Diagrammatic Theories of 1- and 2- Dimensional Knots

BY

JONATHAN SCHNEIDER
B.A., Swarthmore College, 2005
M.S., University of Illinois at Chicago, Chicago, 2012

THESIS

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Defense Committee:

Louis H. Kauffman, Chair and Advisor
Marc Culler
David Radford
Ramin Takloo-Bighash
Arthur L. Licht, Physics

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Jonathan Schneider

University of Illinois Chicago

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Dedicated to my parents.

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Contents

SUMMARY

A diagrammatic knot theory is a set of rules for drawing and manipulating knot diagrams in dimension n. The rules are selected from a universal set that depends on n. Classical knot theory is an example with $n = 1$; virtual and welded knot theory are extensions of the classical theory. Analogous theories to these are also defined for $n = 2$. In this paper we explore these theories in detail and relations between them.

Wherever possible, we describe a topological model for each diagrammatic theory— that is, a class of objects associated to diagrams such that equivalent objects are associated to equivalent diagrams. For example, topological knots are a model for the classical theory with n=1, and topologically knotted surfaces serve for the analogous theory in $n = 2$. Models are also described for many other diagrammatic theories, with special focus on the virtual and welded theories in $n = 1$ and 2.

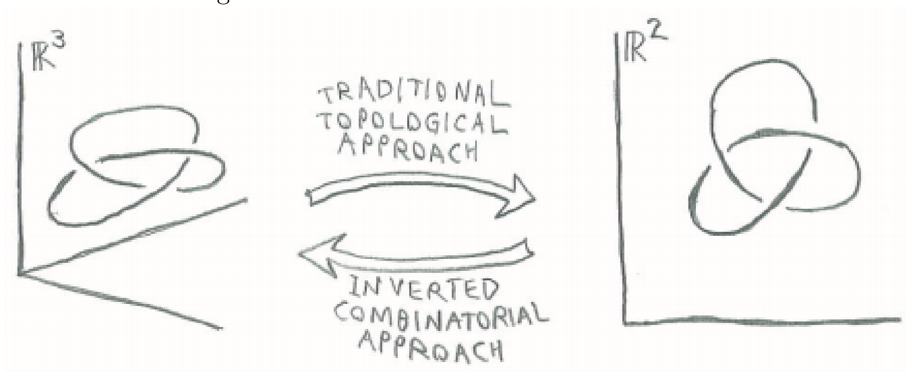
Kauffman proved that closed curves in thickened stabilized surfaces model virtual knots; in this paper we debut an analogous model for the virtual theory in $n = 2$.

Satoh proved that ribbon-knotted toral surfaces model welded knots, and Rourke reformulated Satoh's description by adding a fiber-structure to the toral surfaces and surrounding 4-space. We prove Rourke's formulation is indeed isotopic to Satoh's. We also question the invariance of the fiber-structure in Rourke's model, and debut a new diagrammatic theory, "rotational welded theory", that avoids this problem. We also debut an analogous theory to this in $n = 2$ and define a Rourke-like model for it.

The paper concludes with an overview of diagrammatic theories in arbitrary dimensions.

0 Introduction

Knot theory is usually presented as a topological theory (concerning isotopy classes of embeddings), which admits a combinatorial model (knot diagrams and Reidemeister moves). This organization can be turned around, so that the combinatorial objects are fundamental and the topological description is merely a model of the diagrams.



An advantage of this inverted approach is that we may, by altering the rules for drawing and moving the diagrams, invent new knot theories very different from the classical theory. One of the first and most important occurrences of this was the invention of virtual knot theory. In (Kauffman, 1999), Kauffman debuted the diagrammatic definition of virtual knots and gave them a topologial interpretation as classical knots in thickened stabilized surfaces. That interpretation was refined in (Kuperberg, 2003). Welded knot theory, another theory defined in terms of diagrams, was developed by Kauffman based on (Fenn, Rimanyi, & Rourke, 1997); a topological model appeared in (Satoh, 2000) and was refined in (Rourke, 2006). Moving up a dimension, virtual surface-knot diagrams first appeared in (Takeda, 2012) and earlier in talks I gave at the AMS sectional meeting, U.Kansas, 2012. Blake Winter has suggested other higher-dimensional diagrammatic knot theories in (Winter, 2015).

In **chapter 1** I present a general framework for defining diagrammatic theories, focusing on dimensions 1 and 2 but hinting at higher dimensions as well, building on work by Roseman. In **chapter 2** I survey seven different diagrammatic theories in dimension 1, giving topological models for several. Special attention is paid to welded knot theory. In **chapter 3** I describe the surface-knot analogs for those seven theories, and give topological models for several. Higher dimensions are briefly surveyed in **chapter 4**.

0.1 Main results

1. **Theorem (section 2.6.6).** Rourke and Satoh's models for welded knots are isotopic. (This was stated, but not proven, in (Rourke, 2006).)

2. Rourke's enhancement of Satoh's welded-knot invariant seems to cause invariance to fail **(section 2.6.7)**. Plausible counterexamples are proposed without proof. However–

   Theorem (section 2.7) Rourke's construction is an invariant of rotational welded knot theory, which is a refinement of welded knot theory introduced in this paper.

3. **Theorem (section 3.5)** Let D, D' be virtual 2-knot diagrams. Let $S(D), S(D')$ be any classical 2-knot diagrams in 3-manifolds derived from D, D' via the construction described. If $D \sim D'$ (where $\sim$ is virtual 2-knot equivalence) then $S(D) \sim S(D')$ (where $\sim$ is equivalence via classical Roseman moves and stabilization).

4. **Theorem (section 3.7)** Let D, D' be rotational welded 2-knot diagrams. Let $R(D), R(D')$ be the fiberwise-embedded 3-manifolds derived from D, D' via Rourke's construction. If $D \sim D'$ (where $\sim$ is rotational welded 2-knot equivalence) then $R(D) \sim R(D')$ (where $\sim$ is equivalence via a fiberwise $\mathbb{R}^5$ isotopy).

0.2 Other results

Various minor results are scattered throughout this paper, including an enumeration of the 66 signed Roseman moves **(section 1.2.2)**, and various endomorphisms and epimorphisms of diagrammatic knot theories (see "relation to other theories" for each theory in chapters 2 and 3).

1 Theoretical framework

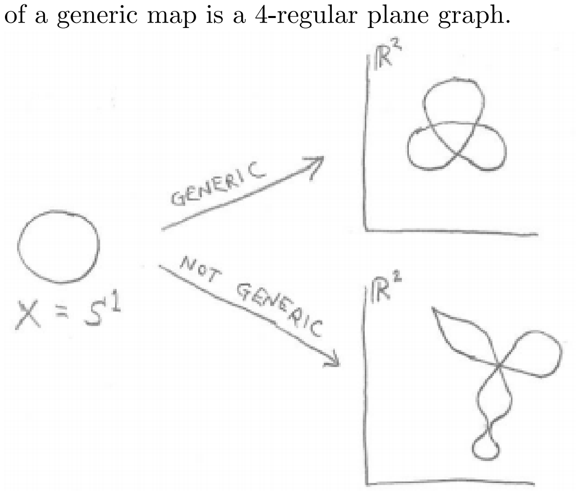
Terminology: In this paper, the term "knot" refers generally to either knots or links. For the purposes of defining knot equivalences, we assume all knot domains are unoriented with unlabeled components, and the ambient space around a knot or knot diagram is oriented.

A **knot diagram** is a smooth generic map $f : X^n \to \mathbb{R}^{n+1}$ with **crossing data** at the crossings. A **diagrammatic theory** consists of a set of knot diagrams, called the theory's **universe**, and a set of **moves**, which determines an equivalence relation on the diagrams. The equivalence classes are the theory's **knot types**. We define these concepts in detail.

1.1 1-knot theories

1.1.1 Universes

Let X^1 be a circle, or a disjoint union of finitely many circles. Let Y^2 be a smooth surface (usually $\mathbb{R}^2$). A C^∞ map $f : X \to Y$ is **generic** if it is an immersion and is one-to-one except at a discrete set of degree-two transverse crossings. The map is **tame** if there are only finitely many crossings. The image of a generic map is a 4-regular plane graph.



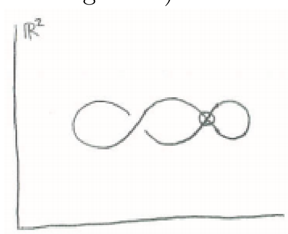
Let $f_0 : X_0 \to \mathbb{R}^2$ and $f_1 : X_1 \to \mathbb{R}^2$ be generic maps. The following three conditions are equivalent.

- There exists an orientation-preserving homeomorphism $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathrm{im}(f_1) = \mathrm{im}(\psi \circ f_0)$.
- There are homeomorphisms $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ and $\phi : X_1 \to X_0$, with ψ orientation-preserving, satisfying $f_1 = \psi \circ f_0 \circ \phi$.
- The images of f_0 and f_1 are isomorphic as plane graphs (where the isomorphism preserves the orientation of the plane).

If these (equivalent) conditions are met, we say that f_0 and f_1 are **isotopic**. This is an equivalence relation on generic maps $X \to \mathbb{R}^2$. The third version of the condition tells us that the isotopy type of a generic map is completely encoded in its graph structure and may therefore be understood as a combinatorial object.

(In the second version of the condition above, if we take $X_0 = X_1$ and add the requirement that ϕ be orientation-preserving or component-preserving, we get a stronger relation called **isotopic respecting orientation** or **components**. For example, there is only one crossing-free generic map of a circle, up to isotopy, but there are two such maps up to isotopy respecting orientation. This notion will not be used in this paper; we will assume henceforth that X is unoriented, and its components are interchangeable.)

We can impose **crossing data** at each crossing of a generic map $f : X \to \mathbb{R}^2$. This means the crossing is designated either "classical" or "virtual"; at classical crossing, we designate the two strands "over" and "under". We signify this graphically by drawing a "break" in the understand. A virtual crossing is drawn without a break, but with a small circle. A generic map endowed with crossing data is called a **knot diagram**. (For convenience, we will use the term "diagram" to refer either to the generic map f or its image, together with its crossing data.)



The **universe** of a knot theory is the set of all knot diagrams considered valid in that theory. (The domains of the diagrams vary over the universe, so some will have one circle component, others two, etc.) In general, the universe of an n-knot theory is determined by selecting which singularity types of generic maps are allowed, and how crossing data may be arranged there. In the case of 1-knot theories, these choices are as follows:

- Are crossings permitted?
- If so, what types of crossing data are accepted— classical, virtual, or both?

Thus, there are four possible universes for 1-knot theories. We call them the **simple, classical, virtual, and mixed** universes.

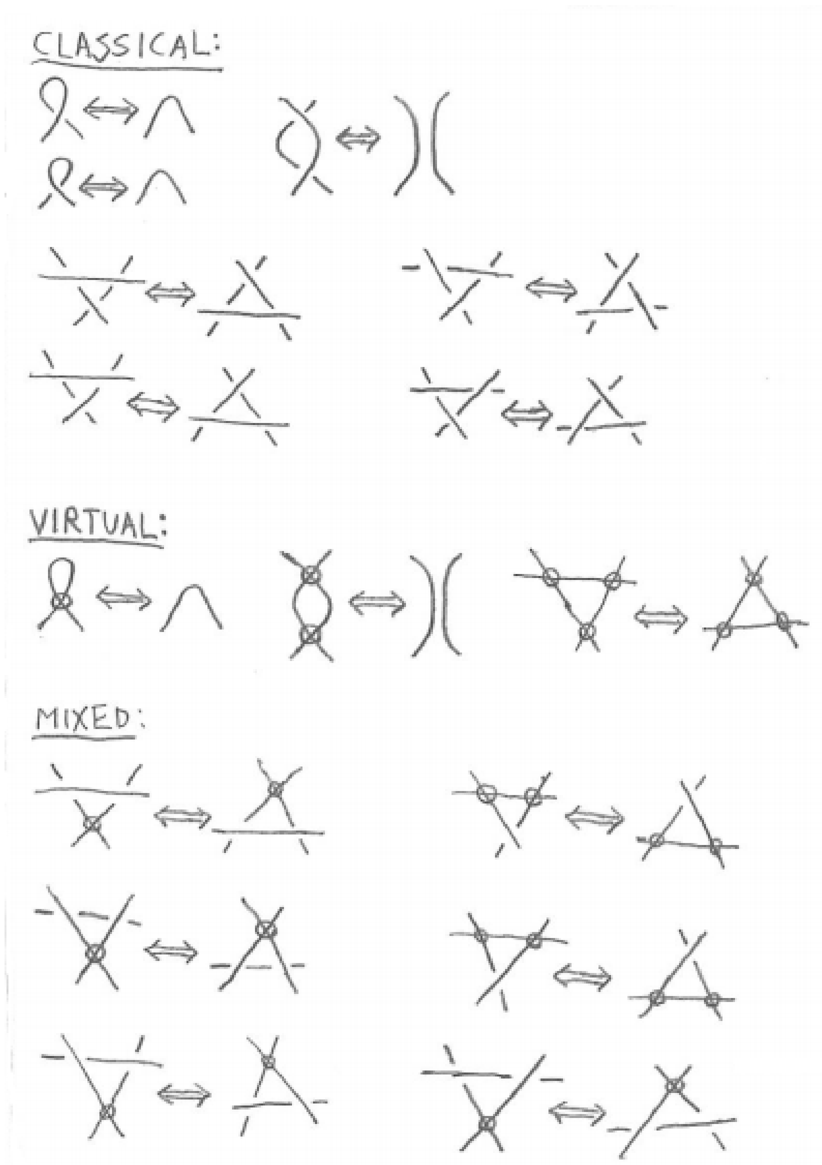
The singularity types of generic maps in dimension n are called **features** of n-knot diagrams. Degree-two transverse crossings are the only feature that occurs in generic maps of curves in the plane. In general for dimension n, the list of features is more diverse (though always finite). The list must be known before one can declare the universe for an n-knot theory. In this paper, the lists of features for $n = 1, 2, 3$ are described. (The list for $n = 0$ is empty.) Instructions for generating this list for higher n can be found in (Roseman, 2000).

Dimension n	0	1	2	3
# of features	0	1	3	5

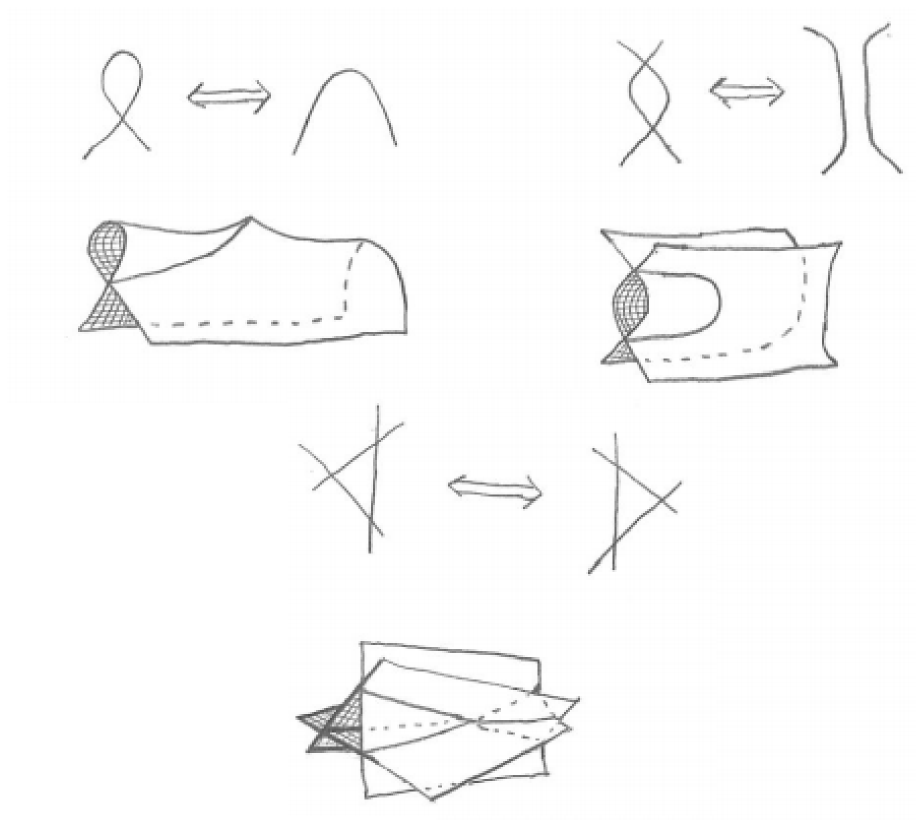
1.1.2 Move-sets

A generic map (without crossing data) can be transformed by applying one of the three unsigned **Reidemeister moves** to a small disk in the plane containing part of the image. (The word 'unsigned' refers to the absence of crossing data.) A move is a homotopy of the generic map. If the generic map is known only up to isotopy, so it is encoded as a plane graph, then the move may be understood combinatorially as an operation on this graph.

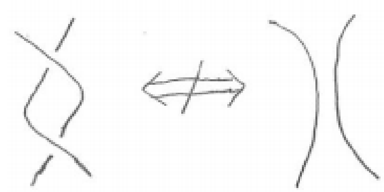
Each unsigned Reidemeister move can be enhanced with crossing data in various combinations. This gives us the following complete list of possible moves. (Note: We assume that $\mathbb{R}^2$ is oriented, so the left- and right-handed versions of the I-move and III-move are considered distinct.) The list is organized into three groups, depending on which of the four universes the moves apply to.



A move applied to a diagram can be interpreted as a homotopy $\{f_t : X \to \mathbb{R}^2\}$, which can be written as a level-preserving map $F : X \times I \to \mathbb{R}^2 \times I$. The three Reidemeister moves are precisely the homotopies whose level-preserving maps F are generic maps of the surface $X \times I$ into 3-space. (Generic maps of surfaces are discussed in the next section.)



Furthermore, the crossing data on f_t must extend continuously to the crossings of F (which are curves). For example, the following “false move” is not included in our list, because it fails this criterion:



A **move-set** is any subset of the list.

Relative to a given universe, a move is **valid** if it involves only features that are allowed in the universe, and crossing data in allowed arrangements. For example, the classical universe allows classical crossings but forbids virtual crossings, so only the seven pure-classical moves are valid there, while the other nine moves are invalid. A move-set is **valid** if it contains only valid moves.

1.1.3 Diagrammatic theories

A diagrammatic 1-knot theory consists of a universe and a valid set of moves. Any valid subset of the above list of moves can be used. Thus there are

- One simple 1-knot theory
- 2^7 classical 1-knot theories
- 2^3 virtual 1-knot theories
- 2^{16} mixed 1-knot theories.

Two diagrams in a given knot theory are called **equivalent diagrams** if one can be transformed into the other by applying a sequence of moves from the theory's move-set. The equivalence classes are called the **knot types** of the theory.

1.1.4 Relations between theories

Given two theories A and B, we say A **embeds into** B if

- every knot type of A is a subset of a knot type of B, and
- no two knot types of A are in the same knot type of B.

Thus two diagrams in A are equivalent in A if and only if they are equivalent in B. Theory B may have a larger universe than A.

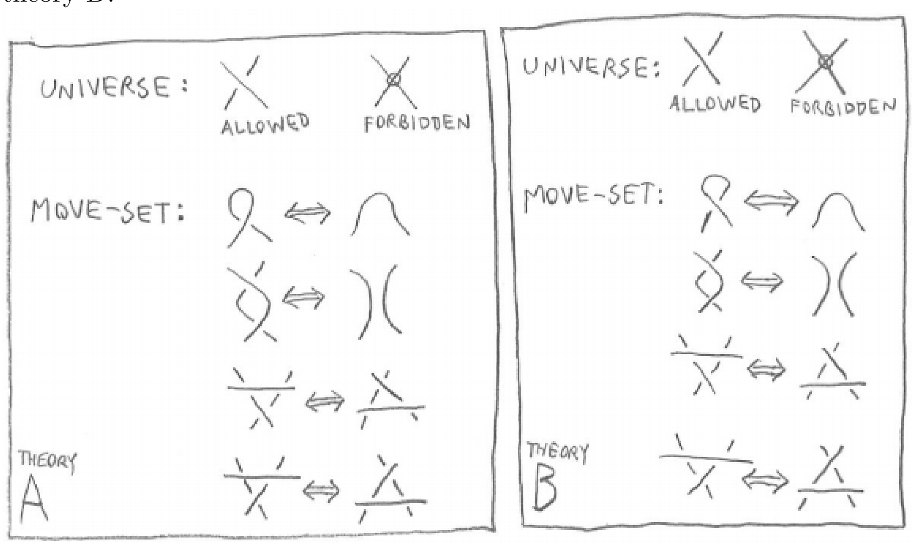
Given two theories A and B, we say A **maps onto** B if

- they have the same universe, and
- every knot type of B is the (disjoint) union of knot types of A.

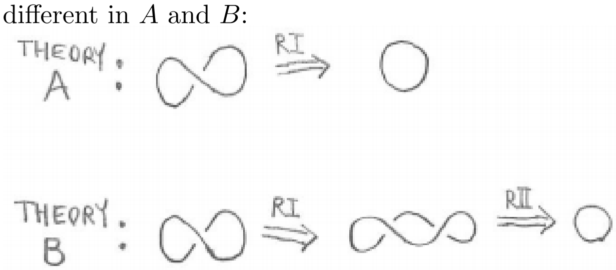
Thus if two diagrams are equivalent in A, then they are equivalent in theory B.

Two theories are **isomorphic** if they have precisely the same knot types. That is, A both embeds into and maps onto B.

EXAMPLE. Theories A and B, given below, both use the classical universe but have distinct move-sets. However, the theories are isomorphic. Two diagrams are equivalent in theory A if and only if they are also equivalent in theory B.



The sequence of moves used to transform one diagram into another may be different in A and B:



The 2^{16} definable 1-knot theories can be classified up to isomorphism. This classification has not yet been investigated.

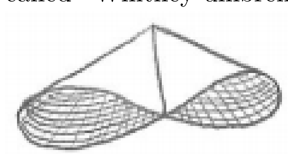
1.2 2-knot theories

Terminology. In this writing we use the term **2-knot diagram** to refer to an image of *any* closed surface (not necessarily connected, not necessarily spheres, not necessarily orientable) under a **generic map** (defined below). In the literature, the term *surface knot diagram* is sometimes preferred for this notion, while the term *2-knot diagram* refers specifically to an image of S^2. That will not be our usage here.

