Dynamics of Equicontinuous Group Actions on Cantor Sets

BY

JESSICA CELESTE DYER
B.S., University of California, Santa Barbara, 2007
M.A.St., University of Cambridge, UK, 2009
M.S., University of Illinois at Chicago, 2011

THESIS
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Defense Committee:
      Steven Hurder, Chair and Advisor
      Olga Lukina
      Alexander Furman
      Christian Rosendal
      Ilie Ugarcovici, DePaul University

Contribution of Authors

This thesis is wholly my own writing, with some editorial advice from Professor Steven Hurder and Dr. Olga Lukina. The results in this thesis began with a program of study of minimal group actions on Cantor sets, and the construction of generalized Bratteli diagrams to model these actions, undertaken under the direction of my advisor, Professor Steven Hurder. This work led me to the results and examples in Chapters 2 and 3 of the thesis.

Professor Hurder suggested that I consider the problem of the automorphism groups of minimal Cantor actions, and to study a published work of Professors Fokkink and Oversteegen [**11**]. This led directly to the results in Chapters 4, 5 and 6 of the thesis.

Dr. Lukina suggested that I consider the role of a special inverse limit group that is associated to a group chain, as this group was mentioned in passing in prior published work of M.-I. Cortez and S. Petite. I explored the properties of this group, which was named the discriminant group in my thesis. Chapters 7 and 8 are the results of this study. Chapter 9 contains my calculations of the group N_∞ for each of the examples.

My work on this thesis was discussed as it developed in a weekly seminar with Professors Hurder and Lukina. I contributed key ideas to this work during these collaborative meetings, and wrote all the theorems and their proofs, and calculated the multiple examples in the thesis. The preprint [**10**] coauthored by Dyer, Hurder, and Lukina, which has been submitted for publication, is a partial presentation of the results of the discussions in this seminar, and contains some of the results from this thesis.

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<div align="right">JCD</div>

Summary

A Vietoris solenoid ([**23**], [**1**]) is the inverse limit of $n-to-1$ covering maps over the torus, which are all homogeneous. In [**17**], McCord introduced generalized weak solenoids and showed that they are homogeneous if the monodromy action is defined by a normal chain. Schori in [**21**] showed by construction that non-homogeneous weak solenoids exist. Rogers and Tollefson in [**20**] showed that there exists a weak solenoid that is not defined by a normal chain, but is homogeneous. They also constructed in [**19**] a non-homogeneous solenoid given by covering maps which are regular from level i to $i-1$, but whose composition onto the base space is non-regular. Fokkink and Oversteegen in [**11**] gave a criterion in terms of defining group chains for a weak solenoid to be homogeneous, i.e. for the monodromy action to be regular. In this work, we investigate further the properties of the dynamics of group chains. We show that each group chain yields a minimal equicontinuous Cantor dynamical system. Conversely, we use a method of Kakutani-Rokhlin partitions to show that minimal equicontinuous Cantor dynamical systems can be represented by group chains. We then use their associated chains to classify minimal equicontinuous Cantor dynamical systems as regular, weakly regular, or irregular. We show that this classification is an invariant of the cardinality of the set of orbits of the Automorphism group. We consider the set $\mathfrak{G}_\phi$ of all group chains associated to a dynamical system, and show that the classification as regular, weakly regular, or irregular is an invariant of the number of equivalence classes of chains in $\mathfrak{G}_\phi$. We introduce a new invariant of a dynamical system called the *discriminant group*, and show that its cardinality is related to the degree of non-homogeneity of the system. We give new proofs using group chains of the irregularity of the Schori and Rogers and Tollefson solenoids, and we introduce new examples of group chains which are weakly regular and have either finite or infinite discriminant group.

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CHAPTER 1

Introduction

Let G be a finitely generated group and let X be a Cantor set, that is, a space that is perfect, metrizable, compact, and totally disconnected. Suppose G acts on X by homeomorphisms, that is, there is a homomorphism $\phi : G \to Homeo(X)$ that associates to each group element $g \in G$ a homeomorphism $\phi_g : X \to X$. We will also use the notation $g \cdot x = \phi_g(x)$ where convenient. Let the action be minimal, that is, every orbit $G \cdot x = \{g \cdot x \mid g \in G\}$ is dense in X. The action ϕ of G on X gives a *dynamical system*, which we denote by (X, G, ϕ) (or just (X, G) if the action is clear). If we distinguish a basepoint $x \in X$, then we call (X, G, ϕ, x) (or just (X, G, x)) a *pointed dynamical system*.

We then ask, how to classify such dynamical systems? Classification problems deal with finding invariants to classify dynamical systems up to various levels of equivalence. There are several different types of equivalence that may result in stronger or weaker classification theorems. The strongest equivalence relation between dynamical systems is topological conjugacy; two dynamical systems (X, G, ϕ) and (Y, H, ψ) are said to be *(topologically) conjugate* if there exists a homeomorphism $z : X \to Y$ such that $z(g \cdot x) = g \cdot z(x)$, i.e. such that $z(\phi_g(x)) = \psi_g(z(x))$. Two pointed dynamical systems, (X, G, ϕ, x) and (Y, H, ψ, y) are said to be *pointed conjugate* if there exists a homeomorphism $z : X \to Y$ such that $z(g \cdot x) = g \cdot z(x)$ and such that $z(x) = y$. A weaker form of equivalence is orbit equivalence. The systems (X, G, ϕ) and (Y, H, ψ) are *orbit equivalent* if there is a homeomorphism $z : X \to Y$ which maps orbits of ϕ into orbits of ψ, without necessarily preserving the time parametrization of orbits. A group action is *equicontinuous* if for all $\varepsilon > 0$ there exists $\delta = \delta_\varepsilon > 0$ such that for all $g \in G$ and any $x, y \in X$, if $d(x, y) < \delta_\varepsilon$, then $d(\phi_g(x), \phi_g(y)) < \varepsilon$. This means that if we start with two points $x, y \in X$ that are "close together", and apply our action to them, the resulting points $\phi_g(x), \phi_g(y)$ are also "close together" in a precise way.

In this thesis, we confine ourselves to the study of Cantor dynamical systems that are minimal and equicontinuous, and we concentrate on the case where the group G is finitely generated, and is not required to be abelian. There are many situations where such Cantor dynamical systems arise naturally. One important source of examples which we consider here, arises as the monodromy

actions of weak solenoids. A *weak solenoid* M_∞ is the inverse limit of closed manifolds $\{M_i \mid i \geq 0\}$ which are proper finite-to-one covering spaces of the base space M_0. The projection of M_∞ onto every manifold in the sequence is a fiber bundle, and the fiber $\mathfrak{C}$ of the inverse limit space M_∞ over a point in M_0 is a Cantor set. The monodromy action of the fundamental group of M_0 on the fiber $\mathfrak{C}$ is minimal and equicontinuous. For example, if the solenoid M_∞ is defined by a tower of coverings of a compact surface with genus ≥ 2, then the fundamental group of M_0 is non-abelian and the monodromy action on the fiber satisfies the criteria above. The minimal Cantor actions which arise in this way are examples of dynamical systems for which this thesis applies.

The *regular solenoids* form a special subclass of the weak solenoids. These spaces were introduced and studied by McCord in [**17**], and so are sometimes also called McCord solenoids. If for every $i > 0$ the composition of covering maps

$$p_0^i = p_{i-1}^i \circ p_{i-2}^{i-1} \circ \cdots p_0^1 \colon M_i \to M_0$$

is a regular, or normal covering, then the weak solenoid M_∞ a regular solenoid.

McCord showed in [**17**] that a regular solenoid M_∞ is *homogeneous*, meaning that for every $x, y \in M_\infty$, there is a homeomorphism $h : M_\infty \to M_\infty$ with $h(x) = y$. Rogers and Tollefson in [**20**] showed by example that there exists a weak solenoid with every covering map p_0^i is non-regular, but for which the inverse limit space M_∞ homogeneous. In contrast to this, the Schori solenoid constructed in [**21**] is an inverse limit of covering spaces over a surface of genus 2, which is a non-homogeneous weak solenoid. Also, the Rogers and Tollefson solenoid constructed in [**19**] is an inverse limit of covering spaces over the Klein bottle which is non-homogeneous.

Rogers and Tollefson asked in [**20**], how can we tell if a weak solenoid is homogeneous or not?

Fokkink and Oversteegen gave a solution to this question in the work [**11**]. Associated to a weak solenoid is a nested chain of subgroups, defined by letting $G = \pi_1(M_0, m)$ be the fundamental group of the base space, and $G_i = (p_0^i)_*(\pi_1(M_i, m_i)) \subset G$ be the projection of the fundamental group of the i-th space onto the base space. They introduced the class of *weakly normal* group chains, and showed that a solenoid M_∞ is homogeneous if and only if its associated group chains is weakly normal. We explain this concept in more detail next.

Fix a finitely-generated group G. A *group chain* (G_i) in G is an infinite nested chain of finite index subgroups

$$G = G_0 > G_1 > G_2 > \ldots$$

where each $G_i < G_{i-1}$ is a proper subgroup. The intersection $K = \bigcap\limits_i G_i$ is called the *kernel* of the group chain (G_i) and is a subgroup of G, which is not assumed to be trivial.

Since G_i has finite index in G, G/G_i is a finite coset space. We consider the inverse limits of chains of coset spaces from group chains, with bonding maps given by coset inclusion. This inverse limit $G_\infty = \varprojlim \{G/G_i \to G/G_{i-1}\}$ is a Cantor set, and the group G acts on it minimally and equicontinuously by component-wise multiplication, giving the dynamical system (G_∞, G). There is a natural choice of basepoint in G_∞, the sequence (eG_i) with every entry the coset of the identity. Then $(G_\infty, G, (eG_i))$ is a minimal equicontinuous pointed dynamical system.

Given an equicontinuous minimal Cantor dynamical system, we can associate to it a (non-unique) group chain, using a method of nested partitions. It is standard to use nested clopen partitions called Kakutani-Rokhlin partitions to represent actions on Cantor sets. For example, such partitions were used by Herman, Putnam, and Skau in [14] to build Bratteli diagrams classifying a minimal Cantor dynamical system defined by an action of $\mathbb{Z}^n$ up to orbit equivalence, and similar partitions were used by Cortez and Petite in [8] to study group actions on a Cantor set of a finitely generated group G. In this work, we use a method of coding based on work by Clark and Hurder in [4] which applies for all minimal equicontinuous actions of a group G.

Given a minimal equicontinuous pointed Cantor dynamical system (G, X, ϕ, x), we build a nested sequence of clopen partitions $\mathcal{P}_i = \{P_1^i, P_2^i, ..., P_{m_i}^i\}$ called an Almost Finite (AF) Presentation, with the property that the action of G permutes the elements of the partition, so each $P_k^i = g \cdot P_1^i$ for some $g \in G$, and as $i \to \infty$, the diameter of each partition element approaches zero, and with the basepoint $x \in P_1^i$ for all i. We then let $G_i = \{g \in G \mid g \cdot P_1^i = P_1^i\}$ be the isotropy group of P_1^i. This gives a group chain (G_i) *associated to* the system (G, X, ϕ, x). We show that in this case there is a homeomorphism $X \to G_\infty$ that preserves basepoints and is equivariant with respect to the G-actions on X and on G_∞.

In this thesis, we study the set $\mathfrak{G}$ of group chains of a finitely-generated group G, and develop algebraic invariants for group chains which determine dynamical and geometric properties of the associated Cantor minimal systems (G_∞, G). We introduce the notions of equivalence and conjugacy between group chains, which extend ideas introduced in Fokkink and Oversteegen in [11]. We show that two AF Presentations for the same pointed system yield equivalent group chains, by the notion of group chain equivalence defined below, which we will show corresponds to pointed conjugacy of the associated dynamical systems. Thus, we use AF Presentations to show that any minimal

equicontinuous Cantor system can be represented by a group chain dynamical system, justifying writing the rest of our work primarily in the language of group chains.

There is one important difference between our setting and that of weak solenoids. A chain of covering spaces $\{M_i\}$ yields the same (up to homeomorphism) weak solenoid if we remove a finite number of levels. In particular, if we remove the first n levels, we still have the same solenoid, up to homeomorphism. However, the fundamental group of the new base space $G_n = \pi_1(M_n, m_n)$ is not isomorphic to $G_0 = \pi_1(M_0, m_0)$, since the proper coverings yield proper subgroups of the fundamental group. In the setting of group chains, we require that $G = G_0$ is fixed.

In order to state our main results, we recall a notion of group chain equivalence from [20].

DEFINITION 1.1. *The group chains* $(G_i), (H_i)$ *in* G *are* equivalent *if, for every* i, *there is a* j *such that* $G_i > H_j$ *and* $H_i > G_j$.

We will prove that group chain equivalence corresponds to pointed conjugacy of dynamical systems.

THEOREM 1.2. *Let* (G_i) *and* (H_i) *be group chains in* G, *with associated inverse limits*

$$G_\infty = \varprojlim \{G/G_i \to G/G_{i-1}\}$$

$$H_\infty = \varprojlim \{G/H_i \to G/H_{i-1}\}.$$

Then, the following are equivalent:

(1) (G_i) *and* (H_i) *are associated to the same pointed dynamical system* (X, G, ϕ, x).
(2) *The systems* $(G_\infty, G, (eG_i))$ *and* $(H_\infty, G, (eH_i))$ *are pointed conjugate.*
(3) (G_i) *and* (H_i) *are equivalent group chains.*

We recall from [11] a notion of conjugate equivalence between group chains.

DEFINITION 1.3. *The group chains* $(G_i), (H_i)$ *in* G *are* conjugate equivalent *if there exists a sequence of elements* $(g_i) \in G$, *with* $g_i H_i = g_j H_i$ *for all* $j \geq i$, *such that the chain* $(g_i H_i g_i^{-1})$ *is equivalent to* (G_i). *We then write* $(G_i) \sim (g_i H_i g_i^{-1})$

We then have the general result for topological conjugacy without preserving basepoints.

THEOREM 1.4. *Let* (G_i) *and* (H_i) *be group chains in* G, *with associated inverse limits*

$$G_\infty = \varprojlim \{G/G_i \to G/G_{i-1}\}$$

$$H_\infty = \varprojlim \{G/H_i \to G/H_{i-1}\}.$$

Then, the following are equivalent:

(1) (G_i) *and* (H_i) *are associated to the same dynamical system* (X, G, ϕ).

(2) *The systems* (G_∞, G) *and* (H_∞, G) *are topologically conjugate.*

(3) *There exists a sequence* (g_i) *in* G *with* $g_i H_i = g_j H_i$ *for all* $j \geq i$ *such that* $(g_i H_i g_i^{-1})$ *is equivalent to* (G_i).

The conclusions from Theorem 1.2 is that algebraic invariants of a group chain $(G_i) \in \mathfrak{G}$ which are invariant under group chain equivalence, provide pointed conjugacy invariants of the associated dynamical system. Likewise, Theorem 1.4 implies that algebraic invariants of a group chain $(G_i) \in \mathfrak{G}$ which are invariant under conjugate group chain equivalence, provide topological conjugacy invariants of the associated dynamical system.

With this philosophy in mind, we introduce several classes of group chains that we will study.

DEFINITION 1.5. *Let* G *be a finitely-generated group, and* $(G_i) \in \mathfrak{G}$ *a group chain with kernel* $K(G_i)$. *Then* (G_i) *is:*

(1) <u>normal</u> *if every* G_i *is a normal subgroup of* G.

(2) <u>almost normal</u> *if there is some subgroup* $N < G$ *and an index* i_0 *so that every* G_i *for* $i \geq i_0$ *is normal in* N.

(3) <u>regular</u> *if it is* equivalent *to a normal chain* (N_i) *with* $N_i \lhd G$ *for all* i *(but the* G_i *may not themselves be normal in* G*).*

(4) <u>weakly regular</u> *if there is some* $N < G$ *and some* i_0 *such that the chain* $(G_i)_{i \geq i_0}$ *is regular inside* N.

(5) <u>irregular</u> *if it is not weakly regular (and therefore not regular or normal).*

Condition (1.5.4) is a modification of the definition of weakly normal chains as defined by Fokkink and Oversteegen in [**11**].

One of the main tools for the study of a group chain $(G_i) \in \mathfrak{G}$ is the construction of its associated core chain $(Core_G(G_i))$, where

$$C_i = Core_G(G_i) = \bigcap_{g \in G} g G_i g^{-1}$$

is the normal core of (G_i) in G. Observe that C_i is a normal subgroup of G, and has finite index. Then define:

DEFINITION 1.6. *The* <u>core chain</u> *of* $(G_i) \in \mathfrak{G}$ *is the normal chain* $(Core_G(G_i)) \in \mathfrak{G}$.

We give equivalent conditions for regular and weakly regular group chains, following [**11**].

THEOREM 1.7 ([**11**]). *For a group chain* (G_i) *in* G, *the following are equivalent:*

(1) (G_i) *is regular.*

(2) (G_i) *is equivalent to the chain* $(Core_G(G_i))$.

(3) *For every sequence* $\{g_i\}$ *in* G *with* $g_j G_i = g_i G_i$ *for every* $j \geq i$, *we have* $(G_i) \sim (g_i G_i g_i^{-1})$.

We have similar equivalent conditions for the weakly regular case.

THEOREM 1.8. *For a group chain* (G_i) *in* G, *the following are equivalent:*

(1) (G_i) *is weakly regular.*

(2) *There exists an index* i_0 *and a subgroup* $N < G$ *such that for* $i \geq i_0$, $G_i < N$, *and such that* $(G_i)_{i \geq i_0}$ *is equivalent to the chain* $(Core_N(G_i))_{i \geq i_0}$.

(3) *There exists an index* i_0 *and a subgroup* $N < G$ *such that for* $i \geq i_0$, $G_i < N$, *and such that for every sequence* $\{h_i\}$ *in* N *with* $h_j G_i = h_i G_i$ *for every* $j \geq i$, *we have* $(G_i)_{i \geq i_0} \sim (h_i G_i h_i^{-1})_{i \geq i_0}$.

We show that being regular, weakly regular, or irregular, is not just a property of the particular group chain, but a shared property of all group chains associated to a given system. Thus, we can refer to a dynamical system as regular, weakly regular, or irregular. We first show this for the case of pointed dynamical systems.

PROPOSITION 1.9. *Let* (G_i) *and* (H_i) *be group chains associated to the same pointed dynamical system. Then,*

(1) (G_i) *is regular if and only if* (H_i) *is regular.*

(2) (G_i) *is weakly regular if and only if* (H_i) *is weakly regular.*

(3) (G_i) *is irregular if and only if* (H_i) *is irregular.*

We then show the same invariance for topological conjugacy (without preserving basepoints).

THEOREM 1.10. *Let (G_i) and (H_i) be group chains associated to the same (non-pointed) dynamical system. Then the following holds.*

    (1) (G_i) is regular if and only if H_i is regular.

    (2) (G_i) is weakly regular if and only if H_i is weakly regular.

    (3) (G_i) is irregular if and only if H_i is irregular.

Thus, weak regularity is an invariant of dynamical systems up to conjugacy.

We study the automorphism group of a dynamical system. An *automorphism* of the Cantor dynamical system (X, G, ϕ) is a homeomorphism $h : X \to X$ which is equivariant with respect to the group action, that is, $g \cdot h(x) = h(g \cdot x)$. Let $Aut(X, G, \phi)$ be the automorphism group of the system. In [**11**], Fokkink and Oversteegen asked whether the group of automorphisms of a weak solenoid can be classified up to isotopy. In [**3**], Clark and Fokkink studied the automorphism group in the case where (G_i) is a regular group chain with $G_i \lhd G$ for all i. We prove the following classification theorem:

THEOREM 1.11. *Let (X, G, ϕ) be a minimal equicontinuous Cantor dynamical system.*

    (1) (X, G, ϕ) is regular if and only if $Aut(X, G, \phi)$ acts transitively on X.

    (2) (X, G, ϕ) is weakly regular if and only if $Aut(X, G, \phi)$ has a finite number of orbits in X.

    (3) (X, G, ϕ) is irregular if and only if $Aut(X, G, \phi)$ has an infinite number of orbits in X.

We now introduce a new invariant of a class of group chains associated to a dynamical system. Let $C_i = Core_G(G_i) = \cap_{g \in G} g G_i g^{-1}$ be the *(normal) core* of G_i in G, and let $D_i = G_i/C_i$, which is a group since C_i is normal in G_i. We define the *discriminant group* of a group chain (G_i) to be $D_\infty = \varprojlim \{D_i \to D_{i-1}\}$, where the bonding maps are given by inclusion. D_∞ is a profinite group.

We show that in the regular and weakly regular case, the group itself is an invariant of the system. Whether this is also true in the irregular case we leave as an open question.

THEOREM 1.12. *Let (X, G, ϕ) be a weakly regular minimal equicontinuous Cantor dynamical system. Let $(G_i), (H_i)$ be two group chains in G associated to (X, G, ϕ). Let D_∞ be the discriminant group of (G_i), and let D'_∞ be the discriminant group of (H_i). Then, D_∞ and D'_∞ are isomorphic as topological groups.*

We show that in the regular case, the discriminant group is trivial.

THEOREM 1.13. *Let (X, G, ϕ) be a minimal equicontinuous Cantor dynamical system with associated group chain (G_i) and discriminant group D_∞. Then (X, G, ϕ) is regular if and only if D_∞ is the trivial group.*

We also study the weakly regular case. We show that if the discriminant group is finite, then the system must be weakly regular. However, we show by example that there exist weakly regular systems with either finite or infinite discriminant groups, so the converse of that theorem does not hold.

THEOREM 1.14. *Let $D_\infty = \varprojlim \{G_i/C_i \to G_{i-1}/C_{i-1}\}$ be the discriminant group of a chain (G_i). If D_∞ is finite, then (G_i) is weakly regular.*

We construct an example given by a semi-direct product of $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ which is weakly regular but with infinite discriminant group, so the converse of Theorem 1.14 does not hold.

We also give new proofs that the Schori and Rogers and Tollefson weak solenoids are irregular.

A *Vietoris solenoid* is given by a sequence of p_i to one covering spaces over the torus, where $\{p_i\}$ is a sequence of primes ([**23**] [**1**]). Since the fundamental group of the torus is abelian, in both of these cases all associated subgroups are normal, so all group chains are normal, and thus regular, and thus all have trivial discriminant group.

Two Vietoris solenoids given by two sequences of primes, $\{p_i\}$ and $\{q_i\}$ are homeomorphic if we can delete a finite number of primes from each sequence so that the resulting sequences have each prime appear the same number of times in each sequence [**1**].

Writing $\mathbb{Z}/p\mathbb{Z} = \mathbb{Z}_p$, a Vietoris solenoid S_p with associated sequence of primes $\{p_i\}$ gives a group chain given by $G_i = \mathbb{Z}_{p_1 p_2 \ldots p_i}$, and a solenoid S_q with sequence $\{q_i\}$ has associated group chain $H_i = \mathbb{Z}_{q_1 q_2 \ldots q_i}$. Notice that $\mathbb{Z}_{p_1 p_2 \ldots p_i}$ is a subgroup of $\mathbb{Z}_{q_1 q_2 \ldots q_i}$ if and only if $p_1 p_2 \ldots p_i$ divides $q_1 q_2 \ldots q_j$. Since p_k, q_k are all prime, this means that the list $\{p_1, p_2, \ldots, p_i\}$ is a subset of the list $\{q_1, q_2, \ldots, q_j\}$. So, we have group chain equivalence, that is, for every i there is a j so that $G_j < H_i$ and $H_j < G_i$, implies that S_p and S_q are equivalent as solenoids. Thus, the group chain equivalence definition of Fokkink and Oversteegen extends the idea of equivalence between Vietoris solenoids.

An *odometer*, or *adding machine*, is the action of $\mathbb{Z}$ on a sequence of finite quotients $\mathbb{Z}/p^i\mathbb{Z}$. Cortez and Petite consider in [**8**] a generalization of this notion. They study group chains (G_i) with the condition that $\cap_i G_i = \{e\}$. If (G_i) is a normal chain, they called this system a $G-$odometer, and if G_i need not be a normal subgroup, they call the system a $G-$subodometer. We give examples

of group chains with G being the discrete Heisenberg group, but we do not require $\cap_i G_i = \{e\}$. If the kernel $K = \cap_i G_i$ is normal in G, we can mod out by K to obtain a quotient chain (G_i/K) and thus reduce to the G-odometer case. As we allow K to be a non-normal subgroup of G, our approach is more general. The structure of subgroups of the discrete Heisenberg group was given by Littlewood, Şahin, and Ugarcovici in [**15**], where they classify G−odometers given by group chains with G_i normal in G, by whether they can be represented as a product of 1-dimensional odometers. We consider further examples of group chains in the Heisenberg group where the subgroups need not be normal.

THEOREM 1.15. *There exist group chains that are regular, weakly regular but not regular, and irregular.*

Let $G = (\mathbb{Z}^3, *)$ *be the discrete Heisenberg group, with $*$ given by $(x, y, z) * (x', y', z') = (x + x', y + y', z + z' + xy')$. Let $(G_n = M_n\mathbb{Z}^2 \times m_n\mathbb{Z})$ be a group chain in G.*

(1) *The chain with $G_n = M_n\mathbb{Z}^2 \times m_n\mathbb{Z}$, where $M_n = \begin{pmatrix} p^n & 0 \\ 0 & p^n \end{pmatrix}$ and $m = p$ is regular.*

(2) *The chain with $M_n = \begin{pmatrix} qp^n & pq^n \\ p^{n+1} & q^{n+1} \end{pmatrix}$ and $m_n = p$ is weakly regular, but not regular.*

(3) *The chain with $M_n = \begin{pmatrix} p^n & 0 \\ 0 & q^n \end{pmatrix}$ and $m_n = p^n$ is irregular.*

We construct examples to show that the discriminant group of a weakly regular system can be either finite or infinite.

THEOREM 1.16. *There exist weakly regular systems with finite and infinite discriminant groups.*

(1) *Let H be a finite simple group, Γ a finitely generated group, $G = H \times \Gamma$, (Γ_i) a normal chain in Γ, K a nontrivial subgroup of H, and $G_i = K \times \Gamma_i$. Then (G_i) is not regular but is weakly regular, and has finite discriminant group.*

(2) *Let $\Gamma = \mathbb{Z}^2$, and let p, q be distinct primes. Let $\Gamma_i = p^i\mathbb{Z} \times q^i\mathbb{Z}$, and let $H = \mathbb{Z}/2\mathbb{Z} = \{1, t \mid t^2 = 1\}$. Let $\theta : H \to Aut(\Gamma)$ be the homomorphism given by letting θ_1 be the identity map, and θ_t be the transpose map. Let $G = \Gamma \rtimes_\theta H \cong \mathbb{Z}^2 \rtimes_\theta \mathbb{Z}/2\mathbb{Z}$. Let $G_i = \Gamma_i \times \{1\}$. Then (G_i) is weakly regular but not regular in G, and has infinite discriminant group.*

The rest of the thesis is structured as follows.

Chapter 2 shows that any group chain yields a minimal equicontinuous Cantor dynamical system, and presents the construction of our main examples of group chains, which we will refer to throughout the thesis as we explore their various properties.

Chapter 3 defines AF Presentations, shows they exist, and uses them to show that, given a minimal equicontinuous Cantor dynamical system, we can always represent it by an associated group chain system.

In Chapter 4, we prove theorems relating group chain equivalence and conjugacy, and discuss the role of changing or preserving basepoints.

In Chapter 5, we define regular, weakly regular, and irregular group chains, and prove the related classification theorems. We also extensively revisit our examples and classify them as regular, weakly regular, or irregular.

In Chapter 6, we define the automorphism group of a dynamical system, and prove Theorem 1.11.

In Chapter 7, we define the Discriminant group, and prove it is an invariant of a weakly regular system.

In Chapter 8, we consider the Discriminant group of weakly regular systems. We show that if the Discriminant group is finite, then the system must be weakly regular, but the converse does not hold and there do exist weakly regular systems with infinite discriminant groups.

In Chapter 9, we define the action of N_∞ and explore our examples in this setting.

Some of the results in this thesis are included in the paper [10] by the author, S. Hurder and O. Lukina, which has been submitted for publication.

CHAPTER 2

Construction of Examples

Recall from the introduction that a *group chain* (G_i) in G is an infinite chain of proper finite index nested subgroups of G,

$$G = G_0 > G_1 > G_2 > \cdots$$

In this chapter, we first show that each such group chain yields a minimal equicontinuous Cantor dynamical system. Then, we construct our main examples of group chains. We will refer back to these examples and their associated dynamical systems throughout the subsequent chapters.

1. Group Chains to Dynamical Systems

We begin by showing that a group chain yields an associated minimal equicontinuous Cantor dynamical system. (We will show the converse of this statement in Theorem 3.1). We then construct examples of group chains whose associated systems we will classify throughout the thesis.

We first recall the definition of an inverse limit, and some basic properties of an inverse limit of finite sets.

Consider a sequence of finite sets X_i, each equipped with the discrete topology. Their product, $\prod_i X_i$, is given the product topology. Suppose we also have a sequence of maps $f_{i-1}^i : X_i \to X_{i-1}$, which we call *bonding maps*. For $m < n$, we denote their composition as $f_m^n = f_{n-1}^n \circ f_{n-2}^{n-1} \circ ... \circ f_m^{m-1}$.

DEFINITION 2.1. *The* inverse limit X_∞ *of the sequence of maps and spaces* $f_{i-1}^i : X_i \to X_{i-1}$ *is the compact topological subspace*

$$X_\infty = \varprojlim \{X_i, f_{i-1}^i\} = \{(x_0, x_1, x_2, \ldots) \mid f_{i-1}^i(x_i) = x_{i-1}\} \subset \prod_{i \geq 0} X_i.$$

With the subspace topology from the product topology on $\prod_{i \geq 0} X_i$.

DEFINITION 2.2. *A* Cantor set *is any set that is metrizable, totally disconnected, compact, and perfect.*

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PROPOSITION 2.3. *Let $\{X_i\}$ be a sequence of finite sets with bonding maps $f_{i-1}^i : X_i \to X_{i-1}$. Then, the inverse limit $X_\infty = \varprojlim \{X_i, f_{i-1}^i\}$ is a Cantor set with a clopen basis for its topology given by cylinder sets.*

For proofs of these properties, see for example [**18**].

We now show that, given any group chain, there is an associated inverse limit Cantor set on which G acts minimally and equicontinuously.

THEOREM 2.4. *Given a group chain (G_i) in G, there is an associated Cantor set G_∞ with a minimal equicontinuous $G-$action.*

PROOF. Let $X_i = G/G_i$, which is a finite, nonempty set of points (and not a singleton, since the subgroups are proper). Note that so far we have made no assumption of normality, so X_i is a coset space but may not be a group. We have bonding maps $f_{i-1}^i : X_i \to X_{i-1}$ given by coset subset inclusion; that is, $gG_i \subset hG_{i-1}$ if and only if $f_{i-1}^i(gG_i) = hG_{i-1}$. We will refer to this type of map as a *coset inclusion map*. Notice that, if $gG_i \subset hG_{i-1}$, then hG_{i-1} can also be written gG_{i-1} (due to ambiguity in coset representatives). So, we can equivalently define the map as $f_{i-1}^i(gG_i) = gG_{i-1}$. We will deal with several coset inclusion maps in this chapter, and we will use these characterizations interchangeably.

Let $X_\infty = \varprojlim \{X_i, f_{i-1}^i\}$ be the inverse limit (as in Definition 2.1). By Proposition 2.3, X_∞ is a Cantor set. Since each $x_i \in X_i = G/G_i$ is a coset, we can also use the notation

$$G_\infty = \varprojlim \{G/G_i, f_{i-1}^i\} = \{(g_0G_0, g_1G_1, g_2G_2, \ldots) \mid f_{i-1}^i(g_iG_i) = g_{i-1}G_{i-1} \}.$$

So then $G_\infty = X_\infty$.

