$ over $U_3$ are projective spaces of dimension $\binom{n}{k} - 2k(n-k) + n - 3$. The locus

$U_3$ consists of pairs of points $([W_1],[W_2])$ such that the line spanned by them is contained in $G(k,n)$. Hence, $\dim(U_3) = 2k + (k+1)(n-k-1)$.

Note that if a hyperplane $H$ is tangent to $G(k,n)$ at both $[W_1]$ and $[W_2]$, then it is tangent at all points along the line spanned by $[W_1]$ and $[W_2]$. We claim this implies that the fibers of $\pi_2$ over $\pi_2(\pi_1^{-1}(U_3))$ have dimension at least two. Let $H$ be an element of $\pi_2(\pi_1^{-1}(U_3))$. Then view $\pi_2^{-1}(H)$ as the incidence correspondence

$$\pi_2^{-1}(H) = \{([W_1],[W_2]) \mid \mathbb{T}_{[W_1]}, \mathbb{T}_{[W_2]} \subset H, \dim(W_1 \cap W_2) = k-1\}$$

$$p_1 \swarrow \qquad\qquad \searrow p_2$$

$$G(k,n) \qquad\qquad\qquad G(k,n).$$

Suppose that $\dim \pi_2^{-1}(H) = 0$. Then there are finitely many pairs $([W_1],[W_2])$ such that $\mathbb{T}_{[W_1]}, \mathbb{T}_{[W_2]} \subset H$. But this contradicts the fact that $\mathbb{T}_{[W]} \subset H$ for any $[W]$ on the line spanned by $[W_1]$ and $[W_2]$. Now suppose $\dim \pi_2^{-1}(H) > 0$ and consider the first projection $p_1$. If $[W_1] \in p_1(\pi_2^{-1}(H))$, then $p_1^{-1}([W_1])$ has dimension at least one since otherwise there are only finitely many points $[W_2]$ such that $H$ is tangent to $G(k,n)$ at $[W_2]$. But again this contradicts that $\mathbb{T}_{[W]} \subset H$ for any $[W]$ on the line spanned by $[W_1]$ and $[W_2]$. This proves the claim. By the Theorem on the Dimension of Fibers, the codimension of $\pi_2(\pi_1^{-1}(U_3))$ will be less than two if $2k + (k+1)(n-k-1) - 2k(n-k) + n - 2 > 0$.

Rewriting this inequality, $0 > (k-2)n - k^2 + 3$. Using $n \geq 2k$, we immediately see that this inequality cannot be satisfied if $k \geq 4$. When $k = 3$, the inequality becomes $6 > n$.

Hence, we conclude that the inequality is not satisfied for $k \geq 3$ and $n \geq 2k$. It follows that if $n - 2 > k > 2$, $G(k,n)^*$ is a hypersurface and a general tangent hyperplane is tangent at a unique point. We have proved the following well-known fact for which we could not find a convenient reference.

**Proposition 2.3.8.** *If $2 < k < n-2$, then $G(k,n)^*$ in $\mathbb{P}^*(\bigwedge^k V)$ is a hypersurface. Furthermore, a general hyperplane parameterized by $G(k,n)^*$ is tangent to $G(k,n)$ at one point.* $\square$

# CHAPTER 3

# CONDITIONS FOR SURJECTIVITY ONTO $\mathbb{P}^*(\bigwedge^2 V)$

In this chapter, we prove Theorem 1.0.2 and discuss its generalizations to $G(k,n)$.

*Proof of Theorem 1.0.2.* Let $\Sigma_{a,b}(F_{n-1-a} \subset F_{n-b})$ be a Schubert variety with class $\sigma_{a,b}$ in $G(2,n)$. Suppose that $H$ is a hyperplane in $\mathbb{P}(\bigwedge^2 V)$ containing $\Sigma_{a,b}(F_{n-1-a} \subset F_{n-b})$. Notice that $\Sigma_{a,b}(F_{n-1-a} \subset F_{n-b}) \subset G(2, F_{n-b})$. There are two possibilities. Either $G(2, F_{n-b}) \subset H$; or $H \cap G(2, F_{n-b})$ is a hyperplane section of $G(2, F_{n-b})$ that contains $\Sigma_{a,b}(F_{n-1-a} \subset F_{n-b})$. We will now analyze each of these possibilities.

First, assume that $H \cap G(2, F_{n-b})$ is a hyperplane section of $G(2, F_{n-b})$. A linear embedding $V' \hookrightarrow V$ induces an embedding $G(2, V') \hookrightarrow G(2, V)$. The following lemma analyzes the relation between the singular loci of $H \cap G(2, V)$ and $H \cap G(2, V')$.

**Lemma 3.0.9.** *Let $G(2,n) \hookrightarrow G(2,n+1)$ be the embedding induced by the embedding of $V_n \hookrightarrow V_{n+1}$. Let $H \cap G(2,n)$ be a linear section of $G(2,n)$ in $\mathbb{P}(\bigwedge^2 V_n)$ with singular locus $\Sigma_{2r,2r}$. Let $H'$ be a general hyperplane in $\mathbb{P}(\bigwedge^2 V_{n+1})$ such that $H' \cap G(2,n+1)$ restricts to $H \cap G(2,n)$. Then the singular locus of $H' \cap G(2,n+1)$ is $\Sigma_{2(r+1),2(r+1)}$.*

*Proof.* Pick a basis $e_1, \dots, e_{n+1}$ of $V_{n+1}$ such that $V_n$ is spanned by the first $n$ vectors and the singular locus of $H \cap G(2,n)$ parameterizes two-dimensional subspaces contained in the span $F_{n-2r}$ of the first $n-2r$ vectors. Then $H$ is defined by a linear equation $L(p_{i,j}) = 0$, where $L$ is a linear combination of the Plücker coordinates $p_{i,j}$ for $i < j$ and $n-2r < j \leq n$. A hyperplane in $\mathbb{P}(\bigwedge^2 V_{n+1})$ that contains $H$ may be expressed as $L(p_{i,j}) + \sum_{i=1}^n a_i p_{i,n+1} = 0$.

By Bertini's Theorem (Hartshorne, 1977, II.8.18), the singular locus of $H' \cap G(2, n+1)$ for a general hyperplane containing $H$ is contained in $H \cap G(2,n)$. Let $W$ be the $(n-2r-1)$-dimensional linear space cut out on $F_{n-2r}$ by the linear equation $\sum_{i=1}^n a_i x_i = 0$, where the $x_i$ form a basis for $V^*$. Then $H' \cap G(2, n+1)$ contains the tangent space to $G(2, n+1)$ at any two-dimensional space contained in $W$. At a point, $u \wedge v$ with $u, v \in W$, the tangent space is spanned by replacing at most one of $u$ or $v$ by elements of a basis. All the Plücker coordinates defining $H'$ clearly vanish at all these points. Hence $H' \cap G(2, n+1)$ is singular along two-dimensional subspaces contained in $W$. We conclude that the singular locus of $H' \cap G(2, n+1)$ contains a $\Sigma_{2(r+1),2(r+1)}$ of two-dimensional subspaces contained in $W$. Conversely, for a two-dimensional space not contained in that hyperplane, there exists a vector $v$ such that $\sum a_i v_i \neq 0$. Hence, the point $v \wedge e_{n+1}$ is not contained in $H'$, but it is contained in the tangent space to a point $w \wedge v$. Hence, the singular locus does not contain all of $\Sigma_{2r,2r}$. The lemma follows. $\square$

We are now ready to prove the theorem in the case $H$ does not contain $G(2, F_{n-b})$. There are two cases that we need to analyze separately. First, assume that $a = n-2$. Since the Grassmannian contains linear spaces of the form $\Sigma_{n-2,0}$, any hyperplane section contains linear spaces $\Sigma_{n-2,1}$ of one smaller dimension. Hence, $\pi_2$ is surjective for $\lambda = (n-2, i)$ when $i > 0$. We now have to analyze the case $\lambda = (n-2, 0)$. In this case, the flag variety $F(1, n; n)$ is isomorphic to $\mathbb{P}^{n-1}$. Hence, $\dim(\mathcal{I}(n-2, 0)) = \binom{n}{2} - 1$. If $n$ is even, then the general singular hyperplane section $X$ of $G(2,n)$ is singular along a point $[\Lambda] \in G(2,n)$. Furthermore, in this case the dual variety $G(2,n)^*$ is a hypersurface, hence has dimension $\binom{n}{2} - 2$. By Lemma 2.3.2, if $F_1 \subset \Lambda$, then every two-dimensional subspace containing $F_1$ is contained in $X$. Since the space of one-dimensional subspaces of $\Lambda$ is isomorphic to $\mathbb{P}^1$, the general fiber of $\pi_2$ over $G(2,n)^*$ has dimension greater than or equal to one. By the Theorem on the Dimension of Fibers, $\dim(\pi_2^{-1}(G(2,n)^*) \geq \binom{n}{2} - 1$. However, since $\pi_2^{-1}(G(2,n)^*) \subset \mathcal{I}(n-2,0)$, $\dim(\pi_2^{-1}(G(2,n)^*) \leq \binom{n}{2} - 1$. We conclude that $\pi_2^{-1}(G(2,n)^*) = \mathcal{I}(n-2,0)$ and consequently, $\pi_2$ is not surjective.