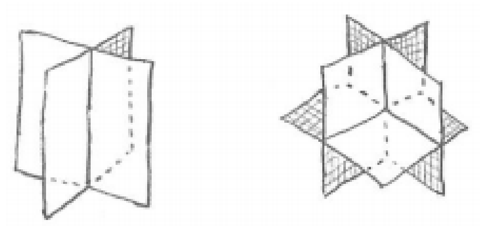
1.2.1 Universes

Let X be any closed surface, not necessarily connected. A C^∞ map $f : X \to \mathbb{R}^3$ is **generic** (and tame) if:

- it is an immersion at all but finitely many points of X, called **branch points**, where the image locally looks like the cone over a figure-8 (a so-called "Whitney umbrella"),

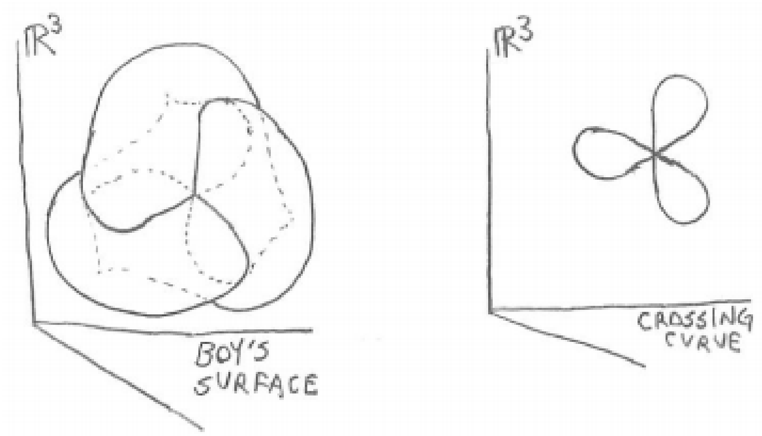
  

- it is one-to-one everywhere except on finitely many immersed curves in X, where the map behaves locally like the transverse crossing of two or three coordinate planes in Cartesian $\mathbb{R}^3$. These curves are called **crossings** of f. They may be immersed circles or immersed open-ended intervals (whose ends are the aforementioned branch points). There can be only finitely many triple points.



Crossing curves, triple points, and branch points comprise a complete list of the **features**— that is, singularity types— of generic maps of surfaces. This list can be recovered from the unsigned Reidemeister moves of 1-knot theory. Viewed as level preserving maps, the Reidemeister I-move is a Whitney umbrella with a branch point; the Reidemeister II-move is a crossing curve; and the Reidemeister III-move contains a triple point. The complete list of singularity types in n-knot diagrams can, in general, be recovered from the moves of $(n-1)$-knot diagrams, although for $n > 2$ the correspondence is not one-to-one.

The word **crossing** is often used to refer to an *entire* crossing curve— either an immersed circle, or an immersed interval running from branch point to branch point. When a crossing passes through a triple point, it does so in the same manner as a coordinate axis passes through the intersection of the three coordinate planes at the origin in Cartesian $\mathbb{R}^3$. For example, Boy's surface contains only one crossing, a circle with a triple point.



The notion of **isotopic** generic maps $X \to \mathbb{R}^3$ is analogous to the definition given for 1-knots. Let f_0 and f_1 be generic maps $X \to \mathbb{R}^3$. The following two conditions are equivalent.

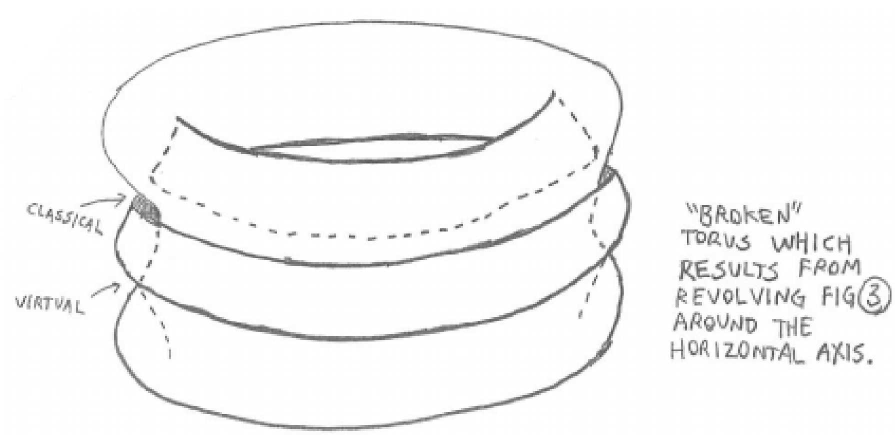
- There exists an orientation-preserving homeomorphism $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\mathrm{im}(f_1) = \mathrm{im}(\psi \circ f_0)$.
- There are homeomorphisms $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ and $\phi : X \to X$, with ψ orientation-preserving, satisfying $f_1 = \psi \circ f_0 \circ \phi$.

If these conditions are met, we say f_0 and f_1 are **isotopic**; this defines an equivalence relation on generic maps $X \to \mathbb{R}^3$. The relations **isotopic respecting orientations** or **components** are also defined similarly as they were for 1-knot diagrams.

In the case of 1-knots, the images of generic maps $X^1 \to \mathbb{R}^2$ are plane graphs, so the isotopy type of a such a map can be encoded as the isomorphism type of a plane graph. For 2-knots, however, it is more complicated to give a combinatorial description. The images of generic maps $X^2 \to \mathbb{R}^3$ are, in a sense, a higher-dimensional analog of plane graphs, but there are some complications: The crossing curves may be topologically knotted in $\mathbb{R}^3$, as may handles of the surface. Nonetheless, the essential topological structure of a generic map

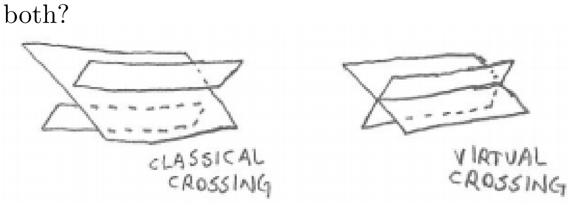
$X^2 \to \mathbb{R}^3$, taken up to isotopy, contains only a finite amount of data. The question of how to encode that data will not be considered here.

We can impose **crossing data** at each crossing curve of a generic map $f : X^2 \to \mathbb{R}^3$. This means the crossing is designated either "classical" or "virtual"; at a classical crossing, we designate the two intersecting sheets "over" and "under" in a continuous manner over the length of the crossing. We signify this assignment graphically by drawing a "break" in the undersheet. A virtual crossing is drawn without a break or any other decoration. A generic map endowed with crossing data is called a **2-knot diagram**, and a drawing of its image, so decorated, is called a **broken surface diagram**.

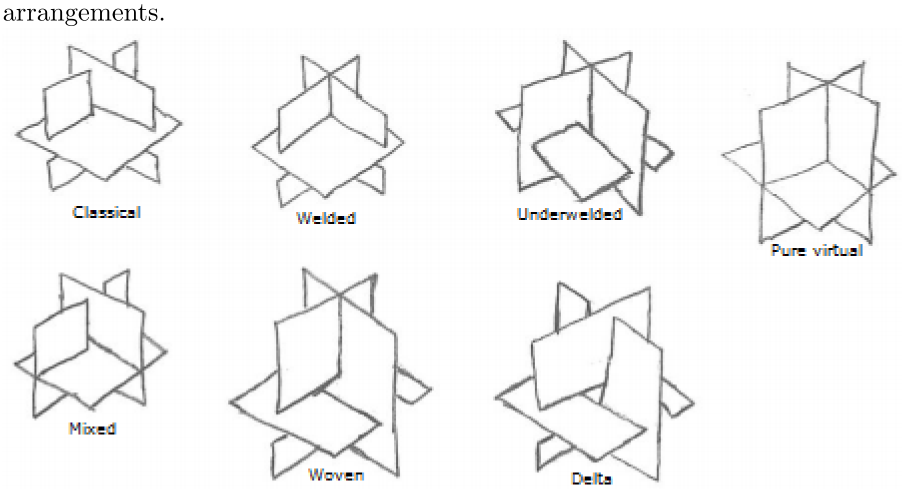


The **universe** of a 2-knot theory is a subset of the set of all diagrams, determined by the following choices.

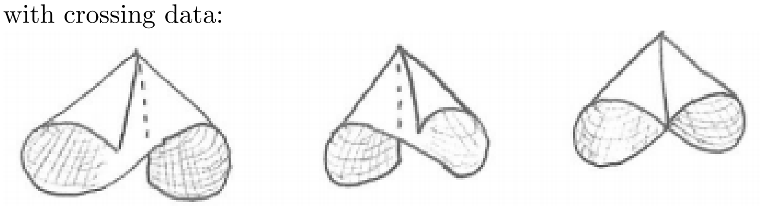
- Are crossings permitted?
- If so, what types of crossing data are permitted— classical, virtual, or both?



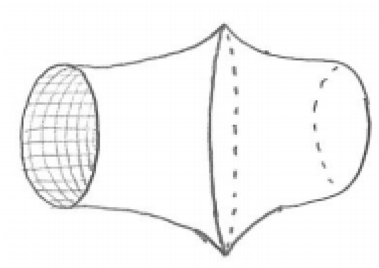
- Are triple points permitted? If so, what combinations and arrangements of crossing data are permitted at triple points? There are seven possible arrangements.



- Are branch points permitted? If so, what types of crossing data are permitted at branch points? There are three ways to decorate a branch point with crossing data:



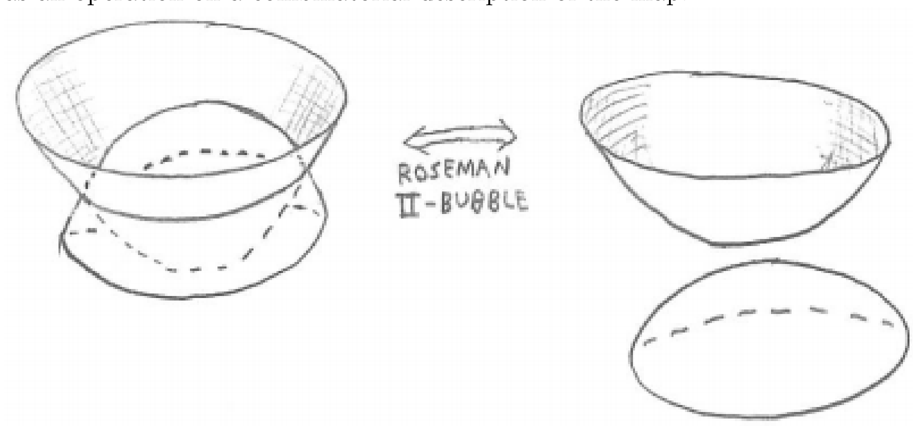
Branch points always occur in pairs. If X is orientable, then every left-handed classical branch point must be paired with a right-handed one, and vice-versa.

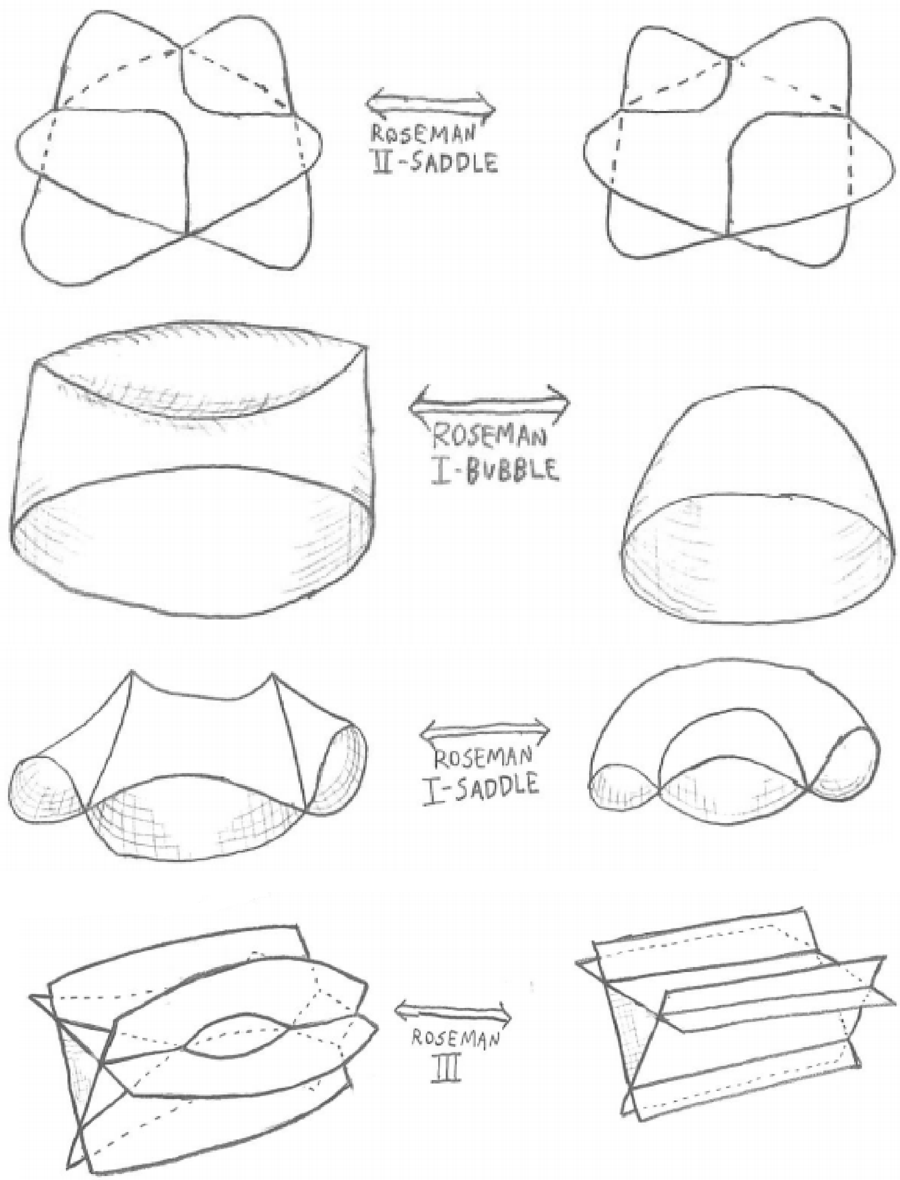


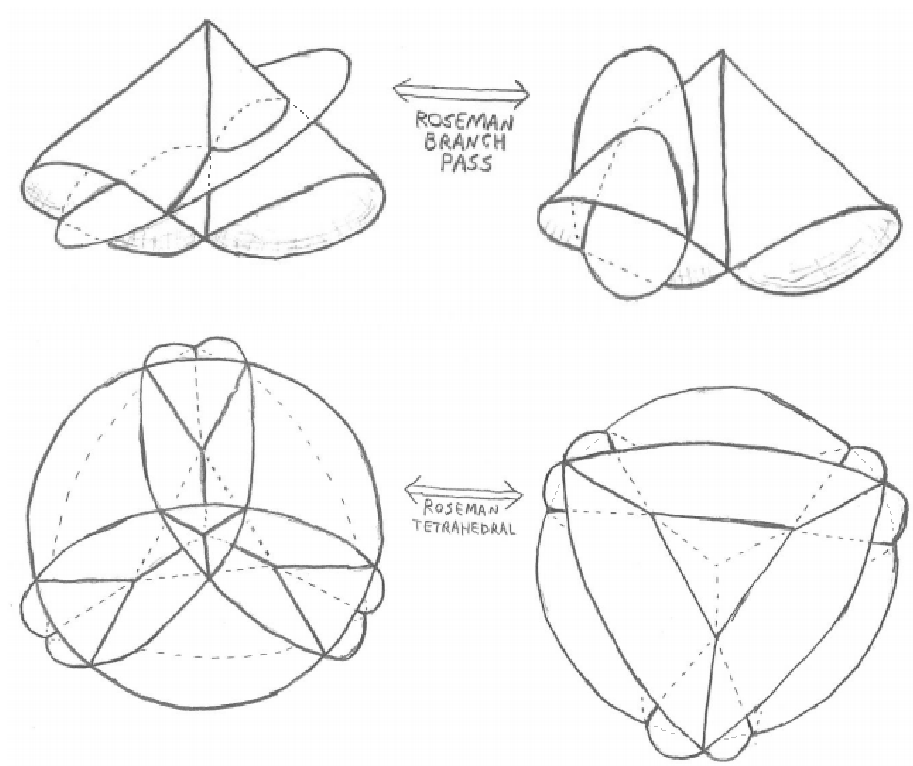
There are $(1 + 2^4 + 2^2 + 2^{10}) = 1045$ possible universes for 2-knot theories. These universes fall into four broad categories— **simple, classical, virtual,** and **mixed**— depending on which types of crossing data are allowed.

1.2.2 Move-sets

A generic map (without crossing data) can be transformed by applying one of the seven unsigned **Roseman moves** to a small region of its image. (Roseman, 1998) A move can be can be understood as a homotopy of the generic map, or as an operation on a combinatorial description of the map.




ROSEMAN
II-SADDLE
ROSEMAN
I-BUBBLE
ROSEMAN
I-SADDLE
ROSEMAN
III



Each unsigned Roseman move can be enhanced with crossing data. This gives a much longer list of 66 enhanced moves. The chart gives the number of ways each unsigned move can be enhanced with crossing data. A list of the 42 tetrahedral moves is given in the appendix.

move	classical	virtual	mixed	total
I-bubble	1	1	-	2
I-saddle	1	1	-	2
II-bubble	1	1	-	2
II-saddle	1	1	-	2
III	2	1	4	7
branch-pass	4	1	4	9
tetrahedral	4	1	37	42

A **move-set** is any subset of these 66 moves.

For a given universe, a move is **valid** if it involves only features that are allowed in the universe. For example, if virtual branch points are forbidden, then the virtual I-bubble and I-saddle moves are invalid, as well as three of the nine types of branch pass. A move-set is **valid** if it consists only of valid moves.

1.2.3 Diagrammatic theories

A 2-knot theory consists of a universe and a move-set. There are are a large, but finite, number of such theories. Two diagrams are **equivalent** in a theory if one can be transformed into the other via a series of isotopies and moves. The equivalence classes are called the theory's **knot types**.

1.2.4 Relations between theories

The notions of **embeds into**, **maps onto**, and **is isomorphic to** are the same as they were for 1-knot theories. The problem of classifying 2-knot theories by isomorphism type has not yet been explored.

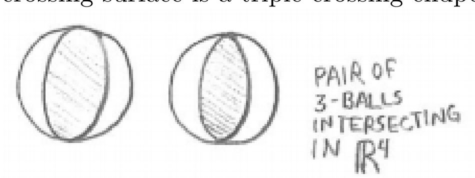
1.3 3-knot theories

This section briefly summarizes the framework for defining 3-dimensional diagrammatic knot theories.

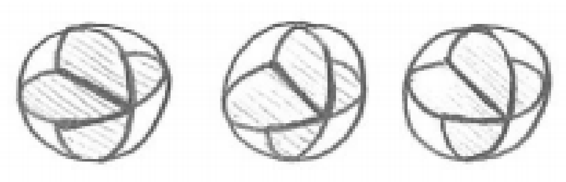
1.3.1 Universes

Let X^3 be a closed 3-manifold. A C^∞ map $f : X^3 \to \mathbb{R}^4$ is **generic** if it is one-to-one and an immersion everywhere, except at singularities of the following five types, the **features** of a generic map. (The descriptions that follow refer to the image of f.)

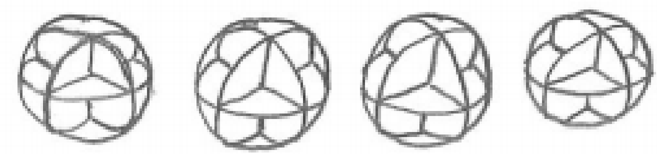
- **Double-crossing surface.** This is the intersection of two hyperplanes. It is an immersed surface, possibly with boundary, self-intersection, and "Whitney umbrellas". Three double-crossing surfaces can intersect along a *triple-crossing curve.* The boundary components of a double-crossing surface are branch circles. The vertex of a "Whitney umbrella" in a double-crossing surface is a triple-crossing endpoint.



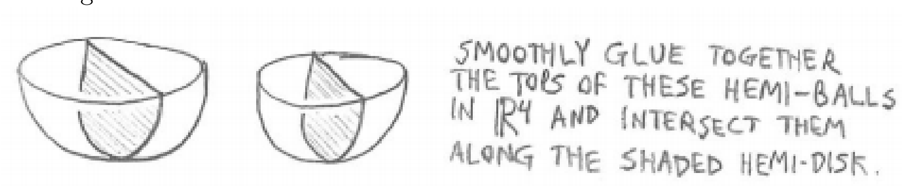
- **Triple-crossing curve.** This is an immersed curve, the intersection of three double-crossing surfaces. (To clarify: Within each of the three hyperplanes which meet at this curve, there are two double-crossing surfaces that intersect transversely.) A triple-crossing curve may be an immersed circle or an immersed interval. If it is an interval, each endpoint lies on a branch circle. Triple-crossing curves can have degree-4 intersections, like the coordinate axes in 4-space.



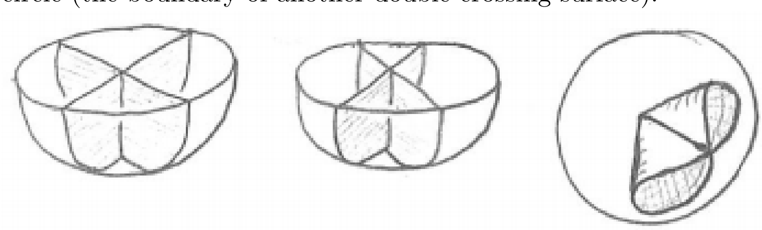
- **Quadruple-crossing point.** This is an isolated point. It occurs at the intersection of four triple-crossing curves, and six double-crossing surfaces.



- **Branch circle.** This is an embedded circle, the boundary of a double-crossing surface. Branch circles do not touch each other.



- **Triple-crossing endpoint.** This is an isolated point, the vertex of a double-crossing surface's "Whitney umbrella". It also lies on a branch circle (the boundary of another double-crossing surface).



The above list can be obtained from the Roseman moves for 2-knot diagrams. Regard each Roseman move as a homotopy acting locally on a diagram $X^2 \to \mathbb{R}^3$, and represent this homotopy as a level-preserving map $F : X^2 \times I \to \mathbb{R}^3 \times I$. Then (ignoring boundary effects and ignoring the levels) F is a generic map of a three-manifold into four-space, containing one or more of the 3-knot features listed above. In fact:

- The II-bubble move and II-saddle move both correspond to a double crossing surface (with two different choices of "height" function added to the diagram).
- The II-move corresponds to a triple-crossing curve.
- The tetrahedral move corresponds to a quadruple-crossing point.
- The I-bubble move and I-saddle move both correspond to an arc of a branch circle (with two different choices of "height" function added to the diagram).
- The branch-pass move corresponds to a triple-crossing endpoint.

We may apply crossing-data to each of the five features in various ways. Every double-crossing surface is either virtual or classical with designated over- and under-hypersheet. Virtual and classical crossing data combines in various spatial arrangements in the other four features. The **universe** of a 3-knot theory is set by declaring which arrangements are and are not allowed in the knot diagrams.