The inverse limit G_∞ can be pictured as a tree (Figure 1), with each element an infinite branch. The nodes at the nth level of the tree are the cosets G/G_n, with a single point G/G at level 0. The basic open sets are cylinder sets, which are given by fixing the nth level of a branch, and then taking all infinite branches which emanate from that node. Given a finite branch $(g_1G_1, ...g_nG_n)$, we write the cylinder set as $C[(g_1G_1, ...g_nG_n)] = \{(h_iG_i) \in G_\infty \mid h_i = g_i \forall i \leq n\}$.

There is a natural left action of G on G_∞ given by component-wise multiplication:

$$h \cdot (g_0G_0, g_1G_1, g_2G_2, \ldots) = (hg_0G_0, hg_1G_1, hg_2G_2, \ldots).$$

LEMMA 2.5. *The action of G on X_∞ is equicontinuous.*
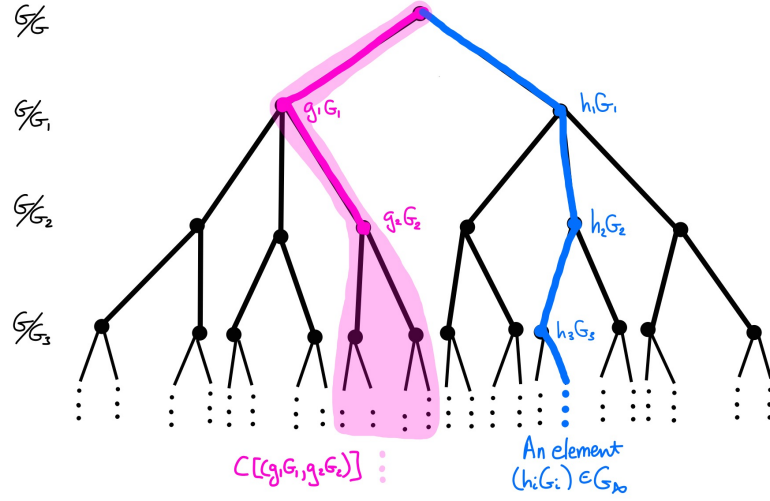
FIGURE 1. Graph of G_∞
The pink shaded region on the left denotes a clopen cylinder set. The blue shaded path on the right denotes a single element.

Recall the metric defined on X_∞ in the proof of Proposition 2.3. Let $x = (x_0, x_1, x_2, ...)$ and $y = (y_1, y_2, y_3, ...)$ be elements of X_∞. Suppose x and y diverge at level n, that is, we have $x_i = y_i$ for $i < n$, but $x_n \neq y_n$. Then we define the distance between x and y to be $d(x, y) = \frac{1}{2^n}$.

PROOF. By the definition of the metric d on X_∞, distances in X_∞ can only be of the form $\frac{1}{2^n}$ for some n. Suppose $d((g_i G_i), (h_i G_i)) = \frac{1}{2^n} < \delta$, and let $g \in G$. The assumption that $d((g_i G_i), (h_i G_i)) = \frac{1}{2^n}$ means that $(g_i G_i), (h_i G_i)$ first differ at level n, so $g_i G_i = h_i G_i$ for $i \leq n$. Then, $gg_i G_i = gh_i G_i$ for $i \leq n$, which implies that $d(g \cdot (g_i G_i), g \cdot (h_i G_i)) = \frac{1}{2^n}$. Thus, taking $\varepsilon = \delta$ shows equicontinuity. $\qquad\square$

LEMMA 2.6. *The action of G on X_∞ is minimal.*

PROOF. An action is minimal if every orbit is dense. An orbit is dense in X_∞ if it intersects every open set of X_∞, or equivalently if it intersects every basic open set of X_∞.

The basic open sets of X_∞ are cylinder sets of the form $\{(h_i G_i) \mid h_i G_i = g_i G_i \text{ for } 0 \leq i \leq n\}$. Observe that the action of G on each finite level G/G_k is transitive. Consider the orbit of a point $(h_i G_i)$ in X_∞, and the cylinder set $C[(g_1 G_1, ... g_n G_n)]$. By transitivity of the action of G on G/G_n, there exits a $\gamma \in G$ such that $\gamma h_n G_n = g_n G_n$. By the structure of the inverse limit, this implies that $\gamma h_i G_i = g_i G_i$ for all $i \leq n$ as well. Thus, $\gamma \cdot (h_i G_i) \subset \{(h_i G_i) \mid h_i G_i = g_i G_i \text{ for } 0 \leq i \leq n\}$, so the orbit of $(h_i G_i)$ intersects every cylinder set. Thus, the orbit is dense and the action is minimal. $\quad\square$

Thus, we obtain a minimal equicontinuous Cantor dynamical system (G_∞, G). □

In the setting of group chains, we have a canonical basepoint given by (eG_i), the group chain with the coset of the identity at each level. So, given a group chain (G_i), we have an associated pointed dynamical system $(G_\infty, G, (eG_i))$.

In the remainder of this Chapter, we present a collection of examples of group chains which illustrate the definitions above, and provide models for the theory developed in the following Chapters. We will refer to and classify these examples throughout the thesis.

2. Heisenberg Group Chains

Let G be the discrete Heisenberg Group, presented in the form $G = (\mathbb{Z}^3, *)$ with the group operation $*$ given by $(x, y, z) * (x', y', z') = (x + x', y + y', z + z' + xy')$. Note that this is standard addition in the first two coordinates, but addition with a twist in the last coordinate. Hence, we think about G as $\mathbb{Z}^2 \times \mathbb{Z}$, where the $\mathbb{Z}^2$ part is abelian, and the $\mathbb{Z}$ part is not.

We consider subgroups of the form $G_n = M\mathbb{Z}^2 \times m\mathbb{Z}$ where $M = \begin{pmatrix} i & j \\ k & l \end{pmatrix}$ is a 2 by 2 matrix with integer entries and m is an integer, as in [**15**]. Then $g \in G$ is of the form $g = (ix + jy, kx + ly, mz)$ for some $x, y, z \in \mathbb{Z}$.

LEMMA 2.7. $G_n = M\mathbb{Z}^2 \times m\mathbb{Z}$ is a subgroup only if m divides both entries of one of the rows of M.

PROOF. Let $g, g' \in G_n$. We will check when the resulting element $g * g'$ is an element of G_n, that is, we check closure under the group operation. Then

$$g = (ix + jy, kx + ly, mz),$$

$$g' = (ia + jb, ka + lb, mc),$$

so

$$g * g' = (i(x + a) + j(y + b), k(x + a) + l(y + b), m(z + c) + (ix + jy)(ka + lb)).$$

Note the first two coordinates are already of the required form to be in G_n, so we need only check the third coordinate, which is $m(z + c) + ikxa + ilxb + jkya + jlyb$. In order for $g * g'$ to be an element of G_n, this last term must be divisible by m. So the condition on M is that m must divide

simultaneously ik, il, jk, jl. This will occur if m divides either i and j or k and l. That is, m must divide both entries of one of the rows of M, and then we have G_n a subgroup of G. $\square$

LEMMA 2.8. *Given* $\gamma = (ix+jy, kx+ly, mz) \in G_n$, $h = (a, b, c) \in H$ *arbitrary,* $h*\gamma*h^{-1} \in N(G_n)$ *iff* m *divides* $mz + akx + aly - ixb - jyb$.

PROOF. Let $h = (a, b, c) \in G$ arbitrary. We will conjugate an arbitrary element $\gamma \in G_n$ by h and see what conditions are needed on (a, b, c) in order for $h * \gamma * h^{-1}$ to be an element of G_n, and thus what the normalizer $N(G_n)$ looks like. Under the Heisenberg group law, $h^{-1} = (-a, -b, -c + ab)$.

$$
\begin{aligned}
h * \gamma * h^{-1} &= (a, b, c) * (ix + jy, kx + ly, mz) * (-a, -b, -c + ab) \\
&= (a + ix + jy, b + kx + ly, c + mz + a(kx + ly)) * (-a, -b, -c + ab) \\
&= (ix + jy, kx + ly, c + mz + akx + aly - c + ab + (a + ix + jy)(-b)) \\
&= (ix + jy, kx + ly, c + mz + akx + aly - c + ab - ab - ixb - jyb) \\
&= (ix + jy, kx + ly, mz + akx + aly - ixb - jyb)
\end{aligned}
$$

Note that the first two entries have trivially stayed in the form of G_n since that part is abelian, so we ignore those and focus on the last entry, $mz + akx + aly - ixb - jyb$. In order for h to be in the normalizer of G_n, this term must be a multiple of m. $\square$

We will consider three main examples of group chains in the Heisenberg group.

EXAMPLE 2.9. Let G be the Heisenberg group, and let

$$
G_i = \begin{pmatrix} p^i & 0 \\ 0 & p^i \end{pmatrix} \mathbb{Z}^2 \times p\mathbb{Z}.
$$

We will show in Example 5.2 that this chain (G_i) is regular.

EXAMPLE 2.10. Let G be the Heisenberg group, and let

$$
G_n = \begin{pmatrix} qp^n & pq^n \\ p^{n+1} & q^{n+1} \end{pmatrix} \mathbb{Z}^2 \times p\mathbb{Z}.
$$

We will show in Example 5.10 that this chain (G_i) is weakly regular but not regular.

EXAMPLE 2.11. Let G be the Heisenberg group, and let

$$G_n = \begin{pmatrix} p^n & 0 \\ 0 & q^n \end{pmatrix} \mathbb{Z}^2 \times p^n\mathbb{Z} = p^n\mathbb{Z} \times q^n\mathbb{Z} \times p^n\mathbb{Z}.$$

We will show in Example 5.14 that this chain (G_i) is irregular.

We will refer back to these examples in future chapters.

3. Weak Solenoid Examples

We first recall some standard background about weak solenoids, and then construct the examples we will use in this thesis.

A *Vietoris solenoid* is an inverse limit of $n-to-1$ covering maps of the circle $\mathbb{S}^1$. A *weak solenoid* is a generalization of this construction, where the circle $\mathbb{S}^1$ is replaced by a closed compact manifold M of dimension greater than 1. The properties of weak solenoids are discussed further in the works [1] [2, 4, 5, 6, 17, 19, 20, 21].

For the study of minimal Cantor actions , it suffices to consider the case where M is a compact connected surface of genus 1 or higher, without boundary, with basepoint $m \in M$ and fundamental group $G = \pi_i(M, m)$. Consider a sequence of covering spaces over M,

$$... \to M_2 \to M_1 \to M,$$

with covering maps given by $p_{i-1}^i : M_i \to M_{i-1}$. We write the composition as $p_m^n = p_{n-1}^n \circ p_{n-2}^{n-1} \circ ... \circ p_m^{m-1}$ (for $m < n$).

DEFINITION 2.12. *Let M be a compact connected manifold of genus 1 or higher, with basepoint $m \in M$ and fundamental group $G = \pi_i(M, m)$. Consider a sequence of covering spaces over M,*

$$... \to M_2 \to M_1 \to M,$$

with covering maps given by $p_{i-1}^i : M_i \to M_{i-1}$.

Then, the inverse limit $M_\infty = \varprojlim\{p_{i-1}^i : M_i \to M_{i-1}\}$ is called a (2-dimensional) weak solenoid. There are then projection maps $p_i : M_\infty \to M_i$ for $i \geq 0$.

Note that this representation is not unique - there could be more than one sequence of covering spaces that give homeomorphic weak solenoids.

There is a natural minimal equicontinuous action of $G = \pi_i(M, m)$ on F, the *Monodromy action*, which is defined in the following way. Let $[g_0] \in G$ be an equivalence class of a closed loop g_0 in M_0 based at m_0, and let $(m_0, m_1, m_2, ..)$ be an element of F. Then, for each $i \geq 1$, there is a unique lift g_i of g_0 with $g_i(0) = m_i$ (where $g_i(0)$ is the starting point of the path g_i, and $g_i(1)$ is the endpoint of the path g_i). Define the action $G \times F \to F$ by $g_0 \cdot (m_0, m_1, m_2, ..) = (g_0(1), g_1(1), g_2(1), ...)$ (which is an element of F by covering space theory, e.g. from [**16**].) So, intuitively, the action is taking each point m_i in the fiber of m_0, and applying the loop g_0 to it, which means applying the lift of g_0 in M_i to m_i, taking us to another point of the fiber in M_i.

The following Theorem has been called The Fundamental Theorem of Covering Spaces, and a proof can be seen for example in [**16**].

THEOREM 2.13 ([**16**]). *There is a one to one correspondence between finite index subgroups G_n of $G_0 = \pi_1(M_0, m_0)$ and (finite-to-one) covering spaces M_n of M_0. Furthermore, G_n is a normal subgroup of G_0 if and only if M_n is a regular cover of M_0.*

This correspondence is given in the following way. For a given subgroup $H < G$, define M_H to be the covering space such that $p_*(\pi_1(M_H, m_H)) = H$, for a suitable basepoint $m_H \in M_H$.

Given a weak solenoid, we can find an associated group chain. Choose a basepoint $m_\infty \in M_\infty$, and set $m_i = p_i^\infty(m_\infty) \in M_i$ with $m = m_0$. Let $(p_{i-1}^i)_* : \pi_1(M_i, m_i) \to \pi_1(M_{i-1}, m_{i-1})$ be the induced homomorphism of fundamental groups. Let $G_i = (p_0^i)_*(\pi_1(M_i, m_i))$, which is a subgroup of $G = \pi_i(M, m)$. This gives us a group chain $G = G_0 > G_1 > G_2 > ...$. Then the inverse limit $G_\infty = \{G/G_i \to G/G_{i-1}\}$, where the bonding maps are inclusion maps, is a Cantor set, and by Theorem 2.4, G acts minimally and equicontinuously on G_∞ by component-wise multiplication. This action is equivalent to the monodromy action of G on F.

This group chain is not unique, since the sequence of covering spaces associated to a weak solenoid (up to homeomorphism) is not unique. However, for the following two examples we will have standard group chains associated to these constructions, that we refer to consistently throughout this thesis.

EXAMPLE 2.14. [Rogers and Tollefson Klein Bottle Example]

The Rogers and Tollefson solenoid is constructed as follows, as in [**11**]. The torus T can be represented as $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$, and then the Klein bottle K can be represented as the quotient of the torus under $(x, y) \to (x + 1/2, -y)$. A double cover of the torus by itself can be given by $(x, y) \to (x, 2y)$. Since these two maps commute, they induce a double cover of the Klein bottle by itself, which we

denote by $p : K \to K$. We can then take an infinite sequence of iterations of this map on Klein bottles. Then the surface and map at each level are the same, but to keep track of the levels we denote them by $K_0, K_1, ...$, with $p_{i-1}^i : K_i \to K_{i-1}$.

Denote the composition by

$$p_0^k = p_{k-1}^k \circ p_{k-2}^{k-1} \circ \cdots \circ p_0^1 .$$

Then

$$K_\infty = \varprojlim \{K_i, p_{i-1}^i\}$$

is the Rogers and Tollefson solenoid.

Choose a basepoint $x_0 \in K_0$, and points $x_i \in K_i$ such that $p_{i-1}^i(x_i) = x_{i-1}$. Choose loops a, b representing the generators of the fundamental group $\pi_1(K_0, x_0)$. The fundamental group of the Klein bottle K_0 has standard presentation

$$G = G_0 = \pi_1(K_0, x_0) = \langle a, b \mid bab^{-1} = a^{-1} \rangle.$$

Let p_{0*}^i be the induced homomorphism from $\pi_1(K_i, x_i)$ onto $\pi_1(K_0, x_0)$.

$$G_i = p_{0*}^i \pi_1(K_i, x_i)$$

be the image of the fundamental group of K_i onto G under i compositions of p_{0*}, the homomorphism induced by the first level projection map p_0. Then the subgroups $G_0 > G_1 > G_2 > ...$ form an infinite nested chain.

The induced homomorphism $p_{0*}^1 : \pi_1(K_1, x_1) \to \pi_1(K_0, x_0)$ is given by $a \to a^2, b \to b$. Applying p_{0*}^1 to $\pi_1(K_0, x_0)$, we see that

$$G_1 = \langle a^2, b \mid bab^{-1} = a^{-1}, ba^2b^{-1} = a^{-2} \rangle .$$

Continuing inductively,

$$G_i = \langle a^{2^i}, b \mid bab^{-1} = a^{-1}, ba^2b^{-1} = a^{-2}, ba^4b^{-1} = a^{-4}..., ba^{2^i}b^{-1} = a^{-2^i} \rangle .$$
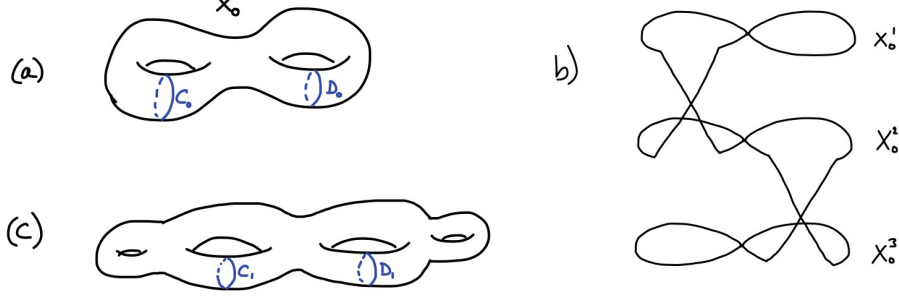
FIGURE 2. Coverings of the genus 2 surface for the Schori solenoid
Figure (a) is the genus 2 surface X_0 with curves C_0 and D_0. Figure (b) is three copies of X_0 which are cut along C_0 and D_0 and glued according to the pattern shown. Figure (c) is the resulting surface X_1, with genus 4.

This (G_i) will be referred to as the group chain associated to the Rogers and Tollefson Solenoid. We will show in Example 5.22 that this group chain is irregular. Rogers and Tollefson showed that this solenoid is not regular, but we give a new proof using group chains in Example 5.22.

EXAMPLE 2.15. [The Schori Solenoid]

The Schori solenoid is a weak solenoid over a genus 2 surface, with bonding maps that are 3-to-1 covering maps constructed according to a particular cutting and gluing process which we will recall from [21].

Let X_0 be a genus 2 surface, which is a union of two 1-handles H_0 and F_0, glued along their boundaries. Let C_0 be a simple closed curve in H_0, and D_0 a simple closed curve in F_0. We take three copies of X_0, denoted $\tilde{X}_0^1, \tilde{X}_0^2, \tilde{X}_0^3$ and cut each of them along the curves C_0 and D_0. We then glue those three surfaces together along those cuts according to the pattern shown in Figure 2(c). The resulting surface will be labelled X_1.

We now continue by induction. Consider X_{k-1}. It has two k-handles H_k and F_k. Let C_k, D_k be simple closed curves in H_k and F_k respectively. To construct X_k, take three copies of X_{k-1}, denoted by $\tilde{X}_{k-1}^1, \tilde{X}_{k-1}^2, \tilde{X}_{k-1}^3$. Let $\tilde{F}_k^i, \tilde{H}_k^i$ be the k-handles in $\tilde{X}_{k-1}^i$, with simple closed curves $\tilde{C}_k^i, \tilde{D}_k^i$ in $\tilde{F}_k^i, \tilde{H}_k^i$ respectively. Cut each $\tilde{X}_{k-1}^i$ along the curves $\tilde{C}_k^i, \tilde{D}_k^i$, and glue them together according to the same pattern as before to obtain X_k. Continuing in this manner, we see that X_k is a surface of genus $3^k + 1$, with a shape shown in Figure 3. Define the map $f_k^{k+1} : X_{k+1} \to X_k$ by sending a point $(x, i) \in \bar{X}_k^i \subset X_{k+1}$ to $x \in X_k$. This map is a $3 - 1$ covering map.

The Schori solenoid is now given by $X_\infty = \varprojlim \{X_k, f_k^{k+1}\}$
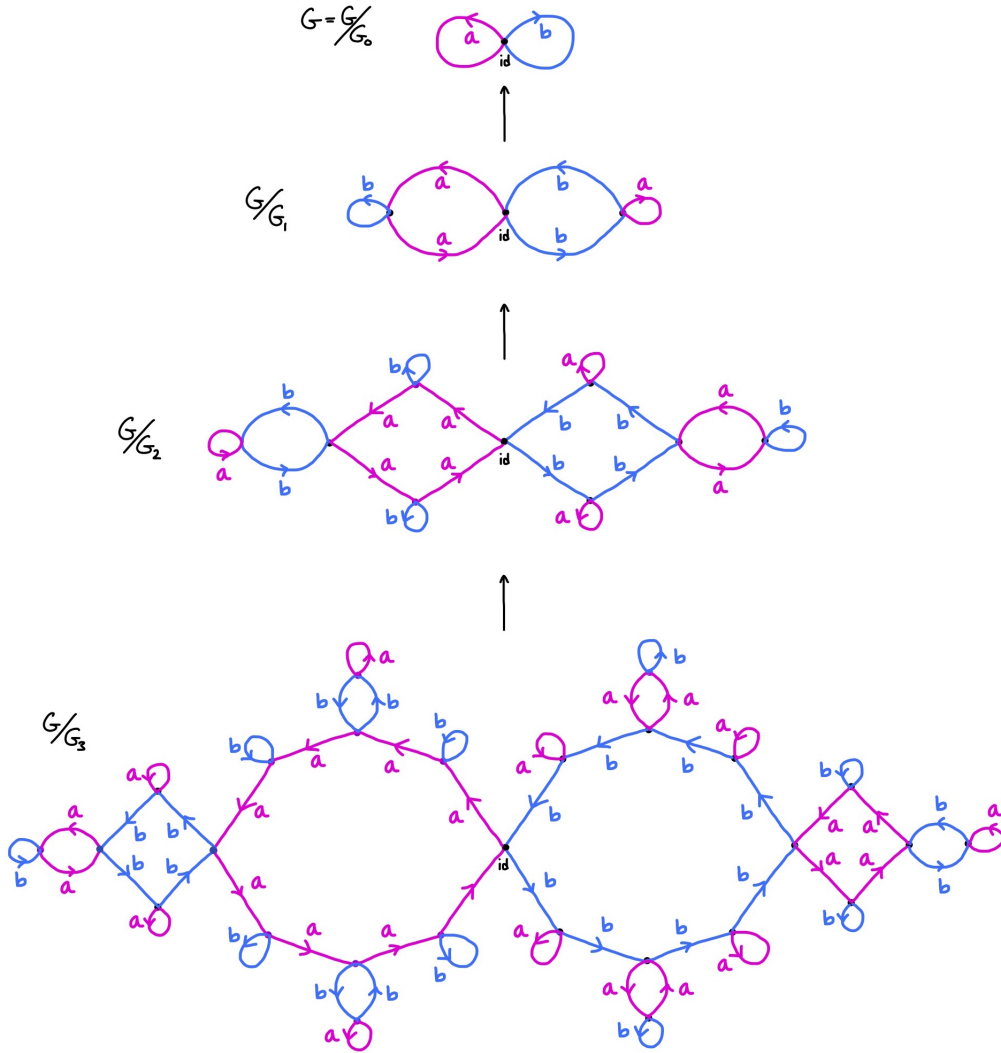
FIGURE 3. Tower of Schreier graphs for the Schori solenoid.
The diagrams illustrate the Schreier graphs for the sets G/G_i. The corresponding surfaces X_i are obtained by "thickening up" each edge into a tube.

Let $x_0 \in H_0 \cap F_0$ be a fixed basepoint in X_0, and let a, b, α, β be loops representing the generators of the fundamental group $\pi_1(X_0, x_0)$.

For each $k > 0$, there are three points x_k^1, x_k^2, x_k^3 that project down to x_0, that is, such that

$$f_0^k(x_k^i) = f_{k-1}^k \circ f_{k-2}^{k-1} \circ \cdots \circ f_0^1(x_k^i) = x_0.$$

One of these is a unique point $x_k \in X_k$ that projects down to x_0 and also such that $x_k \in H_k \cap F_k$. We will consider this x_k as the basepoint of X_k.

Denote $G_0 = \pi_1(X_0, x_0)$, and let $G_k = f_{0*}^k \pi_1(X_k, x_k)$. The groups G_k were computed explicitly in [2], where they obtained the following explicit description of G_k.

We have loops $S = \langle a, b, \alpha, \beta \rangle$ representing the generators of the fundamental group $\pi_1(X_0, x_0)$ with a relation $\mathrm{rel}_0 = [\mathrm{a}, \alpha][\mathrm{b}, \beta] = \mathrm{a}\alpha\mathrm{a}^{-1}\alpha^{-1}\mathrm{b}\beta\mathrm{b}^{-1}\beta^{-1}$, where the operation is the usual multiplication (concatenation) of paths.

For $k = 0$, distinguish the following subsets of generators of G_0,

(1a) $\qquad\qquad\qquad S_{0ab} = \emptyset, \qquad\qquad\qquad\qquad S_{0ba} = \emptyset$

(1b) $\qquad\qquad\qquad S_{0a} = \{a, \alpha\}, \qquad\qquad\qquad\qquad S_{0b} = \{b, \beta\}.$

For $k \geq 1$ define

$$S_{kab} = S_{(k-1)ab} \cup a^{2^{k-1}} S_{(k-1)ab} a^{-2^{k-1}} \cup a^{2^{k-1}} S_{(k-1)b} a^{-2^{k-1}},$$

$$S_{kba} = S_{(k-1)ba} \cup b^{2^{k-1}} S_{(k-1)ba} b^{-2^{k-1}} \cup b^{2^{k-1}} S_{(k-1)a} b^{-2^{k-1}},$$

(1c) $\qquad\qquad S_{ka} = \{a^{2^k}, \alpha\} \cup S_{kab},$

$$S_{kb} = \{b^{2^k}, \beta\} \cup S_{kba}.$$

Then for $k \geq 0$, [2] calculate that

$$G_k = \langle a^{2^k}, \alpha, b^{2^k}, \beta, S_{kab}, S_{kba} \mid \mathrm{rel}_k = \mathrm{id}, \mathrm{rel}_0 = \mathrm{id}\rangle,$$

where rel_k is the corresponding relation for the m_k-genus surface.

Since f_k^{k+1} is a degree 3 covering map, G_{k+1} has index 3 in G_k, and thus G_k has genus 3^k in G_0.

Notice that the loops α or β lift to a loop in X_i for every i, and so the action of the corresponding elements of G_i on $\varprojlim G/G_k$ is trivial.

Thus, we can give the Schreier diagram for the coset space G/G_k with respect only to the edges a, b. Figure 3 gives the Schreier diagrams for G/G_i for $i = 0, 1, 2, 3$. It is illuminating to notice that the relations for G_k can be seen in the Schreier diagram for G/G_k, making the origin of the above formulas more clear. For example, we can see in Figure 4 the relations $a^4 = 1$ and $b^2 a^2 b^{-2} = 1$ highlighted in the diagram for G/G_2, starting at the center basepoint id_2.
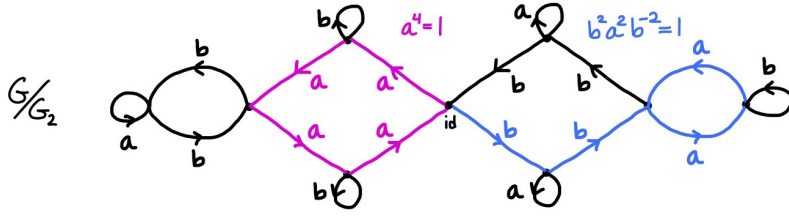
FIGURE 4. Schreier diagram for Schori coset space G/G_2
The shaded paths illustrate the two relations.

Thus, we have a group chain (G_i) associated to the Schori solenoid. Schori showed that this solenoid is irregular, but we give a new proof using group chains in Example 5.21.

4. Direct and Semi-Direct Product Examples

We can also build new group chains from other group chains using direct or semi-direct products.

EXAMPLE 2.16. Let Γ be a finitely generated group, and let (Γ_i) be a normal group chain in Γ, that is, $\Gamma_i \lhd \Gamma$ for each i. Let H be a finite simple group and let K be a nontrivial subgroup of H. Let $G = H \times \Gamma$, and let $G_i = K \times \Gamma_i$, so we have obtained a new group chain (G_i) in G. We will show in Example 5.19 and Example 8.3 that (G_i) is weakly regular with finite discriminant group.

EXAMPLE 2.17. Let $\Gamma = \mathbb{Z}^2 = \{(a, b) \mid a, b \in \mathbb{Z}\}$, and let p, q be distinct primes. Let

$$\Gamma_i = p^i \mathbb{Z} \times q^i \mathbb{Z} = \{(ap^i, bq^i) \mid a, b \in \mathbb{Z}\}.$$

Let $H = \mathbb{Z}/2\mathbb{Z} = \{1, t \mid t^2 = 1\}$. We will form a semi-direct product of Γ and H (notice that we write the operation in H as multiplication and the operation in $\Gamma = \mathbb{Z}^2$ as addition).

Let $\theta : H \to Aut(\Gamma)$ be the homomorphism defined as follows:

$$\theta : H \to Aut(\Gamma)$$

$$1 \to \theta_1 : (a, b) \to (a, b) \text{ (i.e., } \theta_1 \text{ is the identity map)}$$

$$t \to \theta_t : (a, b) \to (b, a) \text{ (i.e., } \theta_t \text{ is the transpose map)}$$

Then we can form the semi-direct product $G = \Gamma \rtimes_\theta H \cong \mathbb{Z}^2 \rtimes_\theta \mathbb{Z}/2\mathbb{Z}$. Recall that the semi-direct product $\Gamma \rtimes_\theta H$ is the set $\Gamma \times H$ with the operation given by $(\gamma_1, h_1) * (\gamma_2, h_2) = (\gamma_1 \theta_{h_1}(\gamma_2), h_1 h_2)$, i.e., $((a_1, b_1), h_1) * ((a_2, b_2), h_2) = ((a_1, b_1)\theta_{h_1}((a_2, b_2), h_2)), h_1 h_2)$, where $h_1, h_2 \in \mathbb{Z}/2\mathbb{Z}$ are 1 or t,

and $(a, b) \in \mathbb{Z}^2$. Recall also that the inverse of an element in the semi-direct product is given by $(\gamma, h)^{-1} = (\theta_h(\gamma^{-1}), h^{-1})$, i.e., $((a, b), h)^{-1} = (\theta_h((-a, -b)), h^{-1})$.

We then form a new group chain in G by letting $G_i = \Gamma_i \times \{1\}$.

We will show in Examples 5.20 and 8.4 that this (G_i) is weakly regular but not regular, with infinite discriminant group.

AF Presentations: Cantor Dynamical Systems to Group Chains and vice versa

The goal of this chapter is to prove the following theorem:

THEOREM 3.1. *If (X, G, x) be a minimal equicontinuous pointed dynamical system, then there is an associated group chain (G_i) and a pointed conjugacy between the minimal equicontinuous Cantor systems (X, G, x) and $(G_\infty, G, (eG_i))$.*

This shows that, given any pointed minimal equicontinuous Cantor dynamical system (X, G, ϕ, x), we can find an associated group chain (G_i) and corresponding dynamical system $(G_\infty, G, (eG_i))$ so that the systems (X, G, ϕ, x) and $(G_\infty, G, (eG_i))$ are pointed conjugate. That is, we can represent any of our dynamical systems in terms of a group chain dynamical system. This justifies the usage of group chains throughout the rest of the thesis.

We prove Theorem 3.1 using a nested sequence of a special type of Kakutani-Rokhlin partitions, that we call an Almost Finite (AF) Presentation. We will show that AF Presentations exist for minimal, equicontinuous Cantor dynamical systems, and we will show how they can be used to construct an associated group chain and vice versa.

In Section 1, we define AF Presentations and prove that they exist for minimal equicontinuous systems. In Section 3, we prove Theorem 3.1.