If $n$ is odd, then the dual variety $G(2,n)^*$ has codimension 3, or dimension $\binom{n}{2} - 4$. The general singular hyperplane section $X$ of $G(2,n)$ is singular along a plane $\Sigma_{n-3,n-3}(F_2 \subset F_3)$. If $F_1$ is a one-dimensional subspace such that $F_1 \subset F_3$, then $\Sigma_{n-2,0}(F_1 \subset F_n) \subset X$. Conversely, we would like to show that any Schubert variety $\Sigma_{n-2,0}(F_1 \subset F_n)$ contained in $X$ must have $F_1 \subset F_3$. Suppose to the contrary that $F_1 \not\subset F_3$. Then $F_4 = Span(F_1, F_3)$ is a four-dimensional vector space. We will show that any two-dimensional subspace intersecting $F_4$ non-trivially is contained in $X$. Let $G_2$ be a two-dimensional subspace intersecting $F_4$ in a one-dimensional subspace $G_1$. Then we can find a two-dimensional subspace, namely $G_2' = Span(G_1, F_1)$, such that $G_2'$ intersects $F_3$. Let $G_3' = Span(G_2', G_2)$. We claim that the two-dimensional subspaces contained in $G_3'$, and in particular $G_2$, are all contained in $X$. The two-dimensional subspaces contained in $G_3'$ form a plane in the Plücker embedding of $G(2,n)$. Hence a hyperplane section either is a line or contains the entire plane. By assumption, the Schubert variety $\Sigma_{n-2,n-3}(G_1 \subset G_3')$ is contained in $X$. Similarly, the Schubert variety $\Sigma_{n-2,n-3}(G_2' \cap F_3 \subset G_3')$ is contained in $X$. Hence, the entire family of two-dimensional subspaces contained in $G_3'$ has to be contained in $X$. Observe that any linear space contained in a hyperplane section must be contained in its singular locus, as the tangent space to a point of this linear space will be contained in the hyperplane $H$. We conclude that the singular locus of $X$ is larger than $\Sigma_{n-3,n-3}(F_2 \subset F_3)$, contrary to assumption. Hence, the general fiber of $\pi_2$ over $G(2,n)^*$ has dimension 2 and $\dim(\pi_2^{-1}(G(2,n)^*)) \leq \binom{n}{2} - 2$. We conclude that the image of $\pi_2$ must contain a hyperplane not contained in $G(2,n)^*$. Since any two smooth hyperplane sections of $G(2,n)$ are equivalent under the action of $\mathbb{P}GL(n)$, we conclude that $\pi_2$ is surjective.

Now we can discuss the case $\Sigma_{a,0}$ with $a < n-2$. If $a$ is odd, then the singular locus of a general hyperplane contains $\Sigma_{a+1,a+1}$. Conversely, a linear section whose singular locus is $\Sigma_{a+1,a+1}$ contains a Schubert variety of the form $\Sigma_{a,0}$. We conclude that $\pi_2(\mathcal{I}(a,0)) = S_{(a+1)/2}$. If $a$ is even, then the singular locus of a hyperplane section containing $\Sigma_{a,0}$ contains $\Sigma_{a+1,a+1}$. However, since the singular loci have to be of the form $\Sigma_{2k,2k}$, it follows that the singular locus has to contain a Schubert variety of the form $\Sigma_{a,a}$. Conversely, a hyperplane section whose singular locus has the form $\Sigma_{a,a}$ contains a Schubert variety of the form $\Sigma_{a,0}$. We conclude that the image of $\pi_2$ is $S_{a/2}$.

Returning to the original argument, if $b > 0$, then $\Sigma_{a,b}$ is a Schubert variety with class $\sigma_{a-b,0}$ in $G(2,n-b)$. Hence, any hyperplane section of $G(2,n-b)$ containing $\sigma_{a-b,0}$ is singular along a Schubert variety of the form $\Sigma_{a-b+1,a-b+1}$ if $a-b$ is odd or $\Sigma_{a-b,a-b}$ if $a-b$ is even. Using Lemma 3.0.9 $b$-times, we conclude that if $a-b$ is even, then the general hyperplane containing $\Sigma_{a,b}$ is smooth if $a+b > n-3$ or singular along a Schubert variety of the form $\Sigma_{a+b+1,a+b+1}$ when $a+b \leq n-2$. Similarly, when $a-b$ is odd, then a hyperplane section of $G(2,n-b)$

containing $\Sigma_{a-b,0}$ is singular along $\Sigma_{a-b,a-b}$. Using Lemma 3.0.9 $b$-times, we conclude that a general hyperplane containing $\Sigma_{a,b}$ is smooth when $a+b > n-2$ or singular along $\Sigma_{a+b,a+b}$ when $a+b \leq n-2$.

Finally, we analyze the cases when the hyperplane contains $G(2, n-b)$ or when $a = b$. The first observation is that the only hyperplanes containing a Schubert variety of the form $\Sigma_{1,1}(F_{n-2} \subset F_{n-1})$ are Schubert varieties $\Sigma_1(G_{n-2} \subset G_n)$. The flag variety $F(n-1;n) \cong (\mathbb{P}^{n-1})^*$, hence has dimension $n-1$. The fiber dimension of $\pi_1$ over a point in $F(n-1;n)$ is $n-2$. Hence the dimension of $\mathcal{I}(1,1)$ is $2n-3$. The locus of Schubert varieties in $\mathbb{P}^*(\bigwedge^2 V)$ of the form $\Sigma_1$, which we denote by $S_1$ according to the notation of Donagi (Donagi, 1977), has dimension $2(n-2)$ because a choice of $\Sigma_1(F_{n-2} \subset F_n)$ is equivalent to a choice of $[F_{n-2}]$ in $G(n-2,n)$. If $F_{n-1}$ contains $G_{n-2}$, then $\Sigma_{1,1}(F_{n-2} \subset F_{n-1}) \subset \Sigma_1(G_{n-2} \subset G_n)$. Hence, the fiber of $\pi_2$ over a hyperplane corresponding to a Schubert variety has dimension at least one. We conclude that $\dim(\pi_2^{-1}(S_1)) = 2n-3 = \dim(\mathcal{I}(1,1))$. Hence, $\pi_2(\mathcal{I}(1,1)) = S_1$ and every hyperplane containing a Schubert variety $\Sigma_{1,1}$ is a Schubert variety $\Sigma_1$. Applying Lemma 3.0.9 $(b-1)$-times, we conclude that a general hyperplane section containing $\Sigma_{b,b}$ is smooth if $2b > n-2$ or singular along a Schubert variety of the form $\Sigma_{2b,2b}$ if $2b \leq n-2$. This also concludes the discussion of the case $a \neq b$. Let $H$ and $H'$ be two hyperplanes containing $\Sigma_{a,b}$. If $G(2, F_{n-b}) \subset H$ and $G(2, F_{n-b}) \not\subset H'$, then the dimension of the singular locus of $G(2,n) \cap H$ is greater than or equal to the dimension of the singular locus of $H' \cap G(2,n)$. This concludes the proof of the theorem. $\square$

Since the proof of Proposition 3.0.10 uses similar techniques, we include it in this chapter.

**Proposition 3.0.10.** *Let $\lambda$ be a partition for $G(k,n)$ such that $\lambda_1 < n-k$ and $\lambda_k = 0$. Then the image of the second projection $\pi_2(\mathcal{I}(\lambda))$ is contained in $G(k,n)^*$, in particular, it is not surjective. On the other hand, let $\lambda$ be a partition such that either $\lambda_{k-1} = n-k$ and $\lambda_k > 0$; or $\lambda_1 = n-k$ and $\lambda_k = n-k-1$ . Then $\pi_2(\mathcal{I}(\lambda))$ is surjective.*

*Proof.* Let $\lambda$ be a partition of the form $\lambda_1 = \lambda_{k-1} = n-k$ and $\lambda_k > 0$, then the Plücker image of $\Sigma_\lambda$ is a linear space. Since the Grassmannian contains linear spaces with cohomology class $\sigma_\mu$, where $\mu = ((n-k)^{k-1}, 0)$, every hyperplane section contains linear spaces with cohomology class $\sigma_\lambda$. The same argument applies for a partition $\lambda$ with $\lambda_1 = n-k$ and $\lambda_k \geq n-k-1$ by considering linear spaces with cohomology class $\sigma_\nu$, where $\nu = ((n-k-1)^k)$. This proves the second part of the proposition.

To prove the first part of the proposition, we will show that if $\lambda$ is a partition such that $\lambda_1 < n-k$ and $\lambda_k = 0$, then any hyperplane $H$ containing $\Sigma_\lambda$ is singular. Fix a basis $e_1, \ldots, e_n$ of $V$. Let $F_\bullet$ be the flag where the flag element $F_i$ is the span of the basis vectors $e_1, \ldots, e_i$. Let $H$ be a hyperplane containing $\Sigma_\lambda(F_\bullet)$. Then the equation defining $H$ must be a linear combination of the Plücker coordinates defining $\Sigma_\lambda(F_\bullet)$. Recall that the Plücker coordinates vanishing on $\Sigma_\lambda(F_\bullet)$ are $p_{i_1,\ldots,i_k}$ with $i_1 < \cdots < i_k$ such that $i_j > n-k+j-\lambda_j$ for at least one $j$. Since by assumption $\lambda_k = 0$ and we cannot have $i_k > n$, there must exist $j < k$ such that $i_j > n-k+j-\lambda_j$.

