Logarithmic Potentials and Quasiconformal Flows on the Heisenberg Group

BY

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MMath, University of Warwick, 2010

THESIS

Submitted as partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Chicago, 2016

Chicago, Illinois

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For Roxana and Phoebe.

ACKNOWLEDGMENTS

I would like to thank my advisor, Jeremy Tyson, for his readiness to take on a student from another campus, for introducing me to some wonderful mathematics, and for his advice and support over the years.

I thank David Dumas, who acted as my local advisor, and who was therefore burdened with some of the boring parts, and none of the interesting parts of being an advisor. I also learned a lot from David working together as organizers of the Undergraduate Mathematics Symposium.

I thank my other defense committee members, Fabrice Baudoin, Marianna Csörnyei, and Kevin Whyte, for taking the time to perform this duty.

I would also like to thank Mario Bonk, and Leonid Kovalev for useful conversations relevant to this thesis.

My wife Roxana Hadad is an inspiration, and my daughter Phoebe Austin, born while completing this thesis, is a source of great motivation. I thank them both.

I thank my parents, Wendy Jordan, and Allan Nutt, who gave me the curiosity, and determination, necessary to complete this project. I also thank my wife's parents Marcia Hadad and Jorge Hadad who took care of Phoebe while important parts of this thesis were being written.

Finally, I acknowledge funding from the National Science Foundation, grant NSF DMS-1201875 'Geometric mapping theory in sub-Riemannian and metric spaces'.

A.D.A.

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SUMMARY

The introduction contains some history and motivation for the problem, as well as useful information about the Heisenberg group, and the necessary notation.

The second section introduces all the needed facts about quasiconformal mappings on the Heisenberg group, most can be found in the literature, however, we prove some elementary results that we could not find.

In the third section, we begin by extending the flow method of generating quasiconformal mappings on the Heisenberg group, first developed by Korányi and Reimann. We then give a result linking the Jacobian of a flow mapping so generated with the horizontal divergence of the vector field.

Section four is technical, containing the multi-layered construction of a potential that will give rise to a vector field with horizontal divergence that approximates an admissible quasi-logarithmic potential in a suitable way. Here, and in the remainder of this summary, we use 'admissible' as shorthand for 'satisfying the requirements of the relevant result'.

The constructions of section four, along with the results of section three are used in section five to produce the quasiconformal mapping with Jacobian (almost everywhere) comparable to (the exponential of twice) a given admissible quasi-logarithmic potential.

We conclude in section six by showing how the comparability results of section five give rise to bi-Lipschitz equivalence results for certain metric spaces conformally equivalent to the sub-Riemannian Heisenberg group.

1 Introduction

1.1 Statement of Main Result

The quasiconformal Jacobian problem on $\mathbb{R}^n$ asks, given $\omega \in L^1_{\text{loc}}(\mathbb{R}^n)$, $\omega \geq 0$, when does there exist $C \geq 1$ and quasiconformal mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, such that

$$\frac{1}{C}\omega \leq \det(Df) \leq C\omega \tag{1.1}$$

almost everywhere? This problem, still open, has generated a lot of wonderful mathematics.

In [4] important progress was made. The authors construct an intricate machine for the production of quasiconformal mappings of $\mathbb{R}^n$, using it to prove

Theorem 1.1 (Bonk, Heinonen, Saksman)**.** *Given $n \geq 2$, and $K \geq 1$, there exist $\epsilon > 0$ and $C, K' \geq 1$ such that, if u is a quasi-logarithmic potential on $\mathbb{R}^n$, $u = \Lambda_\mu \circ g$ almost everywhere, with $\|\mu\| < \epsilon$, and g a K-quasiconformal mapping of $\mathbb{R}^n$, then there is K'-quasiconformal $f : \mathbb{R}^n \to \mathbb{R}^n$ with $\det(Df)$ and $\omega = e^{nu}$ comparable almost everywhere, as in* (1.1)*, with constant C.*

If $M = (X, d, \nu)$ is a metric measure space, a quasiconformal mapping of M is a homeomorphism, $f : M \to M$, such that

$$H_f(p) = \limsup_{r\to 0} \frac{\sup_{d(p,q)\leq r} d(f(p), f(q))}{\inf_{d(p,q)\geq r} d(f(p), f(q))} \tag{1.2}$$

is bounded independently of p. If H_f is not only bounded, but essentially bounded by $K \geq 1$, then we say f is a K-quasiconformal mapping.

For f a quasiconformal mapping, we define the Jacobian of f as

$$J_f(p) = \limsup_{r\to 0} \frac{\nu(fB(p, r))}{\nu(B(p, r))}. \tag{1.3}$$

Let p_0 be a distinguished point of M (0 in the case of $M = \mathbb{R}^n$). If μ is a finite, signed Radon measure on M with

$$\int \log^+ d(p_0, q)\, \mathrm{d}|\mu|(q) < \infty, \tag{1.4}$$

we call it an admissible measure. If $u : M \to [-\infty, \infty]$ is equal almost everywhere to

$$\Lambda_\mu(p) := -\int \log d(p, q)\, \mathrm{d}\mu(q), \tag{1.5}$$

for some admissible measure μ, then we call u a logarithmic potential on M.

If u is equal almost everywhere to $\Lambda_\mu \circ g$, for some admissible measure μ, and a quasiconformal mapping $g : M \to M$, then we say u is a quasi-logarithmic potential on M.

Consequently, the quasiconformal Jacobian problem can be posed for any metric measure space, and in particular, it is sensible to ask whether something like Theorem 1.1 holds.

In this paper, we begin an investigation of the quasiconformal Jacobian problem on the first Heisenberg group. This is the metric measure space $\mathbb{H} = (\mathbb{R}^3, d, m)$. The metric d is given by

$$d(p, q) = \|q^{-1} \star p\|, \tag{1.6}$$

where

$$(x_1, y_1, t_1) \star (x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + 2(x_2 y_1 - x_1 y_2)), \tag{1.7}$$

$$\|(x, y, t)\| = \left((x^2 + y^2)^2 + t^2\right)^{\frac{1}{4}}, \tag{1.8}$$

and

$$(x, y, t)^{-1} = (-x, -y, -t). \tag{1.9}$$

(From now on, for $p, q \in \mathbb{H}$, we will write $pq = p \star q$.) The measure m, is the Lebesgue measure on $\mathbb{R}^3$, and for $E \subset \mathbb{H}$, we will write $|E| = m(E)$.

With regard to logarithmic potentials on $\mathbb{H}$, the distinguished point is 0, and so $d(0,q) = \|q\|$.

We include a lengthier discussion of the Heisenberg group later in this introduction.

We are able to prove the exact analog of Theorem 1.1, our main result is

Theorem 1.2. *Given $K \geq 1$, there exist $\epsilon > 0$, and $C, K' \geq 1$, such that, if u is a quasi-logarithmic potential on $\mathbb{H}$, $u = \Lambda_\mu \circ g$ almost everywhere, with $\|\mu\| < \epsilon$, and g a K-quasiconformal mapping of $\mathbb{H}$, then there is a K'-quasiconformal mapping $f : \mathbb{H} \to \mathbb{H}$ such that*

$$\frac{1}{C} e^{2u} \leq J_f \leq C e^{2u}$$

almost everywhere.

The beautiful paper [4] containing Theorem 1.1 was the direct inspiration for this work, and we follow its overall scheme, using the flow method of constructing quasiconformal mappings. We extend the work of Korányi and Reimann ([19], [20]), who first developed the method in the Heisenberg setting and so established the principal means of constructing quasiconformal mappings of $\mathbb{H}$. With considerable current interest in $\mathbb{H}$ as a testing ground for the development of analysis in metric spaces, of which quasiconformal analysis has been one of the success stories, our relevant results have independent interest. Rather than state them here, we direct the interested reader to Propositions 3.3 and 3.10. The central development of the former is to remove the compact support assumption of Theorem H of [20], including instead some natural growth conditions on the vector field.

Before moving on, we note that Theorem 1.1 led to some very interesting results in conformal geometry. We hope that our Theorem 1.2 might have similar applications to CR geometry. We encourage the reader to consult section 1.5 of this introduction for a brief discussion of this fascinating topic that includes a geometric interpretation of our main result.

1.2 The Quasiconformal Jacobian Problem

The inception of the quasiconformal Jacobian problem, on $\mathbb{R}^n$ and in general, is the paper [13] of David, and Semmes, in which they write

'For instance, the Jacobian of a quasiconformal homeomorphism on $\mathbb{R}^n$ is always strongly A_∞ by an argument of Gehring. We do not know if any reasonable converse to this statement holds.'

Call $\omega \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $\omega \geq 0$ a weight. For $x, y \in \mathbb{R}^n$, let $B_{x,y}$ denote the smallest ball containing both x and y. A weight ω is said to be strongly A_∞, or a strong-A_∞ weight, if, for all measurable $E \subset \mathbb{R}^n$, $E \mapsto \int_E \omega(x)\,\mathrm{d}x$ defines a doubling measure, and

$$\tilde{d}_\omega(x,y) = \left(\int_{B_{x,y}} \omega(x)\,\mathrm{d}x \right)^{\frac{1}{n}} \tag{1.10}$$

is comparable to a metric.

If ω is a strong-A_∞ weight, we will write d_ω for some choice of metric comparable to $\tilde{d}_\omega$, and call $(\mathbb{R}^n, d_\omega)$ a David-Semmes deformation (of $\mathbb{R}^n$).

If a weight ω is comparable to a quasiconformal Jacobian, then $\mathbb{R}^n$ and $(\mathbb{R}^n, d_\omega)$ are bi-Lipschitz equivalent. The argument is identical to the one we give for $\mathbb{H}$ in Section 6.

The first results showed that any converse, if reasonable, was not going to be straightforward. In [32] Semmes showed that there is a strong-A_∞ weight on $\mathbb{R}^3$ which cannot be comparable to a quasiconformal Jacobian. Indeed, the weight was not only strong-A_∞ but also continuous. It was then shown by Laakso in [24] that there is a strong-A_∞ weight on $\mathbb{R}^2$ which cannot be comparable to a quasiconformal Jacobian. The two dimensional situation was further clarified by Bishop in [2] who showed that there is an A_1 weight on $\mathbb{R}^2$ that is not comparable to any quasiconformal Jacobian. The A_1 condition is that of Muckenhoupt. Every A_1 weight is strongly A_∞, but this is not so for every A_p weight with $p > 1$.

Nevertheless, results followed that were a testament to the flexibility of quasiconformal mappings. In [23] Kovalev and Maldonado focus not on a weight, but a set, and the question is whether there exists a quasiconformal mapping with Jacobian singular on that set (either zero or infinite everywhere on that set in a precise limiting sense). They show that this is true of any set $E \subset \mathbb{R}^n$ so long as the Hausdorff dimension of E is less than 1. Then in [22], Kovalev, Maldonado, and Wu, prove that each weight in a certain class of Riesz potentials is comparable to a quasiconformal Jacobian.

While the relationship between bi-Lipschitz equivalence and the quasiconformal Jacobian problem had long been known, in [3] Bonk, Heinonen, and Saksman explain that, in $\mathbb{R}^2$, the two things are more or less equivalent. This follows from deep work of Bonk and Kleiner [5] on quasisymmetric parameterizations of $\mathbb{R}^2$. They go on to show that if u is a locally integrable function on $\mathbb{R}^2$ with distributional gradient in $L^2(\mathbb{R}^2)$, then e^{2u} is comparable to a quasiconformal Jacobian. They deduce this from a theorem of Fu ([15]) (strengthened by Bonk and Lang in [6]), which says that if the integral curvature of a complete Riemannian 2-manifold homeomorphic to $\mathbb{R}^2$ is small, then it is bi-Lipschitz equivalent to $\mathbb{R}^2$.

This brings us to [4], already discussed above. Here the authors go in the other direction to that of [3], directly showing that a class of weights are comparable to quasiconformal Jacobians (weights of the form $\omega = e^{nu}$, with u a quasi-logarithmic potential on $\mathbb{R}^n$), and use this to prove a result on bi-Lipschitz equivalence. This last was the fore-runner of Theorem 1.3 below, which can be viewed as a four dimensional analog of Fu's theorem, in that bi-Lipschitz equivalence results from small integral Q-curvature (in [4] the authors actually give a result for all even dimensions).

To our knowledge, this thesis is the first look at the quasiconformal Jacobian problem outside the Euclidean setting. Some of what we do depends, though that dependence is hidden, on a capacity estimate for the Heisenberg group. Such a capacity estimate is the key property of a metric measure space in order for it to support a fruitful quasiconformal analysis. In the landmark work [17] of Heinonen and Koskela, the authors axiomatize this capacity lower bound, it being the defining property of the Loewner spaces they introduce. They go on to show that on such spaces, many aspects of quasiconformal mappings that make them both useful and tractable in the Euclidean setting continue to hold. They show, for example, that the defining infinitesimal property in (1.2) is equivalent to the global quasisymmetric property, and that (for complete, geodesic, Ahlfors-regular, Loewner spaces) the Jacobian satisfies a reverse Hölder inequality (though as stated this relies on the later work of Keith and Zhong in [18]). We note in passing that the Loewner condition was shown, in the same paper, to be equivalent (under appropriate assumptions) to the existence of a Poincaré inequality. It was, to some extent, this latter criterion that came to be the focus, the so called PI-spaces.

Given a PI-space, the Heinonen-Koskela theory just discussed guarantees that it supports a class of quasiconformal mappings for which much of the Euclidean theory transfers across. The question would remain, however, as to how rich a family of mappings is being described. One set of PI-spaces amenable to analysis are the Carnot groups. Pansu proved in [27] that in Carnot groups corresponding to the boundaries of quaternionic and Cayley hyperbolic spaces, all quasiconformal mappings are in fact 1-quasiconformal, and so no results similar to those we prove here could be expected to be true.

One class of Carnot groups that do support a comparative wealth of quasiconformal mappings are the Heisenberg groups, $\mathbb{H}^n$ (these correspond to the boundaries of the complex hyperbolic spaces, as the Euclidean spaces correspond to the boundaries of the real hyperbolic spaces). This was conclusively demonstrated by Korányi and Reimann in [19] and [20] where they identify conditions on a vector field so that the corresponding flow is a one parameter family of quasiconformal mappings of $\mathbb{H}^n$. Similar flows are the subject of this thesis, though we currently restrict our attention to the first Heisenberg group only.

1.3 The Heisenberg Group

$\mathbb{H}$ is a Lie group, the group product given by (1.7). It plays an important role in harmonic analysis and sub-Riemannian geometry.

Our first choice label for a point in $\mathbb{H}$ is p with coordinates (x, y, t). If several points are in play, any mention of x, y, or t always refers to the point labeled p.

A basis of left-invariant vector fields, and so a basis of the Lie algebra, $\mathfrak{h}$, is given by

$$X\big|_p = X_p = \partial_x + 2y\partial_t \qquad Y\big|_p = Y_p = \partial_y - 2x\partial_t \qquad T\big|_p = T_p = \partial_t. \tag{1.11}$$

Note that $[X, Y]\big|_p = -4T_p$ and so the vector fields X, Y satisfy Hörmander's condition. It follows that $\mathbb{H}$ is a Carnot group. Let $H\mathbb{H} \subset T\mathbb{H}$ be defined by $H_p\mathbb{H} = \mathrm{span}(X_p, Y_p)$. We call $H\mathbb{H}$ the horizontal layer of the tangent bundle. If $b > 0$, call a continuous mapping $\gamma : [0, b] \to \mathbb{H}$ a horizontal curve if $\gamma \in C^1((0, b))$ with $\gamma'(s) \in H_{\gamma(s)}\mathbb{H}$ for all $s \in (0, b)$. Define an inner product g_0 on each $H_p\mathbb{H}$ by

$$g_0(X_p, X_p) = 1 \qquad g_0(X_p, Y_p) = 0 \qquad g_0(Y_p, Y_p) = 1. \tag{1.12}$$

We will refer to g_0 as the canonical sub-Riemannian metric on $\mathbb{H}$. It gives rise to a Carnot-Carathéodory distance function

$$\rho(p, q) = \inf_{\gamma} \int_0^b \sqrt{g_0(\gamma'(s), \gamma'(s))}\, \mathrm{d}s \tag{1.13}$$

where the infimum is taken over all piecewise horizontal curves. Note that $g_0(\gamma'(s), \gamma'(s)) = \gamma_1'(s)^2 + \gamma_2'(s)^2$. If $V \in H_p\mathbb{H}$ for some p, we will sometimes write $|V|_H = \sqrt{g_0(V, V)}$.

We may also identify the Lie algebra with the tangent space at the origin: if V is a left-invariant vector field, we identify V with V_0. The previously given basis corresponds to the basis $X_0 = \partial_x$, $Y_0 = \partial_y$, and $T_0 = \partial_t$. The bracket is then defined as $[V_0, W_0] = [V, W]\big|_0$. It is usually this identification and basis that we have in mind when we involve the exponential mapping. It is easy to see that the unique one-parameter subgroup γ satisfying $\gamma(0) = 0$, $\gamma'(0) = W_0$ is given by $s \mapsto (sw_1, sw_2, sw_3)$ where the w_i are defined by $W_0 = w_1X_0 + w_2Y_0 + w_3T_0$. It follows that $\exp : \mathfrak{h} \to \mathbb{H}$ is given by $\exp(w_1X_0 + w_2Y_0 + w_3T_0) = (w_1, w_2, w_3)$. We call $\mathrm{span}(X_0, Y_0)$ the horizontal layer of the Lie algebra.

Often a left-invariant vector field V will be treated as a differential operator, and if $F : \mathbb{H} \to \mathbb{R}$, then VF is shorthand for $p \mapsto V_pF(p)$. This might indicate a classical or distributional derivative, it will be made clear in each instance.

In practice, we typically work with the metric d defined in (1.6), though ρ, and weighted versions of it, are the subject of Section 6. It is true, in any case, that $\rho \simeq d$. We define the length of a continuous curve, $\gamma : [0, b] \to \mathbb{H}$, with respect to the metric d as

$$l_d(\gamma) = \limsup_{m\to\infty} \sum_{i=1}^{m} d(\gamma(s_i), \gamma(s_{i-1})), \tag{1.14}$$

with $s_i = ib/m$. It is shown in [8] that if $\gamma \in C^1((0, b))$ then $l_d(\gamma)$ coincides with $\int_0^b |\gamma'|_H$ if γ is horizontal, and is infinite otherwise.

We will refer to $\|\cdot\|$ as in (1.8) as the Korányi gauge. It is an example of a homogeneous norm, so called because there is a family of dilations

$$\delta_r(p) = (rx, ry, r^2t),$$

for which $\|\delta_r(p)\| = r\|p\|$. We will be consistent in our use of δ for these dilations and nothing else. Writing $B(p, r)$ for the ball of center p and radius r with respect to the metric d, and L_p for left translation by the point p, it follows from $B(p, r) = L_p(\delta_r(B(0, 1)))$, and the standard change of variable formula, that $\mathbb{H}$ is Ahlfors 4-regular,

$$|B(p,r)| = Cr^4$$

for some constant $C > 0$. In that it is both self-similar, and has Hausdorff dimension greater than its topological dimension, $\mathbb{H}$ qualifies as a fractal.

We require two more well known facts about the Heisenberg group. The first is that, given a compact set $\Omega \subset \mathbb{H}$, there is a $C = C(\Omega) > 0$ such that

$$\frac{1}{C}|p-q| \leq d(p,q) \leq C|p-q|^{\frac{1}{2}} \tag{1.15}$$

where $|p-q|$ is the Euclidean distance between the points $p, q \in \mathbb{H}$ treated as points of $\mathbb{R}^3$.

The second is a polar coordinate integration formula, see [14],

$$\int_{\mathbb{H}} f(p)\,\mathrm{d}p = \int_{S(1)} \int_0^\infty f(\delta_r(q)) r^3\,\mathrm{d}r \mathrm{d}\nu(q), \tag{1.16}$$

with ν an appropriate measure on the unit sphere $S(1)$ with respect to d, valid for all $f \in L^1(\mathbb{H})$.

1.4 Outline

In Section 2 we take a brisk look at the required features of quasiconformal mappings. Most is well known. We develop some elementary results that, if known, are harder to find, but nothing that will surprise an expert.

Section 3 contains our first true innovations. There are two subsections. The first extends the flow method of Korányi and Reimann for generating quasiconformal mappings on the Heisenberg group. In [19] existence of the flow is assumed, and the vector field is stipulated to be in $C^2(\mathbb{H})$. In [20], existence of the flow is proved, and only minimal regularity is assumed, however, the vector fields are compactly supported. In Proposition 3.3, we prove existence of the flow, retain minimal regularity, but introduce some growth conditions on the vector field so that it may have unbounded support. These growth conditions should not be considered restrictive, they correspond to similar conditions imposed in the Euclidean case in [28], which in two dimensions are known to be necessary for quasiconformal flow. Proposition 3.3 puts quasiconformal flows on $\mathbb{H}$ on roughly the same footing as those on $\mathbb{R}^n$, $n \geq 3$. The second subsection identifies (Proposition 3.10) a means of linking the Jacobian of the flow mappings with the horizontal divergence of the vector field.

The constructions of Section 4 are made with a twofold purpose in mind, the vector fields should satisfy the requirements of Section 3 so that those results may be applied, and the horizontal divergence should approximate the quasi-logarithmic potential in a suitable way. When reading the details of the construction, it is useful to keep the following in mind. Suppose we have a quasi-logarithmic potential, equal (almost) everywhere to

$$-\int \log d(g(p), q)\, \mathrm{d}\mu(q),$$

with g to the identity, and μ twice the Dirac measure centered at the origin. The quasi-logarithmic potential has reduced to

$$-2 \log \|p\|. \tag{1.17}$$

Multiply this by t and call it

$$\phi(t) := -2t \log \|p\|. \tag{1.18}$$

It is this ϕ, used as a potential to generate a vector field as in Section 3, that Miner identified in [26] as having time-s flow mappings that are 'essentially' $f_s(p) = p\|p\|^{e^s-1}$. These flow mappings later appeared as the radial stretch mappings of Balogh, Fässler, and Platis in [1], where they are identified as being the correct analog (in terms of their extremal properties) of the Euclidean radial stretch mappings. Radial stretch mappings appear frequently in the Euclidean quasiconformal Jacobian problem as they are simple examples of quasiconformal mappings with explosive volume change at the origin (that is, the Jacobian is infinite there). A vector field generated (see Section 3) by ϕ as in (1.18) has horizontal divergence $-2\log\|p\| + \zeta(p)$, where ζ is a bounded function. Consequently, the horizontal divergence nicely approximates the logarithmic potential (1.17). Much of the work of Section 4 is dedicated to the more general case that g is not the identity. Other measures are taken care of rather easily.