1. Definition and Existence of AF Presentations

A *Kakutani-Rokhlin (KR) partition* for a group action on a Cantor set is a partition of the Cantor set into clopen sets, arranged into "towers" in a way that is compatible with the group action. For the case where $G = \mathbb{Z}$ and the action is generated by one homeomorphism $\phi : X \to X$, Herman, Putnam, and Skau define a KR partition using sets of equal return times under the generating homeomorphism ([**14**]) and show that they can be constructed for minimal Cantor $\mathbb{Z}-$actions.

DEFINITION 3.2 (Kakutani-Rokhlin partition for a $\mathbb{Z}-$action, as defined in [**14**]). *Let $(X, \mathbb{Z}, \phi)$ be a minimal Cantor dynamical system. Let $x \in X$ be a basepoint, and let Z be a clopen subset of X that contains x. A Kakutani-Rokhlin partition of X based on Z is a set of clopen sets $Z(k, j)$, with $k = 1, ..., K, j = 1, ...J(k)$, satisfying:*

(1) $\cup_k Z(k, J(k)) = \cup_k Z(k, 0) = Z$

(2) $\cup_k Z(k, 1) = \phi(Z)$

(3) $\phi(Z(k, j)) = Z(k, j + 1)$ for $a \le j \le J(k)$

(4) $\{Z(k, j) : k = 1, ..., K, j = 1, ...J(k)\}$ is a finite clopen partition of X.

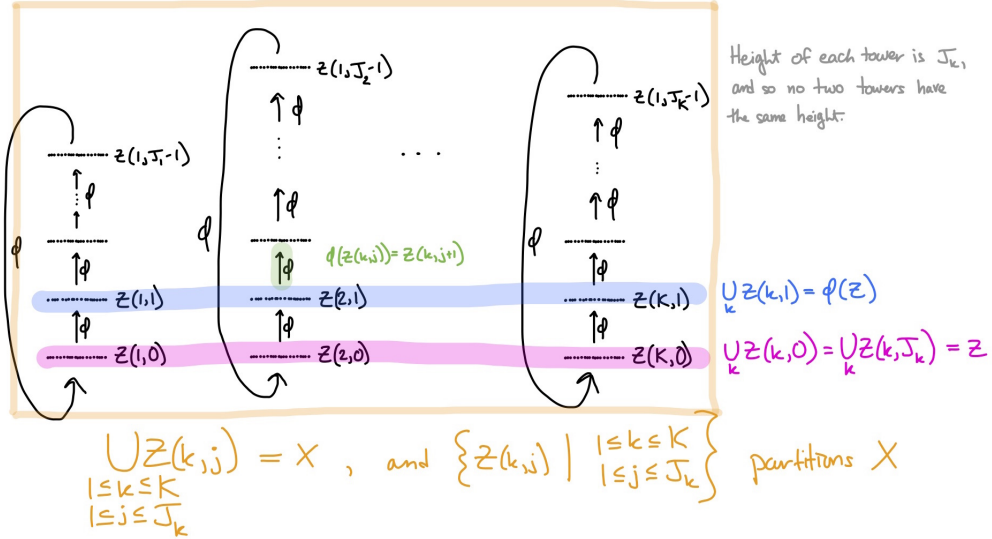This definition is illustrated in Figure 5.

FIGURE 5. Kakutani-Rokhlin partition

For the case where G is a finitely generated group but not necessarily $\mathbb{Z}$, a KR partition can still be defined. Forrest gave such a construction for the case of minimal $\mathbb{Z}^n$ actions in the works [**12, 13**]. Cortez and Petite gave a more general construction in [**9**], where they do this using a set of return times rather than a single return time under one map. They also use coding to construct their KR partitions, but it is a bit different than our coding method. These previous Kakutani-Rokhlin partitions do not include metric properties, while our construction takes advantage of the equicontinuous assumption to obtain that the sizes of the diameters of the sets in the partition are shrinking in a controlled way. Additionally, our partitions have a single tower over one base.

Let X be a metric space with metric d. Given two partitions P, Q of X, we say that P *refines* Q or is a *refining* partition of Q if for every $U \in P$ there is a $V \in Q$ such that $U \subset V$. Define the *diameter* of any subset $Y \subset X$ as

$$diam(Y) = sup\{d(x, y) \mid x, y \in Y\}.$$

Given two subsets $W, Y \subset X$, we define the *distance* between W and Y as

$$dist(W, Y) = inf\{d(w, y) \mid w \in W, y \in Y\}.$$

Recall that a group action of G on X is a homomorphism $\phi : G \to Homeo(X)$ that takes $\phi \in g$ to $\phi(g) : X \to X$, and we write $\phi(g)(x) = \phi_g(x) = g \cdot x$. Recall that ϕ is *minimal* if every orbit is dense in X, and ϕ is *equicontinuous* if for all $\varepsilon > 0$ there exists $\delta = \delta_\varepsilon > 0$ such that for all $g \in G$ and any $x, y \in X$, if $d(x, y) < \delta_\varepsilon$, then $d(\phi(g)(x), \phi(g)(y)) < \varepsilon$.

DEFINITION 3.3. *Let $(X, G, \phi, \tilde{x})$ be a pointed Cantor dynamical system. An* Almost Finite Presentation $\{\tilde{x}, \mathcal{P}_i, \mathcal{A}_i, \eta_i, \psi_i, k_i\}$ *of $(X, G, \phi, \tilde{x})$ is an infinite nested collection of finite clopen partitions $\mathcal{P}_i = \{P_1^i, P_2^i, ..., P_{m_i}^i\}$ of X with the following properties:*

(1) *There is a sequence $\{\varepsilon_i\}$, $\varepsilon_i > 0$, such that $\lim_{i \to \infty} \varepsilon_i = 0$, and for each P_k^i in $\mathcal{P}_i$, we have $diam(P_k^i) < \varepsilon_i$.*

(2) *There is a finite collection of finite sets $\mathcal{A}_i$ and maps $\eta_i : X \to \mathcal{A}_i$ such that if $x, y \in P_k^i$, $z \in P_l^i$ with $l \neq k$, then $\eta_i(x) = \eta_i(y) \neq \eta_i(z)$.*

(3) *For each $g \in G$ and each $\mathcal{P}_i$, there is a homomorphism $\psi_i : G \to Perm(\mathcal{A}_i)$ such that $\eta_i(\phi_g(x)) = \psi_i(g)(\eta_i(x)) \; \forall x \in X$.*

(4) *$\tilde{x} \in P_1^i$ for all i.*

(5) *$\mathcal{P}_{i+1}$ refines $\mathcal{P}_i$.*

(6) *There are maps $k_i : \mathcal{A}_i \to \mathcal{A}_{i-1}$ such that $k_i \circ \eta_i(x) = \eta_{i-1}(x) \; \forall x \in X$.*

(7) *For each i, $k_i(\psi_i(g))(a) = \psi_{i-1}(g)(k_i(a))$.*

We will often refer to the AF Presentation $\{\tilde{x}, \mathcal{P}_i, \mathcal{A}_i, \eta_i, \psi_i, k_i\}$ simply as $\{\mathcal{P}_i\}$ if we do not need to refer to $\mathcal{A}_i, \eta_i, \psi_i, k_i$.

Property (1) of this definition says that the diameters of the elements of the partitions shrink to zero. Property (2) says that we have a collection of finite sets $\mathcal{A}_i$ ("alphabets" or "index sets") such that each $\mathcal{A}_i$ indexes $\mathcal{P}_i$, and a collection of continuous surjective "indexing" maps $\eta_i : X \to \mathcal{A}_i$ such that η_i is constant on each piece P_k^i of the partition $\mathcal{P}_i$, and such that η_i is distinct on distinct

elements of the partition $\mathcal{P}_i$. Property (3) says that each homeomorphism ϕ_g permutes the elements of $\mathcal{P}_i$, i.e. the group action permutes the elements of the partition. Property (6) says that the maps $k_i : \mathcal{A}_i \to \mathcal{A}_{i-1}$ are compatible with the indexing maps η_i, and we refer to the maps k_i as "bonding maps", for future use in an inverse limit. The last property of the definition says that the bonding maps between alphabets are compatible with the permutations induced by the group action. Thus, these properties put together describe a special type of Kakutani-Rokhlin partition that is compatible with our group action and that has the size of the elements of the partition shrinking to zero in a controlled way, with only one tower.

THEOREM 3.4 (Existence of AF Presentations). *Let (X, G, ϕ) be a minimal equicontinuous Cantor dynamical system with basepoint $\tilde{x} \in X$. Then there exists an Almost Finite Presentation $\{\tilde{x}, \mathcal{P}_i, \mathcal{A}_i, \eta_i, \psi_i, k_i\}$ of (X, G, ϕ).*

The main idea of the proof is the following: we will start with an arbitrary clopen partition of X, and use a method of coding of orbits to obtain a clopen partition that is compatible with the group action. Then, we use an inductive process to get a nested chain of such clopen partitions with the desired properties. This proof requires several lemmas. The coding argument a special case of the method from [**4**].

PROOF OF THEOREM 3.4. Fix a montone decreasing sequence $\{\varepsilon_i > 0 \mid i \geq 0\}$ such that $\lim_{i \to \infty} \varepsilon_i = 0$.

Choose a finite clopen partition $\mathcal{W}_1 = \{W_1^1, W_2^1, ..., W_{n_1}^1\}$ of X, arbitrary except that for all $q \leq k \leq n_1$, we require that $diam(W_k^1) < \varepsilon_1$.

For each $x \in X$, we define a coding function $C_x^1 : G \to \{1, ..., n_1\}$ by

$$C_x^1(g) = k, \quad \text{if } \phi_g(x) \in W_k^1.$$

That is, the coding function based on the partition $\mathcal{W}^1$ with respect to the point x tells us into which set of the partition $\mathcal{W}^1$ the action of $g \in G$ takes x.

For each $x \in X$, consider the set of points in X which have the same coding function as x, that is,

$$P_x^1 = \{y \in X \mid C_y^1 = C_x^1\}.$$

We will show in a moment that the collection $\mathcal{P}_1 = \{P_x^1\}_{x \in X}$ forms a finite clopen partition of X.

In this situation, we say that the partition $\mathcal{P}_1$ is *obtained from the partition $\mathcal{W}_1$ by coding (with respect to the G action)*.

LEMMA 3.5. *For any $x \in X$, P_x^1 is an open set.*

PROOF. Recall that we assumed $diam(W_k^1) < \varepsilon_1 \ \forall k$. Since the partition $\mathcal{W}_1$ is a finite disjoint collection of compact sets, there exists $\tilde{\varepsilon}_1 > 0$ such that

$$\tilde{\varepsilon}_1 < \min\{\varepsilon_1, \mathrm{dist}(W_i^{(1)}, W_j^{(1)}) \mid i \neq j, 1 \leq i, j \leq n_1\}.$$

Since the action ϕ of G on X is equicontinuous, there exists an equicontinuity constant $\tilde{\delta}_1$ for $\tilde{\varepsilon}_1$. That is, there exists $\tilde{\delta}_1 > 0$ such that for all $g \in G$ and any $x, y \in X$, if $d(x, y) < \tilde{\delta}_1$, then $d(\phi_g(x), \phi_g(y)) < \tilde{\varepsilon}_1$.

We will show that for any $x \in P_x^1$, P_x^1 contains an open ball around x. Suppose $y \in X$ such that $d(x, y) < \tilde{\delta}_1$, and suppose $g \in G$.

Then by equicontinuity, we have that $d(\phi_g(x), \phi_g(y)) < \tilde{\varepsilon}_1$. We have $\phi_g(x)$ in one of the sets of the partition $\mathcal{W}_1$, say $\phi_g(x) \in W_k^1$. By assumption on $\tilde{\delta}_1$ we have that $d(\phi_g(x), \phi_g(y)) < \tilde{\varepsilon}_1$ which is less than the distance between any two distinct sets in $\mathcal{W}_1$, so we must also have $\phi_g(y) \in W_k^1$. Thus, $C_x^1(g) = C_y^1(g) = k$, so $y \in P_x^1$. Thus, P_x^1 is open. □

LEMMA 3.6. $\mathcal{P}_1 = \{P_x^1\}_{x \in X}$ *forms a partition of X.*

PROOF. We will show that if $P_x \cap P_y \neq \emptyset$, then $P_x = P_y$. Suppose there is some $z \in P_x \cap P_y$, and let $g \in G$. Since $z \in P_x$, we have

$$C_x^1(g) = C_z^1(g),$$

and since $z \in P_y$, then we have

$$C_y^1(g) = C_z^1(g).$$

Therefore, $C_x^1(g) = C_y^1(g)$, which implies that $P_x = P_y$. Thus, the collection $\mathcal{P}_1 = \{P_x^1\}_{x \in X}$ forms a partition of X. □

LEMMA 3.7. *The partition $\mathcal{P}_1$ is finite.*

PROOF. The collection $\mathcal{P}_1 = \{P_x^1\}_{x \in X}$ is a cover of X by open sets, so by compactness of X, it must have a finite subcover. But, since the sets are disjoint, the only subcover that still covers all of X is $\{P_x^1\}_{x \in X}$ itself, so it must be a finite collection of sets. □

LEMMA 3.8. *For any $x \in X$, P_x^1 is a closed set.*

PROOF. For each P_x^1, its complement in X is a finite union of open sets, which is thus an open set, so P_x^1 is closed. □

Thus, the collection $\mathcal{P}_1 = \{P_x\}_{x \in X}$ forms a finite clopen partition of X. Since the partition is finite, we can adjust our notation to make these sets easier to keep track of. Let m_1 be the number of sets in $\mathcal{P}_1$, and renumber the sets as $\mathcal{P}_1 = \{P_1^1, P_2^1, ..., P_{m_1}^1\}$, arbitrarily except that we require the basepoint $\tilde{x} \in P_1^1$.

LEMMA 3.9. *$\mathcal{P}_1$ is a refining partition of $\mathcal{W}_1$.*

PROOF. Let $x, y \in X$, and consider the coding functions of x, y applied to the identity group element $id \in G$. We have $c_x^1(id) = k$ if $\phi_{id}(x) \in W_k^1$, but $\phi_{id}(x) = x$ because ϕ_{id} is the identity homeomorphism on X. So, $c_x^1(id) = k$ for $x \in W_k^1$. Thus, x, y can only have the same coding function if they lie in the same element W_k^1 of the partition $\mathcal{W}_1$. Since $\mathcal{P}_1$ is made up of subsets of X that have the same coding function, this shows that $\mathcal{P}_1$ is a refining partition of $\mathcal{W}_1$. □

LEMMA 3.10. *For each k, $diam(P_k^1) < \varepsilon_1$.*

PROOF. This follows directly from Lemma 3.9. We have that $W_k^1 < \varepsilon_1$ for all k, and each $P_k^1 \subset W_l^1$ for some l. □

LEMMA 3.11. *The action ϕ of G permutes the sets of $\mathcal{P}_1$.*

PROOF. Suppose $x, y \in P_i^1$, and let $g \in G$. We claim that the action by every $g \in G$ maps x, y into the same P_j^1. Suppose not, so suppose that there exists a $g \in G$ such that $\phi_g(x) \in P_n^1, \phi_g(y) \in P_m^1$, $m \neq n$.

By the definition of P_i^1, $\phi_g(x), \phi_g(y)$ are in the same set W_k^1 (of our original, arbitrary partition). We have

$$\phi_g(x) \in P_n^1 = \{z : C_{\phi_g(x)} = C_z\},$$

and

$$\phi_g(y) \in P_m^1 = \{w : C_{\phi_g(y)} = C_w\}.$$

If $P_n^1 \neq P_m^1$, then $C_{\phi_g(x)} \neq C_{\phi_g(y)}$ as functions, meaning there is some $g' \in G$ such that $\phi_{g'}(\phi_g(x))$ and $\phi_{g'}(\phi_g(y))$ are in different elements of the partition $\mathcal{W}^1$, which contradicts the assumption that x, y are in the same P_i^1. So, the lemma holds by contradiction. $\qquad\square$

Now that we have shown that $\mathcal{P}^1$ is a finite clopen partition that is permuted by the action of G, we explicitly define indexing sets and maps, and permutation maps, and check that the properties in Definition 3.3 hold.

Recall m_1 is the cardinality of $\mathcal{P}_1 = \{P_1^1, P_2^1, ..., P_{m_1}^1\}$, and let $\mathcal{A}^1 = \{1, 2, ..., m_1\}$ be the "alphabet" or "indexing set" associated to $\mathcal{P}_1$. Define the associated index map $\eta_1 : X \to \mathcal{A}_1$ as $\eta_1(x) = k$ if $x \in P_k^1$. That is, η_1 tells us which element of the partition $\mathcal{P}_1$ each point of X is in. It will be useful to note that this means that $\eta_1^{-1}(k) = P_k^1$. Notice that η_i is continuous, because for each $k \in \mathcal{A}_i$, $\eta_1^{-1}(k) = P_k^1$ is an open set.

Let $Perm(\mathcal{A}^1)$ be the group of permutations of $\mathcal{A}^1 = \{1, 2, ..., m_1\}$. We define a map $\psi_1 : G \to Perm(\mathcal{A}_1)$ in the following way. We have shown that each $g \in G$ permutes the elements of the partition $\mathcal{P}_1$, so for $a \in \mathcal{A}_1$, we have $\phi_g(P_a^1) = P_b^1$ for some $b \in \mathcal{A}_1$. Define $\psi_1(g)$ to be the permutation of $\mathcal{A}_1$ such that $\psi_1(g)(a) = b$ if and only if $\phi_g(P_a^1) = P_b^1$. From this definition it is clear that ψ_1 is a group homomorphism, and that $\eta_i(\phi_g(x)) = \psi_i(g)(\eta_i(x)) \ \forall x \in X$.

We have constructed a partition $\mathcal{P}_1$ of X, and shown that it has the properties necessary to be the first partition in the desired sequence from Definition 3.3. We will now inductively define a nested sequence of partitions, starting from $\mathcal{P}_1$.

PROPOSITION 3.12. *Suppose we have a finite clopen partition of X:*

$$\mathcal{P}_{i-1} = \{P_1^{i-1}, P_2^{i-1}, ..., P_{m_{i-1}}^{i-1}\}$$

such that:

(1) *For each P_k^{i-1} in $\mathcal{P}_{i-1}$, we have $diam(P_k^{i-1}) < \varepsilon_{i-1}$.*

(2) *The basepoint $\tilde{x} \in P_1^{i-1}$.*

(3) *There is an index set $\mathcal{A}_{i-1} = 1, 2, ..., m_{i-1}$, where m_{i-1} is the cardinality of $\mathcal{P}_{i-1}$, and there is an index map $\eta_{i-1} : X \to \mathcal{A}_{i-1}$ given by $\eta_{i-1}(x) = k$ if $x \in P_k^{i-1}$.*

(4) *There is a homomorphism $\psi_{i-1} : G \to Perm(\mathcal{A}_{i-1})$ such that $\eta_{i-1}(\phi_g(x)) = \psi_{i-1}(g)(\eta_{i-1}(x))$ for all $x \in X$.*

Then, there is a finite clopen partition $\mathcal{P}_i = \{P_1^i, P_2^i, ..., P_{m_i}^i\}$ of X with the following properties:

(1) *The basepoint $\tilde{x} \in P_1^i$.*

(2) *For each P_k^i in $\mathcal{P}_i$, we have $diam(P_k^i) < \varepsilon_i$.*

(3) *There is an index set $\mathcal{A}_i = 1, 2, ..., m_i$, where m_i is the cardinality of $\mathcal{P}_i$, and there is an index map $\eta_i : X \to \mathcal{A}_i$ given by $\eta_i(x) = k$ if $x \in P_k^i$.*

(4) *We have a homomorphism $\psi_i : G \to Perm(\mathcal{A}_i)$ such that $\eta_i(\phi_g(x)) = \psi_i(g)(\eta_i(x)) \ \forall x \in X$. (So the action of G permutes the sets of $\mathcal{P}_i$.)*

(5) *$\mathcal{P}_i$ refines $\mathcal{P}_{i-1}$, as partitions of X.*

(6) *There are maps between the alphabets $k_i : \mathcal{A}_i \to \mathcal{A}_{i-1}$ that are compatible with the indexing maps in the following way: $k_i \circ \eta_i(x) = \eta_{i-1}(x) \ \forall x \in X$.*

(7) *The maps between alphabets are compatible with the permutations induced by the group action, that is, $k_i(\psi_i(g))(a) = \psi_{i-1}(g)(k_i(a))$.*

To prove this proposition, we will construct the partition $\mathcal{P}_i = \{P_1^i, P_2^i, ..., P_{m_i}^i\}$. We will see that, based on our construction, properties 1, 2, 3, 4, and 5 will be clear. We will then define the necessary maps to check properties 6, 7. This construction will proceed in the same manner, using coding, as the construction of $\mathcal{P}_1$.

PROOF. Choose a clopen partition $\mathcal{W}_i = \{W_1^i, ... W_{n_i}^i\}$ such that $diam(W_k^i) < \varepsilon_i$ for all $1 \leq k \leq n_i$ and such that $\mathcal{W}_i$ is a refining clopen partition of $\mathcal{P}_{i-1}$. That is, for each $1 \leq k \leq n_i$ there exists $1 \leq j_k \leq n_{i-1}$ such that $W_k^i \subset P_{j_k}^{i-1}$.

For each $x \in X$, we define the coding function (based on $\mathcal{W}_i$)

$$C_x^i : G \to \{1, ..., n_i\}$$

$$C_x^i(g) = k, \quad \text{if } \phi_g(x) \in W_k^i.$$

Let

$$P_x^i = \{y \in X \mid C_y^i = C_x^i\}.$$

The collection $\mathcal{P}_i = \{P_x^i\}_{x \in X}$ forms a finite clopen partition of X. The proof of this claim is essentially the same as the proofs of Lemmas 3.5, 3.6, 3.7, and 3.8.

The partition $\mathcal{P}_i$ refines $\mathcal{P}_{i-1}$ because $\mathcal{W}_i$ refines $\mathcal{P}_{i-1}$.

Since the collection $\mathcal{P}_i = \{P_x^i\}_{x \in X}$ forms a finite partition, we will renumber it. Let m_i be the cardinality of $\mathcal{P}_i$, and renumber this collection as $\mathcal{P}_i = \{P_1^i, P_2^i, ..., P_{m_i}^i\}$, arbitrarily ordered except that we require the basepoint $\tilde{x} \in P_1^i$.

The proof of property 2 is essentially the same as the same argument for $\mathcal{P}_1$.

We define the index sets and map, and permutation map, in the same way as we did for $\mathcal{P}_1$:

Let $\mathcal{A}^i = \{1, 2, ..., m_i\}$ be the "alphabet" or "indexing set" associated to $\mathcal{P}_i$. Define the associated index map $\eta_i : X \to \mathcal{A}_i$ as $\eta_i(x) = k$ if $x \in P_k^i$.

We define the permutation map $\psi_i : G \to Perm(\mathcal{A}_i)$ in the following way: For $g \in G$, $a \in \mathcal{A}_i$, we have $\phi_g(P_a^i) = P_b^i$ for some $b \in \mathcal{A}_1$. Define $\psi_i(g)$ to be the permutation of $\mathcal{A}_i$ such that $\psi_i(g)(a) = b$ if and only if $\phi_g(P_a^i) = P_b^i$.

Properties 3 and 4 are clear from these definitions.

We now define the map $k_i : \mathcal{A}_i \to \mathcal{A}_{i-1}$ in the following way. Let $k_i(a) = b$ if $P_a^i \subset P_b^{i-1}$. This map is surjective, but not injective. From this definition, it is clear that $k_i \circ \eta_i(x) = \eta_{i-1}(x) \ \forall x \in X$, and $k_i(\psi_i(g))(a) = \psi_{i-1}(g)(k_i(a))$.

This completes the proof of Proposition 3.12. $\qquad\square$

It follows from the construction of the partition $\mathcal{P}_1$ and the inductive step Proposition 3.12, that an Almost Finite Presentation with the properties of Definition 3.3 exists. This completes the proof of Theorem 3.4. $\qquad\square$

EXAMPLE 3.13. Recall that in Theorem 2.4, we showed that a group chain (G_i) yields a Cantor set G_∞ and a minimal equicontinuous action of G on G_∞. For the resulting dynamical system (G_∞, G), there is a canonical construction of an AF Presentation using cylinder sets. Recall $G_\infty = \varprojlim \{\theta_{i-1}^i G/G_{i+1} \to G/G_i\}$, where the bonding maps θ_{i-1}^i are coset inclusion maps. Recall also that the canonical basepoint is (eG_i), so we have a pointed dynamical system $(G_\infty, G, (eG_i))$.

In this case, the partitions can be given by, for each $i \geq 1$,

$$P_1^i = \{(h_k G_k) \in G_\infty \mid h_k G_k = eG_k \text{ for } k \leq i\}.$$

$$\mathcal{P}_i = \{g \cdot P_1^i \mid g \in G\}.$$

2. Further Properties

We now give some further important properties of the AF Presentations constructed in Section 1.

Let $G_i = \{g \in G \mid g(P_1^i) = P_1^i\}$ be the isotropy subgroup of the action of G on the set $P_1^i \subset X$.

PROPOSITION 3.14. *The isotropy subgroup G_i acts minimally on P_1^i.*

PROOF. Let $x \in P_1^i$, and consider the orbit $G_i(x) = \{g \cdot x \mid g \in G_1\}$. To show that $G_i(x)$ is dense in P_1^i, we show that every open subset $U \subset P_1^i$ intersects $G_i(x)$. Since G acts minimally on x, there is some $g \in G$ so that $g \cdot x \in U$. Since $x \in P_1^i$ and $g \cdot x \in U$, we have $P_1^i \cap g(P_1^i) \neq \emptyset$, which by Lemma 3.6 implies that $P_1^i = g(P_1^i)$. Then, by definition of G_i, we have $g \in G_i$, so $g \cdot x \in G_i(x)$, so $U \cap G_i(x) \neq \emptyset$. Thus, the orbit of x under G_i intersects every open subset of P_1^i, so $G_i(x)$ is dense in P_1^i, so G_i acts minimally on P_1^i. □

Recall ψ_i is the permutation map previously defined.

PROPOSITION 3.15. *The isotropy subgroup G_i of P_1^i is a normal subgroup of G if and only if $G_i = ker(\psi_i)$.*

PROOF. Denote the isotropy group of P_k^i by $iso(P_k^i) = \{g \in G \mid g(P_k^i) = P_k^i\}$ (so $G_i = iso(P_1^i)$). For the forward direction, assume $G_i \trianglelefteq G$. We claim this implies that the isotropy subgroup of any other element of the partition is also G_i, i.e., $iso(P_k^i) = iso(P_1^i) = G_i$. This holds by the following argument:

First, to show $G_i \subset iso(P_k^i)$, let $G \in G_i$, and let $g_k \in G$ be such that $g_k(P_k^i) = P_1^i$, so then $g_k^{-1}(P_1^i) = P_k^i$. Since $G_i \trianglelefteq G$, we have $g_k G g_k^{-1} \in G_i$. So, $g_k G g_k^{-1} = P_i^i$. But, $g_k G g_k^{-1} = g_k G(P_k^i)$, so $g_k G(P_k^i) = P_1^i$, so $G(P_k^i) = g_k^{-1}(P_1^i) = P_k^i$, so $G \in iso(P_k^i)$. Thus, $G_i \subset iso(P_k^i)$.

To show the reverse inclusion, $iso(P_k^i) \subset G_i$, choose $h_k \in G$ such that $h_k(P_1^i) = P_k^i$, and follow the same argument as above.

Thus, if $G_i \trianglelefteq G$, the isotropy subgroup of any partition element P_k^i is also G_i. Thus, for any $G \in G_i$, we have that $\psi_i(G)$ is the identity permutation on $\mathcal{A}_i$, i.e., $G_i \subset ker(\psi_i)$.

For the backwards direction, assume that $g \in ker(\psi_i)$. This implies that g is in the isotropy subgroup of any partition element P_k^i, so by a similar argument to the above, G_i must be normal in G. □

3. Correspondence between Group Chains and AF Presentations

We now show that, given any minimal equicontinuous Cantor dynamical system, we obtain an associated group chain, and given any group chain, we obtain an associated minimal equicontinuous Cantor dynamical system.

Let G be a finitely generated group. Recall that a *group chain* is an infinite nested sequence of finite index proper subgroups

$$G = G_0 > G_1 > G_2 > ... = (G_i)$$

We showed in Theorem 2.4 that a group chain (G_i) yields a Cantor set G_∞ on which G acts minimally and equicontinuously, with basepoint (eG_i).

Recall that the dynamical systems (X, G) and (G_∞, G) are *(topologically) conjugate* if there is a homeomorphism $z : X \to G_\infty$ that is equivariant with respect to the group actions, i.e., $\gamma \cdot z(x) = z(\gamma \cdot x)$ for all $\gamma \in G$. Recall also that the systems (X, G, x) and $(G_\infty, G, (eG_i))$ are *pointed conjugate* if there is an equivariant homeomorphism z that also preserves basepoints, i.e. $z(x) = (eG_i)$.

We now re-state and prove Theorem 3.1.

THEOREM 3.1. *If (X, G, x) be a minimal equicontinuous pointed dynamical system, then there is an associated group chain (G_i) and a pointed conjugacy between (X, G, x) and $(G_\infty, G, (eG_i))$.*

PROOF. Given a minimal, equicontinuous pointed Cantor system $(X, G, \phi, \tilde{x})$, Theorem 3.4 gives us an associated AF Presentation $\{\tilde{x}, \mathcal{P}_i, \mathcal{A}_i, \eta_i, \psi_i, k_i\}$ for the system.

We then define a group chain by letting $G_i = \{g \in G \mid g(P_1^i) = P_1^i\}$ be the isotropy subgroup of the action of G on the set $P_1^i \subset X$. Then we have $G = G_0 > G_1 > G_2 > ...$ as the associated group chain.

Let $f_{i-1}^i : G/G_i \to G/G_{i-i}$ be coset inclusion maps.

LEMMA 3.16. *There is a bijective map $\tilde{\zeta}_i : \mathcal{P}_i \to G/G_i$.*

PROOF. Define $\tilde{\zeta}_i : \mathcal{P}_i \to G/G_i$ by $\tilde{\zeta}_i(P_k^i) = gG_i$ if and only if $g \cdot P_1^i = P_k^i$. If h is another coset representative of gG_i, so $h \in gG_i$, then $g^{-1}h \in G_i$, so $g^{-1}h \cdot P_1^i = P_1^i$. Then $g \cdot P_1^i = h \cdot P_1^i$. Thus, the map is well defined. □

Notice that this bijection implies that $card(G/G_i) = card(\mathcal{P}_i)$, so the alphabet $\mathcal{A}_i$ that indexes $\mathcal{P}_i$ also indexes G/G_i.

Define a map $\iota_i : X \to \mathcal{P}_i$ by $\iota_i(x) = P_k^i$ if and only if $x \in P_k^i$. Let $\zeta_i = \tilde{\zeta}_i \circ \iota_i : X \to G/G_i$. Then the maps $\{\zeta_i\}$ form a collection of mappings that are compatible with the bonding maps f_{i-1}^i, so by [18] there is a map $\zeta : X \to G_\infty$.

The equicontinuity of the action implies that this map is injective. To see this, let $x, y \in X$ with $x \neq y$ and choose $i \geq 1$ such that $d(x, y) > \varepsilon_i$. Since $diam(P_k^i) < \varepsilon_i$ for all k, x, y are in different sets of $\mathcal{P}_i$, so $\zeta_i(x) \neq \zeta_i(y)$, so $\zeta(x) \neq \zeta(y)$.

The action of G permutes the sets of $\mathcal{P}_i$, so ζ is equivariant with respect to the action of G. By construction, $\zeta(x) = (eG_i)$, since $x \in P_1^i$ for all i. $\qquad\square$

In Example 3.13, we gave a standard way to build an AF Presentation for a group chain dynamical system. So, given a minimal equicontinuous Cantor dynamical system, we get a group chain, and given a group chain, we get a minimal equicontinuous Cantor dynamical system. We should note, however, that this is not a 1-1 correspondence because these choices are not unique.