It follows that $H \cap G(k,n)$ is singular at the point $p = e_1 \wedge e_2 \wedge \cdots \wedge e_k$. The tangent space to $G(k,n)$ at $p$ is spanned by Plücker coordinates $p_{i_1,\ldots,i_k}$ where the set $\{i_1, \ldots, i_k\}$ differs from $\{1, \ldots, k\}$ in at most one element. On the other hand, the Plücker coordinates occurring in the equation of $H$ have indices that differ from $\{1, \ldots, k\}$ in at least two elements. Hence, $H$ vanishes at all the points spanning the tangent space to $G(k, n)$ at $p$. We conclude that $H \cap G(k, n)$ is singular at $p$. This concludes the proof of the proposition. □

**Corollary 3.0.11.** *Let $\lambda$ be the partition $\lambda_1 = \cdots = \lambda_{k-1} = n - k - 1$ and $\lambda_k = 0$. Then $\pi_2(\mathcal{I}(\lambda))$ surjects onto $G(k, n)^*$.*

It is very rare to have an explicit, concrete resolution of singularities of a variety. Corollary 3.0.12 provides such a resolution for the dual of the Grassmannian in its Plücker embedding.

**Corollary 3.0.12.** *Let $n - 2 > k > 2$. Let $\lambda$ be the partition $\lambda_1 = \cdots = \lambda_{k-1} = n - k - 1$ and $\lambda_k = 0$. Let $N = \binom{n}{k} - k(n-k) - 2$. Then the incidence correspondence $\mathcal{I}(\lambda)$ is a $\mathbb{P}^N$ bundle over $G(k, n)$. The map $\pi_2(\mathcal{I}(\lambda))$ is birational onto $G(k, n)^*$ and gives a resolution of singularities of $G(k, n)^*$.*

When $\lambda$ is the partition $\lambda_1 = \cdots = \lambda_{k-1} = n - k - 1$ and $\lambda_k = 0$, then, by Proposition 3.0.10, for any hyperplane $H$ containing $\Sigma_\lambda$ the hyperplane section $H \cap G(k, n)$ is singular at a point. Conversely, if $H \cap G(k, n)$ is singular at a point $p = e_1 \wedge \cdots \wedge e_k$, then by Lemma 2.3.1 the Schubert variety $\Sigma_\lambda$ parameterizing $k$-dimensional subspaces that intersect $Span(e_1, \ldots, e_k)$ in a subspace of dimension at least $k - 1$ is contained in $H$. In this case, we conclude that the image of $\pi_2(\mathcal{I}(\lambda))$ is precisely the dual variety.

Note that $h^0(I_{\Sigma_\lambda}(1)) = \binom{n}{k} - k(n-k) - 1 = N$. Hence, the incidence correspondence $\mathcal{I}(\lambda)$ is a projective space bundle over $G(k, n)$ with fibers of dimension $N - 1$. In particular,

$\dim(\mathcal{I}(\lambda)) = \binom{n}{k} - 2$. When $n - 2 > k > 2$, the dual variety $G(k, n)^*$ is a hypersurface and the general tangent hyperplane to $G(k, n)$ is tangent at a unique point. Therefore, $\pi_2$ is a birational map. Hence, $\pi_2 : \mathcal{I}(\lambda) \to G(k, n)^*$ gives a resolution of singularities of $G(k, n)^*$. This concludes the proofs of Corollary 3.0.11 and Corollary 3.0.12.

# CHAPTER 4

# PARAMETER SPACES OF SCHUBERT VARIETIES IN HYPERPLANE SECTIONS

In this chapter, we prove Theorem 1.0.3 and discuss some generalizations to $G(k,n)$. Recall that the parameter space of Schubert varieties $\Sigma_\lambda$ for fixed $\lambda$ in a given hyperplane section $H$ is precisely the $\pi_2$-fiber of $[H]$ over $\mathbb{P}^*(\bigwedge^2 V)$.

*Proof of Theorem 1.0.3.* Let $H$ be a hyperplane in $\mathbb{P}(\bigwedge^2 V)$ such that $[H] \in S_r \setminus S_{r-1}$. Then $H \cap G(2,n)$ is singular along a Schubert variety $\Sigma_{2r,2r}$ parameterizing two-dimensional subspaces of $V$ contained in a linear subspace $F_{n-2r}$. First, suppose that $a \neq b$. Let $(V_{n-a-1} \subset V_{n-b})$ be the partial flag defining a Schubert variety $\Sigma_{a,b} \subset H \cap G(2,n)$. Suppose that $\dim(V_{n-a-1} \cap F_{n-2r}) = j$. Then clearly

$$0 \leq j \leq \min(n-a-1, n-2r).$$

Consider the restriction of $H$ to $G(2, V_{n-b})$. Either $H$ identically vanishes on $G(2, V_{n-b})$; or $H$ defines a hyperplane section of $G(2, V_{n-b})$.

If $H$ identically vanishes on $G(2, V_{n-b})$, then both $V_{n-a-1}$ and $V_{n-b}$ are $Q_H$-isotropic. Hence, trivially $V_{n-a-1} \subset V_{n-b} \subset V_{n-a-1}^{\perp}$. Take a linear space $S_{2r}$ of dimension $2r$ complementary to $F_{n-2r}$. Then the restriction of $Q_H$ to $S_{2r}$ is non-degenerate. Since $Span(V_{n-a-1}, F_{n-2r}) \cap S_{2r}$ is isotropic with respect to the restriction of $Q_H$ to $S_{2r}$, its dimension $n-a-1-j$ must be less than or equal to $r$. Equivalently, $n-a-1-r \leq j$. Similarly, since $V_{n-b}$ is isotropic, $n-b \leq n-r$. In particular, $b \geq r$. Hence, the inequality $n-a-1-\min(r,b) \leq j$ holds.

Next, suppose that $H$ defines a hyperplane section of $G(2, V_{n-b})$. By our assumption that $\Sigma_{a,b}(V_{n-a-1} \subset V_{n-b}) \subset H \cap G(2,n)$, we must have that $[W] \in H \cap G(2,n)$ for every two-dimensional subspace $W$ that intersects $V_{n-a-1}$ non-trivially and is contained in $V_{n-b}$. In particular, $[W]$ is contained in $H \cap G(2,n)$ for every two-dimensional subspace $W$ contained in $V_{n-a-1}$. We conclude that the skew-symmetric form $Q_H$ vanishes identically on $V_{n-a-1}$. Hence, $V_{n-a-1}$ is $Q_H$-isotropic. Hence, $Span(V_{n-a-1}, F_{n-2r})$ is also $Q_H$-isotropic. The dimension of this vector space, which by assumption is $n-a-1+n-2r-j$, has to be less than or equal to $n-r$. We conclude that $n-a-1-r \leq j$.

Finally, since the restriction of $Q_H$ to $V_{n-b}$ must contain $V_{n-a-1}$ in its kernel, we must have that $V_{n-b} \subset V_{n-a-1}^{\perp}$. By assumption, the dimension of $V_{n-a-1}^{\perp}$ is $n-1-a-j$. Hence, $n-a-1-j \leq b$. Combining all these inequalities, yields the inequality

$$\max(0, n-a-1-\min(b,r)) \leq j \leq \min(n-a-1, n-2r).$$

Note that by assumption $2r \leq a+b+1$, so for $j$ satisfying the assumptions of the theorem, these inequalities hold.

Conversely, suppose $j$ satisfies the inequalities

$$\max(0, n-a-1-\min(b,r)) \leq j \leq \min(n-a-1, n-2r).$$

Then every Schubert variety $\Sigma_{a,b}(V_{n-a-1} \subset V_{n-b})$ is contained in $H \cap G(2,n)$ provided $V_{n-a-1}$ is $Q_H$ isotropic and $V_{n-b} \subset V_{n-a-1}^{\perp}$. This is clear since the kernel of $Q_H$ restricted to $V_{n-a-1}^{\perp}$ contains $V_{n-a-1}$. Hence, every two-dimensional space intersecting $V_{n-a-1}$ non-trivially is $Q_H$ isotropic.

Furthermore, there exists flags $(V_{n-a-1} \subset V_{n-b})$ such that $\dim(V_{n-a-1} \cap F_{n-2r}) = j$. To construct such a flag, let $S_{2r}$ be a linear space complementary to $F_{n-2r}$. Pick a $Q_H$ isotropic subspace $W$ of dimension $n-a-1-j$ in $S_{2r}$. This is possible since $n-a-1-j \leq r$. Pick a $j$-dimensional subspace $W'$ of $F_{n-2r}$. Let $V_{n-a-1} = Span(W, W')$. Then $V_{n-a-1}$ is isotropic and has dimension $n-a-1$. Next, consider $V_{n-a-1}^{\perp}$, which has dimension $a+1+j$. Since by assumption $n-a-1-b \leq j$, $n-b \leq a+1+j$. Therefore, there exists $(n-b)$-dimensional subspaces of $V_{n-a-1}^{\perp}$ containing $V_{n-a-1}$.