1.3.2 Move-sets

Roseman cataloged the unsigned moves for 3-knot diagrams. There are twelve of them. The unsigned moves can be further refined by adding crossing-data. A move is considered **valid** relative to a given universe if it involves only features included in that universe, with only crossing data allowed in that universe. A set of valid moves forms a **move-set** for a **diagrammatic 3-knot theory**. A move-set determines an equivalence relation on the universe, the **knot-types**. Relations of 3-knot theories are the same as defined in the preceding sections.

2 Examples of 1-knot theories

In this section, we describe some popular diagrammatic 1-knot theories using the format laid out in section 1.1.

For each theory, we give a **topological invariant**, that is, an algorithm that takes as input a knot diagram D from the theory, and outputs an object $h(D)$ of some kind (for example, a topological embedding). The object $h(D)$ represents an equivalence class $[h(D)]$ (for example, its isotopy class of embeddings) which also contains the output object $h(D')$ for any diagram D' that is equivalent to D within the knot theory being considered.

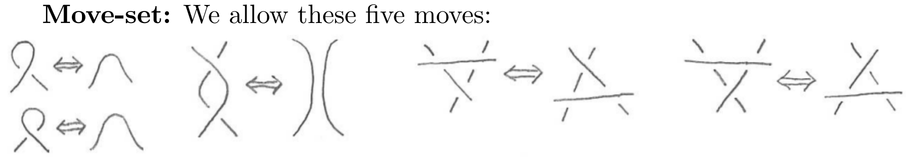
Some topological invariants are **complete**. This means that the class $[h(d)]$ does not contain the output object $h(D'')$ for any diagram D'' that is *not* equivalent to D.

2.1 Classical knot theory

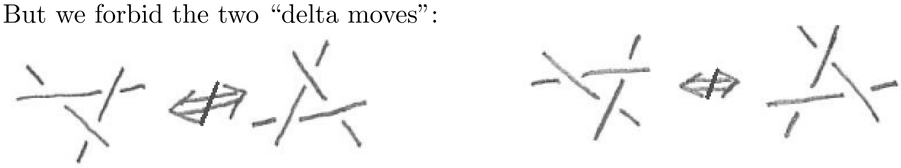
The most familiar of all knot theories.

Universe: We allow classical crossings, but forbid virtual crossings.

Move-set: We allow these five moves:



But we forbid the two "delta moves":

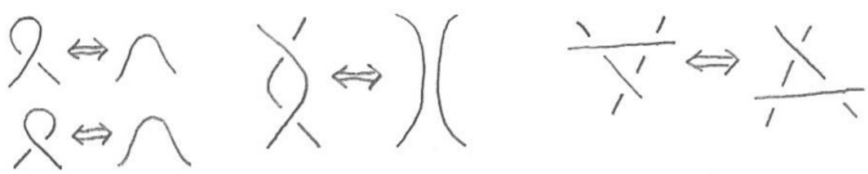


Relation to other theories: Classical knot theory is isomorphic to a number of other theories because its move-set contains some redundancies— it is unnecessary to include both the left- and right-handed versions of the I-move or

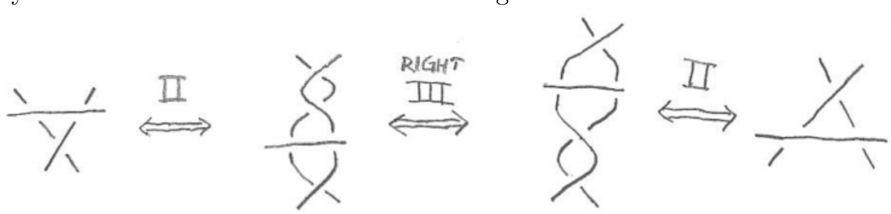
III-move, so long as the II-move is included. The following restricted move-set induces the same equivalence relation on diagrams as the move-set listed above.

In this restricted theory, although the left-handed I-move is not included, it can be emulated by a combination of the II-move and the right-handed I-move. Thus the diagram equivalence relations are the same for both move-sets, but the number of moves required to transform one diagram into another may differ.

Here is another restricted move-set that gives a theory isomorphic to classical knot theory:



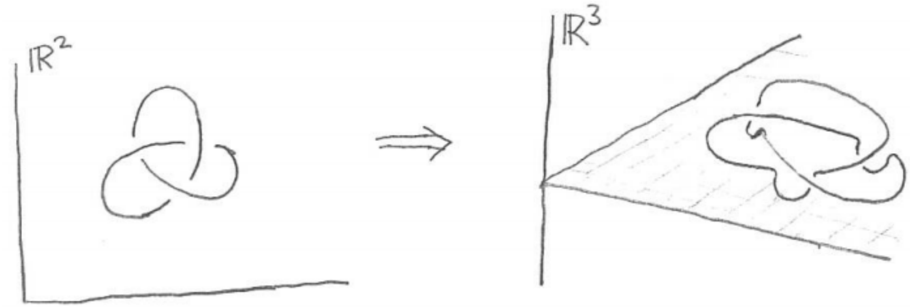
In this theory we have omitted the left-handed III-move, which can be emulated by a combination of the II-move and the right-handed III-move.



Due to these redundancies, there are nine distinct move-sets which all give theories isomorphic to classical knot theory. These same redundancies will appear again in delta knot theory, in Kauffman's virtual knot theory, and in welded knot theory.

Classical knot theory embeds in Kauffman's virtual knot theory and welded knot theory, but not in delta knot theory.

Topological invariant: To each diagram D (whose domain X is a collection of circles), we associate a smooth embedding $k : X \to \mathbb{R}^3$, as follows. Let $f : X \to \mathbb{R}^2$ be the generic map underlying D, and let $i : \mathbb{R}^2 \to \mathbb{R}^3$ be the inclusion of the coordinate xy-plane into xyz-space. Now modify the immersion $i \circ f$ in a small neighborhood of each under-crossing point in X, by adding a smooth negative bump to the z-coordinate of the image there. (Think of this operation as "filling in" the broken arc conventionally drawn in the knot diagram.) The result is an embedding k whose isotopy type is a complete invariant of the knot type of D, as proven by Reidemeister.

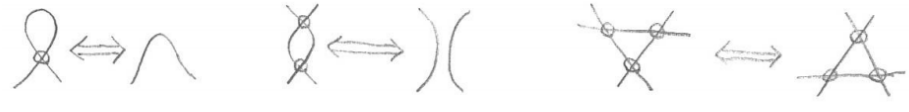


2.2 Pure-virtual knot theory

Here, all crossing data must be virtual. This theory is "trivial" in the sense that only one knot type exists for each X— every knot can be unknotted.

Universe: We allow virtual crossings, but forbid classical crossings.

Move-set: We allow all three pure-virtual moves. There are no forbidden moves.



Relation to other theories: This theory is unique— any theory using the purely virtual universe but a different move-set is non-isomorphic to this one. In fact, the 2^3 theories using this universe are all distinct, because it is impossible to emulate any one of the three pure-virtual moves using some combination of the other two.

Pure-virtual knot theory is related to only one other theory in this chapter: Rotational pure-virtual knot theory maps onto it.

Topological invariant: In this theory, the knot types are homotopy classes of smooth maps $X \to \mathbb{R}^2$. This is because any smooth map $X \to \mathbb{R}^2$ is homotopic to a generic map, and any homotopy can be deformed into one which is a sequence of unsigned Reidemeister moves. Since $\mathbb{R}^2$ is simply connected, there is only one homotopy class of such maps, so the knot types are determined entirely by X, which in turn is determined entirely by its number of component circles. Thus, our "topological" invariant is just an integer, the number of components of X, and this invariant is complete.

2.3 Rotational pure-virtual knot theory

This is a refinement of pure-virtual knot theory, formed by restricting that theory's move-set. The word *rotational* (coined in [Kauffman, *New Ideas*] means that the homotopies underlying virtual moves are regular— at no moment in time can the map f_t contain a point, local (in spacetime) to a virtual crossing, where the derivative vanishes. The virtual Reidemeister I-move forces such a singularity. This theory forbids virtual I-moves, hence the name.

Universe: We allow only virtual crossings.

Move-set: We allow only these two moves:

Relation to other theories: Rotational pure-virtual knot theory is not isomorphic to any other theory. It maps onto "ordinary" pure-virtual knot theory.

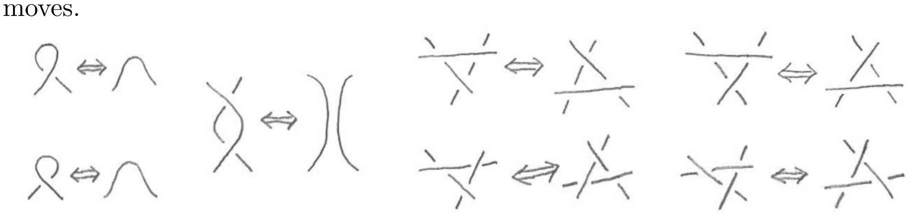
Topological invariant: The regular homotopy class of a circle immersed in the plane is determined by the circle's turning number. Our knot diagrams consist of unoriented immersed circles, so the knot type of a diagram is determined by the (absolute value of) the turning numbers of its components.

2.4 Delta knot theory

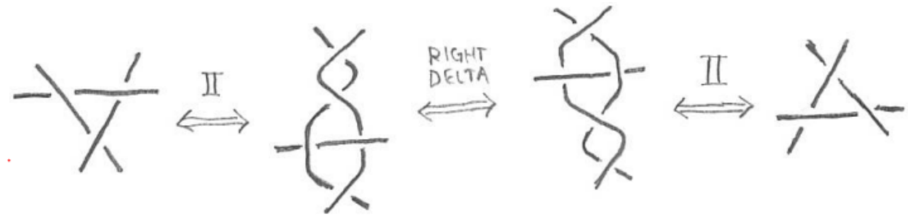
So-called because it includes the two "delta moves" that are forbidden in classical knot theory. Like pure-virtual knot theory, this theory is "trivial" since all knots are unknotted (proven by Murakami and Nakanishi (1989)). However, the two theories are unrelated (neither embeds into, nor maps onto, the other).

Universe: We allow only classical crossings.

Move-set: We allow all seven classical moves. There are no forbidden moves.



Relation to other theories: As noted above, it is redundant to include both versions of the I-move or both versions of the III-move when the II-move is included. Additionally, it is redundant to include both the left- and right-handed delta moves when the II-move is included:



There are no other redundancies; any restriction of the move-set besides those just described results in strictly finer diagram equivalence. Thus there are 27 distinct move-sets that all give theories isomorphic to delta knot theory.

Clearly, classical knot theory maps onto delta knot theory.

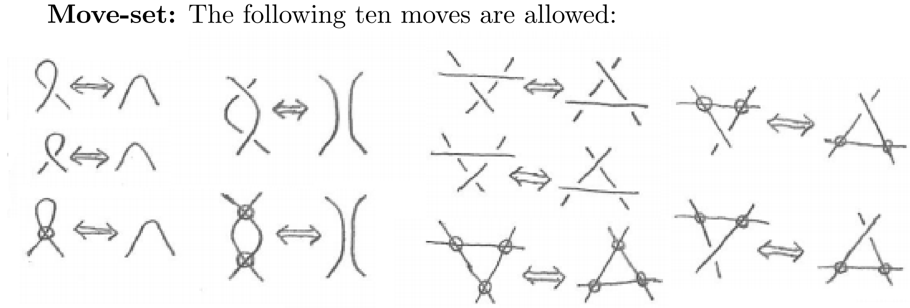
Topological invariant: Since this theory, like pure-virtual knot theory, is trivial, the invariant is, again, just the number of components of X, and this invariant is complete.

2.5 Kauffman's virtual knot theory

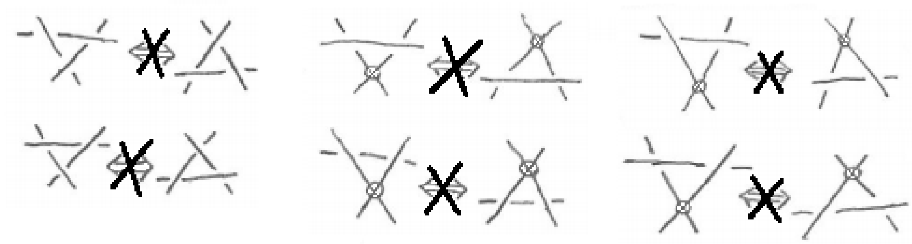
First defined in (Kauffman, 1999), this is the most-studied extension of classical knot theory that exists.

Universe: We use the "mixed" universe. All diagrams are allowed. Crossings may be classical or virtual.

Move-set: The following ten moves are allowed:



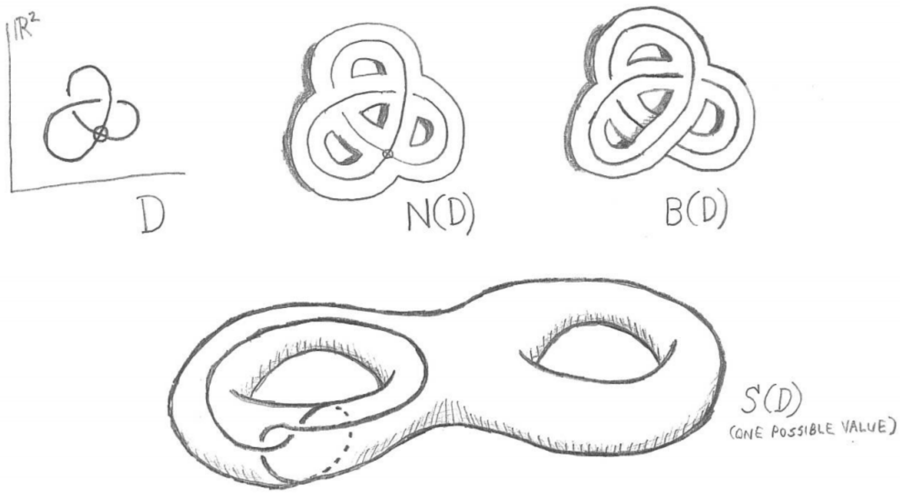
The following six moves are forbidden:



Relation to other theories: As previously noted, the inclusion of both classical I-moves and III-moves is redundant; besides these, there are no other redundancies in the move-set. Classical knot theory embeds into virtual knot theory; this was first shown in (Kauffman, 1999) and (Goussarov, Polyak, & Viro, 2000) for one-component virtual knots, later extended to the general case in (Kuperberg, 2003). Virtual knot theory maps onto welded knot theory, proven in (Rourke, 2006).

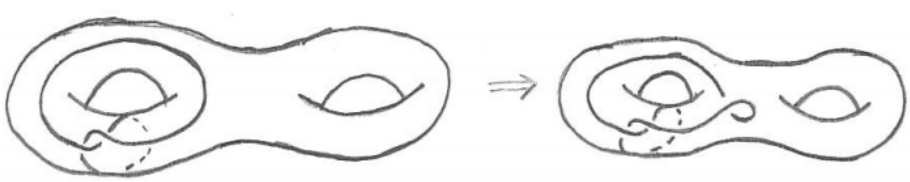
Topological invariant: The idea is to think of a virtual knot diagram as a classical diagram drawn on a closed orientable surface. We then define an equivalence relation on these objects that extends classical move-equivalence and allows the surface to vary.

Take as input a virtual knot diagram D. Let $N(D)$ be a regular neighborhood of the diagram, that is, a thickened plane graph. At each virtual crossing of D, double the square-shaped junction of $N(D)$ to create overlapping "bands". Call this surface $B(D)$; it has a knot diagram drawn on it with no virtual crossings. Now embed $B(D)$ into any closed orientable surface (not necessarily connected). The result, called $S(D)$, is a surface containing a classical knot diagram. The particular choice of embedding does not matter, because all the possible choices are equivalent under the following relation.

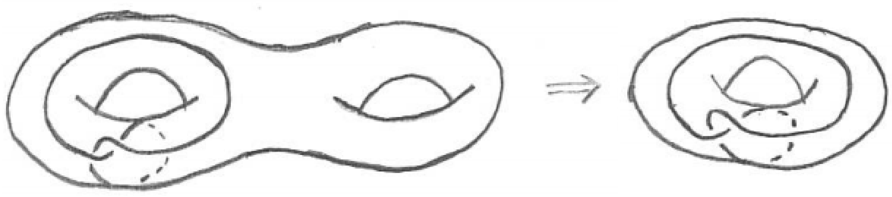


Two orientable surfaces with classical knot diagrams drawn on them are **equivalent** when one can be transformed into the other via a sequence of the following two operations.

- **Classical Reidemeister moves:** Take a disk in the surface containing part of the knot diagram. Modify the disk just as one would modify portions of a classical knot diagram, ie., by a boundary-fixing self-homeomorphism of the disk, or by any of the five classical knot moves if applicable. The result is the same surface, but with a slightly different knot diagram drawn on it.

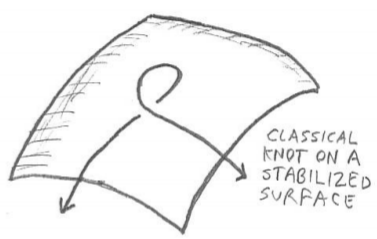


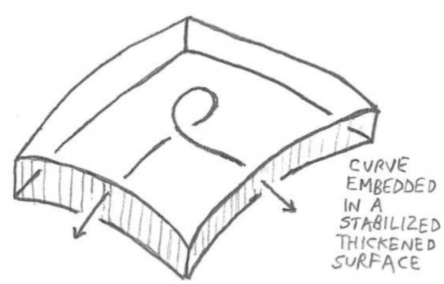
- **Stabilization:** Take a neighborhood of the knot diagram in the surface, and embed it in another closed orientable surface. The result is a new surface, but with the same classical knot diagram drawn on it.



These operations define an equivalence relation on surfaces with classical diagrams drawn on them. An equivalence class is called a **classical knot on a stabilized surface**. It is an invariant of virtual knot type. That is, if D and D' are equivalent virtual knot diagrams, then $S(D)$ and $S(D')$ are equivalent under the above operations. The converse is also true, so the invariance is complete, as proven by (Kauffman, 1999) and (Carter, Kamada, & Saito, 2002).

Another topological invariant of virtual knots is **curves embedded in stabilized thickened surfaces**. The definition is the same as the one just given, except the surfaces are now thickened (product with an interval), the knot diagrams are embedded curves, the Reidemeister moves are 3-dimensional isotopies, and stabilization involves surgery on the thickened surface (so when you take a neighborhood, you cut through the thickened surface).





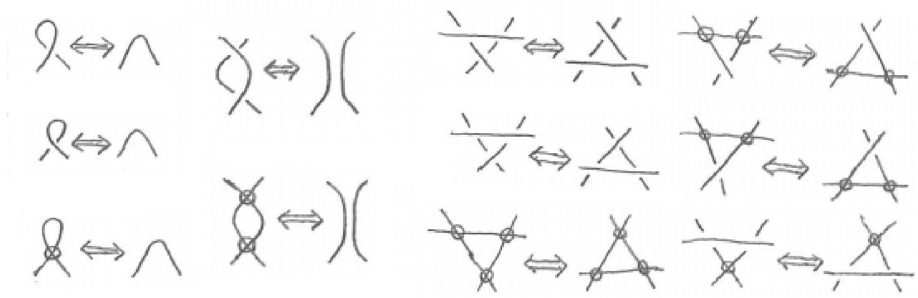
The two models correspond in much the same way as diagrammatic classical knot theory corresponds to its topological invariant, curves embedded in $\mathbb{R}^3$.

2.6 Welded knot theory

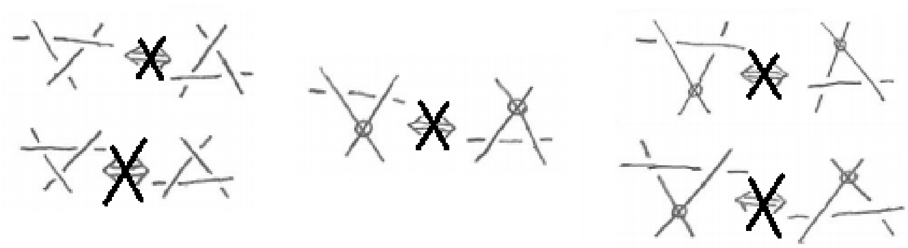
Welded knot theory is an interesting coarsening of Kauffman's virtual knot theory, formed by including one of that theory's "forbidden moves".

Universe: The universe is the same as that for Kauffman's virtual knot theory. That is, we allow diagrams with classical or virtual crossings.

Move-set: The following eleven moves are allowed:

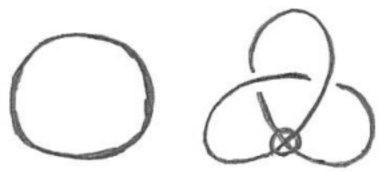


The following five moves are forbidden:



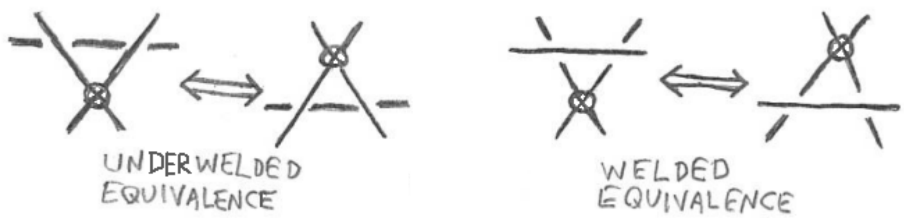
Relation to other theories: This move-set contains the same redundancies noted above, concerning the classical left- and right-handed I-moves and III-moves.

Obviously, Kauffman's virtual knot theory maps onto welded knot theory, since all the Kauffman-virtual moves are included in welded knot theory. Less obvious is that the theories are not isomorphic. Here are a pair of diagrams that are welded-equivalent but not Kauffman-virtual-equivalent:



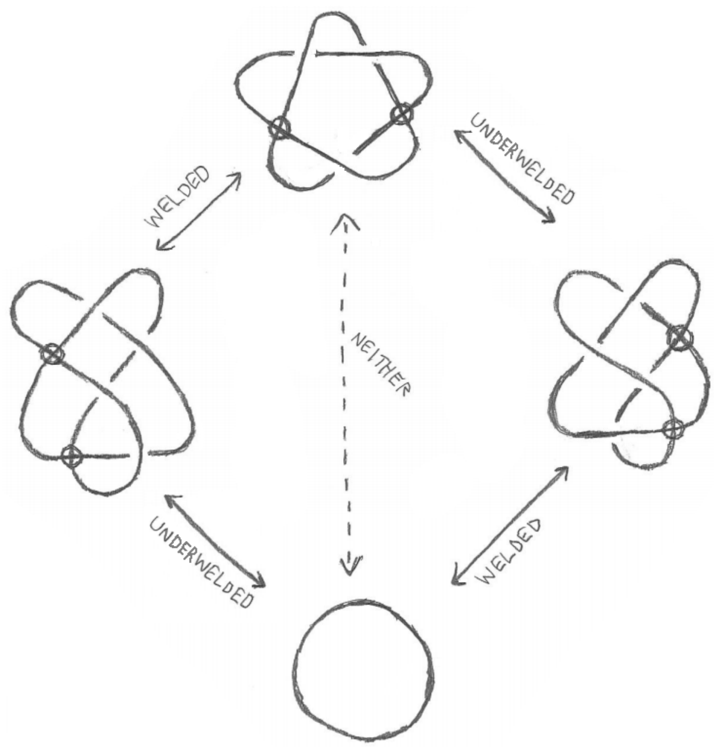
Classical knot theory embeds into welded knot theory. It is impossible to transform one classical knot diagram into another via welded moves, unless this can be done solely via classical moves.