We now consider when two AF Presentations give distinct but equivalent group chains. We introduce a notion of group chain equivalence, which we will discuss further in the next chapter.

DEFINITION 3.17. *Let (G_i) and (H_i) be group chains in a finitely generated group G. We say (G_i) and (H_i) are* equivalent *if for every i there is a j $(i \leq j)$ such that $G_i > H_j$ and $H_i > G_j$.*

PROPOSITION 3.18. *If we have two distinct AF Presentations for the same group action with the same basepoint, then their associated group chains are equivalent.*

PROOF. Recall Definition 3.3, and consider two distinct AF presentations for the same action ϕ of G on X, with the same basepoint $\tilde{x} \in X$. Let one AF presentation be given by an infinite nested collection of finite clopen partitions $\mathcal{P}_i = \{P_1^i, P_2^i, ..., P_{n_i}^i\}$ of X, with a sequence $\varepsilon_i > 0$ such that $\{\varepsilon_i\}$ is decreasing to 0, and such that $diam(P_k^i) < \varepsilon_i$ for all i and for $1 \leq k \leq n_i$. Let the other AF presentation be given by an infinite nested collection of finite clopen partitions $\mathcal{Q}_i = \{Q_1^i, Q_2^i, ..., Q_{m_i}^i\}$ of X, such that $\{\varepsilon_i\}$ is decreasing to 0 and such that $diam(Q_k^i) < \varepsilon_i$ for all i and for $1 \leq k \leq m_i$. We suppose both partitions have the same basepoint, so $\tilde{x} \in P_k^i$ and $\tilde{x} \in Q_k^i$ for all i.

We associate group chains to each of these presentations as in the proof of Theorem 3.1, that is, let $G_i = \{g \in G \mid g(P_1^i) = P_1^i\}$, and let $H_i = \{g \in G \mid g(Q_1^i) = Q_1^i\}$.

Since the sequence $\{\varepsilon_i\}$ is decreasing to 0, for every $\varepsilon_i > 0$, there exists a j such that $\varepsilon_j < \frac{\varepsilon_i}{2}$. Then, we have that $diam(P_1^j) < \frac{1}{2}diam(Q_1^i)$ and $diam(Q_1^j) < \frac{1}{2}diam(P_1^i)$. Since both sequences $\{P_1^i\}, \{Q_1^i\}$ are shrinking to the same point $\tilde{x}$, this implies that $P_1^j \subset Q_1^i$ and $Q_1^j \subset P_1^i$.

Then, since $P_1^j \subset Q_1^i$ and $Q_1^j \subset P_1^i$, we have $G_j < H_i$ and $H_j < G_i$, thus, the group chains (G_i) and (H_i) are equivalent. $\qquad\square$

We can also consider two AF Presentations for the same system with different basepoints. We will show in Theorem 4.4 that this corresponds to conjugating the associated group chains.

Equivalence and Conjugacy of Group Chains

1. Introduction and Main Results

Recall that we defined a *group chain* in G as an infinite nested sequence of finite index proper subgroups

$$G = G_0 > G_1 > G_2 > ... = (G_i).$$

In Section 3, we discussed the relationship between AF partitions and group chains. In Theorem 3.1, we showed that, given a minimal equicontinuous pointed dynamical system (X, ϕ, G, x), there is a (non-unique) associated group chain (G_i) and a homeomorphism $\zeta : X \to G_\infty$ which is equivariant with respect to the actions of G on X and on G_∞, so that G_∞ represents (X, ϕ, G, x). In this chapter, we will explore further properties of group chains.

Since the choice of a group chain to represent a given group action is not unique, we study different chains associated to the same action (up to conjugacy). For this, we need a notion of equivalence of group chains, and properties that are invariant under this equivalence. The following definition comes from the work of Rogers and Tollefson in [20].

DEFINITION 4.1 (Group Chain Equivalence). *Let (G_i) and (H_i) be group chains in a finitely generated group G. We say (G_i) and (H_i) are* equivalent *if for every i there is a j $(i \leq j)$ such that $G_i > H_j$ and $H_i > G_j$. We will write $(G_i) \sim (H_i)$ to denote group chain equivalence between (G_i) and (H_i).*

That is, $(G_i) \sim (H_i)$ if and only if for each level i there is a level j so that we have the following diagram, where the arrows represent coset inclusion maps.

In this chapter, we consider the two dynamical systems arising from two group chains $(G_i), (H_i)$, both in the group G. As before,

$$(2) \qquad G_\infty \quad = \quad \varprojlim \{G/G_i \to G/G_{i-1}\}$$

$$(3) \qquad H_\infty \quad = \quad \varprojlim \{G/H_i \to G/H_{i-1}\}.$$

In both inverse limits, the bonding maps are coset inclusion maps. We consider the two dynamical systems (G_∞, G) and (H_∞, G), where the action of G on each of G_∞ and H_∞ is given by component-wise multiplication. Recall that we showed in Theorem 2.4 that G_∞ and H_∞ are Cantor sets and the action of G on them by component-wise multiplication is minimal and equicontinuous. The natural basepoint in G_∞ (respectively H_∞) is the sequence with every entry the coset of the identity, so, writing e for the identity element in G, we also consider the pointed dynamical systems $(G_\infty, G, (eG_i))$ and $(H_\infty, G, (eH_i))$.

Recall that the dynamical systems (G_∞, G) and (H_∞, G) are *(topologically) conjugate* if there is a homeomorphism $z : G_\infty \to H_\infty$ that is *equivariant* with respect to the G-actions on G_∞ and on H_∞, i.e., $\gamma \cdot z((g_iG_i)) = z(\gamma \cdot (g_iG_i))$ for all $\gamma \in G$. Such a map z is called a *conjugating homeomorphism* or a *G-map homeomorphism*. Recall also that the systems $(G_\infty, G, (eG_i))$ and $(H_\infty, G, (eH_i))$ are *pointed conjugate* if there is an equivariant homeomorphism z that also preserves basepoints, i.e. $z((eG_i)) = (eH_i)$.

Note that all Cantor sets are, by definition, homeomorphic to each other, but those arbitrary homeomorphisms may not conjugate the actions or preserve basepoints.

We also have a notion of conjugacy of group chains, resulting from conjugating each subgroup by an element of the group. We caution the reader to take care with this terminology, as the same word "conjugate" here is used to mean two different things - conjugacy of subgroups (and hence of group chains), and topological conjugacy of dynamical systems. The subsequent theorems will make the relationship between the two types of conjugacy clear.

DEFINITION 4.2. *Two group chains (G_i), (H_i) in G are said to be* conjugate *if there exists a sequence of elements $\{g_i\} \in G$, with $g_iG_i = g_jG_i$ for all $j \geq i$, such that $H_i = g_iG_ig_i^{-1}$ for all i.*

Notice that the condition $g_iG_i = g_jG_i$ for all $j \geq i$ simply ensures that $(g_iG_ig_i^{-1})$ is in fact a group chain - without that condition, we might not have a subgroup relationship between $g_iG_ig_i^{-1}$ and $g_jG_jg_j^{-1}$. In particular, if we take a constant sequence $g_i = g$ for all i, then (gG_ig^{-1}) is a conjugate chain to (G_i). Conversely, if $(g_iG_ig_i^{-1})$ is a group chain, then $g_iG_i = g_jG_i$ for all $j \geq i$.

Fokkink and Oversteegen in [**11**] showed that weak solenoids with equivalent group chains are home-omorphic by a basepoint preserving homeomorphism. In the context of group chains, which is more general than that of weak solenoids, we consider pointed conjugacy of the associated dynamical systems, and obtain an analogous result.

THEOREM 4.3. *Let (G_i) and (H_i) be group chains in G, with associated inverse limits G_∞ defined by (2) and H_∞ defined by (3). Then, the following are equivalent:*

(1) *(G_i) and (H_i) are associated to the same pointed dynamical system (X, G, ϕ, x).*

(2) *The systems $(G_\infty, G, (eG_i))$ and $(H_\infty, G, (eH_i))$ are pointed conjugate.*

(3) *(G_i) and (H_i) are equivalent group chains.*

So, equivalence of group chains corresponds to *pointed* conjugacy of dynamical systems. We will give an example to show that two non-equivalent group chains can be associated to the same dynamical system, but with different basepoints. If we allow our basepoints to change, we obtain the following theorem:

THEOREM 4.4. *Let (G_i) and (H_i) be group chains in G, with associated inverse limits G_∞ defined by (2) and H_∞ defined by (3). Then, the following are equivalent:*

(1) *(G_i) and (H_i) are associated to the same dynamical system (X, G, ϕ).*

(2) *The systems (G_∞, G) and (H_∞, G) are topologically conjugate.*

(3) *There exists a sequence (g_i) in G with $g_i H_i = g_j H_i$ for all $j \geq i$ such that $(g_i H_i g_i^{-1})$ is equivalent to (G_i).*

Note that the third condition in Theorem 4.4 says that the group chains (G_i) and (H_i) are *conjugate equivalent*, not necessarily equivalent. Examples 4.10 and 4.14 give examples of conjugate equivalent group chains that are not equivalent.

2. Proofs of Results

We now prove some intermediate properties and results we need in order to prove the main theorems of this chapter.

LEMMA 4.5. *Group chain equivalence is an equivalence relation.*

PROOF. It is clear that group chain equivalence is symmetric and reflexive, so we check tran-sitivity. Let $(G_i), (H_i), (K_i)$ be group chains in G such that $(G_i) \sim (H_i)$ and $(H_i) \sim (K_i)$. Since

$(G_i) \sim (H_i)$, for every i, there is a j such that $H_j < G_i$ and $G_j < H_i$. Since $(H_i) \sim (K_i)$, there is an n so that $K_n < H_j$ and $H_n < K_j$. Then again because $(G_i) \sim (H_i)$, there is an m so that $G_m < H_n$. So then we have $K_n < H_j < G_i$ and $G_m < H_n < K_j < K_i$. Since we also have $K_m < K_n < H_j < G_i$, we now have $K_m < G_i$ and $G_m < K_i$. So, $(G_i) \sim (K_i)$. □

We will make repeated use of Lemma 1.1.16 from [**18**]. For clarity, we first translate this lemma into our notation:

LEMMA 4.6 ([**18**]). *Let $G_\infty = \varprojlim \{G/G_i \to G/G_{i-1}\}$, with projection maps $p_i : G_\infty \to G/G_i$. Let w_i be a continuous mapping from G_∞ onto a discrete finite space Y_i. Then w_i factors through the projection map p_k for some k, that is, there exists some k and some continuous map $v_k^i : G/G_k \to Y_i$ such that $w_i = v_k^i \circ p_k$.*

LEMMA 4.7. *Let (G_{i_k}) be a subsequence of (G_i), and let $G'_\infty = \varprojlim \{G/G_{i_k} \to G/G_{i_{k-1}}\}$. Then $(G'_\infty, G, (eG_{i_k}))$ is pointed conjugate to $(G_\infty, G, (eG_i))$.*

PROOF. The homeomorphism follows from standard properties of inverse limits ([**18**]), and the conjugacy is clear since (G_{k_i}) is a subsequence of (G_i). □

Recall that we say a group chain (G_i) is *associated to* a dynamical system (X, G, ϕ) if there is an AF Presentation $\{\mathcal{P}_i\}$ for (X, G, ϕ) with $G_i = iso(P_1^i)$.

PROPOSITION 4.8. *Let (G_i) and (H_i) be group chains in G, with associated inverse limits G_∞ defined by (2) and H_∞ defined by (3).*

(1) *If (G_∞, G) and (H_∞, G) are conjugate, then $(G_i), (H_i)$ are both associated to the same dynamical system.*

(2) *If $(G_\infty, G, (eG_i))$ and $(H_\infty, G, (eH_i))$ are pointed conjugate, then $(G_i), (H_i)$ are both associated to the same pointed dynamical system.*

PROOF. Let $z : G_\infty \to H_\infty$ be a conjugating homeomorphism, i.e., $\gamma \cdot z((g_i G_i)) = z(\gamma \cdot (g_i G_i))$. Partition G_∞ by standard cylinder sets, that is

$$
\begin{aligned}
P_1^n &= \{(g_i G_i) \mid g_i G_i = eG_i \; \forall i \leq n\} \\
\mathcal{P}_i &= \{g \cdot P_1^i \mid g \in G\}.
\end{aligned}
$$

Similarly, partition H_∞ by standard cylinder sets, that is

$$Q_1^n \quad = \quad \{(g_i H_i) \mid g_i H_i = e H_i \; \forall i \leq n\}$$

$$\mathcal{Q}_i \quad = \quad \{g \cdot Q_1^i \mid g \in G\}.$$

Then $G_i = iso(P_1^i)$ and $H_i = iso(Q_1^i)$. Since $\{\mathcal{Q}_i\}$ is an AF Presentation of H_∞ with isotropy groups H_i, the group chain (H_i) is clearly associated to $X = H_\infty$. We will show that (G_i) is also associated to H_∞.

Consider the isotropy group of $z(P_1^i)$. We have $g \in iso(z(P_1^i))$ if and only if $g \cdot z(P_1^i) = z(P_1^i)$, and by the equivariance of z, $g \cdot z(P_1^i) = z(g \cdot (P_1^i))$. So, $g \in iso(z(P_1^i))$ if and only if $z(g \cdot P_1^i) = z(P_1^i)$, which since z is a homeomorphism, is true if and only if $g \cdot P_1^i = P_1^i$. So, $g \in iso(z(P_1^i))$ if and only if $g \in iso(P_1^i) = G_i$. So, (G_i) is a group chain associated to the partitions $z(P_1^i)$ of H_∞.

For the second statement, assume further that z preserves basepoints, so $z((eG_i)) = (eH_i)$. By definition of an AF Presentation, we have $(eG_i) \in P_1^i$ for all i, and $(eH_i) \in Q_1^i$. Then since z preserves basepoints, we have the same conclusion that (G_i) is a group chain associated to the partitions $z(P_1^i)$ of H_∞, and further that $(eH_i) \in z(P_1^i)$ for all i. So (G_i) is also associated to $(H_\infty, G, (eH_i))$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

We now show that pointed conjugacy of dynamical systems corresponds to group chain equivalence.

THEOREM 4.9. *Two group chains (G_i) and (H_i) in G are equivalent if and only if the associated pointed dynamical systems $(G_\infty, G, (eG_i))$ and $(H_\infty, G, (eH_i))$ are pointed conjugate.*

PROOF. First, assume $(G_i) \sim (H_i)$. Denote the coset inclusion bonding maps by

$$\theta_{i-1}^i : G/G_i \to G/G_{i-1} \quad \text{and} \quad \sigma_{i-1}^i : G/H_i \to G/H_{i-1}.$$

That is, $\theta_{i-1}^i(gG_i) = hG_{i-1}$ if and only if $gG_i \subset hG_{i-1}$, and $\sigma_{i-1}^i(gH_i) = hH_{i-1}$ if and only if $gH_i \subset hH_{i-1}$. Notice that this also means we can choose the same coset representative and write $\theta_{i-1}^i(gG_i) = gG_{i-1}$ and $\sigma_{i-1}^i(gH_i) = gH_{i-1}$.

We will inductively construct maps between subsequences of G_∞ and H_∞. For the base case, let $i_0 = j_0 = 0$, and since $G_0 = H_0 = G$, G/G_0 and H/H_0 are each a single point. Let $\tau_0 : G/G_0 \to G/H_0$ be the map of the point to the point. Let $j_1 = 1$. Since $G_1 < G_0 = H_0$, we have $G_1 < H_0$ so there is a map $\tau_1 : G/G_{j_1} \to G/H_{j_0}$. Similarly, $H_1 < H_0 = G_0$, so there is a map $\nu_1 : G/H_{j_1} \to G/G_{j_0}$. Trivially, $\tau_1(eG_{j_1}) = eH_{j_0}$ and $\nu_1(eH_{j_1}) = eG_{j_0}$.

Now for the inductive step, consider level j_k, and assume there are mappings $\tau_{j_k} : G/G_{j_k} \to G/H_{j_{k-1}}$ and $\nu_{j_k} : G/H_{j_k} \to G/G_{j_{k-1}}$.

By hypothesis, $(G_i) \sim (H_i)$, which means that for every j_k there exists a j_{k+1} such that $G_{j_{k+1}} < H_{j_k}$ and $H_{j_{k+1}} < G_{j_k}$. Let $\tau_{j_{k+1}} : G/G_{j_{k+1}} \to G/H_{j_k}$ be the coset inclusion map from the subgroup relation $G_{j_{k+1}} < H_{j_k}$, and let $\nu_{j_{k+1}} : G/H_{j_{k+1}} \to G/G_{j_k}$ be the coset inclusion map from the subgroup relation $H_{j_{k+1}} < G_{j_k}$. Since these are coset inclusion maps and the bonding maps are also coset inclusion maps, they are compatible. That is, we have the following diagram, where the maps τ_{j_k} are labeled, and the maps ν_{j_k} are represented by dashed lines.

Since we have defined the maps τ_{j_k} only for the subsequences (G_{j_k}) and (H_{j_k}) of (G_i) and (H_i), let

$$G'_\infty = \varprojlim \{G/G_{j_k} \to G/G_{j_{k-1}}\} \quad \text{and let} \quad H'_\infty = \varprojlim \{G/H_{j_k} \to G/H_{j_{k-1}}\}.$$

Let $p_i : G_\infty \to G/G_i$, $\pi_i : H_\infty \to G/H_i$ be the standard projection maps. We have the following diagram, where the vertical arrows are the bonding maps:

The maps τ_{j_k} are surjective and can be composed with the projection maps $p_i : G_\infty \to G/G_i$ to produce maps $\tau_{j_k} \circ p_{j_k} : G_\infty \to G/H_{j_{k-1}}$. So, by [18] Lemma 1.1.16, there is a continuous surjective map $\tau : G'_\infty \to H'_\infty$.

We verify that τ is injective. Let $(g_{j_k} G_{j_k}), (f_{j_k} G_{j_k}) \in G'_\infty$ be distinct, which means $g_{j_n} G_{j_n} \neq f_{j_n} G_{j_n}$ for some n. By compatibility with the bonding maps, we have $\nu_{j_{k+1}} \circ \tau_{j_{k+2}} = \theta_{j_k}^{j_{k+2}}$. Since

$g_{j_n} G_{j_n} \neq f_{j_n} G_{j_n}$, we must have $\theta_{j_n}^{j_{n+2}}(g_{j_{n+2}} G_{j_{n+2}}) \neq \theta_{j_n}^{j_{n+2}}(f_{j_{n+2}} G_{j_{n+2}})$, so $\tau_{j_{n+2}}(g_{j_{n+2}} G_{j_{n+2}}) \neq \tau_{j_{n+2}}(f_{j_{n+2}} G_{j_{n+2}})$, and so $\tau(g_{i_k} G_{i_k}) \neq \tau(f_{i_k} G_{i_k})$. So, τ is 1-1.

Thus τ is a continuous surjective injective map of compact spaces, so τ is a homeomorphism. By construction τ preserves basepoints. Since all the mappings in the construction are coset inclusion maps, τ conjugates the actions of G on G'_∞ and H'_∞. We have constructed a pointed conjugacy between sublimits of G_∞ and H_∞. By Lemma 4.7, $(G_\infty, G, (eG_i))$ and $(H_\infty, G, (eH_i))$ are pointed conjugate.

For the converse, suppose $\tau : G_\infty \to H_\infty$ is a conjugating homeomorphism with $\tau((eG_i)) = (eH_i)$. Let $p_i : G_\infty \to G/G_i$, $\pi_i : H_\infty \to G/H_i$ be the standard projection maps. Let $\tilde{\tau}_i = \pi_i \circ \tau : G_\infty \to G/H_i$, and let $\tilde{\nu}_i = p_i \circ \tau^{-1} : H_\infty \to G/G_i$. Then by [18] Lemma 1.1.16, for each i there exists an index k and a map $\tau_k^i : G/G_k \to G/H_i$ such that $\tau_k^i \circ p_k = \tilde{\tau}_i$. Similarly by by [18] Lemma 1.1.16, for each i there exists an index l and a map $\nu_l^i : G/H_l \to G/G_i$ such that $\nu_l^i \circ \pi_i = \tilde{\nu}_i$.
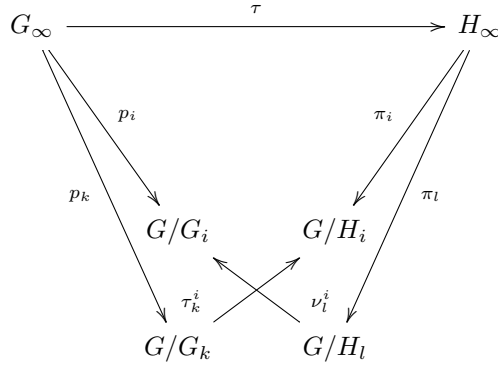
Since $p_k((eG_i)) = eG_k$ and $\tau((eG_i)) = (eH_i)$ so $\tilde{\tau}_i((eG_i)) = eG_i$, we have $\tau_k^i(eG_k) = eH_i$, so τ_k^i is a coset inclusion map and thus $G_k < H_i$. Similarly, applying the same argument to the map $\tilde{\nu}u_i = p_i \circ \tau^{-1}$, we get that there is some l so that $H_l < G_i$. Taking $j = max(k,l)$, we get $G_j < H_i$ and $H_j < G_i$. Thus, the chains (G_i) and (H_i) are equivalent. $\qquad\square$

Recall we showed in Proposition 3.18 that if $(G_i), (H_i)$ are group chains in G associated to the same pointed dynamical system (X, G, ϕ, x), then (G_i) and (H_i) are equivalent group chains. Putting together Propositions 3.18, 4.8 and Theorem 4.9, we have proven Theorem 4.3.

We now consider the role of changing basepoints and the role of conjugate group chains. First, we note that conjugate group chains need not be equivalent. We give an example of a group chain (G_i) that is not equivalent to a conjugate group chain (hG_ih^{-1}).

EXAMPLE 4.10. Let $G = (\mathbb{Z}^3, *)$ be the Heisenberg group, discussed in Section 2, and recall Example 2.10, with

$$G_n = \begin{pmatrix} qp^n & pq^n \\ p^{n+1} & q^{n+1} \end{pmatrix} \mathbb{Z}^2 \times p\mathbb{Z},$$

where p, q are distinct primes.

Let $\gamma = (qp^n x + pq^n y, p^{n+1}x + q^{n+1}y, pz) \in G_n$, and let $h = (a, b, c) \in G$. Then the last coordinate of $h * \gamma * h^{-1}$ is

$$ap^{n+1}x + aq^{n+1}y - bpq^n y - bp^n qx + pz.$$

If we choose $h = (a, b, c)$ such that a does not divide p, for example $h = (q, 1, 1)$, then the term $aq^{n+1}y$ is not divisible by p, so the last coordinate of $h * \gamma * h^{-1}$ is not divisible by p. Thus, $h * \gamma * h^{-1}$ is not an element of G_n for any n, so hG_ih^{-1} is not a subset of G_n for any n, so (G_i) can't be equivalent to the conjugate chain (hG_ih^{-1}).

These non-equivalent group chains represent the same action, but with different basepoints. Since they represent the same action, there should be a conjugating homeomorphism between them, but it need not preserve basepoints. The next proposition gives such a non-basepoint-preserving conjugating homeomorphism.

PROPOSITION 4.11. *Let (G_i) be a group chain in G, and let $(H_i) = (g_iG_ig_i^{-1})$ for a sequence $g_i \in G$ such that $g_iG_i = g_jG_i$ for all $j \geq i$. Then (G_∞, G) and (H_∞, G) are conjugate via a conjugating homeomorphism $\tau : G_\infty \to H_\infty$ with $\tau(eG_i) = (g_i^{-1}H_i)$.*

Notice that this proposition does not contradict Theorem 4.3, because the conjugating homeomorphisms obtained in the proofs of the theorems are distinct. Two dynamical systems can be conjugate via more than one homeomorphism, and the homeomorphism obtained in this theorem is not necessarily unique.

PROOF. We choose the following natural partitions to give an AF Presentation for the space G_∞. Let $\mathcal{P}_i = \{P_1^i, ..., P_{n_1}^i\}$, where P_1^i is the cylinder set of the basepoint up to level i. That is,

$$P_1^i = \{(h_kG_k) \in G_\infty \mid h_kG_k = eG_k \text{ for } k \leq i\}.$$

Then the other partition elements $P_l^i \in \mathcal{P}_i$ can be written as

$$P_l^i = \gamma_l \cdot P_1^i = \{(h_kG_k) \in G_\infty \mid h_kG_k = \gamma_lG_k \text{ for } k \leq i\},$$

for some $\gamma_l \in G$.

By hypothesis, $H_i = g_i G_i g_i^{-1}$ for a sequence of elements $\{g_i\}$ in G such that $g_i G_i = g_j G_i$ for all $j \geq i$. Then, for each level i, $g_i \cdot P_1^i$ is one of the elements of $\mathcal{P}_i$.

Now, the isotropy group of P_1^i is $iso(P_1^i) = \{g \in G \mid g \cdot P_1^i = P_1^i\} = G_i$. Consider the isotropy group of $g_i \cdot P_1^i$. This is $iso(g_i \cdot P_1^i) = g_i G_i g_i^{-1} = H_i$, because $g \in iso(g_i \cdot P_1^i)$ iff $g g_i P_1^i = g_i P_1^i$, i.e. $g_i^{-1} g g_i P_1^i = P_1^i$.

We have $H_i = g_i G_i g_i^{-1}$, so H_i is a conjugate subgroup of G_i.

So, we now have $G_i = iso(P_1^i)$ and $H_i = iso(g_i \cdot P_1^i)$. That is, P_1^i corresponds to G_i and $g_i \cdot P_1^i$ corresponds to H_i. Since the action of G permutes the elements of $\mathcal{P}_i$, there is an equivariant bijective map $\tau_i : G/G_i \to G/H_i$ such that for each $h \in G$, hG_i and $\tau_i(hG_i)$ correspond to the same element of $\mathcal{P}_i$. Then in particular $\tau_i(eG_i) = g_i^{-1} H_i$ (since we act by g_i to get from G_i to H_i, we must act by g_i^{-1} to get back).

We now have a collection of maps $\tau_i : G/G_i \to G/H_i$, which are compatible with the projection maps $p_i : G_\infty \to G/G_i$. So, by [18], there is a map $\tau : G_\infty \to H_\infty$. That τ conjugates the actions and that $\tau(eG_i) = (g_i^{-1} H_i)$ follows from the properties of τ_i. $\qquad\square$

We now prove the last two parts of Theorem 4.4.

PROPOSITION 4.12. *If* $(G_i), (H_i)$ *are associated to the system* (X, G, ϕ), *then there exists a sequence of elements* $(g_i) \in G$, *with* $g_i H_i = g_j H_i$ *for all* $j \geq i$, *such that the chain* $(g_i H_i g_i^{-1})$ *is equivalent to* (G_i).

PROOF. First, suppose $(G_i), (H_i)$ are both group chains associated to the system (X, G, ϕ).

Let $\{\mathcal{P}_i\}$ be an AF Presentation associated to the group chain (G_i), that is, $G_i = iso(P_i^i)$ and $\mathcal{P}_i = \{P_1^i, ..., P_{n_i}^i\} = \{g \cdot P_1^i \mid g \in G\}$ and $\{x\} = \cap_i P_1^i$. Similarly, let $\{\mathcal{Q}_i\}$ be an AF Presentation associated to the group chain (H_i), that is, $H_i = iso(Q_i^i)$ and $\mathcal{Q}_i = \{Q_1^i, ..., Q_{m_i}^i\} = \{g \cdot Q_1^i \mid g \in G\}$ and $\{y\} = \cap_i Q_1^i$.

Since $\mathcal{Q}_i$ partitions X, x is in one of the elements of $\mathcal{Q}_i$, say $x \in Q_k^j = g_i \cdot Q_1^i$. For $j > i$, let $Q_k^j = g_j \cdot Q_1^j$ be the element of the partition such that $x \in g_j \cdot Q_1^j$. Then $g_j \cdot Q_1^j \subset g_i Q_1^i$ since $\mathcal{Q}_j$ refines $\mathcal{Q}_i$. This implies $g_j H_i = g_i H_i$ for all $j > i$, so $g_i H_i g_i^{-1} < g_j H_j g_j^{-1}$ for $i < j$, so $(g_i H_i g_i^{-1})$ is a group chain. For all i, we have $iso(Q_i) = g_i H_i g_i^{-1}$. So, we take the chain $(g_i H_i g_i^{-1})$, and we will now show it is equivalent to (G_i).

By Definition 3.3 of AF Presentations, we have $diam(P_1^i), diam(Q_1^i) < \frac{1}{2^i}$. Since P_1^i, Q_1^i are clopen sets and X is totally disconnected, there is a $\delta_i > 0$ such that

$$dist(P_1^i, X - P_1^i) > \delta_i \qquad \text{and} \qquad dist(Q_1^i, X - Q_1^i) > \delta_i.$$

Let $j > i$ be large enough so that $\frac{1}{2^j} < \delta_j$. Then $diam(P_1^j < \frac{1}{2^j})$, so $P_1^j \subset Q_1^i$, so $G_j < g_i H_i g_i^{-1}$. Similarly, $diam(Q_1^j < \frac{1}{2^j})$, so $Q_1^j \subset P_1^i$, so $g_j H_j g_j^{-1} < G_i$. Thus, $(G_i) \sim (g_i H_i g_i^{-1})$. $\square$

PROPOSITION 4.13. *Let $(G_i), (H_i)$ be group chains in G. If there exists a sequence of elements $(g_i) \in G$, with $g_i H_i = g_j H_i$ for all $j \geq i$, such that the chain $(g_i H_i g_i^{-1})$ is equivalent to (G_i), then the systems (G_∞, G) and (G_∞, G) are topologically conjugate.*

PROOF. Assume $(G_i) \sim (g_i H_i g_i^{-1})$. Let $H'_\infty = \varprojlim \{G/g_i H_i g_i^{-1} \rightarrow G/g_{i-1} H_{i-1} g_{i-1}^{-1}\}$. By Proposition 4.11, there exists a conjugating homeomorphism $\tau_1 : H_\infty \rightarrow H'_\infty$ with $\tau_1(eH_i) = (g_i^{-1} H_i)$. By Theorem 4.3, $(H'_\infty, G, (g_i H_i g_i^{-1}))$ is pointed conjugate to $(G_\infty, G, (eG_i))$, so there is a conjugating homeomorphism $\tau_2 : H'_\infty \rightarrow G_\infty$ such that $\tau_2((g_i H_i g_i^{-1})) = (eG_i)$. So, $\tau = \tau_2 \circ \tau_1 : H_\infty \rightarrow G_\infty$ is a conjugating homeomorphism (that does not preserve basepoints). So, (H_∞, G) and (G_∞, G) are conjugate dynamical systems. $\square$

Putting together Propositions 4.12, 2, and 4.8, we have proven Theorem 4.4.

We now give an example of two non-equivalent conjugate group chains associated to the same action. The dynamical systems generated by the two chains are topologically conjugate but are not pointed conjugate.

EXAMPLE 4.14. Recall the Schori solenoid, constructed in Example 2.15. Let (G_i) be the group chain associated to the Schori solenoid, where a and b represent the generators of the fundamental group of the base space. Let $H_i = a G_i a^{-1}$, so (H_i) is a group chain that is conjugate to (G_i).

Let $G_\infty = \varprojlim \{G/G_i \rightarrow G/G_{i-1}\}$ and $H_\infty = \varprojlim \{G/H_i \rightarrow G/H_{i-1}\}$, so (G_∞, G) is the system associated to G_∞, and (H_∞, G) is the system associated to H_∞. In both systems, the action of G on the inverse limit is given by component-wise multiplication. Theorem 4.4 tells us that (G_∞, G) and (H_∞, G) are topologically conjugate. $=$

We now consider the *pointed* dynamical systems $(G_\infty, G, (eG_i))$ and $(H_\infty, G, (eH_i))$, and we claim they are not *pointed* conjugate.