Let $Z_j^0$ denote the locus of two-step flags $(V_{n-a-1} \subset V_{n-b})$ in $F(n-a-1, n-b; n)$ such that $V_{n-a-1}$ is $Q_H$ isotropic, $\dim(V_{n-a-1} \cap F_{n-2r}) = j$ and $V_{n-b} \subset V_{n-a-1}^{\perp}$. Let $Z_j$ denote the closure of $Z_j^0$. It is clear from the construction in the previous paragraph that $Z_j$ is irreducible. We have also shown that

$$X((a,b),H) = \bigcup_{j=M}^{\min(n-a-1,n-2r)} Z_j$$

and in this range each $Z_j^0$ is non-empty. Finally, there remains to check that $Z_j$ is an irreducible component of $X((a,b),H)$ if $j \leq n-r-\frac{a+b+1}{2}$ and $X((a,b),H) = \bigcup_{j=M}^{N} Z_j$.

The dimension $\dim(V_{n-a-1} \cap F_{n-2r})$ is an upper-semi-continuous function. Consequently, if $j_1 > j_2$, then linear spaces intersecting $F_{n-2r}$ in a $(j_1)$-dimensional subspace cannot specialize to linear spaces intersecting $F_{n-2r}$ in a $j_2$-dimensional subspace. Therefore, $Z_{j_2}$ cannot be contained in $Z_{j_1}$. On the other hand, $\dim(V_{n-b} \cap F_{n-2r})$ is also an upper-semi-continuous function. By construction, for a general point $(V_{n-a-1}, V_{n-b})$ in $Z_j$, $\dim(V_{n-b} \cap F_{n-2r}) = \max(j, 2n-2r-a-b-1-j)$ since $V_{n-b}$ is an arbitrary linear space containing $V_{n-a-1}$ and contained in the $(a+j+1)$-dimensional space $V_{n-a-1}^{\perp}$. Suppose $n - r - \frac{a+b+1}{2} \geq j_1 > j_2$, then the dimension of $V_{n-b} \cap F_{n-2r}$ for a general point in $Z_{j_1}$, respectively, $Z_{j_2}$ is given by $2n-2r-a-b-1-j_1 < 2n-2r-a-b-1-j_2$. Hence, $Z_{j_1}$ cannot be contained in $Z_{j_2}$. We conclude that for $M \leq j \leq N$, $Z_j$ form irreducible components of $X((a,b),H)$.

There remains to show that when $2j > 2n-2r-a-b-1$, then $Z_j$ is contained in $Z_{j-1}$. Let $(V_{n-a-1} \subset V_{n-b})$ be a point of $Z_j$ such that $\dim(V_{n-a-1} \cap F_{n-2r}) = \dim(V_{n-b} \cap F_{n-2r}) = j$. Let $E$ be a codimension one linear space in $V$ containing the vector space $Span(V_{n-b}, F_{n-2r})$. By assumption,

$$\dim(Span(V_{n-b}, F_{n-2r})) = 2n - 2r - b - j < a + 1 + j \leq n.$$

Hence, we can always find a codimension one linear space $E$ containing $Span(V_{n-b}, F_{n-2r})$. Since a non-degenerate skew-symmetric form can only exist in an even-dimensional vector space, the dimension of the kernel of $Q_H$ restricted to $E$ has to have dimension greater than or equal to $n - 2r + 1$. Denote this kernel by $K_E$. Let $V_{a+1-b}$ be a general subspace in $V_{n-b}$ complementary to $V_{n-a-1}$. Pick a pencil of linear spaces $V_{n-a-1}(t)$ such that $V_{n-a-1}(0) = V_{n-a-1}$, $V_{n-a-1}(t) \subset K_E$ and $V_{n-a-1}(t) \not\subset F_{n-2r}$ for $t \neq 0$. Consider the pencil of flags $(V_{n-a-1}(t) \subset Span(V_{n-a-1}(t), V_{a+1-b}))$. First, notice that when $t = 0$, this is simply $(V_{n-a-1} \subset V_{n-b})$. Hence, except for finitely many $t$, these flags are contained in $F(n-a-1, n-b; n)$. By construction, $\dim(V_{n-a-1}(t) \cap F_{n-2r}) = j-1$. Since $V_{n-a-1}(t) \subset K_E$, $Span(V_{n-a-1}(t), V_{a+1-b}) \subset V_{n-a-1}(t)^{\perp}$. Hence, the general member of this family is contained in $Z_{j-1}$. We conclude that $Z_j \subset Z_{j-1}$.

The computation of the dimension of $Z_j$ is standard. We have to choose a $Q_H$ isotropic subspace $V_{n-a-1}$ that intersects the kernel of $Q_H$ in a subspace of dimension $j$. The reader can easily check that the dimension of the space of such isotropic subspaces is

$$\frac{(n-a-1)(3a+j-n+4)}{2} - j\frac{(4r+3a+3j-3n+4)}{2}.$$

Then we need to choose an $(n-b)$-dimensional subspace in the $(a+j+1)$-dimensional subspace $V_{n-a-1}^{\perp}$ containing $V_{n-a-1}$. The dimension of the space of such linear spaces $V_{n-b}$ is

$$(a+1-b)(a+b+j-n+1).$$

This immediately yields the dimension formula for $Z_j$.

Next, suppose that $a = b$. In this case, the Schubert variety is determined by one flag element $V_{n-a}$. Since $\Sigma_{a,a} \subset H \cap G(2,n)$, $V_{n-a}$ is $Q_H$ isotropic. Conversely, if $V_{n-a}$ is $Q_H$-isotropic, then $[W] \in H \cap G(2,n)$ for every two-dimensional subspace $W \subset V_{n-a}$. We conclude that $X((a,a), H)$ is the space of $Q_H$-isotropic linear spaces of dimension $n-a$. It is standard that this space is irreducible and has the claimed dimension. $\square$

The corollaries are obtained by specializing the numbers $a$ and $b$.

**Corollary 4.0.13.** *Let $[H] \in S_r \setminus S_{r-1}$. Then $X((r,r),H)$ is isomorphic to the Lagrangian Grassmannian $SG(r,2r)$. In particular, $X((r,r),H)$ is irreducible of dimension $\binom{r+1}{2}$.*

*Proof.* When $a = b = r$, we are in Case (2) of Theorem 1.0.3. $X((a,a),H)$ parameterizes $(n-a)$-dimensional isotropic subspaces of $Q_H$. These are maximal dimensional isotropic subspaces, hence they all contain the kernel $F_{n-2r}$ of $Q_H$. Passing to the quotient $V/F_{n-2r}$, we see that $X((a,a),H)$ parameterizes $r$-dimensional isotropic subspaces of a $2r$-dimensional vector space under a non-degenerate skew-symmetric form. We conclude that $X((a,a),H)$ is isomorphic to $SG(r,2r)$. This variety is irreducible of dimension $\binom{r+1}{2}$. □

**Corollary 4.0.14.** *Let $[H] \in S_r \setminus S_{r-1}$ and $a+b+1 = 2r$, then $X((a,b),H)$ is isomorphic to the isotropic Grassmannian $SG(b,2r)$. In particular, $X((a,b),H)$ is irreducible of dimension $\frac{b(2a-b+3)}{2}$.*

*Proof.* When $a+b+1 = 2r$, we are in Case (1) of Theorem 1.0.3. The integers $a$ and $b$ must satisfy the inequalities $b < r \leq a$. Hence $n-a-b-1 = n-2r \leq j \leq n-r-\frac{a+b+1}{2} = n-2r$. We conclude that $j = n-2r$ and that $X((a,2r-a-1),H)$ is irreducible. The linear space $V_{n-a-1}$ must contain the kernel of $Q_H$, which by assumption has dimension $n-2r = j$. Furthermore, $\dim(V_{n-a-1}^{\perp}) = n-2r+a+1 = n-b$. Hence, $V_{n-b} = V_{n-a-1}^{\perp}$. Therefore, $X((a,2r-a-1),H)$ can be identified with $SG(b,2r)$. □

**Corollary 4.0.15.** *Let* $[H] \in S_r \setminus S_{r-1}$ *and* $a+1 \geq 2r$*. Then* $X((a,0),H)$ *is isomorphic to the Grassmannian* $G(n-a-1, n-2r)$*, hence it is irreducible of dimension* $(n-a-1)(a+1-2r)$*.*

*Proof.* When $b=0$, we are in Case (1) of Theorem 1.0.3. In this case, $n-a-1 \leq j \leq n-a-1$. Hence, there is only one component and $V_{n-a-1}$ is contained in $F_{n-2r}$. Therefore, in this case, $X((a,0),H)$ parameterizes linear spaces $V_{n-a-1}$ contained in $F_{n-2r}$. This is the Grassmannian $G(n-a-1, n-2r)$, which has dimension $(n-a-1)(a+1-2r)$. □

Finally, we prove Proposition 4.0.16, which clearly specializes to Corollary 4.0.17 when $k=2$.

