Optimal Power Allocation Strategies in Two-Hop X-duplex Multi-source Relay Channel

BY

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B.S., Politecnico di Torino, Turin, Italy, 2016

THESIS

Submitted as partial fulfillment of the requirements
for the degree of Master of Science in Electrical and Computer Engineering
in the Graduate College of the
University of Illinois at Chicago, 2018

Chicago, Illinois

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ACKNOWLEDGMENTS

I would like to express my gratitude to my supervisors, which guided me and spent their time to teach and listen to me. I would like also to thank my loved ones, who supported me during my entire academic career and who were always there, through good times and bad times

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LIST OF ABBREVIATIONS

AF	Amplify-and-Forward
CF	Compress-and-Forward
DF	Decode-and-Forward
FD	Full-Duplex.
HD	Half-Duplex
MARC	Multiple Access Relay Channel
RSU	Road Side Unit
RX	Receiver
SINR	Signal-to-Interference-plus-Noise Ratio
SNR	Signal-to-Noise Ratio
TX	Transmitter
V2I	Vehicle-to-Infrastructure
XD	X-Duplex

SUMMARY

The aim of this thesis is to find an optimal power allocation strategy for the transmitting nodes in a network scenario which comprises two sources, a single relay and a single destination. The main problem which is present in such network configurations is the self-interference phenomenon created by the relay, which attempts to transmit and receive at the same time on the same frequency band. Finding the correct power level (both at sources and relay) which allows the overall rate maximization is a theme of great interest, since many wireless applications present this problem and the use of relays is expected to increase in the near future. The strategy found in this work contemplates the use of Half-Duplex or Full-Duplex transmission modes at the relay. Moreover it is proved that such strategy has interesting performances with respect to other solutions provided in similar scenarios, but which require more precise information about the relay to be known at the sources.

The novelty introduced here is the presence of multiple sources, which raises issues of fairness. The fairness concern is addressed and integrated into the system model provided. All these topics are mathematically described and solved, and the theoretical part is supported with numerical results which confirm the validity of the provided model.

CHAPTER 1

INTRODUCTION

Communications channels which include relays are becoming increasingly important in the wireless field. They may find applications in the next generation mobile communication standards like $5G$: a relay can extend the coverage of an antenna by repeating its signal in areas in which holes are present, so where the conformation of the territory makes the signal weak or inexistent, as well as making stronger the connection for users who are positioned far way from the base station of a particular cell. Placing a relay instead of an antenna is less expensive and furthermore it does not modify the cell grid of an area. Another application can be V2I communications, where vehicles can communicate directly with each others but they can also rely on roadside units (RSUs) which act exactly as a relay. In addition the RSUs can add useful information to the original transmitted signal, so this type of relay needs to modify the received signal. All these examples fall in the area of multi-hop multi-user communications.

Such an importance placed the relay at the center of many studies. There are different cooperative schemes for a relay, which make them more suitable for different environments. Here they are listed in order of increasing complexity:

- Amplify-and-Forward (AF): the relay acts as a simple repeater, so it takes the incoming signal and transmits it towards the destination. A drawback for this forwarding technique is that it amplifies everything that it receives, so when there is a lot of noise added to the useful information, the noise is amplified too. This may not be the best solution for too much noisy channels, and even when the noise level is low, a decode-and-forward (DF) relaying technique is preferred in terms of performance.

- Compress-and-Forward (CF): also known as Estimate-and-Forward or Quantize-and-Forward. The received signal is estimated without being decoded, then the estimate is compressed (through quantization) and forwarded to the destination. There is more complexity inside a relay of this type than for AF, since some signal processing is performed. The estimation of the incoming signal is more accurate when the channel conditions are good, hence there is high SNR. In the high SNR region the CF relay tends to behave as a DF relay, while in the low SNR region the performances are comparable to an AF relay.

- Decode-and-Forward (DF): this cooperative scheme is the most complex one. The relay decodes the signal received, applying signal processing tools to recover information, hence it more robust to noise, since the relay acts as an intermediate destination which makes decisions on the received bits. After having decoded the signal that comes from the source, the relay re-encodes the information and it sends that to the destination. The destination then decodes the received signal. The decision made at the relay is critical, since it determines the performances of the overall link. As in the case of the other relaying strategies, the quality of the channel greatly impacts the outcomes of the transmission.

There are some "hybrid" relays which adapt themselves to channel conditions and change accordingly their cooperative scheme, but of course they are more complex. There are also

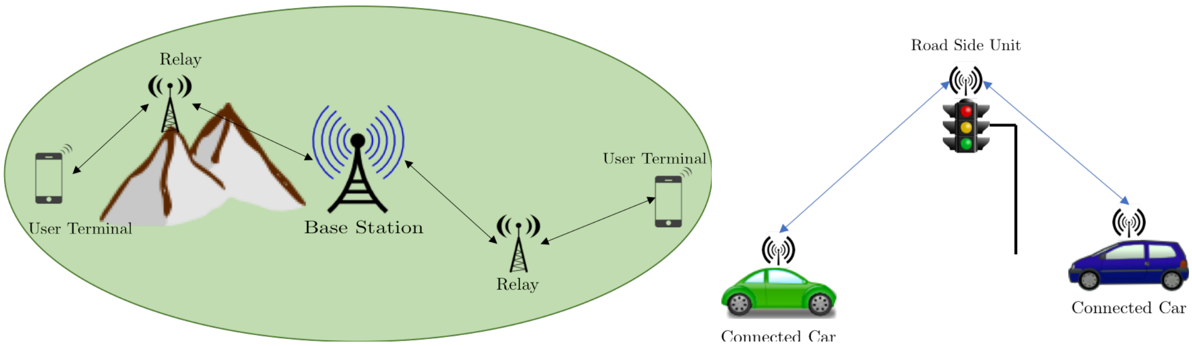

Figure 1: Examples of relay applications

examples in which a combination of modalities is used, as in [11], where CF and DF are mixed and they turn out to be efficient in a particular type of network.

By allowing those intermediate transmitters to manipulate in some way the signal, security issues may arise, since the link between source and destination is split into several links and information is ”seen” by relays which try to decode data or the signal is simply eavesdropped by other relays. Some of this problems are addressed and analyzed in [6, 10].

Besides the employed forwarding techniques, a relay can also work in three different transmission modes:

- Half-Duplex (HD): the relay either transmits or receives, but it cannot perform both the operations at the same time. It is the simpler case to treat, since the relay cannot create interference with itself.
- Full-Duplex (FD): the relay can transmit and receive at the same time, in the same frequency band. FD relays are more complex than HD relays since their transmitted signal

may interfere with the received one, creating the so called *self-interference* phenomenon. This case is the one of interest in this work since it is the key concept around which this thesis revolves.

- X-Duplex (XD): simply it is a combination of the two previously described modes, in which the relay is able to work in both FD and HF modes based on the needs of the transmission or the networks status. Actually this is the kind of relay considered in this thesis, since it offers more flexibility inside a network such as the ones illustrated as examples in the beginning of the introduction.

Having assumed an additive white Gaussian noise model for the channel, a lot of work has been developed towards finding optimal transmission strategies and bounds with those three cooperative schemes. In particular the HD transmission mode has been studied a lot because of its minor complexity, since by definition does not have the problem of the self-interference [3,5,19–21]. Considering the FD case, a number of variants have been studied. The simplest case is the one in which self-interference is neglected, as in [4], that derived the capacity of the FD channel in such conditions. Instead, when self-interference is taken into account, the problem is way more complex and needs proper models. Self-interference is of great interest nowadays, since the technology now allows powerful methods to suppress it without compromising the integrity of the useful information [9, 15]. The signal emitted from the relay loops back to its input, and it is modeled as a Gaussian additive noise which has a variance proportional to the instantaneous transmit power of the relay [12]. In particular, in [14] the aim is to maximize the instantaneous and average spectral efficiency of a two-hop network, where the relay can dynamically and optimally change its transmission mode (HD or FD) with both the AF and DF relaying protocols. That means that the relay switches mode by observing the channel conditions, and it selects HD mode to improve the quality of the transmission by removing self-interference. In [7] a relay with multiple transmitting antennas and a single receiving one is considered, working with AF cooperative scheme. When the relay is subject to constraints on average transmit power, it is found that this configuration outperforms the HD-AF relay in terms of end-to-end signal-to-interference-plus-noise ratio (SINR). In [1], Gaussian inputs are assumed and then achievable rate and bound are derived for both CF and DF relaying protocols. In [12] the case of a two-hop X-duplex channel was analyzed in detail, and a complete description of optimal power allocation strategies was given. That scenario was comparable to the one investigated in [22], where the residual self-interference was modeled as a Gaussian random variable with a variance directly proportional the to amplitude of the transmitted symbol at the relay. The constraints imposed in [22] are the limitations of the average transmit power at source and relay up to certain maximum values. The results of this work is that for what concerns the source, the optimal conditional probability distribution of the source input, given the relay input, is Gaussian, while for the relay the input can be either finite (described by delta functions) or Gaussian, where the latter case occurs only when the relay-destination link is the bottleneck link. Those distribution allow the system to achieve capacity, assuming that the source knows exactly at each instant which symbol the relay is transmitting. To this purpose a buffer can be used, which maintains synchronization between source and relay by holding the data decoded by the relay. This data is re-encoded by the relay in the next available channel use, and if the source has the same encoder, it can predict which symbol will be transmitted by the relay. The novelty of the scheme proposed in [12] consider a more practical case in which there is no total awareness at the source of the symbols transmitted at the relay. In fact, a DF network operating in X-Duplex mode is considered, with a residual self-interference with variance dependent on the relay transmit power. Now the source is only aware of the transmit power distribution adopted by the relay in a precise time frame [12]. This solution is more practical and useful in those scenarios in which the physical-layer or the link-layer are modified by the relay in order to add some information or encryption. The relay still decodes data from the source and forwards it to the destination, but the synchronicity between source and relay cannot be maintained. Since the source knows the power distribution at the relay, the relay operational mode can be controlled and switched by simply setting the correct power level at the source. A Gaussian input distribution is assumed at both source and relay, with a defined maximum variance. An optimization problem is formulated for this scenario, and it aims to maximize the achievable data rate. The system has different solutions based on the operational region in which the relay works, and those regions are delimited by power thresholds. The main result is that the optimal probability density function of the relay transmit power is discrete, composed of either one or two delta functions [12]. Those delta functions give all the necessary information to the system to set the correct power level at source and relay, and also the duration of such setting. A delta function consists of two parameters: the coefficient and the position in which it is centered. That is very few data to exchange between nodes. The resulting optimal communication strategy consists of time frames in which the relay works in FD or HD for a given fraction of time. The obtained performance closely approaches that of communication strategies which assume perfect knowledge at the source of the relay transmitted symbols.

What this thesis aims to do is to extend and generalize the work developed in [12]. In fact it adapts such analysis to the case of multiple sources, which is a more realistic scenario. Considering multiple sources means that the synchronicity between sources and relay cannot be maintained, even without considering transformations applied to the data by the relay, of which the source is not aware. In fact the relay receives information from a set of sources and then such information is re-transmitted taking into consideration data priority, First-Come-First-Served policy or any other scheduling policy. Each source is not aware of the data transmitted by the other sources, so it cannot predict which symbol will be transmitted by the relay in a certain time slot. For such reason it is proposed a strategy in which only the power allocation policy at the relay is known at the sources. The work focuses on the particular case of two sources, but the results can be extended to n sources with more complex mathematical models. The system model and the optimization problem are formulated in Chapter 2. The power allocation strategy found in Chapter 3 is completely similar to the one in [12], in fact it is described by a discrete pdf, which means FD/HD operational modes allocated during precise time frames. In Chapter 4 a new parameter is introduced, since it is related to multiple sources scenarios. Such parameter describes fairness among sources, which is a very important issue: there may be scenarios in which some user must have priority on others, for QoS reasons, and others in which there must be total equity. In Chapter 5, the network model described in Chapter 2 (in the particular case of two sources) is simulated with Matlab, and various plots and performances are shown in order to validate the mathematical results found in Chapters 3 and 4.

To summarize, the thesis is divided into two main parts: the first part, in which the model is presented and an optimal mathematical solution is provided, and the second part, in which simulation of the model assumed are implemented in order to provide graphs and numerical results.

CHAPTER 2

OVERVIEW AND METHODOLOGY

2.1 System Model

This model analyzes a more complex scenario with respect to the case described in [12]. We consider a network scenario with 3 different type of nodes: source nodes s_i, relay node r and destination node d. The communication channel connecting the source to the destination can be divided in two parts: the first hop is the channel which connects source(s) and relay, while the second hop is the one connecting relay and destination. This network model is of paramount importance in real world applications, since a relay can receive information by multiple users who want to reach the same destination.

The existence of the relay is of primary importance, since there is no direct link between the sources and the destination, so they must rely on it to establish communication.
All the channels, both the ones from the different sources to the relay and the one from the relay to the destination, are independent, memoryless block fading channels, and all of them are considered to be subject to additive Gaussian noise. We consider a generic set of channels, where each channel from the source s_i to the relay r has channel gain h_i, while the channel from r to d has gain h_0.

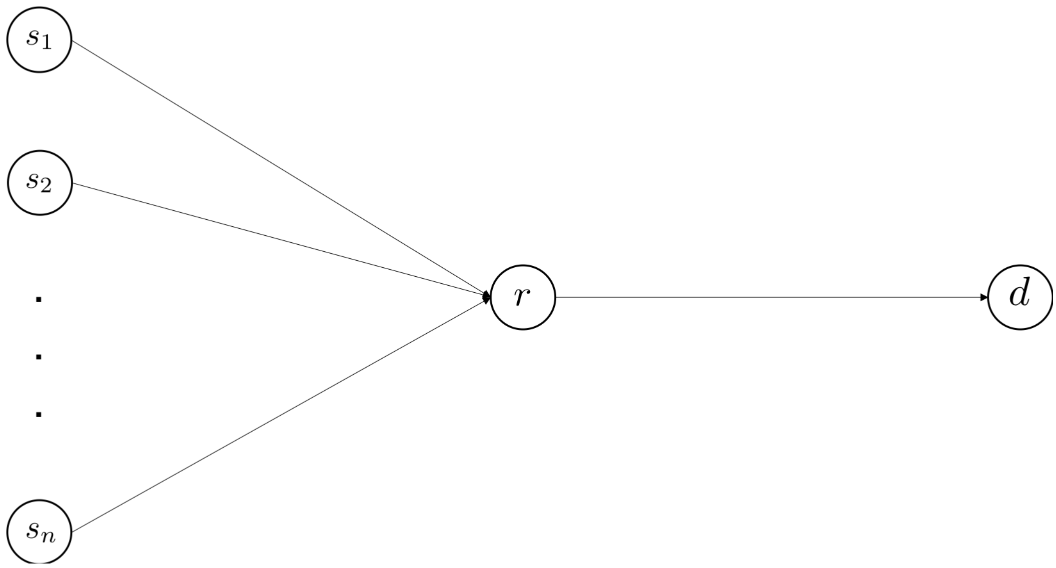

Figure 2: System model overview

Sources and relay work on a frame-by-frame basis of length T, during which the channel gains are assumed to be constant. The relay has three operational modes:

- HD-RX mode: while the sources transmit, the relay receives only.
- HD-TX mode: the relay transmits towards the destination, while the source is silent.
- FD mode: the sources transmit, while the relay transmits and receives at the same time

The most interesting case actually is the last one, since when the relay is transmitting it creates self-interference which is summed to the signal received from the multiple sources. Those signals can be formalized as follows: the signal received by the relay from the sources is denoted by y and is given by

$$y = \sum_{i=1}^{n} \sqrt{P_i} h_i x_i + \nu + n_r \tag{2.1}$$

while similarly the signal received by the destination node is

$$z = \sqrt{p} h_0 x_0 + n_d \tag{2.2}$$

where

- h_i is the channel gain of the source i-relay link, while h_0 is the channel gain of the relay-destination link. As previously stated, the nodes in the system communicate on a per-frame basis, which lasts a specific time T, where T is chosen sufficiently small in order to guarantee static channel conditions.
- x_i and x_0 are respectively the symbols transmitted by the source i and the relay. The symbols transmitted at both sources and relay are assumed to be Gaussian distributed, with zero mean and unit variance, which means that $\mathbb{E}[|x_i|^2] \leq 1$ and $\mathbb{E}[|x_0|^2] \leq 1$. p_i is defined as the transmit power at the source i, while p is the relay transmit power. From that, it derives the instantaneous power of the sources, $p_i|x_i|^2$, and the instantaneous power of the relay, $p|x_0|^2$. We assume p_i and p to have support in the ranges $[0, p_i^{max}]$ and $[0, p^{max}]$, respectively. The transmit power is assumed to be independent of the transmitted symbol, for all the transmitting nodes in the system.
- n_r and n_d represent a Gaussian noise with zero mean and variance N_0

- ν is the instantaneous residual self-interference at the relay. Typically ν is modeled as a Gaussian noise with variance proportional to the instantaneous power of the relay p. So $\nu = \sqrt{\beta p} \cdot G$, where β is the self-interference attenuation factor at the relay and $G \sim \mathcal{N}(0,1)$. This interference model is linear and it is the worst case one, and it is the same assumed also in [1, 17, 18, 22]. The self-interference is called residual because it is the one which still persists after digital and analog suppression.