The key to this example is to take care with the group operation in G/G_i versus in G/H_i, since the operation on coset representatives is different. We can perform these operations using the algebraic

generators and relations calculated for the coset spaces, but we can also see these operations more clearly on the Schreier diagrams, as illustrated in Figure 6. Since $H_i = aG_i a^{-1}$, the chain (H_i) has a different basepoint on the diagrams, as shown in Figure 6.
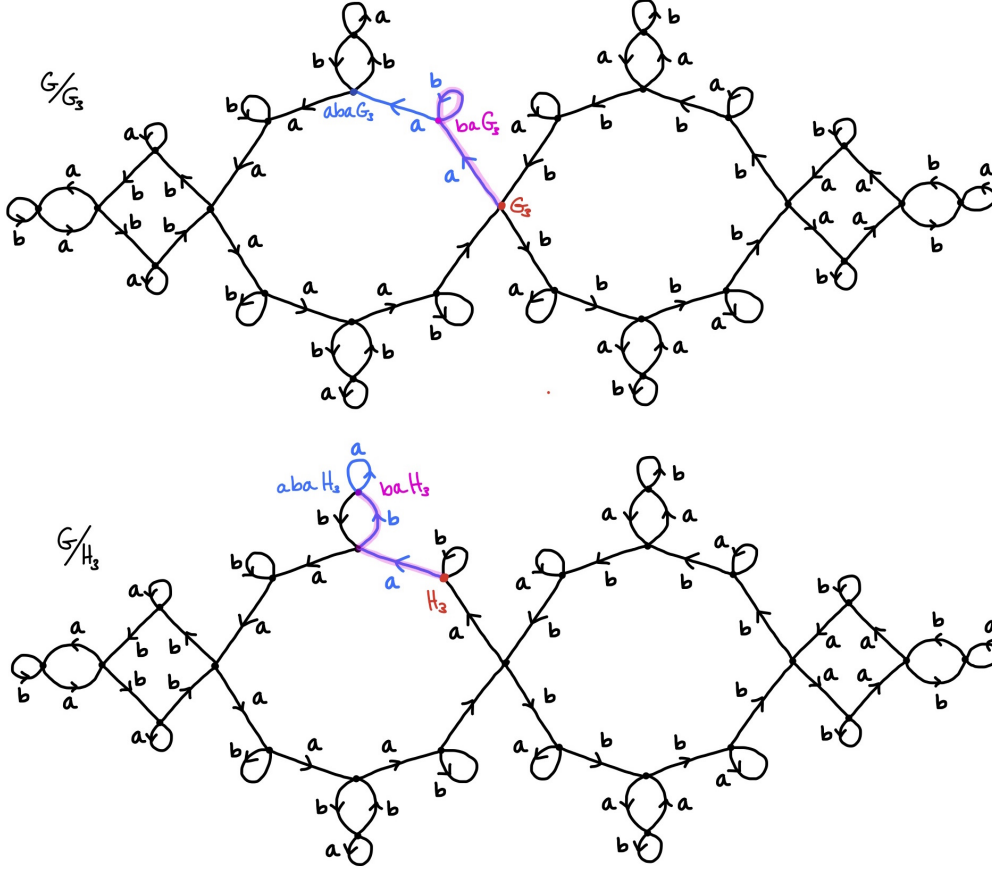

FIGURE 6. Schreier diagrams for the Schori solenoid with two different basepoints.

So, in G/G_i, we have baG_i and $abaG_i$ as distinct cosets, while in G/H_i we have $baH_i = abaH_i$.

Now, suppose there was a pointed conjugacy between $(G_\infty, G, (eG_i))$ and $(H_\infty, G, (eH_i))$. Then we have a homeomorphism $z : G_\infty \to H_\infty$ with $z((eG_i)) = (eH_i)$ and $z(g \cdot (g_i G_i)) = g \cdot z(g_i G_i)$ for all $g \in G$. Let $g = ba$, so then we have

$$z(baG_i) = ba \cdot z((eG_i)) = baH_i = abaH_i = aba \cdot z((eG_i)) = z(abaG_i)$$

So, $z(baG_i) = z(abaG_i)$, but since $baG_i \neq abaG_i$, this contradicts the injectivity of a homeomorphism z. So, no such homeomorphism can exist, and the systems cannot be *pointed* conjugate.

CHAPTER 5

Regular and Weakly Regular Group Chains

In this chapter, we classify group chains and their associated dynamical systems into three categories: regular, weakly regular, and irregular. We give the definitions and equivalent conditions of these categories, and we show that they are invariant under group chain equivalence and under group chain conjugacy, which implies they are invariant under dynamical system conjugacy.

DEFINITION 5.1. *Let (G_i) be a group chain in G. We say that (G_i) is* normal *if every G_i is a normal subgroup of G.*

We begin with an example of a normal group chain - a chain (G_n) where every $G_n \triangleleft G$.

EXAMPLE 5.2. Let G be the Heisenberg group, discussed in Section 2, and recall Example 2.9, with

$$G_i = \begin{pmatrix} p^i & 0 \\ 0 & p^i \end{pmatrix} \mathbb{Z}^2 \times p\mathbb{Z},$$

where p is prime. Then, $G_n \triangleleft G$ for each n, as shown in [15]. Indeed, if $\gamma = (p^n x, p^n y, pz) \in G_n$ and $h = (a, b, c) \in G$, then $h * \gamma * h^{-1}$ has third term $pz + ap^n y - p^n xb$, which is divisible by $m = p$ no matter what a, b, c are. Therefore, all elements $h \in G$ normalize G_n, so $G_n \triangleleft G$. So, (G_n) is a normal chain.

EXAMPLE 5.3. A Vietoris solenoid has G the fundamental group of the torus, which is abelian. Therefore, all subgroups in G are normal, and so all group chains associated to any Vietoris solenoid are normal chains.

The definition of regular and weakly regular chains is a modification of the corresponding definitions in [11].

DEFINITION 5.4. *Let (G_i) be a group chain in G. We say that (G_i) is* regular *if (G_i) is equivalent to a group chain (N_i) such that every N_i is a normal subgroup of G.*

That is, a chain (G_i) is regular if it is equivalent to a normal chain, even though each G_i might not itself be a normal subgroup of G.

By definition, a normal chain is regular. In [20], Rogers and Tollefson constructed a weak solenoid with a group chain that is regular but not normal, so in general regularity does not imply normality.

DEFINITION 5.5. *Let (G_i) be a group chain in G. We say that (G_i) is* weakly regular *if there exist an index i_0 and a finite index subgroup $N < G$ such that the chain $(G_i)_{i \geq i_0}$ is regular in N, that is, for $i \geq i_0$, we have $G_i < N$, and there is a chain (N_i) with $N_i \triangleleft N$ such that (G_i) is equivalent to (N_i).*

Notice that regular implies weakly regular. If a group chain is not weakly regular (and therefore not regular either), we call it *irregular*.

DEFINITION 5.6. *The* (normal) core *of a subgroup H of G is*

$$Core_G(H) = \cap_{g \in G} gHg^{-1}.$$

The normal core $Core_G(H)$ of H in G is the maximal subgroup of H that is normal in G.

LEMMA 5.7. *The normal cores of a group chain form a nested chain, i.e., if (G_i) is a group chain, then $Core_G(G_i) < Core_G(G_{i-1})$, so $(Core_G(G_i))$ is a group chain.*

Our first two theorems in this chapter give equivalent conditions for regularity and weak regularity. This theorem comes from Fokkink and Oversteegen in [11].

THEOREM 5.8 ([11]). *For a group chain (G_i) in G, the following are equivalent:*

(1) *(G_i) is regular.*
(2) *(G_i) is equivalent to the chain $(Core_G(G_i))$.*
(3) *For every sequence $\{g_i\}$ in G with $g_j G_i = g_i G_i$ for every $j \geq i$, we have $(G_i) \sim (g_i G_i g_i^{-1})$.*

Recall as mentioned in the last chapter, the condition $g_j G_i = g_i G_i$ for every $j \geq i$ just ensures that the chain $(g_i G_i g_i^{-1})$ is in fact a nested group chain. So the third condition here says that (G_i) is equivalent to all of its conjugate chains.

PROOF. We will use the notation $G_i^g = g G_i g^{-1}$.

(2) $\Rightarrow$ (1) is clear, because the normal cores $Core_G(G_i)$ are each normal in G, so the chain $(Core_G(G_i))$ plays the role of (N_i) in Definition 5.4.

The argument for $(3) \Rightarrow (2)$ is similar to the proof of Theorem 18 in [**11**]. Suppose that (G_i) is equivalent to $(g_i G_i g_i^{-1})$, for every sequence $\{g_i\}$ in G such that $g_j G_i = g_i G_i$ for every $j \geq i$.

It is always true that $Core_G(G_i) \leq G_i$, so in order to show group chain equivalence, we need to show that for every i there is some j so that $G_j \leq Core_G(G_i)$. G_i has finite index in G, say $[G : G_i] = n$, so there are finitely many conjugacy classes $G_i^{g_1}, G_i^{g_2}, ..., G_i^{g_n}$. Hence, $Core_G(G_i) = \cap_{g \in G} g G_i g^{-1} = G_i^{g_1} \cap G_i^{g_2}, ..., \cap G_i^{g_n}$. We assumed that (G_i) is equivalent to $(G_i^{g_k})$, hence for each k, there is some j_k such that $G_{j_k} \leq (G_i^{g_k})$. Since all of these G_{j_k} are nested, we can take the largest of the j_k to be j. Then we have $G_j \leq Core_G(G_i)$, and thus the chains (G_i) and $(Core_G(G_i)$ are equivalent.

We now consider $(1) \Rightarrow (3)$. Suppose (G_i) is equivalent to a group chain (N_i) such that every N_i is normal in G. This means that for every i there is a k such that $G_i \geq N_k$ and $N_i \geq G_k$, and for every k there is a j such that $G_k \geq N_j$ and $N_k \geq G_j$. Thus, we have the following diagram, where arrows indicate coset inclusion maps.

$$
\begin{array}{ccc}
G_i & & N_i \\
\uparrow \diagdown\diagup \uparrow & & \\
G_k & & N_k \\
\uparrow \diagdown\diagup \uparrow & & \\
G_j & & N_j
\end{array}
$$

Since $G_j \leq N_k$ and N_k is normal, for any $g_j \in G$ we have $g_j G_j g_j^{-1} \leq N_k$. We also have $N_k \leq G_i$, so $g_j G_j g_j^{-1} \leq G_i$. This is half of the equivalence - we now must show that $G_j \leq g_i G_i g_i^{-1}$.

We have $N_k \leq G_i$, so $g_i N_k g_i^{-1} \leq g_i G_i g_i^{-1}$. But, N_k is normal in G, so for any $g_i \in G$, $g_i N_k g_i^{-1} = N_k$. So, we have $N_k \leq g_i G_i g_i^{-1}$. We also have $G_j \leq N_k$, thus $G_j \leq g_i G_i g_i^{-1}$. Thus, for every i there is a j such that $g_j G_j g_j^{-1} \leq G_i$ and $G_j \leq g_i G_i g_i^{-1}$, i.e., (G_i) is equivalent to $(g_i G_i g_i^{-1})$. $\square$

It is worth noting a useful property that arose in the previous proof. We always require the subgroups in our group chains to have finite index in G, so we then always have $Core_G(G_i) = \cap_{g \in G} g G_i g^{-1} = G_i^{g_1} \cap G_i^{g_2}, ..., \cap G_i^{g_n}$, the intersection of a finite number of conjugacy classes.

THEOREM 5.9. *For a group chain (G_i) in G, the following are equivalent:*

   (1) (G_i) is weakly regular.

(2) *There exists an index i_0 and a subgroup $N < G$ such that for $i \geq i_0$, $G_i < N$, and such that $(G_i)_{i \geq i_0}$ is equivalent to the chain $(Core_N(G_i))_{i \geq i_0}$.*

(3) *There exists an index i_0 and a subgroup $N < G$ such that for $i \geq i_0$, $G_i < N$, and such that for every sequence $\{h_i\}$ in N with $h_j G_i = h_i G_i$ for every $j \geq i$, we have $(G_i)_{i \geq i_0} \sim (h_i G_i h_i^{-1})_{i \geq i_0}$.*

PROOF. This proof is the same as the proof of Theorem 5.8, but simply restricting to the subgroup N instead of G, and to indices $i \geq i_0$ instead of $i \geq 0$. We proceed through the details for completeness.

(2) $\implies$ (1) is clear, since the chain $(Core_N(G_i))_{i \geq i_0}$ has every $Core_N(G_i) \lhd N$ and $(G_i)_{i \geq i_0} \sim (Core_N(G_i))_{i \geq i_0}$.

For (3) $\implies$ (2), let $i \geq i_0$. We always have $Core_N(G_i) < G_i$, so we need to show there is a $j \geq i$ so that $G_j < Core_N(G_i)$. Indeed, G_i has finite index in G and thus in N, so there a finite number of conjugacy classes $h_1 G_i h_1^{-1}, ..., h_n G_i h_n^{-1}$ with $h_i \in N$. Hence,

$$Core_N(G_i) = \cap_{h \in N} h G h^{-1} = h_1 G_i h_1^{-1} \cap ... \cap h_n G_i h_n^{-1}.$$

We assumed that $(G_i)_{i \geq i_0}$ is equivalent to all of its conjugate chains within N, so for each h_k in the list of conjugates above, there is some j_k such that $G_{j_k} < h_1 G_i h_1^{-1}$. Since there are a finite number of these, we can take $j = max\{j_k\}$ and then we have $G_j < Core_N(G_i)$. So, then we have $(G_i)_{i \geq i_0} \sim (Core_N(G_i))_{i \geq i_0}$.

For (1) $\Rightarrow$ (3), suppose $(G_i)_{i \geq i_0}$ is equivalent to a group chain $(N_i)_{i \geq i_0}$ such that every N_i is normal in N. This means that for every $i \geq i_0$ there is a k such that $G_i \geq N_k$ and $N_i \geq G_k$, and for every k there is a j such that $G_k \geq N_j$ and $N_k \geq G_j$ (see diagram).
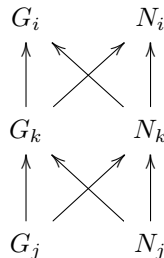
Since $G_j \leq N_k$ and N_k is normal in N, for any $h_j \in N$ we have $h_j G_j h_j^{-1} \leq N_k$. We also have $N_k \leq G_i$, so $h_j G_j h_j^{-1} \leq G_i$. This is half of the equivalence - we now must show that $G_j \leq h_i G_i h_i^{-1}$.

We have $N_k \leq G_i$, so $h_i N_k h_i^{-1} \leq h_i G_i h_i^{-1}$. But, N_k is normal in N, so for any $h_i \in N$, $h_i N_k h_i^{-1} = N_k$. So, we have $N_k \leq h_i G_i h_i^{-1}$. We also have $G_j \leq N_k$, thus $G_j \leq h_i G_i h_i^{-1}$. Thus, for every i there is a j such that $h_j G_j h_j^{-1} \leq G_i$ and $G_j \leq h_i G_i h_i^{-1}$, i.e., (G_i) is equivalent to $(h_i G_i h_i^{-1})$. $\square$

We now give an example of a chain that is weakly regular but is not regular.

EXAMPLE 5.10. Let G be the Heisenberg group, discussed in Section 2, and recall Example 2.10, with

$$G_n = \begin{pmatrix} qp^n & pq^n \\ p^{n+1} & q^{n+1} \end{pmatrix} \mathbb{Z}^2 \times p\mathbb{Z},$$

where p, q are distinct primes. Recall that in Example 4.10, we showed that (G_i) is not equivalent to the conjugate chain $(hG_i h^{-1})$, where $h = (q, 1, 1)$. So, by Theorem 5.8, (G_i) is not regular, since it is not equivalent to all of its conjugate chains in G.

But, we can show that it is weakly regular. We calculate the normalizer of G_n:

Let $\gamma = (qp^n x + pq^n y, p^{n+1} x + q^{n+1} y, pz) \in G_n$, and let $h = (a, b, c) \in G$. Then by Lemma 2.8, the last coordinate of $h * \gamma * h^{-1}$ is

$$ap^{n+1}x + aq^{n+1}y - bpq^n y - bp^n qx + pz.$$

In order for h to be in $N_G(G_n)$, we want this last term to be divisible by p. We see that the only term that does not already have a factor of p is $aq^{n+1}y$. The prime q is fixed as a prime and therefore is not divisible by p, and y is an arbitrary integer that can change, so in order for this term to be divisible by p, we must have p divides a. Therefore, in order for $h = (a, b, c)$ to be in the normalizer of G_n, its first entry must be divisible by p and the other two entries can be anything. So, $N(G_n) = p\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ for all $n \geq 1$.

Now, taking $N = N_G(G_n) = p\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ shows that $(G_i)_{i \geq 1}$ is regular in N, and thus (G_i) is weakly regular.

PROPOSITION 5.11. *Let (G_i) and (H_i) be equivalent group chains. Then,*

  (1) (G_i) is regular if and only if (H_i) is regular.

  (2) (G_i) is weakly regular if and only if (H_i) is weakly regular.

  (3) (G_i) is irregular if and only if (H_i) is irregular.

PROOF. This is clear by definition, since group chain equivalence is an equivalence relation by Lemma 4.5. So, if (G_i) is equivalent to (N_i) and (G_i) is equivalent to (H_i), then (H_i) is also equivalent to (N_i). For the weakly regular case, suppose (G_i) is weakly regular. Then for some i_0, $(G_i)_{i \geq i_0}$ is regular inside of some subgroup $N < G$. That is, there is a chain $(N_i)_{i \geq i_0}$ with $N_i \lhd N$ such that $(N_i)_{i \geq i_0} \sim (G_i)_{i \geq i_0}$. Since $(G_i) \sim (H_i)$, we also have $(G_i)_{i \geq i_0} \sim (H_i)_{i \geq i_0}$. So, since group chain equivalence is an equivalence relation by Lemma 4.5, we have $(N_i)_{i \geq i_0} \sim (G_i)_{i \geq i_0} \sim (H_i)_{i \geq i_0}$. We may not have $H_{i_0} < N$, but since $(H_i) \sim (G_i)$, there is some j so that $H_k < G_{i_0} < N$ for all $k > j$. So, (H_i) is also regular in N for $i \geq j$ and thus weakly regular. Since the first two statements are if and only ifs, the third statement follows. □

THEOREM 5.12. *Let (G_i) and (H_i) be group chains such that (H_i) is equivalent to $(g_i G_i g_i^{-1})$, for a sequence $\{g_i\}$ with $g_j G_i = g_i G_i$ for every $j \geq i$. Then:*

(1) *(G_i) is regular if and only if (H_i) is regular.*

(2) *(G_i) is weakly regular if and only if (H_i) is weakly regular.*

(3) *(G_i) is irregular if and only if (H_i) is irregular.*

Before proving this theorem, we note an important consequence. Recall Theorem 4.4, which said that the dynamical systems (G_∞, G) and (H_∞, G) are conjugate if and only if there exists a sequence of elements $(g_i) \in G$, with $g_i H_i = g_j H_i$ for all $j \geq i$, such that the chain $(g_i H_i g_i^{-1})$ is equivalent to (G_i). Theorem 4.4 along with Theorem 5.12 shows that if a group chain associated to a dynamical system is regular (respectively weakly regular, irregular), then all group chains associated to that system are regular (respectively weakly regular, irregular). So, we can classify dynamical systems as regular, weakly regular, or irregular, and the following definition is well defined.

DEFINITION 5.13. *Let (X, G, ϕ) be an equicontinuous minimal Cantor dynamical system. Then we say*

(1) *(X, G, ϕ) is regular if its associated group chains are regular.*

(2) *(X, G, ϕ) is weakly regular if its associated group chains are weakly regular.*

(3) *(X, G, ϕ) is irregular if its associated group chains are irregular.*

PROOF OF THEOREM 5.12. For (1), suppose (G_i) is regular. Then, by Theorem 5.8, $(G_i) \sim (g_i G_i g_i^{-1})$. So, we have $(G_i) \sim (g_i G_i g_i^{-1}) \sim (H_i)$ so $(G_i) \sim (H_i)$ since group chain equivalence is an equivalence relation by Lemma 4.5. Then since equivalence preserves regularity by Proposition 5.11, (G_i) being regular implies (H_i) is regular. Conversely, if (H_i) is regular, then by Proposition

5.11 we have $(g_iG_ig_i^{-1})$ regular. Then $(g_iG_ig_i^{-1})$ is equivalent to all of its conjugates by Theorem 5.8, one of which is $(g_i^{-1}g_iG_ig_i^{-1}g_i) = (G_i)$, so again $(H_i) \sim (g_iG_ig_i^{-1}) \sim (G_i)$ and thus (G_i) is also regular.

For (2), suppose (G_i) is weakly regular. Then, there is an $N < G$ and an index i_0 so that for $i \geq i_0$, (G_i) is regular inside N. Notice that we may not have (G_i) equivalent to $(g_iG_ig_i^{-1})$, since g_i may not be in N.

Without loss of generality, we can assume $N = G_{i_0}$. By Definition 5.5, there exists a chain (N_i) so that $N_i \lhd G_{i_0}$ and $(G_i)_{i \geq i_0} \sim (N_i)_{i \geq i_0}$.

Let $H = g_{i_0}G_{i_0}g_{i_0}^{-1}$. Then H contains the chains $(g_{i_0}N_ig_{i_0}^{-1})_{i \geq i_0}, (g_{i_0}G_ig_{i_0}^{-1})_{i \geq i_0}$ and $(g_iG_ig_i^{-1})_{i \geq i_0}$.

We have that $(G_i)_{i \geq i_0} \sim (N_i)_{i \geq i_0}$. This means that for every $i \geq i_0$, there is a j so that $N_j < G_i$, and $G_j < N_i$. Conjugating by g_{i_0}, we get $g_{i_0}N_jg_{i_0}^{-1} < g_{i_0}G_ig_{i_0}^{-1}$, and $g_{i_0}G_jg_{i_0}^{-1} < g_{i_0}N_ig_{i_0}^{-1}$, so $(g_{i_0}G_ig_{i_0}^{-1})_{i \geq i_0} \sim (g_{i_0}N_ig_{i_0}^{-1})_{i \geq i_0}$. Since $N_i \lhd G_{i_0}$, we have $g_{i_0}N_ig_{i_0}^{-1} \lhd g_{i_0}G_{i_0}g_{i_0}^{-1} = H$. So, $(g_{i_0}G_ig_{i_0}^{-1})_{i \geq i_0}$ is regular inside H and thus $(g_{i_0}G_ig_{i_0}^{-1})$ is weakly regular.

We now show that $(g_iG_ig_i^{-1}) \sim (g_{i_0}G_ig_{i_0}^{-1})$. Since $g_jG_i = g_iG_i$ for every $j \geq i$, we have $g_{i_0}G_{i_0} = g_iG_{i_0}$ for every $i \geq i_0$, i.e. g_i and g_{i_0} are in the same coset of g/G_{i_0}, namely $g_{i_0}G_{i_0}$. Since $g_i \in G_{i_0}$, g_i can be written as $g_i = g_{i_0}h_i$ for some $h_i \in G_{i_0}$. Then we have $g_iG_ig_i^{-1} = g_{i_0}h_iG_ih_i^{-1}g_{i_0}^{-1}$. By hypothesis, (G_i) is regular inside of G_{i_0}, so since $h_i \in G_{i_0}$, we have $(G_i) \sim (h_iG_ih_i^{-1})$. Conjugating by g_{1_0}, we get $(g_{i_0}G_ig_{i_0}^{-1}) \sim (g_{i_0}h_iG_ih_i^{-1}g_{i_0}^{-1}) = (g_iG_ig_i^{-1})$. So, $(g_iG_ig_i^{-1}) \sim (g_{i_0}G_ig_{i_0}^{-1})$. Since we showed $(g_{i_0}G_ig_{i_0}^{-1})$ is weakly regular, by Proposition 5.11, we have that $(g_iG_ig_i^{-1})$ is weakly regular. Then, since $(g_iG_ig_i^{-1}) \sim (H_i)$, again by Proposition 5.11, (H_i) is weakly regular. The converse of part (2) can be shown by the same proof, since if (H_i) is weakly regular and is equivalent to $(g_iG_ig_i^{-1})$, we have $(g_i^{-1}g_iG_ig_i^{-1}g_i) = (G_i)$ as a conjugate chain to $(g_iG_ig_i^{-1})$. So, this completes part (2) of the proof.

Part (3) follows since parts (1) and (2) are if and only ifs, the only possibility left is that (G_i) is irregular if and only if (H_i) is irregular. $\qquad \square$

We now give an example of a group chain that is irregular.

EXAMPLE 5.14. Let G be the Heisenberg group, discussed in Section 2, and recall Example 2.11, with

$$G_n = \begin{pmatrix} p^n & 0 \\ 0 & q^n \end{pmatrix} \mathbb{Z}^2 \times p^n\mathbb{Z} = p^n\mathbb{Z} \times q^n\mathbb{Z} \times p^n\mathbb{Z},$$

where p, q are distinct primes.

If (G_i) were a weakly regular group chain, then we would have a subgroup $N < G$ and an index i_0 such that for $i > i_0$, $G_i < N$, and for all $h \in N$, (G_i) equivalent to (hG_ih^{-1}). To contradict this characterization, we will show that for every i_0, there is an $h \in G_{i_0}$ such that $(G_i)_{i \geq i_0}$ is not equivalent to $(hG_ih^{-1})_{i \geq i_0}$. Since $G_i < N$ for all $i > i_0$, this implies that there cannot be any N such that for all $h \in N$, (G_i) equivalent to (hG_ih^{-1}).

Fix i_0, and let $i > i_0$. Let $h = (p^{i_0}, q^{i_0}, p^{i_0}) \in G_{i_0}$.

We will show that hG_ih^{-1} does not contain G_j for any $j > i$. Let $g = (p^i x, q^i y, p^i z)$ for some $x, y, z \in \mathbb{Z}$, so g is a generic element of G_i. We calculate $h * g * h^{-1}$:

$$
\begin{aligned}
h * g * h^{-1} &= (p^{i_0}, q^{i_0}, p^{i_0}) * (p^i x, q^i y, p^i z) * (-p^{i_0}, -q^{i_0}, -p^{i_0} + p^{i_0} q^{i_0}) \\
&= (p^{i_0} + p^i x, q^{i_0} + q^i y, p^{i_0} + p^i z + p^{i_0} q^i y) * (-p^{i_0}, -q^{i_0}, -p^{i_0} + p^{i_0} q^{i_0}) \\
&= (p^i x, q^i y, p^{i_0} + p^i z + p^{i_0} q^i y - p^{i_0} + p^{i_0} q^{i_0} + (p^{i_0} + p^i x)(-q^{i_0})) \\
&= (p^i x, q^i y, p^{i_0}(p^{i-i_0} + q^i y - p^{i-i_0} x q^{i_0}))
\end{aligned}
$$

We see that the last term is divisible by p^{i_0}, but is not divisible by p^j for any $j > i_0$. Hence, $h * g * h^{-1}$ is not an element of G_j for any $j > i_0$, and therefore the chains $(G_i)_{i \geq i_0}$ and $(hG_ih^{-1})_{i \geq i_0}$ cannot be equivalent.

Since we can find such an $h \in G_{i_0}$ for every i_0, this shows that the chain (G_i) cannot be weakly regular, and thus is irregular.

1. Almost Normal Group Chains

Definition 5.5 and the equivalent conditions given in Theorem 5.9 give criteria for weak regularity that all involve equivalence between two group chains. This means that in practice, in order to check if a specific example of a group chain is weakly regular, we must consider all possible group chains equivalent to it in all possible subgroups of G. This can be impractical in some cases, and so we now introduce a related definition that is much easier to check by direct computation.

DEFINITION 5.15. *We call a group chain (G_i) in G almost normal if there exists an index i_0 and a finite index subgroup $N < G$ such that for all $i \geq i_0$, $N < N_G(G_i)$.*

This says that there is a subgroup $N < G$ so that, for all $i \geq i_0$, G_i is a normal subgroup of N. The advantage of this definition is that it deals only with the group chain (G_i) directly, not with other equivalent group chains, so this definition is much easier to work with in specific examples.

EXAMPLE 5.16. Let G be the Heisenberg group, discussed in Section 2, and recall Example 2.10, with

$$G_n = \begin{pmatrix} qp^n & pq^n \\ p^{n+1} & q^{n+1} \end{pmatrix} \mathbb{Z}^2 \times p\mathbb{Z},$$

where p, q.

Recall that in Example 5.10, we calculated that $N(G_n) = p\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ for all $n \geq 1$. Note that this is nontrivial and does not depend on n, therefore there is a stable chain of normalizers and this example is almost normal.

EXAMPLE 5.17. Let G be the Heisenberg group, discussed in Section 2, and recall Example 2.11, with

$$G_n = \begin{pmatrix} p^n & 0 \\ 0 & q^n \end{pmatrix} \mathbb{Z}^2 \times p^n\mathbb{Z} = p^n\mathbb{Z} \times q^n\mathbb{Z} \times p^n\mathbb{Z},$$

where p, q are distinct primes. In Example 5.14, we showed this group chain is irregular, and thus cannot be almost normal. But, we can show this directly as well, by calculating the normalizer of G_n :

Let $G = (p^n x, q^n y, p^n z) \in G_n$, $h = (a, b, c)$. Then, we have

$$h * G * h^{-1} = (a + p^n x, b + q^n y, c + p^n z + aq^n y) * (-a, -b, -c + ab)$$

$$= (p^n x, q^n y, c + p^n z + aq^n y - c + ab + (a + p^n x)(-b))$$

$$= (p^n x, q^n y, p^n(z - bx) + aq^n y).$$

In order for the last coordinate to be divisible by p^n, we need p^n to divide a. So, $N_G(G_n) = p^n\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, so $N_G(G_n) > N_G(G_{n+1})$. That is, there is a strictly descending chain of normalizers. This implies that this example is not almost normal.

LEMMA 5.18. *Let (G_i) be a group chain and let N be a finite index subgroup such that $N < G_i$ for all i. Then, the chain (G_i) must stabilize; that is, there is an index i_0 such that, for all $i \geq i_0$, $G_i = G_{i+1}$.*

PROOF. Consider the chain resulting from modding out by N:

$$G/N > G_1/N > G_2/N > ... > N/N = \{1\}.$$

Since G/N is finite, this chain must stabilize at some point, which implies that the original chain must stabilize as well. □

In general, given two subgroups $K < H < G$, there may be no subgroup relationship between $N_G(K)$ and $N_G(H)$, so in general the normalizers of a group chain may not themselves form a group chain. Instead, we consider the chain of successive intersections of normalizers:

$$N_G(G_0) > N_G(G_0) \cap N_G(G_1) > N_G(G_0) \cap N_G(G_1) \cap N_G(G_2)... > \cap_{0 \leq k \leq i} N_G(G_k) > ...$$

If (G_i) is a regular group chain, then the chain of successive intersections of normalizers must stablize at some point, since N is contained in every $\cap_{0 \leq k \leq i} N_G(G_k)$.

We now consider the relationship between the properties weakly regular and almost normal. By definition, almost normal implies weakly regular. An almost normal chain is analogous to a normal chain, while a weakly regular chain is analogous to a regular chain. In [20], Rogers and Tollefson constructed a group chain that is regular but not normal. Since one characterization of a weakly regular chain is that it is regular inside a subgroup, and almost normality also means normality inside a subgroup, this example also shows that weakly regular does not imply almost normal in general.