**Proposition 4.0.16.** *Let* $H$ *be a hyperplane in* $\mathbb{P}(\bigwedge^k V)$ *of the form*

$$\Sigma_1(F_{n-k} \subset F_{n-k+2} \subset \cdots \subset F_n).$$

*Let* $\lambda$ *be a partition of the form* $\lambda = (\mu_1^{i_1}, \ldots, \mu_t^{i_t})$*. Let* $\delta$ *denote the Krönecker delta function. Then* $X(\lambda, H)$ *has* $t - \delta_{0,\mu_t}$ *components, where, for* $1 \leq j \leq t - \delta_{0,\mu_t}$*, the component* $Z_j$ *is the Schubert variety in* $F(n-k+k_1-\mu_1, \ldots, n-\mu_t; n)$ *parameterizing flags*

$$(V_{n-k+k_1-\mu_1} \subset \cdots \subset V_{n-\mu_t})$$

*such that* $\dim(V_{n-k+k_j-\mu_j} \cap F_{n-k}) \geq n-k-\mu_j+1$*.*

*Proof.* Let $H = \Sigma_1(F_{n-k} \subset F_{n-k+2} \subset \cdots \subset F_n)$. A Schubert variety $\Sigma_\lambda$ is contained in $H$ if and only if every $k$-dimensional subspace parameterized by $\Sigma_\lambda$ intersects $F_{n-k}$ non-trivially. Let

$$V_{n-k+k_1-\mu_1} \subset V_{n-k+k_2-\mu_2} \subset \cdots \subset V_{n-\mu_t}$$

be the linear spaces defining $\Sigma_\lambda$. Let $W$ be any $k$-dimensional subspace such that $[W] \in \Sigma_\lambda$. If for some $j$, $\dim(V_{n-k+k_j-\mu_j} \cap F_{n-k}) \geq n-k-\mu_j+1$, then we can estimate $\dim(W \cap F_{n-k} \cap V_{n-k+k_j-\mu_j})$ as follows. $\dim(W \cap V_{n-k+k_j-\mu_j}) \geq k_j$ since $[W] \in \Sigma_\lambda$. Hence, $\dim(W \cap F_{n-k} \cap V_{n-k+k_j-\mu_j}) \geq k_j + n - k - \mu_j + 1 - (n-k+k_j-\mu_j) = 1$. We conclude that $[W] \in H \cap G(k,n)$, hence $\Sigma_\lambda \subset H \cap G(k,n)$.

Note that if $\mu_t = 0$, then the condition $\dim(V_{n-\mu_t} \cap F_{n-k}) \geq n-k+1$ is impossible to satisfy. Therefore, that case has to be treated separately.

Conversely, suppose that $\dim(V_{n-k+k_j-\mu_j} \cap F_{n-k}) = n-k-\mu_j$ for every $1 \leq j \leq t$. Then there exists a $k_1$-dimensional subspace in $V_{n-k+k_1-\mu_1}$ that does not intersect $F_{n-k}$. This can be extended to a $k_2$-dimensional subspace in $V_{n-k+k_2-\mu_2}$ that does not intersect $F_{n-k}$. Continuing this way, we construct a $k$-dimensional subspace $W$ such that $[W] \in \Sigma_\lambda$, but $[W] \notin H \cap G(k,n)$.

Let $S_j$ be the Schubert variety in the flag variety $F(n-k+k_1-\mu_1, \ldots, n-\mu_t; n)$ defined by

$$S_j = \{(V_{n-k+k_1-\mu_1} \subset \cdots \subset V_{n-\mu_t} \mid \dim(V_{n-k+k_j-\mu_j} \cap F_{n-k}) \geq n-k-\mu_j+1\}.$$

We have shown that $X(\lambda, H) = \cup_{i=1}^{t-\delta_{0,t}} S_j$. Since the Schubert varieties $S_j \not\subset S_i$ for $i \neq j$, we conclude that the $t - \delta_{0,t}$ Schubert varieties $S_j$ form the irreducible components of $X(\lambda, H)$. This concludes the proof of the proposition. □

**Corollary 4.0.17.** *Let* $[H = \Sigma_1(F_{n-2} \subset F_n)] \in S_1$ *and* $a > b > 0$. *Then* $X((a,b), H)$ *is the union of the following two Schubert varieties in* $F(n-a-1, n-b; n)$

1. $\{(V_{n-a-1} \subset V_{n-b}) \mid V_{n-a-1} \subset F_{n-2}\}$,
2. $\{(V_{n-a-1} \subset V_{n-b}) \mid \dim(V_{n-b} \cap F_{n-2}) \geq n-b-1\}$.

# CHAPTER 5

# FURTHER RESEARCH

Directions for future research include considering: (1) the geometry of intersections of $G(2, n)$ with higher codimension linear spaces; (2) the geometry of intersections of $G(2, n)$ with higher degree hypersurfaces of $\mathbb{P}^N$ (Griffiths and Harris (Griffiths and Harris, 1978) do this when $n = 4$ and the degree is 2); and (3) the extent to which these or similar results hold for $G(k, n)$ when $k$ is greater than two.

Currently we consider the Grassmannian over the complex numbers. However, I am interested in the generalization of my research to arbitrary rings. Ravi Vakil (Vakil, 2006) has described many cases in which intersection theory over Grassmannians can be done over arbitrary commutative rings. Over the course of my career I would like to explore noncommutative algebraic geometry and use my current research as a starting point of investigation.

**APPENDICES**

# Appendix A

# BASIC ALGEBRAIC GEOMETRY FACTS

## A.1 Dual Varieties and Singular Hyperplane Sections

Let $Y \subset \mathbb{P}^r$ be a projective variety, $y \in Y$. If $\mathfrak{m}_y$ is the maximal ideal corresponding to the point $y$, then the projective tangent space $\mathbb{T}_y Y$ is the projective closure of the tangent space $(\mathfrak{m}_y/\mathfrak{m}_y^2)^*$ to the point $y$. If $\dim Y = q$ and $y$ is a smooth point of $Y$, then $\dim \mathbb{T}_y Y = q$. A *tangent hyperplane* to a variety $Y$ is a hyperplane in $\mathbb{P}^r$ that contains the projective tangent space to at least one point $y \in Y$.

$\mathbb{P}^{r*}$ is the set of hyperplanes in projective space of dimension $r$. Given a smooth variety $Y \subset \mathbb{P}^r$, the *dual variety* $Y^*$ in $\mathbb{P}^{r*}$ is the set of tangent hyperplanes to $Y$. One can also view this as the set of singular hyperplane sections of $Y$, as $H \cap Y$ is singular at $y$ iff $\mathbb{T}_y Y \subset H$.

Thus the dual Grassmannian is the subvariety of $\mathbb{P}^*(\bigwedge^k V)$ parameterizing singular hyperplane sections of $G(k, n)$. For more facts about dual varieties, see (Ein, 1986).

**Theorem A.1.1** ((Bertini's Theorem)). *Let $Y$ be a smooth closed subvariety of $\mathbb{P}^r$. Then there exists a hyperplane $H \subset \mathbb{P}^r$ not containing $Y$ such that $H \cap Y$ is smooth, and furthermore the locus of such hyperplanes in $\mathbb{P}^{r*}$ is a dense open subset.*

*Idea of Proof.* Construct an incidence correspondence of points in $Y$ and "bad" hyperplanes, namely hyperplanes $H$ such that either $H \supseteq Y$ or $H \cap Y$ is singular. See (Hartshorne, 1977) or (Shafarevich, 1994) for complete proofs. □

## Appendix A (Continued)

Since $G(2, n)$ is an irreducible, smooth subvariety of $\mathbb{P}^N$, the Bertini Theorem tells us that a general hyperplane section is smooth. A useful fact in classifying the *singular* hyperplane sections of $G(2, n)$ is the following.

**Proposition A.1.2.** *((Shafarevich, 1994)) Let $Y$ be a nondegenerate smooth projective subvariety of $\mathbb{P}^n$ of dimension $m$, $H \subset \mathbb{P}^n$ a hyperplane, and $p \in H \cap Y$. Then $p \in H \cap Y$ is a singular point iff $H \supset T_pY$.*

*Idea of Proof.* If $H \supseteq T_pY$, then $\dim(H \cap Y)$ is one less than $\dim Y$, but $\dim T_p(H \cap Y) = \dim T_pY$. $\square$

# Appendix B

## THE CASE OF $G(2,4)$ IN MORE DETAIL

The goal of this thesis has been to study the geometry of $G(k,n)^*$, the dual variety to the Grassmannian. We focused mainly on the case $k = 2$. In order to study the geometry of the dual of a variety, we must characterize singular hyperplane sections of that variety; to study the geometry of any variety, we can examine moduli spaces of subvarieties. By simultaneously examining the possible subvarieties of smooth hyperplane sections, we can see which types of subvarieties force a hyperplane section to be singular. For the dual Grassmannian a natural place to begin is to investigate moduli spaces of Schubert varieties in hyperplane sections of the Grassmannian, as Schubert classes generate the cohomology ring of the $G(k,n)$. We will construct incidence correspondences, the fibers of whose second projection maps will be precisely the moduli spaces we seek.

The purpose of this appendix is to answer in detail the above questions for $G(2,4)$, the smallest Grassmannian that is not isomorphic to a projective space. We will show that there are only two types of hyperplane sections of $G(2,4)$: those that are smooth and those with singular locus consisting of one point. Also, the largest linear subspace of $\mathbb{P}(\bigwedge^2 V)$ that can be contained in a smooth hyperplane section of $G(2,4)$ is a line in the Plücker embedding.

**Proposition B.0.3.** *The only type of singular hyperplane section of* $G(2,4)$ *is a* $\Sigma_{1,0}$*, which contains only one singular point.*

**Appendix B (Continued)**

*Proof.* Let $X := H \cap G(2,4)$, where $H$ is a hyperplane of $\mathbb{P}(\bigwedge^2 V)$. Suppose $[\Lambda] \in X^{sing}$. By 2.3.1, $\Sigma_{1,0}(\Lambda) \subset X$. But for dimension reasons and since both are irreducible, $\Sigma_{1,0}(\Lambda) = X$, and by (Coskun, 2010), the singular locus of $\Sigma_{1,0}(\Lambda)$ is $\Sigma_{2,2}(\Lambda) = \{[\Lambda]\}$, the result of adding a hook to the tableau of $\Sigma_{1,0}(\Lambda)$.