We define $f(p)$, as the probability density function which represents the power allocation distribution at the relay. While $p_i(p)$ represents the power allocation distribution at the source i, but as it was previously stated, it is directly linked to power set at the relay. We define the average transmit power both at the sources and at the relay, constraining them to some target values:

$$\bar{p} = \mathbb{E}_p \mathbb{E}_{x_0}[p|x_0|^2] = \mathbb{E}_p[p] = \int_0^{p^{max}} pf(p)\mathrm{d}p \tag{2.3}$$

$$\bar{p_i} = \mathbb{E}_p \mathbb{E}_{x_i}[p_i|x_i|^2] = \mathbb{E}_p[p_i] = \int_0^{p^{max}} p_i(p)f(p)\mathrm{d}p \tag{2.4}$$

where $\mathbb{E}[\cdot]$ is the average operator. As anticipated before, the power level at the relay is limited to a maximum value. The above expressions have been derived assuming that the transmit power and the transmitted symbol are independent and that the variance of the transmitted symbol is unitary.

Such system model is the generalization of the single user case provided in [12].

2.2 Bounds for capacity region

The model described in the previous section is also called MARC (Multiple-Access Relay Channel) and it is described in [16]. In [16] some capacity bounds are given for such network model. We will use those bounds as a starting point. Over those bounds some ulterior assumptions will be made, which are present in our model, as it will be developed later on. In order to do that, it is useful to define a system model equivalent from the one of the previous section, but described from an information theory point of view. We will identify the relay with the number 0, while the sources are associated to numbers $1, 2, ..., K$. Time is defined by $t \in [n]$, which are the channel uses.

For a complex-valued two-hop degraded AWGN MARC we have:

- $W_k \in 2^{nR_k}$, $\forall k \in [K]$ is the message set, uniform and independent across users
- $X_{k,t} = \text{enc}_{k,t}(W_k)$, $\forall k \in [K]$ is the encoder belonging to source k. Each encoder is subject to a power constraint given by $\frac{1}{n}\sum_{t\in[n]} |X_{k,t}(W_k)|^2 \leq 1$, $\forall k \in [K]$. This constraint is equivalent to the one in Equation 2.4.
- $Y_t = \sqrt{|X_{0,t}(Y^{t-1})|^2}\nu + \sum_{k\in[K]} h_k X_{k,t}(W_k) + N_{1,t}$ is the output at the relay, where the first term is the self-interference while the second one is the useful information received by the sources, weighted by the channel gain (which is different for each source-relay link). The last term is the noise present on the first hop. Also the relay encoder is subject to the power constraint $\frac{1}{n}\sum_{t\in[n]} |X_{k,t}(W_k)|^2 \leq 1$. This constraint is equivalent to the one in Equation 2.3. ν is the self-interference which was already defined before.

- $Z_t = h_0 X_{0,t}(Y^{t-1}) + N_{2,t}$ is the output at the destination. The first term represents the information transmitted by the relay weighted by the channel gain of the channel between relay and destination, while the second term is the noise level on the second hop.
- $(\widehat{W}_1, ..., \widehat{W}_K) = \text{dec}(Z^n)$ is the joint decoder at the destination.
- $C = \text{ConvHull}\left\{(\rho_{11}, ..., \rho_{1K}) : \lim_{n\to\infty} \text{P}\left[\cup_{k\in[K]}\{\widehat{W}_k \neq W_k\} = 0\right]\right\}$ is the capacity region (set of rate-tuples), assuming $N_i \sim (0, \sigma_i^2)$ and static channel gains. Rate ρ_{1i} is the rate between source i and the relay.

Furthermore we assumed a degraded channel, which can be modeled as a Markov chain in which each state information knowledge is based only on the information coming from the previous state. In fact

$$f_{Y,Z|X_0,X_1,...,X_K}(y, z|x_0, x_1, .., x_K) = \frac{1}{\pi(\sigma_1^2 + \beta|x_0|^2)} e^{-\frac{|y - \sum_{k\in[K]} h_k x_k|^2}{\sigma_1^2 + \beta|x_0|^2}} \frac{1}{\pi(\beta)} e^{-\frac{|z - h_0 x_0|^2}{\sigma_2^2}}$$
$$\iff f_{Y|X_0,X_1,...,X_K} f_{Z|X_0,Y} \quad (2.5)$$

2.2.1 Single user case

It is useful to derive capacity and related bound for the single user case, because it is more immediate and allows a better understanding when it comes to the multi user case. Since the relay channel with a single source ($K = 1$) belongs to the class of degraded memoryless channels, according to [12, 22], we have that

$$
\begin{aligned}
C(\beta) &= \sup_{P_{X_1,X_0}} \min(I(X_1;Y|X_0), I(X_0;Z)) && (2.6)\\
&= \sup_{P_{X_0}, 0\le p_1(\cdot):\mathbb{E}[p_1(X_0)]\le 1} \min\left(\mathbb{E}_{X_0}\left[\log\left(1+\frac{|h_1|p_1(X_0)}{\sigma_1^2+\beta|X_0|^2}\right)\right], I(X_0;Z)\right) && (2.7)\\
&= \sup_{P_{X_0}} \min\left(\mathbb{E}_{X_0}\left[\log^+\left(\eta\frac{|h_1|^2}{\sigma_1^2+\beta|X_0|^2}\right)\right], I(X_0;Z)\right) && (2.8)
\end{aligned}
$$

The capacity is given by the superior of the minimum between the mutual information on the first hop given what the relay transmits (there is knowledge of the symbols transmitted by the relay at the source) and the mutual information on the second hop. P_{X_0} is the power distribution at the relay, while P_{X_1} is the one at the source. $p_1(\cdot)$ and $p_0(\cdot)$ are the transmission power at the single source and the transmission power at the relay respectively. Using Shannon capacity formula we obtain Equation 2.7, under the constraint of the average power and also considering power values greater or equal than zero (practical values in real applications). Equation 2.8 comes from the Lagrange optimization method, necessary to solve the problem. The solution coming from the Lagrangian gives an expression for the power used by the source to transmit, knowing the relay transmitted symbol X_0

$$
p_1(X_0) = \left[\eta - \frac{\sigma_1^2+\beta|X_0|^2}{|h_1|^2}\right]^+ : \mathbb{E}_{X_0}\left[\left[\eta - \frac{\sigma_1^2+\beta|X_0|^2}{|h_1|^2}\right]^+\right] \le 1 \qquad (2.9)
$$

Such system is capacity achieving. Capacity can be upper bounded when we consider the ideal case in which there is no self-interference. In fact

$$C(\beta) = \sup_{P_{X_0}, 0 \le p_1(\cdot): \mathbb{E}[p_1(X_0)] \le 1} \min \left(\mathbb{E}_{X_0} \left[\log \left(1 + \frac{|h_1|^2 p_1(X_0)}{\sigma_1^2 + \beta |X_0|^2} \right) \right], I(X_0; Z) \right) \tag{2.10}$$

$$\le \sup_{P_{X_0}, 0 \le p_1(\cdot): \mathbb{E}[p_1(X_0)] \le 1} \min \left(\mathbb{E}_{X_0} \left[\log \left(1 + \frac{|h_1|^2 p_1(X_0)}{\sigma_1^2 + 0} \right) \right], I(X_0; Z) \right) \tag{2.11}$$

$$\le \sup_{P_{X_0}, 0 \le p_1(\cdot): \mathbb{E}[p_1(X_0)] \le 1} \min \left(\log \left(1 + \frac{|h_1|^2 \mathbb{E}[p_1(X_0)]}{\sigma_1^2 + 0} \right), I(X_0; Z) \right) \tag{2.12}$$

$$= \sup_{P_{X_0}} \min \left(\log \left(1 + \frac{|h_1|^2}{\sigma_1^2} \right), I(X_0; Z) \right) \tag{2.13}$$

$$\le \min \left(\log \left(1 + \frac{|h_1|^2}{\sigma_1^2} \right), \log \left(1 + \frac{|h_0|^2}{\sigma_2^2} \right) \right) = C(0) \tag{2.14}$$

As it was anticipated in the introduction, we consider a practical case in which the symbols transmitted by the relay are not known at the sources. Gaussian inputs are assumed, as it was done in the system model chapter. Furthermore it is introduced a time sharing random variable Q, which is an auxiliary variable and it is not part of the channel variables. In our specific case $Q = p$, the transmit power level at the relay. For this reason, from now on $p_0(p) = p$. In addition we have conditionally independent gaussian inputs, thus

$$P_{X_1, X_0, p} = P_p P_{X_0|p} P_{X_1|p} = P_p \cdot \mathcal{N}(X_0; 0, p) \cdot \mathcal{N}(X_1; 0, p_1(p))$$

$$\iff X_j = \sqrt{p_j(p)} G_j : G_j \sim \mathcal{N}(0,1) \quad iid, \quad 0 \le p_j(\cdot) : \mathbb{E}_p[p_j(p)] \le 1 \tag{2.15}$$

Such assumptions allows us to write a lower bound for capacity, which is

$$C = \sup_{P_{X_1,X_0 p}} \min(I(X_1;Y|X_0,p), I(X_0;Z|p)) \tag{2.16}$$

$$\geq \sup_{P_p, 0\leq p_j(\cdot):\mathbb{E}[p_j(p)]\leq 1, j\in[0:1]} \min\left(\mathbb{E}_{p,G_0}\left[\log\left(1+\frac{|h_1|^2 p_1(p)}{\sigma_1^2+\beta p|G_0|^2}\right)\right],\right. \tag{2.17}$$

$$\left.\mathbb{E}_p\left[\log\left(1+\frac{|h_0|^2 p}{\sigma_2^2}\right)\right]\right)$$

$$\geq \sup_{P_p, 0\leq p_j(\cdot):\mathbb{E}[p_j(p)]\leq 1, j\in[0:1]} \min\left(\mathbb{E}_{p}\left[\log\left(1+\frac{|h_1|^2 p_1(p)}{\sigma_1^2+\beta p\mathbb{E}[|G_0|^2]}\right)\right],\right. \tag{2.18}$$

$$\left.\mathbb{E}_p\left[\log\left(1+\frac{|h_0|^2 p}{\sigma_2^2}\right)\right]\right)$$

$$= \sup_{P_p, 0\leq p_0(\cdot):\mathbb{E}[p]\leq 1} \min\left(\mathbb{E}_p\left[\log^+\left(\eta\frac{|h_1|^2 p_1(p)}{\sigma_1^2+\beta p}\right)\right], \mathbb{E}_p\left[\log\left(1+\frac{|h_0|^2 p}{\sigma_2^2}\right)\right]\right) \tag{2.19}$$

Equation 2.17 is obtained assuming Gaussian inputs and their related properties. We can further lower bound with Jensen's inequality, which allows to take inside the first term the average operator regarding G_0, and this is done in Equation 2.18. In the end the problem can be solved again trough Lagrange method, which is formulated in Equation 2.19. The solution is given for

$$p_1(p) = \left[\eta - \frac{\sigma_1^2+\beta p}{|h_1|^2}\right]^+ : \mathbb{E}_p\left[\left[\eta - \frac{\sigma_1^2+\beta p}{|h_1|^2}\right]^+\right] \leq 1 \tag{2.20}$$

Such solution corresponds to a classical waterfilling policy. We can notice how the solution for the power allocation at the source does not depend anymore on the symbol transmitted at the relay, which is not known, but it depends on the new variable p. Furthermore the solution for the power distribution of the source in [12] is coincident with the derivation just provided.

2.2.2 MARC capacity bounds

Now we can extend the reasoning done for the single user case to a multiple user environment. Let $\mathcal{P}^{(out)}$ be the set of input distribution (as in [16]) of the form

$$\mathcal{P}^{(out)} := \left\{ P_{X_0,X_1,\dots,X_K,p} = P_p \cdot \prod_{k\in[K]} P_{X_k|p} \cdot P_{X_0|X_1,\dots,X_K,p},\ |p| \leq 2(2^K - 1) \right\} \tag{2.21}$$

From this input distribution we can derive an outer bound for capacity, since it is assumed that sources know exactly what relay is transmitting. We already discussed how in a multiple sources scenario such assumption is not realistic, since it is necessary to have all the transmitters synchronized and aware of the scheduling applied at the relay. An outer bound comes from Theorem 1 in [16], which uses cut-sets. From that we have

$$C \subseteq \cup_{\mathcal{P}^{(out)}} \begin{cases} \rho_{\mathcal{S}} \leq I(X_0, X_{\mathcal{S}}; Z|p, X_{\mathcal{S}^c}), \\ \rho_{\mathcal{S}} \leq I(X_{\mathcal{S}}; Z, Y|p, X_{\mathcal{S}^c}, X_0), \\ \forall \mathcal{S} \subseteq [K] \end{cases} \tag{2.22}$$

$$= \cup_{\mathcal{P}^{(out)}} \begin{cases} \rho_{\mathcal{S}} \leq I(X_0; Z|p, X_{\mathcal{S}^c}), \\ \rho_{\mathcal{S}} \leq I(X_{\mathcal{S}}; Y|p, X_{\mathcal{S}^c}, X_0), \\ \forall \mathcal{S} \subseteq [K] \end{cases} \tag{2.23}$$

$\rho_{\mathcal{S}}$ represent the union of the rate-tuple of the sources belonging to $\mathcal{S} \subseteq [K]$, a subset of active sources. The first inequality is related to the second hop, while the second inequality is related to the first hop. $X_{\mathcal{S}^c}$ is the complementary set of $X_{\mathcal{S}}$ in $\mathcal{S}$. This passage from Equation 2.22 to Equation 2.23 is done keeping in mind that we are considering a degraded channel, so in the first inequality the dependency on $X_{\mathcal{S}}$ disappears, as well as Z in the second inequality.

An inner bound on the capacity can be found too. Now we do not assume anymore that sources know what relay is transmitting. To this end we define again a set of input distribution, which comes from the DF region described by equation 8 in [16]:

$$\mathcal{P}^{inDF} = \left\{ P_{X_0,X_1,\ldots,X_K,V_1,\ldots,V_K,p} = P_p \cdot \prod_{k\in[K]} P_{V_k|p} P_{X_k|p,V_k} \cdot P_{X_0|p,V_1,\ldots,V_K} \right\}; \tag{2.24}$$

In this first set it appears the auxiliary random variable V_i, which allows cooperation between the $i-th$ source and the relay. In our system model we do not have such variables, so we need to eliminate them.

$$\mathcal{P}^{inDFnoV} = \left\{ P_{X_0,X_1,\ldots,X_K,p} = P_p \cdot \prod_{k\in[K]} P_{X_k|p} \cdot P_{X_0|p} \right\}; \tag{2.25}$$

$$\mathcal{P}^{inDFnoVGauss} = \left\{ P_{X_0,X_1,\ldots,X_K,p} = P_p \cdot \prod_{k\in[K]} \mathcal{N}(X_k; 0, p_k(p)) \cdot \mathcal{N}(X_0; 0, p), \right. \tag{2.26}$$

$$\left. 0 \le p_i(\cdot) : \mathbb{E}[p_i(\cdot)] \le 1,\, i \in [0:K] \right\};$$

In the last step we assumed again to use Gaussian inputs. Then we have

$$C \supseteq \cup_{\mathcal{P}(inDF)} \begin{cases} \rho_{\mathcal{S}} \leq I(X_0, X_{\mathcal{S}}; Z|p, X_{\mathcal{S}^c}, V_{\mathcal{S}^c}), \\ \rho_{\mathcal{S}} \leq I(X_{\mathcal{S}}; Z, Y|p, X_{\mathcal{S}^c}, V_{[K]}, X_0), \\ \forall \mathcal{S} \subseteq [K] \end{cases} \tag{2.27}$$

$$= \cup_{\mathcal{P}(inDF)} \begin{cases} \rho_{\mathcal{S}} \leq I(X_0; Z|p, X_{\mathcal{S}^c}, V_{\mathcal{S}^c}), \\ \rho_{\mathcal{S}} \leq I(X_{\mathcal{S}}; Y|p, X_{\mathcal{S}^c}, V_{[K]}, X_0), \\ \forall \mathcal{S} \subseteq [K] \end{cases} \tag{2.28}$$

$$\supseteq \cup_{\mathcal{P}(inDFnoV)} \begin{cases} \rho_{\mathcal{S}} \leq I(X_0; Z|p, X_{\mathcal{S}^c}), \\ \rho_{\mathcal{S}} \leq I(X_{\mathcal{S}}; Y|p, X_{\mathcal{S}^c}, X_0), \\ \forall \mathcal{S} \subseteq [K] \end{cases} \tag{2.29}$$

$$\supseteq \cup_{\mathcal{P}(inDFnoVGauss)} \begin{cases} \rho_{\mathcal{S}} \leq \mathbb{E}_p \left[\log \left(1 + \frac{|h_0|^2 p}{\sigma_2^2} \right) \right], \\ \rho_{\mathcal{S}} \leq \mathbb{E}_{p,X_0} \left[\log \left(1 + \frac{\sum_{k \in \mathcal{S}} |h_k|^2 p_k(p)}{\sigma_1^2 + \beta |X_0|^2} \right) \right], \\ \forall \mathcal{S} \subseteq [K] \end{cases} \tag{2.30}$$

$$\supseteq \cup_{\mathcal{P}(inDFnoVGauss)} \begin{cases} \rho_{\mathcal{S}} \leq \mathbb{E}_p \left[\log \left(1 + \frac{|h_0|^2 p}{\sigma_2^2} \right) \right], \\ \rho_{\mathcal{S}} \leq \mathbb{E}_p \left[\log \left(1 + \frac{\sum_{k \in \mathcal{S}} |h_k|^2 p_k(p)}{\sigma_1^2 + \beta p} \right) \right], \\ \forall \mathcal{S} \subseteq [K] \end{cases} \tag{2.31}$$

Equation 2.27 is obtained assuming $\mathcal{P}^{inDF}$ distribution. Equation 2.28 comes from considering a degraded channel again, so the mutual information on the first hop is measured between $X_{\mathcal{S}}$ and Y (second inequality) and the one on the second hop is measured between X_0 and Z (first inequality). Equation 2.29 eliminates the presence of the auxiliary variables V, while Equation 2.30 comes from the Gaussian inputs assumption. Ultimately Equation 2.31 is obtained through Jensen's inequality, which allows to state $\mathbb{E}_{X_0|Q}[|X_0|^2] = p_0(p) = p$. The inequalities in Equation 2.31 are a generalization of the single user case. In order to obtain the single user case, so $K = 1$, we must consider equation Equation 2.28 and substitute $V_1 = X_0$. In fact when there is a single source, there is knowledge at each instant of the transmitted symbol at the relay, so we can consider as the unique auxiliary variable the transmitted symbol itself.