Let (G_i) be a weakly regular chain. Then, by definition, there is a subgroup $N < G$ and an index i_0 such that, for $i \geq i_0$, (G_i) is equivalent to the chain $(Core_N(G_i))$. The normal core of G_i with respect to N, $Core_N(G_i)$, is normal in N, so $N < N_G(Core_N(G_i))$ for all $i \geq i_0$. Thus, the chain $(Core_N(G_i))$ is almost normal. So, a weakly regular chain (G_i) may not be almost normal itself, but it is equivalent to an almost normal chain, namely, the chain of its normal cores inside N (with N coming from the definition of weakly regular).

We can construct further almost normal and weakly regular examples out of normal examples, as follows.

EXAMPLE 5.19. Recall the construction given in Example 2.16: Let Γ be a finitely generated group, and let (Γ_i) be a normal group chain in Γ, that is, $\Gamma_i \triangleleft \Gamma$ for each i. Let H be a finite simple group, that is, H has no nontrivial normal subgroups. Let K be a nontrivial subgroup of H.

Now, let $G = H \times \Gamma$, and let $G_i = K \times \Gamma_i$, so (G_i) is a group chain in G. Since K is not normal in H, G_i is not normal in G. Further, since there are no normal subgroups of H, there cannot be any equivalent chain that is normal in G. So, (G_i) is not a regular chain. But, (G_i) is normal inside $K \times \Gamma < G$, since K is normal in itself and Γ_i is normal in Γ. So, (G_i) is almost normal and thus weakly regular.

We now see that the group chain in Example 2.17, using a semi-direct product, is weakly regular.

EXAMPLE 5.20. Recall the construction in Example 2.17: Let $\Gamma = \mathbb{Z}^2 = \{(a,b) \mid a, b \in \mathbb{Z}\}$, and let p, q be distinct primes. Let

$$\Gamma_i = p^i \mathbb{Z} \times q^i \mathbb{Z} = \{(ap^i, bq^i) \mid a, b \in \mathbb{Z}\}.$$

Let $H = \mathbb{Z}/2\mathbb{Z} = \{1, t \mid t^2 = 1\}$.

Let $\theta : H \to Aut(\Gamma)$ be the homomorphism defined as follows:

$$\theta : H \to Aut(\Gamma)$$

$$1 \to \theta_1 : (a, b) \to (a, b) \text{ (i.e., } \theta_1 \text{ is the identity map)}$$

$$t \to \theta_t : (a, b) \to (b, a) \text{ (i.e., } \theta_t \text{ is the transpose map)}$$

Let $G = \Gamma \rtimes_\theta H \cong \mathbb{Z}^2 \rtimes_\theta \mathbb{Z}/2\mathbb{Z}$.

Let $G_i = \Gamma_i \times \{1\}$, so (G_i) is a group chain in G.

It is easy to see by direct calculation (done in Example 8.4) that G_i is not normal in G. We will see further in Example 8.4 that (G_i) is not regular in G.

Since $\Gamma = \mathbb{Z}^2$ is abelian, each Γ_i is a normal subgroup of Γ, so G_i is normal in $\Gamma \times \{1\} < G$, which shows that the chain (G_i) is weakly regular in G.

Thus, there are group chains that are weakly regular but not regular. On the other hand, Fokkink and Oversteegen showed in [11] that weak regularity does imply regularity at the level of homeomorphism of weak solenoids. The distinction is that in the setting of weak solenoids, removing the first n levels does not change the homeomorphism class. In the setting of group chains, we require that our initial group G remain the same. So, at the level of group chain equivalence, and equivalently, topological conjugacy, weak regularity does not imply regularity.

2. Schori and Rogers and Tollefson Solenoids are Irregular

The Schori and Rogers and Tollefson examples have previously been shown to be nonhomogeneous and thus their associated group chains must be irregular. Here, we give direct proofs that the group chains are irregular.

EXAMPLE 5.21. Recall Example 2.15, and let (G_i) be the group chain constructed in that example, associated to the Schori solenoid.

We claim (G_i) is not a weakly regular group chain. Suppose for contradiction that it is, so then there is an index i_0, a subgroup $N < G$, and a group chain (H_i) such that for all $i \geq i_0$, (G_i) is equivalent to (H_i) and $H_i \triangleleft N$.

We have

$$G_k = \langle a^{2^k}, \alpha, b^{2^k}, \beta, S_{kab}, S_{kba} \mid \mathrm{rel}_k = \mathrm{id}, \mathrm{rel}_0 = \mathrm{id} \rangle.$$

By the definition of group chain equivalence, for every i there is a i' and $\tilde{i}$ such that $G_{\tilde{i}} > H_i > G_{i'}$. Without loss of generality, choose $i', \tilde{i}$ to be the closest such indices to i that satisfy $G_{\tilde{i}} > H_i > G_{i'}$. That is, let i' be the smallest index such that $G_{i'} < H_i$, and let $\tilde{i}$ be the largest index such that $H_i < G_{\tilde{i}}$. Then, there must be an H_j with $j > i$ such that $G_{i'+1} < H_j$.
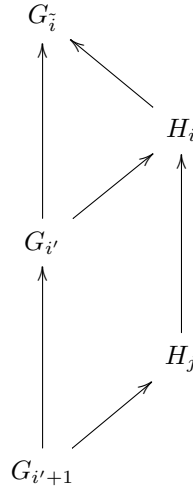
By the construction of the Schori subgroups (see Example 2.15), we have $s = a^{2^{i'}} \in G_{i'} \subset H_i$, and $m = a^{2^{i'}} b^{2^{i'}} a^{-2^{i'}} \in G_{i'+1} \subset H_j$. Since we chose i' to be the smallest index such that $G_{i'} < H_i$, we have that $H_j \neq H_i$ and $b^{2^{i'-1}} \notin H_j$.

Now $s^{-1}ms = b^{2^{i'-1}} \notin H_j$. So, we have conjugated $m \in H_j$ by s and gotten an element that is not in H_j, so $s \notin N_G(H_j)$. We now reach our contradiction. If $H_i \triangleleft N$ for all i, then $N \subset \cap_i N_G(H_i)$. But, we have $s \notin N_G(H_j)$, but also $s \in H_i$, so H_i can't be contained in $\cap_i N_G(H_i)$, which is a contradiction. So, there cannot exist such an (H_i) and N, so (G_i) can't be weakly regular, and thus is irregular.

EXAMPLE 5.22. Recall Example 2.14, and let (G_i) be the group chain constructed in that example, associated to the Rogers and Tollefson solenoid.

Suppose (G_i) is weakly regular. Then there is an index i_0, a group chain (H_i), and a group $N \in G$ such that (H_i) is equivalent to (G_i) and $H_i \triangleleft N$ for all $i > i_0$.

The group chain equivalence between (H_i) and (G_i) implies that for every i there is a i' and $\tilde{i}$ such that $G_{\tilde{i}} > H_i > G_{i'}$. Without loss of generality, choose $i', \tilde{i}$ to be the closest such indices to i that satisfy $G_{\tilde{i}} > H_i > G_{i'}$. That is, let i' be the smallest index such that $G_{i'} < H_i$, and let $\tilde{i}$ be the largest index such that $H_i < G_{\tilde{i}}$.

Since $G_{i'} < H_i$, we must have $b, a^{2^{i'}} \in H_i$. Now choose an index $j > i$ large enough so that $a^{2^{i'+1}}$ is not an element of $G_{\tilde{j}}$ (this j must exist by the definition of the chain (G_i)). This also implies that $a^{2^{i'}} \notin G_{\tilde{j}}$. So, $a^{2^{i'+1}}, a^{2^{i'}} \notin H_j \subset G_{\tilde{j}}$.
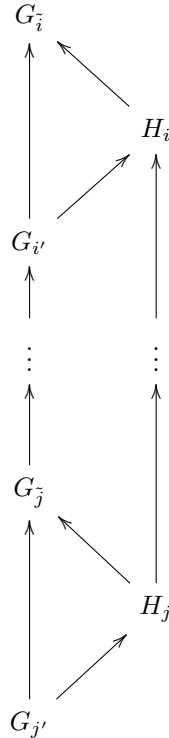
Now we claim that $a^{2^{i'}}$ is not an element of $N_G(H_j)$. We will conjugate $b \in H_j$ by $a^{2^{i'}}$. We have $H_j < G_{i'}$, so in H_j we have the relation $ba^{2^{i'}}b^{-1} = a^{-2^{i'}}$, which implies $a^{-2^{i'}}b = ba^{2^{i'}}$. Thus, we have $a^{-2^{i'}}ba^{2^{i'}} = ba^{2^{i'}}a^{2^{i'}} = ba^{2*2^{i'}} = ba^{2^{i'+1}}$. Since we chose j large enough that $a^{2^{i'+1}}$ is not an element of H_j, we also have that $ba^{2^{i'+1}}$ is not an element of H_j. So, $a^{2^{i'}}$ is not an element of $N_G(H_j)$, which implies that $a^{2^{i'}}$ is not an element of $\cap_i N_G(H_i)$.

Now, we arrive at our contradiction. If $H_i \lhd N$ for all i, then $N < \cap_i N_G(H_i)$. We have that $a^{2^{i'}}$ is an element of H_i, but we also have that $a^{2^{i'}}$ is not an element of $\cap_i N_G(H_i)$. This shows that H_i cannot be a subgroup of $\cap_i N_G(H_i)$, which is a contradiction to the assumption that $H_i \lhd N < \cap_i N_G(H_i)$.

Thus, by this contradiction, there cannot be any choice of (H_i) and N that make (G_i) weakly regular, so (G_i) can't be weakly regular, and thus is irregular.

Notice that since each covering map p_{i-1}^i is of degree 2, and all subgroups of order 2 are normal, each map p_{i-1}^i is regular. However, the composition of maps p_0^i is not regular ([**19**]), so this is not a contradiction.

This concludes Example 5.22.

We note also that since almost normal implies weakly regular, irregular implies not almost normal. Therefore, both of these solenoids are not almost normal.

We notice that all of our examples of irregular chains (G_i) have the property that the normalizers $N_G(G_i)$ form a strictly descending chain. We conjecture that any chain with this property is irregular.

CONJECTURE 5.23. *A group chain (G_i) with a strictly descending chain of normalizers is irregular.*

We have not been able to prove this conjecture, since investigation of all possible equivalent chains is nontrivial in general. We can, however, show that such an example is not almost normal, which is an easier definition to work with since it does not involve equivalent chains.

PROPOSITION 5.24. *Let (G_i) be a group chain in G with a strictly descending chain of normalizers, i.e., for each i, $N_G(G_{i-1})$ is strictly contained in $N_G(G_i)$. Then, (G_i) is not almost normal.*

PROOF. Suppose (G_i) were almost normal. Then, we would have a finite index subgroup $H < G$ so that, for all i greater than or equal to some i_0, $G_i \lhd H$. This implies that $H \subseteq N_G(G_i)$ for all

$i \geq i_0$, so $H \subset \cap_i N_G(G_i)$. Then we have a chain (using $\supset$ to mean *strict* containment):

$$N_G(G_{i_0}) \supset N_G(G_{i_0-1}) \supset ... \supset \cap_i N_G(G_i) \supseteq H \supseteq G_{i_0} \supset G_{i_0-1} \supset ...$$

Since H is finite index, we can mod out by H to get a finite subgroup $N_G(G_{i_0})/H$, and a chain in it:

$$N_G(G_{i_0})/H \supset N_G(G_{i_0-1})/H \supset ...$$

This is now a chain in a finite group, and so it must stabilize. So, it is not possible for this chain to be strictly descending forever, and we have obtained our contradiction. Therefore, such an H cannot exist, and (G_i) cannot be almost normal. $\square$

CHAPTER 6

The Automorphism Group

In this chapter, we relate the definitions for group chains of the previous chapter with the geometry of the associated group action. We will show that the number of orbits of the automorphism group determines the classification of the associated group chains as regular, weakly regular, or irregular.

DEFINITION 6.1. *Let (X, G, ϕ) be a Cantor dynamical system. An* automorphism *of X is a homeomorphism $h : X \to X$ that is equivariant with respect to the $G-$action on X, that is, $g \cdot h(x) = h(g \cdot x)$ for all $g \in G$.*

Let $Aut(X, G, \phi)$ be the group of automorphisms of the system (X, G, ϕ).

Let (X, G, ϕ) be a minimal equicontinuous Cantor dynamical system, and let (G_i) be an associated group chain, which exists by Theorem 3.1. Also by Theorem 3.1, there is an equivariant homeomorphism $z : X \to G_\infty$. Given $h \in Aut(X, G, \phi)$, there is a corresponding automorphism $\tilde{h} \in Aut(G_\infty, G)$, given by $\tilde{h} = z \circ h \circ z^{-1}$.

$$\begin{array}{ccc} X & \xrightarrow{\ h\ } & X \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ G_\infty & \dashrightarrow{\ \tilde{h}\ } & G_\infty \end{array}$$

The following result gives a standard form for realizing automorphisms of (G_∞, G) under a technical condition.

THEOREM 6.2. *Let (X, G, ϕ) be minimal equicontinuous a Cantor dynamical system, let (G_i) be an associated group chain, and let (G_∞, G) be the corresponding Cantor dynamical system.*

Let $h \in Aut(X, G, \phi)$ be an automorphism of X, and let $\tilde{h} \in Aut(G_\infty, G)$ be the corresponding automorphism of G_∞. Then:

    (1) there exists a collection of equivariant maps $\tau_i : G/G_{k_i} \to G/G_i$ such that $\tilde{h} = \varprojlim \tau_i$;

    (2) we can choose $k_i = i$ for all $i \geq 0$ if and only if $a_i \in \cap_{j \leq i} N_G(G_j)$ for all i, where $a_i G_i = \tau_i(e_i G_i)$.

PROOF. For part (1), let $p_i : G_\infty \to G/G_i$ be the projection maps. Then, $p_i \circ \tilde{h} : G_\infty \to G/G_i$ is a surjective map onto a finite discrete space, so by [**18**] Lemma 1.1.16, there is an index k_i and a map $\tau_i : G/G_{k_i} \to G/G_i$ such that $p_i \circ \tilde{h} = \tau_i \circ p_{k_i}$, as in the diagram.

$$
\begin{array}{ccc}
G_\infty & \xrightarrow{\ \tilde{h}\ } & G_\infty \\
& & \downarrow{\scriptstyle p_i} \\
\downarrow{\scriptstyle p_{k_i}} & & G/G_i \\
& {\scriptstyle \tau_i} \nearrow & \\
G/G_{k_i} & &
\end{array}
$$

The map $\tilde{h}$ is equivariant because it is an automorphism, and p_i is equivariant because it is a projection, and τ_i is a map of cosets (but not necessarily an inclusion map), so τ_i is also equivariant. Then $\tilde{h} = \varprojlim \tau_i$ ([**18**]).

Suppose $\tau_i(eG_{k_i}) = aG_i$. Then, since τ_i is equivariant, for any $g \in G$ we have

$$\tau_i(gG_{k_i}) = g \cdot \tau_i(eG_{k_i}) = g \cdot aG_i = gaG_i.$$

So, each map τ_i is determined by a single element - in order to know what the maps τ_i are, we only need to know where τ_i sends the coset of the identity, eG_{k_i}.

Since τ_i is equivariant, we have that $G_{k_i} \subseteq aG_i a^{-1}$.

LEMMA 6.3. *Let $j > i$, and $g_i, g_j \in G$ be such that $g_j G_i = g_i G_i$. Suppose $g_i \in N_G(G_i)$ and $g_j \in N_G(G_j)$. Then $g_j \in N_G(G_i) \cap N_G(G_j)$.*

PROOF. If $g_j G_i = g_i G_i$, then $g_j \in g_i G_i$, so there is $\gamma \in G_i$ such that $g_j = g_i \gamma$. Then, since $\gamma, g_i \in G_i$, we have

$$g_j G_i g_j^{-1} = g_i \gamma G_i \gamma^{-1} g_i^{-1} = g_i G_i g_i^{-1} = G_i.$$

So, $g_j \in N_G(G_i)$. $\qquad\square$

Now, suppose we have $k_i = i$ for all i, that is, $\tau_i : G/G_i \to G/G_i$ with $\tau_i(eG_i) = a_i G_i$.

Suppose we have an AF Presentation $\mathcal{P}_i = \{P_1^i, ..., P_{n_i}^i\}$ with G_i the isotropy group of P_1^i. Then the isotropy group of $a_i P_1^i$ is $a_i G_i a_i^{-1}$, and since τ_i is equivariant, we must have $a_i G_i a_i^{-1} = G_i$. So, $a_i \in N_G(G_i)$ for all i, so by Lemma 6.3, $a_i \in \cap_{j \leq i} N_G(G_j)$.

Conversely, assume $a_i \in \cap_{j \leq i} N_G(G_j)$ and define $\tau_i : G/G_i \to G/G_i$ by $\tau_i(eG_i) = a_i G_i$. We will show that $a_i \in \cap_{j \leq i} N_G(G_j)$ implies that these maps are well defined.

Let γ_1, γ_2 be two coset representatives of $\gamma_1 G_i$, so $\gamma_1, \gamma_2 \in \gamma_1 G_i$. Then, there is an element $g_1 \in G_i$ so that $\gamma_2 = \gamma_1 g_1$. Since $a_i \in N_G(G_i)$ and left and right cosets are the same with respect to elements of the normalizer, we have $a_i G_i = G_i a_i$, so there is some $g_2 \in G_2$ so that $g_1 a_i = a_i g_2$.

Then we have $\tau_i(\gamma_1 G_i) = \gamma_1 a_i G_i$ and

$$\tau_i(\gamma_2 G_i) = \gamma_2 a_i G_i$$

$$= \gamma_1 g_1 a_i G_i$$

$$= \gamma_1 a_i \gamma_2 G_i$$

$$= \gamma_1 a_i G_i \text{ since } \gamma_2 \in G_i.$$

So, $\tau_i(\gamma_2 G_i) = \gamma_1 a_i G_i = \tau_i(\gamma_1 G_i)$ so the map is well defined. $\qquad\square$

COROLLARY 6.4. *Let (G_i) be a group chain, and (G_∞, G) be the associated dynamical system. A homeomorphism $\tilde{h} : G_\infty \to G_\infty$ with $\tilde{h}(eG_i) = (a_i G_i)$ is an automorphism if and only if $(G_i) \sim (a_i G_i a_i^{-1})$.*

PROOF. From the proof of Theorem 6.2, we have that $G_{k_i} \subseteq a G_i a^{-1}$. That is, for every i there exists a k_i so that $G_{k_i} < a_i G_i a_i^{-1}$. We can apply the same argument to the inverse homeomorphism $\tilde{h}^{-1}$ to get that for every i there is a l_i so that $a_{l_i} G_{l_i} a_{l_i}^{-1}$. Taking $j = max\{k_i, l_i\}$ gives the group chain equivalence $(G_i) \sim (a_i G_i a_i^{-1})$. $\qquad\square$

Recall that an action is *transitive* if every orbit is dense.

THEOREM 6.5. *Let (X, G, ϕ) be a minimal equicontinuous Cantor dynamical system. Then*

(1) *The system (X, G, ϕ) is regular if and only if $Aut(X, G, \phi)$ acts transitively on X.*

(2) *The system (X, G, ϕ) is weakly regular if and only if $Aut(X, G, \phi)$ has a finite number of orbits in X.*

(3) *The system (X, G, ϕ) is irregular if and only if $Aut(X, G, \phi)$ has infinite number of orbits on X.*

PROOF. Proof of part (1): First, suppose the system (X, G, ϕ) is regular. Since the system is regular, we can find an associated chain G_∞ with $G_i \triangleleft G$ for all i (if we have an associated chain

without that property, then by the definition of regularity, it is equivalent to some chain that does have each subgroup normal in G, so we can look at that chain instead). Then, by [**17**], the space G_∞ is homogeneous. This means hat $Aut(G_\infty G)$ acts transitively on G_∞, and thus on X.

Conversely, suppose $Aut(X, G, \phi)$ acts transitively on X. Let $\mathcal{P}_i = \{P_1^i, ..., P_{n_i}^i\}$ be an AF Presentation for the system, and let (G_i) be an associated group chain with $x \in X$ corresponding to (eG_i) and G_i the isotropy group of P_1^i. Then for every $y \in X$, there is an automorphism $h \in Aut(X, G, \phi)$ such that $h(x) = y$. Then, if y corresponds to $(a_i G_i)$, the corresponding automorphism of $\tilde{h} \in Aut(G_\infty, G)$ has $\tilde{h}(eG_i) = (a_i G_i)$. Then, by Corollory 6.4, since $\tilde{h}$ is an automorphism with $\tilde{h}(eG_i) = (a_i G_i)$, we have $(G_i) \sim (a_i G_i a_i^{-1})$. Since we can do this for every point $y \in X$, we will get such an automorphism for every conjugate chain $(g_i G_i g_i^{-1})$ in G_∞, so this means that (G_i) is equivalent to all of its conjugate chains, and thus by Theorem 5.8, (G_i) is regular, so the associated system (X, G, ϕ) is regular.

Proof of part (2): Suppose the system (X, G, ϕ) is weakly regular. Then by definition there exists a subgroup $N < G$ and index i_0 so that $(G_i)_{i \geq i_0}$ is regular inside N. Without loss of generality, we can take $N = G_{i_0}$. By Theorem 5.9, for all chains $(g_i G_i g_i^{-1})_{i \geq i_0}$ with $g_i \in G_{i_0}$ and $g_i G_i = g_j G_i$ for all $j \geq i$, we have $(g_i G_i g_i^{-1})_{i \geq i_0} \sim (G_i)_{i \geq i_0}$. By Corollary 6.4, whenever we have $(g_i G_i g_i^{-1})_{i \geq i_0} \sim (G_i)_{i \geq i_0}$, there is an automorphism $\tilde{h}_{g_i} : G_\infty \to G_\infty$ with $\tilde{h}_{g_i}(eG_i) = (g_i G_i)$, i.e. the point $(g_i G_i) \in G_\infty$ is in the orbit of (G_i) under the action of $Aut(G_\infty, G)$.

Since every (G_i) has finite index in G_{i_0}, there are a finite number of conjugate subgroups $g_i G_i g_i^{-1}$ with $g_i \in G_{i_0}$, and hence a finite number of orbits in G_∞ under the action of $Aut(G_\infty, G)$. Since this action corresponds to the action of $Aut(X, G, \phi)$ on X, this also means there are a finite number of orbits of $Aut(X, G, \phi)$ in X.

Conversely, suppose $Aut(X, G, \phi)$ has a finite number of orbits in X. Let $\mathcal{P}_i = \{P_1^i, ..., P_{n_i}^i\}$ be an AF Presentation for the system, and let (G_i) be an associated group chain with $x \in X$ corresponding to (eG_i) and G_i the isotropy group of P_1^i. Since $Aut(X, G, \phi)$ has a finite number of orbits in X, choose a representative of each orbit, letting the first one of them be our basepoint x, and list them as $x, x_1, ..., x_n$. Then we can list the corresponding chains respectively as

(4) $$(G_i), (g_i^{(1)} G_i (g_i^{(1)}))^{-1}), ..., (g_i^{(n)} G_i (g_i^{(n)}))^{-1})$$

for some elements $g_i^{(1)}, ..., g_i^{(n)} \in G$.

Since each of these chains are in distinct orbits, by Corollory 6.4, we know none of these chains are equivalent to each other. In general if two chains $(G_i), (H_i)$ are not equivalent, then by definition that means that there exists an i so that for every $j > i$, there is either an element in H_j that is not in G_i, or an element in G_j that is not in H_i. Similarly, for every $k > i$ and every $j > k$, there is either an element in H_j that is not in G_k, or an element in G_j that is not in H_k. Since we have a finite number of group chains in the list (4), there must be some level m such that all of these group chains have distinct entries at level m. Fix this m.

To show (G_i) weakly regular, we will show that $(G_i)_{i \geq m}$ is regular inside of G_m. Let $\{s_i\}$ be a sequence inside G_m (so $s_i \in G_m$ for all i) such that $s_i G_i = s_j G_i$ for all $j \geq i$. We claim that $(G_i)_{i \geq m} \sim (s_i G_i s_i^{-1})_{i \geq m}$. For contradiction, suppose not. Then, $(s_i G_i s_i^{-1})_{i \geq m}$ must be equivalent to one of the other chains in the list (4), because $(s_i G_i s_i^{-1})_{i \geq m}$ must be in one of the finite number of orbits of the action. Say $(s_i G_i s_i^{-1})_{i \geq m} \sim (g_i^{(k)} G_i (g_i^{(k)}))^{-1})_{i \geq m}$. Then, by definition of group chain equivalence, for the fixed level m, there is a level j so that

$$s_j G_j s_j^{-1} \subset g_m^{(k)} G_m (g_m^{(k)}))^{-1}$$

and

$$g_j^{(k)} G_j (g_j^{(k)}))^{-1} \subset s_m G_m s_m^{-1}.$$

But, since $s_m \in G_m$, we have $s_m G_m s_m^{-1} = G_m$, so then we have

$$g_j^{(k)} G_j (g_j^{(k)}))^{-1} \subset s_m G_m s_m^{-1} = G_m,$$

so $g_j^{(k)} G_j (g_j^{(k)}))^{-1} \subset G_m$. But, since the chains are nested, this would mean that $g_m^{(k)} G_m (g_m^{(k)}))^{-1} = G_m$, which contradicts the fact that the entries of these chains at level m are distinct conjugates.

So, we have a contradiction, and we must have $(G_i)_{i \geq m} \sim (s_i G_i s_i^{-1})_{i \geq m}$. So, (G_i) is equivalent to every conjugate chain inside of G_m, and thus is weakly regular.

Since the first two parts are if and only if statements, part (3) follows as the only option left for an irregular system. $\square$

Recall $\mathfrak{G}$ is the space of all group chains in a group G. Let (G_i) be a group chain in G, and $G_\infty = \varprojlim \{G/G_i \to G/G_{i-1}\}$ as usual. We then write $\mathfrak{G}_\phi$ for the space of all group chains in G that are associated to the system (G, G_∞, ϕ). By Theorem 4.4,

$$\mathfrak{G}_\phi = \{(H_i) \mid (H_i) \text{ is conjugate equivalent to } (G_i)\}.$$

Then there is a nice corollory to Theorem 6.5.

COROLLARY 6.6. *Let (G, X, ϕ) be a minimal equicontinuous Cantor dynamical system. Then:*

(1) *(G, X, ϕ) is regular if and only if $\mathfrak{G}_\phi$ has only one equivalence class of group chains.*

(2) *(G, X, ϕ) is weakly regular if and only if $\mathfrak{G}_\phi$ has a finite number of equivalence classes of group chains.*

(3) *(G, X, ϕ) is irregular if and only if $\mathfrak{G}_\phi$ has an infinite number of equivalence classes of group chains.*

PROOF. This follows from Theorem 6.5 and the results of Chapter 4. $\square$

CHAPTER 7

The Discriminant Group

In this chapter, we introduce a new invariant of a minimal equicontinuous pointed dynamical system, the *discriminant group*. We will show that if the system is weakly regular, the discriminant groups for all basepoints are isomorphic, and thus in this case the discriminant group is also an invariant of the non-pointed dynamical system, up to isomorphism.

We will prove that a minimal equicontinuous Cantor dynamical system is regular if and only if the associated discriminant group is trivial. In the weakly regular case the behavior is more complicated; we will prove that if the discriminant group is finite, then the system is weakly regular. However, we will show by example that there do exist weakly regular systems with infinite discriminant group, so an if and only if for this statement is not possible.

Let (X, G, ϕ, x) be a pointed dynamical system with associated group chain (G_i). Let

$$C_i = Core_G(G_i) = \bigcap_{g \in G} gG_ig^{-1}$$

be the normal core of G_i in G. Since G_i has finite index in G, C_i is an intersection of finitely many conjugate subgroups, so C_i also has finite index. Recall that by Lemma 5.7, we have $C_i < C_{i-1}$, so (C_i) is a group chain. We then have maps $\sigma_i^{i+1} : G/C_{i+1} \to G/C_i$ given by coset inclusion, i.e., $\sigma_i^{i+1}(gC_{i+1}) = gC_i$. Since $C_i \triangleleft G$, G/C_i is a group. The maps σ_i^{i+1} are group homomorphisms, because

$$\sigma_i^{i+1}(aC_{i+1})\sigma_i^{i+1}(bC_{i+1}) = aC_ibC_i = abC_i = \sigma_i^{i+1}(abC_{i+1}) = \sigma_i^{i+1}(aC_{i+1}bC_{i+1}).$$

So, we can form the inverse limit of the groups G/C_i under the bonding maps σ_i^{i+1}, which we call the *limit core* and we denote by $Core_\infty$ or C_∞. That is,

$$C_\infty = Core_\infty = \varprojlim \{\sigma_i^{i+1} : G/C_{i+1} \to G/C_i\}$$

exists and is a profinite group.

We first note that C_∞ is an invariant of the dynamical system (X, G) up to topological isomorphism, and does not depend on a choice of basepoint.

PROPOSITION 7.1. *Let $(G_i), (H_i)$ be group chains in G associated to the same minimal equicontinuous Cantor dynamical system (X, G). Let $C_i = Core_G(G_i)$ and $M_i = Core_G(H_i)$. Let*

$$C_\infty = \varprojlim \{\theta_i^{i+1} : G/C_{i+1} \to G/C_i\},$$

$$M_\infty = \varprojlim \{\delta_i^{i+1} : G/M_{i+1} \to G/M_i\}.$$

Then, C_∞ and M_∞ are isomorphic as topological groups.

PROOF. Case 1: If $(G_i), (H_i)$ are associated to the same *pointed* dynamical system, then by Theorem 4.3, (G_i) and (H_i) are equivalent group chains. That is, for every i there is a $j \geq i$ such that $G_j < H_i$ and $H_j < G_i$. Then since $M_j < H_j < G_i$, we have $M_j < G_i$. But, since M_j is normal in G and C_i is the maximal normal subgroup of G_i, we must have $M_j < C_i$. Similarly, $C_j < G_j < H_i$ and M_i is the maximal normal subgroup of H_i, so $C_j < M_i$. Thus, the group chains (C_i) and (M_j) are equivalent.

Case 2: If $(G_i), (H_i)$ are associated to the same (non-pointed) dynamical system, then by Theorem 4.4, there exists a sequence of elements g_i in G so that $(H_i) \sim (g_i G_i g_i^{-1})$. Since $C_i = Core_G(G_i) = \bigcap_{g \in G} gG_i g^{-1}$, we have $Core_G(g_i G_i g_i^{-1}) = Core_G(G_i) = C_i$. So, by Case 1, again $(C_i) \sim (M_i)$.

So, in both cases we have $(C_i) \sim (M_i)$. Without loss of generality, we can renumber so that we have $M_{i+1} < C_i$ and $C_{i+1} < M_i$ for all i. These inclusions induce coset inclusion maps $\tau_i : G/M_i \to G/C_i$ and $\nu_i : G/C_{i+1} \to G/M_i$. Since C_i and M_i are normal subgroups of G, the maps τ_i, ν_i are well-defined group homomorphisms. Let $p_i : C_\infty \to G/C_i$ and $\pi_i : M_\infty \to G/M_i$ be the projection maps. Then there are surjective group homomorphisms $\phi_i = \nu_{i+1} \circ p_{i+1} : C_\infty \to G/M_i$. So we have the following diagram.

From this we can see that the maps ϕ_i are compatible with the projection and bonding maps in the sense that $\phi_i = \delta_i^{i+1} \circ \phi_{i+1}$ and $\tau_i \circ \nu_{i+1} = \theta_i^{i+1}$. So, by [**18**], there is a continuous surjective homomorphism $\phi = \varprojlim \phi_i : C_\infty \to M_\infty$.

To see that $C_\infty \cong M_\infty$, we show that ϕ is injective. Let $(g_i C_i), (h_i C_i) \in C_\infty$ be distinct. Then, there is some level m so that $g_m C_m \neq h_m C_m$. Then, $g_{m+1} C_{m+1} \neq h_{m+1} C_{m+1}$ are also distinct. So $\theta_m^{m+1}(g_{m+1} C_{m+1}) \neq \theta_m^{m+1}(h_{m+1} C_{m+1})$. Since we have $\tau_i \circ \nu_{i+1} = \theta_i^{i+1}$, this means that also $\nu_m^{m+1}(g_{m+1} C_{m+1}) \neq \nu_m^{m+1}(h_{m+1} C_{m+1})$, which means that the images $\phi((g_i C_i)), \phi((h_i C_i))$ in M_∞ are distinct.