With regard to our last statement, we should be careful so as not to give the wrong impression. We arrived at our prototypical ϕ above by considering the measure that is twice the Dirac measure centered at the origin, $\mathrm{d}\mu(q) = \delta_0(q)\,\mathrm{d}q$. It is a curious fact that, despite being useful in this way, this has $\|\mu\| = 2 \geq \epsilon$ where $\epsilon > 0$ is as in Theorem 1.2. We know this, not because we give an explicit value for ϵ (which we do not), but because $e^{2(-2\log\|p\|)} = \|p\|^{-4}$ which is not locally integrable at the origin, and so cannot be comparable to a quasiconformal Jacobian.

In Section 5 we use the constructions of Section 4, along with the results of Section 3, to construct quasiconformal mappings with prescribed Jacobian. To do so, we adapt the machine of [4], Section 6, finding our desired mapping in the limit of a sequence (f_m), with each f_m the composition of m (normalized) time-$1/m$ flow mappings. Listing the adaptations made would not serve this outline well, however, the reason for making them is illuminating. The main difficulty was that $\mathbb{H}$ has a somewhat less flexible family of conformal mappings as compared to $\mathbb{R}^n$. In $\mathbb{R}^n$ there are the translations, dilations and rotations (we ignore the inversions as they are deemed not useful in this situation). In $\mathbb{H}$ all three of these are present, however, the rotations are reduced to those about the group center. This meant that a normalization strategy present in [4] was not available, and we had to find other means of achieving the necessary compactness results for the sequences of quasiconformal mappings we construct.

The arguments we use in Section 6 have become standard in the Euclidean case, however, we may be writing them down for the first time in the case of the Heisenberg group. This is suggested by our use of the very recent [21]. Curve families controlled in measure as in [30] were suspected (or known by indirect arguments) to exist in $\mathbb{H}$, however, [21] is the first explicit construction to our knowledge. We use them at a crucial step in our bi-Lipschitz equivalence result (Theorem 1.4 below), using a David-Semmes deformation of $\mathbb{H}$ as an auxiliary space.

1.5 Geometric Applications

An interesting class of sub-Riemannian manifolds is given by the conformal equivalence class of the sub-Riemannian Heisenberg group, the set of all $(\mathbb{H}, e^u g_0)$, with g_0 the canonical sub-Riemannian metric on $\mathbb{H}$, and $u : \mathbb{H} \to \mathbb{R}$ a continuous function. Let ρ be the (Carnot-Carethéodory) distance function associated to g_0, and ρ_u that associated to $e^u g_0$ (see Section 6). It is useful to know when one of the metrics $e^u g_0$ is bi-Lipschitz equivalent to g_0, that is, when there exists $L \geq 1$ and homeomorphism $f : \mathbb{H} \to \mathbb{H}$ such that for all $p, q \in \mathbb{H}$,

$$\frac{1}{L}\rho(p,q) \leq \rho_u(f(p), f(q)) \leq L\rho(p,q), \tag{1.19}$$

as then $(\mathbb{H}, e^u g_0)$ has many of the geometric and analytic properties of $(\mathbb{H}, g_0)$ itself. One goal of the program initiated here, is the sub-Riemannian analog of

Theorem 1.3 (Bonk, Heinonen, Saksman, Wang)**.** *Suppose $(\mathbb{R}^4, e^{2u} g_E)$ is a complete Riemannian manifold with normal metric. If the Q-curvature satisfies*

$$\int |Q| \, \mathrm{dvol} < \infty,$$

and

$$\frac{1}{4\pi^2}\int Q\,\mathrm{dvol} < 1,$$

then $(\mathbb{R}^4, e^{2u}g_E)$ *and* $(\mathbb{R}^4, g_E)$ *are bi-Lipschitz equivalent.*

Here g_E is the canonical Euclidean metric. In these circumstances the Q-curvature satisfies $\Delta^2 u = 2Qe^{2u}$ (the Paneitz operator associated to g_E reduces to the biharmonic operator Δ^2). A metric $e^{2u}g_E$ on $\mathbb{R}^4$ is normal if at all $x \in \mathbb{R}^4$

$$u(x) = -\frac{1}{4\pi^2}\int \log\frac{|x-y|}{|y|}Q(y)e^{4u(y)}\,\mathrm{d}y + C$$

with C a constant. In other words, u is essentially a logarithmic potential with respect to the measure $\mathrm{d}\mu(y) = Q(y)e^{4u(y)}\,\mathrm{d}y$. In this thesis we take what is hopefully a substantial step toward a sub-Riemannian counterpart, proving

Theorem 1.4. *There exists* $\epsilon > 0$ *such that, if* u *is a continuous logarithmic potential on* $\mathbb{H}$*,*

$$u(p) = -\int \log\|q^{-1}p\|\,\mathrm{d}\mu(q)$$

for a finite, signed, Radon measure μ *with*

$$\int \mathrm{d}|\mu| < \epsilon$$

and

$$\int \log^+\|q\|\,\mathrm{d}|\mu|(q) < \infty,$$

then $(\mathbb{H}, g_0)$ *and* $(\mathbb{H}, e^u g_0)$ *are bi-Lipschitz equivalent.*

The reason for our note of caution, is that in the Heisenberg setting, there is currently no notion of normal metric to take aim at. In a way, we are working backwards; in the Euclidean setting, normal metrics were known, and known to be interesting, prior to Theorem 1.3.

For example, the Q-curvature can be thought of as a higher-dimensional version of the Gaussian curvature of a surface, and Chang, Qing, and Yang, prove in [11] something like a Gauss-Bonnet theorem for manifolds satisfying the hypotheses of Theorem 1.3. In the same paper, they show that normal metrics are not unusual: if $(\mathbb{R}^4, e^{2u}g_E)$ is complete, has integrable Q-curvature, and the scalar curvature is non-negative at infinity, then the metric is normal.

There is, however, cause for optimism, and we take the viewpoint that our results suggest the investigation of a potentially rich thread in sub-Riemannian / CR geometry. There was already evidence to suggest that similar phenomenon should exist. The correct definition of sub-Riemannian normal metric will likely exploit, then strengthen, what Case and Yang in [10] call the 'deep analogy between the study of three dimensional CR manifolds, and four dimensional conformal manifolds' (the Heisenberg group is an example of a three dimensional CR manifold). Suitable objects for such an investigation were only recently made available, the Paneitz-type operator, and Q-like curvature introduced for the CR-sphere and Heisenberg group by Branson, Fontana, and Morpurgo in [7]. Case and Yang were abstracting these to the more general CR setting in [10]. This is a fascinating area, with many strands to pursue, however, we say no more about it here.

The passage from Theorem 1.2 to Theorem 1.4 is straightforward, though non-trivial; indeed we currently rely on a construction found in [21]. A stronger version of Theorem 1.4 involving quasi-logarithmic potentials also holds, it is stated as Theorem 6.8.

Theorem 1.3 as stated is Wang's, it can be found in [34]. Wang was building on the work of Bonk, Heinonen, and Saksman in [4]. As already mentioned, they include a bi-Lipschitz equivalence result, however, like ours, it is something of a corollary to the main result. The primary contribution of Wang to Theorem 1.3 was to give the sharp constants on the size of the measure (which translate into the integral bounds on the Q-curvature). Given how the story went in the Euclidean case, with the Dirac measure identifying the end point, following on from our comments in Section 1.4, it is tempting to conjecture that our main theorem is true for all quasi-logarithmic potentials with admissible μ such that $\|\mu\| < \infty$, and $\mu(\mathbb{H}) < 2$.

1.6 Notation

If $p \in \mathbb{H}$ then $\|p\|$ is the Korányi gauge applied to p, as above, and $|p|$ is the Euclidean norm of the point of $\mathbb{R}^3$ with which p is identified.

If μ is an admissible measure, with Jordan decomposition $\mu = \mu_- - \mu_+$, then $\|\mu\|$ is the total variation, $\|\mu\| = \mu_-(\mathbb{H}) + \mu_+(\mathbb{H})$.

We write $M_n(\mathbb{R})$ for the $n \times n$ matrices with real entries. If $A \in M_n(\mathbb{R})$ for some n, then $|A|$ is the operator norm, $|A| = \sup_{v \in \mathbb{R}^n, |v|=1} |Av|$, and $\det A$, $\operatorname{tr} A$, A^T are respectively the determinant, trace and transpose of A. I_n is the $n \times n$ identity matrix.

$B(p,r)$ is the ball of center p and radius r, $B(p,r) = \{q \in \mathbb{H} : d(p,q) \leq r\}$. We write $B(r) = B(0,r)$. $S(p,r)$ is the sphere of center p and radius r, $S(p,r) = \partial B(p,r)$, and we write $S(r) = S(0,r)$.

The spaces $L^r(\mathbb{H})$ (with norm $\|\cdot\|_r$) and $L^r_{\text{loc}}(\mathbb{H})$, $1 \leq r \leq \infty$, have their usual definition, and as they are defined with respect to the Lebesgue measure on $\mathbb{R}^3$, they are identical to their Euclidean counterparts $L^r(\mathbb{R}^3)$ and $L^r_{\text{loc}}(\mathbb{R}^3)$.

As they indicate differentiation using the smooth manifold structure (given by the set identity of $\mathbb{R}^3$ as global chart), the spaces $C^k(\mathbb{H})$ and $C^k_0(\mathbb{H})$, $1 \leq k \leq \infty$ are identical with $C^k(\mathbb{R}^3)$ and $C^k_0(\mathbb{R}^3)$.

$HC^1(\mathbb{H})$ is the space of continuous functions $F : \mathbb{H} \to \mathbb{R}$ such that the classical horizontal derivatives XF, YF exist and are continuous everywhere (functions continuously differentiable in the horizontal directions).

$HW^{1,r}_{\mathrm{loc}}(\mathbb{H})$ is the space of locally integrable functions $F : \mathbb{H} \to \mathbb{R}$ with distributional derivatives $XF, YF \in L^r_{\mathrm{loc}}(\mathbb{H})$ (the first horizontal Sobolev spaces).

If F is a function of several components on $\mathbb{H}$ and each component is real valued, we will write $F \in HC^1(\mathbb{H})$ (respectively $F \in HW^{1,r}_{\mathrm{loc}}(\mathbb{H})$) if each component is in $HC^1(\mathbb{H})$ (respectively in $HW^{1,r}_{\mathrm{loc}}(\mathbb{H})$).

$\mathbb{1}_E$ is the indicator function of the set E.

We make heavy use of the notation $\lesssim$, $\gtrsim$, and $\simeq$, writing $A \lesssim B$ for there exists $C > 0$ such that $A \leq CB$, $A \gtrsim B$ for $B \lesssim A$, and $A \simeq B$ for there exists $C > 0$ with

$$\frac{1}{C} B \leq A \leq CB.$$

If A or B are functions then the implied C is a constant in that it does not depend on any variables. It may depend on parameters. Our convention is to identify dependence on pertinent parameters in the statement of a result, using $A \lesssim_{P_1,\dots,P_k} B$ for $A \leq CB$, with $C = C(P_1, \dots, P_k) > 0$ a constant dependent on the parameters $P_1, \dots, P_k$. Similarly for $\gtrsim$ and $\simeq$. Typically we do not indicate dependence on parameters in the proofs of statements. Whenever we say that A and B are comparable, we mean that $A \simeq B$.

2 Quasiconformal Mappings

We aim for an efficient summary of the key aspects of the theory, therefore, in this section we will generally not include citations in the body text, but provide some bibliographical notes at the end. Definitions given here are intended to supersede any given in the first paragraphs of the introduction.

Let $U, U' \subset \mathbb{H}$ be open connected sets. A homeomorphism $f : U \to U'$ is said to be a quasiconformal mapping if the quantity

$$H_f(p) := \limsup_{r \to 0} \frac{\max_{d(p,q)=r} d(f(p), f(q))}{\min_{d(p,q)=r} d(f(p), f(q))},$$

is bounded independently of $p \in U$. For us, a quasiconformal mapping is always a homeomorphism, $f : \mathbb{H} \to \mathbb{H}$. $H_f(p)$ is called the dilatation of f at p, and we will call f a K-quasiconformal mapping if the dilatation is not only bounded, but also essentially bounded by K (necessarily $1 \leq K < \infty$). We will then call such a K the essential dilatation (or simply the dilatation) of f. It is convenient to define $K(f) = \operatorname{ess\,sup} H_f$ for quasiconformal f.

A quasiconformal mapping, f, is Pansu-differentiable ($\mathcal{P}$-differentiable) at almost every $p \in \mathbb{H}$, which means that at such a p, the mappings

$$q \mapsto \delta_s^{-1}\left[f(p)^{-1} f(p\,\delta_s(q))\right], \quad q \in \mathbb{H},$$

converge locally uniformly as $s \to 0$ to a homomorphism $q \mapsto h_p f(q)$ of $\mathbb{H}$. Using the exp mapping, such a homomorphism gives rise to a Lie algebra homomorphism that we will denote $(h_p f)_*$. At a point p of $\mathcal{P}$-differentiability, and suppressing dependence on p, it can be shown that the horizontal partial derivatives Xf_1, Yf_1, Xf_2, Yf_2 exist, and that $(hf)_*$

acts on $\mathfrak{h}$ with respect to the basis $\{X_0, Y_0, T_0\}$ via the matrix

$$\begin{pmatrix} Xf_1 & Yf_1 & 0 \\ Xf_2 & Yf_2 & 0 \\ 0 & 0 & Xf_1Yf_2 - Xf_2Yf_1 \end{pmatrix}.$$

We denote the matrix of $(h_p f)_*$ by $\mathcal{P}f(p)$, and define a matrix $D_H f(p)$, which we call the horizontal differential of f at p, by the relationship

$$\mathcal{P}f = \begin{pmatrix} D_H f & 0 \\ 0 & \det D_H f \end{pmatrix}. \tag{2.1}$$

The Jacobian of a quasiconformal mapping f, is

$$J_f := \det \mathcal{P}f,$$

and so exists at almost every $p \in \mathbb{H}$. This agrees, at points of existence, with the definition given in (1.3) of the introduction (the Jacobian as volume derivative). Note that $J_f = (\det D_H f)^2$. If f is a K-quasiconformal mapping, then

$$|D_H f|^4 \leq K^2 J_f \tag{2.2}$$

almost everywhere. Indeed, if f is quasiconformal, then f is K-quasiconformal if and only if

$$|D_H f|^2 \leq K \det D_H f \tag{2.3}$$

almost everywhere.

If $f : \mathbb{H} \to \mathbb{H}$ is $\mathcal{P}$-differentiable at $p \in \mathbb{H}$ then f is contact at p, that is, Xf_3, Yf_3 also exist at p, with

$$Xf_3 = 2f_2Xf_1 - 2f_1Xf_2, \tag{2.4}$$

$$Yf_3 = 2f_2Yf_1 - 2f_1Yf_2. \tag{2.5}$$

Consequently a quasiconformal mapping is weakly contact, in that it is contact almost everywhere. This is a prerequisite for a mapping to act in a constrained manner with respect to the Heisenberg geometry. Suppose a mapping $f : \mathbb{H} \to \mathbb{H}$ is differentiable at a point p, in the Euclidean sense, and contact at that point. Then the Euclidean differential Df maps $\mathrm{H}_p\mathbb{H}$ (the horizontal layer at p) to $\mathrm{H}_{f(p)}\mathbb{H}$. Actually, if f is $\mathcal{P}$-differentiable at p, then the restriction of f to $p\exp[\mathrm{span}(X_0, Y_0)]$ is differentiable in the Euclidean sense at p, and this derivative is given by hf_* restricted to the horizontal layer of $\mathfrak{h}$. The matrix of this restriction is given by D_Hf. We will discuss the Sobolev regularity of quasiconformal mappings briefly in Section 3.

We now record two properties of quasiconformal mappings that will be important to us. The first is well known (which is not to say that the argument is brief).

Lemma 2.1. *If f is a K-quasiconformal mapping, then f^{-1} is also K-quasiconformal.*

The next seems less well known, and so we provide a proof. Note, however, as elsewhere in this section, we are relying on deeper results that we gloss over.

Lemma 2.2. *If f_1, f_2 are, respectively, K_1, K_2-quasiconformal mappings, then $f_1 \circ f_2$ is a K_1K_2-quasiconformal mapping.*

Proof. From either the geometric or quasisymmetric characterizations of quasiconformal mappings (we do not discuss the geometric characterization, see below for the quasisymmetric), it is easy to see that $f_1 \circ f_2$ is quasiconformal. The only question is with regard to the essential bound on the dilatation. For this we use the analytic characterization of the essential dilatation (2.3). First of all, for $E \subset \mathbb{H}$,

$$|E| = 0 \iff |f_2 E| = 0.$$

There is, therefore, a set E with $|\mathbb{H} \setminus E| = 0$, at each p of which, $f_1 \circ f_2$ is $\mathcal{P}$-differentiable, f_2 is $\mathcal{P}$-differentiable, and such that f_1 is $\mathcal{P}$-differentiable at $f_2(p)$. A calculation similar to that of the Euclidean case, shows that we have the following chain rule:

$$h_p(f_1 \circ f_2) = h_{f_2(p)} f_1 \circ h_p f_2.$$

As

$$(h_{f_2(p)} f_1 \circ h_p f_2)_* = (h_{f_2(p)} f_1)_* \circ (h_p f_2)_*,$$

then

$$\mathcal{P}(f_1 \circ f_2) = \mathcal{P} f_1(f_2) \mathcal{P} f_2,$$

and consequently

$$D_H(f_1 \circ f_2) = D_H f_1(f_2) D_H f_2.$$

We have then,

$$\begin{aligned} |D_H(f_1 \circ f_2)|^2 &\leq |D_H f_1(f_2)|^2 |D_H f_2|^2 \\ &\leq K_1 K_2 \det D_H f_1(f_2) \det D_H f_2 \\ &= K_1 K_2 \det D_H(f_1 \circ f_2). \end{aligned}$$

It follows from (2.3) that $f_1 \circ f_2$ is $K_1 K_2$-quasiconformal. ∎

Let f be a K-quasiconformal mapping, and $p \in \mathbb{H}$. Consider the quantity

$$H_{f,p}(r, s) := \frac{\max_{d(p,q)=r} d(f(p), f(q))}{\min_{d(p,q)=s} d(f(p), f(q))},$$

whenever $0 < s \leq r < \infty$. It requires some work, however, it can be shown that

$$H_{f,p}(r,s) \lesssim_K (r/s)^{K^{\frac{2}{3}}}. \tag{2.6}$$

In particular, let $p, q, u \in \mathbb{H}$ be distinct points such that $d(p,q) \leq d(p,u)$. Then

$$d(f(p), f(q)) \leq \max_{d(p,w)=d(p,u)} d(f(p), f(w)) \lesssim \min_{d(p,w)=d(p,u)} d(f(p), f(w)) \leq d(f(p), f(u)).$$

That is, f is weakly-quasisymmetric, with constant dependent on K only. It happens to be true that f is also quasisymmetric: there exists a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ such that, for distinct $p, q, u \in \mathbb{H}$,

$$\frac{d(f(p), f(q))}{d(f(p), f(u))} \leq \eta\left(\frac{d(p,q)}{d(p,u)}\right).$$

Let us deduce some easy consequences of (2.6). First of all, there exists $C = C(K) > 0$ such that, for all $p \in \mathbb{H}$ and $r > 0$, there is $s > 0$ with

$$B(f(p), s) \subset fB(p,r) \subset B(f(p), Cs).$$

Indeed, it is frequently useful that

$$|fB(p,r)|^{1/4} \simeq_K d(f(p), f(q)), \tag{2.7}$$

when q is any point on $S(p,r) = \partial B(p,r)$.

Now consider g, also K-quasiconformal, but with $g(0) = 0$. Then (2.6) leads easily to

$$\|g(p)\| \lesssim_K \|g(q)\| \left(1 + \left(\frac{\|p\|}{\|q\|}\right)^{K^{\frac{2}{3}}}\right)$$

for all $p, q \in \mathbb{H}$. Now suppose g_i, $i \in I$, I some index set, is a family of K-quasiconformal mappings (the same K for each i) each of which fixes 0. Furthermore, suppose there exist $D, D' > 0$ such that for each i there is $q_i \in \mathbb{H}$ with $\|q_i\| \geq D$, $\|g(q_i)\| \leq D'$. Then we have a uniform distortion estimate for the g_i,

$$\|g_i(p)\| \lesssim_{K,D'} 1 + \left(\frac{\|p\|}{D}\right)^{K^{2/3}}. \tag{2.8}$$

We will typically use this estimate with $D = D' = 1$. For this reason we introduce the notation $\mathcal{Q}_0(K)$ for the family of K-quasiconformal mappings that fix the origin, and leave invariant the norm of at least one point on the unit sphere (the point may depend on the mapping). We have proved,

Lemma 2.3. *Given $R > 0$, there exists $R' = R'(K, R) > 0$ such that, for all $g \in \mathcal{Q}_0(K)$,*

$$gB(R) \subset B(R').$$

Quasiconformal mappings are locally Hölder continuous. Given K-quasiconformal mapping f and $R > 0$, let $R' > 0$ be such that $fB(3R+1) \subset B(R')$. Then there exists $\alpha = \alpha(K) > 0$ such that

$$d(f(p), f(q)) \lesssim_{K,R,R'} d(p,q)^{\alpha}. \tag{2.9}$$

If we combine this with Lemma 2.3 we have the following.

Lemma 2.4. *Given $R > 0$, there exists $\alpha = \alpha(K) > 0$ such that, for all $g \in \mathcal{Q}_0(K)$,*

$$d(g(p), g(q)) \lesssim_{K,R} d(p,q)^{\alpha}$$

for all $p, q \in B(R)$.

Crucially in the previous lemma, the implied constant is dependent on g only through its dependence on K.