2.3 Maximum achievable rate on first hop

In the previous section we found bounds for the capacity of our MARC model. From Equation 2.31 it is possible to understand that the rate over the network is the minimum between the rate on the first hop and the rate on the second hop. The rates are upper bounded by the capacity values on the related links, which are averaged over p. We are going to write explicitly such expected values in Equation 2.31, but first we introduce some terminology related to rates, which is used throughout the thesis. We start by defining the instantaneous rates, with relay and sources power fixed, can be defined:

$$\rho_{1i} \leq C_{1i} = \log\left(1 + \frac{p_i(p)|h_i|^2}{N_0 + \beta p}\right) \tag{2.32}$$

is the instantaneous rate between source i and relay, as if no other sources were present. Instead

$$\rho_2 = \log\left(1 + \frac{p|h_0|^2}{N_0}\right) \tag{2.33}$$

is the instantaneous rate on the relay-destination link. It is also necessary to define the total instantaneous rate from the sources to the relay, which is:

$$\rho_1 \le C_1 = \log\left(1 + \frac{\sum_{i=1}^{n} |h_i|^2 p_i(p)}{N_0 + \beta p}\right) \tag{2.34}$$

What is interesting for us is to find the achievable rate region on the first hop (which is the critical part in the system), which comprehends all the possible rate combinations for a given set of sources. This region is expected to be a n-dimensional convex polytope, defined by a set of $2^n - 1$ equations. To prove that we start by considering n sources:

- $\mathcal{S} \subseteq \mathcal{N}$: $\mathcal{S}$ could be any subset of $\mathcal{N}$, which is the set that comprehends all the n sources. It is realistic to assume that there may be some sources which are active and some others which are not.
- $\mathcal{S} \neq \emptyset$: it is the case in which all the sources are inactive, which is excluded since it is useless.
- $\rho_{\mathcal{S}} = \sum_{i \in \mathcal{S}} \rho_{1i}$: $\rho_{\mathcal{S}}$ is the sum of the instantaneous rates coming from each source belonging to $\mathcal{S}$. Each rate is computed by ignoring the presence of the other sources belonging to the subset, as if the considered source and the relay are not aware of the other ones.

- $C_{\mathcal{S}} = \log\left(1 + \frac{\sum_{i\in\mathcal{S}} |h_i|^2 p_i}{N_0+\beta p}\right)$: $C_{\mathcal{S}}$ is the capacity of the subset $\mathcal{S}$

The following condition then must be satisfied:

$$\forall \mathcal{S} \subseteq \mathcal{N},\ \mathcal{S} \neq \emptyset \Longrightarrow \rho_{\mathcal{S}} \leq C_{\mathcal{S}} \tag{2.35}$$

Such condition guarantees that, considering any possible subset of $\mathcal{N}$ (excluding the empty set), the sum of the rates coming from each source separately never overcomes the capacity of the whole subset. Since that condition must be satisfied for any possible combination of the n sources, except one, a set of $2^n - 1$ constraints on the rate is created. Such set completely describes the convex polytope, which allows to understand the possible performances of the system.

Such set of constraints shapes the convex polytope represented in Figure 3. A couple (ρ_{11}, ρ_{12}) defines a point in the achievable rate region. Working on any red line means that the sum of the two rates remains constant. The aim is to reach the dominant face of the region (blue edge).

From now on, it is useful to define the average rates, which are average over the relay transmit power p. The average rate for a single source is given by

$$R_{1i}(f, p_i) \leq \int_0^{p^{max}} f(p)\rho_{1i}\,\mathrm{d}p = \int_0^{p^{max}} f(p)\log\left(1 + \frac{p_i(p)|h_i|^2}{N_0 + \beta p}\right)\mathrm{d}p \tag{2.36}$$

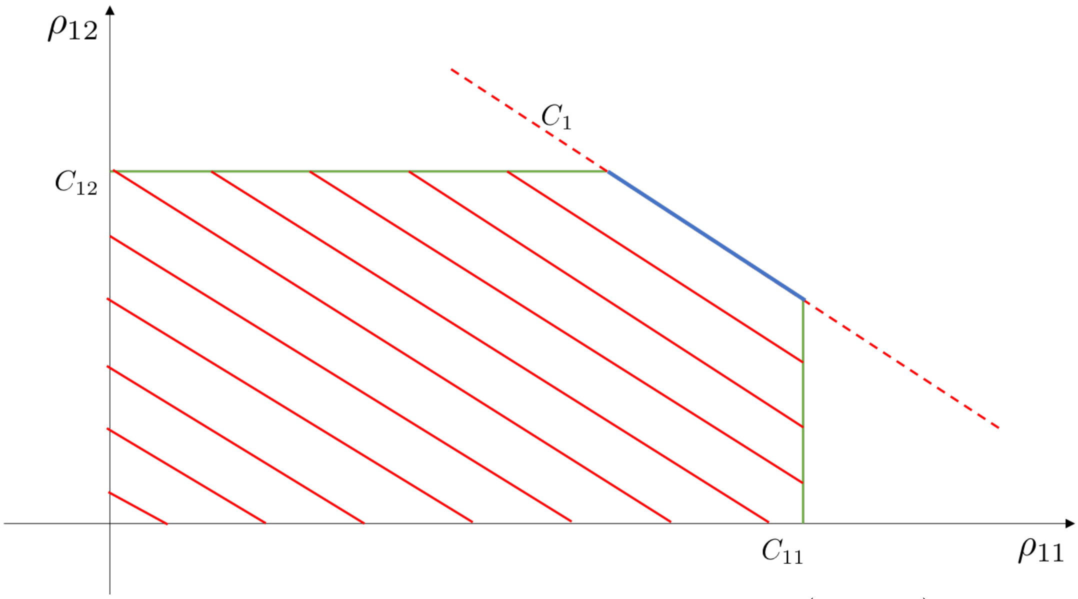

Figure 3: Multiuser channel achievable rate region (first hop)

while the average rate on the relay-destination link is

$$R_2(f) = \int_0^{p^{max}} f(p)\rho_2 \,\mathrm{d}p = \int_0^{p^{max}} f(p) \log\left(1 + \frac{p|h_0|^2}{N_0}\right) \mathrm{d}p. \tag{2.37}$$

$R_{\mathcal{S}}$, which appears in Equation 2.31, is defined as

$$R_{\mathcal{S}}(f, p_i \,\forall i \in \mathcal{S}) \leq \int_0^{p^{max}} f(p) \log\left(1 + \frac{\sum_{i \in \mathcal{S}} |h_i|^2 p_i(p)}{N_0 + \beta p}\right) \mathrm{d}p \tag{2.38}$$

We are interested in the particular case in which all the sources are active and want to transmit, so we have that $\mathcal{S} = \mathcal{N}$. Consequently the total average rate from the sources to the relay is:

$$R_1(f, p_1, ..., p_n) \leq \int_0^{p^{max}} f(p)\rho_1 \,\mathrm{d}p = \int_0^{p^{max}} f(p) \log\left(1 + \frac{\sum_{i=1}^{n} |h_i|^2 p_i(p)}{N_0 + \beta p}\right) \mathrm{d}p \qquad (2.39)$$

CHAPTER 3

OPTIMAL POWER ALLOCATION AT THE SOURCES AND AT THE RELAY

3.1 Problem Formulation

The aim of this thesis is to find the optimal power allocation, both at the multiple sources and at the relay, which ensures the highest achievable rate from the sources to the destination node. Because of the self-interference that occurs at the relay, the rates over the sources-relay links depend on the relay transmit power. The overall rate is the minimum between the rate on the relay-destination link and the sum rate achieved over the sources-relay links. In particular we are going to focus on the multiuser case which has two sources.

The maximum achievable rate is then defined as follows:

$$R \triangleq \max_{f(\cdot),p_1(\cdot),p_2(\cdot)} \min\{R_1(f,p_1,p_2), R_2(f)\} \tag{3.1}$$

$$= \max_{f(\cdot)} \min \left\{ \max_{p_1(\cdot),p_2(\cdot)} R_1(f,p_1,p_2), R_2(f) \right\} \tag{3.2}$$

where the last equality comes from the fact that only R_1 depends on p_1 and p_2.

In order to solve this problem it is necessary to maximize R with respect to p_1 and p_2 in the first place, and then maximize it over $f(\cdot)$. In order to find the power allocation policies in the achievable region in Equation 2.31, we need to divide the optimization into two separated steps.

3.2 Optimal power allocation at the sources

The first step is to maximize the rate over the power allocation at the two sources, according to Equation 3.2. The maximization over $f(\cdot)$ will be done later on, so we start by maximizing over p_1 and p_2, considering $f(\cdot)$ as given.

A similar problem was analyzed in [8], where a multiuser channel is considered. Such channel corresponds exactly to the first hop of our system model. However there are some differences which lead to a different solution. In [8] the channel conditions are constantly measured in order to obtain a precise SNR for each channel. The channel with the best SNR has the right to transmit until another channel becomes the best one. If the received power from a source is below a certain threshold, then such source cannot transmit (it may be that all the sources cannot transmit). The adopted policy can be interpreted as water-filling, since more power is allocated for the channels with good conditions.

In our system model we have static channel gains, so they do not change over time. The channel conditions vary over time because of the different power level at the relay, which entails different amounts of self-interference. Such variation is "proportional" for all the channels, since they all will be affected by the relay transmission. This is the main difference from [8], even tough the scenario is very similar. In fact in [8] the Lagrange method allows to find the set of inequalities described by Equation 9 inside such paper. The fact that the channel gain are random variables guarantees that the average power constraints are met over time (that means that all the sources except one can remain silent), while in our case the channel gains are static but what changes is the power level at the relay, which affects all the channels in the same way. Such proportionality can be eliminated trough Lagrange derivation, and the only way to met the average power constraints is to allow all the sources to transmit at the same time. All of this can be observed in the Lagrange derivation, which is provided in the following.

Then the maximization problem w.r.t. p_1 and p_2 can be formulated as follows:

$$\begin{aligned}
\mathbf{P0}: \quad & R_1(f) = \max_{p_1(\cdot),p_2(\cdot)} R_1(f,p_1,p_2) \quad \text{s.t.} \\
& (a) \quad \int_0^{p^{max}} p_1(p) f(p) \mathrm{d}p = \bar{P}_1; \qquad (b) \quad 0 \le p_1(p) \le p_1^{max} \\
& (c) \quad \int_0^{p^{max}} p_2(p) f(p) \mathrm{d}p = \bar{P}_2; \qquad (d) \quad 0 \le p_2(p) \le p_2^{max}
\end{aligned}$$

recalling that $R_1(f,p_1,p_2) = \int_0^{p^{max}} f(p) \log\left(1 + \frac{|h_1|^2 p_1 + |h_2|^2 p_2}{N_0 + \beta p}\right) \mathrm{d}p$. The constraints imposed by (a), (b), (c) and (d) are the same ones that we imposed in the system model section. We reported them here to stress their importance.

Since the solution of this problem is not trivial, we apply the Lagrange multipliers technique. Since the optimization problem contains inequality constraints we also impose the KKT conditions [2]. The KKT approach is a generalization of the Lagrange's one, since it allows also inequality constraints. The procedure is explained in detail in appendix A.

As said previously, it is better to focus on the sum of the powers. Recalling from the Appendix A the definition

$$P_s(p) \triangleq |h_1|^2 p_1(p) + |h_2|^2 p_2(p) \tag{3.3}$$

in the end we have

$$P_s(p) = \frac{|h_1|^2}{\lambda_1} - (N_0 + \beta p) \tag{3.4}$$

with $P_s(p) \geq 0$ and $P_s(p) \leq |h_1|^2 p_1^{max} + |h_2|^2 p_2^{max} \triangleq P_s^{max}$. λ_1 is the Lagrange multiplier defined in A.

The solution for $P_s(p)$ is:

$$P_s(p) = \min\left\{ \left[\frac{|h_1|^2}{\lambda_1} - (N_0 + \beta p)\right]^+, |h_1|^2 p_1^{max} + |h_2|^2 p_2^{max} \right\} \tag{3.5}$$

$$= \beta \min\left\{ \left[\frac{|h_1|^2}{\beta\lambda_1} - \frac{N_0}{\beta} - p\right]^+, \frac{P_s^{max}}{\beta} \right\} \tag{3.6}$$

$$= \beta \min\{[\omega - p]^+, \mathcal{P}_s^{max}\} \tag{3.7}$$

where the operator $[\cdot]^+ \triangleq \max\{0, \cdot\}$. Furthermore some new parameters are defined:

$$\mathcal{P}_s^{max} \triangleq \frac{P_s^{max}}{\beta} \tag{3.8}$$

is the maximum power sum weighted by $\frac{1}{\beta}$ and

$$\omega \triangleq \frac{|h_1|^2}{\beta\lambda_1} - \frac{N_0}{\beta} \tag{3.9}$$

is the a function of the Lagrange multiplier λ_1. If $f(\cdot)$ is given, and that is the assumption we made in this first part, the optimal value of ω can be found by substituting Equation 3.7 into **P0**-(a).

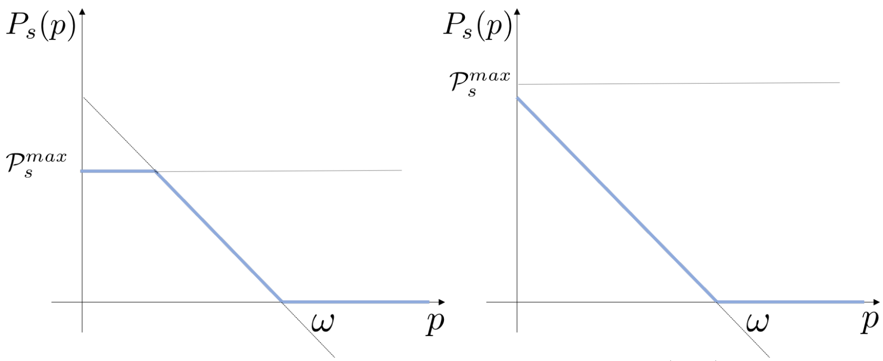

Figure 4: Transmission power sum of the sources: $\omega \geq \mathcal{P}_s^{max}$ (left) and $\omega < \mathcal{P}_s^{max}$ (right)

The two case are illustrated in Figure 4. Having fixed ω, for an increasing relay transmit power the power at the sources is lowered since the self-interference at the relay is higher and higher. After the critical point $\omega = p$ the self-interference is too high and the most convenient thing to do for the sources it to not transmit at all. For simplicity, p_1^{max} and p_2^{max} are assumed as very large values, such that Equation 3.7 can be reduced to

$$P_s(p) = \beta[\omega - p]^+ \tag{3.10}$$

Such assumption has practical relevance when the sources don't have strict limits on transmit power, so when they are macro-cell Base Station (BS) for example.

Since a parameter of fundamental interest is the average rate, by substituting the optimal solution in the expression of the average rate $R_1(f, p_1, p_2)$ (which comes from Equation 3.2) the dependency on ω is highlighted:

$$R_1(f) = \int_0^{p^{max}} f(p) \log\left(1 + \frac{\beta[\omega - p]^+}{N_0 + \beta p}\right) = \int_0^{p^{max}} f(p) \log\left(1 + \frac{\beta_0[\omega - p]^+}{1 + \beta_0 p}\right) \tag{3.11}$$

where $\beta_0 \triangleq \frac{\beta}{N_0}$.