There is another diagrammatic knot theory, called **underwelded knot theory**, which is the same as welded knot theory except that it uses the "under" forbidden move instead of the "over" forbidden move[1]:



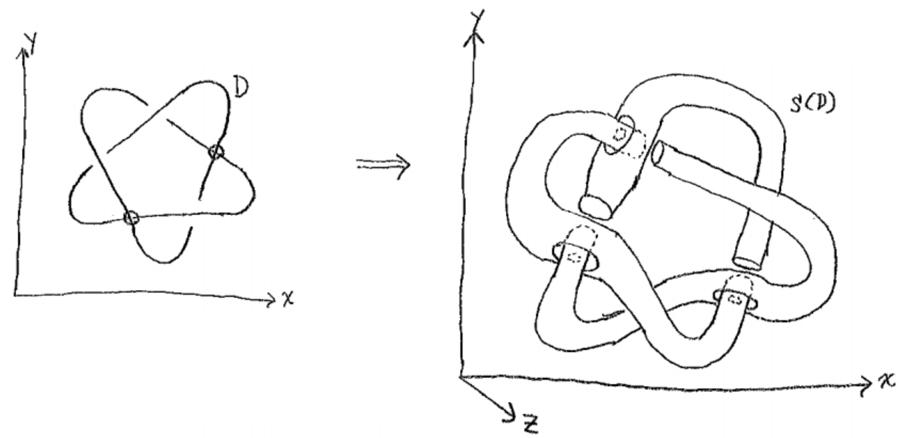
The operation of **mirror imaging** a diagram— that is, reversing the roles of under- and overstrands at every classical crossing— defines a bijection between the welded and underwelded universes, which carries welded knot types to underwelded knot types and vice-versa. However, underwelded knot theory neither embeds into nor maps onto welded knot theory. Each theory contains a pair of equivalent diagrams that are inequivalent in the other:

[1]Rourke used the word **unwelded** for the "under" forbidden move. We do not use that word in this paper; our underwelded move is the same as his unwelded move. Rourke also described an **unwelded knot theory** which, *unlike* underwelded knot theory, allows *both* forbidden moves. That theory is much coarser than underwelded knot theory— knot types are determined by their "linking numbers"; in particular, the only knot type is the unknot when X is a circle. Unwelded theory will not be discussed further in this paper.



Topological invariant: Topological interpretations of welded knot theory were put forward by (Satoh, 2000) and (Rourke, 2006). Satoh's invariant associates to any welded knot diagram D a particular surface $S(D)$ embedded in 4-space, such that welded-equivalent diagrams give rise to isotopically embedded surfaces. It is not known whether Satoh's invariant is complete. Rourke's invariant is constructed differently from Satoh's, but the result $R(D)$ is isotopic to $S(D)$. (Rourke did not explicitly prove the equivalence; I do in section 2.6.6.) Rourke's construction also includes a fiber-structure which Satoh lacks. He claims this enhancement makes his construction a *complete* invariant of welded knots, but I challenge this, and suggest (without proof) that Rourke's fiber-enhanced construction is not an invariant of welded knots at all.

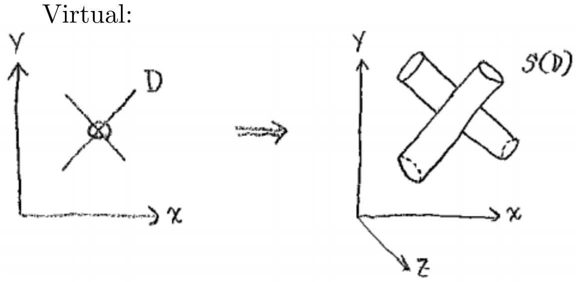
2.6.1 Satoh



I now describe the welded-knot invariant due to (Satoh, 2000).

Begin with a welded diagram D in the xy-plane. Give D an orientation, arbitrarily. (If D has k components, then there are 2^k possible orientations. All of them will result in isotopic tori, so the choice doesn't matter, and no generality is lost.) Let $S(D)$ be a union of k embedded tori in $xyzw$-space; each torus is a circular tube whose core curve is a component of D (except near virtual crossings), confined to the $w = 0$ hyperplane (except near classical crossings). Crossings are handled as follows.

Virtual:

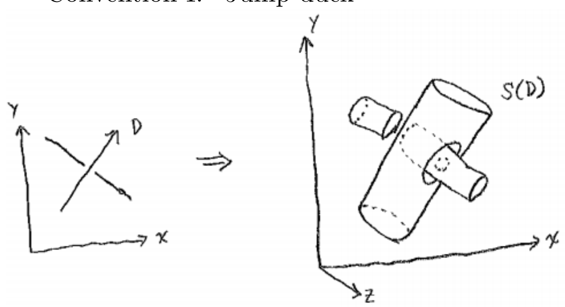


The handling of a virtual crossing does not depend on the orientation on D. The image on the right is the $w = 0$ hyperplane of $xyzw$-space. Note that the apparently arbitrary decision to place one tube higher than the other (in the

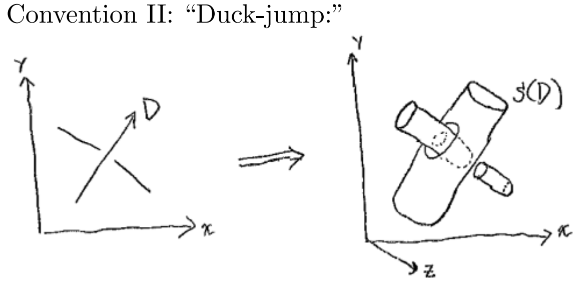
z-direction) doesn't matter— the two possibilities are isotopic in $xyzw$-space.

Classical: There are two possible conventions for handling classical crossings. It doesn't matter which convention is used, so long as it is used exclusively and consistently at every classical crossing. The two conventions produce isotopically embedded tori.

Convention I: "Jump-duck"



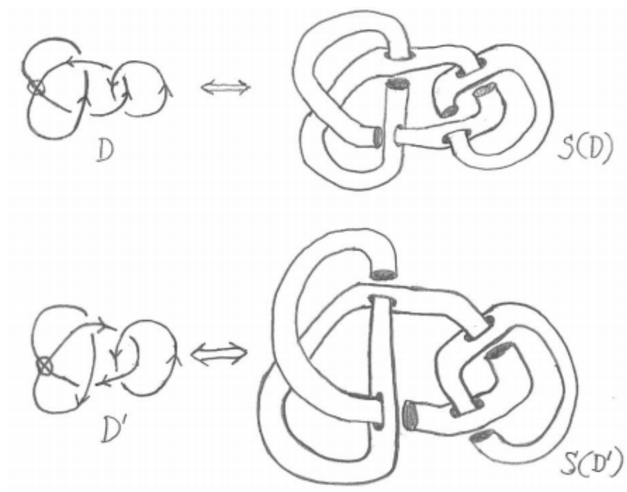
Convention II: "Duck-jump:"



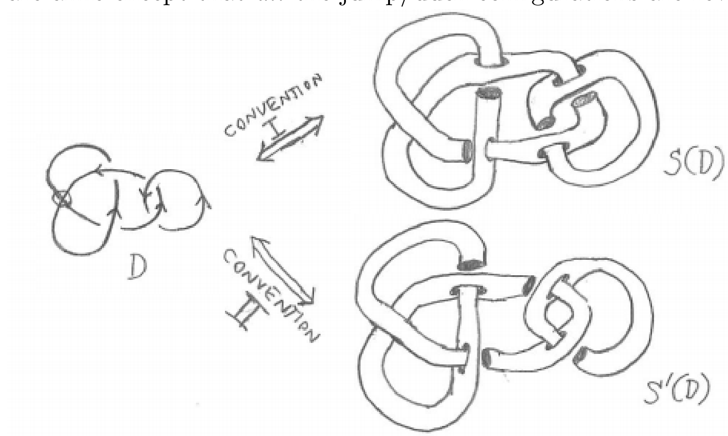
The handling of classical crossings uses the local orientation of the over-strand in D. The image on the right is a "broken surface diagram" of $S(D)$. The drawing "breaks" wherever $S(D)$ "ducks" in the negative w-direction.

2.6.2 Knot orientation and jump/duck convention

Suppose D and D' are copies of the same knot diagram, but with the opposite orientations on one or more of their components. Then the toral surfaces $S(D)$ and $S(D')$ (both constructed using the same convention, say, convention I) look alike, except that the "jump/duck" configuration is different wherever the orientation changed at a classical overcrossing.



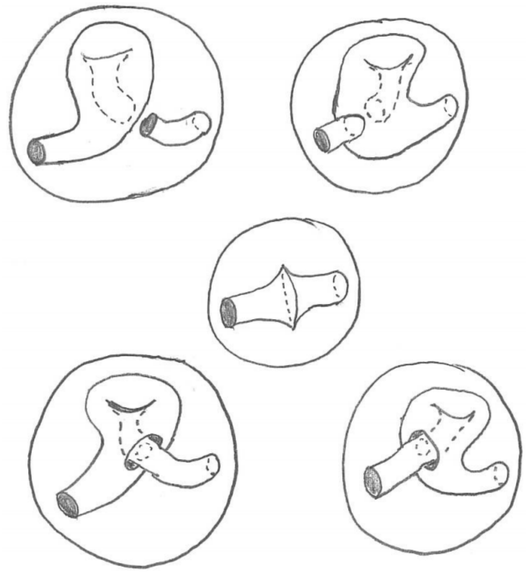
Now suppose D is an oriented diagram, and $S(D)$ and $S'(D)$ are the tori-unions constructed using conventions I and II, respectively. These two constructions are alike except that *all* the jump/duck configurations are reversed.



As it turns out, $S(D)$, $S(D')$, and $S'(D)$ are all isotopic in 4-space. The isotopies relating them can be realized using a system called **wen calculus**. We use broken-surface diagrams to describe this system pictorally; regarding these as classical 2-knot diagrams, we can use the classical Roseman moves to prove each equivalence listed below.

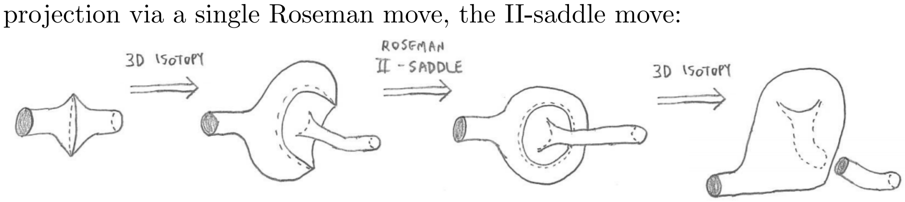
Wen calculus.

1. The following five pictures are projections of 4-balls with embedded annuli, with the annulus boundary embedded in the ball boundary:

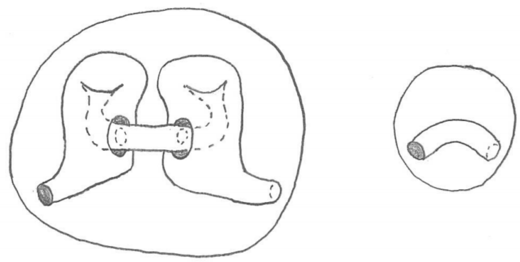


All four are isotopic. (That is, there is a boundary-preserving homeomorphism from each ball to any of the others, carrying one annulus to the other.) The annuli are called **wens**. They are similar to the "neck" of a klein bottle: They represent a tube turning inside-out in 4-space.

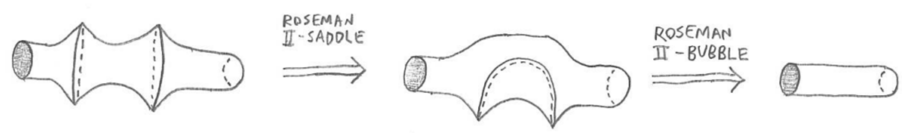
Each of the four regular projections can be obtained from the branched projection via a single Roseman move, the II-saddle move:



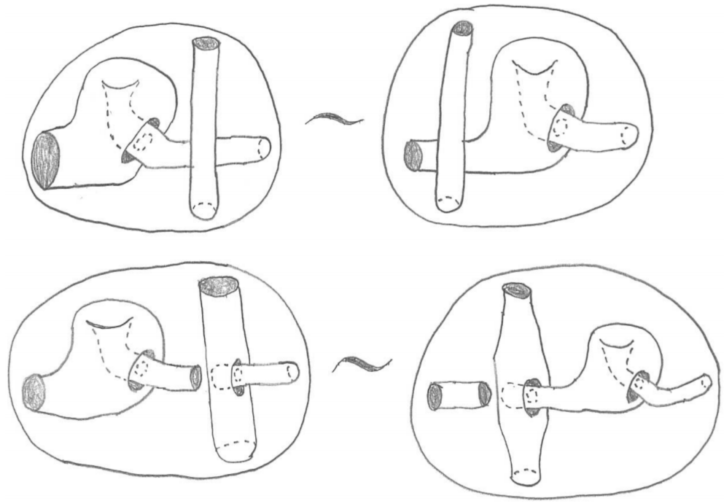
2. Wens cancel in pairs, or can be generated in pairs. These balls are isotopic:



This is proven via Roseman moves, using the “branched” version of wens (by rule #1 all the versions of wens are freely interchangeable).

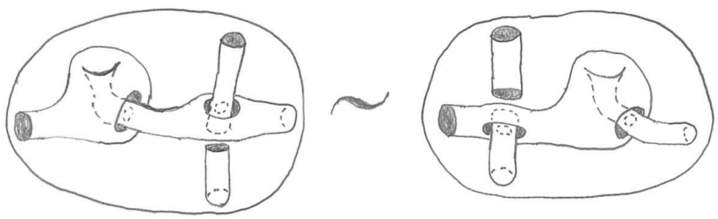


3. Wens can slide along tubes, and have no effect as they pass through the Satoh-construction of a virtual crossing or classical underpass:



The virtual case is obvious— the required 4D isotopy is just an extension of the evident 3D isotopy in the picture. The classical case is also obvious— simply move the wen through 4-space so that it ducks into, then jumps out of, the cross-tube.

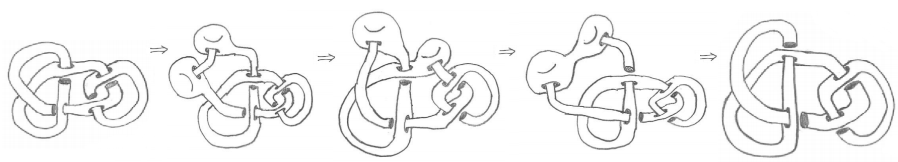
4. When a wen slides through a classical overpass, however, the jump/duck arrangement there is reversed:



   As the wen passes by, the overstrand turns inside-out, so the jump and duck swap places.

We apply this calculus to realizing isotopies between $S(D)$, $S(D')$, and $S'(D)$.

To convert $S(D)$ to $S(D')$, apply the following procedure to each tube whose corresponding component of D had its orientation reversed in D'. First, use rule 2 to create a pair of wens at some point on the tube, away from crossings. Next, use rules 3 and 4 to slide one of the wens (it doesn't matter which one) along the tube until it has made almost a full circuit. As it travels, it reverses the jump/duck arrangement at each overcrossing of this component. Last, the moving wen returns to its sibling and cancels with her by rule 2.



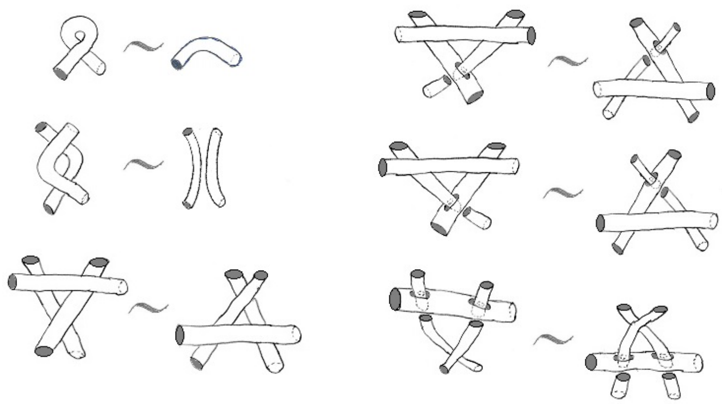
Note that $S'(D)$ is the same as $S(D'')$, where D'' is the result of switching the orientations of *every* component of D. Thus the above procedure, carried out on *every* tube of $S(D)$, produces $S'(D)$.

2.6.3 Invariance

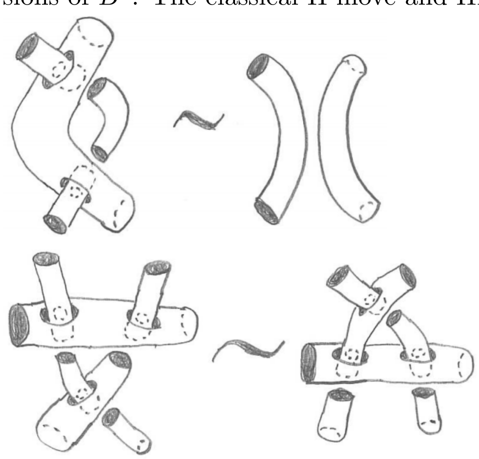
If two diagrams D and D' are welded-equivalent— that is, D can be transformed into D' by applying a series of welded moves— then $S(D)$ is isotopic to $S(D')$.

It will suffice to check this locally for every welded-move. For each move, we check that there is an isotopy of a 4-ball carrying the tubes constructed from the "before" position to the "after" position.

For the three pure-virtual moves and the three mixed moves, the required isotopies of B^4 are simply extensions of the obvious isotopies of B^3 shown here:

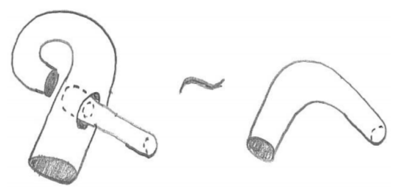


For the five pure-classical moves, the required isotopies involve all four dimensions of B^4. The classical II-move and III-move are accomplished as follows:



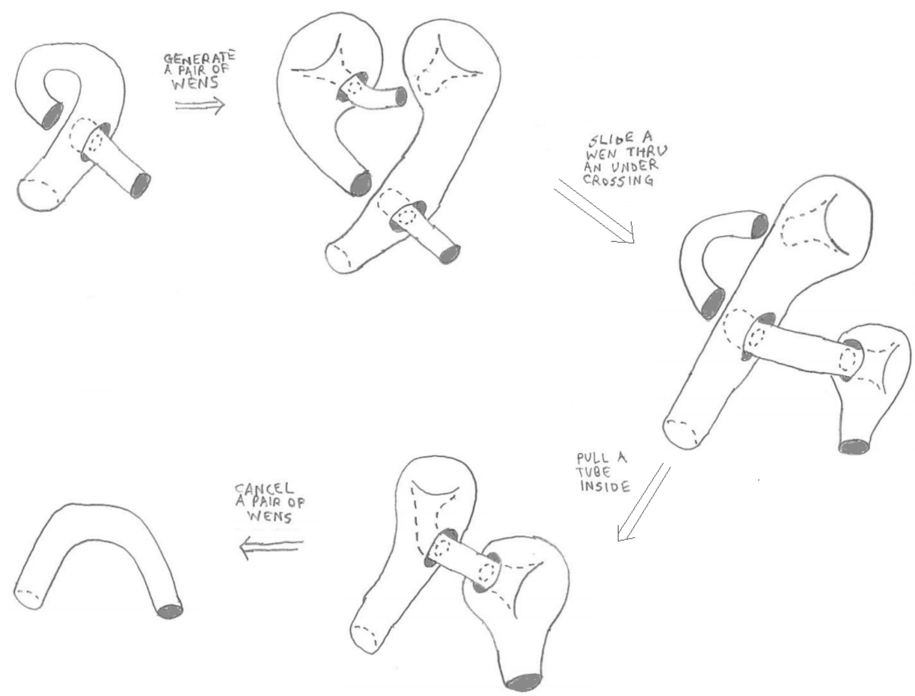
For both of these moves, the 4D isotopies are easy to see: A section of tubing simply ducks into a fat tube, then jumps out the other side.

The classical I-move is easiest to visualize using movie moves, or using wen calculus. Here is what the move is supposed to accomplish:



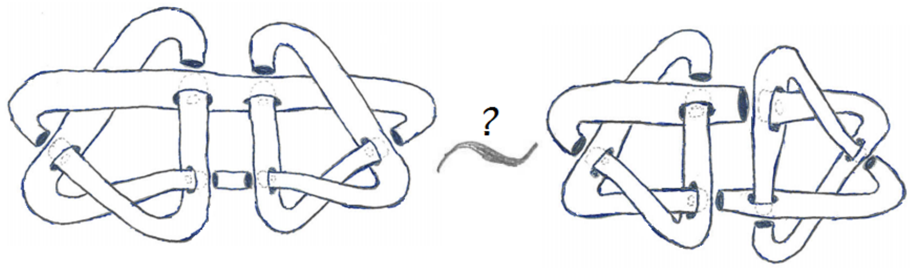
Here is a demonstration of the move accomplished using movie moves. Note that exactly four of the steps are Roseman moves.

Here is a demonstration of the move using wen calculus:

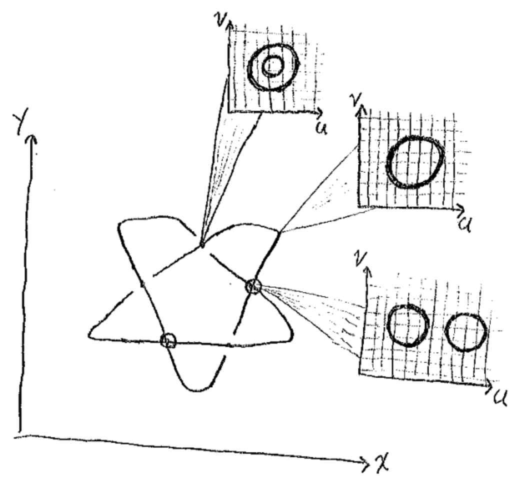


Invariance is proved. It is unknown whether Satoh's invariant is complete. There may exist two diagrams D and D' which are not welded-equivalent, but whose Satoh-tubes $S(D)$ and $S(D')$ are nonetheless istopic. The isotopy of $\mathbb{R}^4$ which carries $S(D)$ to $S(D')$ would necessarily involve maneuvers different from those depicted in the above proof, and contain intermediate states of the surface which do not look like the Satoh-tubes for any diagram. It is unknown if such an example exists. Here is a plausible example[2]:

[2] For illustrative purposes only. The author takes full responsibility if this "example" turns out to be false.