Now suppose $[\Lambda_1], [\Lambda_2] \in X^{sing}$. Then $\mathbb{T}_{[\Lambda_1]} \subset H$ and $\mathbb{T}_{[\Lambda_2]} \subset H$. But $\dim \mathbb{T}_{[\Lambda_i]} = 4 = \dim H$, so $\mathbb{T}_{[\Lambda_1]} = H = \mathbb{T}_{[\Lambda_2]} \Rightarrow [\Lambda_1] = [\Lambda_2]$. □

This means that every element of $G(2,4)^*$ is a Schubert variety of the form $\Sigma_{1,0}(F_2)$ for some $F_2$. Thus to choose an $F_2$ defining this $\Sigma_{1,0}(F_2)$ is equivalent to choosing a point of $G(2,4)^*$, so $G(2,4)^*$ is isomorphic to $G(2,4)$.

Now we calculate the moduli spaces of Schubert varieties of the form $\Sigma_{1,1}, \Sigma_{2,0}$, and $\Sigma_{2,1}$ in the two types of hyperplane sections of $G(2,4)$. Note that by 2.2.6, each of these is a linear space: $\Sigma_{1,1}$ and $\Sigma_{2,0}$ are each isomorphic to a projective plane, and $\Sigma_{2,1}$ is isomorphic to a line in $\mathbb{P}^5$.

**Proposition B.0.4.** *If $X$ is a smooth hyperplane section of $G(2,4)$, then it contains no planes.*

We first recall a useful result of intersection theory that will allow us to construct an argument by contradiction using excess intersections.

*Fact* B.0.5. (Fulton, Proposition 7.1 and Lemma 7.1) If $V$ and $X$ are schemes of codimension $d$ and dimension $k$, respectively, in $Y$ and

$$V \cap X = W_1 \cup \cdots \cup W_r,$$

## Appendix B (Continued)

then $k - d \leq \dim W_i \leq k$. If $\dim W_i = k - d$, then $[V \cap X] = \sum a_i[W_i]$, $a_i \geq 1$. $\square$

In particular, this means that if the intersection is proper, then $[V \cap X] = [V] \cdot [X]$.

*Proof of Proposition.* Since every projective plane in $G(2,4)$ is either a $\Sigma_{1,1}$ or a $\Sigma_{2,0}$ by **[Harris Ex. 6.9]**, we consider the following two cases.

Suppose $X \supset \Sigma_{1,1}(R)$ for some $\mathbb{P}R \subset \mathbb{P}^3$. Then we will show $X = \Sigma_1(L)$ for some line $\mathbb{P}L$.

[Insert picture.]

Let $p := \mathbb{P}F_1$ be a point not contained in $\mathbb{P}R$. Consider $\Sigma_2(F_1)$, the lines in $\mathbb{P}^3$ that pass through $p$. This is also a plane in the Grassmannian. Since $p$ was chosen generally with respect to $\mathbb{P}R$ (which in this case simply means that we chose $p$ not contained in $\mathbb{P}R$), $\Sigma_2(F_1)$ is general for the action of $GL(4)$ on $G(2,4)$. Thus by the Kleiman-Bertini Theorem, $\Sigma_2(F_1) \cap X$ is proper. Note that $[X] = \sigma_1$, $[\Sigma_2(F_1)] = \sigma_2$, and $\sigma_1 \cdot \sigma_2 = \sigma_{2,1}$. This in particular means that we expect this intersection to be irreducible and reduced if proper. In $G(2,4)$ a representative of $\sigma_{2,1}$ is of dimension 1; its degree is given by $\sigma_1 \cdot \sigma_{2,1} = 1$. Thus, $\sigma_{2,1}$ is the class of a line. But a line in $G(2,4)$ consists of lines meeting in a point and contained in a plane, so $\Sigma_2(F_1) \cap X$ in $G(2,4)$ is a $\Sigma_{2,1}(F_1 \subset Q)$ for some plane $\mathbb{P}Q$.

[Insert picture.]

Let $\mathbb{P}L = \mathbb{P}Q \cap \mathbb{P}R$. We want to show that $\Sigma_1(L) \subset X$. Choose a point $q := \mathbb{P}F_1' \in \mathbb{P}L$ and consider $\Sigma_2(F_1') \cap X$. If this is a proper intersection, then by the above reasoning we expect this also to be a $\Sigma_{2,1}$. But

**Appendix B (Continued)**

$$\Sigma_2(F_1') \cap X \supset \Sigma_{2,1}(F_1' \subset R) \cup \{[\overline{F_1, F_1'}]\}.$$

[Insert picture of $\mathbb{P}R$ containing $q$ with lines in $\mathbb{P}R$ passing through $q$ and one line off $\mathbb{P}R$ through $q$ passing through $p$.]

So we have the extra component containing $[\overline{F_1, F_1'}]$. By the Fact, $[\overline{F_1, F_1'}]$ is part of a component of dimension at least 1, so

$$[\Sigma_2(F_1') \cap X] = \sigma_{2,1} + a\sigma_{2,1}, \qquad a \geq 1,$$

since the only basic class of dimension 1 in $H^*(G(2,4))$ is $\sigma_{2,1}$.

This contradicts what we expect if the intersection is proper, namely that the intersection will be irreducible and reduced, so $\dim(\Sigma_2(F_1') \cap X) = 2$ (the only possibilities were 1 or 2 here), which means $\Sigma_2(F_1') \cap X = \Sigma_2(F_1')$, or in other words, $\Sigma_2(F_1') \subset X$. Since we chose $q$ arbitrarily on $\mathbb{P}L$, we have actually shown that every line meeting $\mathbb{P}L$ is contained in $X$, that is, $\Sigma_1(L)$ is contained in $X$. But they are both irreducible and of the same dimension, so $X = \Sigma_1(L)$, hence $X$ is not smooth.

Now suppose $X$ contains a plane of the form $\Sigma_2(F_1)$ for some $p := \mathbb{P}F_1 \in \mathbb{P}^3$. Choose a plane $\mathbb{P}R \subset \mathbb{P}^3$, $\mathbb{P}R \not\ni p$. Consider lines in $\mathbb{P}R$, $\Sigma_{1,1}(R)$. We expect $[\Sigma_{1,1}(R) \cap X] = \sigma_{1,1} \cdot \sigma_1 = \sigma_{2,1}$.

**Appendix B (Continued)**

Since $\mathbb{P}R$ has been chosen generally (*i.e.*, not containing $p$), again by the Kleiman-Bertini Theorem the intersection is proper. So,

$$\Sigma_{1,1}(R) \cap X = \Sigma_{2,1}(q \subset R)$$

for some $q := \mathbb{P}F_1' \in \mathbb{P}R$. In order to construct an argument similar to the previous case, we have to find something "non-general" and intersect its Schubert variety with $X$. We connect $p$ with $q$ and choose a plane $\mathbb{P}Q$ containing $\mathbb{P}\left(\overline{F_1, F_1'}\right)$. If $\Sigma_{1,1}(Q) \cap X$ were proper, then we would have that the intersection is precisely $\Sigma_{2,1}(F_1 \subset Q)$. But $[L] = [Q \cap R]$ is also contained in the intersection: $\mathbb{P}L$ passes through $q$ and is contained in $\mathbb{P}R$, and it is a line contained in $\mathbb{P}Q$, but it does not pass through $p$, so that

$$\Sigma_{1,1}(Q) \cap X \supset \Sigma_{2,1}(F_1 \subset Q) \cup \{[L]\}.$$

As before, $[L]$ belongs to a component of positive dimension, which contradicts what we expect the intersection to be. We conclude that the intersection is in fact not proper. Hence $\Sigma_{1,1}(Q) \subset X$ for any plane $\mathbb{P}Q$ containing $\mathbb{P}\left(\overline{F_1, F_1'}\right)$, so every line meeting $\mathbb{P}\left(\overline{F_1, F_1'}\right)$ is contained in $X$. By irreducibility and for dimension reasons, $X = \Sigma_1(\overline{F_1, F_1'})$, which means $X$ is not smooth. $\square$

## Appendix B (Continued)

*Remark* B.0.6. There is another way to prove this proposition using incidence correspondences and the theorem on the dimension of fibers; this is a technique that will appear frequently in this work. For example, let

$$\begin{array}{ccc} & \mathscr{I}_{1,1} = \{(F_3, H) \mid \Sigma_{1,1}(F_3) \subset H\} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ G(3,4) & & \mathbb{P}^{5*} \supset G(2,4)^*. \end{array}$$

Given $[F_3] \in G(3,4)$, $\dim \pi_1^{-1}(F_3) = 2$ since for a hyperplane $H$ to contain a $\mathbb{P}^2 \cong \Sigma_{1,1}(F_3) = G(2, F_3)$ imposes 3 conditions on $\mathbb{P}^{5*}$. The map $\pi_1$ is surjective because given an $F_3$ we take the linear span of $\Sigma_{1,1}(F_3)$ and choose a hyperplane $H$ containing that linear span. Since both $G(3,4)$ and the fiber over a general point $[F_3]$ are irreducible, $\mathscr{I}_{1,1}$ is irreducible of dimension 5.