Furthermore also the average sum power is of interest for us, and the dependency on ω can be found by substituting $P_s(p)$ in **P0**-(a) (or **P0**-(b), it is the same)

$$\int_0^{p^{max}} f(p) \frac{\beta[\omega - p]^+ - |h_2|^2 p_2(p)}{|h_1|^2} \mathrm{d}p = \bar{P}_1 \tag{3.12}$$

the fraction is then divided into two terms

$$\int_0^{p^{max}} f(p)\beta[\omega - p]^+ \mathrm{d}p - \int_0^{p^{max}} f(p)|h_2|^2 p_2(p) \mathrm{d}p = |h_1|^2 \bar{P}_1 \tag{3.13}$$

and on the second term constraint (c) is used

$$\int_0^{p^{max}} f(p)\beta[\omega - p]^+ \mathrm{d}p - |h_2|^2 \bar{P}_2 = |h_1|^2 \bar{P}_1 \tag{3.14}$$

and finally with simple math operations

$$\int_0^{p^{max}} f(p)[\omega - p]^+ \mathrm{d}p = \frac{|h_1|^2 \bar{P}_1 + |h_2|^2 \bar{P}_2}{\beta} \triangleq \bar{\mathcal{P}}_s \tag{3.15}$$

After the maximization of the rate R_1 w.r.t. p_1 and p_2 we need to optimize the rate w.r.t. $f(\cdot)$. To this end we rewrite the maximization problem **P0** as:

$$
\begin{aligned}
\mathbf{P1}: \quad & R = \max_{f(\cdot)} \min\{R_1(f), R_2(f)\} \quad \text{s.t.} \\
(a) \quad & R_1(f) = \int_0^{p^{max}} f(p) \log\left(1 + \frac{\beta_0[\omega - p]^+}{1 + \beta_0 p}\right) \mathrm{d}p \\
(b) \quad & R_2(f) = \int_0^{p^{max}} f(p) \log(1 + vp) \mathrm{d}p \\
(c) \quad & \int_0^{p^{max}} f(p)[\omega - p]^+ \mathrm{d}p = \bar{\mathcal{P}}_s \\
(d) \quad & \int_0^{p^{max}} f(p) \mathrm{d}p = 1; \quad \int_0^{p^{max}} p f(p) \mathrm{d}p = \bar{p}; \quad 0 \leq p \leq p^{max}
\end{aligned}
$$

where $v \triangleq \frac{|h_2|^2}{N_0}$. The constraints of the problem are:

- (a) is the average rate on the first part of the communication link, therefore between the sources and the relay, which was derived in Equation 3.11.
- (b) is the average rate on the second part of the communication link, therefore between the relay and the destination.
- (c) represents the constraint on the average power of the sources, which was found in Equation 3.15.
- (d) simply defines $f(\cdot)$ as a distribution with mean value $\bar{p}$ and support in $[0, p^{max}]$.

ω is the free parameter in the problem, but also $f(\cdot)$ need to be chosen in an optimal way. Due to the presence of the non linear operator $[\cdot]^+$ in (a) and (c) we have to consider two separate cases:

(i) The easier case is for $\omega \geq p^{max}$, because it allows to get rid of the $[\cdot]^+$ operator, since its argument is always greater than (or equal to) zero. In fact, according to Figure 5, the non linearity of the function is avoided, letting the problem to be easier.

(ii) When $\omega < p^{max}$ the situation is more complex, from a mathematical point of view, and some additional assumptions must be done. In this thesis such particular case is not treated, but it can be analyzed in future work.

The two previously mentioned cases are illustrated in Figure 5. As it is shown, for

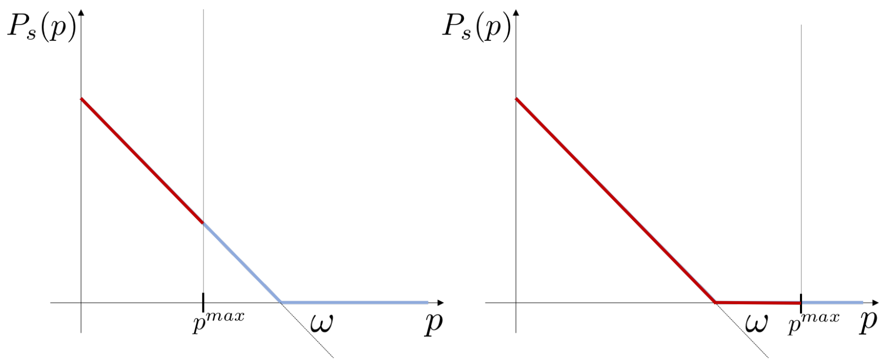

Figure 5: Transmission power sum of the sources: $\omega \geq p^{max}$ (left) and $\omega < p^{max}$ (right)

$\omega \geq p^{max}$ the function which describes the sources power is linear and easy to deal with, while in the other case the discontinuity is still present and a different approach is required to deal with that.

3.3 Optimal power allocation at the relay when $\omega \geq p^{max}$

As it was said in the previous section, by assuming $\omega \geq p^{max}$ we can remove the non-linear $[\cdot]^+$ operator from Equation 3.10. First of all, by using the definitions provided in **P1**-(d) and plugging them into **P1**-(c), ω can be written as:

$$\int_0^{p^{max}} \omega f(p)\mathrm{d}p - \int_0^{p^{max}} pf(p)\mathrm{d}p = \bar{\mathcal{P}}_s \tag{3.16}$$

then by using the constraints coming from (d)

$$\omega - \bar{p} = \bar{\mathcal{P}}_s \tag{3.17}$$

and finally

$$\omega = \bar{\mathcal{P}}_s + \bar{p} \tag{3.18}$$

By substituting this value for ω in Equation 3.11 we obtain a new expression for the rate:

$$\begin{aligned} R_1(f) &= \int_0^{p^{max}} f(p)\log\left(\frac{1+\beta_0 p + \beta_0(\omega - p)}{1+\beta_0 p}\right)\mathrm{d}p && (3.19)\\ &= \int_0^{p^{max}} \log(1+\beta_0(\bar{\mathcal{P}}_s+\bar{p}))f(p)\mathrm{d}p - \int_0^{p^{max}} f(p)\log(1+\beta_0 p)\mathrm{d}p && (3.20)\\ &= \log(1+\beta_0(\bar{\mathcal{P}}_s+\bar{p})) - \int_0^{p^{max}} f(p)\log(1+\beta_0 p)\mathrm{d}p && (3.21)\end{aligned}$$

Moreover an equivalent way to write the condition $\omega \geq p^{max}$ is $\bar{\mathcal{P}}_s \geq p^{max} - \bar{p} \triangleq \mathcal{P}_0$, which comes from Equation 3.18. This is an important definition, since it defines a power threshold. The next step is to lower and upper bound the two rates $R_1(f)$ and $R_2(f)$, , whose expressions are given by Equation 3.21 and **P1**-(b), so that three different regions of operability will be found.

3.3.1 Bounding the rates

To find the bounds the following lemma is used, which is exactly the same lemma used in [12]:

Lemma 4.1. *Let $\phi(p)$ be a continuous concave function and $f(p)$ be a probability distribution, both with support in $[a, b]$. Let $\int_a^b pf(p)\,dp = m$. Then,*

$$\frac{b-m}{b-a}\phi(a) + \left(1 - \frac{b-m}{b-a}\right)\phi(b) \leq \int_a^b f(p)\phi(p)\,dp \leq \phi(m) \tag{3.22}$$

The lower bound holds with equality when $f(p) = \frac{b-m}{b-a}\delta(p-m) + \left(1 - \frac{b-m}{b-a}\right)\delta(p-b)$, while the upper bound holds with equality when $f(p) = \delta(p-m)$, where $\delta(\cdot)$ is the Dirac delta.

The proof of this lemma is reported in Appendix B.

By observing that $f(p)$ is a distribution with mean $\bar{p}$ and that in general a function of the type $\log(1+cp)$, with $c > 0$ is concave, the following bounds can be found. At first the lower bound is applied to Equation 3.21 and **P1**-(b):

$$R_1 \leq \log(1+\beta_0(\bar{\mathcal{P}}_s+\bar{p})) - \frac{\bar{p}}{p^{max}}\log(1+\beta_0 p^{max}) \triangleq r_1^{max} \tag{3.23}$$

$$R_2 \geq \frac{\bar{p}}{p^{max}}\log(1+vp^{max}) \triangleq r_2^{min} \tag{3.24}$$

Both Equation 3.23 and Equation 3.24 hold with equality when

$$f(p) = \left(1 - \frac{\bar{p}}{p^{max}}\right)\delta(p) + \frac{\bar{p}}{p^{max}}\delta(p-p^{max}). \tag{3.25}$$

In a similar way, by using the upper bound on the two rates we obtain:

$$R_1 \geq \log(1+\beta_0(\bar{\mathcal{P}}_s+\bar{p})) - \log(1+\beta_0\bar{p}) \triangleq r_1^{min} \tag{3.26}$$

$$R_2 \leq \log(1+v\bar{p}) \triangleq r_2^{max} \tag{3.27}$$

with the equality that holds when

$$f(p) = \delta(p-\bar{p}). \tag{3.28}$$

3.3.2 P1 solution

The four bounds of the rates which were obtained in the previous section allow a subdivision of the problem **P1** into three distinct working regions.

1) $r_2^{min} \geq r_1^{max}$: in this case $R = r_1^{max}$. The optimal relay power distribution is the one found in Equation 3.25, so

$$f^{\star}(p) = \left(1 - \frac{\bar{p}}{p^{max}}\right) \delta(p) + \frac{\bar{p}}{p^{max}} \delta(p - p^{max}). \tag{3.29}$$

Furthermore, from the inequality $r_2^{min} \geq r_1^{max}$, the power threshold which defines this region can be found, along with the average rate value

$$r_2^{min} \geq r_1^{max}$$

by substituting Equation 3.24 and Equation 3.23 in the inequality

$$\frac{\bar{p}}{p^{max}} \log(1 + vp^{max}) \geq \log(1 + \beta_0(\bar{\mathcal{P}}_s + \bar{p})) - \frac{\bar{p}}{p^{max}} \log(1 + \beta_0 p^{max})$$

and by solving it for $\bar{\mathcal{P}}_s$

$$\bar{\mathcal{P}}_s \leq \mathcal{P}_1 \triangleq \frac{1}{\beta_0} [(1 + \beta_0 p^{max})(1 + vp^{max})]^{\frac{\bar{p}}{p^{max}}} - \frac{1 - \beta_0 \bar{p}}{\beta_0} \tag{3.30}$$

So R is equal to Equation 3.23:

$$R = \log(1 + \beta_0(\bar{\mathcal{P}}_s + \bar{p})) - \frac{\bar{p}}{p^{max}} \log(1 + \beta_0 p^{max}) \tag{3.31}$$

2) $r_1^{min} \geq r_2^{max}$: now $R = r_2^{max}$. The optimal relay power distribution comes from Equation 3.28 and it is

$$f^\star(p) = \delta(p - \bar{p}). \tag{3.32}$$

Similarly to case 1), from the condition that defines this case is obtained

$$r_1^{min} \geq r_2^{max}$$

by substituting Equation 3.26 and Equation 3.27 in the inequality

$$\log(1 + \beta_0(\bar{\mathcal{P}}_s + \bar{p})) - \log(1 + \beta_0 \bar{p}) \geq \log(1 + v\bar{p}) \triangleq r_2^{max}$$

and by solving for $\bar{\mathcal{P}}_s$

$$\bar{\mathcal{P}}_s \geq \mathcal{P}_2 \triangleq \bar{p}v \frac{1 + \beta_0 \bar{p}}{\beta_0} \tag{3.33}$$

along with the rate, which is equal to Equation 3.27

$$R = \log(1 + \bar{p}v). \tag{3.34}$$

3) If the two previous conditions do not hold, we need to solve the problem by setting $R = R_1 = R_2$. This problem is more difficult to solve. Its solution provides the expression for the rates in the range $\mathcal{P}_1 \leq \bar{\mathcal{P}}_s \leq \mathcal{P}_2$. We therefore rewrite **P1** as:

$$
\begin{aligned}
&\mathbf{P2}: \quad R = \max_{f(\cdot)} R_1(f) = \log(1 + \beta_0(\bar{\mathcal{P}}_s + \bar{p})) - \min_{f(\cdot)} \int_0^{p^{max}} f(p) \log(1 + \beta_0 p) \mathrm{d}p \quad \text{s.t.} \\
&\quad (a) \quad R_2(f) = R_1(f) = \int_0^{p^{max}} f(p) \log(1 + vp) \mathrm{d}p \\
&\quad (b) \quad \int_0^{p^{max}} f(p) \mathrm{d}p = 1; \quad \int_0^{p^{max}} p f(p) \mathrm{d}p = \bar{p}; \quad 0 \leq p \leq p^{max}
\end{aligned}
$$

Constraint (a) can be rewritten as

$$
\int_0^{p^{max}} f(p) \log[(1 + \beta_0 p)(1 + pv)] \mathrm{d}p = \log(1 + \beta_0(\bar{\mathcal{P}}_s + \bar{p})) \tag{3.35}
$$

In order to find an optimal solution to this problem it is necessary to apply the following theorem, which again is the same used in [12].

Theorem 4.1. *Consider the following constrained minimization problem:*

$$R = \min_{f(\cdot)} \int_a^b f(p)\phi(p)\,dp \quad s.t. \tag{3.36}$$

$$(a) \quad \int_a^b f(p)\psi(p)\,dp = c$$

$$(b) \quad \int_a^b pf(p)\,dp = m$$

$$(c) \quad \int_a^b f(p)\,dp = 1$$

$$(d) \quad f(p) \geq 0, \forall p \in [a, b]$$

where $\phi(p) = \log(1 + \gamma_1 p)$*,* $\eta(p) = \log(1 + \gamma_2 p)$*,* $\psi(p) = \phi(p) + \eta(p)$ *and* $f(p)$ *is a probability distribution with support in* $[a, b]$*,* $a > 0$*. Moreover,* $\gamma_1 > 0$*,* $\gamma_2 > 0$*,* $m \in [a, b]$ *and* c *are constant parameters. Then, the minimizer has the following expression*

$$f^\star(p) = \begin{cases} \frac{p_B - m}{p_B - a}\delta(p - a) + \frac{m - a}{p_B - a}\delta(p - p_B) & \text{if } \gamma_1 > \gamma_2 \\ \\ \frac{b - m}{b - p_A}\delta(p - p_A) + \frac{m - p_A}{b - p_A}\delta(p - b) & \text{if } \gamma_1 \leq \gamma_2 \end{cases} \tag{3.37}$$

where $p_A \in [a, m]$ *and* $p_B \in [m, b]$ *are obtained by replacing Equation 3.37 in Equation 3.36-(a).*

The proof of the theorem is given in C. By using Theorem 4.1 in the specific case of **P2**, the condition $\gamma_1 \leq \gamma_2$ becomes $v \geq \beta_0$, which leads to the optimal power distribution at the relay:

$$f^\star(p) = \frac{p^{max} - \bar{p}}{p^{max} - p_A}\delta(p - p_A) + \frac{\bar{p} - p_A}{p^{max} - p_A}\delta(p - p^{max}) \tag{3.38}$$

where p_A is a value of p which can be obtained by replacing $f^\star(p)$ defined in Equation 3.37 into **P2**-(a), which means solving the following equation for p_A

$$\left[\frac{(1+\beta_0 p_A)(1+p_A v)}{(1+\beta_0 p^{max})(1+p^{max} v)}\right]^{\frac{p^{max}-\bar{p}}{p^{max}-p_A}} = \frac{1+\beta_0(\bar{\mathcal{P}}_s+\bar{p})}{(1+\beta_0 p^{max})(1+p^{max} v)}. \tag{3.39}$$

Similarly, when $v < \beta_0$:

$$f^\star(p) = \frac{p_B - \bar{p}}{p_B}\delta(p) + \frac{\bar{p}}{p_B}\delta(p - p_B) \tag{3.40}$$

and by applying the same substitution done before, the following equation must be solved to find p_B

$$[(1+\beta_0 p_B)(1+p_B v)]^{\frac{\bar{p}}{p_B}} = 1+\beta_0(\bar{\mathcal{P}}_s+\bar{p}). \tag{3.41}$$

The most important deduction that can be done from these results is that the optimal power allocation at the relay $f^\star(p)$ is *discrete*. This means that a *time division strategy* can be adopted: in fact $f^\star(p)$ is always made of one or two delta functions. In particular, the value in which the delta function is centered defines the power level of the relay, while the coefficient establish the duration of the transmission with such power level. In the case of two probability masses, the transmission time is divided into two phases whose time fractions are given by the coefficient of the deltas. A graphical representation is given in Figure 6.

Once the relay power allocation is chosen, the correspondent $f^\star(p)$ is placed into Equation 3.15, to obtain the sources optimal power level. Thus, the relay needs to communicate to the sources only those few parameters which define the delta functions, making it easy from

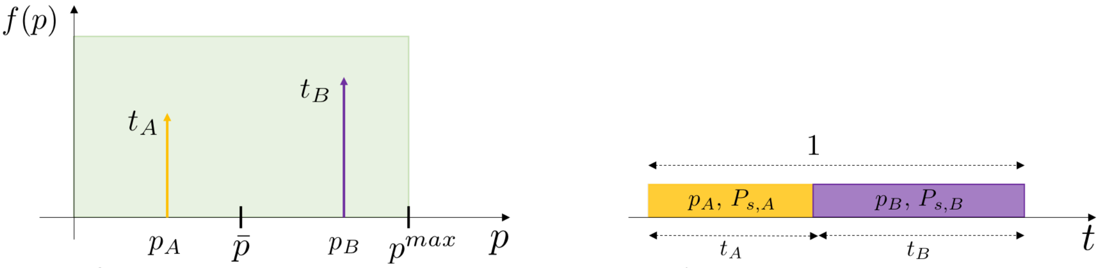

Figure 6: Optimal communication strategy. The time frame is divided into two phases (A and B), with the associated parameters: p_A and p_B define the relay transmit power of the corresponding phase, while t_A and t_B the duration of the phase

a practical implementation point of view: such quantities will be sent to sources through a very small overhead (of course it is fundamental the synchronization at the frame level between sources and relay). Furthermore, we observe that the expressions for $f(\cdot)$ only depend on the channel gain h_0, while the power allocation at the sources depends exclusively on the channel gains h_1 and h_2.

In Table I all the results are collected for the case $\bar{\mathcal{P}}_s \geq \mathcal{P}_0$ according to the regions defined by the power thresholds, and they can be summarized as follows:

- for $\bar{\mathcal{P}}_s \leq \mathcal{P}_1$ the communication strategy adopted at the relay is FD during phase A, while HD-RX in phase B. That means that in phase A both sources and relay are transmitting, while in phase B the relay receives only.