2.6.4 Rourke



I now describe the variant of Satoh's invariant, due to (Rourke, 2006).

Begin with an (unoriented) welded knot diagram D in the xy-plane of $xyuv$-space. In the uv plane above each point $(x, y) \in D$, draw a circle. If (x, y) is a crossing of D, draw two circles, one corresponding to each strand of the crossing. The circles should vary continuously as the point (x, y) moves along the strand of D, so that their union is a torus $R(D)$ in $xyuv$-space. At a virtual crossing, the two circles should be unnested. At a classical crossing, the circles should be nested, with the understrand's circle nested inside the overstrand's.

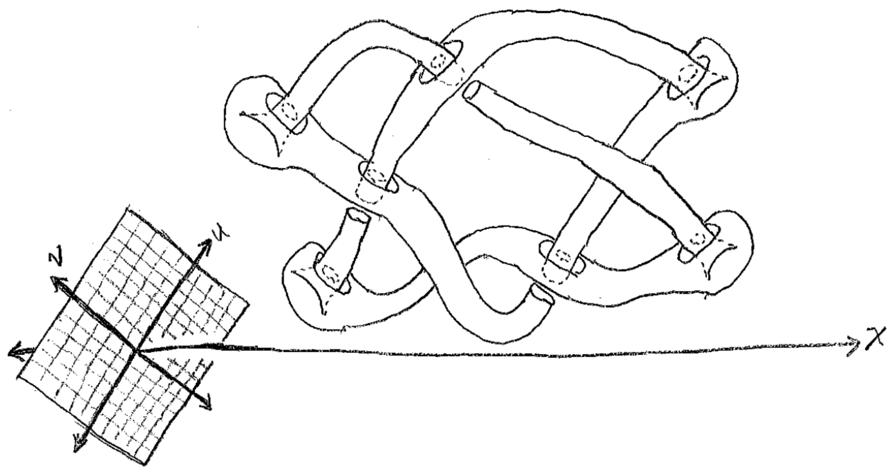
The location and size of each uv-circle doesn't matter, so long as they vary continuously as we move along the strand of D, since every possible choice gives the same torus (up to isotopy of 4-space). Also, if D and D' are isotopic diagrams in the plane (ie., they are isomorphic as plane graphs with crossing data), then of course $R(D)$ and $R(D')$ are isotopic tori in 4-space (since the isotopy of the xy-plane can be extended to all of $xyuv$-space).

2.6.5 y-projection

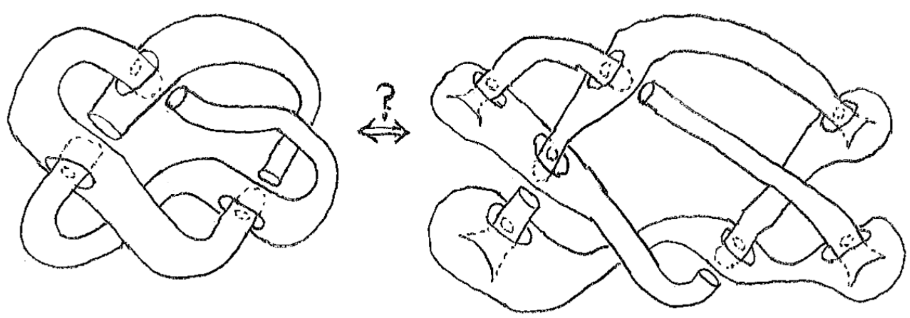
The above description, though complete, is somewhat difficult to visualize. To draw a clearer picture of Rourke's surface, we will arrange the uv-circles so that the surface projects neatly into 3-space. Without loss of generality, make the following assumptions.

- The diagram D is vertical (aligned with the y-direction) at only finitely many points, all of which are local x-extrema (not inflections).
- The v-coordinate of every circle's center equals the y-coordinate of the corresponding point in D.
- The u-coordinate of every circle's center is zero except near virtual crossings.
- The radius of every circle is the same except near classical crossings and x-extrema of D.
- Near classical crossings, the overstrand's circles should grow a bit larger.
- Near x-extrema, the cirlces should grow a bit larger y-below the extremum.

With these assumptions, we now project $R(D)$ into the xuv-hyperplane by collapsing the y-dimension. The result is a broken-surface diagram, where the "breaks" are determined by the surface's y-height before projection. Note the presence of a wen at each x-extrema.



2.6.6 Equivalence



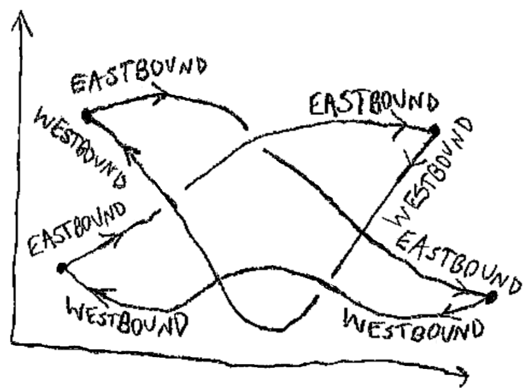
The visible differences between $S(D)$ and $R(D)$ are the presence of wens in $R(D)$ and the disagreement of "jump/duck" patterns at some (but maybe not all) of the classical crossings. We will describe an isotopy of $R(D)$ which eliminates all the wens and reverses the jump/duck pattern at the "bad" crossings, so that the isotoped surface matches $S(D)$.

Theorem.

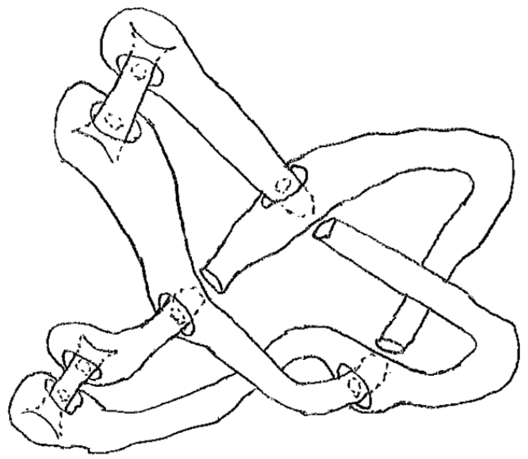
The surfaces $S(D)$ and $R(D)$ are isotopic in 4-space.

Proof.

Suppose we've chosen a particular orientation on D and we've chosen to use convention I when constructing $S(D)$. Then D can be divided into "eastbound" and "westbound" strands, whose termina are the x-extrema. Classical crossings whose overstrands are eastbound will look the same in $R(D)$ and $S(D)$, because in $R(D)$ the understrand "jumps" these crossings on the westside (where its y-height is greater than the overstrand's) and "ducks" them on the eastside (where its y-height is lower than the overstrand's). However, classical crossings whose overstrands are westbound will look different in $R(D)$ and $S(D)$.



We need an ambient isotopy of $R(D)$ which kills all the wens and reverses the jump/duck pattern at the classical crossings whose overstrands were westbound. This can be done by sliding the wens on the east side of the diagram (corresponding to x-maxima of D) westward along the westbound tubes. As they move across the surface, they pass through the overstrand of each "bad" crossing, once each, and reverse its jump/duck pattern (by rule 4 of wen calculus). When the wens reach the other side, they each find their sibling-wen and cancel with it (by rule 2 of wen calculus). The result is a surface identical to $S(D)$.



2.6.7 Complete invariance?

As a surface in 4-space, Rourke's construction $R(D)$ is isotopic to Satoh's construction $S(D)$ and, as noted above, it is unknown whether this surface is a complete invariant of welded knots. However, Rourke's construction provides a fiber-structure on the surface which Satoh's construction lacks. *Rourke claimed the fibered surface is a complete invariant of welded knots, but I suggest that the fibered surface is not an invariant of welded knots at all.*

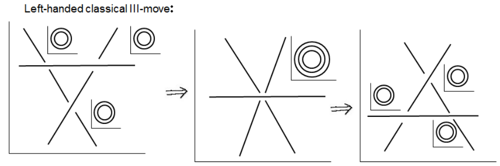
The 4-space in which Rourke's tori are embedded is fibered as $\mathbb{R}^2 \times \mathbb{R}^2$. The first factor is the xy-plane (the base space) and the second factor is the uv-plane (the fiber). This structure restricts to a fibering of each torus component of $R(D)$, as $S^1 \times S^1$. The first S^1 factor is a component of the diagram D in the xy-plane (the base space), and the second S^1 factor is a circle drawn in the uv-plane (the fiber).

If there is an isotopy of 4-space which carries $R(D)$ to $R(D')$, where D and D' are not welded-equivalent, then the isotopy cannot respect the fiber-structure of $\mathbb{R}^4$. Any isotopy respecting this structure must preserve the fiber-structure of $R(D)$, which determines the welded knot type. Therefore, *if* Rourke's fibered surface were a welded knot invariant, then it would be a complete invariant.

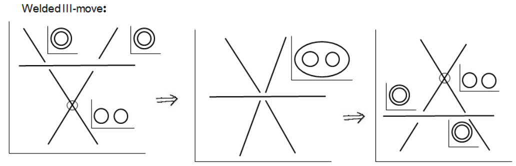
To check whether the fibered surface is a welded knot invariant, we check each welded Reidemeister move to see whether it translates to a fiberwise isotopy of $\mathbb{R}^2 \times \mathbb{R}^2$. This can be done for each of the eleven moves. The answer is obviously yes for all the moves except for the virtual-I move, where the answer is *apparently* no. If the answer is ‘no’, then Rourke’s fibered toral surface is not an invariant of welded knots.

Checking invariance follows the same routine for each Reidemeister move, so I present only a few examples rather than all eleven of them. In every version of the II-move and III-move, the uv-plane circles move in a smooth and continuous manner:

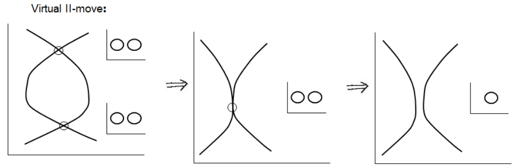
Left-handed classical III-move:



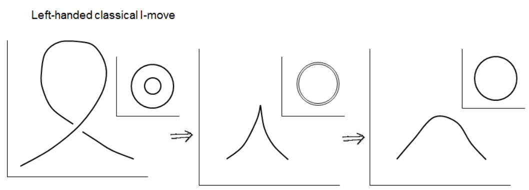
Welded III-move:



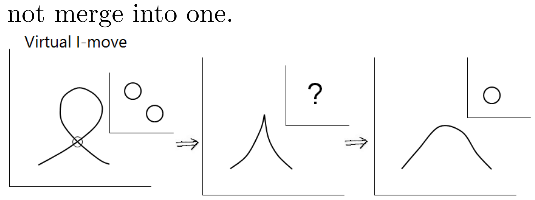
Virtual II-move:



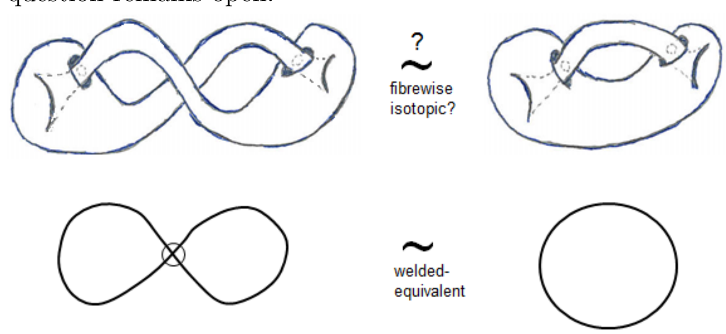
In the classical I-move, the concentric circles which share the uv-plane over the crossing gradually change size until they merge into one.



In the virtual I-move, these circles are side-by-side rather than nested, and cannot merge into one.



Therefore, there may be welded-equivalent diagrams D and D' such that any isotopy of $\mathbb{R}^4$ relating $R(D)$ to $R(D')$ fails to be fiberwise. For example, there may be no fiberwise isotopy relating these two fibered surfaces (their y-projection is shown here), even though both were derived from the welded unknot. The question remains open.



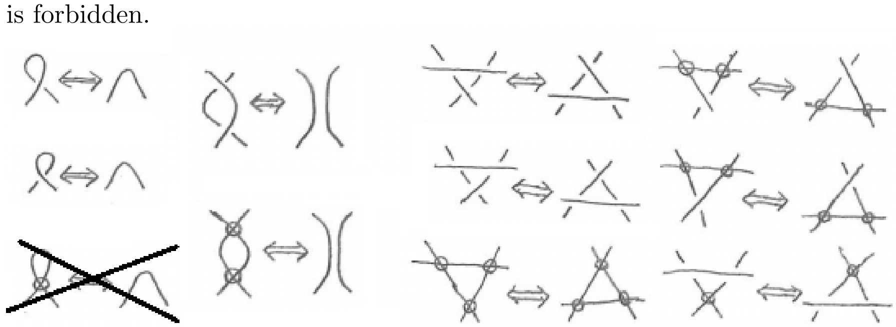
In the next section, we eliminate the problematic virtual I-move from the theory so that Rourke's construction will definitely be invariant (though it still might not be a complete invariant).

2.7 Rotational welded 1-knot theory

This is a refinement of welded knot theory, the result of eliminating the virtual I-move.

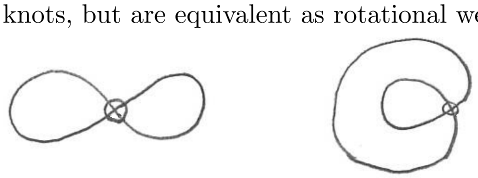
Universe: We allow both classical and virtual crossings.

Move-set: We allow all the welded moves except the virtual I-move, which is forbidden.

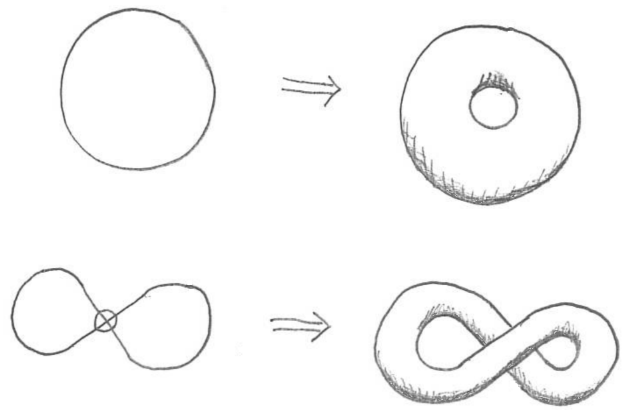


Relation to other theories: Rotational welded knot theory maps onto welded knot theory. Every welded knot type is the union of rotational welded knot types. The theories are not isomorphic, however. Here is a pair of diagrams that are equivalent as welded knots, but are inequivalent as rotational welded knots.

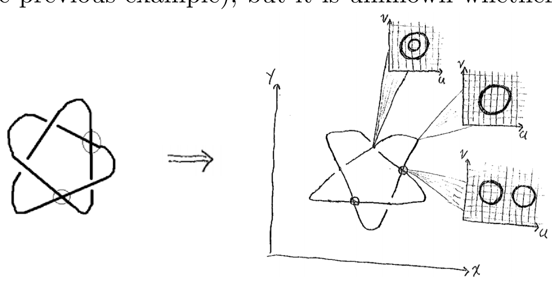
Rotational pure-virtual knot theory does not embed into rotational welded knot theory. Here is a pair of diagrams that are inequivalent as rotational pure-virtual knots, but are equivalent as rotational welded knots.



Invariant: The Satoh invariant defined for welded works just fine for rotational welded, although now it is *definitely* not a complete invariant. For example, here are two inequivalent rotational welded diagrams with isotopic Satoh tori.



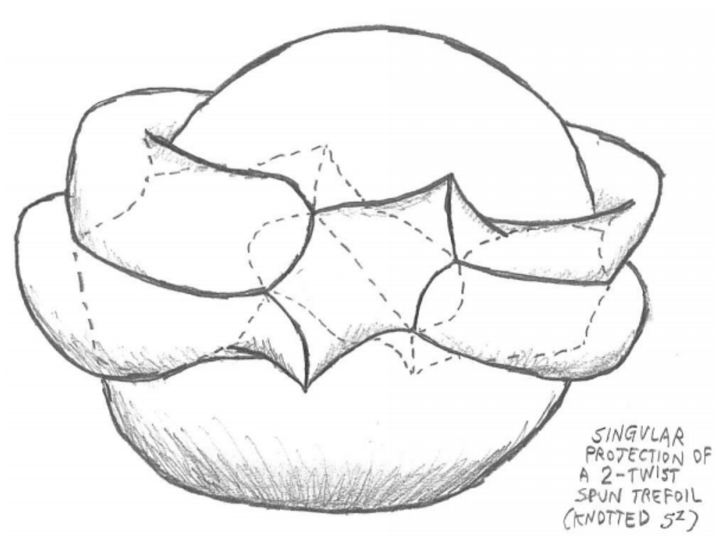
Rourke's fibered version of Satoh's surfaces is also an invariant of rotational welded. It may be stronger than Satoh's invariant (for example, distinguishing the previous example), but it is unknown whether it is complete.



Theorem: Rourke's construction is an invariant of rotational welded knot theory.

Proof: Invariance can be checked for every move, as described in **section 2.6.7**.

3 Examples of 2-knot theories

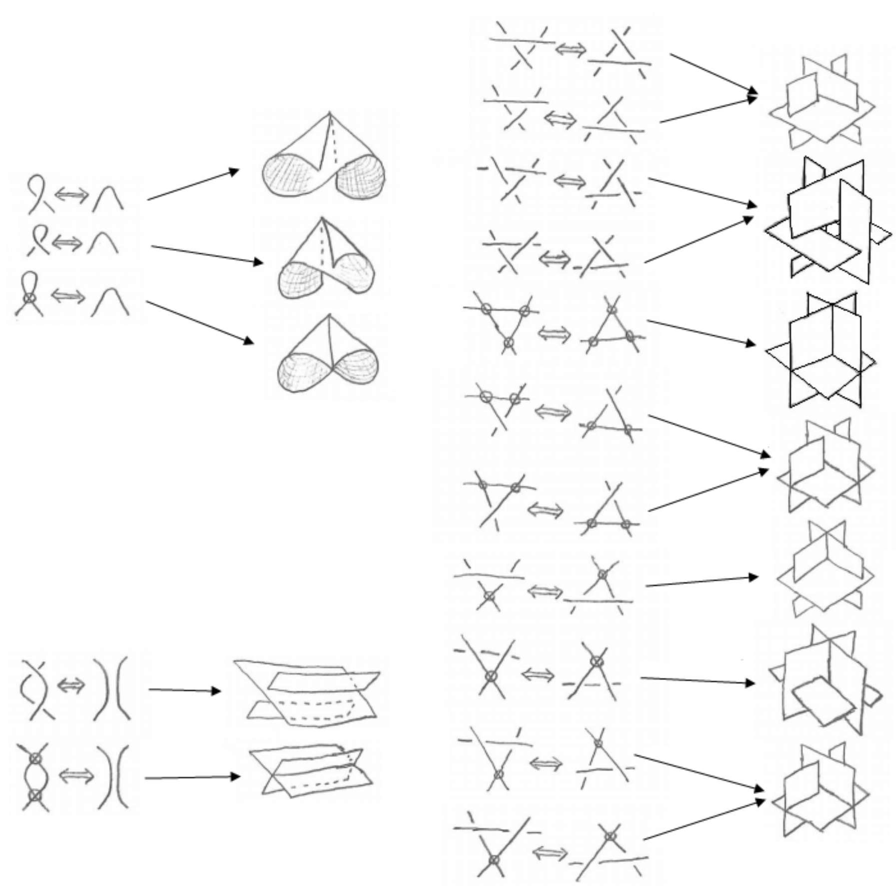


In this section, we describe some popular 2-knot theories using the format laid out in section 1.2. A topological invariant is given for each theory.

In the literature, the term *2-knot* sometimes is used to refer only to the special case where X is a 2-sphere, and the term *surface knot* is preferred when X can be any closed surface. We do not adopt that usage here.

The examples defined below are named after the 1-knot theories from the preceding chapter. In fact, there is a natural one-to-one mapping J from the set of 1-knot theories to the set of 2-knot theories. If *Th* is a 1-knot theory, then $J(Th)$ is the 2-knot theory whose universe is determined from the move-set of *Th* by regarding each permitted Reidemeister move in *Th* as a permitted diagrammatic feature in $J(Th)$, as indicated in the figure below. The move-set of $J(Th)$ is the largest one valid— that is, there are no forbidden moves in $J(Th)$.

The map J will be considered further in section 4.2.

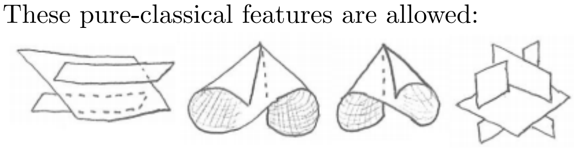


3.1 Classical 2-knot theory

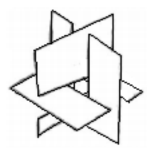
The most familiar of all 2-knot theories.

Universe: We allow classical crossings only; virtual crossings are forbidden.