We now record the various results that are pertinent to our focus on the quasiconformal Jacobian. On numerous occasions we use that, if f is a quasiconformal mapping, it satisfies the following change of variable formula

$$\int_{f\Omega} u = \int_{\Omega} (u \circ f) J_f, \tag{2.10}$$

valid for all non-negative, measurable functions $u : \mathbb{H} \to \mathbb{R}$, and measurable $\Omega \subset \mathbb{H}$ (and with the necessary measurability of $(u \circ f)J_f$ part of the result).

The result just recorded relies on the fact that

$$J_f > 0$$

almost everywhere, and $J_f \in L^1_{\mathrm{loc}}(\mathbb{H})$. Actually more is true, the Jacobian of a quasiconformal mapping f satisfies a reverse Hölder inequality, the power of which will be ably demonstrated by multiple appearances at crucial moments later. To be precise, if f is a K-quasiconformal mapping, then there exists $r > 1$ such that, if $B \subset \mathbb{H}$ is a ball,

$$\left(\frac{1}{|B|} \int_B J_f^r \right)^{\frac{1}{r}} \lesssim \frac{1}{|B|} \int_B J_f \tag{2.11}$$

independently of B. Indeed the exponent and implied coefficient can be taken to depend on K only. That J_f satisfies a reverse Hölder inequality implies that it is an A_∞ weight, as in [33]. It is also true that the inequality can be shown to imply the A_p condition for some $1 \leq p < \infty$ (the calculation can be found in [33]). We do not record the A_p condition here, but observe that it has the following easy implication: there exists $\alpha > 0$ such that, if $B \subset \mathbb{H}$ is a ball,

$$\frac{1}{|B|} \int_B J_f^{-\alpha} \lesssim \left(\frac{|B|}{|f(B)|} \right)^{\alpha} \tag{2.12}$$

independently of B.

Ultimately, the mapping we construct that has Jacobian comparable to a given weight, will be found in the limit of a sequence of mappings. We therefore need to be able to say something useful about the limit of the Jacobians. Presently we are only able to prove the following weak convergence result which suffices for our purpose.

Lemma 2.5. *Suppose (f_m) is a sequence of quasiconformal mappings converging locally uniformly to a quasiconformal mapping f. Suppose also that the f_m^{-1} converge pointwise to f^{-1}. Then, given $\xi \in C_0^\infty(\mathbb{H})$, $\xi \geq 0$,*

$$\lim_{m\to\infty} \int \xi J_{f_m} = \int \xi J_f.$$

Proof. Let $R > 0$ be such that $\text{support}(\xi) \subset B(R)$. As the f_m converge locally uniformly, there exists $R' > 0$ such that $f_m B(R) \subset B(R')$ for all m. Note, therefore, that

$$\text{support}(\xi \circ f_m^{-1}) \subset B(R')$$

for all m. Using the change of variable formula (2.10),

$$\begin{aligned} \int \xi J_{f_m} &= \int (\xi \circ f_m^{-1})(f_m) J_{f_m} \\ &= \int \xi \circ f_m^{-1}. \end{aligned}$$

As $|\xi \circ f_m^{-1}| \leq \max(|\xi|)\mathbb{1}_{B(R')}$ for all m, then the Dominated Convergence Theorem applies, and we conclude that

$$\lim_{m\to\infty} \int \xi J_{f_m} = \int \xi \circ f^{-1} = \int \xi J_f.$$ ∎

We end this section with some instances in which a sequence (f_m) of quasiconformal mappings converges to a quasiconformal mapping f. These are based on well known results, however, we tailor the statements to our purpose.

Lemma 2.6. *Suppose (f_m) is a sequence of K-quasiconformal mappings, and there exists*

$p_0 \in \mathbb{H}$, $\|p_0\| = 1$, such that for all m, $f_m(0) = 0$, and $\|f(p_0)\| = 1$. Then the f_m subconverge, locally uniformly, to a K-quasiconformal mapping f. Furthermore, any convergent subsequence, (f_k), has the f_k^{-1} converging pointwise to f^{-1}.

Proof. Local uniform subconvergence of the f_m to a quasiconformal mapping is standard in these circumstances. That the essential dilatation of the limit mapping is the same as those of the sequence is somewhat less expected, a proof can be found in [20]. We are left then to prove the statement regarding the inverses, which one would think should be automatic, but we currently have no better argument.

Let (f_k) be a convergent subsequence as in the statement. By Lemma 2.1, each f_k^{-1} is K-quasiconformal. It is also true that $f_k^{-1}(0) = 0$ for all k. Furthermore, our assumption regarding the existence of p_0 implies that for each k, there exists p_k, with $\|p_k\| = 1$, and $\|f_k^{-1}(p_k)\| = \|p_0\| = 1$. It follows that $(f_k^{-1}) \subset \mathcal{Q}_0(K)$.

Choose some $q \in \mathbb{H}$, and let p be such that $f(p) = q$ (where $f = \lim f_k$). There is an $0 < R < \infty$ such that $f(p) \in B(R)$, and $f_k(p) \in B(R)$ for all k. As in Lemma 2.4, let $\alpha > 0$ be such that

$$d(f_k^{-1}(u_1), f_k^{-1}(u_2)) \lesssim d(u_1, u_2)^\alpha$$

for all $u_2, u_2 \in B(R)$ independently of k. Then

$$\begin{aligned} d(f_k^{-1}(q), p) &= d(f_k^{-1}(f(p)), f_k^{-1}(f_k(p))) \\ &\lesssim d(f(p), f_k(p))^\alpha. \end{aligned}$$

Consequently, $\lim f_k^{-1}(q) = p = f^{-1}(q)$, as required. ∎

Once it is known the f_k^{-1} converge pointwise, in these circumstances local uniform convergence follows, but we did not require this for the proof of Lemma 2.5.

Lemma 2.7. *Suppose (f_m) is a sequence of K-quasiconformal mappings, $f_m(0) = 0$ for all* *m, and*

$$\int_{B(1)} J_{f_m} \simeq 1$$

independently of m. Then the f_m subconverge, locally uniformly, to a K'-quasiconformal mapping. Furthermore, if (f_k) is a convergent subsequence, then the f_k^{-1} converge pointwise to f^{-1}.

Proof. Fix a point $p_0 \in S(1)$. It follows from (2.7) and (2.10) that

$$\int_{B(1)} J_{f_m} \simeq_K d(f_m(0), f_m(p_0))^4,$$

independently of m. Given our assumption on the size of the integral, we therefore have

$$d(f_m(0), f_m(p_0)) \simeq 1,$$

which, coupled with $f_m(0) = 0$ for all m is enough to conclude the locally uniform subconvergence using well known compactness properties of quasisymmetric mappings (in other words we have essentially reduced to the hypotheses of Lemma 2.6).

As for the pointwise convergence of f_k^{-1} to f^{-1} for a convergent subsequence (f_k), again, the argument is essentially the same as in Lemma 2.6, we just need to work a little harder. We have the existence of $0 < R \leq R' < \infty$, such that for each k, there exists a point p_k with $R \leq \|p_k\| \leq R'$, and such that $\|f_k^{-1}(p_k)\| = \|p_0\| = 1$. We have, therefore, a uniform distortion estimate for the f_k as in (2.8), and we can use this to derive Hölder continuity with uniform constants (as in (2.9)) on a useful ball, and proceed as in the proof of Lemma 2.6. ∎

Notes to Section 2: The primary reference for the results of this section is [20]. For the almost everywhere differentiability of quasiconformal mappings see [27]. For the matrix of the $\mathcal{P}$-differential see [12]. The analytic criterion (2.3) can be found in [20], along with

Lemma 2.1. The case $r = s$ of (2.6) is in [20], the general form is in [9]. It should be observed that the proof of (2.6) rests on a suitable capacity estimate, as proved in [29]. The local Hölder continuity of quasiconformal mappings is in [20]. The change of variable formula (2.10) is in [12]. The reverse Hölder inequality is proved in [20]. Lemma 2.6 is in [20], but these things hold for quasisymmetric mappings in a more general setting, with the results nicely stated in [16].

3 Quasiconformal Flows

The measurable Riemann mapping theorem guarantees a plentiful supply of quasiconformal mappings $f : \mathbb{C} \to \mathbb{C}$. It is a consequence of that theorem that any quasiconformal mapping of the complex plane embeds as the time-s flow mapping of a suitably well behaved vector field.

While quasiconformal mappings of the Heisenberg group satisfy a 'Beltrami system' of equations, no similar results on the existence of solutions are known. We may, however, identify suitable conditions on a vector field $v : \mathbb{H} \to \mathrm{T}\mathbb{H}$ such that the flow is quasiconformal.

Such conditions were first identified by Korányi and Reimann in [19] and [20]. The results of [19] are for reasonably smooth flows. In [20] the main relevant result requires significantly less regularity, but demands that the vector field be compactly supported. See the introduction for more discussion.

Our results require both low regularity, and unbounded support, and it is the purpose of the first part of this section to remove the assumption of compact support from the theorem of [20]. In its place we make stipulations on the growth of the vector field, then use a cut off argument to reduce to the compactly supported case.

Remember that a quasiconformal mapping of $\mathbb{H}$ is almost everywhere contact. It is a theorem of Liebermann ([25]) that in order for a vector field to generate contact flow it must be of the form

$$v = v_\phi = -\frac{1}{4} Y\phi X + \frac{1}{4} X\phi Y + \phi T, \tag{3.1}$$

for a function $\phi : \mathbb{H} \to \mathbb{R}$. We will call such a ϕ, to be used in this way, a contact generating potential, or simply a potential. Whenever a potential is in play and we write v_ϕ we mean the above expression. As in the work of Korányi and Reimann mentioned above, we will typically work at the level of the potential, deducing from its properties the properties of the flow. Indeed, Section 4 is all about constructing a potential suitable for our purposes.

If v is a vector field, as above, we will on occasion have need to discuss component functions of v. Perhaps the obvious choice would be to define these with respect to the basis X_p, Y_p, T_p of $T_p\mathbb{H}$, however in order to be consistent with something that comes later, we let v_1, v_2, v_3 be defined by

$$v = v_1 \partial_x + v_2 \partial_y + v_3 \partial_t,$$

with the obvious identifications needed taken as implicit.

The second part of this section is dedicated to proving a variational equation that links (the logarithm of) the Jacobian of the flow mapping to the horizontal divergence of the vector field. If ϕ is a potential, it will be apparent that the the horizontal divergence of v_ϕ is given by $T\phi$. Consequently, part of the work of Section 4 is in designing a ϕ such that the T derivative resembles a given logarithmic potential. The variational equation is then the key stepping stone linking Jacobian to weight. The results of this part follow a similar sequence of results in [4].

Before moving on, some notation. In Section 2 we defined the horizontal differential $D_H f$ of a quasiconformal mapping using the Pansu-derivative. From here on, so long as a function $F : \mathbb{H} \to X$, where $X = \mathbb{R}^3$ as a set, has $F_1, F_2 \in HW^1_{\mathrm{loc}}(\mathbb{H})$, then we will write $D_H F$ for the equivalence class of matrices

$$\begin{pmatrix} XF_1 & YF_1 \\ XF_2 & YF_2 \end{pmatrix},$$

though in practice we will typically work with a particular representative.

Note that we do not require a function F to be contact (not even weakly so) in order to discuss $D_H F$. At this level of regularity, we will refer to $D_H F$ as a formal horizontal differential (of F) if we are talking about a representative, and the formal horizontal differential if we are talking about the equivalence class.

3.1 Vector Fields with Unbounded Support

The following proposition is Theorem H of [20].

Proposition 3.1. *Suppose $\phi : \mathbb{H} \to \mathbb{R}$ is in $HC^1(\mathbb{H})$, is compactly supported, and the distributional derivative $ZZ\phi \in L^\infty(\mathbb{H})$ with*

$$\sqrt{2}\|ZZ\phi\|_\infty \leq c$$

for some $0 \leq c < \infty$. Then for each $p \in \mathbb{H}$, the flow equation for v_ϕ at p,

$$\gamma'(s) = v(\gamma(s)), \quad \gamma(0) = p,$$

has exactly one solution, $\gamma_p : \mathbb{R} \to \mathbb{H}$. Furthermore, for $s \geq 0$, the time-s flow mapping,

$$f_s : \mathbb{H} \to \mathbb{H},$$
$$f_s(p) = \gamma_p(s),$$

is a K-quasiconformal homeomorphism, where K satisfies $K + K^{-1} \leq 2e^{cs}$.

As mentioned in the introduction to this section, we intend to adapt this theorem, identifying suitable means of removing the assumption of compact support. First we have a smaller, but still important, adaptation to make. The proof of Proposition 3.1 makes use of the square, or Frobenius, norm on $D_H f_s$, which leads to the form of the bound on K. Unfortunately, this bound is not suitable for our purposes as it does not behave well in a later limiting argument. We first, therefore, rework part of the proof in the smooth case, using the operator norm in place of the square norm. We need only the smooth case, as it is this that feeds into the proof of Proposition 3.1 in an approximation argument. We thank Jeremy Tyson for improving the proof of the following lemma.

Lemma 3.2. *Suppose $\phi \in C_0^\infty(\mathbb{H})$, and $\sqrt{2}\|ZZ\phi\|_\infty \leq c$ for some $0 \leq c < \infty$. Then v_ϕ generates a smooth flow of homeomorphisms, and each time-s flow mapping, $0 \leq s < \infty$, is K-quasiconformal, with $K \leq e^{cs}$.*

Proof. That $\phi \in C_0^\infty(\mathbb{H})$ is already enough for existence and uniqueness of solutions to the flow equation for v_ϕ, and the time-s flow mappings are well defined C^∞-smooth homeomorphisms of $\mathbb{H}$.

$ZZ\phi$ should be considered the Heisenberg equivalent of what, in the Euclidean case, is sometimes called the Ahlfors conformal strain of the vector field. Actually, in this proof we work with an even more direct analog of the Ahlfors conformal strain. Writing $v = v_\phi$, let

$$\mathcal{S}_H v := \frac{1}{2}\begin{pmatrix} Xv_1 - Yv_2 & Xv_2 + Yv_1 \\ Xv_2 + Yv_1 & Yv_2 - Xv_1 \end{pmatrix}.$$

More generally, the Ahlfors conformal strain of a 2×2 matrix M is

$$\mathcal{S}(M) := \tfrac{1}{2}(M + M^T) - \tfrac{1}{2}(\operatorname{tr} M)I_2.$$

This is the symmetric, trace-free part of M. In our situation, $\mathcal{S}_H v = \mathcal{S}(D_H v)$. Note that (or see [19]), if $\|M\| = \sqrt{\operatorname{tr}[MM^T]}$ is the square norm of M, then

$$\sqrt{2}|ZZ\phi| = 2\|\mathcal{S}_H v\|.$$

As $|\mathcal{S}_H v| \leq \|\mathcal{S}_H v\|$, our assumed bound on $|ZZ\phi|$ translates to

$$2\|\mathcal{S}_H v\|_\infty \leq c,$$

where, at the risk of confusion, we write $\|\mathcal{S}_H v\|_\infty$ for $\sup_{p\in\mathbb{H}} |\mathcal{S}_H v(p)|$. Let f_s be the time-s flow mapping generated by v. From the integral formula for solutions to the flow equation, the smoothness, and the contact equations 2.4, it is immediate that

$$(D_H f_s)' = D_H v(f_s) D_H f_s. \tag{3.2}$$

For notational convenience, let $A := D_H f_s$, $B := D_H v$. Then (3.2) becomes

$$A' = B(f_s)A,$$

which we rewrite in the form

$$B(f_s) = A'A^{-1}.$$

It follows that

$$\mathcal{S}(B(f_s)) = \tfrac{1}{2}A'A^{-1} + \tfrac{1}{2}(A^{-1})^T(A')^T - \tfrac{1}{2}\operatorname{tr}(A'A^{-1})I_2.$$

For our smooth quasiconformal mappings, f_s, the dilatation H_{f_s} has an analytic expression (cf. (2.3))

$$H_{f_s} = \frac{|D_H f_s|^2}{\det D_H f_s} = \frac{|A|^2}{\det A}. \tag{3.3}$$

Consequently, we need only show that everywhere in $\mathbb{H}$,

$$\frac{|A|^2}{\det A} \leq \exp(cs)$$

holds true and we will be done. To this end, recall that $|A|^2$ is equal to the larger eigenvalue λ of the matrix $A^T A$. For each $s \geq 0$ there is a unit eigenvector $v(s)$ for the eigenvalue $\lambda(s)$, with

$$\lambda = \langle A^T Av, v\rangle = |Av|^2.$$

Differentiating with respect to s gives

$$\lambda' = 2\langle A'v, Av\rangle + 2\langle Av', Av\rangle. \tag{3.4}$$

As $|v(s)|^2 = 1$ for all s, then v' and v are orthogonal:

$$0 = (|v|^2)' = \langle v, v\rangle' = 2\langle v', v\rangle.$$

It follows that the second term of (3.4) is zero, indeed

$$\langle Av', Av\rangle = \langle v', A^T Av\rangle = \langle v', \lambda v\rangle = \lambda \cdot 0.$$

Consequently, using the standard formula

$$(\det M)' = (\det M)\,\mathrm{tr}(M'\,M^{-1}),$$

we have

$$\left(\log \frac{|A|^2}{\det A}\right)' = \frac{\lambda'}{\lambda} - \frac{(\det A)'}{\det A} = \frac{2\langle A'v, Av\rangle}{|Av|^2} - \mathrm{tr}(A'A^{-1}).$$

Set $w = Av$. Then, evaluating at some point $p \in \mathbb{H}$,

$$\begin{aligned}
\left(\log \frac{|A|^2}{\det A}\right)' &= 2\frac{\langle A'A^{-1}w, w\rangle}{|w|^2} - \mathrm{tr}(A'A^{-1}) \\
&= \left\langle \frac{w}{|w|}, \frac{A'A^{-1}w}{|w|} + \frac{(A^{-1})^T(A')^T w}{|w|} - \mathrm{tr}(A'A^{-1})\frac{w}{|w|} \right\rangle \\
&\leq \frac{|A'A^{-1}w + (A^{-1})^T(A')^T w - (\mathrm{tr}\,A'A^{-1})w|}{|w|} \\
&\leq |A'A^{-1} + (A^{-1})^T(A')^T - (\mathrm{tr}\,A'A^{-1})I_2| \\
&= 2|\mathcal{S}(B(f_s))|.
\end{aligned}$$

As p was arbitrary, $s \mapsto A(s)$ has $A(0) = I_2$, $\mathcal{S}(B(f_s)) = \mathcal{S}_H v(f_s)$, and $2|\mathcal{S}_H v(f_s)| \leq$

$2\|\mathcal{S}_H v\|_\infty \leq c$, it follows from (3.3) that

$$K(f_s) \leq e^{cs}$$

as desired. ∎

A careful check of the proof of Proposition 3.1 shows that the quasiconformal mappings it promises have dilatation bounded in the same manner as the smooth mappings of Lemma 3.2. We are now ready to formulate a new version of Proposition 3.1 with the assumption of compact support replaced by some natural growth conditions. The proof uses some ideas from Reimann's work in the Euclidean setting [28].

Proposition 3.3. *Suppose $\phi : \mathbb{H} \to \mathbb{R}$ is in $HC^1(\mathbb{H})$ and the distributional derivative $ZZ\phi \in L^\infty(\mathbb{H})$ with*

$$\sqrt{2}\|ZZ\phi\|_\infty \leq c$$

for some $0 \leq c < \infty$. Further suppose that, at each $p \in \mathbb{H}$,

$$|\phi(p)| \lesssim \|p\|^2 \log \|p\|,$$

$$|Z\phi(p)| \lesssim \|p\| \log \|p\|,$$

$$|v_\phi(p)| \lesssim 1 + |p|(1 + \log |p|),$$

independently of p.

Then for each $p \in \mathbb{H}$, the flow equation for v_ϕ at p,

$$\gamma'(s) = v_\phi(\gamma(s)), \quad \gamma(0) = p,$$

has exactly one solution, $\gamma_p : \mathbb{R} \to \mathbb{H}$. Furthermore, for $s \geq 0$, the time-s flow mapping,

$$f_s : \mathbb{H} \to \mathbb{H},$$
$$f_s(p) = \gamma_p(s),$$

is a K-quasiconformal homeomorphism, $K \leq e^{cs}$.

Remark 3.4. Note the appearance of the Euclidean norm in the growth condition on v_ϕ, not only on the vector field, but on points of the Heisenberg group. We admit this is unappealing. It is useful, as we will see below, due to the Euclidean nature of integral solutions to the flow equation (as we are currently writing them). In the statement of the proposition it is likely the Euclidean norm on the points could be replaced with the Korányi gauge relatively easily. It is also possible that we could be consistent with this desire in the proof, making the solutions look more group-like using, e.g. the exponential map etc. This is possibly specious, in that the flow equation involves the Euclidean derivative on the curve. If the answer to this is that we should consider $\mathbb{R}$ also as Carnot group, and use the Pansu-derivative on the curve, we have a problem: for a curve to be almost everywhere Pansu-differentiable it should be almost everywhere horizontal, and the integral curves of contact flow are not horizontal. We may revisit this in the future, however, for the time being our attitude in the first part of the proof is that $\mathbb{H}$ is a manifold with a global chart to $\mathbb{R}^3$, to be used as is convenient.

Proof of Proposition 3.3. Write $v = v_\phi$. Let $u \in \mathbb{H}$ be given. As v is continuous, solutions to the flow equation for v exist at u, at least on some interval $(-s_0, s_0)$, $s_0 > 0$. Consider one of them, call it γ_0.

For all $s \in (-s_0, s_0)$, γ_0 satisfies

$$\gamma_0(s) = u + \int_0^s v(\gamma_0(\sigma))\,\mathrm{d}\sigma,$$

and so, by assumption, there exist $C_1, C_2 > 0$ such that

$$|\gamma_0(s)| \leq |u| + C_1 s_0 + C_2 \int_0^s |\gamma_0(\sigma)|(1 + \log|\gamma_0(\sigma)|)\, \mathrm{d}\sigma.$$

For $s \in (-s_0, s_0)$, define

$$\lambda(s) = |u| + C_3 + C_2 \int_0^s |\gamma_0(\sigma)|(1 + \log|\gamma_0(\sigma)|)\, \mathrm{d}\sigma.$$

where $C_3 = e^{-1} + C_1 s_0 + C_2 e^{-2} s_0$ (the function $s \mapsto s + s\log s$, $s \geq 0$, has a minimum value of $-e^{-2}$, therefore, this choice of C_3 ensures that $1 + \log\lambda(s) > 0$, and $|\gamma_0(s)| \leq \lambda(s)$, for all $s \in (-s_0, s_0)$).