- when $\mathcal{P}_1 < \bar{\mathcal{P}}_s < \mathcal{P}_2$ two case must be distinguished. If $v \geq \beta_0$, then the relay works in FD in both the two phases, although with different paramters. If $v < \beta_0$ the relay works in FD during phase A, but in HD-RX during phase B.
- for $\bar{\mathcal{P}}_s \geq \mathcal{P}_2$ the relay works in FD mode all time, maintaining the same parameters for the whole duration of the transmission time allocated. In particular both sources and relay transmit at their average power level.

TABLE I: OPTIMAL POWER ALLOCATION FOR $\bar{\mathcal{P}}_s \geq \mathcal{P}_0$.

$v \geq \beta_0$	Phase A			Phase B		
	t_A	P_A	p_A	t_B	P_B	p_B
$\bar{\mathcal{P}}_s \in [\mathcal{P}_0, \mathcal{P}_1]$	$\frac{\bar{p}}{p^{max}}$	$\beta(\bar{\mathcal{P}}_s + \bar{p} - p^{max})$	p^{max}	$1 - \frac{\bar{p}}{p^{max}}$	$\beta(\bar{\mathcal{P}}_s + \bar{p})$	0
$\bar{\mathcal{P}}_s \in (\mathcal{P}_1, \mathcal{P}_2)$	$\frac{\bar{p} - p_A}{p^{max} - p_A}$	$\beta(\bar{\mathcal{P}}_s + \bar{p} - p^{max})$	p^{max}	$\frac{p^{max} - \bar{p}}{p^{max} - p_A}$	$\beta(\bar{\mathcal{P}}_s + \bar{p} - p_A)$	p_A
$\bar{\mathcal{P}}_s \in [\mathcal{P}_2, \infty)$	–	–	–	1	$\beta\bar{\mathcal{P}}_s$	$\bar{p}$
$v < \beta_0$	Phase A			Phase B		
	t_A	P_A	p_A	t_B	P_B	p_B
$\bar{\mathcal{P}}_s \in [\mathcal{P}_0, \mathcal{P}_1]$	$\frac{\bar{p}}{p^{max}}$	$\beta(\bar{\mathcal{P}}_s + \bar{p} - p^{max})$	p^{max}	$1 - \frac{\bar{p}}{p^{max}}$	$\beta(\bar{\mathcal{P}}_s + \bar{p})$	0
$\bar{\mathcal{P}}_s \in (\mathcal{P}_1, \mathcal{P}_2)$	$\frac{\bar{p}}{p_B}$	$\beta(\bar{\mathcal{P}}_s + \bar{p} - p_B)$	p_B	$1 - \frac{\bar{p}}{p_B}$	$\beta(\bar{\mathcal{P}}_s + \bar{p})$	0
$\bar{\mathcal{P}}_s \in [\mathcal{P}_2, \infty)$	1	$\beta\bar{\mathcal{P}}_s$	$\bar{p}$	–	–	–

TABLE II: OPTIMAL RATE FOR $\bar{\mathcal{P}}_s \geq \mathcal{P}_0$.

	Rate R
$\bar{\mathcal{P}}_s \in [\mathcal{P}_0, \mathcal{P}_1]$	$\log(1+\beta_0(\bar{\mathcal{P}}_s+\bar{p})) - \frac{\bar{p}}{p^{max}}\log(1+\beta_0 p^{max})$
$\bar{\mathcal{P}}_s \in (\mathcal{P}_1, \mathcal{P}_2);\ v \geq \beta_0$	$\log(1+\beta_0(\bar{\mathcal{P}}_s+\bar{p})) - \frac{p^{max}-\bar{p}}{p^{max}-p_A}\log(1+\beta_0 p_A)+$ $-\frac{\bar{p}-p_A}{p^{max}-p_A}\log(1+\beta_0 p^{max})$
$\bar{\mathcal{P}}_s \in (\mathcal{P}_1, \mathcal{P}_2);\ v < \beta_0$	$\log(1+\beta_0(\bar{\mathcal{P}}_s+\bar{p})) - \frac{\bar{p}}{p_B}\log(1+\beta_0 p_B)$
$\bar{\mathcal{P}}_s \in [\mathcal{P}_2, \infty);\ v < \beta_0$	$\log(1+\bar{p}v)$

CHAPTER 4

FAIRNESS AMONG SOURCES

An aspect which has not been discussed yet is fairness among sources. It is important inside a network to control the relations between sources, giving a sort of priority to them. There may be scenarios in which some sources should transmit at their maximum rate, and we are not interested in maximizing the rate sum. To this end different operability regions will be highlighted in the achievable rate polytope from Figure 3. Moreover, as we will discuss later in this Chapter, total fairness among sources does not imply optimality, and we will propose an approach to insert such α parameter in our already existing system model. In the end the optimal power allocation, both at sources and relay, will be of the same form found in Chapter 3, but some extra constraints related to α may restrict the acceptability of such solutions.

4.1 General considerations

In our considered network mode, fairness can be set by introducing a parameter,α, which binds the instantaneous rates as follows

$$\rho_{11} = \frac{1}{\alpha}\rho_{12} \tag{4.1}$$

The relation can be visualized as a straight line of slope α, which passes through the origin of the Cartesian plane.

Some values of α bound specific regions in the convex polytope in Figure 3. Such regions are highlighted in Figure 7.

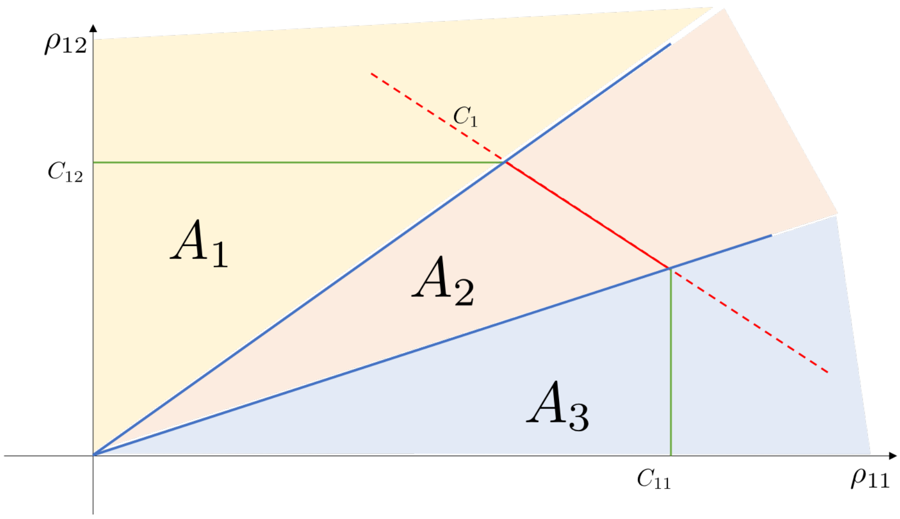

Figure 7: Working regions defined by α

The three regions are (if we consider only the possible values inside the polytope):

- A_1 is the yellowish region: its upper green edge represents the set of points in which S_2 works at its maximum rate C_{12} while S_1 works at a rate value $\rho_{11} < C_1 - C_{12}$. Hence S_1 is penalized with respect to S_2.
- A_2 is the reddish region. Along its red edge the sum rate is maximized. In general here sources do not work at their maximum rate, except when the extremes of this region are considered.

- A_3 is the blueish region. Only along its green edge S_1 works at its maximum rate C_{11} while S_2 works at a rate value $\rho_{12} < C_1 - C_{11}$. Hence here S_2 is penalized with respect to S_1.

Observe that $\alpha = 1$ imply perfect fairness among the two sources. The line $\rho_{12} = \rho_{11}$, however can belong to any of the three regions A_1, A_2 and A_3, whose shape depend on the system parameters. Such dependency is related in particular to the channel gains. The next step is to find the corresponding range of values of α for each region, once the remaining system parameters are fixed. It is expected that for α that tends to 0 also R_{12} tends to zero and $R_{11} = C_{11}$, while for α going to ∞, R_{11} goes to 0 and $R_{12} = C_{12}$. Starting with region A_1, the following system of equations must be introduced:

$$\begin{cases} \rho_{11} = \frac{1}{\alpha}\rho_{12} \\ \rho_{12} = C_{12} \\ \rho_{11} + \rho_{12} = C_1 \end{cases} \tag{4.2}$$

recalling that $C_{11} = \log\left(1 + \frac{|h_1|^2 p_1}{N_0 + \beta p}\right)$, $C_{12} = \log\left(1 + \frac{|h_2|^2 p_2}{N_0 + \beta p}\right)$ and $C_1 = \log\left(1 + \frac{|h_1|^2 p_1 + |h_2|^2 p_2}{N_0 + \beta p}\right)$. This system is necessary to find the slope α of the straight line which passes through the boundary point between A_1 and A_2. The other boundary value of α for A_1 is trivial and it is ∞. The solution of Equation 4.2 is

$$\alpha_1 = \frac{C_{12}}{C_1 - C_{12}} = \frac{\log\left(1 + \frac{|h_2|^2 p_2(p)}{N_0+\beta p}\right)}{\log\left(\frac{1+\frac{|h_1|^2 p_1(p)+|h_2|^2 p_2(p)}{N_0+\beta p}}{1+\frac{|h_2|^2 p_2(p)}{N_0+\beta p}}\right)} = \frac{\log\left(1 + \frac{|h_2|^2 p_2(p)}{N_0+\beta p}\right)}{\log\left(1 + \frac{|h_1|^2 p_1(p)}{|h_2|^2 p_2(p)+N_0+\beta p}\right)} \tag{4.3}$$

A similar reasoning can be done for A_3, which has a trivial bound for α that is 0 and the other one can be found by solving the following system:

$$\begin{cases} \rho_{12} = \alpha\rho_{11} \\ \rho_{11} = C_{11} \\ \rho_{11} + \rho_{12} = C_1 \end{cases} \tag{4.4}$$

which has solution

$$\alpha_2 = \frac{C_1 - C_{11}}{C_{11}} = \frac{\log\left(\frac{1+\frac{|h_1|^2 p_1(p)+|h_2|^2 p_2(p)}{N_0+\beta p}}{1+\frac{|h_1|^2 p_1(p)}{N_0+\beta p}}\right)}{\log\left(1 + \frac{|h_1|^2 p_1(p)}{N_0+\beta p}\right)} = \frac{\log\left(1 + \frac{|h_2|^2 p_2(p)}{|h_1|^2 p_1(p)+N_0+\beta p}\right)}{\log\left(1 + \frac{|h_1|^2 p_1(p)}{N_0+\beta p}\right)} \tag{4.5}$$

In the end the three regions are associated with the following values of α:

- $A_1 \Rightarrow \alpha_1 < \alpha < \infty$
- $A_2 \Rightarrow \alpha_2 \leq \alpha \leq \alpha_1$
- $A_3 \Rightarrow 0 < \alpha < \alpha_2$

where α_1 and α_2 are defined respectively in Equation 4.3 and in Equation 4.5

Now, if α is chosen so that it belongs to region A_1 for example, it is possible to notice how the rate of the source S_1 is significantly lower than its maximum. That means that much of the power transmitted by source S_1, $p_1(p)$, is wasted. In order to avoid this energy waste, the source powers should be adjusted and optimized depending on the choice of the parameter α.

4.2 Bounds on source power

A better approach to find an optimal solution is to fix the α coefficient, and consequently the related straight line, which defines the desired fairness conditions. Then the polytope previously described can be reshaped in order to let the straight line fall into the region A_2, where the sum of the rates is constant (Figure 8). In such conditions the rate achieved on the sources-relay link is described by:

$$\rho_1(p_1, p_2, p) = \log\left(1 + \frac{p_1(p)|h_1|^2 + p_2(p)|h_2|^2}{N_0 + \beta p}\right) \tag{4.6}$$

while the average rate is given by $R_1(f, p_1, p_2) = \int_0^{p^{max}} \rho_1(p_1, p_2, p)\, \mathrm{d}p$. We also recall that the source transmit power should satisfy the constraints

$$\int_0^{p^{max}} f(p) p_i(p) \mathrm{d}p = \bar{P}_i \tag{4.7}$$

To solve the problem of maximization of R_1 the following variable is defined

$$P_s(p) = p_1(p)|h_1|^2 + p_2(p)|h_2|^2 \tag{4.8}$$

so that the maximization problem becomes

$$\textbf{P2:}\quad R_1(f) = \max_{p_1, P_s} \int_0^{p^{max}} f(p) \log\left(1 + \frac{P_s(p)}{N_0 + \beta p}\right) \mathrm{d}p \qquad s.t. \tag{4.9}$$

$$(a) \quad \int_0^{p^{max}} f(p) P_s(p) \mathrm{d}p = \int_0^{p^{max}} f(p)|h_1|^2 p_1(p) + \int_0^{p^{max}} f(p)|h_2|^2 p_2(p) = \bar{P}_s \tag{4.10}$$

$$(b) \quad \int_0^{p^{max}} f(p) p_1(p) \mathrm{d}p = \bar{P}_1 \tag{4.11}$$

$$\tag{4.12}$$

where $\bar{P}_s = \bar{P}_1|h_1|^2 + \bar{P}_2|h_2|^2$. The constraint (a) is simply the extension of the average power constraints on single sources to their power sum. The maximization problem is solved in a totally equivalent way as it was done in the previous section, by following the procedure in appendix A. Now the Lagrangian depends only on a single variable and has the form

$$\mathcal{L}(P_s) = f(p) \log\left(1 + \frac{P_s(p)}{N_0 + \beta p}\right) - \mu f(p) P_s(p) \tag{4.13}$$

and can be solved for P with the equation $\frac{\partial \mathcal{L}(P_s)}{\partial P_s} = 0$. The solution obtained is

$$P_s(p) = \frac{1}{\mu} - N_0 - \beta p \tag{4.14}$$

Considering that $P \geq 0$, similarly to subsection 3.2, the final solution is:

$$P_s(p) = \left[\frac{1}{\mu} - N_0 - \beta p\right]^+ = \beta[\omega - p]^+ \tag{4.15}$$

where $\omega = \frac{1}{\mu\beta} - \frac{N_0}{\beta}$, i.e., $\frac{1}{\mu} = N_0 + \beta\omega$. Again there are two cases to consider: $\omega \geq p^{max}$ and $\omega < p^{max}$. Since the former case avoids the non linearity part of the function, it is simpler to treat. The behavior of $P_s(p)$ is the same as that reported in Figure 5 .

4.2.1 $\omega \geq p^{max}$

In this case it it possible to get rid of the operator $[\cdot]^+$, since the working power region of the relay belongs to the linear part of the function $P_s(p)$. Then we have

$$P_s(p) = \beta(\omega - p) \tag{4.16}$$

and

$$R_1(f) = \log(N_0 + \beta\omega) - \min_{p_1} \int_0^{p^{max}} f(p)\log(N_0 + \beta p)\mathrm{d}p \tag{4.17}$$

is obtained by substituting Equation 4.16 into the expression of R_1. By substituting Equation 4.16 into Equation 4.10, the average power constraint becomes

$$\int_0^{p^{max}} f(p)P(p)\mathrm{d}p = \int_0^{p^{max}} f(p)\beta(\omega - p)\mathrm{d}p = \beta(\omega - \bar{p}) = \bar{P}_s \tag{4.18}$$

which provides

$$\omega = \frac{\bar{P}_s}{\beta} + \bar{p} \tag{4.19}$$

We now need to impose constraints on p_1 and p_2. It is important to remember that we always work on the dominant face of the achievable rate region (for the first hop) where the sum rate is maximized, in order to avoid energy waste. In this region the total rate is maximized. As it is shown if Figure 8, a certain $\alpha = c$ is chosen. Such slope α does not allow the corresponding straight line to fall inside the optimality region. Instead of changing α, the region is reshaped, and that can be done differently, based on the availability of network resources.

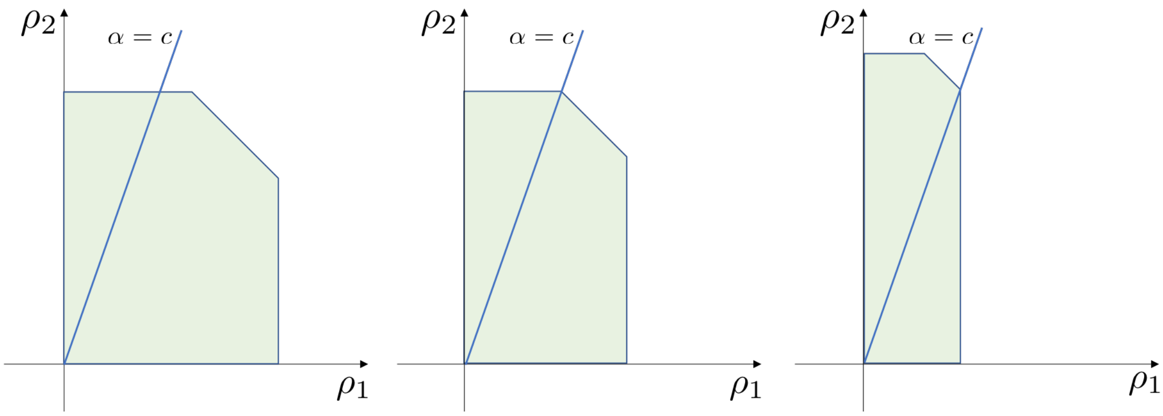

Figure 8: Region reshape once α is fixed.