So, ϕ is a continuous injective surjective homomorphism and thus is an isomorphism of topological groups. So, we have $C_\infty \cong M_\infty$. $\qquad\square$

This means that, in particular, we can refer to the limit core C_∞ of a system (X, G) even if we have not yet specified a particular group chain (G_i), since C_∞ will be invariant up to topological isomorphism for any associated group chain.

We now proceed with our construction of the discriminant group of a system. Since each C_i is a normal subgroup of G, we also have C_i normal in G_i, so each G_i/C_i is a group, which we denote by D_i. The inverse limit of these finite groups is a profinite group.

LEMMA 7.2. *The inclusion maps induce group homomorphisms $\iota_i^{i+1} : D_{i+1} \to D_i$. Thus, we can define the profinite group D_∞ by setting:*

$$(5) \qquad\qquad D_\infty = \varprojlim \{\iota_i^{i+1} : D_{i+1} \to D_i\}.$$

PROOF. The group $D_i = G_i/C_i$ is naturally identified with a subgroup of G/C_i, and by Noether's Third Isomorphism Theorem, we have

$$(G/C_i)/(G_i/C_i) \cong G/G_i.$$

We denote the quotient map

$$q_i : G/C_i \to (G/C_i)/(G_i/C_i) \cong G/G_i.$$

Then we have

$$
\begin{array}{ccc}
G/C_{i-1} & \xrightarrow{q_{i-1}} & G/G_{i-1} \\
\uparrow & & \uparrow \\
G/C_i & \xrightarrow{q_i} & G/G_i \\
\uparrow & & \uparrow \\
G/C_{i+1} & \xrightarrow{q_{i+1}} & G/G_{i+1}
\end{array}
$$

where the arrows pointing up are inclusion maps.

We have $\sigma_i^{i+1} : G/C_{i+1} \to G/C_i$ inclusion maps. Since $G_{i+1} < G$, we can restrict σ_i^{i+1} to G_{i+1}/C_{i+1}, and we denote the restriction by $\iota_i^{i+1} : G_{i+1}/C_{i+1} \to G/C_i$. We claim that $\iota_i^{i+1}(G_{i+1}/C_{i+1}) \subset G_i/C_i$, so then ι_i^{i+1} is actually a map from G_{i+1}/C_{i+1} to G_i/C_i. Indeed, this is true because ι_i^{i+1} is an inclusion map since it is a restriction of the inclusion map σ_i^{i+1}. So, since $G_{i+1} \subset G_i$ and $C_{i+1} \subset C_i$, we do not get any cosets outside of G_i/C_i in the image.

So, we have $\iota_i^{i+1} : G_{i+1}/C_{i+1} \to G_i/C_i$, i.e., $\iota_i^{i+1} : D_{i+1} \to D_i$, a coset inclusion map. Since ι_i^{i+1} is a restriction of σ_i^{i+1}, which is a group homomorphism, we have ι_i^{i+1} also a group homomorphism. Since ι_i^{i+1} is a coset inclusion map, we have $\iota_{i-1}^{i+1} = \iota_{i-1}^i \circ \iota_i^{i+1}$. So, with $\iota_i^{i+1} : D_{i+1} \to D_i$ as bonding maps, we can form the inverse limit $D_\infty = \varprojlim \{\iota_i^{i+1} : D_{i+1} \to D_i\}$, which is a profinite group since D_i are groups and ι_i^{i+1} are group homomorphisms ([**18**]). $\qquad\square$

DEFINITION 7.3. *Let (X, G, x) be a pointed minimal equicontinuous Cantor dynamical system. Let (G_i) be a group chain associated to (X, G, x), and let $C_i = Core_G(G_i)$. Then*

$$
\text{(6)} \qquad\qquad D_\infty = \varprojlim \{\iota_i^{i+1} : D_{i+1} \to D_i\} \ \subset \ C_\infty,
$$

where the bonding maps are coset inclusion maps, is called the discriminant group *of (X, G, x).*

The definition of D_∞ depends on the particular group chain (G_i), and thus on the basepoint $x \in X$ associated to that group chain. We will use the notation $D = D_\infty = D_{(G_i)} = D_x$, where $x \in X$ is the basepoint corresponding to $(eG_i) \in G_\infty$.

Recall that C_∞, up to isomorphism, is an invariant of (X, G) that does not depend on the basepoint. However, $D_\infty = D_{(G_i)} = D_x$ does depend on the basepoint $x \in X$ corresponding to the group chain (G_i). Let $\mathcal{C}$ be the set of all subgroups of C_∞. Then, we can view D_x as a function from X into $\mathcal{C}$.

DEFINITION 7.4. *We define the* discriminant function

$$D_x : X \to \mathcal{C}$$

$$x \to D_x$$

where $D_x = D_{(G_i)} = D_\infty$ for a group chain (G_i) with basepoint $x \in X$.

We will see that in the regular and weakly regular case, the discriminant groups associated to different basepoints in the same system will be isomorphic. However, notice that this does not mean D_x is a constant function; the discriminant groups at different basepoints may be isomorphic but are distinct as elements of C_∞.

We first show that equivalent group chains yield isomorphic discriminant groups.

THEOREM 7.5. *Let (G_i) be a group chain in G with associated discriminant group D_∞, and let (H_i) be a group chain in G with associated discriminant group D'_∞. If $(G_i) \sim (H_i)$, then D_∞ and D'_∞ are isomorphic as topological groups.*

PROOF. First, we establish our notation. Let $C_i = Core_G(G_i)$ and $M_i = Core_G(H_i)$. Let $D_i = G_i/C_i$, and $D'_i = H_i/M_i$. Then $D_\infty = \varprojlim \{\iota^i_{i-1} : D_i \to D_{i-1}\}$, and $D'_\infty = \varprojlim \{\eta^i_{i-1} : D'_i \to D'_{i-1}\}$, where the bonding maps ι^i_{i-1} and η^i_{i-1} are coset inclusion maps.

Suppose $(G_i) \sim (H_i)$. Then, for every i there is a j so that $G_j < H_i$ and $H_j < G_i$. We also have $C_j \triangleleft G_j$ and $M_j \triangleleft H_j$. So, $M_j < G_i$ and $C_j < H_i$. Since C_j, M_j are normal in G, they are normal in any subgroup of G, so we have $M_j \triangleleft G_i$ and $C_j \triangleleft H_i$. But, C_i is the normal core of G_i and so is the maximal normal subgroup in G_i, so another normal subgroup M_j of G_i must be contained in C_i. So, we have $M_j < C_i$, and similarly, $C_j < M_i$. Then, coset inclusion gives us a well-defined map $\tau_j : H_j/M_j \to G_i/C_i$, i.e. $\tau_j : D'_j \to D_i$. Similarly, there is a coset inclusion map $\nu_j : D_j \to D'_i$. The maps τ_j, ν_j are well-defined but may not be one to one or onto. Since τ_j, ν_j are inclusion maps, they are group homomorphisms.

So, we have the following diagram, where the maps τ_j are represented by solid arrows, and the maps ν_j are represented by dashed arrows:

Let $p_i : D_\infty \to D_i$ and $p'_i : D'_\infty \to D'_i$ be the projection maps. Thus we have the following diagram, where the maps pointing up are the bonding maps:

$$
\begin{array}{ccc}
D_\infty & & D'_\infty \\
\searrow^{p_j} & G_i/C_i \quad H_i/M_i & \swarrow_{p'_j} \\
& \uparrow \; \times \; \uparrow & \\
& G_j/C_j \;\; {}_{\nu_j} \;\; {}_{\tau_j} \;\; H_j/M_j &
\end{array}
$$

Define $z_j : D_\infty \to D'_i$ by $z_i = \nu_j \circ p_j$. Since ν_j is induced by inclusion of cosets, z_j is compatible with the bonding maps $\eta^i_{i-1} : D'_i \to D'_{i-1}$, so by [**18**] there is an induced group homomorphism $z : D_\infty \to D'_\infty$.

We have constructed maps between levels i, j with an index j given for each i by the definition of group chain equivalence. Let $\{j_k\}$ be a subsequence of the natural numbers so that we have maps $\nu_{j_k} : D_{j_k} \to D'_{j_{k-1}}$ and $\tau_{j_k} : D'_{j_k} \to D_{j_{k-1}}$.

We follow the same argument as in the proof of Theorem 4.3 to show that z is injective. Let $(g_{j_k} C_{j_k}), (h_{j_k} C_{j_k}) \in D_\infty$ be distinct, which means that there is an n so that $g_{j_n} C_{j_n} \neq h_{j_n} C_{j_n}$. By compatibility of bonding maps, we have $\tau_{j_{k+1}} \circ \nu_{j_{k+2}} = \eta^{j_{k+2}}_{j_k}$. Since $g_{j_n} C_{j_n} \neq h_{j_n} C_{j_n}$, we have

$$
\eta^{j_{n+2}}_{j_n}(g_{j_{n+2}} C_{j_{n+2}}) \neq \eta^{j_{n+2}}_{j_n}(h_{j_{n+2}} C_{j_{n+2}}),
$$

so $\nu_{j_{n+2}}(g_{j_{n+2}} C_{j_{n+2}}) \neq \nu_{j_{n+2}}(h_{j_{n+2}} C_{j_{n+2}})$, so z is one to one.

To see that z is surjective, let $(h_{j_k} M_{j_k}) \in D'_\infty$. We have $\tau_{j_k}(h_{j_k} M_{j_k}) \in G_{j_{k-1}}/C_{j_{k-1}}$, and then

$$
\nu_{j_{k-1}} \circ \tau_{j_k}(h_{j_k} M_{j_k}) \in H_{j_{k-1}}/M_{j_{k-1}}.
$$

Since $\nu_{j_{k-1}} \circ \tau_{j_k} = \eta^{j_k}_{j_{k-2}}$, we see that $(h_{j_k} M_{j_k})$ has a preimage in D_∞, so the map z is onto.

So, $z : D_\infty \to D'_\infty$ is a group homomorphism that is 1-1 and onto, hence a group isomorphism. Thus, we have shown that $D_\infty \cong D'_\infty$. $\qquad\qquad\square$

It is now easy to show that for a regular system, the discriminant group is an invariant of the system, which is independent of basepoint, up to topological isomorphism.

COROLLARY 7.6. *Let (X, G, ϕ) be a regular minimal equicontinuous Cantor dynamical system. Let $(G_i), (H_i)$ be two group chains in G associated to (X, G, ϕ). Let D_∞ be the discriminant group of (G_i), and let D'_∞ be the discriminant group of (H_i). Then, D_∞ and D'_∞ are isomorphic as topological groups.*

PROOF. Since $(G_i), (H_i)$ are associated to the same dynamical system, Theorem 4.4 tells us that there exists a sequence of elements $g_i \in G$, with $g_i G_i = g_j G_i$ for all $j \geq i$, such that $(g_i G_i g_i^{-1}) \sim (H_i)$. Since (G_i) is regular, we have $(G_i) \sim (g_i G_i g_i^{-1}) \sim (H_i)$, so $(G_i) \sim (H_i)$. Then, by Theorem 7.5, D_∞ and D'_∞ are isomorphic. □

The following theorem says that it is also true for weakly regular systems that the discriminant groups associated to different basepoints are isomorphic.

THEOREM 7.7. *Let (X, G, ϕ) be a weakly regular minimal equicontinuous Cantor dynamical system. Let $(G_i), (H_i)$ be two group chains in G associated to (X, G, ϕ). Let D_∞ be the discriminant group of (G_i), and let D'_∞ be the discriminant group of (H_i). Then, D_∞ and D'_∞ are isomorphic as topological groups.*

PROOF. Since $(G_i), (H_i)$ are associated to the same dynamical system, Theorem 4.4 tells us that there exists a sequence of elements $g_i \in G$, with $g_i G_i = g_j G_i$ for all $j \geq i$, such that $(g_i G_i g_i^{-1}) \sim (H_i)$.

Let E_∞ be the discriminant group of the chain $(g_i G_i g_i^{-1})$. By Theorem 7.5, since $(g_i G_i g_i^{-1}) \sim (H_i)$, their discriminant groups E_∞ and D'_∞ are isomorphic. We will show E_∞ and D_∞ are isomorphic, which them implies that D_∞ and D'_∞ are isomorphic, as desired.

By Definition 5.5, weak regularity means there is a subgroup $N < G$ such that $(G_i), (H_i)$ are both regular inside of N, for $i \geq i_0$ for some $i_0 > 0$. Without loss of generality, we can take $N = G_{i_0}$.

We cannot proceed as directly as in the proof of Corollory 7.6, because the sequence g_i may not be inside of G_{i_0}, as we do not assume that $(g_i G_i g_i^{-1}) \sim (G_i)$. However, we do have that the truncated chains satisfy $(G_i)_{i \geq i_0} \sim (h_i G_i h_i^{-1})_{i \geq i_0}$ for any sequence $h_i \in G_{i_0}$, for $i \geq 1$, such that $h_i G_i = h_j G_i$ for $j \geq i$. We have $g_i G_i = g_j G_i$ for all $j \geq i$, so in particular $g_i G_{i_0} = g_{i_0} G_{i_0}$ for all $i \geq i_0$, so we can find a sequence $h_i \in G_{i_0}$ for $i \geq 1$ such that $g_i = g_{i_0} h_i$ for $i \geq i_0$.

Let $S_i = h_i G_i h_i^{-1}$ for $i \geq 1$, and let E'_∞ be the discriminant group of (S_i). Since $(G_i)_{i \geq i_0} \sim (S_i)_{i \geq i_0}$, the proof of Theorem 7.5 shows that their discriminant groups D_∞ and E'_∞ in G are isomorphic.

Let $M_i = g_i G_i g_i^{-1} = g_{i_0} h_i G_i h_i^{-1} g_{i_0}^{-1} = g_{i_0} S_i g_{i_0}^{-1}$. Note that

$$Core_G(M_i) = Core_G(S_i) = Core_G(G_i) = C_i,$$

since g_i, h_i are elements of G. So, we can form the quotient groups M_i/C_i and S_i/C_i. Since $M_i = g_{i_0} S_i g_{i_0}^{-1}$, the conjugation by g_{i_0} induces a group isomorphism $\tau_i : M_i/C_i \to S_i/C_i$. We also have inclusion maps $M_{i+1}/C_{i+1} \to M_i/C_i$. So, we have the following diagram:

$$
\begin{array}{ccccccccc}
\cdots & \longleftarrow & M_i/C_i & \longleftarrow & M_{i+1}/C_{i+1} & \longleftarrow & M_{i+2}/C_{i+2} & \longleftarrow & \cdots \\
& & \downarrow{\scriptstyle \tau_i} & & \downarrow{\scriptstyle \tau_{i+1}} & & \downarrow{\scriptstyle \tau_{i+2}} & & \\
\cdots & \longleftarrow & S_i/C_i & \longleftarrow & S_{i+1}/C_{i+1} & \longleftarrow & S_{i+2}/C_{i+2} & \longleftarrow & \cdots
\end{array}
$$

The maps τ_i commute with the coset inclusion maps, since $aC_{i+1} \subset aC_i \to^{\tau_i} g_{i_0} a g_{i_0}^{-1} C_i$, and $aC_{i+1} \to^{\tau_i} g_{i_0} a g_{i_0}^{-1} C_{i+1} \subset g_{i_0} a g_{i_0}^{-1} C_i$. So, by [**18**], there is an induced group homomorphism $\tau : E_\infty \to E'_\infty$. Since τ_i are injective and surjective, so is τ (again by [**18**]). So, τ is a group isomorphism, and we have $D_\infty \cong E'_\infty \cong E_\infty \cong D'_\infty$ so $D_\infty \cong D'_\infty$ as desired. □

Notice that the method of this proof does not work if the system is irregular. We have not shown that the isomorphism class of the discriminant group is an invariant for irregular systems, but we leave that as an open question.

QUESTION 7.8. *For an irregular system, does the isomorphism class of the discriminant group depend on the choice of basepoint?*

So, we have shown that D is a function from X to $\mathcal{C} = \{$subgroups of $C_\infty\}$. In the weakly regular case, we have shown that the isomorphism class of D_x does not depend on the choice of the basepoint, that is, for $x, y \in X$, $D_x \cong D_y$. But, notice that this does not mean D_x and D_y are equal as subgroups of C_∞.

We conclude this chapter by discussing the case when D_∞ is trivial.

THEOREM 7.9. *Let (X, G, ϕ, x) be a minimal equicontinuous Cantor dynamical system with associated group chain (G_i) and discriminant group D_∞. Then, (X, G, ϕ, x) is regular if and only if D_∞ is the trivial group.*

PROOF. We first prove a lemma.

LEMMA 7.10. $G_{i+1}/C_{i+1} < ker(\iota_i^{i+1})$ *if and only if there exists a normal subgroup* $N_i \triangleleft G$ *such that* $G_{i+1} < N_i < G_i$.

PROOF OF LEMMA. If such an N_i exists, then since C_i is the maximal subgroup of G_i that is normal in G, we must have $N_i < C_i$, so we have $G_{i+1} < N_i < C_i$ so $G_{i+1} < C_i$. Then $G_{i+1}/C_{i+1} < ker(\iota_i^{i+1})$.

Conversely, let $G_{i+1}/C_{i+1} < ker(\iota_i^{i+1})$. Recall that ι_i^{i+1} was induced by the inclusion map $\sigma_i^{i+1} : G/C_{i+1} \to G/C_i$, so $ker(\iota_i^{i+1}) \subset ker(\sigma_i^{i+1})$. So, we have $G_{i+1}/C_{i+1} < ker(\sigma_i^{i+1})$. Since σ_i^{i+1} is a group homomorphism, its kernel $ker(\sigma_i^{i+1})$ is a normal subgroup of G/C_{i+1}. Let $N = \bigcup\{aC_{i+1} \mid aC_{i+1} \in ker(\sigma_i^{i+1})\}$. Then N is a normal subgroup of G such that $G_{i+1} < N < G_i$. □

First, assume that (X, G, ϕ, x) is regular. Notice that if $G_i \triangleleft G$ for all i, then $C_i = G_i$, so each $D_i = G_i/C_i$ is trivial, and thus D_∞ is clearly trivial. So, we must consider the situation when (G_i) is a regular chain in the sense of Definition 5.4, but each G_i may not itself be normal in G.

Now suppose (G_i) is a regular chain, so by Theorem 5.8, (G_i) equivalent to (C_i). This means that for every i, there is a $j \geq i$ so that $G_j < C_i$ (and we always have $C_j < G_j < G_i$). So, for this i, j we have $C_j < G_j < C_i < G_i$. Let $g_j C_j \in G_j/C_j = D_j$. Then $g_j \in G_j < C_i$, so $g_j \in C_i$. Then $g_j C_j \subset C_i$, so $\iota_j^i(g_j C_j) = C_i \in G_i/C_i = D_i$. This holds for every $g_j C_j \in D_j$. Thus, all cosets in D_j are mapped into the single coset of the identity $C_i \in D_i$. This means that there are no sequences in D_∞ that do not have the identity coset C_i at the ith level. Since we have that for every i there is such a j that makes this true, we see that at every level, an element of D_∞ must have the coset of the identity as its sequence entry. Thus, the only element of D_∞ is $(eC_1, eC_2, ...)$, and so D_∞ is the trivial group with one element (eC_i).

Now, for the converse, suppose D_∞ is the trivial group with one element. Let $aC_i \neq eC_i \in G_i/C_i$. If aC_i were in the image of the bonding maps ι_i^j for every $j \geq i$, then we would have a non-trivial element of D_∞, which contradicts the assumption that D_∞ is trivial. So, there must be some level $j_a > i$ so that $aC_{j_a} \notin image(\iota_i^{j_a})$. Since G_i/C_i is finite, there are a finite number of elements $aC_i \neq eC_i \in G_i/C_i$, so we can take the maximum of all such j_a. Let

$$j > max\{j_a \mid aC_i \notin image(\iota_i^{j_a}), aC_i \in G_i/C_i, aC_i \neq eC_i\}.$$

Then every element of G_j/C_j maps to eC_i under ι_i^j, so $G_j/C_j \subset ker(\iota_i^j)$. Then by Lemma 7.10, there is a normal subgroup $N_i \triangleleft G$ with $G_{i+1} < N_i < G_i$. Since we can do this for every level i, we get a normal chain (N_i) that is interlaced with, and thus equivalent to, (G_i). So, (G_i) is regular. $\square$

The following technical result allows us, given a group chain (G_i), to choose an equivalent chain $(H_i) \sim (G_i)$ in a "canonical form".

PROPOSITION 7.11. *Let (G_i) be a group chain with core chain (C_i) and discriminant group D_∞. Then there is a group chain (H_i) with core chain (M_i) such that $(H_i) \sim (G_i)$ and the bonding maps $H_i/M_i \to H_{i-1}/M_{i-1}$ are surjective.*

PROOF. As before, $D_\infty = \varprojlim \{\theta :_{i-1}^i G_i/C_i \to G_{i-1}/C_{i-1}\}$, where the bonding maps $\theta :_{i-1}^i$ are inclusion maps. Let $p_i : D_\infty \to G_i/C_i$ be the projection map, and let $D_i = p_i(D_\infty)$, which is a subgroup of G_i/C_i. Then D_i is isomorphic to a subgroup of G_i which we call E_i. Let

$$H_i = E_i C_i = \{\alpha\gamma \mid \alpha \in E_i, \gamma \in C_i\}.$$

Since C_i is normal in G_i, the product H_i is a subgroup of G_i. We claim that for every i, there is a $j \geq i$ such that G_j is a subgroup of H_i. The condition $G_j < H_i$ is equivalent to saying $G_j/C_j \subset E_i/C_i$, that is, $\theta_i^j(G_j/C_j) = E_i/C_i$.

Suppose there is no j such that $\theta_i^j(G_j/C_j) = E_i/C_i$. Then, for all $j \geq i$, we have $\theta_i^j(G_j/C_j) \neq E_i/C_i$. This means that D_∞ contains an element (G_iC_i) so that $p_i(g_iC_i) \notin D_i$, which contradicts our definition of D_i. So, such a j must exist, and then we have $(G_i) \sim (H_i)$.

Let $M_i = Core_G(H_i)$, and let $\delta_{i-1}^i : H_i/M_i \to H_{i-1}/M_{i-1}$ be the bonding maps for $D_{(H_i)}$, which is isomorphic to $D_{(G_i)} = D_\infty$. We claim the bonding maps δ_{i-1}^i are surjective. Note that the bonding maps are surjective if and only if the projections $\pi_i : D_{(H_i)} \to H_i/M_i$ are surjective.

We have shown that $H_i < G_i$ and that for every i there is a j so that $\theta_i^j(G_j/C_j) = E_i/C_i$. Thus, for any $h_iM_i \in H_i/M_i$, there is a $h_jM_j \in H_j/M_j$ so that $\delta_j^i(h_jM_j) = h_iM_i$. Since this holds for every index i, this means that there is an element of $D_{(H_i)}$ that projects onto each $h_iM_i \in H_i/M_i$. So, the projection maps are surjective and thus the bonding maps are surjective. $\square$

By Theorem 7.5, equivalent group chains yield isomorphic discriminant groups. So, this lemma allows us to assume without loss of generality that a discriminant group $D_\infty = \varprojlim \{G_i/C_i \to G_{i-1}/C_{i-1}\}$ has surjective bonding and projection maps, which will make some calculations in the subsequent proofs simpler.

CHAPTER 8

The Discriminant Group for Weakly Regular Systems

In this chapter, we study some special properties of the weakly regular systems. We will prove that if the discriminant group of a system is finite, then the system is weakly regular. We will then give an example to show that the converse does not hold.

We begin with a general result about systems with finite discriminant groups.

PROPOSITION 8.1. *Let (G, X) be a minimal equicontinuous Cantor dynamical system. The discriminant group D_∞ is finite if and only if there exists an associated group chain (G_i) and an index m such that for all $i \geq m$, we have $G_i \cap C_m = C_i$ (where $C_i = Core_G(G_i)$).*

PROOF. First, suppose the discriminant group $D_\infty = \varprojlim \{\theta :_{i-1}^i G_i/C_i \to G_{i-1}/C_{i-1}\}$ is finite. By Proposition 7.11, we can assume without loss of generality that the projection maps $p_i : D_\infty \to G_i/C_i$ are surjective, so $D_i = p_i(D_\infty) = G_i/C_i$. Since D_∞ is finite, there exists a level m such that the cardinality of D_i is constant for all $i \geq m$. Then, the preimage $p_i^{-1}(eC_m)$ is a single element in D_∞, and all bonding maps after level m are injective. Consider the preimage $(\theta_m^i)^{-1}(eC_m)$. Since the cardinality of D_i is constant for all $i \geq m$, we have $(\theta_m^i)^{-1}(eC_m) = (G_i \cap C_m)/C_i$, and since θ_m^i is one to one, $G_i \cap C_m = C_i$ for all $i \geq m$.

For the converse, suppose we have a group chain (G_i) such that $G_i \cap C_m = C_i$ for some m, and for all $i \geq m$. Since $G_i \cap C_m = C_i$ and $(\theta_m^i)^{-1}(eC_m) = (G_i \cap C_m)/C_i$, this means $(\theta_m^i)^{-1}$ is a single point. Since $(\theta_m^i)^{-1}$ is a single point for every i, $p_i^{-1}(eC_m)$ is a single element of D_∞. Since D_∞ is a group and G_m/C_m is a finite group, D_∞ is finite. □

THEOREM 8.2. *Let $D_\infty = \varprojlim \{G_i/C_i \to G_{i-1}/C_{i-1}\}$ be the discriminant group of a chain (G_i). If D_∞ is finite, then (G_i) is weakly regular.*

PROOF. Suppose $D_\infty = \varprojlim \{G_i/C_i \to G_{i-1}/C_{i-1}\}$ is finite. By Proposition 8.1, we can assume (G_i) is a group chain chosen so that $G_i \cap C_m = C_i$ for some fixed m and for all $i \geq m$. Then, we

have the following diagram

$$
\begin{array}{ccccc}
C_m/C_i & \xrightarrow{=} & C_m/(G_i \cap C_m) & \xrightarrow{\tau_i} & G/G_i \\
\uparrow{\scriptstyle \delta_i^{i+1}} & & \uparrow & & \uparrow{\scriptstyle \theta_i^{i+1}} \\
C_m/C_{i+1} & \xrightarrow{=} & C_m/(G_{i+1} \cap C_m) & \xrightarrow{\tau_{i+1}} & G/G_{i+1}
\end{array}
$$

where the τ_i are induced by the inclusion $C_m \subset G$, δ_i^{i+1} are induced by the inclusion $C_{i+1} \subset C_i$, and θ_i^{i+1} are the bonding maps of the inverse limit $G_\infty = \varprojlim \{G/G_{i+1} \to G/G_i\}$.

Let $C_* = \varprojlim \{C_m/C_{i+1} \to C_m/C_i\}$. Then there is an induced map $\tau = \varprojlim \tau_i : C_* \to G_\infty$ by [18, Lemma 1.1.6]. The image $\tau(C_*)$ is a clopen subset of G_∞ that is homeomorphic to C_*.

Consider the group action of C_m on C_*. Since each C_i is a normal subgroup of G, we have $C_i \triangleleft C_m$ for all i, so the chain (C_i) is normal in C_m, which means the dynamical system (C_m, C_*) is regular.

Consider the isotropy group of the image $\tau(C_*)$,

$$
iso(\tau(C_*)) = \{g \in G \mid g \cdot \tau(C_*) = \tau(C_*)\}.
$$

We claim $iso(\tau(C_*)) = C_m$.

Suppose $g \in G$ is such that for all $(c_i G_i) \in \tau(C_*)$, $c_i \in C_m$, we have $g \cdot (c_i G_i) = (g c_i G_i) \in \tau(C_\infty^1)$.

We have $g c_i G_i \in \tau(C_*)$, so $g c_i G_i$ has a representative in C_m, call it k_m, i.e. $k_m = g c_i g_i$ for some $g_i \in G_i$. So, $g c_i g_i \in C_m$, so $g c_i = g c_i g_i g_i^{-1} \in C_m G_i$. This holds for every sequence in $\tau(C_*)$, so in particular, $g \cdot (e G_i) \in \tau(C_*)$, and so $g \in C_m G_i$ for all $i \geq 0$.

Recall $K = \cap_i G_i$ is the kernel of the group chain (G_i). We have $C_m G_i = \cup\{C_m g \mid g \in G_i\}$, so $\cap C_m G_i = C_m K$. Then $g \in C_m K$. If $K = \{e\}$, then $g \in C_m$.

If $K \neq \{e\}$, then we have $g \in C_m K$, meaning there exists an $h \in C_m$, $\gamma \in K$ so that $g = h\Gamma$. Then $g c_i G_i = h\gamma c_i G_i = h c_i' \gamma G_i$ for some $c_i' \in C_m$ since C_m is a normal subgroup. Then since $\gamma \in K$ means $\gamma \in G_i$ for all i, we have $g c_i G_i = h\gamma c_i G_i = h c_i' \gamma G_i = h c_i' G_i$. So, there is $h \in C_m$ so that $h c_i' G_i = g c_i G_i$ for all i, and $(c_i' G_i)$, $(g c_i G_i)$ are both in $\tau(C_*)$ by hypothesis. So, $h \in C_m$ is in $iso(\tau(C_*))$, so C_m is the isotropy group of $\tau(C_*)$.

Therefore, since $iso(\tau(C_*)) = C_m$, if we restrict the action of G to $\tau(C_*)$, we get simply the action by C_m. Since the system $(C_m, \tau(C_*))$ is regular, Theorem 6.5 tells us that the action of C_m on $\tau(C_*)$ transitive, so all points in the clopen set $\tau(C_*)$ are in the same orbit of $Aut(G_\infty, G)$. Since C_i are normal and thus invariant under conjugation by elements of G, we can repeat this argument for

all group chains (G_i) in G_∞. Therefore, G_∞ is the union of disjoint clopen subsets, which are each one orbit of $Aut(G_\infty, G)$. Since G_∞ is compact, every open cover has a finite subcover, so there are in fact a finite number of orbits of $Aut(G_\infty, G)$. Then by Theorem 6.5, the system is weakly regular. $\qquad\square$

So, we have shown that if a discriminant group is finite, the associated system must be weakly regular. However, the converse of this statement is not true, as we show by example. In the next two examples, we construct weakly regular systems with both finite and infinite discriminant groups. We first calculate the discriminant group for the weakly regular example given in Examples 2.16 and 5.19, and show that it is finite.

EXAMPLE 8.3. Let (G_i) be the group chain given in Example 2.16, with H a finite simple group, Γ a finitely generated group, $G = H \times \Gamma$, (Γ_i) a normal chain in Γ, K a nontrivial subgroup of H, and $G_i = K \times \Gamma_i$. We showed in Example 5.19 that (G_i) is not regular but is weakly regular.