Then λ is differentiable on $(-s_0, s_0)$, with

$$\begin{aligned} \lambda'(s) &= C_2|\gamma_0(s)|(1 + \log|\gamma_0(s)|) \\ &\leq C_2\lambda(s)(1 + \log\lambda(s)). \end{aligned}$$

Equivalently, given our choice of C_3,

$$[\log(1 + \log\lambda(s))]' \leq C_2,$$

and so,

$$1 + \log|\gamma_0(s)| \leq 1 + \log\lambda(s) \leq e^{C_2 s}(1 + \log(|u| + C_3)), \tag{3.5}$$

for all $s \in (-s_0, s_0)$. There exists, therefore, sufficiently large but finite $R > 0$, such that $\gamma_0(s) \in B(R)$, for all $s \in (-s_0, s_0)$, and this is true for any other solution at u when restricted to this same interval.

We now introduce the auxiliary functions that are going to allow us to smoothly truncate our vector field and deduce global properties from a localization that behaves well in the limit. For $3 < l < l' < \infty$, let $\tilde{G}_l : [l, l'] \to [0,1]$ be given by

$$\tilde{G}_l(r) = 1 - l^{-1}(\log\log r - \log\log l).$$

This function decreases from 1 to 0 in a useful way, however, to use it as a cut-off function we need to smoothly extend it to all $r \geq 0$ such that it is constant off the interval $[l, l']$. Consider, therefore, the polynomial $P(z) = 6z^5 - 15z^4 + 10z^3$ (this is known in computer graphics circles as 'smootherstep'). It has $P(0) = 0$, $P(1) = 1$, $P'(0) = 0$, $P'(1) = 0$, $P''(0) = 0$, $P''(1) = 0$. Further, when $0 \leq z \leq 1$, also $0 \leq P(z) \leq 1$. For $l \geq 3$, now define,

$$G_l(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq l \\ P(\tilde{G}_l(r)) & \text{if } l \leq r \leq l' \\ 0 & \text{if } l' \leq r \end{cases}$$

where l' is chosen to make $\tilde{G}_l(l') = 0$ (to be exact $\log l' = e^l \log l$).

The niceness of the polynomial P renders $G_l \in C^2([0,\infty))$. Furthermore, $0 \leq G_l \leq 1$ and, given our assumption on the size of l, we have

$$|G_l'(r)| \lesssim \frac{1}{lr\log r},$$
$$|G_l''(r)| \lesssim \frac{1}{lr^2\log r}.$$

For each suitable l we form the truncated potential,

$$\phi_l(p) = G_l(\|p\|^4)\phi(p).$$

Each ϕ_l has continuous horizontal derivatives, $X\phi_l$, $Y\phi_l$, and is compactly supported. The weak derivative $ZZ\phi_l$ exists and, defining N by $N(p) = \|p\|^4$,

$$ZZ\phi_l = ZZ(G_l \circ N)\phi + 2Z(G_l \circ N)Z\phi + (G_l \circ N)ZZ\phi.$$

Then, applying our assumptions to this last expression, we find that there exist $C_4, C_5, C_6 > 0$ such that

$$\begin{aligned}
\sqrt{2}\|ZZ\phi_l\|_\infty &\leq C_4 \sup_{l\leq\|p\|^4\leq l'} \left[\|p\|^6|G_l''(\|p\|^4)||\phi(p)| + \|p\|^3|G_l'(\|p\|^4)||Z\phi(p)|\right] + c \\
&\leq C_5\frac{1}{l}\sup_{l\leq\|p\|^4\leq l'}\left[\frac{|\phi(p)|}{\|p\|^2\log\|p\|} + \frac{|Z\phi(p)|}{\|p\|\log\|p\|}\right] + c \\
&\leq C_6 l^{-1} + c.
\end{aligned}$$

Making a choice of l so that $(G_l \circ N) \equiv 1$ on $B(R)$ (recall $B(R)$ is home to our solution on $(-s_0, s_0)$), we have that v, and

$$v_l := -\frac{1}{4}Y\phi_l X + \frac{1}{4}X\phi_l Y + \phi_l T,$$

coincide on $B(R)$. It is part of Proposition 3.1 that the flow equation for v_l at u has a unique solution. It follows that γ_0 is the unique solution, on the interval $(-s_0, s_0)$, to the flow equation for v at u.

As u was arbitrary, we have shown that at all $p \in \mathbb{H}$, the solution at p is unique, and remains bounded on any finite time interval. It follows that solutions may be continued unambiguously and therefore exist for all time. Consequently, we find that v has a well defined flow of homeomorphisms, f_s, for all $s \in \mathbb{R}$.

We know turn our attention to the quasiconformality of the time-s flow mapping, f_s, $s \geq 0$ (consider s as fixed in the following). Let $D > 0$ be given. Let f_s^l denote the time-s flow mapping associated to v_l. By the calculation above, using Proposition 3.1 and Lemma 3.2, f_s^l is quasiconformal, with $K(f_s^l) \leq e^{(C_6 l^{-1}+c)s}$.

Let $D' > 0$ be such that $f_s B(D) \subset B(D')$. Choosing l so that $v_l \equiv v$ on $B(D')$, then it follows that the restriction of f_s to $B(D)$, $f_s\big|_{B(D)}$, is quasiconformal, with

$$K\left(f_s\big|_{B(D)}\right) \leq e^{(C_6 l^{-1}+c)s}.$$

We now let $l \to \infty$ to find that, in fact, $K\left(f_s\big|_{B(D)}\right) \leq e^{cs}$. Lastly, as D was arbitrary, we must have that f_s is quasiconformal, with $K(f_s) \leq e^{cs}$, as required. ∎

3.2 A Variational Equation

For the remainder of the section we fix a $\phi : \mathbb{H} \to \mathbb{R}$, and so $v = v_\phi$, that satisfies the hypotheses of Proposition 3.3. We also assume that $X\phi, Y\phi \in HW^{1,r}_{\mathrm{loc}}(\mathbb{H})$ for all $1 \leq r < \infty$. Let $D_H v$ denote a particular choice of representative of the formal horizontal differential of v. Our integrability assumptions on the second distributional horizontal derivatives of ϕ are equivalent to $D_H v$ having the same integrability (with respect to the operator norm).

In the introduction we mentioned that a quasiconformal mapping is $\mathcal{P}$-differentiable almost everywhere. It is also true that, if f is a quasiconformal mapping, then $f \in HW^4_{\mathrm{loc}}(\mathbb{H})$, and the almost everywhere defined classical derivatives determined by the $\mathcal{P}$-derivative may serve as distributional derivatives. More is true, as (2.2), and the reverse Hölder inequality (2.11), imply that there exists $\epsilon > 0$, such that $f \in HW^{4+\epsilon}_{\mathrm{loc}}(\mathbb{H})$. Reserving the notation f_s for the time-s flow mappings of v, let us agree that when we write $D_H f_s$ we mean the representative of the formal horizontal differential of f_s that is defined by the almost everywhere $\mathcal{P}$-differentiability, as in (2.1).

The nature of the argument we give is designed precisely so that our end goal, Proposition 3.10, holds for all values of s in our range of interest (consistently the interval $[0,1]$). It is likely a similar statement could be proved without as much preparation if we aimed only for almost every $s \in [0,1]$.

Lemma 3.5. *For $p \in \mathbb{H}$ and $s \in [0,1]$ the mapping $(p,s) \mapsto f_s(p)$ is continuous. Moreover,* *for each ball $B \subset \mathbb{H}$,*

$$|f_s(B)| \gtrsim 1$$

independently of $s \in [0,1]$.

Proof. We will prove the second statement first. As $s \in [0,1]$, each f_s is e^c-quasiconformal with c such that $\sqrt{2}\|ZZ\phi\|_\infty \leq c$ (so K-quasiconformal with K independent of s). Let $p \in \mathbb{H}$ and $R > 0$ be given. Fix a point $q \in S(p,R)$. It follows from (2.7) that

$$|f_sB(p,R)| \gtrsim_K d(f_s(p), f_s(q))^4.$$

As solutions to the flow equation are continuous, the function on the right hand side, $s \mapsto d(f_s(p), f_s(q))^4$ is continuous, and given that each f_s is injective, has a positive minimum on $[0,1]$.

As for the first statement, let $s \in [0,1]$ be given, and let (p_k, s_k) be a sequence in $\mathbb{H} \times [0,1]$ such that $(p_k, s_k) \to (p, s)$. In particular, $p_k \to p$, and we may assume that $p_k \in B(p,1)$ for all k. It follows from our bound on solutions to the flow equation (3.5), that there exists $R' > 0$ such that for all s_k

$$f_{s_k}B(3(\|p\|+1)+1) \subset B(R').$$

It follows from (2.9) that each f_{s_k} is Hölder continuous on $B(p,1)$ with the coefficient and exponent independent of k. It is now easy to see that $(p,s) \mapsto f_s(p)$ is jointly continuous on $\mathbb{H} \times [0,1]$ as required. ∎

Lemma 3.6. *The mappings,*

$$(p,s) \mapsto D_H v(f_s(p)) \qquad \textit{and} \qquad (p,s) \mapsto D_H v(f_s(p)) D_H f_s(p),$$

are measurable, and integrable on $B \times [0,1]$ whenever B is a ball in $\mathbb{H}$.

Proof. Integrability of a matrix valued function refers to integrability of the operator norm. As already observed, our assumptions imply that $D_H v$ is measurable and locally integrable to the power r for any $1 \leq r < \infty$.

Let $p \in \mathbb{H}$ be a point at which $D_H f_s(p)$ exists in the classical sense. It is the limit of matrices the entries of which are difference quotients. As $f_s(p)$ is jointly continuous in s and p, those difference quotients are measurable functions. It follows that $D_H f_s$ is measurable. Furthermore, as each f_s preserves sets of measure zero, $D_H v(f_s)$ is measurable also.

Observe that the claim will follow if we can show that for each $s \in [0,1]$ the integral of (the norm of) either function over an arbitrary ball is bounded above by a constant independent of s.

Fix a ball $B \subset \mathbb{H}$. Note that, as in Lemma 3.5, for all $s \in [0,1]$, f_s is K-quasiconformal with K independent of s. Consequently, by (2.12) and Lemma 3.5, there exists $\alpha > 0$ such that, for all $s \in [0,1]$,

$$\int_B J_{f_s}^{-\alpha} \lesssim 1 \tag{3.6}$$

independently of s.

Now let $r = 1 + 1/\alpha$. Again by (or as in the proof of) Lemma 3.5, there exists a ball B' such that $f_s B \subset B'$ for all $s \in [0,1]$. Then, by Hölder's inequality for the first estimate, and (3.6), (2.10) for the second,

$$\begin{aligned}
\int_B |D_H v(f_s)| &\leq \left(\int_B |D_H v(f_s)|^r J_{f_s} \right)^{1/r} \left(\int_B J_{f_s}^{-\alpha} \right)^{1/(1+\alpha)} \\
&\lesssim \left(\int_{B'} |D_H v|^r \right)^{1/r} \lesssim 1,
\end{aligned}$$

where the implied constants depend on B, but are independent of s.

Similarly,

$$\begin{aligned}\left(\int_B |D_H v(f_s) D_H f_s|\right)^4 &\lesssim \int_B |D_H v(f_s) D_H f_s|^4 \\ &\lesssim \int_B |D_H v(f_s)|^4 J_{f_s} \\ &\lesssim \int_{B'} |D_H v|^4 \lesssim 1,\end{aligned}$$

where we used (2.2) for the second estimate. Again, the implied constants do not depend on s. ∎

The following can be found on page 46 of [20].

Lemma 3.7. *For each $s \in [0,1]$, $v \circ f_s$ has a formal horizontal differential, and one representative is given by*

$$D_H(v \circ f_s) = D_H v(f_s) D_H f_s.$$

The next lemma presents an alternative representative of the formal horizontal differential of f_s, one that is formally identical to differentiating solutions to the flow equation in the smooth case.

Lemma 3.8. *For each $s \in [0,1]$, the matrix function $F(\cdot, s)$ given by*

$$F(p,s) = I_2 + \int_0^s D_H v(f_\sigma(p)) D_H f_\sigma(p)\, \mathrm{d}\sigma.$$

is a formal horizontal differential of f_s.

Proof. Note that, using Lemma 3.7, we have at almost every $p \in \mathbb{H}$ that

$$F(p,s) = I_2 + \int_0^s D_H(v \circ f_\sigma)(p)\, \mathrm{d}\sigma.$$

We need to show that the components of $F(\cdot, s)$ are weak horizontal derivatives as we claim.

To this end, let $\xi \in C_0^\infty(\mathbb{H})$. Using Lemma 3.6 we have $(D_H(v\circ f_s))_{i,j}\xi$ is in $L^1(\mathbb{H}\times[0,1])$ for each choice of $i,j \in \{1,2\}$. This allows application of Fubini's Theorem. Take, for example, the $(1,1)$-component of F,

$$\begin{aligned}
\int F_{1,1}(p,s)\xi(p)\,\mathrm{d}p &= \int \xi + \int\int_0^s X(v\circ f_\sigma)_1(p)\xi(p)\,\mathrm{d}\sigma\,\mathrm{d}p \\
&= \int \xi - \int_0^s \int (v\circ f_\sigma)_1(p) X\xi(p)\,\mathrm{d}p\,\mathrm{d}\sigma \\
&= \int \xi - \int\int_0^s \frac{\mathrm{d}}{\mathrm{d}\sigma}(f_\sigma)_1(p)X\xi(p)\,\mathrm{d}\sigma\,\mathrm{d}p \\
&= \int \xi - \int (f_s)_1(p)X\xi(p)\,\mathrm{d}p + \int xX\xi(p)\,\mathrm{d}p \\
&= -\int (f_s)_1(p)X\xi(p)\,\mathrm{d}p.
\end{aligned}$$

The other components are similar. ∎

Let F be as in Lemma 3.8. Standard product measure arguments imply that $F(p,s) = D_H f_s(p)$ almost everywhere in $\mathbb{H}\times[0,1]$. It follows that, for almost every $p\in\mathbb{H}$, $D_H f_s(p) = F(p,s)$ for almost every $s\in[0,1]$, something we will use later in conjunction with the next lemma (taken unchanged from [4]).

Lemma 3.9. *Let $F, G : [0,1] \to M_n(\mathbb{R})$ be matrix-valued functions. Suppose that F is continuous, that G is integrable, and that*

$$F(s) = I_2 + \int_0^s G(\sigma)F(\sigma)\,\mathrm{d}\sigma$$

for all $s\in[0,1]$. Then

$$\det(F(s)) = \exp\left(\int_0^s \mathrm{tr}(G(\sigma))\,\mathrm{d}\sigma\right)$$

for all $s\in[0,1]$.

We are now ready to assemble the previous string of results into our variational equation (restating our assumptions for the convenience of the reader).

Proposition 3.10. *Assume $\phi : \mathbb{H} \to \mathbb{R}$ satisfies the hypotheses of Proposition 3.3. Further assume that $X\phi, Y\phi \in HW^{1,r}_{\mathrm{loc}}(\mathbb{H})$ for all $1 \leq r < \infty$. Then for all $s \in [0,1]$, we have*

$$\log J_{f_s}(p) = 2 \int_0^s T\phi(f_\sigma(p))\,\mathrm{d}\sigma$$

at almost every $p \in \mathbb{H}$.

Proof. With the above in place, the proof goes through as in the Euclidean case. Let F be as in Lemma 3.8. Let $p \in \mathbb{H}$ be such that (i) $D_H v(f_s(p))$ is integrable on $[0,1]$, (ii) $D_H v(f_s(p)) D_H f_s(p)$ is integrable on $[0,1]$, (iii) $F(p,s) = D_H f_s(p)$ at almost every $s \in [0,1]$ (these properties hold simultaneously almost everywhere).

Now let $G(s) := D_H v(f_s(p))$. Then by (i) G is integrable on $[0,1]$. Let $s \mapsto F(s)$ be defined by $F(s) = F(p,s)$. Then by (ii) F is continuous on $[0,1]$. Furthermore, (iii) allows us to replace $\sigma \to D_H f_\sigma(p)$ with $\sigma \mapsto F(\sigma)$ in the expression for $F(p,s)$,

$$F(s) = F(p,s) = \int_0^s G(\sigma)F(\sigma)\,\mathrm{d}\sigma.$$

Consequently, G and F so defined satisfy the hypotheses of Lemma 3.9. Let $E \subset \mathbb{H}$ be the set at which properties (i)-(iii) hold. Then by Lemma 3.9, and our preceding observations, at each $s \in [0,1]$,

$$\begin{aligned}
\log J_{f_s}(p) = 2\log[\det D_H f_s(p)] &= 2\log[\det F(s)] \\
&= 2\int_0^s \operatorname{tr} D_H v(f_\sigma)\,\mathrm{d}\sigma \\
&= -2\int_0^s \frac{1}{4}[X,Y]\phi(f_\sigma(p))\,\mathrm{d}\sigma \\
&= 2\int_0^s T\phi(f_\sigma(p))\,\mathrm{d}\sigma,
\end{aligned}$$

almost everywhere in E, hence almost everywhere in $\mathbb{H}$. ∎

We will sometimes refer to $\operatorname{tr} D_H v = T\phi$ as the (formal) horizontal divergence of v, writing $\mathrm{div}_H v$ for the same. Constructing a ϕ so that the horizontal divergence of v_ϕ has special properties, in addition to ϕ satisfying the requirements of the current section, will occupy our efforts in Section 4.

4 Approximation

We will consider ourselves given a logarithmic potential, and construct a contact generating potential ϕ for which the following hold. First, it meets the requirements of Proposition 3.3 so that it generates a quasiconformal flow. Second, $X\phi, Y\phi \in HW^{1,r}_{\mathrm{loc}}(\mathbb{H})$, for all $1 \leq r < \infty$, so that the results of Subsection 3.2 hold, in particular Proposition 3.10. Third, the horizontal divergence of v_ϕ approximates the logarithmic potential in a suitable way, so that we may use Proposition 3.10 to link the logarithmic potential to the Jacobian of the quasiconformal flow mapping.

This section is the most granular, and some of the computations deserve to be described as tedious. We begin, therefore, with some elementary results in order to avoid interrupting the argument with these details later. The first is a mild extension of classical differentiation under the integral, using a horizontal derivative, and tailored to our purpose.

Before stating the lemma, let us make the following agreement that is to hold throughout the section: if $V \in \mathrm{H}\mathbb{H}$, and F is a real valued function whose domain is contained in the n-fold product of $\mathbb{H}$ for some $1 \leq n < \infty$, then VF always refers to differentiation in the first coordinate,

$$VF = V_p F(p, q_1, \ldots, q_{n-1}). \tag{4.1}$$

We will be consistent in our use of p for this first coordinate, and continue our convention that $p = (x, y, t)$.

Lemma 4.1. *Let V, $W \in \{X, Y\}$, and $U \subset \mathbb{H} \times \mathbb{H}$ be open. Let $f : U \times \mathbb{H} \to \mathbb{R}$ be continuous, and such that, for each $(p, q) \in U$, there exists compact $\Omega_{p,q}$ with $f(p, q, u) = 0$ whenever u is outside $\Omega_{p,q}$. Let μ be a measure on $\mathbb{H}$, absolutely continuous with respect to the Lebesgue* *measure. Define $F : U \to \mathbb{R}$ by*

$$F(p,q) = \int f(p,q,u)\,\mathrm{d}\mu(u).$$

Then F is continuous. Furthermore, if Vf, WVf exist and are continuous on $U \times \mathbb{H}$, then VF, WVF exist, are continuous on U, and are given by

$$V_pF(p,q) = \int V_pf(p,q,u)\,\mathrm{d}\mu(u),$$
$$V_pW_pF(p,q) = \int V_pW_pf(p,q,u)\,\mathrm{d}\mu(u).$$

Proof. F is a well defined, real valued function as, for each $(p,q) \in U$, we are integrating a continuous function over a compact set. Fix some $(p,q) \in U$. Let $\Omega_0 \subset U$ be a compact neighborhood of (p,q). Given that f, Vf, and VWf are continuous on the compact set $\Omega_0 \times \Omega_{(p,q)}$, then the absolute value of each achieves a maximum on this set. Let $M > 0$ be the largest of these numbers. Then

$$|f(a,b,u)|,\ |V_af(a,b,u)|,\ |W_aV_af(a,b,u)| \le M\mathbb{1}_{\Omega_{p,q}}(u)$$

for all $(a,b,u) \in \Omega_0 \times \mathbb{H}$. The function on the right is integrable over $\mathbb{H}$ with respect to μ. Let (a_k, b_k) be a sequence in Ω_0 with $(a_k,b_k) \to (p,q)$, and define $f_k(u) = f(a_k,b_k,u)$. As f is jointly continuous in its first two coordinates, an application of the Dominated Convergence Theorem gives that F is continuous on U.

As for the first horizontal derivative, VF, let $h_0 > 0$ be such that for all $h \in [-h_0, h_0]$ we have

$$(p + hV_p, q) \in \Omega_0.$$

For each $u \in \mathbb{H}$ define a function, $G_u : [-h_0, h_0] \to \mathbb{R}$,

$$G_u(s) := f(p + sV_p, q, u).$$

At $s \in (-h_0, h_0)$, should the limit exist, we would have

$$G'_u(s) = \lim_{h \to 0} \frac{f(p + (s+h)V_p, q, u) - f(p + sV_p, q, u)}{h} = V_p f(p + sV_p, q, u),$$

where this is all by definition. A first glance suggests we may be in trouble, as we have only assumed derivatives in the horizontal directions of the first coordinate and, a priori, we do not know that V_p is horizontal at $p + sV_p$. We note, however, that X_p depends only on y, and Y_p depends only on x. Consequently, as $p + sV_p$ does not change the relevant coordinate, we have $V_p = V_{p+sV_p}$. By assumption then, the derivative of G_u does exist at s. Furthermore, if h_k is a sequence in $[-h_0, h_0] \setminus \{0\}$, with $h_k \to 0$ as $k \to \infty$, then by the mean value theorem

$$\left| \frac{f(p + h_k V_p, q, u) - f(p, q, u)}{h_k} \right| \leq \sup_{(-h_0, h_0)} |G'_u(s)| \leq M \mathbb{1}_{\Omega_{p,q}}(u),$$

for all k. Another application of the Dominated Convergence Theorem allows us to differentiate under the integral sign as claimed. Continuity on U follows in a similar way to the continuity of F itself.