Given $P_s(p)$, let $\rho_i(p)$ the rate achieved by source i. By working on the dominant face of the achievable rate region it is needed to impose $\rho_1(p) + \rho_2(p) = \rho_s(p)$ where

$$\rho_s(p) = \log\left(1 + \frac{P_s(p)}{N_0 + \beta p}\right) = \log\left(\frac{N_0 + \beta\omega}{N_0 + \beta p}\right) \tag{4.20}$$

Moreover since it was imposed $\rho_2(p) = \alpha\rho_1(p)$ (fairness condition) it follows that

$$\rho_1(p) = \frac{1}{1+\alpha}\rho_s(p) \tag{4.21}$$

$$\rho_2(p) = \frac{\alpha}{1+\alpha}\rho_s(p) \tag{4.22}$$

Now, since $\rho_i(p) \leq \log(1+T_i)$ (maximum achievable rate for the source S_i), where $T_i = \frac{p_i|h_i|^2}{N_0+\beta p}$, we need to satisfy

$$\log(1+T_1) \geq \frac{1}{1+\alpha}\rho_s(p); \tag{4.23}$$

$$\log(1+T_2) \geq \frac{\alpha}{1+\alpha}\rho_s(p). \tag{4.24}$$

i.e.,

$$T_1 \geq \left(\frac{N_0+\beta\omega}{N_0+\beta p}\right)^{\frac{1}{1+\alpha}} - 1 \tag{4.25}$$

$$T_2 \geq \left(\frac{N_0+\beta\omega}{N_0+\beta p}\right)^{\frac{\alpha}{1+\alpha}} - 1 \tag{4.26}$$

By adding T_1 to the second inequality we obtain

$$T_1 + T_2 \geq \left(\frac{N_0+\beta\omega}{N_0+\beta p}\right)^{\frac{\alpha}{1+\alpha}} - 1 + T_1 \tag{4.27}$$

which can be rewritten as

$$1 + T_1 + T_2 - \left(\frac{N_0 + \beta\omega}{N_0 + \beta p}\right)^{\frac{\alpha}{1+\alpha}} \geq T_1 \tag{4.28}$$

Since $1 + T_1 + T_2 = 1 + \frac{P_s(p)}{N_0+\beta p} = \frac{N_0+\beta\omega}{N_0+\beta p}$ it follows

$$T_1 \leq \frac{N_0 + \beta\omega}{N_0 + \beta p} - \left(\frac{N_0 + \beta\omega}{N_0 + \beta p}\right)^{\frac{\alpha}{1+\alpha}} \tag{4.29}$$

In conclusion Equation 4.25 and Equation 4.26 can be rewritten as

$$\left(\frac{N_0 + \beta\omega}{N_0 + \beta p}\right)^{\frac{1}{1+\alpha}} - 1 \leq T_1 \leq \frac{N_0 + \beta\omega}{N_0 + \beta p} - \left(\frac{N_0 + \beta\omega}{N_0 + \beta p}\right)^{\frac{\alpha}{1+\alpha}} \tag{4.30}$$

By recalling that $T_1 = \frac{p_1(p)|h_2|^2}{N_0+\beta p}$, the inequality in Equation 4.30 can be expressed as

$$\left[\left(\frac{N_0 + \beta\omega}{N_0 + \beta p}\right)^{\frac{1}{1+\alpha}} - 1\right](N_0 + \beta p) \leq p_1(p)|h_1|^2 \leq N_0 + \beta\omega - (N_0 + \beta p)\left(\frac{N_0 + \beta\omega}{N_0 + \beta p}\right)^{\frac{\alpha}{1+\alpha}} \tag{4.31}$$

or

$$(N_0 + \beta\omega)^{\frac{1}{1+\alpha}}(N_0 + \beta p)^{\frac{\alpha}{1+\alpha}} - (N_0 + \beta p) \leq p_1(p)|h_1|^2 \leq N_0 + \beta\omega - (N_0 + \beta\omega)^{\frac{\alpha}{1+\alpha}}(N_0 + \beta p)^{\frac{1}{1+\alpha}} \tag{4.32}$$

For simplicity we define

$$L(p) = (N_0 + \beta\omega)^{\frac{1}{1+\alpha}}(N_0 + \beta p)^{\frac{\alpha}{1+\alpha}} - (N_0 + \beta p) \tag{4.33}$$

and

$$U(p) = N_0 + \beta\omega - (N_0 + \beta\omega)^{\frac{\alpha}{1+\alpha}} (N_0 + \beta p)^{\frac{1}{1+\alpha}}. \tag{4.34}$$

From Equation 4.32 we obtain

$$L(p) \leq p_1(p)|h_1|^2 \leq U(p) \tag{4.35}$$

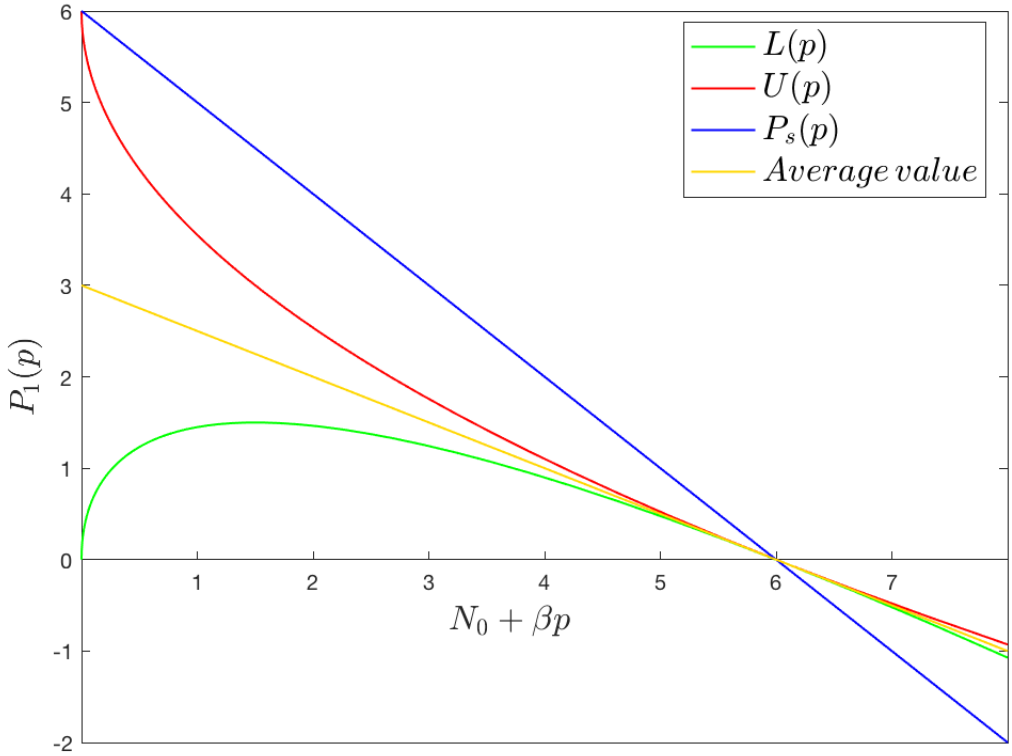

Figure 9: Lower and upper bound for $p_1(p)$

Those bounds are represented in Figure 9, in particular for $\omega = 6$ and $\alpha = 1$. Any value of p_1 selected between the lower bound and the upper bound assures the rate maximization with the specific ratio imposed by α. Depending on the system parameters, there may not be a solutions in some cases, and this happens when the required power falls outside of the bounds. In such cases the fairness constraints cannot be satisfied. Subsequently a procedure completely analogue to what has been developed in Subsections 3.3.1 and 3.3.2 now is applied. As it happened in Chapter 3, different solutions are found depending on which average source power level is applied. As in Subsection 3.3.2, there are three working regions and as many related solutions. Regarding the first two cases, which are the easier ones in mathematical terms, they solutions remain the same: the first one is a discrete probability density function made of two delta functions, one centered in $p = 0$ and the other one in $p = p^{max}$. The second one is again a discrete pdf, but with a single delta function centered in $p = \bar{p}$. The third case, the most complex one, again maintains pretty much the same form, with some minor changes in the formulation of the problem that needs to be maximized (actually it is minimized for the minus in front of the objective function). Now the problem is:

$$\begin{aligned}
\mathbf{P3}: \quad & R = \max_{f(\cdot)} R_1(f) = \log(N_0 + \beta\omega) - \min_{f(\cdot)} \int_0^{p^{max}} f(p)\log(N_0 + \beta p)\mathrm{d}p \quad \text{s.t.} \\
& (a) \quad R_2(f) = R_1(f) = \int_0^{p^{max}} f(p)[\rho_1(p) - \rho_2(p)]\mathrm{d}p \\
& (b) \quad \int_0^{p^{max}} f(p)\mathrm{d}p = 1; \quad \int_0^{p^{max}} pf(p)\mathrm{d}p = \bar{p}; \quad 0 \le p \le p^{max} \\
& (c) \quad \int_0^{p^{max}} f(p)p_1(p)\mathrm{d}p = \bar{P}_1; \quad L(p) \le p_1(p)|h_1|^2 \le U(p)
\end{aligned}$$

This new problem has one more constraint w.r.t. **P2** formulated in Subsection 3.3.2 Moreover it contains the inequality $L(p) \le p_1(p)|h_1|^2 \le U(p)$ which imposes the fairness condition. Such similarity in the formulation of the problem hint that the maximizer $f^\star$ is composed of two delta functions whose positions, p_1 and p_2, are in the range $[0, p^{max}]$. Each delta function carries two degrees of freedom (i.e., position and weight) which are determined by the constraints. Besides this fact, the procedure for solving **P3** follows that outlined in Appendix C.

CHAPTER 5

NUMERICAL RESULTS

In this final chapter, numerical evaluations are provided to support the theory developed in the previous chapters.

One of the parameters of much interest is the rate of course. We want to compare our rate performances with other systems which make different hypotheses on the behavior of the relay, as it was done in [12]. Here they are listed:

- Ideal Full-Duplex: this is the case in which the network always work in FD mode and the relay does not suffer from self-interference. This is only ideal because self-interference cannot be eliminated completely, even with the finest techniques, and for that reason it is considered as an upper-bound to all the other methods. Any other transmission mode adopted could not do better than this.

$$R_{\text{FD-Ideal}} = \min\left\{\log\left(1+\frac{\bar{P}_s}{N_0}\right), \log\left(1+\frac{\bar{p}|h_0|^2}{N_0}\right)\right\} \tag{5.1}$$

- Full-Duplex with the knowledge of instantaneous power (IP) of the relay at the source: Sources always transmit at their average power sum $\bar{P}_s$ and also the relay always transmits at its average power $\bar{p}$. HD is not contemplated in this model, provided in [22]. The main difference with the model proposed in this thesis, to stress it again, is that in our case the source is only aware of the average power of the relay, and also HD mode is used in some cases.

$$R_{\text{FD-IP}} = \min\left\{\int_{-\infty}^{+\infty} \log\left(1+\frac{\bar{P}_s}{N_0+\beta x^2}\right)\frac{e^{-\frac{x^2}{2\bar{p}}}}{\sqrt{2\pi\bar{p}}}\mathrm{d}x,\ \log\left(1+\frac{\bar{p}|h_0|^2}{N_0}\right)\right\} \tag{5.2}$$

- Half-Duplex: the relay either transmits or receives, so this is the worst case considered. This model was also used as comparison in [22] and the transmission is divided into two time fractions, t and $1-t$.

$$R_{\text{HD}} = \max_{\frac{\bar{p}}{p^{max}}\le t\le 1} \min\left\{(1-t)\log\left(1+\frac{\bar{P}_s}{(1-t)N_0}\right),\ t\log\left(1+\frac{\bar{p}|h_0|^2}{tN_0}\right)\right\} \tag{5.3}$$

It must be a lower bound with respect to the system we provide, since in the worst case the X-Duplex solution works in HD.

The scenario that is taken into consideration is analogous to the one assumed in [12]. To evaluate the goodness of all the considerations done previously, we try to obtain the same results found in [12], even if now two sources are assumed instead of one.

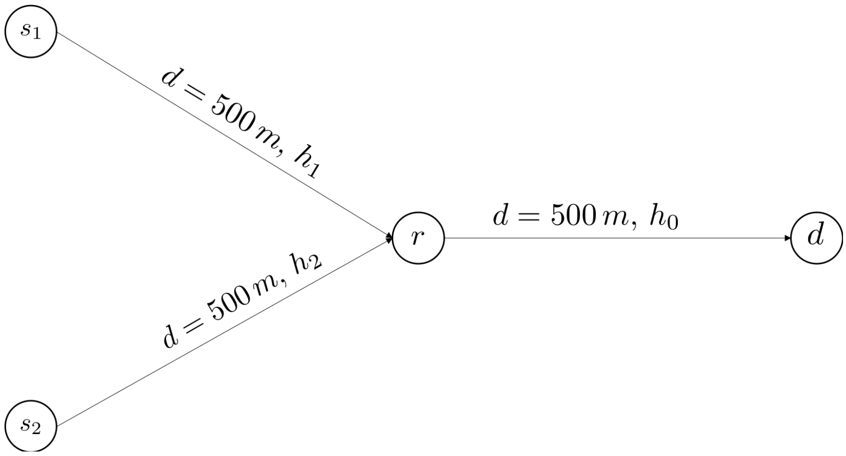

Figure 10: Geometry of the network.

The parameters for the basic scenario, which were used also in [12], are:

- $d = 500$ m is the distance which separates both sources from the relay, and also the relay and the destination node.
- $f_c = 2.4$ GHz is the carrier frequency at which the signal is transmitted.
- $|h_0|^2 = \left(\frac{c}{4\pi f_c}\right)^2 d^{-\gamma}$ is the general formula for the gain associated to a channel, where γ is set to 3. To obtain the same channel conditions that were present in [12], we set $|h_1|^2 = |h_2|^2 = \frac{|h_0|^2}{2}$.
- $N_0 = -151$ dBW is the noise power, which is assumed to be the same for all receivers in the network. The signal bandwidth is $B = 200$ kHz and the power spectral density is 204 dBW/Hz.

- the relay parameters are: $\bar{p} = -10$ dBW and $p^{max} = -7$ dBW, with the self-interference attenuation factor $\beta = -135$ dB.
- the average power at the sources is assumed equal, so $\bar{P}_1 = \bar{P}_2$. To obtain the power thresholds defined in Subsection 3.3.2 it is necessary to remember that $\mathcal{P}_s = \frac{P_s(p)}{\beta} \longrightarrow p_1(p) = p_2(p) = \mathcal{P}_s \frac{\beta}{|h_1|^2+|h_2|^2}$. Since we are considering $P_s(p)$ (made of p_1 and p_2) and not $\mathcal{P}_s$, it is necessary to scale all the power values related to $\mathcal{P}_s$ by $\frac{\beta}{|h_1|^2+|h_2|^2}$. In particular the thresholds values becomes coincident with the ones of [12], which are: $\mathcal{P}_0 = -24$ dBW, $\mathcal{P}_1 = -14.23$ dBW and $\mathcal{P}_2 = -3.04$ dBW.
- the average power at the sources is assumed equal, so $\bar{P}_1 = \bar{P}_2$.

Since $v = \frac{|h_0|^2}{N_0} \approx 30$ dB and $\beta_0 = \frac{\beta}{N_0} \approx 16$ dB, the case $v > \beta_0$ must be considered. Furthermore, since only the situation $\omega \geq p^{max}$ was analyzed in this thesis, the sources power sum used in all the simulations is always $\bar{P}_s \geq \mathcal{P}_0$.

The rate performances are provided in Figure 11, assuming $\alpha = 1$. First of all it is possible to notice how the curve has a different trend inside the three different power regions, which are $[\mathcal{P}_0, \mathcal{P}_1]$, $(\mathcal{P}_1, \mathcal{P}_2]$ and $(\mathcal{P}_2, \infty)$. This is what we expect since for each region the maximizer $f^\star$ assumes a different form, as indicated in Table I. Furthermore, all the rates obtained are identical to the ones that were illustrated in [12]: this is important since it means that even if there are two sources present in the network, the total rate is still maximized and there are no losses regarding the performances. As it was predicted, the ideal FD curve is an upper-bound for all the rates, while the HD curve is a lower-bound. In fact, for very low power levels, our system coincides to the HD one. The system which is aware of the instantaneous power, FD-IP, outperforms the X-Duplex configuration only for power levels greater than $\bar{p} = -10$ dBW, and the gain in performances is not so relevant. Again this is very important, because our proposed solution has outstanding performances for low powers, compared to the other solutions, and for higher power levels it is still a great competitor. In fact the minimal loss in rate performance is balanced by the fact that only the average power of the relay is known at sources, and that requires way less complex transmission mechanisms which are translated in lower costs.