Now we calculate the dimension of $\pi_2^{-1}(G(2,4)^*)$. Let $[\Sigma_{1,0}(F_2')] \in G(2,4)^*$. Note that for a $\Sigma_{1,1}(F_3)$ to be contained in $\Sigma_{1,0}(F_2')$, we need $\mathbb{P}F_2'$ to be contained in $\mathbb{P}F_3$; if $\mathbb{P}F_2' \cap \mathbb{P}F_3$ were only one point, then there would be lines in $\mathbb{P}F_3$ that would miss the point of intersection of $\mathbb{P}F_2'$ and $\mathbb{P}F_3$. Hence

$$\begin{aligned} \pi_2^{-1}(\Sigma_{1,0}(F_2')) &= \{F_3 \mid \Sigma_{1,1}(F_3) \subset \Sigma_1(F_2')\} \\ &= \{F_3 \mid F_2' \subset F_3\}. \end{aligned}$$

**Appendix B (Continued)**

This is the space $G(3-2,4-2) = G(1,2) \cong \mathbb{P}^1$, or more rigorously, this is the Schubert variety $\Sigma_{1,1,0}(F_2')$ in $G(3,4)$, which is isomorphic to $\mathbb{P}^1$ and so is clearly irreducible. Thus $\pi_2^{-1}(G(2,4)^*)$ is an irreducible subvariety of $\mathscr{I}_{1,1}$ of dimension 5, so $\mathscr{I}_{1,1} = \pi_2^{-1}(G(2,4)^*)$. It follows that there does not exist an element of $\mathbb{P}^{5*} \setminus G(2,4)^*$ to which $\pi_2$ maps. We conclude that there does not exist a smooth hyperplane section of $G(2,4)$ that contains a plane of the form $\Sigma_{1,1}$, and the moduli space of $\Sigma_{1,1}$ in a singular hyperplane section is a Schubert variety in $G(3,4)$ isomorphic to $\mathbb{P}^1$.

Similarly we can show that no smooth hyperplane section of $G(2,4)$ contains a $\Sigma_{2,0}$ and that the moduli of such in a singular hyperplane section is a Schubert variety in $G(1,4)$ that is also isomorphic to a projective line.

To calculate the moduli of lines (namely, Schubert varieties of the form $\Sigma_{2,1}$; see (Griffiths and Harris, 1978) and (Harris, 1992)) in a hyperplane section of $G(2,4)$, we use a correspondence involving a partial flag variety. Since a $\Sigma_{2,1}$ depends on an $F_1$ and an $F_3$ where $F_1 \subset F_3$, the parameter space of $\Sigma_{2,1}$ in $G(2,4)$ is isomorphic to $Fl(1,3;4)$, which is of dimension 5.

$$\mathscr{I}_{2,1} = \{(F_1, F_3, H) \mid \Sigma_{2,1}(F_1 \subset F_3) \subset H\}$$

$$\pi_1 \swarrow \qquad\qquad \searrow \pi_2$$

$$Fl(1,3;4) \qquad\qquad \mathbb{P}^{5*} \supset G(2,4)^*.$$

## Appendix B (Continued)

For a hyperplane to contain a line imposes 2 conditions on $\mathbb{P}^{5*}$, so the fiber of $\pi_1$ has dimension 3. Thus $\mathscr{I}_{2,1}$ is irreducible of dimension 8.

In a similar fashion as above, we have that

$$\begin{aligned} \mathcal{F}_{2,1} := \pi_2^{-1}(\Sigma_1(F_2')) &= \{(F_1, F_3) \mid \Sigma_{2,1}(F_1 \subset F_3) \subset \Sigma_1(F_2')\} \\ &= \{(F_1, F_3) \mid F_2' \subset F_3\} \end{aligned}$$

because for all lines in $\mathbb{P}F_3$ passing through the point $\mathbb{P}F_1$ to meet $\mathbb{P}F_2'$, we need that $\mathbb{P}F_2' \subset \mathbb{P}F_3$. We analyze the fiber $\mathcal{F}_{2,1}$ as an incidence correspondence itself:

$$\begin{array}{ccc} & \mathcal{F}_{2,1} & \\ p_1 \swarrow & & \searrow p_2 \\ G(1,4) & & G(3,4) \end{array}$$

Given a general $[F_1] \in G(1,4)$, which means that we choose $F_1$ so that it is not contained in $F_2'$, the vector space $\overline{F_1, F_2'}$ is 3-dimensional, so $\pi_1^{-1}(F_1)$ consists of a single point and the fiber dimension is 0. This shows that $\mathcal{F}_{2,1}$ is irreducible of dimension 3.

*Remark* B.0.7. It is interesting to note that the case $F_1 \subset F_2'$ is parameterized by the 1-dimensional Schubert variety $\Sigma_2$ in $G(1,4)$, and the fiber over such an $[F_1]$ is a $\Sigma_{1,1,0}$ in $G(3,4)$, which is of dimension 1. This is an example of how the fiber dimension may “jump” when points are chosen from closed subvarieties of the space to which the projection morphism maps.

Returning to $\mathscr{I}_{2,1}$, we conclude that $\dim \pi_2^{-1}(G(2,4)^*) = 7 < 8$, so there must exist a point of $\mathscr{I}_{2,1}$ that maps to $\mathbb{P}^{5*} \setminus G(2,4)^*$. In other words, there exists a smooth hyperplane section of $G(2,4)$ containing a line. But since the set of smooth hyperplane sections is homogeneous with respect to the action of $GL(4)$, we have proved the following:

**Proposition B.0.8.** *Every smooth hyperplane section of* $G(2,4)$ *contains a* 3*-dimensional family of lines.* □

# Appendix C

## HIGHER CODIMENSION LINEAR SECTIONS

We show that taking $n$ hyperplane sections of $G(k,n)$ gives a variety with trivial canonical bundle.

**Proposition C.0.9.** *The canonical divisor of the Grassmannian is* $-n\sigma_1$.

*Proof.* Recall the tautological sequence of vector bundles on the Grassmannian $G(k,n)$:

$$0 \to S \to V \otimes \mathscr{O}_{G(k,n)} \to Q \to 0$$

The tangent bundle $T_{G(k,n)}$ is given by $\mathrm{Hom}(S,Q)$ or $S^* \otimes Q$. It is a fact that the canonical divisor $K_X$ of a smooth variety $X$ is $-c_1(T_X)$. Using the splitting principle, suppose $S^* = L_1 \oplus \cdots \oplus L_k$ and $Q = M_1 \oplus \cdots \oplus M_{n-k}$ so that $c_1(S^*) = \alpha_1 + \cdots + \alpha_k$ and $c_1(Q) = \beta_1 + \cdots + \beta_{n-k}$. Then

$$\begin{aligned} c(S^* \otimes Q) = {} & (1+\alpha_1+\beta_1)(1+\alpha_1+\beta_2)\cdots(1+\alpha_1+\beta_{n-k}) \\ & \cdot (1+\alpha_2+\beta_1)(1+\alpha_2+\beta_2)\cdots(1+\alpha_2+\beta_{n-k}) \\ & \cdot \; \cdots \\ & \cdot (1+\alpha_k+\beta_1)(1+\alpha_k+\beta_2)\cdots(1+\alpha_k+\beta_{n-k}) \end{aligned}$$

so that

$$c_1(S^* \otimes Q) = (n-k)(\alpha_1 + \cdots + \alpha_k) + k(\beta_1 + \cdots + \beta_{n-k})$$
$$= (n-k)c_1(S^*) + k\ c_1(Q)$$

Note that $c_1(S^*) = \sigma_1$ because $c_1(S^*)$ is by definition the degeneracy locus of $k$ global sections $s_1, \ldots, s_k$ of $S^*$, in other words, the locus of linear dependence of $k$ linear forms. Locally for a point $\Lambda \in G(k,n)$, this looks like

$$a_1 s_1(\Lambda) + \cdots a_k s_k(\Lambda) = 0, \qquad a_j \text{ not all zero.}$$

This clearly is a homogenous linear equation, so it gives a hyperplane section.

Also, $c_1(Q) = \sigma_1$: if $q_1, \ldots, q_{n-k} \in \Gamma(G(k,n), Q)$, we want the locus of linear dependence of

$$q_1(\Lambda), \ldots, q_{n-k}(\Lambda) \in Q(\Lambda) = V/\Lambda,$$

in other words where

$$b_1 q_1(\Lambda) + \cdots + b_{n-k} q_{n-k}(\Lambda) = 0, \qquad b_\ell \text{ not all zero.}$$

We can view this linear dependence relation of $n-k$ vectors in $V/\Lambda$ as a linear dependence relation of $n$ vectors in $V$ if we include $k$ vectors $\ell_{n-k+1},\ldots,\ell_n$ from $\Lambda$. This gives the equation

$$b_1 q_1(\Lambda)+\cdots+b_{n-k}q_{n-k}(\Lambda)+b_{n-k+1}\ell_{n-k+1}+\cdots+b_n\ell_n=0, \qquad b_\ell \text{ not all zero.}$$

In a vector space of dimension $r$, the locus of $r$ vectors being linearly dependent is a hyperplane. Thus, $c_1(Q)$ is also a hyperplane section $\sigma_1$.