Having arrived at this point, given that $V_p f(p, q, u)$ must also be zero for u outside $\Omega_{p,q}$, the claim regarding WVF follows from a repetition of the same procedures. ∎

The next lemma is very similar to the first, and so we state it without proof.

Lemma 4.2. *Suppose $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ is continuous, and Xf, Yf exist and are continuous on $\mathbb{H} \times \mathbb{H}$. Let*

$$F(p) := \int f(p, q)\psi(q)\,\mathrm{d}q.$$

for some $\psi \in C_0^\infty(\mathbb{H})$.

Then $F \in HC^1(\mathbb{H})$, *and*

$$X_pF(p) = \int X_pf(p,q)\psi(q)\,\mathrm{d}q, \qquad Y_pF(p) = \int Y_pf(p,q)\psi(q)\,\mathrm{d}q.$$

The preceding two lemmas relied on joint continuity (of both function and derivative) to allow differentiation under the integral in the classical sense. In the next lemma, we want to differentiate under the integral, but only weakly so. We retain joint continuity of the function, but swap joint continuity of the derivative for joint integrability of the weak derivative.

Lemma 4.3. *Suppose* $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ *is continuous, and the distributional derivatives* Xf, Yf *are in* $L^r_{loc}(\mathbb{H} \times \mathbb{H})$ *for all* $1 \leq r < \infty$. *Let*

$$F(p) := \int f(p,q)\psi(q)\,\mathrm{d}q,$$

for some $\psi \in C_0^\infty(\mathbb{H})$. *Then* $F \in HW^{1,r}_{loc}(\mathbb{H})$ *for all* $1 \leq r < \infty$, *and*

$$X_pF(p) = \int X_pf(p,q)\psi(q)\,\mathrm{d}q, \qquad Y_pF(p) = \int Y_pf(p,q)\psi(q)\,\mathrm{d}q.$$

Proof. Suppose V is one of X, Y. To clarify our assumption, for almost every $q \in \mathbb{H}$, there exists an (equivalence class of) almost everywhere defined function(s) on $\mathbb{H}$, $p \mapsto V_pf(p,q)$, such that

$$\int f(p,q)V_p\xi(p)\,\mathrm{d}p = -\int V_pf(p,q)\xi(p)\,\mathrm{d}p$$

for all $\xi \in C_0^\infty(\mathbb{H})$. This gives a function, Vf, defined almost everywhere on $\mathbb{H} \times \mathbb{H}$. Let such a ξ be given. Note that $f \in L^1_{\mathrm{loc}}(\mathbb{H} \times \mathbb{H})$ as f is continuous on $\mathbb{H} \times \mathbb{H}$. By assumption

$Vf \in L^1_{\mathrm{loc}}(\mathbb{H} \times \mathbb{H})$. Clearly then, for ψ as in the statement of the lemma,

$$f(p,q)\psi(q)V_p\xi(p),\; V_pf(p,q)\psi(q)\xi(p) \in L^1(\mathbb{H} \times \mathbb{H}).$$

This allows to use the Fubini theorem as we need to. A first application gives

$$\begin{aligned}\int\int f(p,q)\psi(q)\,\mathrm{d}q\,V_p\xi(p)\,\mathrm{d}p &= \int\int f(p,q)\psi(q)V_p\xi(p)\,\mathrm{d}q\,\mathrm{d}p \\ &= \int\int f(p,q)V_p\xi(p)\,\mathrm{d}p\,\psi(q)\,\mathrm{d}q.\end{aligned}$$

Next we use, for almost every q, the existence of the weak derivative Vf, and a second application of Fubini, to find that

$$\begin{aligned}\int\int f(p,q)V_p\xi(p)\,\mathrm{d}p\,\psi(q)\,\mathrm{d}q &= -\int\int V_pf(p,q)\xi(p)\,\mathrm{d}p\,\psi(q)\,\mathrm{d}q \\ &= -\int\int V_pf(p,q)\psi(q)\xi(p)\,\mathrm{d}q\,\mathrm{d}p \\ &= -\int\int V_pf(p,q)\psi(q)\,\mathrm{d}q\,\xi(p)\,\mathrm{d}p.\end{aligned}$$

Putting these two chains together, we find that F, as defined above, has a weak horizontal derivative of the form claimed.

Moving onto the integrability, let $1 \leq r < \infty$, and $\Omega \subset \mathbb{H}$ be some compact set. Applying Hölder's inequality to the finite measure $|\psi(q)|\,\mathrm{d}q$, we find that

$$\int_\Omega \left| \int V_pf(p,q)\psi(q)\,\mathrm{d}q \right|^r \mathrm{d}p \lesssim \int_\Omega \int |V_pf(p,q)|^r|\psi(q)|\,\mathrm{d}q\,\mathrm{d}p,$$

the finiteness of which follows from our higher integrability assumption on Vf. ∎

We now begin the construction in earnest. Throughout the remainder of the section, the letters V, W will always be used to denote an element of $\{X, Y\}$, with the statements independent of the particular choice.

To be precise, we seek to approximate a given *quasi*-logarithmic potential u, which we may assume is of the form $u = \Lambda_\psi \circ g$ almost everywhere, for some quasiconformal mapping $g \in \mathcal{Q}_0(K)$, and $\psi \in C_0^\infty(\mathbb{H})$. As our statements need hold only almost everywhere, we of course work directly with $\Lambda_\psi \circ g$. Fix then $\psi \in C_0^\infty(\mathbb{H})$ and $g \in \mathcal{Q}_0(K)$. Recall this means that g is a K-quasiconformal mapping such that $g(0) = 0$ and $\|g(q)\| = 1$ for at least one point $q \in \mathbb{H}$ with $\|q\| = 1$.

We begin by working with g (ψ comes into play later). First we define a function that gives a suitably smoothed version of $(p, q) \mapsto d(g(p), g(q))$, from which it is possible to extract some useful estimates.

To this end, fix a function $\xi_0 \in C_0^\infty(\mathbb{H})$, $0 \leq \xi_0 \leq 1$, $\xi_0(p) = 1$ for $p \in B(1/4)$, support$(\xi_0) \subset B(1/2)$. For $q \in \mathbb{H}$, let L_q be left translation, $L_q(p) = qp$, and for $u \in \mathbb{H}$ define $\Gamma_u : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$,

$$\Gamma_u(p, q) = \delta_{d(p,q)^{-1}}(L_{u^{-1}} p).$$

Now define $\lambda_g : \mathbb{H} \times \mathbb{H} \to [0, \infty)$,

$$\lambda_g(p, q) = \begin{cases} \left(\int J_g(u) \xi_0(\Gamma_u(p, q)) \, \mathrm{d}u \right)^{\frac{1}{4}} & \text{if } p \neq q \\ 0 & \text{if } p = q. \end{cases} \tag{4.2}$$

The next lemma summarizes the important properties of λ, it is the first building block in the construction of a suitable ϕ. As g is fixed for the remainder of the section, we will write $\lambda = \lambda_g$.

Before proceeding, we remind the reader that convention (4.1) with regard to derivatives is in place.

Lemma 4.4. *λ is continuous, and such that*

(i)

$$\lambda(p, q) \simeq_K d(g(p), g(q)),$$

for all $p, q \in \mathbb{H}$*;*

(ii) *if* $p \neq q$*,* $V\lambda$*,* $WV\lambda$ *exist and are continuous at* (p,q)*, with*

$$\frac{|V\lambda(p,q)|}{\lambda(p,q)} \lesssim_K \frac{1}{d(p,q)}$$
$$\frac{|WV\lambda(p,q)|}{\lambda(p,q)} \lesssim_K \frac{1}{d(p,q)^2}.$$

Proof. For $p \neq q$ we have, using the definition of λ, (2.10), and (2.7), that

$$\lambda_g(p,q) \leq \left(\int_{B(p,d(p,q)/2)} J_g \right)^{\frac{1}{4}} = |gB(p,d(p,q)/2)|^{\frac{1}{4}} \lesssim d(g(p),g(q)),$$

and

$$\lambda_g(p,q) \geq \left(\int_{B(p,d(p,q)/4)} J_g \right)^{\frac{1}{4}} = |gB(p,d(p,q)/4)|^{\frac{1}{4}} \gtrsim d(g(p),g(q)).$$

Together these give the comparability as in statement (i), and the first inequality alone gives that λ_g is continuous on the diagonal of $\mathbb{H} \times \mathbb{H}$.

Continuity of λ off the diagonal, and the existence and continuity of $V\lambda$, and $WV\lambda$ off the diagonal, follow from Lemma 4.1 with measure $\mathrm{d}\mu(u) = J_g(u)\mathrm{d}u$. In order to complete the proof of statement (ii) we make a series of estimates, beginning with the seed at the core of λ, then working our way through each subsequent layer.

Now fix $p, q \in \mathbb{H}$, $p \neq q$.

We develop some estimates under the assumption that $u \in \mathbb{H}$ satisfies $2d(p,u) \leq d(p,q)$. The following statements are to be considered evaluated at the point (p,q). In the following Γ_u^i and $L_{u^{-1}}^i$ refer to the i^{th} component function of Γ_u and $L_{u^{-1}}$ respectively, whereas d^k refers to the k^{th} power of the distance function. The frequent appearances of d^4 arise simply from the chain rule applied to d.

A computation shows that $|Vd^4| \lesssim d^3$, and $|WVd^4| \lesssim d^2$.

For $i = 1, 2$, we have

$$|V\Gamma_u^i| \lesssim \frac{|VL_{u^{-1}}^i|}{d} + \frac{|Vd^4|}{d^4}.$$

The statement for the third coordinate is

$$|V\Gamma_u^3| \lesssim \frac{|VL_{u^{-1}}^3|}{d^2} + \frac{|L_{u^{-1}}^3 Vd^4|}{d^6}.$$

It follows that

$$|V\Gamma_u^i| \lesssim d^{-1} \tag{4.3}$$

for all $i \in \{1, 2, 3\}$.

Further,

$$|WV\Gamma_u^i| \lesssim \frac{|WVL_{u^{-1}}^i|}{d} + \frac{|VL_{u^{-1}}^i Wd^4 + WL_{u^{-1}}^i Vd^4 + L_{u^{-1}}^i WVd^4|}{d^5} + \frac{|L_{u^{-1}}^i Vd^4 Wd^4|}{d^9},$$

when $i = 1, 2$, and

$$|WV\Gamma_u^3| \lesssim \frac{|WVL_{u^{-1}}^3|}{d^2} + \frac{|VL_{u^{-1}}^3 Wd^4 + WL_{u^{-1}}^3 Vd^4 + L_{u^{-1}}^3 WVd^4|}{d^6} + \frac{|L_{u^{-1}}^3 Vd^4 Wd^4|}{d^{10}}.$$

Here the conclusion is

$$|WV\Gamma_u^i| \lesssim d^{-2} \tag{4.4}$$

for all $i \in \{1, 2, 3\}$.

As ξ_0 is smooth and compactly supported, there are bounds on the size of its derivatives. As we have fixed ξ_0, and ξ_0 does not depend on any varying quantity or function we introduce, we may take these bounds to be absolute constants. With this in mind, observe that $V(\xi_0 \circ \Gamma_u) = (\nabla\xi_0)(\Gamma_u) \cdot V\Gamma_u$, where we write $V\Gamma_u$ for $(V\Gamma_u^1, V\Gamma_u^2, V\Gamma_u^3)$, so that, by (4.3),

$$|V(\xi_0 \circ \Gamma_u)| \lesssim d^{-1}. \tag{4.5}$$

Similarly, $WV(\xi_0 \circ \Gamma_u) = \sum_{i=1}^3 \left[((\nabla \partial_i \xi_0)(\Gamma_u) \cdot V\Gamma_u) V\Gamma_u^i + \partial_i \xi_0(\Gamma_u) WV\Gamma_u^i\right]$, so that, this time by (4.4),

$$|WV(\xi_0 \circ \Gamma_u)| \lesssim d^{-2}. \tag{4.6}$$

By the definition of ξ_0, we have $\xi_0(\Gamma_u(p,q)) = 0$ for all u such that $2d(p,u) > d(p,q)$. Consequently, at our fixed (p,q), $p \neq q$,

$$\lambda(p,q) = \left(\int_{B(p,d(p,q)/2)} J_g(u) \xi_0(\Gamma_u(p,q)) \,\mathrm{d}u \right)^{\frac{1}{4}}.$$

Noting by part (i) that $\lambda > 0$ off the diagonal, using Lemma 4.1 to differentiate under the integral when necessary, and applying estimate (4.5),

$$\frac{|V\lambda|}{\lambda} \lesssim \frac{|V\lambda^4|}{\lambda^4} \lesssim \frac{1}{d} \frac{\int_{B(p,d/2)} J_g}{\int_{B(p,d/4)} J_g} = \frac{1}{d} \frac{|gB(p,d/2)|}{|gB(p,d/4)|}.$$

Using (2.7) again, $|gB(p,d(p,q)/2)| \lesssim d(g(p),g(q))^4$, and $|gB(p,d(p,q)/4)| \gtrsim d(g(p),g(q))^4$. Consequently,

$$\frac{|V\lambda|}{\lambda} \lesssim \frac{1}{d}$$

as required.

Similarly, but this time using (4.6) also,

$$\frac{|WV\lambda|}{\lambda} \lesssim \frac{|WV\lambda^4|}{\lambda^4} + \frac{|W\lambda^4 V\lambda^4|}{\lambda^8} \lesssim \frac{1}{d^2} \frac{\int_{B(p,d/2)} J_g}{\int_{B(p,d/4)} J_g} + \frac{1}{d^2} \left(\frac{\int_{B(p,d/2)} J_g}{\int_{B(p,d/4)} J_g} \right)^2,$$

from which

$$\frac{|WV\lambda|}{\lambda} \lesssim \frac{1}{d^2}$$

follows in the same manner. ∎

With λ in hand, we continue to build on top of it. Let $U_g := \{(p,q) \,:\, p \neq g^{-1}(q)\}$. Note, as g^{-1} is continuous, U_g is open. Define $\eta_g : U_g \to \mathbb{R}$,

$$\eta_g(p,q) = -\log \lambda(p, g^{-1}(q)). \tag{4.7}$$

Observe that the curious form of the estimates on the derivatives of λ were because we had logarithmic derivatives in mind. Write $\eta = \eta_g$. It follows from Lemma 4.4 part (i),

$$|\eta(p,q) + \log[d(g(p), q)]| \lesssim_K 1, \tag{4.8}$$

so that

$$|\eta(p,q)| \lesssim_K 1 + |\log[d(g(p), q)]|. \tag{4.9}$$

Using Lemma 4.1 again, we have that $V\eta$, $WV\eta$ exist and are continuous on U_g, with

$$|V\eta(p,q)| = \frac{|V\lambda(p, g^{-1}(q))|}{\lambda(p, g^{-1}(q))} \lesssim_K \frac{1}{d(p, g^{-1}(q))}, \tag{4.10}$$

and

$$|WV\eta(p,q)| \leq \frac{|WV\lambda(p, g^{-1}(q))|}{\lambda(p, g^{-1}(q))} + \frac{|W\lambda(p, g^{-1}(q))V\lambda(p, g^{-1}(q))|}{\lambda(p, q^{-1}(q))^2} \lesssim_K \frac{1}{d(p, g^{-1}(q))^2}. \tag{4.11}$$

We are close to defining the potential ϕ, and choose notation to reflect this. Let $\tilde{\phi}_g : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ be defined by,

$$\tilde{\phi}_g(p,q) = \begin{cases} \eta(p,q)(g^{-1}(q)^{-1}p)_3 & \text{if } p \neq g^{-1}(q) \\ 0 & \text{if } p = g^{-1}(q). \end{cases} \tag{4.12}$$

Here $(g^{-1}(q)^{-1}p)_3$ is the third component of $g^{-1}(q)^{-1}p$.

At various points in the remainder of the section, we will have use for the following elementary lemmas that we state without proof.

Lemma 4.5. *If $u, u' \in \mathbb{H}$ then*

$$|u_i| \le \|u\|,\ i = 1, 2,\ \text{and}\ |u_3| \le \|u\|^2,$$

$$\|u\| \lesssim 1 + |u|,\ \textit{and}$$

$$|u^{-1}u'| \le |u| + |u'| + |u||u'|.$$

Lemma 4.6. *If $s, s_1, s_2, r \in (0, \infty)$, then*

$$\log^+(s_1 s_2) \le \log^+ s_1 + \log^+ s_2,$$

$$\log^+(s_1 + s_2) \le 1 + \log^+ s_1 + \log^+ s_2,$$

$$\log^+(s^r) = r \log^+ s,$$

$$s \lesssim 1 + s \log^+ s,\ \text{and}\ \log^+ s \lesssim 1 + s \log^+ s.$$

We will need several regularity statements on $\tilde{\phi} = \tilde{\phi}_g$ and prefer to break them into small pieces.

Lemma 4.7. *$\tilde{\phi}$ is continuous on $\mathbb{H} \times \mathbb{H}$.*

Proof. If $(p, q) \in \mathbb{H} \times \mathbb{H}$ is such that $p \neq g^{-1}(q)$ then, by Lemma 4.4 part (i), $\lambda(p, g^{-1}(q)) \neq 0$, and so $\tilde{\phi}$ is continuous at (p, q) by the continuity of log away from 0. Suppose, therefore, that $q = g(p)$, so that $\tilde{\phi}(p, q) = 0$. Let (p_k, q_k) be a sequence of points limiting on (p, q), and such that for all k, $(p_k, q_k) \in U_g$ (the presence of points outside U_g would not disturb the argument, as $\tilde{\phi}$ is zero at these points, and we wish to show that $\tilde{\phi}(p_k, q_k)$ converges to zero). Then, for all k,

$$\tilde{\phi}(p_k, q_k) = \eta(p_k, q_k)(g^{-1}(q_k)^{-1} p_k)_3.$$

It follows from (4.9) that

$$|\tilde{\phi}(p_k, q_k)| \lesssim |(g^{-1}(q_k)^{-1} p_k)_3| + |\log[d(g(p_k), q_k)]||(g^{-1}(q_k)^{-1} p_k)_3|.$$

It is obvious, given our assumptions, that the first term on the right hand side tends to zero as $k \to \infty$, therefore, we only need work on the second term. Lemma 4.5 reminds us that for all $u \in \mathbb{H}$ we have $|u_3| \leq \|u\|^2$, and so

$$|\log[d(g(p_k), q_k)]||(g^{-1}(q_k)^{-1}p_k)_3| \leq |\log[d(g(p_k), q_k)]|d^2(p_k, g^{-1}(q_k)).$$

We complete the proof using the local Hölder continuity of quasiconformal mappings. As we may assume the (p_k, q_k) are close to the point (p, q), then there exists $0 < R < \infty$ such that for all k, $g(p_k), q_k \in B(R)$. There exists, therefore, $\alpha > 0$ such that for all k,

$$d(p_k, g^{-1}(q_k)) = d(g^{-1}(g(p_k)), g^{-1}(q_k)) \lesssim d^{\alpha}(g(p_k), q_k).$$

Putting these last two observations together, we get

$$|\log[d(g(p_k), q_k)]||(g^{-1}(q_k)^{-1}p_k)_3| \lesssim \log[d(g(p_k), q_k)]|d^{2\alpha}(g(p_k), q_k)$$

and it is now easy to see that this goes to 0, as $k \to \infty$. ∎

Lemma 4.8. *The derivatives $X\tilde{\phi}$, $Y\tilde{\phi}$ exist and are continuous on $\mathbb{H} \times \mathbb{H}$.*

Proof. Existence and continuity is immediate from previous observations if we are at a point (p, q) such that $q \neq g(p)$. Let us consider these things at a point of the form $(p, g(p))$. We will do the calculations for X_p, those for Y_p are similar.

Working with the definition of the derivative we find

$$\begin{aligned} X_p\tilde{\phi}(p, q) &= \lim_{h \to 0} \frac{\tilde{\phi}(p\delta_h \exp X_0, g(p))}{h} \\ &= \lim_{h \to 0} \frac{\eta(p\delta_h \exp X_0, g(p))(p^{-1}p\delta_h \exp X_0)_3}{h} \\ &= 0, \end{aligned}$$

and so the derivative exists. Let us consider continuity, the argument is similar to that of the continuity of $\tilde{\phi}$ itself. As before, let $(p_k, q_k) \to (p, g(p))$ through points in U_g. Then,

$$X_{p_k}\tilde{\phi}(p_k, q_k) = (X_{p_k}\eta(p_k, q_k))(g^{-1}(q_k)^{-1}p_k)_3 + 2\eta(p_k, q_k)(g^{-1}(q_k)^{-1}p_k)_2.$$

Using (4.9), (4.10), and Lemma 4.5, this leads to,

$$|X_{p_k}\tilde{\phi}(p_k, q_k)| \lesssim d(p_k, g^{-1}(q_k)) + |\log[d(g(p_k), q_k)]| d(p_k, g^{-1}(q_k)).$$

Now use Hölder continuity, as we did in Lemma 4.7, to conclude that the right hand side goes to 0 as $k \to \infty$. It follows that $X_p\tilde{\phi}(p, q)$ is continuous on $\mathbb{H} \times \mathbb{H}$, and the same is true for $Y_p\tilde{\phi}(p, q)$. ∎

Lemma 4.9. *Let $V, W \in \{X, Y\}$. For each $q \in \mathbb{H}$, the distributional derivative $WV\tilde{\phi}$ defines, via $(p, q) \mapsto W_pV_p\tilde{\phi}(p, q)$, an element of $L^r_{loc}(\mathbb{H} \times \mathbb{H})$ for all $1 \leq r < \infty$.*

Proof. To elaborate on the statement of the lemma, our aim is to show that for each $q \in \mathbb{H}$ there is an almost everywhere defined function $p \mapsto W_pV_p\tilde{\phi}(p, q)$, such that for all $\xi \in C_0^\infty(\mathbb{H})$,

$$\int W_pV_p\tilde{\phi}(p, q)\xi(p)\,\mathrm{d}p = -\int V_p\tilde{\phi}(p, q)W_p\xi(p)\,\mathrm{d}p.$$

This defines almost everywhere a function $(p, q) \mapsto W_pV_p\tilde{\phi}(p, q)$ on $\mathbb{H} \times \mathbb{H}$, and we show that $WV\tilde{\phi} \in L^r_{\text{loc}}(\mathbb{H} \times \mathbb{H})$.