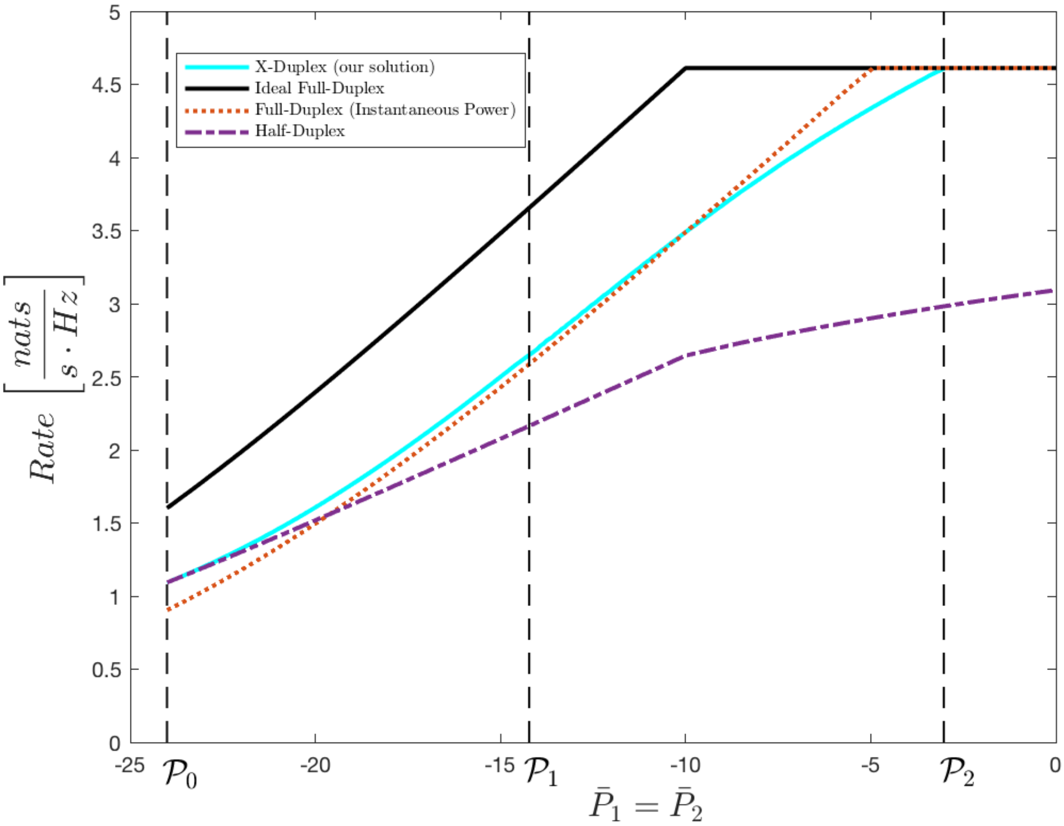

Figure 11: Rate performances with different transmission systems

Similarly Figure 12 shows the optimal powers at sources and relay for the two phases highlighted in Table I, which come along with the rates depicted in Figure 11. In particular here it is plotted the sum of the transmit powers of the two sources. The obtained results are in agreement with the ones found in [12], which confirms one more time the validity of the proposed system for multiple sources.

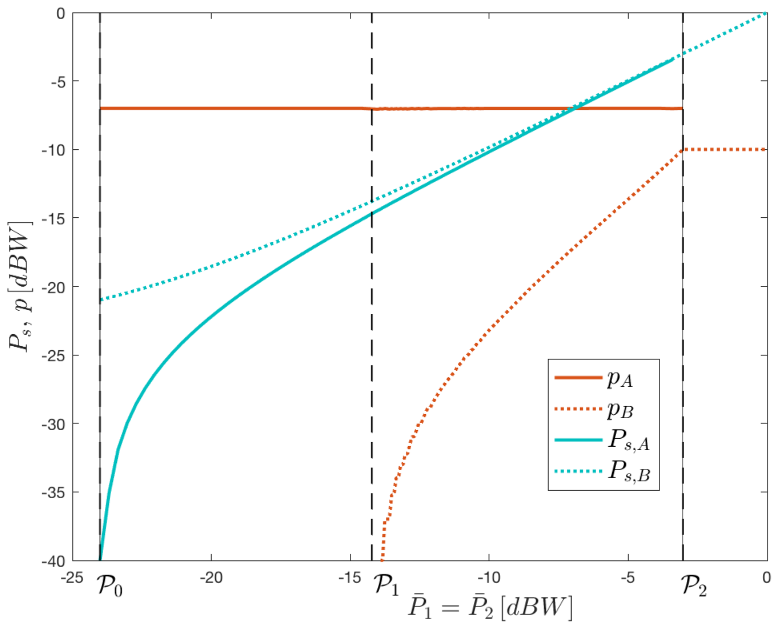

Figure 12: Optimal source and relay transmit powers for phase A (solid lines) and phase B (dashed lines)

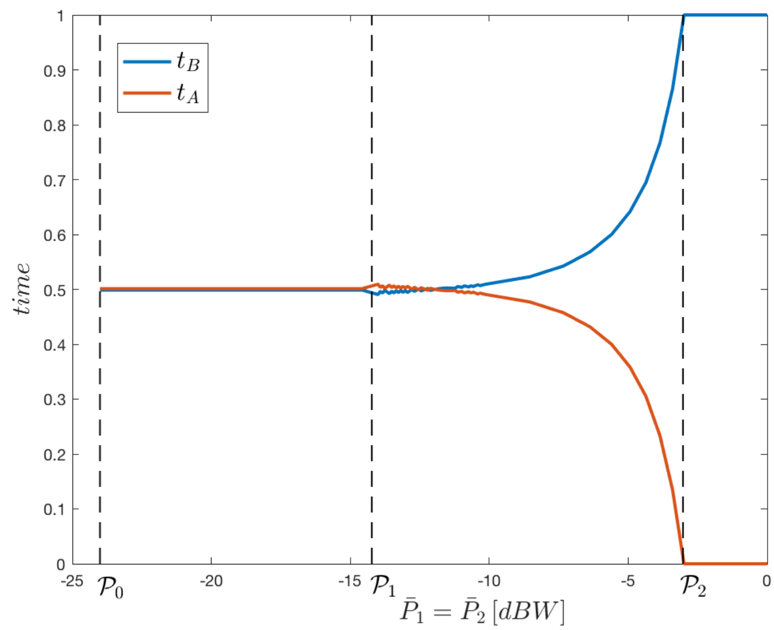

Figure 13: Phase durations

In fact, the total power coming from the sources should not overcome the optimal power in the case of a single source, with the system settings that were imposed previously. All this curves are illustrated in Figure 12. Similarly also the phase durations are shown in Figure 13.

By observing those two graphs, it is possible to observe that for $\mathcal{P}_0 \leq \bar{P}_s \leq \mathcal{P}_1$ the time fraction associated to both phases is equal, while the two sources transmit in all the phases, with the respective powers.

The relay instead transmits only during phase A. When $\mathcal{P}_1 < \bar{P}_s \leq \mathcal{P}_2$, the relay works in FD in both phases, while the sources transmit with a power level which becomes more and more coincident for both the phases. The phase durations start to diverge, one goes to 0 and the other one goes to 1. In fact, when $\bar{P}_s > \mathcal{P}_2$, there is a single phase of duration 1, in which the relay and the sources transmit at their average values.

Now that we have verified that the system works correctly, by doing the comparison with the single source case provided in [12], we can introduce variations on the fairness coefficient.

By varying the fairness coefficient α we change the proportionality between the rates of the two sources, letting one of the two being favored. In Figure 14 it is possible to observe how different αs affect the rate curves.

$\alpha = 1$ and $\alpha = 0.5$ provide the same rate curve, which is the maximum obtainable rate curve. That means that the average power level provided by the sources is always sufficient to guarantee such rate relationship. In the other cases in Figure 14 in which α is smaller or greater than the previous values, it is possible to notice how the rate is zero up to a certain power value, and then it reaches the maximum possible rate (the curve for $\alpha = 1$): that means

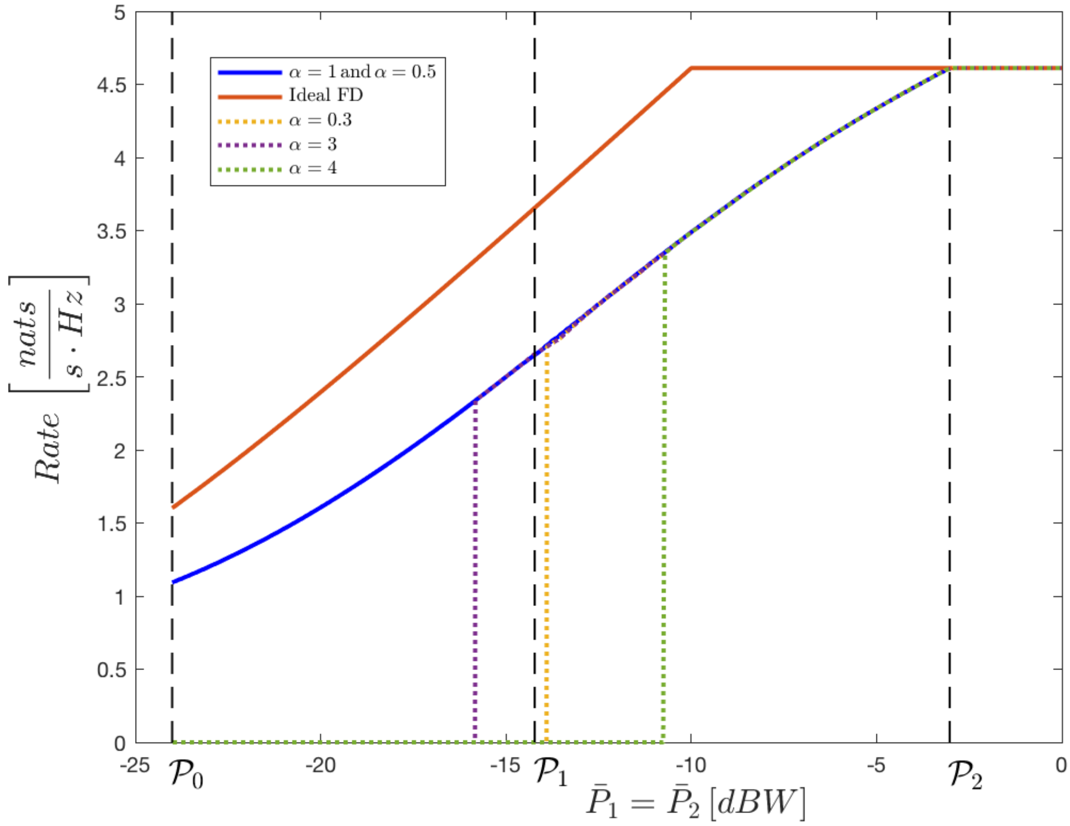

Figure 14: Rate performances with different fairness coefficients

that the lower average power values are not sufficient to ensure the condition defined by α, but when the power level is high enough, the total rate is maximized with such relationship between the two sources. That happens always assuming $\bar{P}_1 = \bar{P}_2$. For example for $\alpha = 0.3$ the threshold value is $\bar{P}_1 \approx -13.5$ dB, for $\alpha = 3$ it is $\bar{P}_1 \approx -16$ dB and for $\alpha = 4$ it is $\bar{P}_1 \approx -9$ dB. A more detailed explanation of the phenomena is given in Figure 15. By converting the α values to the corresponding angles with the arc tangent function, we have on the X axis angle values that go from 0 to $\frac{\pi}{2}$, while on the Y axis the minimum power level required for the corresponding α. When the angle tends to 0, so $\alpha \to 0$, the required power values tends to ∞, as it was already discussed when the fairness coefficient was introduced. Similarly, that happens also when the angle approaches $\frac{\pi}{2}$, so when $\alpha \to \infty$. Intuitively, the graph is symmetric around $\alpha = 1$, which corresponds to the angle $\frac{\pi}{4}$. In fact the relationship defined by α is simply inverted if the angle is x or $\frac{\pi}{2} - x$.

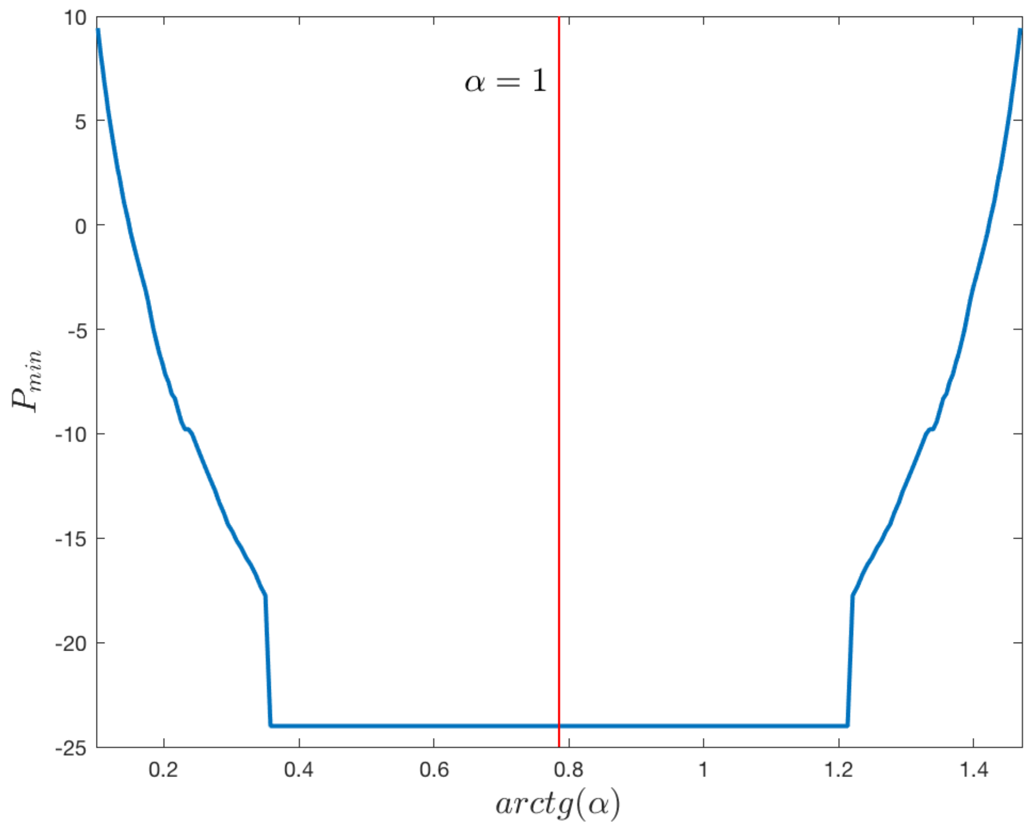

Figure 15: Minimum required power $\bar{P}_1$ as a function of α

In Figure 16, for different fixed average power values, it is shown the range of alphas for which the system has a solution. As expected, by increasing the average power level the rate becomes higher and also the range of supported αs in wider. In our results, when the network cannot support the desired fairness conditions we set to 0 the achievable rate ($R = 0$). Viceversa, if there is availability of power, the power level can be increased in order to obtain the wanted proportionality between sources.

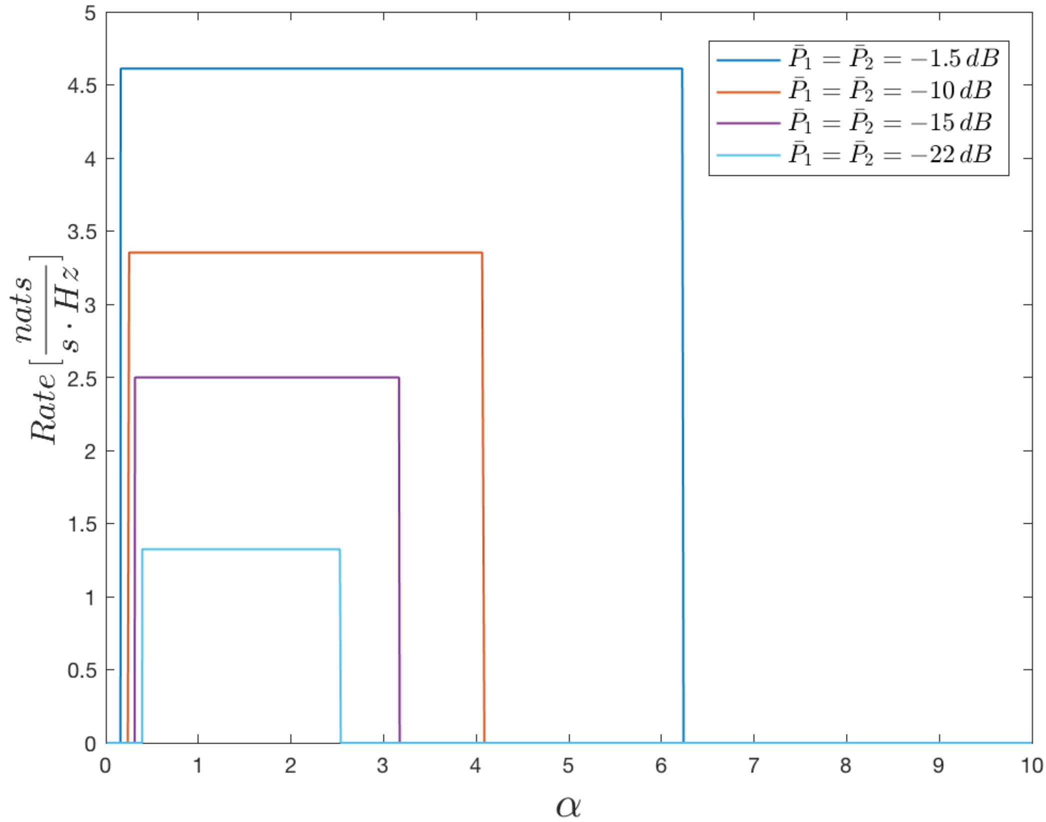

Figure 16: Range of αs for a fixed $\bar{P}_1 = \bar{P}_2$

It is also interesting to observe how performances change as the channel gains vary. To this end we considered random channel gains on the source(i)-relay links and on the relay-destination link. All the other parameters used before remain the same (including $\alpha = 1$), while the channel gains are Gaussian random variables of the form $|\hat{h}_{11}|^2 = |\hat{h}_{12}|^2 = \frac{|h_1|^2}{2} \cdot |\eta|^2$, where η is a Gaussian random complex number with 0 mean and unit variance. In Figure 17 are provided three different plots for three different channel configurations, where the dashed lines are the thresholds, the solid lines are the achieved rates and the dashed-dotted lines are the upper bound rates coming from the ideal FD. Here are the channel configurations listed:

- Red: $|h_1|^2 = 4.1665e^{-13}$, $|h_2|^2 = 1.6733e^{-13}$ and $|h_0|^2 = 1.9251e^{-12}$.
- Green: $|h_1|^2 = 4.3169e^{-14}$, $|h_2|^2 = 3.47623e^{-13}$ and $|h_0|^2 = 7.3278e^{-13}$.
- Blue: $|h_1|^2 = 2.8627^{-13}$, $|h_2|^2 = 7.4921e^{-14}$ and $|h_0|^2 = 4.9413e^{-13}$.