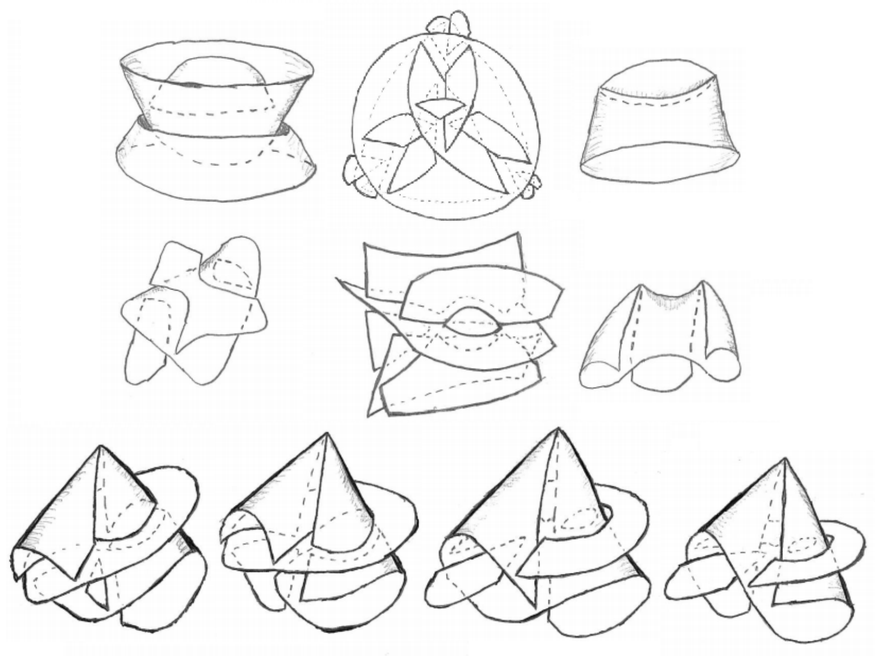
These pure-classical features are allowed:



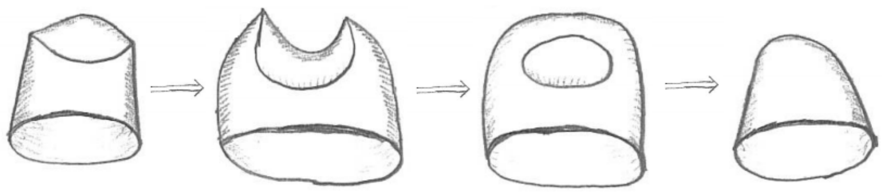
The only forbidden pure-classical feature is the delta triple point:



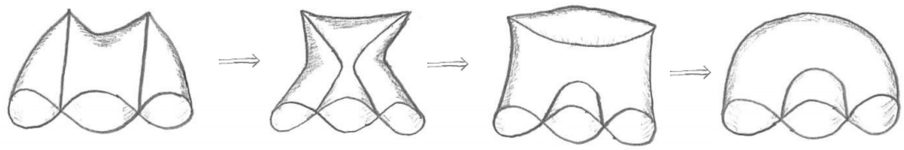
Move-set: There are ten Roseman moves valid in this universe, and all of them are included in the move-set. These are the pure-classical moves not involving a delta triple point. Four of these moves are branch-passes; the other six are all different kinds. (In the illustration, I show only the "before" state of each move, to save space.)



Relation to other theories: The classical 2-knot theory move-set has redundancies. It is possible to restrict the move-set without affecting the knot types. For example, if the Roseman I-bubble move were forbidden, the resulting theory is still isomorphic to classical 2-knot theory as defined here. This is because the I-bubble move can be emulated by a I-saddle followed by a II-bubble move.



As another example, if the Roseman I-saddle move were forbidden, it could be emulated by a II-saddle followed by a I-bubble move:



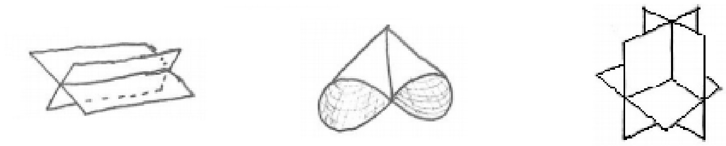
Classical 2-knot theory may or may not embed into virtual and welded 2-knot theory (sections 3.5 & 3.6). The affirmative was true for the analogous 1-dimensional theories, but I don't know whether it's true here. Is it possible, allowing virtual crossings and branches, allowing pure-virtual/mixed/welded triple points, and allowing all the Roseman moves involving those features, that a classical 2-knot diagram might be transformed into another from which it is classically distinct?

Topological invariant: To each diagram D (whose domain X is a closed surface, not necessarily connected), we associate a smooth embedding $k : X \to \mathbb{R}^4$, as follows. Let $f : X \to \mathbb{R}^3$ be the generic map underlying D, and let $i : \mathbb{R}^3 \to \mathbb{R}^4$ be the inclusion of the coordinate xyz-plane into $xyzw$-space. Now modify the immersion $i \circ f$ in a small regular neighborhood of each under-crossing curve in X, by adding a smooth negative bump to the w-coordinate of the image there. (Think of this operation as "filling in" the broken surface drawn in the knot diagram.) Approaching branch points, make this bump continuously diminish to nothing. The result is an embedding k whose isotopy type is a complete invariant of the knot type of D, as proven by Roseman.

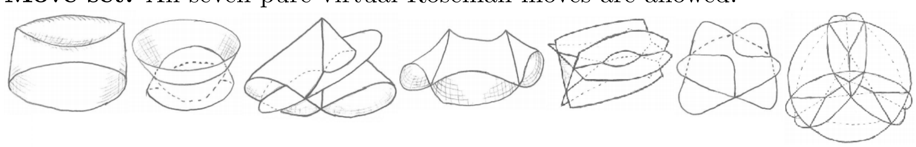
3.2 Pure-virtual 2-knot theory

Here, crossing data must all be virtual.

Universe: We allow virtual crossings only; classical crossings are forbidden. Virtual branch points are allowed, as are pure-virtual triple points.



Move-set: All seven pure-virtual Roseman moves are allowed.



Relation to other theories: Any theory using the purely virtual universe but a different move-set is non-isomorphic to this one. In fact, the 2^7 theories using this universe are all distinct, because it is impossible to emulate any one of the seven pure-virtual moves using some combination of the other six.

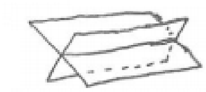
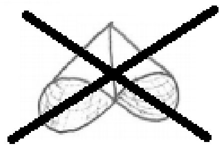
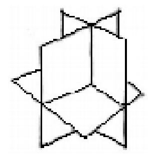
Pure-virtual 2-knot theory is related to rotational pure-virtual 2-knot theory (section 3.3); specifically, rotational pure-virtual 2-knot theory maps onto pure-virtual 2-knot theory.

Topological invariant: Like pure-virtual 1-knot theory, this theory is trivial in the sense that the knot type of a diagram depends only on X. The homeomorphism type of the domain X is a complete invariant.

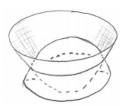
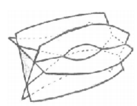
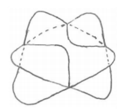
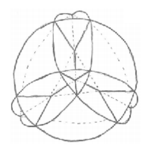
3.3 Rotational pure-virtual 2-knot theory

Whereas rotational pure-virtual 1-knot theory was developed by restricting only the move-set, for rotational pure-virtual 2-knot theory we restrict the universe. In the context of 2-knots, the term "rotational" means that virtual branch points are forbidden. This is different from, but analogous to, the usage of "rotational" in the context of 1-knots, where it means that the virtual Reidemeister I-move is forbidden.

Universe: We allow virtual crossings only. Branch points are forbidden. Triple points are allowed.

Move-set: The move-set consists of the four pure-virtual Roseman moves that don't involve branch points.

Relation to other theories: Rotational pure-virtual 2-knot theory maps onto pure-virtual 2-knot theory.

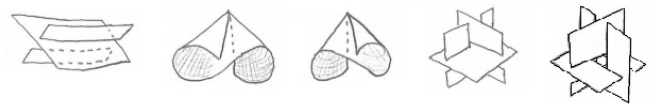
Topological invariant: Knot types in this theory are regular homotopy classes of immersions. This is because the four moves are themselves regular homotopies, and every regular homotopy can be perturbed into a generic form consisting of a sequence of these four moves.

If we assume X is oriented, and that the universe and move-set have no additional restrictions based on orientation, then we get **oriented rotational pure-virtual 2-knot theory**. This can be used, for example, to evert the 2-sphere via Roseman moves. In (Carter, 2011), Carter does this via *chart moves*, which include the Roseman moves as a subset.

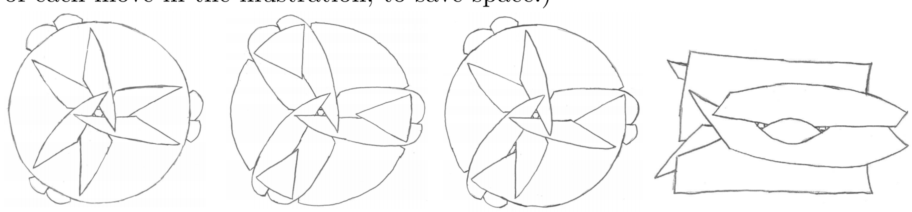
3.4 Delta 2-knot theory

I include this section for completeness, but I do not yet know very much about it. It might be trivial.

Universe: We allow classical crossings only. All types of classical branch point and triple point are allowed, including the delta triple point.



Move-set: All 14 pure-classical Roseman moves are allowed. That includes the ten moves of classical 2-knot theory (section 3.1), plus these additional four moves that involve a delta triple point. (Again, I only show the "before" state of each move in the illustration, to save space.)

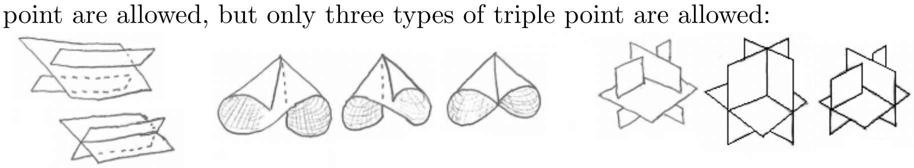


Relation to other theories: Clearly, classical 2-knot theory maps onto delta 2-knot theory.

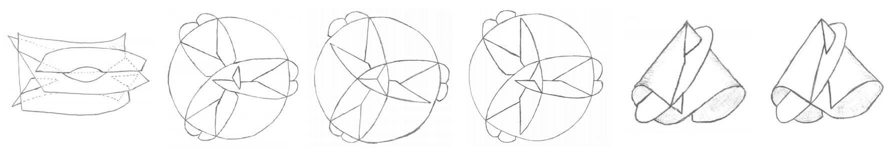
3.5 Virtual 2-knot theory

This is the 2-dimensional analog of Kauffman's virtual knots.

Universe: We allow both classical and virtual crossings. All types of branch point are allowed, but only three types of triple point are allowed:



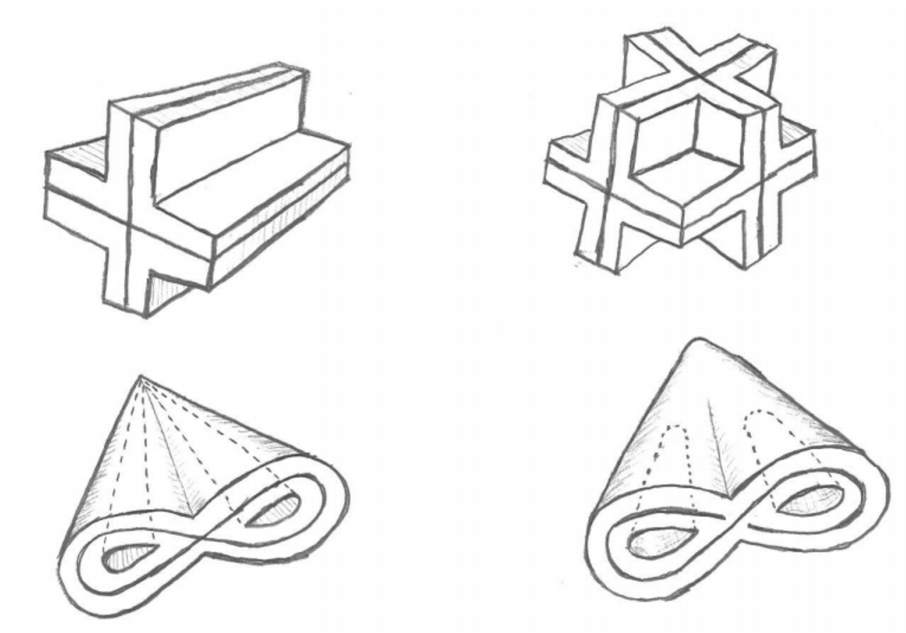
Move-set: There are 23 Roseman moves valid for this universe, and all of them are included in the move-set. These include the ten moves from classical 2-knot theory, the seven moves from pure-virtual 2-knot theory, and these six new moves involving both classical and virtual crossings:



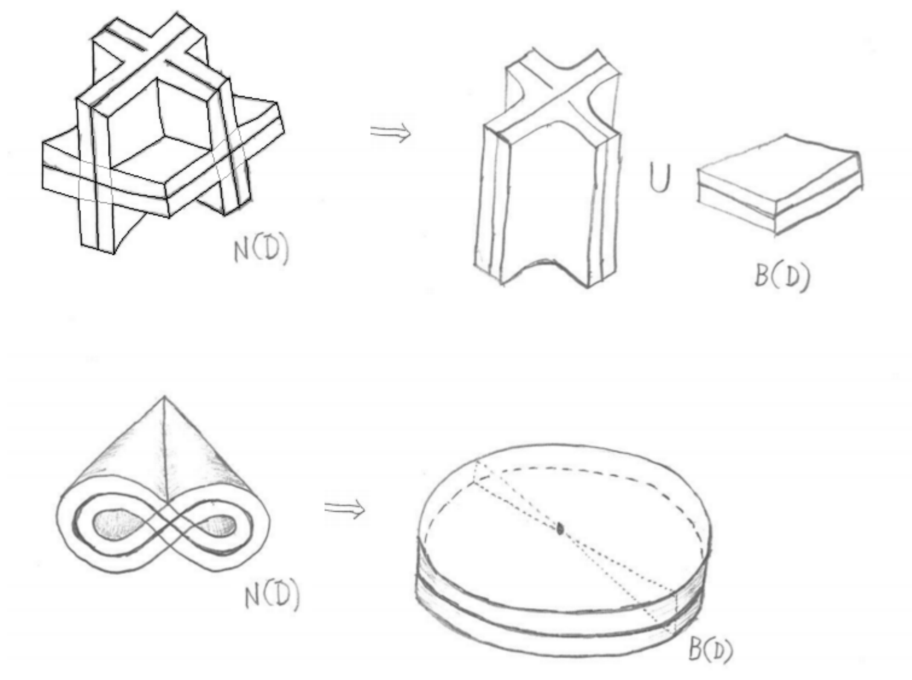
Relation to other theories: Takeda showed that this theory has knot types which do not contain any pure-classical diagrams. It is not known whether classical 2-knot theory embeds into virtual 2-knot theory.

Topological invariant: The development of an invariant for virtual 2-knot theory closely parallels that for virtual 1-knot theory. The idea is to think of a virtual 2-knot diagram as a classical 2-knot diagram "drawn" on a closed 3-manifold. We then define an equivalence relation on these objects that extends classical move-equivalence and allows the 3-manifold to vary.

Take as input a virtual 2-knot diagram D. Let $N(D)$ be a neighborhood of the diagram, which is a regular neighborhood except at virtual branch points, in the following sense: $N(D)$ is formed by thickening D everywhere except at virtual branch points; as you approach virtual branch points, let the thickening gradually diminish to zero, so that near the virtual branch point $N(D)$ looks like the cone over a thickened figure-8.



Along each virtual crossing curve of D, double the square-shaped junction of $N(D)$ to create overlapping "slabs". Call this 3-manifold-with-boundary $B(D)$. It has a purely classical knot diagram in it. (To be precise, $B(D)$ is not technically a 3-manifold-with-boundary at virtual branch points, since the "slab" is pinched to zero thickness at these points.)



Now embed $B(D)$ into any closed orientable 3-manifold (not necessarily connected). The result, called $S(D)$, is a closed 3-manifold containing a classical 2-knot diagram. The particular choice of embedding does not matter, because all the possible choices are equivalent under the following relation.

Two closed orientable 3-manifolds containing classical 2-knot diagrams are **equivalent** when one can be transformed into the other via a sequence of the following two operations.

- **Classical Roseman moves:** Take a ball in the 3-manifold containing part of the knot diagram. Modify the ball just as one would modify portions of a classical 2-knot diagram, that is, by a boundary-fixing self-homeomorphism of the ball, or by any of the ten classical 2-knot moves if applicable. The result is the same 3-manifold, but with a slightly different 2-knot diagram in it.

- **Stabilization:** Take a regular neighborhood of the 2-knot diagram in the 3-manifold, and embed it in another closed orientable 3-manifold. The result is a new 3-manifold, but with the same classical 2-knot diagram in it.

These operations define an equivalence relation on 3-manifolds with classical diagrams in them. An equivalence class is called a **classical 2-knot in a stabilized 3-manifold**. It is an invariant of virtual 2-knot type.

Theorem: Let D, D' be virtual 2-knot diagrams. Let $S(D), S(D')$ be any classical 2-knot diagrams in 3-manifolds derived from D, D' via the construction described. If $D \sim D'$ (where $\sim$ is virtual 2-knot equivalence) then $S(D) \sim S(D')$ (where $\sim$ is equivalence via classical Roseman moves and stabilization).

Proof: The argument has the same form as that used for virtual 1-knots. It suffices to check that each move from the move-set corresponds to an equivalence of 2-knot diagrams in 3-manifolds. For the ten classical Roseman diagram moves, it is possible to build the construction so that the move takes place inside a ball in the 3-manifold. For the 13 remaining moves (which all involve virtual crossings), we can build the same 3-manifold construction from both the 'before' and 'after' diagrams.

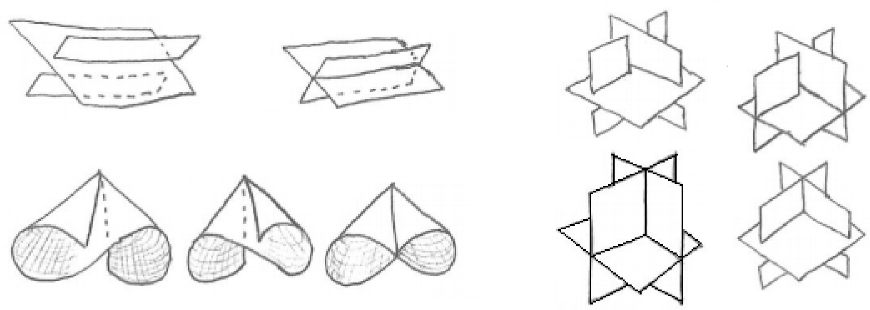
It is unknown whether the converse is true, that is, whether this invariant is complete. Do inequivalent virtual 2-knot diagrams ever give rise to equivalent classical 2-knots in stabilized 3-manifolds? Note that the analogous statement for virtual 1-knots is known to be true (section 2.5).

Another topological invariant of virtual 2-knots is **surfaces embedded in stabilized thickened 3-manifolds**. The definition is the same as the one just given, except the 3-manifolds are now thickened (product with an interval), the 2-knot diagrams are replaced by embedded surfaces, the Roseman moves are relaced by 4-dimensional isotopies, and stabilization involves surgery on the thickened 3-manifold (so when you take a neighborhood of the knot, you cut through the thickened 3-manifold).

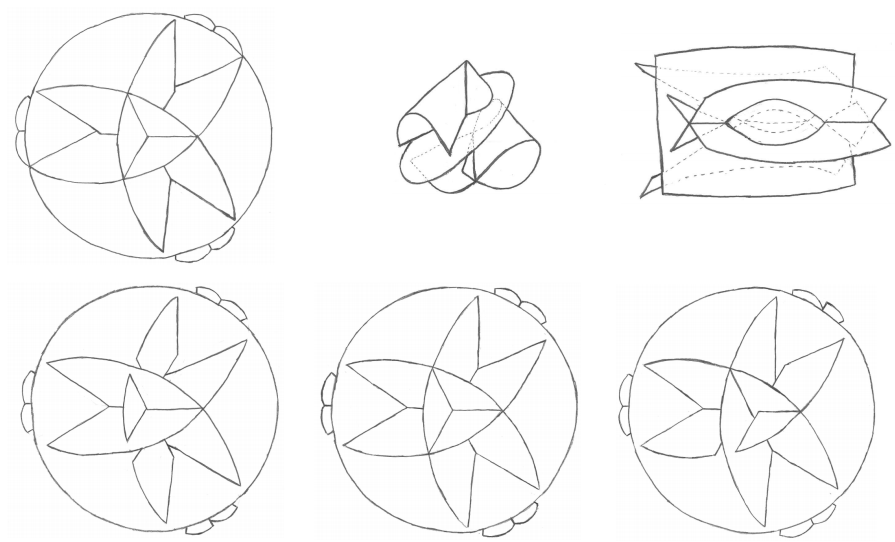
3.6 Welded 2-knot theory

This section presents the 2-dimensional analog of welded 1-knot theory. It includes the surface-feature analog of each of the welded 1-knot moves, including virtual branch points (analog of the virtual Reidemeister I-move). I do not present any topological invariant for welded 2-knots. This is in contrast to welded 1-knots, which had the Satoh toral-surface model (section 2.6.1) as a (possibly incomplete) invariant. If an analogous model exists for welded 2-knots, I have not yet found it. Rourke's model (section 2.6.4) can be applied to welded 2-knots if the virtual branch point is forbidden; this is the subject of section 3.7 on rotational welded 2-knot theory.

Universe: We allow both classical and virtual crossings. All three types of branch point are allowed. Triple points must be one of four types.



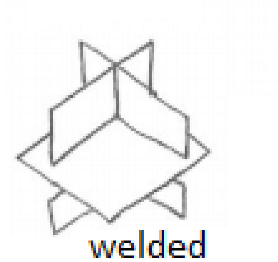
Move-set: There are 29 Roseman moves valid for this universe, and all of them are included in the move-set. These include the 23 moves of virtual 2-knot theory, plus six new moves involving welded triple points.

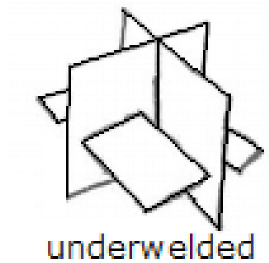


Relation to other theories: Five other theories in this chapter— classical, pure-virtual, rotational pure-virtual, Kauffman-virtual, and rotational welded— use universes and move-sets that are proper subsets of welded's. Which of these theories embed in welded 2-knot theory? I know the answer for a few of them:

- I don't know whether classical or virtual embed in welded, but I conjecture they do. (The statements are true for the analogous 1-dimensional theories.)
- Pure-virtual embeds in welded. This is obvious: There is only one pure-virtual knot type for each X, and the pure-virtual moves also work as welded.
- Rotational pure-virtual does not embed in welded. This is because rotational pure-virtual does not embed in pure-virtual.
- Rotational welded does not embed in welded. This is proven in the rotational welded section below.

There is a theory called **underwelded 2-knot theory**, which is the same as welded 2-knot theory except the welded triple point is forbidden and the underwelded triple point is allowed.





The same alteration is made to the move-set as appropriate (exactly six moves have to be replaced). The operation of **mirror imaging** a diagram— that is, swapping the designations of under- and oversheets along every classical crossing— defines a bijection between the welded and underwelded universes, which carries welded 2-knot types to underwelded 2-knot types and vice-versa. Since each universe contains diagrams that are not present in the other, neither theory can embed into or map onto the other.