Let $q \in \mathbb{H}$ be given. We have seen that $V\tilde{\phi}_q : p \mapsto V_p\tilde{\phi}(p, q)$ is continuous. Indeed, at $p \neq g^{-1}(q)$, $V\tilde{\phi}_q$ is continuously differentiable in all directions. It follows that $V\tilde{\phi}_q$ is absolutely continuous on almost every integral curve of the horizontal, left-invariant, vector field determined by W. Of course, we have not defined the measure on this fibration of $\mathbb{H}$, the details can be found in [20], however, suffice to say $g^{-1}(q)$ lies on only one curve, and a single curve has measure zero.

It follows, see pages 41-42 of [20], that the almost everywhere defined classical derivative $WV\tilde{\phi}_q$ is the distributional derivative. Let us record those derivatives. Let $u := g^{-1}(q)^{-1}p$ so that, for $p \neq g^{-1}(q)$, $\tilde{\phi} = u_3\eta$. Then, for $p \neq g^{-1}(q)$,

$$
\begin{aligned}
XX\tilde{\phi} &= u_3XX\eta + 4u_2X\eta & XY\tilde{\phi} &= u_3XY\eta + 2u_2Y\eta - 2u_1X\eta - 2\eta \\
YX\tilde{\phi} &= u_3YX\eta + 2u_2Y\eta - 2u_1X\eta + 2\eta & YY\tilde{\phi} &= u_3YY\eta - 4u_1X\eta
\end{aligned}
$$

These expressions give measurable, almost everywhere defined functions on $\mathbb{H} \times \mathbb{H}$, and we now consider the local integrability. Given the estimates worked out above, (4.10), (4.11), and using Lemma 4.5, we have in the case $W = V$ that

$$|WV\tilde{\phi}| \lesssim_K 1$$

almost everywhere on $\mathbb{H} \times \mathbb{H}$. In the case $W \neq V$, additionally using (4.9),

$$|WV\tilde{\phi}| \lesssim_K 1 + |\log[d(g(p), q)]|,$$

almost everywhere on $\mathbb{H} \times \mathbb{H}$. Clearly, we need only be concerned about the case $W \neq V$.

Let $1 \leq r < \infty$. Let $\Omega_1 \subset \mathbb{H}$ be compact. We will show that

$$\int_{\Omega_1} |\log[d(g(p), q)]|^r \, \mathrm{d}p \lesssim h(q)$$

for $h \in L^1_{\mathrm{loc}}(\mathbb{H})$. The claimed local integrability to the power r on $\mathbb{H} \times \mathbb{H}$ follows.

To this end, let $r_1 > 1$ be the exponent appearing in the reverse Hölder inequality for g^{-1}, (2.11) , and r_2 the conjugate exponent. Let $0 < R < \infty$ be such that $g\Omega_1 \subset B(R)$. Then

$$\begin{aligned}
\int_{\Omega_1} |\log[d(g(p),q)]|^r \,\mathrm{d}p &= \int_{g\Omega_1} |\log[d(p,q)]|^r J_{g^{-1}}(p)\,\mathrm{d}p \\
&\leq \left(\int_{g\Omega_1} |\log[d(p,q)]|^{rr_2}\,\mathrm{d}p\right)^{\frac{1}{r_2}} \left(\int_{g\Omega_1} J_{g^{-1}}(p)^{r_1}\,\mathrm{d}p\right)^{\frac{1}{r_1}} \\
&\lesssim \left(\int_{g\Omega_1} |\log[d(p,q)]|^{rr_2}\,\mathrm{d}p\right)^{\frac{1}{r_2}} \left(\frac{1}{|B(R)|}\int_{B(R)} J_{g^{-1}}(p)^{r_1}\,\mathrm{d}p\right)^{\frac{1}{r_1}} \\
&\lesssim \left(\int_{g\Omega_1} |\log[d(p,q)]|^{rr_2}\,\mathrm{d}p\right)^{\frac{1}{r_2}},
\end{aligned}$$

where the implied constants depend on a variety of things, but not q. Now let $\Omega_2 \subset \mathbb{H}$ be compact, and observe using Hölder's inequality that

$$\int_{\Omega_2}\left(\int_{g\Omega_1} |\log[d(p,q)]|^{rr_2}\,\mathrm{d}p\right)^{\frac{1}{r_2}}\mathrm{d}q \lesssim \left(\int_{\Omega_2}\int_{g\Omega_1} |\log[d(p,q)]|^{rr_2}\,\mathrm{d}p\,\mathrm{d}q\right)^{\frac{1}{r_2}}.$$

Furthermore,

$$\int_{g\Omega_1} |\log[d(p,q)]|^{rr_2}\,\mathrm{d}p = \int_{g\Omega_1} |\log\|q^{-1}p\||^{rr_2}\,\mathrm{d}p = \int_{q^{-1}g\Omega_1} |\log\|p\||^{rr_2}\,\mathrm{d}p.$$

As Ω_2 is compact, there exists $0 < R' < \infty$ such that $q^{-1}g\Omega_1 \subset B(R')$ for all $q \in \Omega_2$. Consequently, by formula (1.16),

$$\int_{\Omega_2}\int_{g\Omega_1} |\log[d(p,q)]|^{rr_2}\,\mathrm{d}p\,\mathrm{d}q \lesssim \int_{B(R')} |\log\|p\||^{rr_2}\,\mathrm{d}p \lesssim \int_0^{R'} |\log(s)|^{rr_2} s^3\,\mathrm{d}s < \infty. \quad \blacksquare$$

We saw in Proposition 3.3 that one of the requirements for a vector field to have quasiconformal flow was a size constraint (actually basic to the long-time existence of the flow rather than any geometric properties). We take an important step toward establishing such a constraint in the next lemma. Given the nature of that estimate, it is at this point that we start putting the Euclidean norm on points of the Heisenberg group (see the remark preceding the proof of Proposition 3.3).

Lemma 4.10. *Given $R > 0$, for each $q \in B(R)$, we have for all $p \in \mathbb{H}$,*

$$|\tilde{\phi}(p,q) - \frac{1}{2}xX\tilde{\phi}(p,q) - \frac{1}{2}yY\tilde{\phi}(p,q)| \lesssim_{K,R} 1 + |p| \log^{+} |p|.$$

Proof. Let $R > 0$ be given. Let $p, q \in \mathbb{H}$, with $\|q\| < R$. Recall that we are assuming $g \in \mathcal{Q}_0(K)$, and the same is therefore true of g^{-1}. By Lemma 2.3, this means that both $gB(R)$, and $g^{-1}B(R)$ are contained in a ball, the radius of which depends only on K and R. We will often, therefore, be able to replace a dependence on one of $\|q\|$, $\|g(q)\|$, or $\|g^{-1}(q)\|$ with a dependence on K and R. That said, given our aim, dependence of constants on either of K or R will typically not be commented on.

Let

$$F := \tilde{\phi}(p,q) - \frac{1}{2}xX\tilde{\phi}(p,q) - \frac{1}{2}yY\tilde{\phi}(p,q),$$

and $R' > 0$ such that $g^{-1}B(R+1) \subset B(R')$. Such an R' depends only on K and R as guaranteed by Lemma 2.3.

Let $u := g^{-1}(q)^{-1}p$. We calculate,

$$\begin{aligned} F &= (g^{-1}(q)_2 x - g^{-1}(q)_1 y)\eta - \frac{1}{2}u_3(yY\eta + xX\eta) + u_3\eta \\ &= (g^{-1}(q)_2 u_1 - g^{-1}(q)_1 u_2 + u_3)\eta - \frac{1}{2}u_3(u_2 Y\eta + u_1 X\eta + g^{-1}(q)_2 Y\eta + g^{-1}(q)_1 X\eta). \end{aligned}$$

It follows using (4.9), (4.10), and Lemma 4.5, that

$$\begin{aligned} |F| &\lesssim |u|(1 + |\log \|q^{-1}g(p)\||) + \frac{|u_3|}{\|u\|}(\|u\| + 1) \\ &\lesssim 1 + |u| + |u||\log \|q^{-1}g(p)\||. \end{aligned}$$

We break the next part of the proof into two cases.

Case 1: $\|p\| > R'$. If $\|q^{-1}g(p)\| = d(g(p), q) < 1$, then $g(p) \in B(R+1)$, and so $p = g^{-1}(g(p)) \in B(R')$, a contradiction. Therefore, in the case that $\|p\| > R'$ we have

$$|F| \lesssim 1 + |u| + |u| \log^+ \|q^{-1}g(p)\|.$$

Case 2: $\|p\| \leq R'$. Let $R'' > 0$ be such that $gB(R') \subset B(R'')$ (again, ultimately such an R'' depends only on K and R). Then, using Lemma 2.4, there exists $\alpha > 0$ dependent on K only, such that

$$\|g^{-1}(b)^{-1}g^{-1}(a)\| \lesssim \|b^{-1}a\|^{\alpha}$$

whenever a, $b \in B(R'')$.

Assuming, as we may, that $R'' \geq R' \geq R$, we now have p, q, $g^{-1}(q)$, $g(p) \in B(R'')$. It follows from 1.15 that $|u| \lesssim_{R''} \|u\|$, and so

$$|u| \lesssim \|u\| = \|g^{-1}(q)^{-1}g^{-1}(g(p))\| \lesssim \|q^{-1}g(p)\|^{\alpha}.$$

We now subdivide case 2. Case 2 (a): $\|q^{-1}g(p)\| < 1$. In this circumstance, we have

$$|u||\log \|q^{-1}g(p)\|| \lesssim \sup_{\|q^{-1}g(p)\|<1} \|q^{-1}g(p)\|^{\alpha}|\log \|q^{-1}g(p)\| \lesssim 1.$$

Case 2 (b): $\|q^{-1}g(p)\| \geq 1$. This time it is immediate that

$$|u||\log \|q^{-1}g(p)\|| = |u| \log^+ \|q^{-1}g(p)\|.$$

Consequently, in each case, we have

$$|F| \lesssim 1 + |u| + |u| \log^+ \|q^{-1}g(p)\|. \tag{4.13}$$

Using Lemma 4.5, we have

$$|u| = |g^{-1}(q)^{-1}p| \leq |g^{-1}(q)| + |p| + |g^{-1}(q)||p| \lesssim 1 + |p|. \tag{4.14}$$

Furthermore, (2.8) tells us that

$$\|q^{-1}g(p)\| \lesssim 1 + \|g(p)\| \lesssim 1 + \|p\|^{K^{2/3}},$$

which, combined with Lemma 4.6, gives

$$\begin{aligned} \log^+ \|q^{-1}g(p)\| &\lesssim 1 + \log^+ \|p\| \\ &\lesssim 1 + \log^+ |p|. \end{aligned}$$

Putting this last observation together with (4.13) and (4.14), we find

$$\begin{aligned} |F| &\lesssim 1 + |p| + |p| \log^+ |p| \\ &\lesssim 1 + |p| \log^+ |p|. \end{aligned}$$

∎

At this point we bring $\psi \in C_0^\infty(\mathbb{H})$ into the picture. Recall that ψ is the density of the measure associated with our given quasi-logarithmic potential. Let $R > 0$ be such that $\mathrm{support}(\psi) \subset B(R)$.

Define $\phi^1_{g,\psi} : \mathbb{H} \to \mathbb{R}$,

$$\phi^1_{g,\psi}(p) = \int \tilde{\phi}(p,q)\psi(q)\,\mathrm{d}q. \tag{4.15}$$

We established regularity of $\tilde{\phi}$ in such a way that it now transfers easily to $\phi^1 = \phi^1_{g,\psi}$.

Lemma 4.11. $\phi^1 \in HC^1(\mathbb{H})$.

Proof. This follows from Lemmas 4.2, 4.7 and 4.8. ∎

Lemma 4.12. *$X\phi, Y\phi \in HW^{1,r}_{loc}(\mathbb{H})$ for all $1 \le r \le \infty$ and, if $V, W \in \{X, Y\}$,*

$$V_p W_p \phi^1(p) = \int V_p W_p \tilde{\phi}(p,q)\psi(q)\,\mathrm{d}q.$$

Proof. This follows from Lemmas 4.3 and 4.9. ∎

We remind the reader that we write v_ϕ for the vector field generated by potential ϕ as in (3.1). Letting

$$v_{\phi^1} = v_1\partial_x + v_2\partial_y + v_3\partial_t \tag{4.16}$$

define v_1, v_2, v_3, set

$$\phi^2(x,y,t) = c_1 - 4c_2 y + 4c_3 x, \tag{4.17}$$

with

$$(c_1, c_2, c_3) := (v_1(0), v_2(0), v_3(0)).$$

Lastly, define $\phi_{g,\psi} : \mathbb{H} \to \mathbb{R}$,

$$\phi_{g,\psi}(p) = \phi^1(p) - \phi^2(p). \tag{4.18}$$

We have finished adding layers, and are now in a position to demonstrate that this last function has all desired properties of an approximating contact generating potential. We separate this conclusion into two propositions, one summarizing the important properties of the vector fields, and the other the important properties of the flow. As the concluding results of this section, we restate our assumptions in each.

Proposition 4.13. *Given $g \in \mathcal{Q}_0(K)$ for some $K \ge 1$, and $\psi \in C_0^\infty(\mathbb{H})$ with* support$(\psi) \subset$ *$B(R)$ for some $R > 0$, define $\phi = \phi_{g,\psi}$ as in* (4.2), (4.7), (4.12), (4.15), (4.17), *and* (4.18). *Then*

$$|v_\phi(p)| \lesssim_{K,R} \|\psi\|_1(1 + |p|\log^+|p|),$$

for all p, and

$$\mathrm{div}_H v_\phi = \Lambda_\psi \circ g + \zeta$$

with $\zeta \in L^\infty(\mathbb{H})$ *such that* $\|\zeta\|_\infty \lesssim_K \|\psi\|_1$.

Proof. With $R > 0$ as in the statement, we have

$$\phi^1(p) = \int_{B(R)} \tilde{\phi}(p,q)\psi(q)\,\mathrm{d}q.$$

Using Lemma 4.2, with v_1, v_2, v_3 defined as in (4.16), we find that

$$v_3(p) = \int_{B(R)} \left(\tilde{\phi}(p,q) - \frac{1}{2}xX\tilde{\phi}(p,q) - \frac{1}{2}yY\tilde{\phi}(p,q)\right)\psi(q)\,\mathrm{d}q.$$

Applying Lemma 4.10 we have

$$|v_3(p)| \lesssim_{K,R} \|\psi\|_1(1 + |p|\log^+|p|).$$

The computations for v_1, v_2 are similar, but easier, and so we omit them.

As

$$v_{\phi^2} = c_2\partial_x + c_3\partial_y + (c_1 - 2c_2 y + 2c_3 x)\,\partial_t,$$

we find

$$\begin{aligned} |v_\phi| &\le |v_{\phi^1}| + |v_{\phi^2}| \\ &\lesssim \|\psi\|_1(1 + |p|\log^+|p|) + \|\psi\|_1(1 + |p|) \\ &\lesssim \|\psi\|_1(1 + |p|\log^+|p|). \end{aligned}$$

Moving onto the second part of the conclusion, let $p, q \in \mathbb{H}$ be such that $p \neq g^{-1}(q)$. Define

$u = g^{-1}(q)^{-1}p$. Then,

$$\begin{aligned} T\tilde{\phi} &= \eta + u_3 T\eta \\ &= \eta - \frac{1}{4}u_3[X,Y]\eta. \end{aligned}$$

Consequently, using Lemma 4.12,

$$T\phi^1 = \Lambda_\psi \circ g + \zeta,$$

where

$$\zeta(p) = \int \left[\eta(p,q) + \log[d(g(p),q)] - \frac{1}{4}u_3(XY - YX)\eta(p,q)\right] \psi(q)\, \mathrm{d}q.$$

ζ is easily seen to be measurable, and it follows from (4.8), (4.11), and Lemma 4.5 that ζ is essentially bounded,

$$\|\zeta\|_\infty \lesssim_K \|\psi\|_1.$$

As $T\phi^2 = 0$, the proof is concluded by remembering that $\mathrm{div}_H v_\phi = T\phi$. ∎

Proposition 4.14. *Assume the same hypotheses as Proposition 4.13. Then v_ϕ generates a well defined flow of homeomorphisms. Further, if h_s, $0 \leq s < \infty$, are the time-s flow mappings of v_ϕ, then for all s, $h_s(0) = 0$, and h_s is quasiconformal, with $K(h_s) \leq e^{C\|\psi\|_1 s}$, $C \geq 0$ dependent on K only. Lastly, for all $s \in [0,1]$,*

$$\log J_{h_s}(p) = 2\int_0^s T\phi(h_\sigma(p))\, \mathrm{d}\sigma, \tag{4.19}$$

at almost every $p \in \mathbb{H}$.

Proof. The coefficients c_1, c_2, c_3 were chosen precisely so that $v_\phi(0) = 0$, and consequently, $h_s(0) = 0$ for all s also.

Deducing that the flow is quasiconformal, with the required estimates on the dilatation, comes down to verifying that everything is in place in order to invoke Proposition 3.3.

Given the niceness of ϕ^2, continuity of ϕ and its first horizontal derivatives follows from Lemma 4.11.

The size constraint on the vector field itself was contained in Proposition 4.13. Computations that are similar to (but easier than, and so we omit them) Lemma 4.10 give that

$$|\phi(p)| \lesssim \|p\|^2 \log \|p\|,$$

and

$$|Z\phi(p)| \lesssim \|p\| \log \|p\|.$$

Let $u := g^{-1}(q)^{-1}p$. Then

$$\Re ZZ\tilde{\phi} = u_1 Y\eta + u_2 X\eta + u_3(1/4)(XX - YY)\eta,$$

and

$$\Im ZZ\tilde{\phi} = u_1 X\eta - u_2 Y\eta - u_3(1/4)(XY - YX)\eta.$$

Consequently, using Lemmas 4.2 and 4.3, (4.10), (4.11), and Lemma 4.5,

$$\sqrt{2}\|ZZ\phi^1\|_\infty \lesssim_K \|\psi\|_1.$$

Further,

$$\sqrt{2}\|ZZ\phi^2\|_\infty = 0,$$

which should come as no surprise, as ϕ^2 generates a flow of left translation mappings which are conformal. The estimate on the dilatation of the flow mapping follows.

That (4.19) holds follows from Proposition 3.10 so long as we have the required integrability of the weak second horizontal derivatives of ϕ. This follows from Lemma 4.12. ∎

5 Iteration and Convergence

With the large part of the technical work behind us, we are ready to construct the mappings which achieve comparability.

In a first case, the desired mapping f is found in the limit of a sequence of mappings f_m, with each f_m the composition of m mappings, the large part of each generated as the vector flow over a time step of length $1/m$. The arguments consider the competition between an accumulation of a quantity in one direction, verses the contracting effects of a diminishing time step in the other. We will see that, by keeping the size of our measures small enough, we have enough uniformity in our estimates, that the contracting effect of the time step wins every time.

We begin, however, by stating those results on logarithmic potentials we need in the sequel.

5.1 Logarithmic Potentials

Statements analogous to those below were proved in the Euclidean case in [4]. Those proofs go through unchanged (with a like for like replacement of corresponding objects) and so we do not repeat them here.

Recall that, if $\mathrm{d}\mu(q) = \psi(q)\,\mathrm{d}q$ for a measurable function ψ, then we will often write Λ_ψ in place of Λ_μ, where Λ_μ is defined in (1.5).

Lemma 5.1. *Let $\psi \in L^\infty(\mathbb{H})$ with compact support. Then Λ_ψ is Lipschitz continuous.*

Occasionally we will need to smooth a measure. Let $\Psi \in C_0^\infty(\mathbb{H})$ be such that $\mathrm{support}(\Psi) \subset B(1)$, and $\int \Psi = 1$. Now, for each $k \in \mathbb{N}$, let

$$\Psi_k(p) := k^4 \Psi(\delta_k(p)). \tag{5.1}$$

Given a finite, signed Radon measure μ, the k^{th} smooth regularization of μ is

$$\psi_k(p) = \int \Psi_k(q^{-1}p)\,\mathrm{d}\mu(q). \tag{5.2}$$

Lemma 5.2. *Suppose $u = \Lambda_\mu \circ g$ almost everywhere is a quasi-logarithmic potential, g a K-quasiconformal mapping. For each $k \in \mathbb{N}$ define $u_k = \Lambda_{\psi_k} \circ g$ where ψ_k is the k^{th} smooth regularization of μ as in* (5.2)*. Then there exists $\theta = \theta(K) > 0$, such that, for every $0 < \beta < \theta/\|\mu\|$, the function $e^{\beta u}$ is locally integrable, and for every ball $B \subset \mathbb{H}$, we have*

$$\int_B |e^{\beta u_k} - e^{\beta u}| \to 0 \quad as \quad k \to \infty.$$

Lemma 5.3. *Suppose $u = \Lambda_\mu \circ g$ almost everywhere is a quasi-logarithmic potential, g a K-quasiconformal mapping. For each $k \in \mathbb{N}$ define $u_k = \Lambda_{\mu_k} \circ g$ where $\mu_k := \mu\big|_{B(k)}$. Then there exists $\theta = \theta(K) > 0$, such that, for every $0 < \beta < \theta/\|\mu\|$, the function $e^{\beta u}$ is locally integrable, and for every ball $B \subset \mathbb{H}$, we have*

$$\int_B |e^{\beta u_k} - e^{\beta u}| \to 0 \quad as \quad k \to \infty.$$

5.2 Reduction of the Main Theorem

We may reduce Theorem 1.2 to the following proposition.