Hence, $c_1(S^*\otimes Q)=(n-k)\sigma_1+k\sigma_1=n\sigma_1$, which says that $K_{G(k,n)}=-n\sigma_1$. □

Observe that the canonical bundle of $G(k,n)$ has no dependence on $k$. We now specialize to $k=2$ and discuss some examples of the geometry of higher codimension linear sections of $G(2,n)$.

**Corollary C.0.10.** *Five hyperplane sections of $G(2,5)$ gives an elliptic curve; every elliptic curve arises as such.*

*Proof.* Let $H^i$ signify the intersection of $i$ general hyperplanes in the Plücker embedding and define $X_i := H^i \cap G(2,5)$. We use the adjunction formula: if $D \hookrightarrow Y$ is a divisor, then $K_D=(K_Y+D)\mid_D$. So we seek the canonical divisor of $X_5$. Since

$$X_5 \hookrightarrow X_4 \hookrightarrow X_3 \hookrightarrow X_2 \hookrightarrow X_1 \hookrightarrow G(2,5)$$

**Appendix C (Continued)**

where each inclusion is of a divisor, we have

$$\begin{aligned} K_{X_1} &= (K_{G(2,5)} + X_1)\big|_{X_1} \\ &= (-5\sigma_1 + \sigma_1)|_{X_1} = -4\,\sigma_1|_{X_1} \end{aligned}$$

The notation "$\sigma_1|_{X_1}$" is to signify that we view this as a divisor in $X_1$, not in $G(2,5)$. Continued use of the adjunction formula gives

$$\begin{aligned} K_{X_2} &= (K_{X_1} + X_2)|_{X_2} \\ &= (-4\,\sigma_1|_{X_1} + \sigma_1|_{X_1})\big|_{X_2} \\ &= (-3\,\sigma_1|_{X_1})\big|_{X_2} = -3\,\sigma_1|_{X_1\cap X_2} = -3\,\sigma_1|_{X_2} \end{aligned}$$

Similarly we have

$$\begin{aligned} K_{X_3} &= (K_{X_2} + X_3)|_{X_3} = -2\,\sigma_1|_{X_3} \\ K_{X_4} &= (K_{X_3} + X_4)|_{X_4} = -1\,\sigma_1|_{X_4} \\ K_{X_5} &= (K_{X_4} + X_5)|_{X_5} = 0\,\sigma_1|_{X_5} \end{aligned}$$

Notice that since $\dim G(2,5) = 6$, five general hyperplane sections give a smooth curve. Since $0 = \deg K_{X_5} = 2g - 2$, where $g$ is the genus of the curve $X_5$, we have that $g = 1$, *i.e.*, $X_5$ is an elliptic curve. Moreover, all elliptic curves arise in this way (Hartshorne, 1977, IV.4). □

## Appendix C (Continued)

**Corollary C.0.11.** *Seven hyperplane sections of* $G(2,7)$ *give a Calabi-Yau threefold.*

*Proof.* Note that $\dim G(2,7) = 10$ so seven hyperplane sections give a threefold. The canonical divisor of seven hyperplane sections of $G(2,7)$ is trivial, so we have a Calabi-Yau threefold. □

**Corollary C.0.12.** *Similarly, six hyperplane sections of* $G(2,6)$ *give a Del Pezzo surface.* □

**Proposition C.0.13** ((Coskun, 2006))**.** *If* $Y$ *is a smooth surface given by four hyperplane sections of* $G(2,5)$*, then there are 10 lines on* $Y$*. Also,* $Y$ *is the Del Pezzo surface that is the result of blowing up* $\mathbb{P}^2$ *at 4 points. The space of such* $H^4 \cap G(2,5)$ *has no moduli,* i.e., *they are all isomorphic to each other.* □

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Vakil, R.: Schubert induction. Ann. Math., 164(2):489–512, 2006.

# VITA

**Richard J. Abdelkerim**

**Research Interests**

Algebraic Geometry: Grassmannians, flag varieties, and Schubert calculus; enumerative geometry; algebraic combinatorics; commutative and noncommutative algebra.

**Education**

Ph.D. in Mathematics
Thesis Title: Geometry of the Dual Grassmannian.
Thesis Advisor: Izzet Coskun.
May 2011 – University of Illinois at Chicago

M.S. in Mathematics

National Science Foundation GK-12 Fellow 2006-2007, Award Number 0440547.

May 2008 – California State University, Northridge

M.A. in Secondary Education

Emphasis on Mathematics Education.

May 2003 – Loyola Marymount University

B.S. in Applied Mathematics

Graduated *cum laude.* Minor in Chicana and Chicano Studies.

June 2001 – University of California, Los Angeles

**Employment Experience**

**Teaching and Research**

Fall 2010 *Research Assistant,* University of Illinois, Chicago, IL.

Supported by Dr. Izzet Coskun under National Science Foundation Career Grant 0952535.

Summer 2008;2010 *Lecturer,* University of Illinois, Chicago, IL.

Taught an intensive 5-week Intermediate Algebra course to incoming undergraduate students for the Summer Enrichment Workshop. Assigned web-based homework and quizzes. Tutored in the Mathematical Sciences Learning Center.

2009–2010 *Teaching Assistant Coordinator,* University of Illinois, Chicago, IL.

Observed new teaching assistants and provided feedback on teaching practice. Ensured healthy communication between new teaching assistants and their faculty supervisors.

2007–present *Teaching Assistant,* University of Illinois, Chicago, IL.

Facilitate discussion sections for large lecture courses, design and grade quizzes, cooperatively grade exams. Tutor in the Mathematical Sciences Learning Center. Grade and type solutions for homework assignments for advanced undergraduate courses.

- *Introduction to Mathematical Reasoning*, Spring 2011

  *Calculus I*, Spring 2008, Summer 2009, Spring 2010
- *Precalculus Mathematics*, Fall 2008, Fall 2009
- *Abstract Algebra*, Spring 2009
- *Honors Calculus II*, Fall 2007
- *Introduction to Advanced Mathematics*, Fall 2007

2001–2002 *Mathematics Teacher,* Thomas Jefferson High School, Los Angeles, CA.

Devised and taught engaging lessons incorporating visual, auditory, and kinesthetic learning styles for Algebra I and II.

### Student Services

2008–present *Scribe,* University of Illinois, Chicago, IL.

Scribe real analysis exams for student through Disability Resource Center.

2003–2006 *Outreach and Retention Counselor,* Science and Math Student Services Center/Educational Opportunity Program, California State University, Northridge, CA.

Advised College of Science and Mathematics majors academically and helped them become self-sufficient in creating a path toward graduation. Determined reinstatement of student financial aid based on life circumstances, goals for academic progress, and appeal history. Designed and implemented retention workshops; followed up with students regularly. Planned, coordinated, and participated in outreach activities with K-12 schools and community colleges.

## Fellowships, Honors, Memberships

Teaching Recognition University of Illinois, Chicago, IL.

Ranked among the top ten percent of teaching assistants based on student nomination statements and faculty review.

National Science Foundation GK-12 Fellowship, California State University, Northridge, CA.

Designed enrichment curriculum for middle school students of underprivileged backgrounds. Implemented lessons incorporating advanced mathematical topics including modular arithmetic and group theory.

American Mathematical Society

Mathematical Association of America

**Seminar Talks**

Graduate Student Algebraic Geometry Seminar, University of Illinois, Chicago, IL.

Organized Summer 2010 student seminar.

- *Littlewood-Richardson Rule*, August 2010

  *A Brief Introduction to Moduli Spaces*, July 2010
- *The Hilbert Scheme*, February 2010
- *Examples of Contractions of Extremal Rays: Fiber Contraction and Divisorial Contraction*, November 2009
- *The Cone of Curves in the Smooth Case*, September 2009
- *Intersection Products*, March 2009
- *Stable Reduction*, October 2008
- *Tsen's Theorem on Function Fields*, October 2007

Undergraduate Math Club, University of Illinois, Chicago, IL.

*Chaos, Fractals, and Non-Integer Dimensions*, March 2008.
Introduced Hausdorff dimension as a finer analog of quantities such as length, area, and volume and exhibited examples of fractals that have non-integer Hausdorff dimension.

**Conferences and Workshops**

- Wall-Crossing in Mathematics and Physics, Department of Mathematics, University of Illinois at Urbana-Champaign, May 2010

  Midwest Algebra, Geometry and their Interactions Conference (MAGIC), University of Notre Dame, April 2010
- The Ohio State University/University of Illinois at Chicago/University of Michigan Weekend Algebraic Geometry Workshop, University of Michigan, Ann Arbor, MI, November 2009
- Spring School "Fourier-Mukai Functors, Regularity on Abelian Varieties, and Generic Vanishing Theorems," University of Michigan, Ann Arbor, MI, May 2009
- Algebraic Geometry and Commutative Algebra: A Conference to Celebrate Robin Hartshorne's 70th Birthday, University of Illinois, Chicago, IL, April 2008

**Computer Skills**

- MyMathLab (Intermediate Algebra, Precalculus), WebAssign (Calculus I)

  Singular, Macaulay2, 4ti2, LaTeX, C++

**Languages**

- English, natively fluent.

  French, natively fluent.

  Spanish, fluent in speaking, reading, and writing.

  Arabic, some proficiency in speaking, reading, and writing.

**References**

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