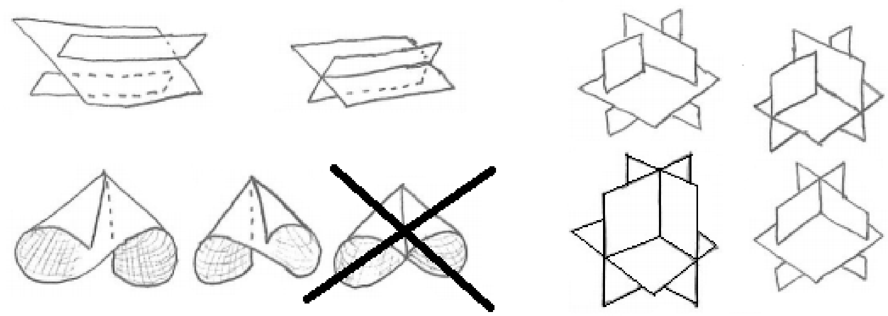
Topological invariant: I do not have a topological invariant for this theory. I attempted to define a 3-dimensional analog of the Satoh's toral-surface model for welded 1-knots (section 2.6.1), but I cannot see how such a model would handle virtual branch points in a welded 2-knot. The trouble is essentially the same as Rourke's failure to remain invariant through a virtual I-move (section 2.6.7).

In the next section, we eliminate virtual branch points. Rourke's construction suits the restricted theory just fine.

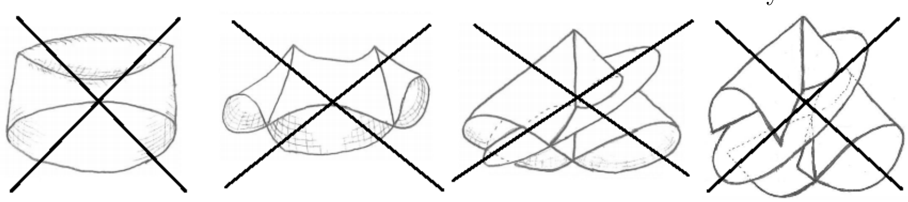
3.7 Rotational welded 2-knot theory

This is a restriction of welded 2-knot theory, forbidding virtual branch points. An invariant similar to Rourke's model for rotational welded 1-knots is possible for this theory.

Universe: The universe is the same as that for welded 2-knot theory, except for the omission of virtual branch points.

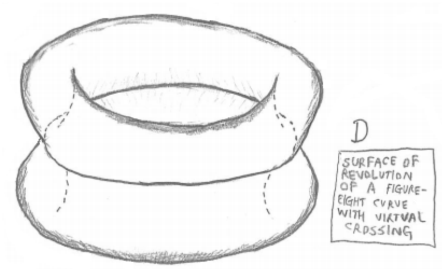


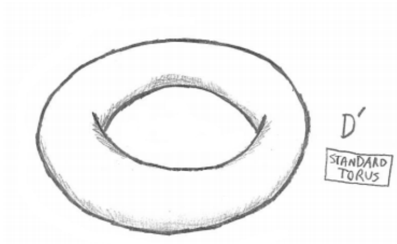
Move-set: The move-set for "ordinary" welded 2-knot theory includes four moves involving virtual branch points. Omitting these four, we are left with 25 moves. These form the move-set of rotational welded 2-knot theory.



Relation to other theories: Does classical 2-knot theory embed into rotational welded 2-knot theory? The answer is yes if classical 2-knot theory embeds into welded 2-knot theory, but this question is open.

Rotational pure-virtual 2-knot theory embeds in rotational welded 2-knot theory. If D and D' are inequivalent as rotational pure-virtual diagrams, then the added power of the rotational welded moves is still insufficient to transform D into D'. For example, consider the following pair of diagrams:





They are inequivalent as rotational pure-virtual diagrams. In order to transform D to D' using rotational welded moves, the circular virtual crossing must somehow be eliminated; the only rotational welded move that can eliminate a crossing is the II-bubble, which requires that the circular crossing be contractible in the surface X. However, the crossing in D is not contractible in X, and there is no rotational welded move that changes the homotopy class in X of a virtual crossing.

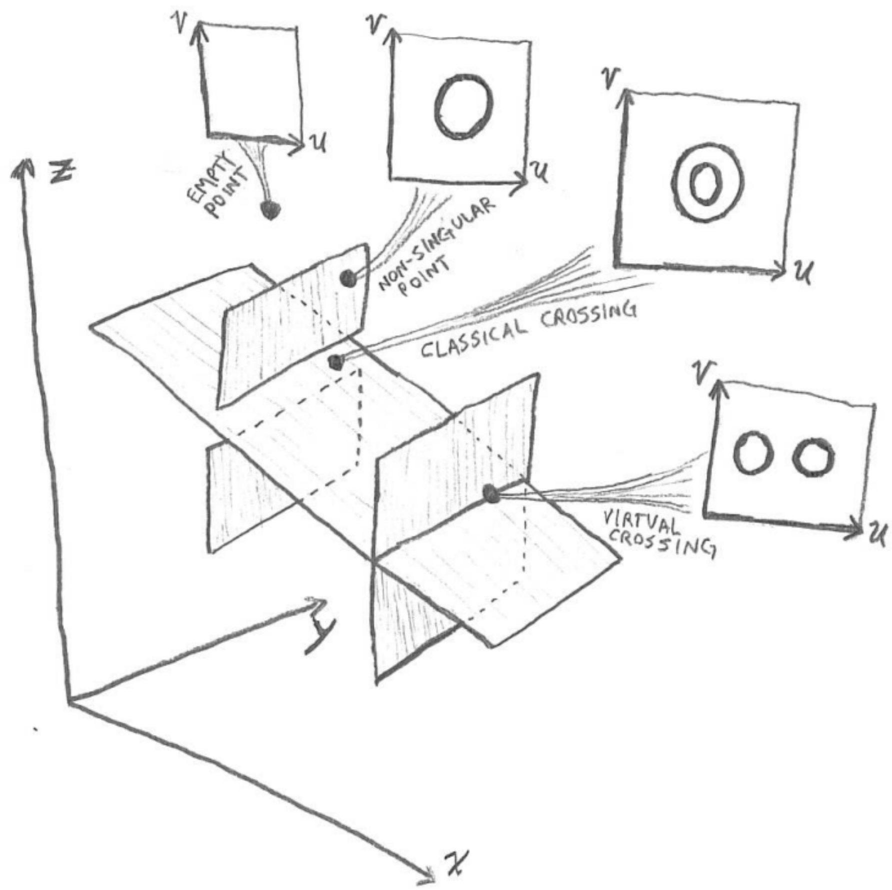
Since rotational pure-virtual 2-knot theory embeds in welded but not rotational welded, we conclude that rotational welded does not embed in welded. The example in the previous paragraph illustrates this.

Topological invariant: The invariant is analogous to Rourke's model for welded 1-knots. To each 2-knot diagram D we associate a 3-manifold $R(D)$ embedded in 5-space, such that equivalent diagrams give rise to isotopically embedded 3-manifolds.

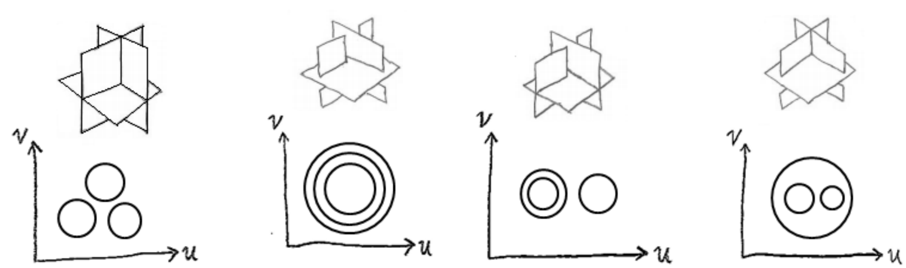
3.7.1 Rourke

Begin with a rotational welded 2-knot diagram D in the xyz-hyperplane of $xyzuv$-space. In the uv-plane above each point $(x, y, z) \in D$, draw a circle. If (x, y, z) lies on a crossing of D, draw two circles, one corresponding to each sheet of the crossing. If (x, y, z) is a triple point of D, draw three circles. The circles drawn in all the copies of the uv-plane should vary continuously as the point (x, y, z) moves around in D, so that their union is the 3-manifold $X \times S^1$

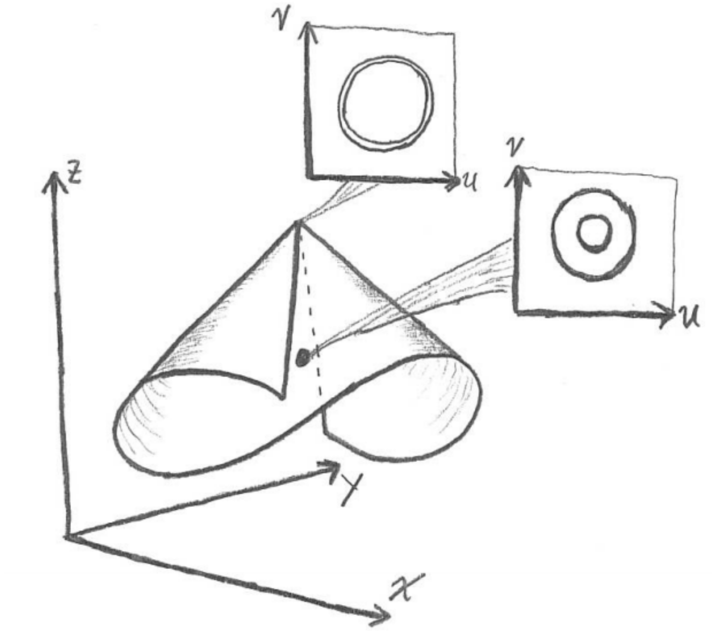
in $xyzuv$-space. The rules for drawing the circles are:



The four allowed types of triple point correspond to the four possible arrangements of three circles in the plane:



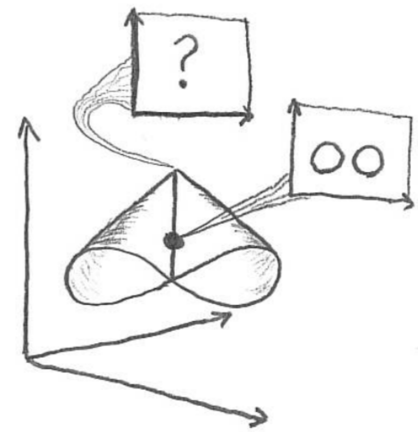
The uv-planes over a classical crossing approaching a branch point contain a pair of nested circles, just like any other classical crossing. However, as the point (x, y, z) moves along this crossing toward the branch point, the two circles gradually converge in size until they coincide exactly.



The location and size of each uv-circle doesn't matter, so long as they vary continuously as we move around within the surface D (always moving transversely through crossings, of course), since every possible choice gives the same 3-manifold (up to isotopy of 5-space). Also, if D and D' are isotopic diagrams in 3-space (ie., some orientation-preserving self-homeomorphism of R^3 carries D to D'), then of course $R(D)$ and $R(D')$ are isotopic 3-manifolds in 5-space (since the isotopy of the xyz-hyperplane can be extended to all of $xyzuv$-space).

3.7.2 Forbidden virtual branch points

The model just described does not extend in any obvious way to welded 2-knot theory with virtual branch points. Moving along a virtual crossing toward a virtual branch point, the uv-planes all contain a pair of unnested circles. But at the branch point itself, there is only one circle. How do the two circles combine into one? Various ideas for pulling this off have failed.



For example, I tried (at the suggestion of Eiji Ogasa) adding a distinguished "point at infinity" to the uv-planes, making them uv-spheres. This distinguished point would only be used at virtual branch points, in order to ensure that the concepts of nested and unnested circles is still meaningful. Unfortunately, this is impossible: Through any point in D over which the distinguished point is used, there must be (at least) a curve in D over each point of which the distinguished point is used also.

Other ideas have also failed. I conjecture that the desired extension of the invariant is impossible.

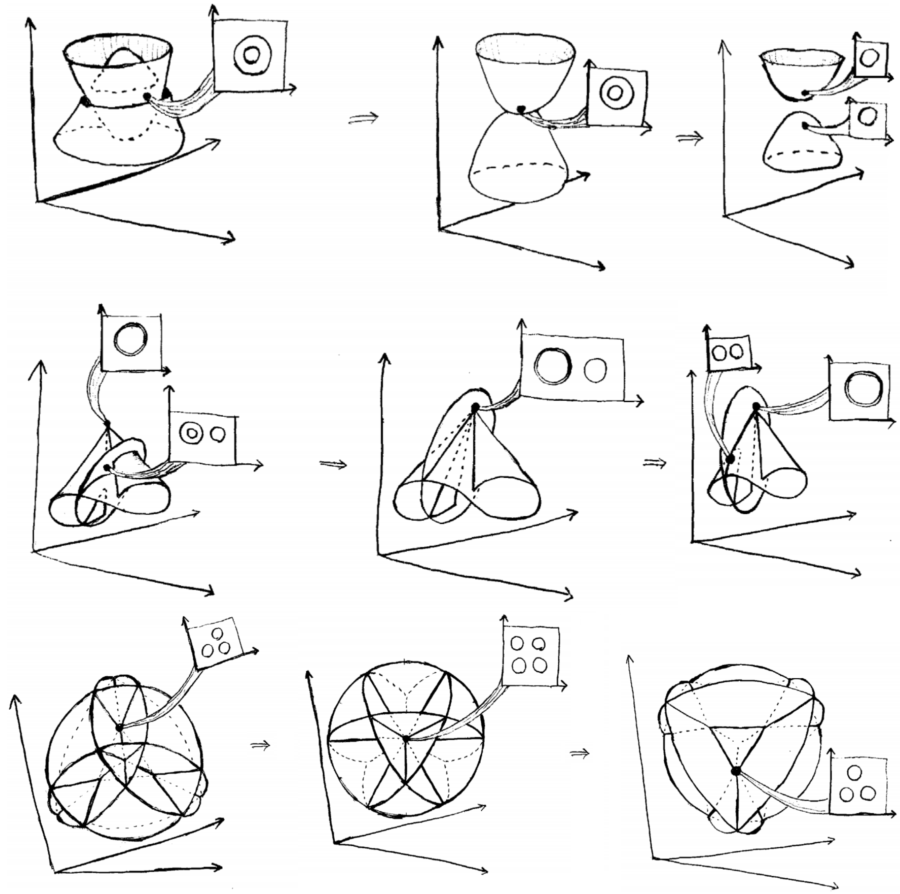
3.7.3 Invariance

The invariance of the Rourke model for rotational 2-knot theory can be checked by regarding each move as a homotopy of the generic map f, and then extending that to a homotopy of the embedding of the 3-manifold in 5-space. This can be done for each of the 25 moves. The 25 verifications all follow essentially the same routine; a few are illustrated in the proof below.

Theorem: Let D, D' be rotational welded 2-knot diagrams. Let $R(D), R(D')$

be the fiberwise-embedded 3-manifolds derived from D, D' via Rourke's construction. If $D \sim D'$ (where $\sim$ is rotational welded 2-knot equivalence) then $R(D) \sim R(D')$ (where $\sim$ is equivalence via a fiberwise $\mathbb{R}^5$ isotopy).

Proof: There are 25 signed Roseman moves in the theory. To verify that a move (interpreted as a generic homotopy) extends to a fiberwise isotopy, we check that the fiber-circles over each crossing pass continuously and smoothly through a valid configuration at the "singular" moment of the move. Three examples are given; the remaining are left as routine exercises.



4 Higher dimensions

4.1 General recipe for declaring a diagrammatic n-knot theory

A diagrammatic n-knot theory can be declared by following this recipe.

1. Know the complete list of (unsigned) diagram features for dimension n.

2. Select which of these to include in your theory. (These selections are not independent. For example, in $n = 2$, triple points can only be included if double-curves are included.)

3. For each feature you chose to include, select which assignments of crossing data will be allowed. (Again, the selections are not independent. For example, pure virtual triple points can only be included if virtual double-curves are included.)

4. Know the complete list of (unsigned) diagram moves for dimension n. Determine which of these are *valid* (involve only features selected in step 2). Determine all the *valid* ways of applying crossing data to these moves (consistent with the choices in step 3).

5. Select which of these moves to include in your theory.

Strictly speaking, steps 1 and 4 are not part of declaring a theory, but rather prior mathematical knowledge needed to declare a theory. The actual declaration properly occurs in steps 2, 3, and 5.

Step 1 of this recipe asks that we know the list of diagram features for dimension n. This is easy if we already know the list of diagram moves for dimension $(n-1)$— simply regard each move as a homotopy, and regard the corresponding level-preserving map as a piece of n-dimensional diagram. There will be some redundancy in this method for $n \geq 3$, but the resulting list of features will be complete.

To restate this more precisely: There exists a map $J_n : M_{(n-1)} \to F_n$, where $M_{(n-1)}$ is the set of all (unsigned) moves in dimension $(n-1)$, and F_n is the set of all (unsigned) diagram features in dimension n. Regard J_n as a "forgetful" map— it "forgets" that the timeline of a move is anything other than just another spatial dimension. The map J_n is onto for all n, but it fails to be one-to-one for $n \geq 3$. A similarly-defined map J_n also exists between sets of moves and features with crossing data (as opposed to unsigned).

n	$\lvert F_n\rvert$	J_n	$\lvert M_n\rvert$
0	0		1
1	1		3
2	3		7
3	5		12

The list of diagram features can also be produced directly using a characterization of generic maps laid out by (Roseman, 2000). In (Roseman, 2004), Roseman also explains how to generate a list of diagrammatic moves for dimension n, useful for step 4 of the recipe.

4.2 Naming $(n+1)$-knot theories after n-knot theories

In step 5 of the recipe, we may declare that *all* valid moves are to be included in the theory. The theory so declared is called **move-complete**. For example, of the 1-knot theories described in chapter 2, only "pure-virtual knot theory" (2.2) and "delta knot theory" (2.4) are move-complete; the five other theories in that chapter each forbid certain valid moves. All seven of the 2-knot theories in chapter 3 are move-complete.

Let L_n be the map which assigns to each move-complete $(n+1)$-knot theory

A the unique n-knot theory $L_n(A) = B$ whose move-set equals the full preimage under J_n of A's permitted diagram features (with crossing data). Note that L_n is one-to-one.

If the theory B has a name (for example, "Kauffman's virtual 1-knot theory") then by convention, theory A should be given the same name ("Kauffman's virtual 2-knot theory"). This convention was followed for every 2-knot theory in chapter 3.

In step 5 of the recipe, we may choose a move-set that is the union of full preimages $J_{n+1}^{-1}(x)$, where x is some diagrammatic feature in dimension $n+1$. The theory so declared is called **move-closed**. For example, a 1-knot theory whose move set contains the left-handed Reidemeister III-move, but not the right-handed III-move, is not move-closed.

The image of L_n is precisely the set of move-closed n-knot theories. For example, a 1-knot theory with the left-handed, but not right-handed, III-move does not get to have a 2-knot theory named after it.

4.3 Characterizing smooth generic maps

Roseman's characterization of of generic maps is an intricate list of geometric criteria. He presents it as a definition, but it has the appearance of a consequence derived from a simpler definition. I propose the following alternative definition for generic maps of smooth manifolds of any dimensions. I conjecture that Roseman's definition will be shown to follow as a special case (the case of codimension 1 maps).

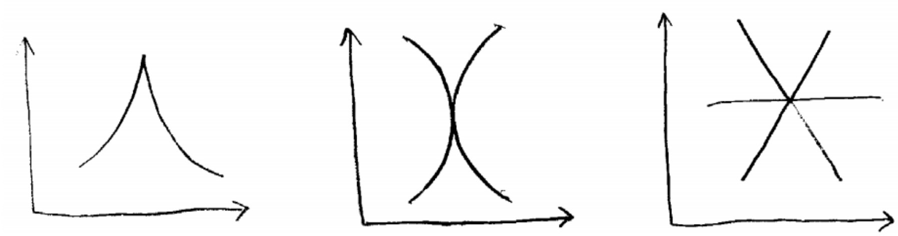
Definition. Let X and Y be smooth manifolds (of any dimensions). A smooth map $f : X \to Y$ is **generic** if for every homotopy $\{f_t\}$, with $f_0 = f$, whose associated level-preserving map $F : X \times I \to Y \times I$ is smooth, there exists a number $T > 0$ and isotopies $\{\alpha_t : X \to X\}$ and $\{\beta_t : Y \to Y\}$ satisfying

$\beta_t \circ f_t \circ \alpha_t = f$ for all $t < T$.

In this definition, we "test" the map f by looking for a small perturbation $\{f_t\}$ which *immediately* alters the topology of the image. The isotopy $\{\beta_t\}$ tries to "correct" the image of the perturbed map and revert it back to image(f). The isotopy $\{\alpha_t\}$ then reparametrizes X so that the final map is exactly equal to f. If there is a perturbation for which these corrections are topologically impossible, then f is not generic.

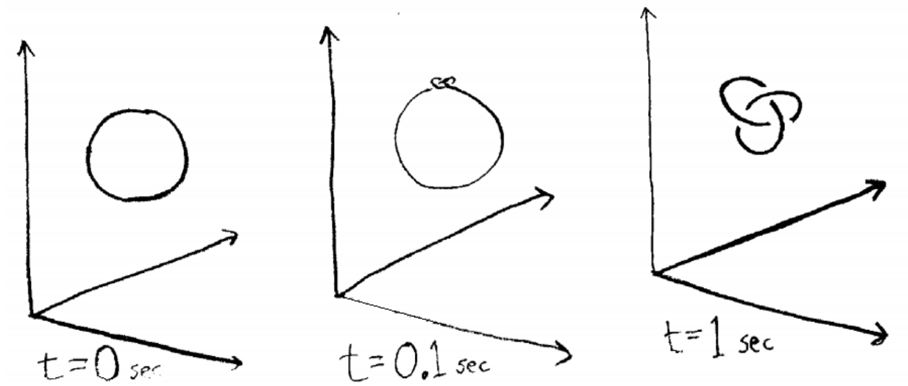
For example, if X is a curve and Y is the plane, then a map f will fail this test if

- the velocity is ever zero,
- there are non-transverse intersections,
- there are intersections of degree ≥ 3.



Each of these non-generic behaviors can be perturbed so that the image *immediately* changes in an essential, topological way.

The definition only allows "test" perturbations that are smooth, in the strong sense of $F : X \times I \to Y \times I$ being smooth. This is to prevent false negatives—if we could use test perturbations that were not smooth in this strong sense, we might find a way to disrupt the topology of a map which we really want to call generic. For example, if X is a circle and Y is 3-space, and f is a map whose image is a geometric circle traversed at unit speed, then f is generic—there is no way to disrupt the topology of the image via a *strongly* smooth perturbation. However, there is perturbation $\{f_t\}$, with f_t smooth for each t, but $F : X \times I \to Y \times I$ not smooth, which changes the topology by growing a knot from a single point on the circle starting at $t = 0$, as shown.



In future work, the author will extend this approach to characterizing and classifying generic moves of generic maps.