Proposition 5.4. *Given $K \geq 1$, there exist $\epsilon = \epsilon(K) > 0$, and $K' = K'(K) \geq 1$, such that, if u is a quasi-logarithmic potential, $u = \Lambda_\mu \circ g$ almost everywhere, with $\|\mu\| \leq \epsilon$, and $g \in \mathcal{Q}_0(K)$, then there is a K'-quasiconformal mapping f with*

$$J_f \simeq_K e^{2u}$$

almost everywhere.

Note that the only difference between this proposition and Theorem 1.2 is that we assume $g \in \mathcal{Q}_0(K)$ as opposed to being simply a K-quasiconformal mapping. Recall that $g \in \mathcal{Q}_0(K)$ if and only if g is a K-quasiconformal mapping such that $g(0) = 0$ and there exists $p_g \in S(1)$ with $g(p_g) \in S(1)$.

Let us assume Proposition 5.4 and explain why it implies Theorem 1.2. Let $K \geq 1$ be given, $\epsilon > 0$ as in Proposition 5.4, and u a quasi-logarithmic potential, $u = \Lambda_\mu \circ g$ almost everywhere, with μ a finite, signed Radon measure such that $\|\mu\| \leq \epsilon$, and g a K-quasiconformal mapping.

Pick $q_0 \in \mathbb{H}$ such that $\|g(g^{-1}(0)q_0)\| = 1$. It is automatic that $q_0 \neq 0$. Let $p_0 := \delta_{\|q_0\|^{-1}}(q_0)$. Note that $\|p_0\| = 1$, and $q_0 = \delta_{\|q_0\|}(p_0)$. Now define

$$h(p) = g(g^{-1}(0)\delta_{\|q_0\|}(p)).$$

As g is K-quasiconformal, so is h. Also, as is easily checked, $h \in \mathcal{Q}_0(K)$, with $p_h = p_0$. Let $u_h = \Lambda_\mu \circ h$ almost everywhere. It follows from Proposition 5.4 that there exists a quasiconformal mapping f_h such that

$$J_{f_h} \simeq_K e^{2u_h} \tag{5.3}$$

almost everywhere, with $K(f)$ dependent on K only.

Given the definition of h, we see that

$$g = h \circ \delta_{\|q_0\|^{-1}} \circ L_{g^{-1}(0)^{-1}}. \tag{5.4}$$

If we define

$$f = \delta_{\|q_0\|} \circ f_h \circ \delta_{\|q_0\|^{-1}} \circ L_{g^{-1}(0)^{-1}},$$

then f is quasiconformal, with essential dilatation equal to that of f_h. It is also true that, at points of existence,

$$J_f(p) = J_{f_h}(\delta_{\|q_0\|^{-1}}(g^{-1}(0)^{-1}p)). \tag{5.5}$$

Given $\delta_{\|q_0\|^{-1}} \circ L_{g^{-1}(0)^{-1}}$ preserves sets of measure zero, it follows from (5.3) that

$$J_{f_h}(\delta_{\|q_0\|^{-1}}(g^{-1}(0)^{-1}p)) \simeq e^{2u_h(\delta_{\|q_0\|^{-1}}(g^{-1}(0)^{-1}p))}$$

at almost every p in $\mathbb{H}$. Using (5.4) and (5.5) this is seen to be equivalent to

$$J_f \simeq e^{2u}$$

almost everywhere, which is the conclusion of Theorem 1.2.

We break the proof of Proposition 5.4 into three cases, of increasing generality, first with u of the form $u = \Lambda_\psi \circ g$ almost everywhere, with $\psi \in C_0^\infty(\mathbb{H})$, then μ compactly supported, and finally general μ.

5.3 $\mathrm{d}\boldsymbol{\mu}(\boldsymbol{q}) = \boldsymbol{\psi}(\boldsymbol{q})\,\mathrm{d}\boldsymbol{q}$, with $\boldsymbol{\psi} \in \boldsymbol{C_0^\infty}(\mathbb{H})$

To be precise, in this subsection, we will prove the following.

Proposition 5.5. *Given $K \geq 1$, there exists $\epsilon = \epsilon(K) > 0$ such that, if u is a quasi-logarithmic potential, $u = \Lambda_\psi \circ g$ almost everywhere, with $\psi \in C_0^\infty(\mathbb{H})$, $\|\psi\|_1 \leq \epsilon$, and $g \in \mathcal{Q}_0(K)$, then there exists a quasiconformal mapping f with*

$$J_f \simeq_K e^{2u}$$

almost everywhere. The dilatation $K(f)$ depends on $K = K(g)$ only.

In this case, identification of the required $\epsilon > 0$ comes from the following lemma, which is essentially Lemma 6.1 of [4].

Lemma 5.6. *Suppose F_s, $s \in [0,1]$, is a family of quasiconformal mappings, G is a continuous, positive, increasing, and locally Lipschitz function, and $\epsilon > 0$ is such that*

$$\epsilon < \int_0^\infty \frac{1}{G(\sigma)} \, \mathrm{d}\sigma.$$

Define

$$\Phi(s) = \sup_{0 \le \sigma \le s} \log K(F_\sigma), \quad 0 \le s \le 1,$$

and assume that for each $m \in \mathbb{N}$, $1 \le j \le m$,

$$\sup_{\frac{j-1}{m} \le \sigma \le \frac{j}{m}} K(F_\sigma) \le \exp\left[\frac{\epsilon}{m} G\left(\Phi\left(\frac{j-1}{m}\right)\right)\right] \sup_{0 \le \sigma \le \frac{j-1}{m}} K(F_\sigma).$$

Then F_1 is K-quasiconformal with K dependent only on G.

Proof. As $\exp\left[\frac{\epsilon}{m} G\left(\Phi\left(\frac{j-1}{m}\right)\right)\right] \ge 1$ we have

$$\sup_{0 \le \sigma \le \frac{j-1}{m}} K(F_\sigma) \le \exp\left[\frac{\epsilon}{m} G\left(\Phi\left(\frac{j-1}{m}\right)\right)\right] \sup_{0 \le \sigma \le \frac{j-1}{m}} K(F_\sigma),$$

which coupled with our assumption gives

$$\sup_{0 \le \sigma \le \frac{j}{m}} K(F_\sigma) \le \exp\left[\frac{\epsilon}{m} G\left(\Phi\left(\frac{j-1}{m}\right)\right)\right] \sup_{0 \le \sigma \le \frac{j-1}{m}} K(F_\sigma).$$

It follows that

$$\Phi\left(\frac{j}{m}\right) \le \Phi\left(\frac{j-1}{m}\right) + \frac{\epsilon}{m} G\left(\Phi\left(\frac{j-1}{m}\right)\right).$$

Given our assumptions on G and the choice of ϵ, the equation

$$\Phi_0'(s) = \epsilon G(\Phi_0(s)), \quad \Phi_0(0) = 0, \quad 0 \le s \le 1,$$

has a unique, finite solution. Note that Φ is increasing. We now show by induction that

$\Phi(j/m) \leq \Phi_0(j/m)$ for all $j = 0, \ldots, m$.

As $\Phi(0) = 0$ it is trivially valid for $j = 0$. Further, if true for some $0 \leq j-1 \leq m-1$ we find

$$\begin{aligned}
\Phi\left(\frac{j}{m}\right) &\leq \Phi\left(\frac{j-1}{m}\right) + \frac{\epsilon}{m} G\left(\Phi\left(\frac{j-1}{m}\right)\right) \\
&\leq \Phi_0\left(\frac{j-1}{m}\right) + \frac{\epsilon}{m} G\left(\Phi_0\left(\frac{j-1}{m}\right)\right) \\
&= \Phi_0\left(\frac{j-1}{m}\right) + \epsilon \int_{(j-1)/m}^{j/m} G\left(\Phi_0\left(\frac{j-1}{m}\right)\right) \mathrm{d}s \\
&\leq \Phi_0\left(\frac{j-1}{m}\right) + \epsilon \int_{(j-1)/m}^{j/m} G(\Phi_0(s)) \, \mathrm{d}s \\
&= \Phi_0\left(\frac{j}{m}\right).
\end{aligned}$$

In conclusion, $\Phi(1) \leq \Phi_0(1)$, and Φ_0 depends only on G. ∎

Let us now fix, for the remainder of this subsection, $K \geq 1$, $\psi \in C_0^\infty(\mathbb{H})$, and $g \in \mathcal{Q}_0(K)$. We will write u for a function equal to $\Lambda_\psi \circ g$ almost everywhere, and let $p_0 \in S(1)$ be a point such that $g(p_0) \in S(1)$.

In what follows, m is always a natural number, and once such an m has been introduced, j is a natural number between 1 and m. We reserve s for our time variable $s \in [0, 1]$.

If $F \in \mathcal{Q}_0(K'')$ for some $K'' \geq 1$, let $\phi(F) = \phi_{F,\psi}$ and $v(F) = v_{\phi(F)}$ be as in Propositions 4.13 and 4.14.

For each m we run the following iterative procedure (omitting instructions setting and increasing a counter on the assumption that the intention is clear). Step 0 is to define $f_{m,0}$

as the identity. Then

$$\text{step } j = \begin{cases} v_{m,j} := v(g \circ f_{m,j-1}^{-1}), \\ h_{m,j} \text{ is defined to be the time-}(1/m) \text{ flow mapping of } v_{m,j}, \\ f_{m,j} := \delta_{\|h_{m,j}(f_{m,j-1}(p_0))\|^{-1}}(h_{m,j} \circ f_{m,j-1}). \end{cases}$$

We define f_m to be the mapping $f_{m,m}$ created by this process. In truth, given our agreed notation, for this algorithm to be well defined, we need to know that each $g \circ f_{m,j-1}^{-1}$ is in $\mathcal{Q}_0(K'')$ for some K''. This observation is included in the proof of the following.

Proposition 5.7. *There exists $\epsilon = \epsilon(K) > 0$ such that, if $\|\psi\|_1 \leq \epsilon$, then the f_m subconverge to a K'-quasiconformal mapping with K' dependent on K only.*

Proof. Obviously the identity is a quasiconformal mapping, fixing both 0 and the unit sphere. We have, by Proposition 4.14, that $h_{m,1}$ is a quasiconformal mapping fixing 0. Consequently, as dilations are quasiconformal, also fix 0, and the dilation in play is designed to make $\|f_{m,1}(p_0)\| = 1$, we have that $f_{m,1} \in \mathcal{Q}_0(K_{m,1})$ for some $K_{m,1} \geq 1$.

Furthermore, as

$$\|g \circ f_{m,1}^{-1}(f_{m,1}(p_0))\| = \|g(p_0)\| = 1,$$

then $g \circ f_{m,1}^{-1} \in \mathcal{Q}(KK_{m,1})$.

Working iteratively, given m, j, we see that $f_{m,j} \in \mathcal{Q}_0(K_{m,j})$ for some $1 \leq K_{m,j} < \infty$, and in particular, this is true of f_m. Define $K_m = K_{m,m}$. Actually, more is true, as for all m we can take the same point, $p_0 \in S(1)$, as the point for which $\|f_m(p_0)\| = 1$.

Given the preceding observations, it follows from Lemma 2.6 that we will have subconvergence if we can demonstrate that there exists K' such that each $K_m \leq K'$, so that each f_m is K'-quasiconformal. This is where Lemma 5.6 comes in.

For each m, define the family of quasiconformal mappings, $f_m(\cdot, s)$, $s \in [0,1]$ as follows,

$$\text{if } s \in \left[\frac{j-1}{m}, \frac{j}{m}\right) \text{ then } f_m(\cdot, s) = h_{m,j,s} \circ f_{m,j-1},$$

where $h_{m,j,s}$ is the time-s flow mapping associated to $v_{m,j}$ (so that, in our algorithm above, $h_{m,j} = h_{m,j,1/m}$).

Given that dilations are 1-quasiconformal, it follows that

$$\begin{aligned}\sup_{\frac{j-1}{m} \leq \sigma \leq \frac{j}{m}} K(f_m(\cdot, \sigma)) &\leq \sup_{0 \leq \sigma \leq \frac{1}{m}} K(h_{m,j,\sigma}) K(f_{m,j-1}) \\ &\leq \sup_{0 \leq \sigma \leq \frac{1}{m}} K(h_{m,j,\sigma}) \sup_{0 \leq \sigma \leq \frac{j-1}{m}} K(f_m(\cdot, \sigma)).\end{aligned}$$

We only, therefore, need express $\sup_{0 \leq \sigma \leq \frac{1}{m}} K(h_{m,j,\sigma})$ in an appropriate form, and we will be ready to invoke the lemma.

First of all, it follows from Proposition 4.14, that for $s \in [0, 1/m]$,

$$K(h_{m,j,s}) \leq e^{C\|\psi\|_1 s} \leq e^{\frac{C\|\psi\|_1}{m}},$$

for some constant $0 \leq C = C(KK_{m,j-1}) < \infty$, with dependence as indicated.

Omitting the details, it can be shown that

$$C(KK_{m,j-1}) \leq A_1 e^{A_2 (KK_{m,j-1})^{\frac{2}{3}}}$$

for absolute constants $A_1, A_2 > 0$. Let us define

$$G(r) = A_1 \exp\left[A_2 K^{\frac{2}{3}} \exp\left(\frac{2}{3} r\right)\right], \quad r \in [0, \infty). \tag{5.6}$$

Then $C(KK_{m,j-1}) \leq G(\log K_{m,j-1})$, and, as G is an increasing function, it follows that

$$C(KK_{m,j-1}) \leq G\left(\sup_{0\leq\sigma\leq\frac{j-1}{m}} \log K(f_m(\cdot,\sigma))\right).$$

Note also that $G > 0$, and $G \in C^1([0,\infty))$, so it is locally Lipschitz. To summarize, G, and $f_m(\cdot, s)$ meet the requirements of Lemma 5.6, with G dependent on K only. We find therefore, that so long as $\|\psi\|_1 \leq \epsilon$, where $\epsilon > 0$ is as in that Lemma, then for all m, f_m is K'-quasiconformal, with K' dependent on K only. ∎

Let us add to our standing assumptions that $\|\psi\|_1 \leq \epsilon$, where $\epsilon = \epsilon(K) > 0$ is as given by Proposition 5.7.

Using Lemma 2.6, Proposition 5.7 gives us a subsequence of the f_m (that we continue to denote f_m) which converge to a K'-quasiconformal mapping f. This mapping f is, modulo a small adjustment later, our candidate for comparability.

We will hereon use the words uniform, and uniformly, to indicate that something is independent of m and j.

The proof of Proposition 5.7 gives that the $g \circ f_{m,j-1}^{-1}$ are uniformly $K'' := KK'$-quasiconformal. This is crucial because it provides uniform estimates on the $v_{m,j}$. To be more precise, recall that by Proposition 4.13, for each $v_{m,j}$ we have

$$u \circ f_{m,j-1}^{-1} = \mathrm{div}_H v_{m,j} + \zeta_{m,j}$$

for an essentially bounded $\zeta_{m,j}$, with $\|\zeta_{m,j}\|_\infty \lesssim_{KK_{m,j-1}} \|\psi\|_1$. With our assumption on $\|\psi\|_1$, and our uniform bound on $KK_{m,j-1}$, we now have,

$$\|\zeta_{m,j}\|_\infty \lesssim_K 1. \tag{5.7}$$

Proposition 4.13 also tells us that

$$|v_{m,j}(p)| \lesssim_{KK_{m,j-1},R} \|\psi\|_1(1 + |p| \log^+ |p|),$$

and it now follows that we have the uniform estimate,

$$|v_{m,j}(p)| \lesssim_{K,R} 1 + |p| \log^+ |p|. \tag{5.8}$$

Proof of Proposition 5.5. We may assume for simplicity that $u = \Lambda_\psi \circ g$ everywhere, for clearly if we prove almost everywhere comparability for such a u, it also holds for any function equal to $\Lambda_\psi \circ g$ almost everywhere. For each m we have, at almost every p, $0 < J_{f_m}(p) < \infty$, with

$$J_{f_m}(p) = \prod_{j=1}^{m} \|h_{m,j}(f_{m,j-1}(p_0))\|^{-4} J_{h_{m,j}}(f_{m,j-1}(p)).$$

Consequently, at those same p,

$$\log(J_{f_m}(p)) = \sum_{j=1}^{m} \log(J_{h_{m,j}}(f_{m,j-1}(p))) - 4\sum_{j=1}^{m} \log(\|h_{m,j}(f_{m,j-1}(p_0))\|).$$

From now on $c_m := -4\sum_{j=1}^{m} \log(\|h_{m,j}(f_{m,j-1}(p_0))\|)$. As above, we write $h_{m,j,s}$ for the time-s flow mapping generated by $v_{m,j}$, and we suppress dependence on the point p. Using Propositions 4.13 and 4.14, we may develop this as

$$\begin{aligned}
\log(J_{f_m}) &= 2\sum_{j=1}^{m} \int_0^{1/m} \mathrm{div}_H v_{m,j}(h_{m,j,\sigma}(f_{m,j-1}))\,\mathrm{d}\sigma + c_m \\
&= 2\sum_{j=1}^{m} \int_0^{1/m} (u \circ f_{m,j-1}^{-1})(h_{m,j,\sigma}(f_{m,j-1})) - \zeta_{m,j}(h_{m,j,\sigma}(f_{m,j-1}))\,\mathrm{d}\sigma + c_m.
\end{aligned}$$

At those same points

$$
\begin{aligned}
&|\log J_{f_m} - 2u - c_m| \\
&\qquad \leq \frac{2}{m}\sum_{j=1}^{m}\Bigg[\sup_{s\in[0,1/m]}|(u\circ f_{m,j-1}^{-1})(h_{m,j,s}(f_{m,j-1})) - u| \\
&\qquad + \sup_{s\in[0,1/m]}|\zeta_{m,j}(h_{m,j,s}(f_{m,j-1}))|\Bigg] \\
&\qquad \leq \frac{2}{m}\left[\sum_{j=1}^{m}\sup_{s\in[0,1/m]}|(u\circ f_{m,j-1}^{-1})(h_{m,j,s}(f_{m,j-1})) - u|\right] + C_1,
\end{aligned}
\tag{5.9}
$$

where $C_1 > 0$ is a constant, dependent only on K, the appearance of which is justified by the uniform essential boundedness of $\zeta_{m,j}$, as in (5.7).

Given that $\psi \in C_0^\infty(\mathbb{H})$, it follows from Lemma 5.1 that Λ_ψ is Lipschitz continuous. We have, therefore,

$$
\begin{aligned}
|(u\circ f_{m,j-1}^{-1})(h_{m,j,s}(f_{m,j-1})) - u| &= |\Lambda_\psi(g(f_{m,j-1}^{-1}(h_{m,j,s}(f_{m,j-1})))) - \Lambda_\psi(g)| \\
&\lesssim d(g(f_{m,j-1}^{-1}(h_{m,j,s}(f_{m,j-1}))), g).
\end{aligned}
\tag{5.10}
$$

Let $\xi \in C_0^\infty(\mathbb{H})$, $\xi \geq 0$, $\int \xi = 1$, and $D > 0$ such that support$(\xi) \subset B(D)$. Now, the $f_{m,j-1}$ are uniformly K'-quasiconformal, and as already noted, satisfy the hypotheses of Lemma 2.3. Consequently, there is a $D' > 0$ such that for all m, j,

$$f_{m,j-1}(B(D)) \subset B(D').$$

Now $h_{m,j,s}$ is generated by $v_{m,j}$, and the uniform estimates on the size of $v_{m,j}$, as in (5.8), mean, using (3.5), that there exists $D'' > 0$ such that for all m, j, s (with $s \in [0, 1/m]$)

$$h_{m,j,s}(B(D')) \subset B(D'').$$

Using Lemma 2.4, we have Hölder continuity uniformly for the $g \circ f_{m,j-1}^{-1}$ on $B(D'')$, and so there exists $\alpha > 0$ such that for all m, j, at every point of $B(D)$

$$\begin{aligned} d(g(f_{m,j-1}^{-1}(h_{m,j,s}(f_{m,j-1}))), g) &= d(g(f_{m,j-1}^{-1}(h_{m,j,s}(f_{m,j-1}))), g(f_{m,j-1}^{-1}(f_{m,j-1}))) \\ &\lesssim d(h_{m,j,s}(f_{m,j-1}), f_{m,j-1})^{\alpha}. \end{aligned} \tag{5.11}$$

Remembering that

$$h_{m,j,s}(p) = p + \int_0^s v_{m,j}(h_{m,j,\sigma}(p))\, \mathrm{d}\sigma,$$

and using that the $v_{m,j}$ are uniformly bounded on $B(D'')$, then we have the Euclidean estimate

$$|h_{m,j,s}(f_{m,j-1}) - f_{m,j-1}| \lesssim \frac{1}{m}$$

on $B(D)$. Using (1.15), we have

$$d(h_{m,j,s}(f_{m,j-1}), f_{m,j-1}) \lesssim_{D''} |h_{m,j,s}(f_{m,j-1}) - f_{m,j-1}|^{\frac{1}{2}}. \tag{5.12}$$

Putting together (5.10), (5.11), and (5.12), we have for points of $B(D)$ that

$$|(u \circ f_{m,j-1}^{-1})(h_{m,j,s}(f_{m,j-1})) - u| \lesssim \left(\frac{1}{m}\right)^{\alpha/2}.$$

Using this in (5.9), we have, therefore, constant $C_2 > 0$ such that, at almost every $p \in B(D)$,

$$|\log J_{f_m} - 2u - c_m| \leq C_2 m^{-\alpha/2} + C_1,$$

or,

$$e^{-C_2 m^{-\alpha/2} - C_1} e^{2u} \leq e^{-c_m} J_{f_m} \leq e^{C_2 m^{-\alpha/2} + C_1} e^{2u}. \tag{5.13}$$

It is worth noting that C_2 depends on the radius of support of ψ, in addition to K, however, C_2 is about to vanish as we take the limit. C_1, which survives the limit, depends on K only, as noted above.