We will calculate the discriminant group. First, we calculate G_∞. We have

$$G/G_i = (H \times \Gamma)/(K \times \Gamma_i) = H/K \times \Gamma/\Gamma_i.$$

So,

$$
\begin{aligned}
G_\infty &= \varprojlim \{G/G_i \to G/G_{i-1}\} \\
&= \varprojlim \{H/K \times \Gamma/\Gamma_i \to H/K \times \Gamma/\Gamma_{i-1}\} \\
&= H/K \times \varprojlim \{\Gamma/\Gamma_i \to \Gamma/\Gamma_{i-1}\} = H/K \times \Gamma_\infty.
\end{aligned}
$$

Since H has no nontrivial normal subgroups, the normal core of K in H, $Core_H(K) = \cap_{h \in H} hKh^{-1}$, must be trivial. We calculate $C_i = Core_G(G_i) = \cap_{g \in G} gG_ig^{-1}$. An element $g \in G = H \times \Gamma$ can be written as $g = (h, \gamma)$ for some $h \in H, \gamma \in \Gamma$. So, we have

$$
\begin{aligned}
C_i = Core_G(G_i) &= \cap_{g \in G} gG_ig^{-1} \\
&= \cap_{(h,\gamma) \in H \times \Gamma}[(h, \gamma)(K \times \Gamma_i)(h^{-1}, \gamma^{-1})] \\
&= \cap_{h \in H, \gamma \in \Gamma}[hKh^{-1} \times \gamma\Gamma_i\gamma^{-1}] \\
&= (\cap_{h \in H} hKh^{-1}) \times \Gamma_i \text{ since } \Gamma_i \lhd \Gamma \\
&= \{e\} \times \Gamma_i \text{ since } \cap_{h \in H} hKh^{-1} \text{ is the trivial normal core of } K \text{ in } H.
\end{aligned}
$$

Now we can calculate the discriminant group of (G_i).

$$D_\infty^{(G_i)} = \varprojlim \{G_i/C_i \to G_{i-1}/C_{i-1}\}$$

$$= \varprojlim \{(K \times \Gamma_i)/(\{e\} \times \Gamma_i) \to (K \times \Gamma_{i-1})/(\{e\} \times \Gamma_{i-1})\}$$

$$= \varprojlim \{K/\{e\} \times \Gamma_i/\Gamma_i \to K/\{e\} \times \Gamma_{i-1}/\Gamma_{i-1}\}$$

$$= \varprojlim \{K \to^{id} K\} \cong K$$

So, the discriminant group of (G_i) is isomorphic to K, and in particular is finite.

We now give an example of a group chain that is weakly regular with infinite discriminant group.

EXAMPLE 8.4. Recall the construction in Example 2.17: Let $\Gamma = \mathbb{Z}^2 = \{(a,b) \mid a, b \in \mathbb{Z}\}$, and let p, q be distinct primes. Let

$$\Gamma_i = p^i\mathbb{Z} \times q^i\mathbb{Z} = \{(ap^i, bq^i) \mid a, b \in \mathbb{Z}\}.$$

Let $H = \mathbb{Z}/2\mathbb{Z} = \{1, t \mid t^2 = 1\}$.

Let $\theta : H \to Aut(\Gamma)$ be the homomorphism defined as follows:

$$\theta : H \to Aut(\Gamma)$$

$$1 \to \theta_1 : (a,b) \to (a,b) \text{ (i.e., } \theta_1 \text{ is the identity map)}$$

$$t \to \theta_t : (a,b) \to (b,a) \text{ (i.e., } \theta_t \text{ is the transpose map)}$$

Let $G = \Gamma \rtimes_\theta H \cong \mathbb{Z}^2 \rtimes_\theta \mathbb{Z}/2\mathbb{Z}$., and let $G_i = \Gamma_i \times \{1\}$.

We showed in Example 5.20 that (G_i) is weakly regular.

We now calculate the core $C_i = Core_G(G_i) = \cap_{g \in G} gG_ig^{-1}$. An element $g \in G$ is of the form $((a,b), 1)$ or $((a,b), t)$, with $(a,b) \in \mathbb{Z}^2$. Since θ_1 is the identity map and the $\mathbb{Z}^2$ part is abelian, conjugating G_i by $((a,b), 1)$ gives G_i again.

Recall that the semi-direct product $\Gamma \rtimes_\theta H$ is the set $\Gamma \times H$ with the operation given by $(\gamma_1, h_1) * (\gamma_2, h_2) = (\gamma_1\theta_{h_1}(\gamma_2), h_1h_2)$, i.e., $((a_1, b_1), h_1) * ((a_2, b_2), h_2) = ((a_1, b_1)\theta_{h_1}((a_2, b_2), h_2)), h_1h_2)$, where $h_1, h_2 \in \mathbb{Z}/2\mathbb{Z}$ are 1 or t, and $(a,b) \in \mathbb{Z}^2$. Recall also that the inverse of an element in the semi-direct product is given by $(\gamma, h)^{-1} = (\theta_h(\gamma^{-1}), h^{-1})$, i.e., $((a,b), h)^{-1} = (\theta_h((-a, -b)), h^{-1})$.

An element of $G_i = \Gamma_i \times \{1\}$ is of the form $(\alpha p^i, \beta q^i, 1)$. We calculate the conjugation by $g = ((a, b), t)$:

$$((a, b), t) * ((\alpha p^i, \beta q^i), 1) * ((-b, -a), t) = ((a, b) + \theta_t(\alpha p^i, \beta q^i), t) * ((-b, -a), t)$$

$$= ((a, b) + (\beta q^i, \alpha p^i), t) * ((-b, -a), t)$$

$$= ((a + \beta q^i, b + \alpha p^i), t) * ((-b, -a), t)$$

$$= ((a + \beta q^i, b + \alpha p^i) + \theta_t(-b, -a), t^2)$$

$$= ((a + \beta q^i, b + \alpha p^i) + (-a, -b), 1)$$

$$= ((\beta q^i, \alpha p^i), 1)$$

So, we have $G_i = \{(ap^i, bq^i) \mid a, b \in \mathbb{Z}\} \times \{1\}$, and $gG_ig^{-1} = \{(aq^i, bp^i) \mid a, b \in \mathbb{Z}\} \times \{1\} = \Gamma_i^T \times \{1\}$ for $g = ((a, b), t)$. In particular, since $gG_ig^{-1} \neq G_i$, (G_i) is not normal in G.

Thus, we have

$$C_i = Core_G(G_i) = \cap_{g \in G} gG_ig^{-1}$$

$$= G_i \cap gG_ig^{-1}$$

$$= \{(ap^iq^i, bp^iq^i) \mid a, b \in \mathbb{Z}\} \times \{1\}$$

So, $(ap^i, bq^i, 1), (cp^i, dq^i, 1) \in G_i$ are equal modulo C_i if and only if $a = c \bmod q^i, b = d \bmod p^i$. So, the cardinality of G_i/C_i is p^iq^i, which tends towards infinity.

Recall the discriminant group of (G_i) is $D_\infty = \varprojlim \{G_i/C_i \to G_{i-1}/C_{i-1}\}$, where the bonding maps are inclusion maps.

As sets, $G_i/C_i = (p^i\mathbb{Z} \times q^i\mathbb{Z} \times 1)/(p^iq^i\mathbb{Z} \times p^iq^i\mathbb{Z} \times 1) = \mathbb{Z}/q^i\mathbb{Z} \times \mathbb{Z}/p^i\mathbb{Z} \times 1$, so it is easy to see that the bonding maps in the inverse limit D_∞ are surjective. Since the maps are surjective and the cardinality of G_i/C_i tends towards infinity, the cardinality of D_∞ is infinite. By Theorem 7.9, (G_i) is not regular.

Thus, this example is weakly regular and has an infinite discriminant group.

The previous two examples show that there exist weakly regular examples with either finite or infinite discriminant groups. This motivates the following definition.

DEFINITION 8.5. *The group chain (G_i) is* tame *if the discriminant group $\mathcal{D}_\infty$ is finite.*

Thus, we have shown that there are weakly regular group chains which are tame, and also which are not tame. Thus, there are weakly regular, equicontinuous, minimal Cantor systems which are tame, and which are not tame.

We denote the *kernel* of the group chain (G_i) by $K = \cap_i G_i$. The following theorem is stated without proof in [**11**]. We give a proof here.

THEOREM 8.6 ([**11**]). *If (G_i) is weakly regular, then the kernel $K = \cap_i G_i$ has a finite number of conjugate subgroups in G.*

PROOF. First, we note that if (G_i) is regular, then $(G_i) \sim (N_i)$ for some chain (N_i) with $N_i \triangleleft G$ for all i. Group chain equivalence means the chains are intertwined, so their intersections are the same, so we have $K = \cap_i G_i = \cap_i N_i$. Since the intersection of normal subgroups is normal, we have $K \triangleleft G$. So, $gKg^{-1} = K$ for all $g \in G$, i.e., K has only one conjugate subgroup, itself.

Now, if (G_i) is weakly regular, then there is a finite index subgroup $N < G$ and a chain (N_i) so that $(G_i)_{i \geq i_0} \sim (N_i)_{i \geq i_0}$ and $N_i \triangleleft N$ for all i. Again, $K = \cap_i G_i = \cap_i N_i$, so $K \triangleleft N$. So, K has only one conjugate inside of N. To get the rest of the conjugates of K inside G, we note that G/N is finite, and claim that only one conjugate comes from each coset G/N.

Let $g, h \in G$ with $gN = hN$. Then there is some $\gamma \in N$ so that $g = h\gamma$. Then we have $gKg^{-1} = h\gamma K \gamma^{-1} h^{-1}$. Since $\gamma \in N$ and $K \triangleleft N$, $\gamma K \gamma^{-1} = K$. So we have $gKg^{-1} = h\gamma K \gamma^{-1} h^{-1} = hKh^{-1}$. Thus, we get at most one distinct conjugate of K from each coset in G/N. Since G/N is finite, K has a finite number of conjugate subgroups in G. $\square$

Example 5.14 gives an example of a group chain (G_i) that is irregular, but where $K = \cap_i G_i = \{e\}$, which has only one conjugate subgroup (itself). So, this shows that the converse of Theorem 8.6 does not hold, and so we cannot get and if and only if statement for this theorem.

CHAPTER 9

Almost Normal Systems

In contrast to the *weakly regular* condition in Definition 5.5, the *almost normal* condition given in Definition 5.15 is more effectively computable in examples. In this chapter, we discuss some properties of almost normal systems. We then show by calculation that this condition is not satisfied for the Schori and Rogers and Tollefson group chains.

Let (G_i) be a group chain in G, and let $N_G(G_i) = \{g \in G \mid gG_ig^{-1} = G_i\}$ be the normalizer of G_i in G. Note that we have $G_i < G_{i-1}$, but in general we may not have any subgroup relationship between $N_G(G_i)$ and $N_G(G_{i-1})$. However, we can form a nested chain of subgroups by taking successive intersections of the normalizers. Let $H_i = \cap_{0 \leq k \leq i} N_G(G_k)$, and then we have $H_i < H_{i-1}$, so (H_i) is a group chain. Since we have $G_i < G_{i-1} < N_G(G_{i-1})$, we always have $G_i < H_i$, and since $H_i < N_G(G_i)$, we have $G_i \triangleleft H_i$.

In many of the examples constructed in Chapter 2, we do in fact have a nested chain of normalizers, with the property that $N_G(G_i) < N_G(G_{i-1})$. Notice that in this case, we get $H_i = \cap_{0 \leq k \leq i} N_G(G_k) = N_G(G_i)$.

Since we have $G_i \triangleleft H_i$, H_i/G_i is a group. In the same manner as in Lemma 7.2, one shows that there are well defined coset inclusion maps which are group homomorphisms $\theta^i_{i-1} : H_i/G_i \to H_{i-1}/G_{i-1}$, given by $\theta^i_{i-1}(h_iG_i) = h_iG_{i-1}$. So, we can form the inverse limit $N_\infty = \varprojlim \{\theta^i_{i-1} : H_i/G_i \to H_{i-1}/G_{i-1}\}$, which is a profinite group.

LEMMA 9.1. *There is an equicontinuous right action of N_∞ on G_∞ that commutes with the left G action.*

PROOF. We define the right action of N_∞ on G_∞ by

$$G_\infty \times N_\infty \to G_\infty$$

$$(g_iG_i) \cdot (h_iG_i) = (g_ih_iG_i)$$

We first check that this action is well-defined, that is, that $(g_i h_i G_i)$ actually is an element of G_∞, i.e., that the bonding maps hold. We assume that $(g_i G_i) \in G_\infty$, i.e. $f_{i-1}^i(g_{i-1}G_{i-1}) = g_i G_i$ for all i, i.e. $g_i G_i \subset g_{i-1} G_{i-1}$, and similarly that $(h_i G_i) \in N_\infty$ for all i, i.e., $t_{i-1}^i(h_{i-1}G_{i-1}) = h_i G_i$ for all i, i.e. $h_i G_i \subset h_{i-1} G_{i-1}$ for all i. Then, we want to show that $g_i h_i G_i \subset g_{i-1} h_{i-1} G_{i-1}$. But, this is clear because g_i, g_{i-1} must be in the same coset of G_i, and $h_i G_i \subset h_{i-1} G_{i-1}$, so then $g_i h_i G_i \subset g_{i-1} h_{i-1} G_{i-1}$.

We see this action is equicontinuous by the same argument we have used previously. By the definition of the metric d in G_∞ (defined in the proof of Proposition 2.3), distances in G_∞ can only be of the form $\frac{1}{2^n}$ for some n. Suppose $(g_i G_i), (\gamma_i G_i) \in G_\infty$, and $(h_i H_i) \in H_\infty$. Suppose $d((g_i G_i), (\gamma_i G_i)) = \frac{1}{2^n} < \delta$. This means that $(g_i G_i), (\gamma_i G_i)$ first differ at level n, so $g_i G_i = G_i \gamma_i$ for $i \leq n$. Then, $g_i h_i G_i = \gamma_i h_i G_i$ for $i \leq n$, which implies that $d(g_i h_i G_i), (\gamma_i h_i G_i)) = \frac{1}{2^n}$. Thus, taking $\varepsilon = \delta$ shows equicontinuity. $\square$

COROLLARY 9.2. *If $N_\infty = G_\infty$, then (G_i) is regular.*

In many examples, the action of N_∞ on G_∞ is not transitive, even if (G_i) is a regular group chain. In fact, it may turn out that that N_∞ is the trivial group.

We next compute N_∞ for the examples presented in Chapter 2.

EXAMPLE 9.3. Recall the Schori solenoid, constructed in Example 2.15. Let (G_i) be the group chain associated to the Schori solenoid.

We claim that $N_G(G_i) = G_i$.

First, we will show that $N_{G_{i-1}}(G_i) = G_i$. Indeed, this follows from the fact that G_i has index 3 in G_{i-1}. By Lagrange's theorem, the index $|N_{G_{i-1}}(G_i) : G_i|$ of G_i in $N_{G_{i-1}}(G_i)$ must divide $|G_{i-1} : G_i| = 3$, so $|N_{G_{i-1}}(G_i) : G_i|$ is 1, or 3. If $|N_{G_{i-1}}(G_i) : G_i| = 1$, then $N_{G_{i-1}}(G_i) = G_i$. But, we claim that G_i is not normal in G_{i-1}. Indeed, there is an element $s = a^{2^{i-1}} \in G_{i-1}$ and $m = a^{2^{i-1}} b^{2^{i-2}} a^{-2^{i-1}} \in G_i$. Conjugating $m \in G_i$ by $s \in G_{i-1}$, we get

$$s^{-1} m s = a^{-2^{i-1}} a^{2^{i-1}} b^{2^{i-2}} a^{-2^{i-1}} a^{2^{i-1}} = b^{2^{i-2}} \notin G_i.$$

These elements can be seen in the algebraic construction of G_i, but it is instructive to see the loops pictured in the Schreier diagram in Figure 7.

So, G_i is not normal in G_{i-1} so $|N_{G_{i-1}}(G_i) : G_i| \neq 1$. Thus, the only possibility left is that $|N_{G_{i-1}}(G_i) : G_i| = 3$, which means $N_{G_{i-1}}(G_i) = G_i$.
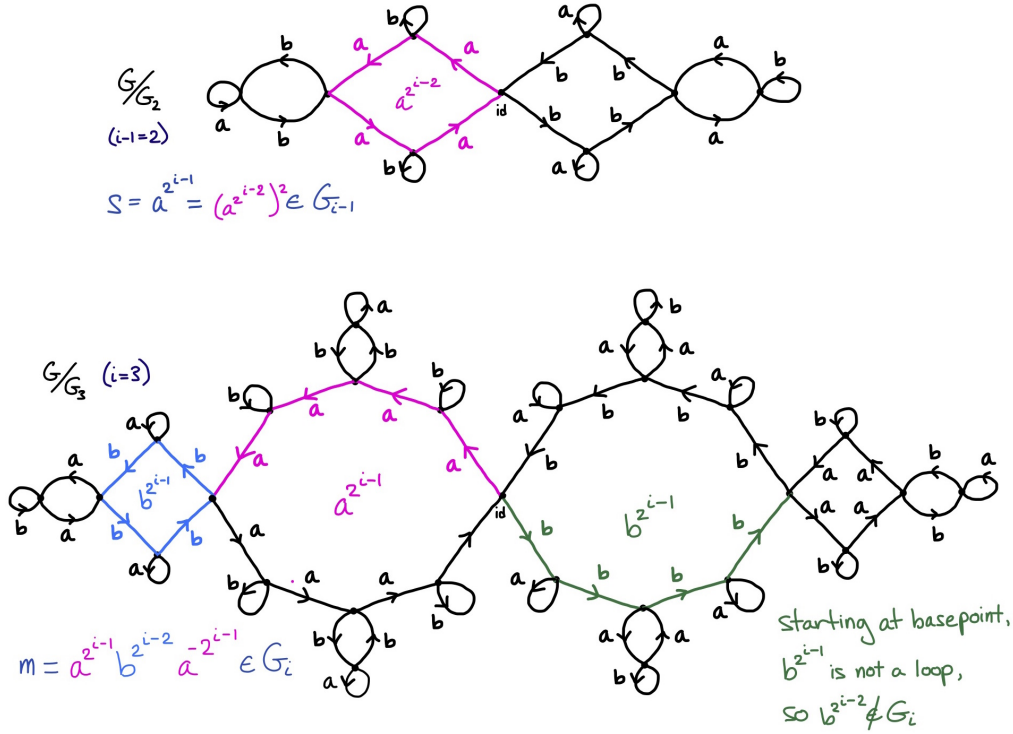
FIGURE 7. Loops in the Schreier diagrams for the Schori solenoid

Since G_{i-1} is the smallest subgroup of G that properly contains G_i, this implies that $N_G(G_i) = G_i$.

Then, $H_i = \cap_{0 \leq k \leq i} N_G(G_k) = G_i$, so $H_i/G_i = G_i/G_i = \{e\}$, so H_∞ is trivial.

EXAMPLE 9.4. Recall the Rogers and Tollefson solenoid, constructed in 2.14, and let (G_i) be the group chain associated to the Rogers and Tollefson solenoid, with

(7) $$G_i = \langle a^{2^i}, b \mid bab^{-1} = a^{-1}, ba^2b^{-1} = a^{-2}, ba^4b^{-1} = a^{-4}, ..., ba^{2^i}b^{-1} = a^{-2^i} \rangle.$$

We claim first that $N_G(G_i) = G_{i-1}$. Since $|G_i : G_{i-1}| = 2$ and all index 2 subgroups are normal, we have that $G_i \lhd G_{i-1}$. To show that that G_i is not normal in G_{i-2}, we conjugate $b \in G_i$ by $a^{-2^{(i-2)}} \in G_{i-2}$. From the relations for G_{i-2}, we have

$$ba^{2^{(i-2)}}b^{-1} = a^{-2^{(i-2)}} \implies ba^{2^{(i-2)}} = a^{-2^{(i-2)}}b$$

so

$$a^{-2^{(i-2)}}ba^{2^{(i-2)}} = ba^{2(2^{i-2})} = ba^{2^{i-1}}.$$

We see that $ba^{2^{i-1}}$ is not in G_i, and thus G_i is not normal in G_{i-2}. By Lagrange's theorem, there cannot be any subgroups in between G_{i-2} and G_{i-1}, so $N_G(G_i) = G_{i-1}$.

So, $H_i = \cap_{0 \leq k \leq i} N_G(G_k) = G_{i-1}$, so $H_i/G_i = G_{i-1}/G_i$.

We consider the two cosets of G_{i-1}/G_i. We have

$$G_i = \langle a^{2^i}, b \mid bab^{-1} = a^{-1}, ba^2b^{-1} = a^{-2}, ba^4b^{-1} = a^{-4}, ..., ba^{2^i}b^{-1} = a^{-2^i} \rangle$$

$$G_{i-1} = \langle a^{2^{i-1}}, b \mid bab^{-1} = a^{-1}, ba^2b^{-1} = a^{-2}, ba^4b^{-1} = a^{-4}, ..., ba^{2^{i-1}}b^{-1} = a^{-2^{i-1}} \rangle$$

We have $|G_{i-1}/G_i| = 2$. One of the cosets must be the coset of the identity, eG_i. Since $b \in G_i$ and $a^{2^{i-1}} \notin G_i$, we see that the other coset can be represented by $a^{2^{i-1}}$. So, we can write $G_{i-1}/G_i = \{eG_i, a^{2^{i-1}}G_i\}$. We now consider the inclusion map $\theta_{i-1}^i : H_i/G_i \to H_{i-1}/G_{i-1}$, which in this example is $\theta_{i-1}^i : G_{i-1}/G_i \to G_{i-2}/G_{i-1}$. Since it is an inclusion map, we have $\theta_{i-1}^i(eG_i) = eG_{i-1}$. We also have $\theta_{i-1}^i(a^{2^{i-1}}G_i) = a^{2^{i-1}}G_{i-1}$. But, $a^{2^{i-1}} \in G_{i-1}$, so $a^{2^{i-1}}G_{i-1} = eG_{i-1}$. So, both cosets in G_{i-1}/G_i are included into the identity coset in G_{i-2}/G_{i-1}. This shows that the map θ_{i-1}^i is trivial, so the inverse limit $N_\infty = \varprojlim \{\theta_{i-1}^i : H_i/G_i \to H_{i-1}/G_{i-1}\}$ is the trivial group.

EXAMPLE 9.5. Let G be the Heisenberg group, and recall the group chain given in Example 2.9:

$$G_i = \begin{pmatrix} p^i & 0 \\ 0 & p^i \end{pmatrix} \mathbb{Z}^2 \times p\mathbb{Z}.$$

We showed in Example 5.2 that $G_i \triangleleft G$, so $N_G(G_i) = G$ for all i. So,

$$N_\infty = \varprojlim \{\theta_{i-1}^i : H_i/G_i \to H_{i-1}/G_{i-1}\} = \varprojlim \{\theta_{i-1}^i : G/G_i \to G/G_{i-1}\} = G_\infty.$$

So, in this example, N_∞ is nontrivial and is in fact equal to G_∞

EXAMPLE 9.6. Let G be the Heisenberg group, and recall the group chain given in Example 2.10:

$$G_n = \begin{pmatrix} qp^n & pq^n \\ p^{n+1} & q^{n+1} \end{pmatrix} \mathbb{Z}^2 \times p\mathbb{Z}.$$

In Example 5.10, we showed that $N_G(G_i) = p\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ for all i.

Then $H_i = p\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ is constant for all i, $N_\infty = \varprojlim \{\theta_{i-1}^i : H_i/G_i \to H_{i-1}/G_{i-1}\}$ is nontrivial.

EXAMPLE 9.7. Let G be the Heisenberg group, and recall the group chain given in Example 2.11:

$$G_n = \begin{pmatrix} p^n & 0 \\ 0 & q^n \end{pmatrix} \mathbb{Z}^2 \times p^n\mathbb{Z} = p^n\mathbb{Z} \times q^n\mathbb{Z} \times p^n\mathbb{Z}.$$

We showed in Example 5.17 that $N_G(G_n) = p^n\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

So, $H_n/G_n = (p^n\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})/(p^n\mathbb{Z} \times q^n\mathbb{Z} \times p^n\mathbb{Z}) = \{0\} \times \mathbb{Z}/q^n\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}$.

$\{0\} \times \mathbb{Z}/q^n\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}$ is a finite group with nontrivial inclusion into $\{0\} \times \mathbb{Z}/q^{n-1}\mathbb{Z} \times \mathbb{Z}/p^{n-1}\mathbb{Z}$, so N_∞ is nontrivial.

EXAMPLE 9.8. Recall the construction given in Example 2.16: Let Γ be a finitely generated group, and let (Γ_i) be a normal group chain in Γ. Let H be a finite simple group and let K be a nontrivial subgroup of H. Let $G = H \times \Gamma$, and let $G_i = K \times \Gamma_i$.

Then, it is easy to check that $N_G(G_i) = N_G(K \times \Gamma_i) = N_H(K) \times N_G(\Gamma_i) = N_H(K) \times \Gamma$, which does not depend on i. So, $H_i = N_H(K) \times \Gamma$, and

$$
\begin{aligned}
N_\infty &= \varprojlim \{H_i/G_i \to H_{i-1}/G_{i-1}\} \\
&= \varprojlim \{(N_H(K) \times \Gamma)/(K \times \Gamma_i) \to (N_H(K) \times \Gamma)/(K \times \Gamma_{i-1})\} \\
&= \varprojlim \{N_H(K)/K \times \Gamma/\Gamma_i \to N_H(K)/K \times \Gamma/\Gamma_{i-1}\} \\
&= \varprojlim \{N_H(K)/K \to N_H(K)/K\} \times \Gamma_\infty
\end{aligned}
$$

which is nontrivial.

EXAMPLE 9.9. Recall Example 2.17, also discussed in Examples 5.20 and 8.4.

Let $\Gamma = \mathbb{Z}^2 = \{(a, b) \mid a, b \in \mathbb{Z}\}$, and let p, q be distinct primes. Let

$$\Gamma_i = p^i\mathbb{Z} \times q^i\mathbb{Z} = \{(ap^i, bq^i) \mid a, b \in \mathbb{Z}\}.$$

Let $H = \mathbb{Z}/2\mathbb{Z} = \{1, t \mid t^2 = 1\}$.

Let $\theta : H \to Aut(\Gamma)$ be the homomorphism defined as follows:

$$\theta : H \to Aut(\Gamma)$$

$$1 \to \theta_1 : (a, b) \to (a, b) \text{ (i.e., } \theta_1 \text{ is the identity map)}$$

$$t \to \theta_t : (a, b) \to (b, a) \text{ (i.e., } \theta_t \text{ is the transpose map)}$$

Let $G = \Gamma \rtimes_\theta H \cong \mathbb{Z}^2 \rtimes_\theta \mathbb{Z}/2\mathbb{Z}$.

An element of $G_i = \Gamma_i \times \{1\}$ is of the form $(\alpha p^i, \beta q^i, 1)$, and an element $g \in G$ is of the form $g = ((a, b), 1)$ or $g = ((a, b), t)$.

We saw in Example 8.4 that

$$((a,b),t) * ((\alpha p^i, \beta q^i), 1) * ((-b,-a),t) = ((\beta q^i, \alpha p^i), 1) \notin G_i$$

However, $((a,b),1)^{-1} = (\theta_1(-a,-b),1)((-a,-b),1) =$ and so

$$((a,b),1) * ((\alpha p^i, \beta q^i), 1) * ((-a,-b),1) = ((\alpha p^i, \beta q^i), 1) \in G_i$$

So, $N_G(G_i) = \Gamma \times \{1\}$, which does not depend on i.

So, $H_i = N_G(G_i) = \Gamma \times \{1\}$, and

$$N_\infty = \varprojlim \{H_i/G_i \to H_{i-1}/G_{i-1}\}$$

$$= \varprojlim \{(\Gamma \times \{1\})/(\Gamma_i \times \{1\}) \to (\Gamma \times \{1\})/(\Gamma_{i-1} \times \{1\})\}$$

$$= \Gamma_\infty \times \{1\}$$

which is nontrivial.

Notice that these calculations also show that the last two examples are almost normal as well as weakly regular, since they have fixed chains of normalizers.

CHAPTER 10

Conclusion

We have shown that we can model any minimal equicontinuous Cantor dynamical system (X, G, ϕ) by an inverse limit of group chains G_∞. Conversely, any group chain can be used to construct a minimal equicontinuous Cantor dynamical system. We have shown that we can classify such systems by three types of properties:

- Regular, weakly regular, or irregular, as in Definition 1.5;
- According to the number of equivalence classes of group chains in $\mathfrak{G}_\phi$;
- According to the number of orbits of the automorphism group $Aut(X, G, \phi)$.

We have then shown in Theorem 6.5 and Corollory 6.6 that these three classifications are equivalent in the following way:

- A system is regular if and only if $\mathfrak{G}_\phi$ has only one equivalence class if and only if $Aut(X, G, \phi)$ has only one orbit.
- A system is weakly regular if and only if $\mathfrak{G}_\phi$ has a finite number of equivalence classes if and only if $Aut(X, G, \phi)$ has a finite number of orbits.
- A system is irregular if and only if $\mathfrak{G}_\phi$ has an infinite number of equivalence classes if and only if $Aut(X, G, \phi)$ has an infinite number of orbits.

We have defined a new invariant of such systems, the discriminant group, and shown that the cardinality of the discriminant group is related to the degree of regularity of the system:

- A system is regular if and only if its associated discriminant group is the trivial group.
- If a system has finite discriminant group, then the system is weakly regular.
- A weakly regular system may have finite or infinite discriminant group.

We have also given examples of systems of each type.

- We gave a new proof using group chains that Example 2.14, the Rogers and Tollefson Klein bottle solenoid, and Example 2.15, the Schori solenoid, are irregular.
- In the Heisenberg group,

- – Example 2.9 is regular.

- – Example 2.10 is weakly regular but not regular.

- – Example 2.11 is irregular.

- Example 2.16, given by a direct product construction, is weakly regular with finite non-trivial discriminant group.

- Example 2.17, given by a semi-direct product construction, is weakly regular with infinite discriminant group.

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Vita

NAME: Jessica Celeste Dyer

EDUCATION: B.S., Mathematics, University of California, Santa Barbara, 2007

M.A.St, Mathematics, University of Cambridge, UK, 2009

M.S., Mathematics, University of Illinios, Chicago, 2011

Ph.D., Mathematics, University of Illinios, Chicago, 2015

HONORS: TA Teaching Award, University of Illinois, Chicago,
Department of Mathematics, Statistics, and Computer Science, 2012

Member Phi Beta Kappa Honors Society, 2006

National Merit Scholar, 2003-2007

AP National Scholar, 2003

AP Scholar With Distinction, 2002

TEACHING: Full Time Lecturer of Mathematics,
Tufts University, 2015-2016

Lecturer of Mathematics,
University of Illinios, Chicago, 2013-2014

Emerging Scholars Program Mathematics Instructor,
University of Illinios, Chicago, 2012-2013

Summer Enrichment Program Mathematics Instructor,
University of Illinios, Chicago, 2010-2011

Teaching Assistant, Mathematics,
University of Illinios, Chicago, 2009-2012

Campus Learning Assistive Services Mathematics Instructor,
University of California, Santa Barbara, 2005-2008

Upward Bound Mathematics Teacher,
University of California, Santa Barbara, 2007

SERVICE: Co-Organizer, Association for Women in Math Student Chapter
                        University of Illinios, Chicago, MSCS Department 2012-2015

                        Chair of Dynamical Systems Session,
                        Midwest Women in Math Symposium,
                        University of Notre Dame, 2014

                        Co-Organizer, Graduate Student Colloquium,
                        University of Illinois, MSCS Department, Chicago, 2013

PUBLICATIONS: J. Dyer, S. Hurder, and O. Lukina,
                        "The Discriminant Invariant of Cantor Group Actions,"
                        Preprint, arXiv:1509.06227, 2015

POSTERS & TALKS   Midwest Women in Math Symposium,
                        Dominican University, River Forrest, Illinois
                        Contributed Talk: "Almost Normal Group Chains," 2015

                        Joint Mathematics Meetings, San Antonio, Texas
                        Poster Presentation: "Bratteli Diagrams for Weak Solenoids", 2015

                        29th Summer Conference on Topology and its Applications,
                        College of Staten Island, New York
                        Invited talk: "Bratteli Diagrams for Weak Solenoids", 2014

                        Midwest Women in Math Symposium,
                        University of Illinios, Chicago
                        "Substitutions, Contributed Talk: Bratelli Diagrams, and Dynamics," 2013

PROFESSIONAL
MEMBERSHIP: American Mathematical Society

                        Association for Women in Mathematics