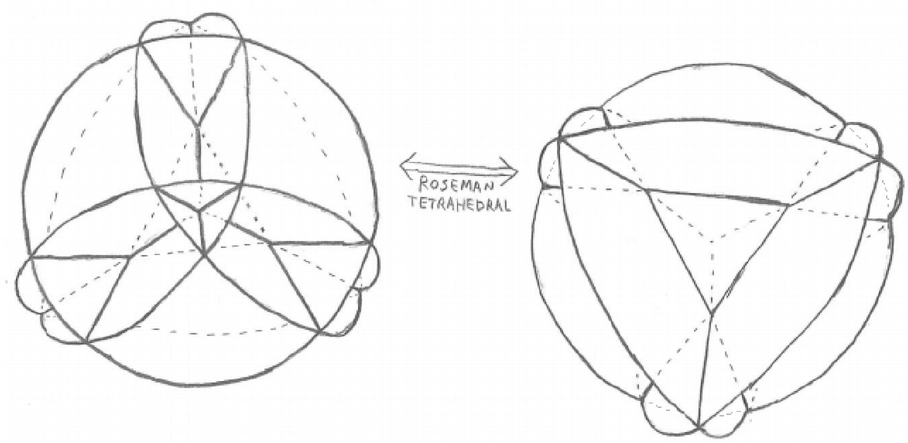
4.4 Conclusion

In this paper, we have established a meta-theoretical framework for defining and classifying diagrammatic knot theories in dimensions 0 through 3; we have found relations between several such theories in dimensions 1 and 2; we have developed topological models for virtual and rotational welded 1-knot and 2-knot theories; and we have laid the groundwork for extending this work into higher dimensions. Future directions of research include:

- Rigorously prove that Rourke's model does not work for non-rotational welded knot theory.
- Define n-dimensional virtual and rotational-welded knot theories.
- Classify all diagrammatic 1-knot and 2-knot theories up to isomorphism.

Appendix:

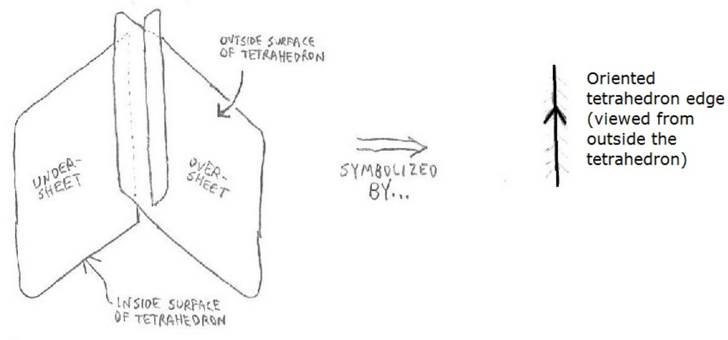
The 42 signed Roseman tetrahedral moves

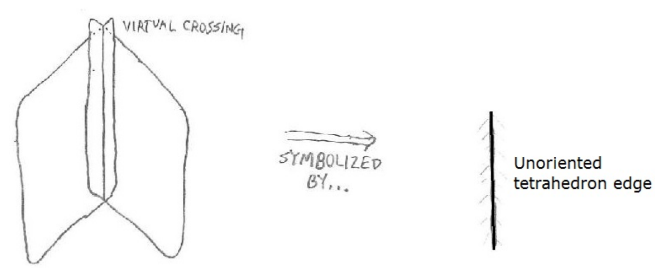


The Roseman tetrahedral move is the 2-knot diagram move which turns a tetrahedron inside-out by passing its four facial planes across a common quadruple point. There are $3^6 = 729$ ways to apply crossing data along each of the six edges of a tetrahedron. Up to rotation, this reduces to only 67 ways. This number is further reduced to 42 ways, by pairing the "before" and "after" pictures of the Roseman move. (Note that 42 is more than half of 67, since some of the "before" and "after" states are alike under rotation.)

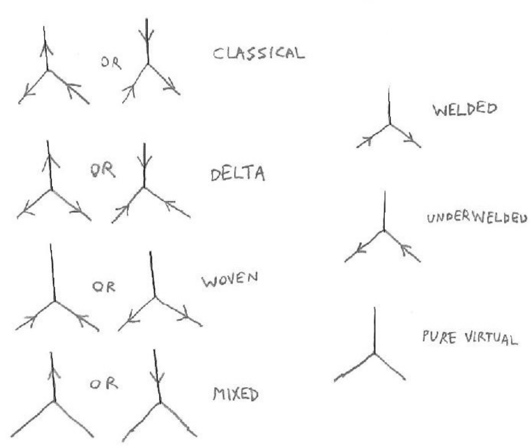
Theorem. There are 42 different signed Roseman tetrahedral moves. Of these, 4 use only classical crossings, 1 uses only virtual crossings, and 37 use a mixture of both.

Proof. We systematically list all 67 signed tetrahedra, drawing the edge-lattice of each tetrahedron seen from birds-eye-view above one of the vertices. (For simplicity, we omit the continuation of the facial planes.) Virtual crossing edges are indicated by plain segments; classical crossings are indicated by an arrow, so that the “under” sheet is to the left and the “over” sheet to the right, seen by a little man standing on the outside surface of the tetrahedron and facing the direction of the arrow.



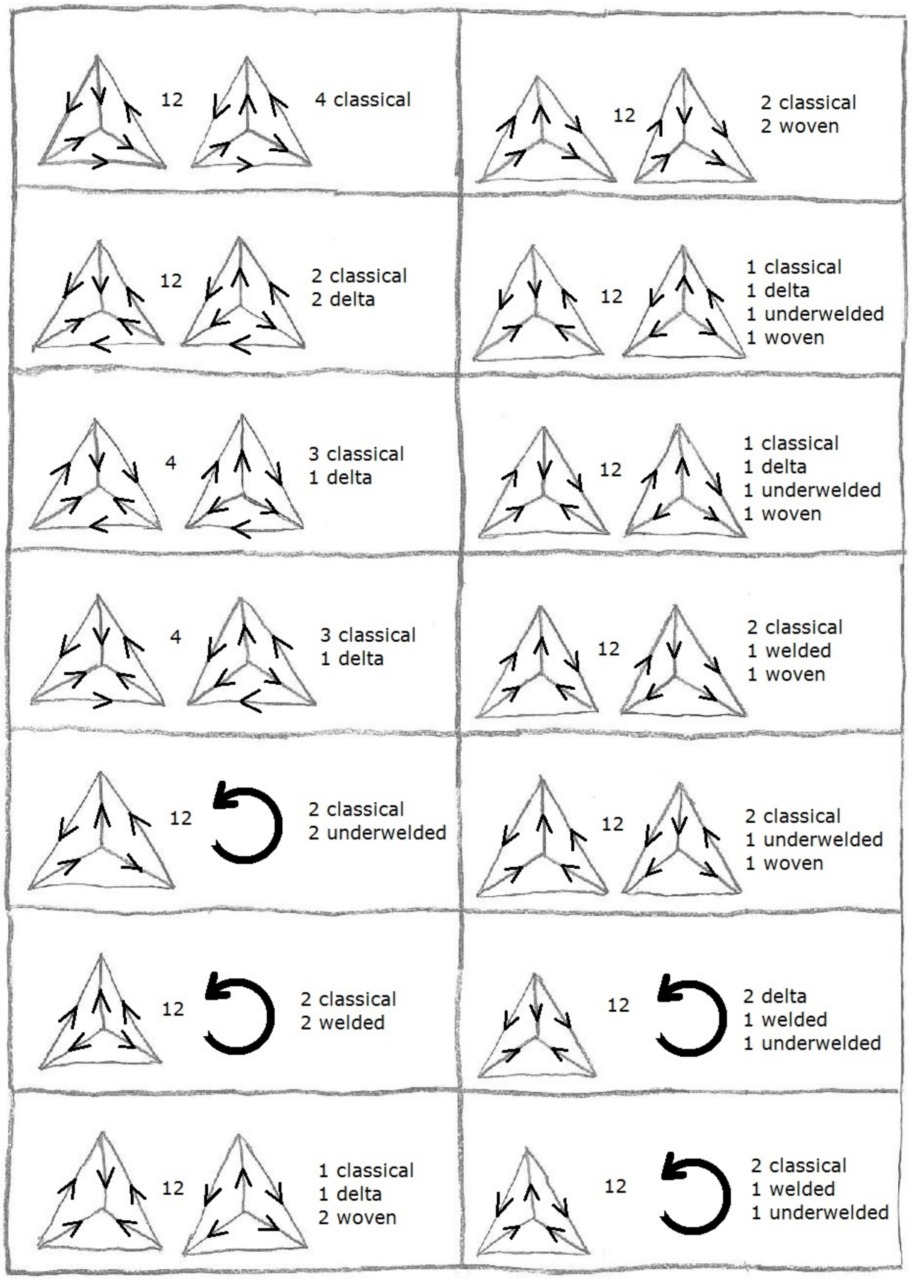


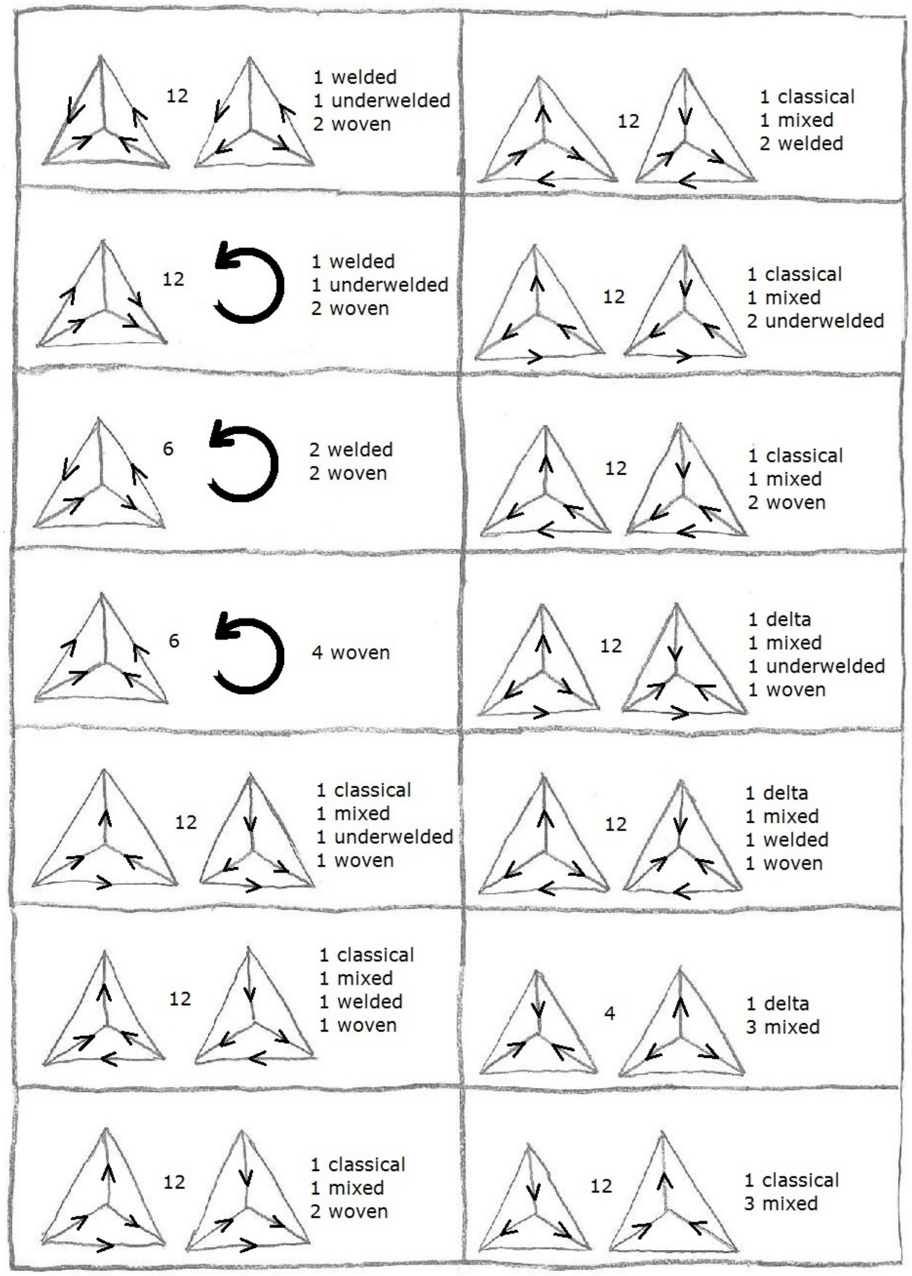
Triple point types are thus indicated by vertices as follows (seen from outside the tetrahedron):

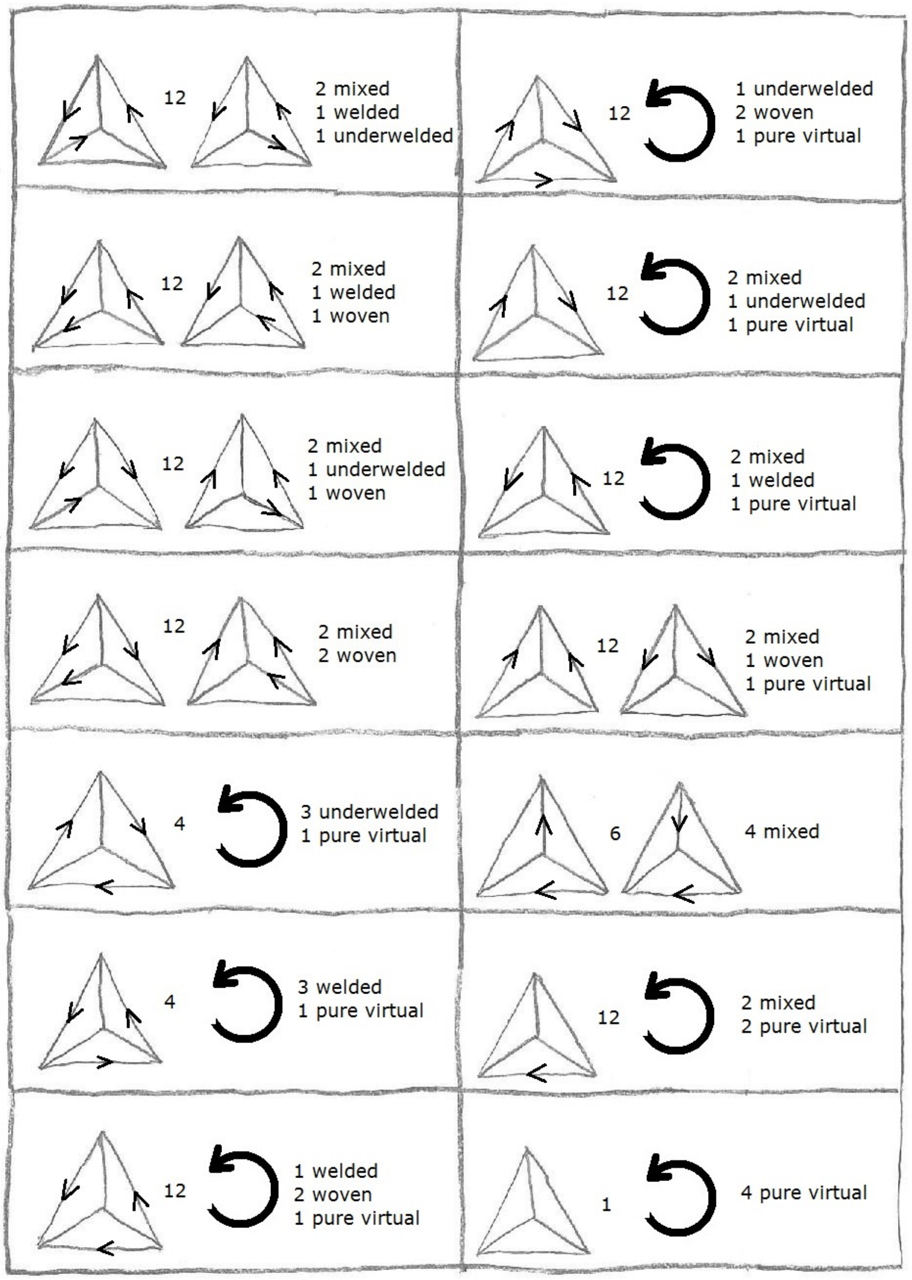


The effect of the Roseman move is to invert a tetrahedron into its dual. In order to pair up "before" and "after" drawings, we combine this inversion with a rotation so that the "before" and "after" drawings are both viewed from above the same triple point. The "after" drawing is obtained by taking the "before" drawing, reflecting it side-to-side, and reversing every arrow. (If the two tetrahedra are alike after a 3D rotation, then only one drawing is included. A curved arrow replaces the "after" drawing.)

For each move, we have indicated a list of the types of triple points involved. Also indicated is the number of "different-looking" rotations of the signed tetrahedron (i.e., the index of the signed tetrahedron's rotation group as a subgroup of the unsigned tetatrahedron's symmetry group). The sum of all these numbers is $3^6 = 729$, as expected (remember to double the numbers whenever the "before" and "after" tetrahedra are distinct).


12
4 classical
12
2 classical
2 woven
12
2 classical
2 delta
12
1 classical
1 delta
1 underwelded
1 woven
4
3 classical
1 delta
12
1 classical
1 delta
1 underwelded
1 woven
4
3 classical
1 delta
12
2 classical
1 welded
1 woven
12
2 classical
2 underwelded
12
2 classical
1 underwelded
1 woven
12
2 classical
2 welded
12
2 delta
1 welded
1 underwelded
12
1 classical
1 delta
2 woven
12
2 classical
1 welded
1 underwelded


12
1 welded
1 underwelded
2 woven
12
1 classical
1 mixed
2 welded
12
1 welded
1 underwelded
2 woven
12
1 classical
1 mixed
2 underwelded
6
2 welded
2 woven
12
1 classical
1 mixed
2 woven
6
4 woven
12
1 delta
1 mixed
1 underwelded
1 woven
12
1 classical
1 mixed
1 underwelded
1 woven
12
1 delta
1 mixed
1 welded
1 woven
12
1 classical
1 mixed
1 welded
1 woven
4
1 delta
3 mixed
12
1 classical
1 mixed
2 woven
12
1 classical
3 mixed


12
2 mixed
1 welded
1 underwelded
12
1 underwelded
2 woven
1 pure virtual
12
2 mixed
1 welded
1 woven
12
2 mixed
1 underwelded
1 pure virtual
12
2 mixed
1 underwelded
1 woven
12
2 mixed
1 welded
1 pure virtual
12
2 mixed
2 woven
12
2 mixed
1 woven
1 pure virtual
4
3 underwelded
1 pure virtual
6
4 mixed
4
3 welded
1 pure virtual
12
2 mixed
2 pure virtual
12
1 welded
2 woven
1 pure virtual
1
4 pure virtual

References

Carter, J. (2011). *An excursion in diagrammatic algebra.* World Scientific.

Carter, J., Kamada, S., & Saito, M. (2002). Stable equivalence of knots on surfaces and virtual knot cobordisms. *Journal of Knot Theory and Its Ramifications*, *11*, 311-322.

Fenn, R., Rimanyi, R., & Rourke, C. (1997). The braid-permutation group. *Topology*, *36*, 123-135.

Goussarov, M., Polyak, M., & Viro, O. (2000). Finite-type invariants of classical and virtual knots. *Topology*, *39*, 1045-1068.

Kauffman, L. (1999). Virtual knot theory. *European Journal of Combinatorics*, *20*, 663-690.

Kuperberg, G. (2003). What is a virtual link? *Algebraic & Geometric Topology*, *3*, 587-591.

Murakami, H., & Nakanishi, Y. (1989). On a certain move generating link-homology. *Mathematische Annalen*, *284*, 75-89.

Roseman, D. (1998). Reidemeister-type moves for surfaces in four-dimensional space. *Banach Center Publications*, *42*, 347-380.

Roseman, D. (2000). Projections of codimension two embeddings. In *Knots in hellas '98.* World Scientific.

Roseman, D. (2004). Elementary moves for higher dimensional knots. *Fundamenta Mathematicae*, *184*, 291-310.

Rourke, C. (2006). What is a welded link? In *Intelligence of low dimensional topology* (p. 263-270). World Scientific.

Satoh, S. (2000). Virtual knot presentation of ribbon torus-knots. *J. Knot Theory Ramifications*, *09*, 531-542.

Takeda, Y. (2012). Introduction to virtual surface-knot theory. *Journal of Knot Theory and Its Ramifications*, *21*.

Winter, B. (2015). Virtual links in arbitrary dimensions. *Journal of Knot Theory and Its Ramifications*, *24*.

Vita

NAME

Jonathan Schneider

EDUCATION

Ph.D., Mathematics, University of Illinois Chicago, **2016**
M.S., Mathematics, University of Illinois Chicago, **2012**
B.A., Mathematics, Swarthmore College, **2005**

TEACHING

Fall 2015: Visiting lecturer, University of Illinois Chicago
Taught Business Calculus

Fall 2013: Adjunct faculty, College of Dupage
Taught Precalculus I, Precalculus II, and Algebra with Applications.
Adjunct faculty, Elgin Community College
Taught Statistics and Quantitative Literacy.

Summer 2013: Adjunct faculty, College of DuPage
Taught Statistics.

Summer 2012: Adjunct faculty, Moraine Valley Community College
Taught Statistics.

Summer 2010 and 2011: Instructor, Universtiy of Illinois Chicago
Taught Intermediate Algebra (course for incoming freshmen)

Summer 2008: Teacher, Center For Talented Youth
Taught Mathematical Modeling (course for gifted 8th graders).

Fall 2006 – Spring 2015: Graduate teaching assistant, UIC
Taught various math courses a T.A.

PRESENTATIONS

2010-present: Quantum Topology Seminar

I regularly give talks on knot theory in this seminar led by Professor Louis Kauffman.

2016: "From Kauffman's virtual model to Satoh's welded model"

Talk at Advances in Quantum and Low-Dimensional Topology. Covered joint work with Eiji Ogasa and Louis Kauffman.

2015: "Welded knots in dimensions 1 and 2"

Talk at the AMS Sectional Meeting, Loyola University. Covered topics related to my graduate dissertation.

2012: "Virtual 2-knots"

Invited talk at the AMS Sectional Meeting, University of Kansas. I explained the object of my graduate research.

2011: "Flat-folded origami"

Guest lecture given to a class at the College of DuPage, IL, on a basic problem in the mathematics of origami.

2008: "A topological model for origami"

Presentation given to the UIC Undergraduate Math Club, on an original model for origami using metric spaces.

2007: "Symmetries of knots"

Talk given at the MAA MathFest in San Jose, CA, on knots with nontrivial 3D symmetry types.

OTHER PUBLIC APPEARANCE

2008: *Between The Folds* (documentary film)

Credited in the film for giving a short interview on the mathematical theory of origami.