Multiplying (5.13) by ξ and integrating, we find

$$e^{-(C_2 m^{-\alpha/2}+C_1)} \int \xi e^{2u} \leq e^{-c_m} \int \xi J_{f_m} \leq e^{C_2 m^{-\alpha/2}+C_1} \int \xi e^{2u}. \tag{5.14}$$

Using Lemmas 2.5 and 2.6, $\limsup_{m\to\infty} \int \xi J_{f_m} = \int \xi J_f$, which given that J_f is locally integrable and almost everywhere greater than zero, is finite and positive. Taking the $\limsup$ as $m \to \infty$ of (5.14), we therefore find that,

$$e^{-C_1} \int \xi e^{2u} \leq \limsup_{m\to\infty}(e^{-c_m}) \int \xi J_f \leq e^{C_1} \int \xi e^{2u}.$$

As $\int \xi J_f < \infty$ and $e^{-C_1} \int \xi e^{2u} > 0$ we must have $\limsup_{m\to\infty}(e^{-c_m}) > 0$. Similarly, given that $e^{C_1} \int \xi e^{2u} < \infty$ and $\int \xi J_f > 0$, we must have $\limsup_{m\to\infty}(e^{-c_m}) < \infty$. Let $c_0 := \limsup_{m\to\infty} e^{-c_m}$.

This being true for all $\xi \in C_0^\infty(\mathbb{H})$, $\xi \geq 0$, $\int \xi = 1$, then it is true for the mollifier $\xi_{q,r}$, of center $q \in \mathbb{H}$, and radius $r > 0$ (we may use the standard 'Euclidean' mollifiers here, there is no need for 'twisted convolution'). As both J_f and e^{2u} are locally integrable, they have Lebesgue points almost everywhere. At a common Lebesgue point q, we have,

$$\int \xi_{q,r} e^{2u}, \int \xi_{q,r} J_f \;\to\; e^{2u(q)},\, J_f(q),$$

respectively, as $r \to 0$. See [35] for these last couple of points. Hence, putting it all together, we find that at almost every point

$$c_0 J_f \simeq e^{2u}.$$

It might seem natural to include c_0 in the implied constant of comparability. It is likely, however, that c_0 depends, not only on K, but on the radius of support of ψ. It is important for the next steps that the constant of comparability does not depend on this radius. This is, however, easy to take care of. Post composing f with the dilation δ_{r_0}, $r_0^4 := c_0$, and calling the result f again, we have a K'-quasiconformal mapping f such that

$$J_f \simeq e^{2u}$$

almost everywhere, as required, and with the implied constant dependent on K only. ∎

5.4 Conclusion of the Proof of Theorem 1.2

Moving from the special case of the preceding subsection to the general case now follows as it does in the Euclidean case. With Proposition 5.5 in place, progress rests principally on Lemmas 5.2 and 5.3. We first prove

Proposition 5.8. *Given $K \geq 1$, there exist $\epsilon = \epsilon(K) > 0$, and $K' = K'(K) \geq 1$, such that, if u is a quasi-logarithmic potential, $u = \Lambda_\mu \circ g$ almost everywhere, with μ compactly supported, $\|\mu\| \leq \epsilon_0$, and $g \in \mathcal{Q}_0(K)$, then there is a K'-quasiconformal mapping f with*

$$J_f \simeq_K e^{2u}$$

almost everywhere.

Proof. Let $u = \Lambda_\mu \circ g$ almost everywhere be a quasi-logarithmic potential, with μ compactly supported, and $g \in \mathcal{Q}_0(K)$. Let ψ_k be a sequence of smooth regularizations of μ, as in (5.2). Note that, given our assumption that μ is compactly supported, the ψ_k are not only smooth, but also compactly supported. Let $u_k := \Lambda_{\psi_k} \circ g$.

Proposition 5.5 tells us there exists $\epsilon' > 0$ such that if $\|\mu\| \leq \epsilon'$, so that each $\|\psi_k\|_1 \leq \epsilon'$, then for each k, there is a quasiconformal mapping f_k, with $f_k(0) = 0$, such that

$$J_{f_k} \simeq e^{2u_k}, \tag{5.15}$$

almost everywhere. The dilatation of f_k and the constant of comparability in (5.15) are both dependent only on K, the dilatation of the given g, hence are each independent of k.

If $\theta = \theta(K) > 0$ is as given by Lemma 5.2, and we let $\epsilon'' > 0$ be defined by $\epsilon'' = \theta/2$, then, if $\|\mu\| \leq \epsilon''$, so that each $\|\psi_k\|_1 \leq \epsilon''$, then $e^{2u} \in L^1_{\text{loc}}(\mathbb{H})$, and for any ball $B \subset \mathbb{H}$,

$$\int_B |e^{2u_k} - e^{2u}| \to 0 \quad \text{as} \quad k \to \infty.$$

It follows that $\int_{B(1)} e^{2u_k} \to \int_{B(1)} e^{2u}$, and so, together with (5.15),

$$J_{f_k} \simeq 1$$

independently of k. Using Lemma 2.7 we may, therefore, pass to a subsequence that converges locally uniformly to a K'-quasiconformal mapping, with $K' = K'(K)$.

Weak convergence of the Jacobians, as in the proof of Proposition 5.5, gives

$$J_f \simeq e^{2u},$$

almost everywhere. The proof is therefore concluded by identifying the required $\epsilon > 0$ as $\epsilon = \min\{\epsilon', \epsilon''\}$. ∎

We are now ready to conclude the proof of Proposition 5.4, and so the proof of Theorem 1.2.

Proof of Proposition 5.4. Let $u = \Lambda_\mu \circ g$ almost everywhere be a quasilogarithmic potential, with $g \in \mathcal{Q}_0(K)$. Given the case just dealt with, it will come as no surprise that we are going to restrict μ, so that its restriction is compactly supported, then show that our desired conclusions hold in the limit, as we let the support grow.

Define, therefore, $\mu_k = \mu\big|_{B(k)}$. Let $u_k := \Lambda_{\mu_k} \circ g$. If $\|\mu\| \leq \epsilon$ where $\epsilon > 0$ is as in

Proposition 5.8, then (obviously) also $\|\mu_k\| \leq \epsilon$ for all k. By Proposition 5.8, there exist K'-quasiconformal mappings, $K' = K'(K)$, f_k, with $f_k(0) = 0$, and

$$J_{f_k} \simeq e^{2u_k}$$

almost everywhere, independently of k. Now use Lemma 5.3 and proceed exactly as in the previous case. ∎

6 Weighted Sub-Riemannian Metrics

In this section we assume that $\omega \geq 0$ is continuous, and comparable to a quasiconformal Jacobian. We then show that $(\mathbb{H}, g_0)$ and $(\mathbb{H}, g)$ are bi-Lipschitz equivalent when $g = \sqrt{\omega} g_0$. Our arguments are inspired by those of [31].

Recall that we write $(\mathbb{H}, g_0)$ for the Heisenberg group equipped with its canonical sub-Riemannian metric, g_0, and associated Carnot-Caratheodory distance function, ρ, as described in Section 1.3. Our use of $(\mathbb{H}, g)$, with $g = \sqrt{\omega} g_0$, is a slight abuse of notation, as (see below) we replace not only the metric, but also the curve families defining the distance function. We are, however, justified in describing the abuse as slight, in that, were we also to replace the curve families in our earlier definition of $(\mathbb{H}, g_0)$ with those used below, then the resulting distance function would be identical to ρ.

The following definition is motivated by our later reliance on the curve families in $\mathbb{H}$ constructed in [21]. Given this reliance, we need curves of this type to be considered among the competitors over which the distance function is to be defined.

Definition 6.1. Let $p, q \in \mathbb{H}$, $b \in [0, \infty)$, $\gamma : [0, b] \to \mathbb{H}$ a continuous mapping, $E \subset [0, b]$ a Lebesgue null set, $N \in \mathbb{N} \cup \{\infty\}$, $\{(a_k, b_k)\}_{k=1}^N$ a collection of open intervals, and $\{\gamma_k : [a_k, b_k] \to \mathbb{H}\}_{k=1}^N$ horizontal curves, such that

1. $\gamma(0) = p$, $\gamma(b) = q$,
2. for all $s \in [0, b] \setminus E$, $\gamma(s) = \sum_{k=1}^N \gamma_k(s) \mathbb{1}_{(a_k, b_k)}(s)$,
3. $\sum_{k=1}^N \int_{a_k}^{b_k} |\gamma_k'(s)|_H \, \mathrm{d}s < \infty$.

Then we say $\left(b, \gamma, E, N, \{(a_k, b_k)\}_{k=1}^N, \{\gamma_k : [a_k, b_k] \to \mathbb{H}\}_{k=1}^N\right)$ is an admissible curve for (p, q).

See Section 1.3 for the definition of horizontal curve. We will call such a collection an admissible curve if it is an admissible curve for some (p, q). Further, we will often denote the admissible curve $(b, \gamma, E, \ldots)$, by γ only.

For γ an admissible curve, and $g : \mathbb{H} \to \mathbb{R}$ continuous, we define

$$\int_\gamma g = \int_{[0,b]\setminus E} g(\gamma(s))|\gamma'(s)|_H \,\mathrm{d}s.$$

Given continuous $\omega : \mathbb{H} \to [0, \infty)$, let

$$\rho_\omega(p, q) := \inf_\gamma \int_\gamma \omega^{\frac{1}{4}},$$

where the infimum is taken over all admissible curves for (p, q). Note that, as things stand, ρ_ω is not necessarily a metric, only a pseudometric, as we have not assumed anything about the set on which ω vanishes.

The goal of the large part of this section is stated precisely as the following proposition.

Proposition 6.2. *Suppose $\omega : \mathbb{H} \to [0, \infty)$ is continuous, and there exist $C > 0$, and K-quasiconformal mapping $f : \mathbb{H} \to \mathbb{H}$, with*

$$\frac{1}{C}\omega \le J_f \le C\omega$$

almost everywhere. Then there exists $L \ge 1$ such that, for all $p, q \in \mathbb{H}$,

$$\frac{1}{L}\rho_\omega(p, q) \le \rho(f(p), f(q)) \le L\rho_\omega(p, q).$$

It follows that, in these circumstances, ρ_ω is a genuine metric, and a rewording of the conclusion is that the quasiconformal f is actually a bi-Lipschitz mapping between the metric spaces $(\mathbb{H}, \rho_\omega)$ and $(\mathbb{H}, \rho)$. Let us write $\rho_f(p, q) = \rho(f(p), f(q))$. We want to show, that with the assumptions of the proposition, $\rho_\omega \simeq \rho_f$.

Proposition 6.2 will be an obvious corollary to the lemmas that occupy us for the rest of the section. Let us fix ω as in the statement of the theorem.

First we define a measure on $\mathbb{H}$,

$$\mu(U) = \int_U \omega.$$

Now introduce the auxiliary function,

$$d_\mu(p,q) := \mu^{\frac{1}{4}}[B_{p,q}],$$

where $B_{p,q} := B(p, d(p,q)) \cup B(q, d(q,p))$. Despite the suggestive notation, this is not in general a metric, but only a quasimetric. The quasimetric space $(\mathbb{H}, d_\mu)$ is called the David-Semmes deformation of $\mathbb{H}$, a fascinating topic that we touch on in the introduction. Before we explain why we find d_μ useful, we need to observe the following nice property of the measure μ.

Lemma 6.3. *μ is doubling, that is, there exists $C > 0$ such that, for all $p \in \mathbb{H}$ and $r > 0$,*

$$\mu[B(p,2r)] \leq C\mu[B(p,r)].$$

Proof. First off, using our assumed comparability of weight and Jacobian, and the change of variable formula for quasiconformal mappings (2.10),

$$\mu[B(p,2r)] = \int_{B(p,2r)} \omega \leq C \int_{B(p,2r)} J_f = C|fB(p,2r)|. \tag{6.1}$$

Some of the arguments that follow are either similar to, or add detail to, remarks made in Section 2. The mapping f is quasisymmetric; let η be the associated control function. With s defined as the minimum of $d(f(p), f(q))$ over $\partial B(p,r)$, and $\eta_2 := \eta(2)$, we have

$$B(f(p), s) \subset fB(p,r) \subset fB(p,2r) \subset B(f(p), \eta_2 s).$$

Using this, and the doubling property of Lebesgue measure on $\mathbb{H}$, it follows that

$$|fB(p,2r)| \leq |B(f(p),\eta_2 s)| \leq C|B(f(p),s)| \leq C|fB(p,r)| \leq C\mu[B(p,r)]. \tag{6.2}$$

Putting together (6.1) and (6.2), we have that μ is doubling as required. ∎

Our reason for introducing d_μ is contained in the following lemma. Note, we do not always state results in the fullest generality, preferring to work with our fixed ω.

Lemma 6.4. $d_\mu \simeq \rho_f$.

Proof. Using the inclusion $B_{p,q} \subset B(p, 2d(p,q))$ and the doubling property of μ,

$$\mu[B_{p,q}] \leq \mu[B(p,2d(p,q))] \leq C\mu[B(p,d(p,q))].$$

It follows that

$$d_\mu(p,q) \simeq \left(\int_{B(p,d(p,q))} J_f\right)^{\frac{1}{4}} \simeq d(f(p),f(q)),$$

where for the last inequality, we use the change of variable and quasisymmetric control in a similar manner to the proof of Lemma 6.3, along with the fact that Lebesgue measure on $\mathbb{H}$ is Ahlfors 4-regular.

As remarked in Section 1.3, it is well known that $d \simeq \rho$, and so $d_\mu \simeq \rho_f$ is now immediate. ∎

Now that we have Lemma 6.4 in place, we may work with d_μ and it remains to show $d_\mu \simeq \rho_\omega$. We will do the two directions separately, in Lemmas 6.5 and 6.7.

Lemma 6.5. $d_\mu \lesssim \rho_\omega$.

Proof. For a horizontal curve $\gamma_k : [a_k, b_k] \to \mathbb{H}$, define the μ-length of γ_k as follows,

$$l_\mu(\gamma_k) = \limsup_{M\to\infty} \sum_{i=1}^{M} d_\mu(\gamma_k(s_{i-1}), \gamma_k(s_i)),$$

where for each $M \in \mathbb{N}$, the s_i partition $[a,b]$ into M equal length intervals, with $s_0 = a$ and $s_M = b$. For an admissible curve, γ, with collection of horizontal curves

$$\{\gamma_k : [a_k, b_k] \to \mathbb{H}\}_{k=1}^N,$$

define

$$l_\mu(\gamma) = \sum_{k=1}^N l_\mu(\gamma_k).$$

Now let $p, q \in \mathbb{H}$, and γ an admissible curve for (p,q) be given. Let us focus for a time on one of the horizontal subcurves, γ_k. Let $\epsilon > 0$. First note that γ_k is uniformly continuous on $[a_k, b_k]$, and there exists a compact set containing the image of γ_k on which ω, and so $\omega^{\frac{1}{4}}$, is uniformly continuous. It follows that there exists an $M_k < \infty$, so that whenever (s_i) is a partition with $|s_i - s_{i-1}| \leq (b_k - a_k)/M_k$,

$$\omega^{\frac{1}{4}}(u) \leq \omega^{\frac{1}{4}}(\gamma_k(s_{i-1})) + \epsilon \tag{6.3}$$

for all $u \in B_{\gamma_k(s_{i-1}),\gamma_k(s_i)}$.

Assume that a partition (s_i) of $[a_k, b_k]$ is sufficiently fine, as in the preceding paragraph, and define $u_{k,i} = \gamma_k(s_i)$, $B_{k,i-1} = B_{\gamma_k(s_{i-1}),\gamma_k(s_i)}$. Then from the definition of d_μ,

$$d_\mu(u_{k,i-1}, u_{k,i}) \leq \left(\sup_{B_{k,i-1}} \omega \right)^{\frac{1}{4}} |B_{k,i-1}|^{\frac{1}{4}}$$

or, equivalently,

$$d_\mu(u_{k,i-1}, u_{k,i}) \leq \sup_{B_{k,i-1}} \left(\omega^{\frac{1}{4}}\right) |B_{k,i-1}|^{\frac{1}{4}}.$$

Using Ahlfors 4-regularity again, and observation (6.3),

$$d_\mu(u_{k,i-1}, u_{k,i}) \lesssim \omega^{\frac{1}{4}}(u_{k,i-1}) d(u_{k,i-1}, u_{k,i}) + \epsilon d(u_{k,i-1}, u_{k,i}).$$

It now follows from (the proof of) Lemma 2.4 in [8] that

$$l_\mu(\gamma_k) \lesssim \int_{a_k}^{b_k} \omega^{\frac{1}{4}}(\gamma_k(s))|\gamma_k'(s)|_H \,\mathrm{d}s + \epsilon l_d(\gamma_k),$$

where l_d is length with respect to d, as defined in Section 1.3. As ϵ was arbitrary, and $l_d(\gamma_k)$ is finite, this improves to

$$l_\mu(\gamma_k) \lesssim \int_{a_k}^{b_k} \omega^{\frac{1}{4}}(\gamma_k(s))|\gamma_k'(s)|_H \,\mathrm{d}s.$$

It follows, using again the continuity of $\omega^{\frac{1}{4}}$ on a compact set containing the image of γ, and property (3) of admissible curves, that

$$l_\mu(\gamma) \lesssim \int_\gamma \omega^{\frac{1}{4}}. \tag{6.4}$$

Using Lemma 6.4, and the observation that concluded its proof, we have $d_f \simeq d_\mu$, where we write $d_f(p,q) = d(f(p), f(q))$. It follows, using the triangle inequality for d, that for any finite collection of points $u_1, u_2, \ldots, u_M \in \mathbb{H}$

$$d_\mu(u_1, u_M) \lesssim d_f(u_1, u_M) \leq \sum_{i=1}^{M} d_f(u_{i-1}, u_i) \lesssim \sum_{i=1}^{M} d_\mu(u_{i-1}, u_i).$$

It is now straightforward that for each horizontal curve, γ_k,

$$d_\mu(\gamma_k(a_k), \gamma_k(b_k)) \lesssim l_\mu(\gamma_k),$$

and so, applying the same argument again,

$$d_\mu(p,q) \lesssim \sum_{k=1}^{N} d_\mu(\gamma_k(a_k), \gamma_k(b_k)) \lesssim \sum_{k=1}^{N} l_\mu(\gamma_k) = l_\mu(\gamma).$$

Consequently, using (6.4),

$$d_\mu(p,q) \lesssim \inf_\gamma l_\mu(\gamma) \lesssim \rho_\omega(p,q),$$

where the infimum is taken over all admissible curves for (p,q). ∎

Before proving the other side of the comparison, we state the proposition of [21] that motivated our definition of admissible curves.

Proposition 6.6. *There exist $\lambda > 1$, $C > 0$, such that, for all $p, q \in \mathbb{H}$, there exists a family Γ of admissible curves for (p,q), and a probability measure α on Γ, with*

$$\int_\Gamma \left(\int_\gamma \omega^{\frac{1}{4}} \right) \mathrm{d}\alpha(\gamma) \lesssim C \int_{B(p,\lambda d(p,q))} \omega^{\frac{1}{4}}(u) \left(\frac{1}{d(p,u)^3} + \frac{1}{d(q,u)^3} \right) \mathrm{d}u.$$

We are now able to conclude the proof of Proposition 6.2 by using, once again, the reverse Hölder inequality for the Jacobian of a quasiconformal mapping, (2.11).

Lemma 6.7. $d_\mu \gtrsim \rho_\omega$.

Proof. Fix $p, q \in \mathbb{H}$. Given the definition of ρ_ω, we have $\rho_\omega(p,q) \le \int_\gamma \omega^{1/4}$ for any admissible γ joining p and q. Let Γ and α be as in Proposition 6.6. Then, as α is a probability measure, that proposition gives us $\lambda > 1$ such that

$$\rho_\omega(p,q) \lesssim \int_{B(p,\lambda d(p,q))} \omega^{\frac{1}{4}}(u) G(u)\, \mathrm{d}u,$$

where $G(u) := \max\{d(p,u)^{-3}, d(q,u)^{-3}\}$.

As ω is comparable to a quasiconformal Jacobian, it also satisfies a reverse Hölder inequality: there exists $s > 1$ such that, if $B \subset \mathbb{H}$ is a ball,

$$\left(\frac{1}{|B|} \int_B \omega^s \right)^{\frac{1}{s}} \lesssim \frac{1}{|B|} \int_B \omega,$$

independently of B.

Let $B := B(p, \lambda d(p,q))$, and $R := d(p,q)$. Let r be the exponent conjugate to $4s$, so that $r < 4/3$. We find

$$\begin{aligned}
\int_B \omega^{\frac{1}{4}} G &\lesssim \left(\int_B \omega^s\right)^{\frac{1}{4s}} \left(\int_B G^r\right)^{\frac{1}{r}} \\
&\lesssim R^4 \left(\frac{1}{|B|}\int_B \omega^s\right)^{\frac{1}{4s}} \left(\frac{1}{|B|}\int_B G^r\right)^{\frac{1}{r}} \\
&\lesssim R^4 \left(\frac{1}{|B|}\int_B \omega\right)^{\frac{1}{4}} R^{-3} \\
&\lesssim \left(\int_B \omega\right)^{\frac{1}{4}} \lesssim d_\mu(p,q),
\end{aligned}$$

as required. ∎

The proof of Lemma 6.7 concludes the proof of Proposition 6.2. That proposition, combined with Theorem 1.2 gives the following, of which Theorem 1.4 is a special case.

Theorem 6.8. *Given $K \geq 1$, there exist $\epsilon = \epsilon(K) > 0$ and $L = L(K) \geq 1$, such that, if u is a continuous quasi-logarithmic potential, $u = \Lambda_\mu \circ g$ almost everywhere, with $\|\mu\| \leq \epsilon$, and g a K-quasiconformal mapping, then $(\mathbb{H}, g_0)$ and $(\mathbb{H}, e^u g_0)$ are L-bi-Lipschitz equivalent.*

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- Special Session on Analysis and Geometry in Nonsmooth Metric Measure Spaces, Joint Mathematics Meetings, Seattle, Jan 7, 2016
- Colloquium, Roosevelt University, Chicago, Oct 23, 2015.

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- Graduate Student Colloquium, Syracuse University, Feb 28, 2014.

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- Undergraduate Research Scholarship, Summer 2009, Summer 2010.

Conference Participation

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- A quasiconformal life, A celebration of F.W. Gehring, University of Helsinki, Aug 2013.
- Rolf Nevanlinna Colloquium, University of Helsinki, Aug 2013.
- Workshop III: Non-smooth Geometry, Interactions between Analysis and Geometry, IPAM, UCLA, Apr 2013.

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- Tutorials, Interactions between Analysis and Geometry, IPAM, UCLA, Mar 2013.
- Metric Measure Spaces: Geometric and Analytic Aspects, SMS, CRM, Jun 2011.