All the plotted rates depends on the channel gain in some way: in fact the thresholds and the achievable rate have different values based on the channel configuration. The red one allows the highest rate among the three, and it assures the equity among sources for all power levels. The green and blue configurations have lower rates and also cannot always guarantee the condition $\alpha = 1$. That happens because the achievable rates on each link are a function of the corresponding channel gain. In fact different channel gains define a particular network configuration, which has its own peculiar achievable rate region (convex-polytope): that happens because the achievable rate region depends on the instantaneous rates of the sources, which are a function of the channel gains. Changing the channel gain is also a synonym of changing the distance between nodes in the network. In Figure 18 the source nodes are placed with increasing

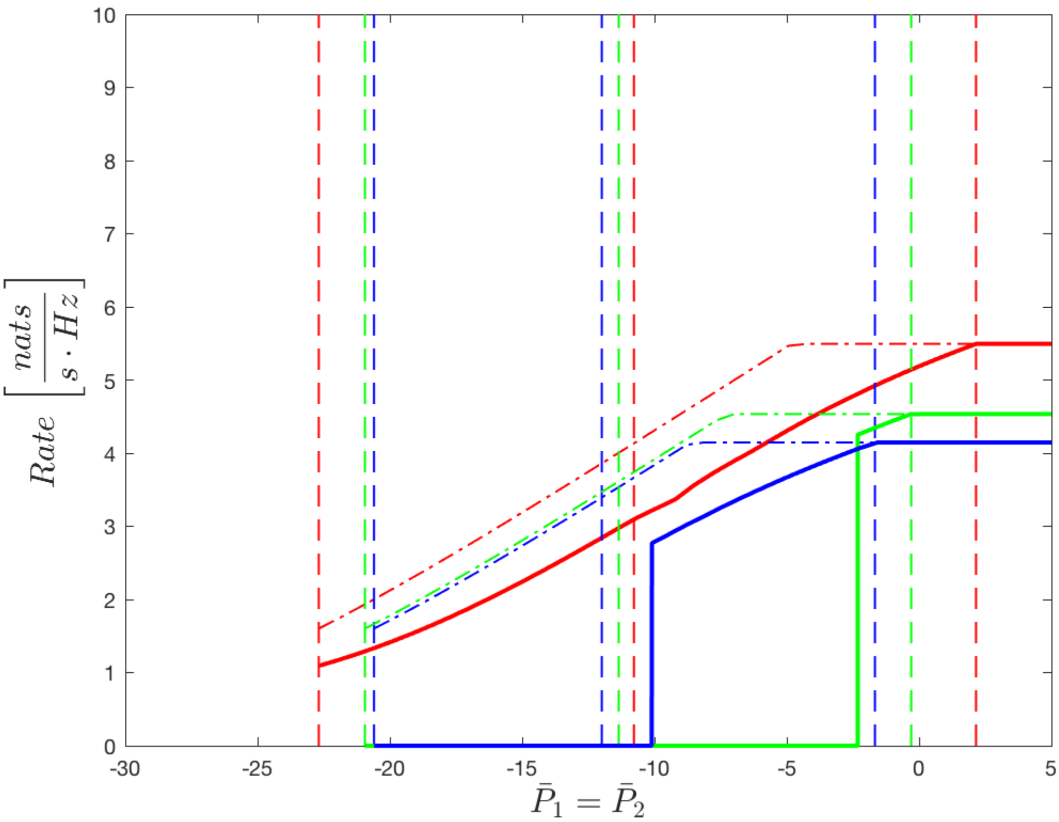

Figure 17: Rates and thresholds for different channel parameters

equal distance from the relay. The plots are done for different fixed average power values at the sources. By increasing that power, the system transmits at its maximum rate for longer distances, and from a certain critical distance value the rate starts to decrease exponentially. The dashed part of the curve represents the rate value for $\omega < p^{max}$, or equivalently $\bar{P}_s < \mathcal{P}_0$. This particular case has not been analyzed in this thesis, but intuitively the rate continues to decrease until at some point it becomes null, since the nodes are too much distant to sense each others.

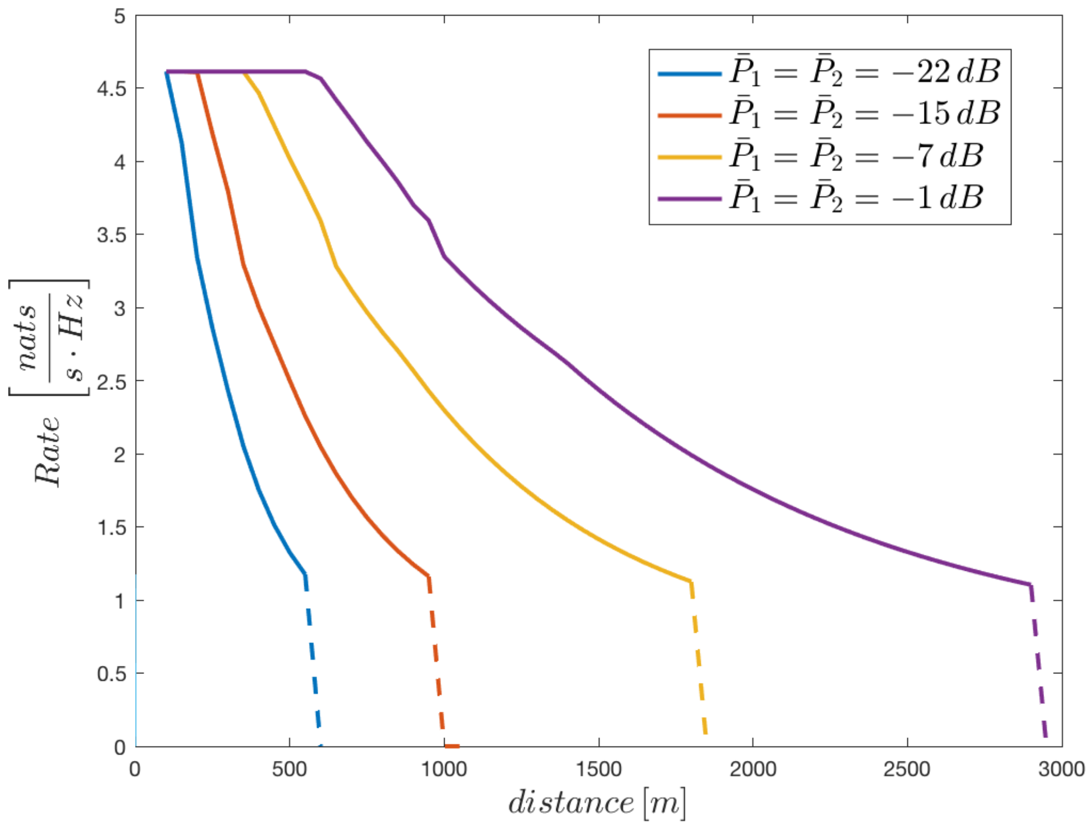

Figure 18: Rate values for increasing distance between sources and relay

Finally, we let the channel gains be random (both the ones belonging to the source-relay links and the one belonging to the relay-destination link, which follow respectively the relationships

$|\hat{h}_{11}|^2 = |\hat{h}_{12}|^2 = \frac{|h_1|^2}{2} \cdot |\eta|^2$ and $|\hat{h}_2|^2 = |h_1|^2 \cdot |\eta|^2$, with η defined previously) for different average power levels at the sources (always assuming $\bar{P}_1 = \bar{P}_2$), the result is the one depicted in Figure 19. It is assumed $\alpha = 1$. The aim is to obtain an average behavior of the network. By increasing the average power at sources the rate, with various different channel configurations, increases, until a certain threshold.

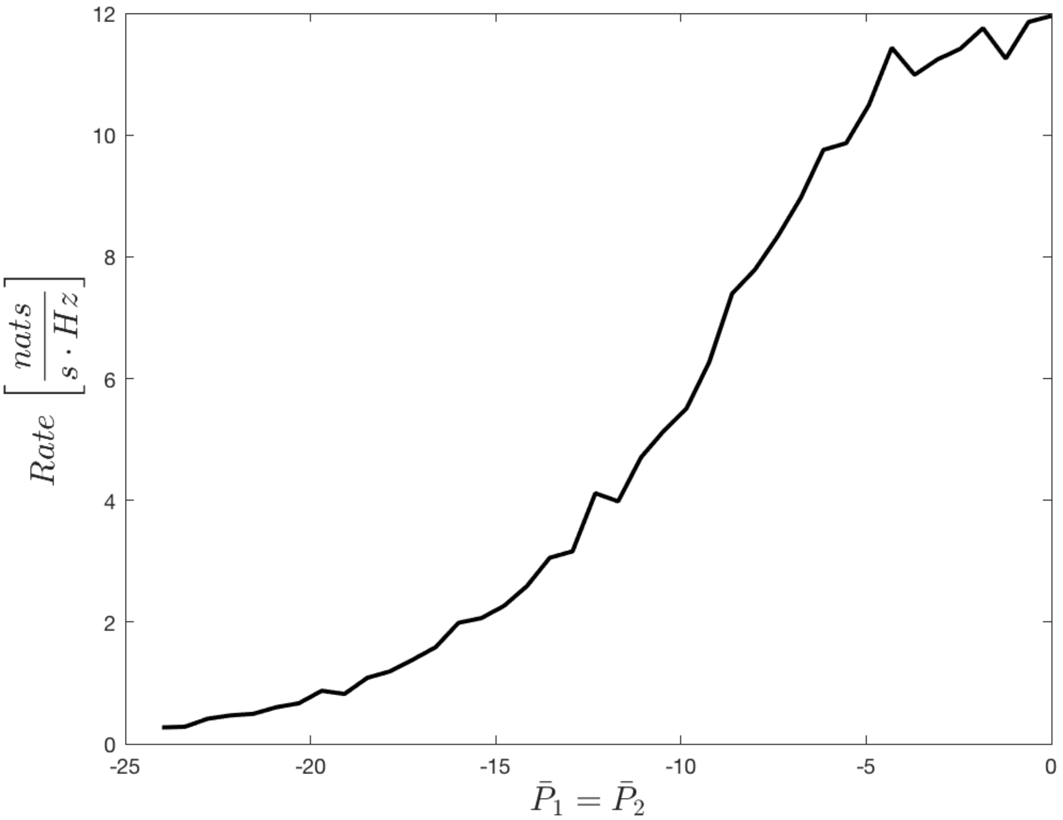

Figure 19: Rate value averaged on $N = 1000$ random realizations of the channels

From the point the curve stops increasing and assumes values in a sort of flat region. The behavior of this quite smooth curve is totally similar to the behavior of any curve with fixed channel parameters. By increasing the number of iterations N the resulting curve would be smoother.

5.1 Conclusion

To conclude this numerical part, in which all the mathematical derivations developed in the first part of the thesis, it is possible to observe that the X-Duplex system proposed and discussed is a very good competitor with respect to all the other solutions provided in the papers which treat similar scenarios. The rates achieved are quite outstanding and inferior to the rate achieved when the instantaneous power of the relay is known at the sources only for high power levels, and moreover that difference can be justified by the lower complexity required for our system, in which only the average power of the relay is known at the sources. Furthermore it has been proved that the system can be adapted to a multiple sources scenario, in which the performances are not affected negatively but reach the same values obtained in the much simpler scenario. Also the fairness concern has been deeply analyzed and the results shown allow to understand how the power provided at the source has a primary role in guaranteeing the condition imposed by the fairness coefficient α.

Some work can still be done: for example the part regarding $\omega < p^{max}$ has not been developed, and by doing that the behavior of the system for any power would be known.

APPENDICES

Appendix A

OPTIMIZATION OF THE SOURCE TRANSMIT POWER

First of all we recall that the problem we want to solve is

$$\begin{aligned}
\mathbf{P0}: \quad & R_1(f) = \max_{p_1(\cdot), p_2(\cdot)} R_1(f, p_1, p_2) \quad \text{s.t.} \\
& (a) \quad \int_0^{p^{max}} p_1(p) f(p) \mathrm{d}p = \bar{P}_1; \qquad (b) \quad 0 \leq p_1(p) \leq p_1^{max} \\
& (c) \quad \int_0^{p^{max}} p_2(p) f(p) \mathrm{d}p = \bar{P}_2; \qquad (d) \quad 0 \leq p_2(p) \leq p_2^{max}
\end{aligned}$$

The Lagrangian is defined as:

$$\begin{aligned}
\mathcal{L}(p_1, p_2) = f(p) \log\left(1 + \frac{|h_1|^2 p_1 + |h_2|^2 p_2}{N_0 + \beta p}\right) - \lambda_1 (f(p) p_1(p) - \bar{P}_1) - \lambda_2 (f(p) p_2(p) - \bar{P}_2) - \\
- \mu_1(p)(p_1(p) - p_1^{max}) + \mu_2(p) p_1(p) - \mu_3(p)(p_2(p) - p_2^{max}) + \mu_4(p) p_2(p) \quad \text{(A.1)}
\end{aligned}$$

where $\mu_1(p), \mu_2(p), \mu_3(p), \mu_4(p) \geq 0$ and λ_1, λ_2 are all KKT multipliers.

The KKT conditions comes from the partial derivatives of the Lagrangian with respect to p_1 and p_2 which are set equal to zero:

$$1. \quad \frac{|h_1|^2}{N_0 + \beta p} \cdot \frac{f(p)}{1 + \frac{|h_1|^2 p_1(p) + |h_2|^2 p_2(p)}{N_0 + \beta p}} - \lambda_1 f(p) - \mu_1(p) + \mu_2(p) = 0 \qquad \text{(A.2)}$$

Appendix A (continued)

2. $$\frac{|h_2|^2}{N_0 + \beta p} \cdot \frac{f(p)}{1 + \frac{|h_1|^2 p_1(p) + |h_2|^2 p_2(p)}{N_0 + \beta p}} - \lambda_2 f(p) - \mu_3(p) + \mu_4(p) = 0 \tag{A.3}$$

while the slackness conditions are:

3. $\mu_1(p)(p_1(p) - p_1^{max}) = 0$

4. $\mu_3(p)(p_2(p) - p_2^{max}) = 0$

5. $\mu_2(p)p_1(p) = 0$

6. $\mu_4(p)p_2(p) = 0$

along with the already existing conditions imposed by (a), (b), (c) and (d).

By looking at the conditions 5 and 6, it is obvious to require $p_1(p)$ and $p_2(p)$ greater than zero, otherwise the source will be inactive. As a consequence $\mu_2(p)$ and $\mu_4(p)$ must be equal to zero. Having said that, we define

$$P_s(p) \triangleq p_1(p)|h_1|^2 + p_2(p)|h_2|^2 \tag{A.4}$$

so that Equation A.2 and Equation A.3 reduce to:

$$f(p) \cdot \left[\frac{1}{N_0 + \beta p + P_s(p)} - \frac{\lambda_1}{|h_1|^2} \right] = 0 \tag{A.5}$$

$$f(p) \cdot \left[\frac{1}{N_0 + \beta p + P_s(p)} - \frac{\lambda_2}{|h_2|^2} \right] = 0 \tag{A.6}$$

Appendix A (continued)

Since $f(p) \neq 0$ is given and the first term inside the brackets is equal for both equations, the solution is:

$$\frac{\lambda_1}{|h_1|^2} = \frac{\lambda_2}{|h_2|^2} = \frac{1}{N_0 + \beta p + P_s(p)} \tag{A.7}$$

From Equation A.7 there is a dependency between the two multipliers, so in the end is possible to use only one multiplier. Ultimately, the solution is

$$P_s(p) = \frac{|h_1|^2}{\lambda_1} - (N_0 + \beta p) \tag{A.8}$$

which is also the optimal sum-power allocation at the sources.

Appendix B

PROOF OF LEMMA 4.1

The upper bound

$$\int_a^b f(p)\phi(p)\mathrm{d}p \leq \phi(m)$$

comes from Jensen's inequality and it is the easiest to prove. In fact from the definition of the Dirac delta function and from its property of translation, the equality holds when

$$f(p) = \delta(p-m). \tag{B.1}$$

For the lower bound, since the function $\phi(p)$ is concave in $p \in [a, b]$, also it is greater than the straight line that connects the points $\phi(a)$ and $\phi(b)$, which has equation $\frac{\phi(b)-\phi(a)}{b-a}(p-a)+\phi(a)$. Therefore we have

$$\begin{aligned}\int_a^b f(p)\phi(p)\mathrm{d}p \geq \int_a^b f(p)\left[\frac{\phi(b)-\phi(a)}{b-a}(p-a)+\phi(a)\right]\mathrm{d}p &= \phi(a)\left(\frac{b-m}{b-a}\right)+\phi(b)\left(\frac{m-a}{b-a}\right)\\ &= \frac{b-m}{b-a}\phi(a)+\left(1-\frac{b-m}{b-a}\right)\phi(b)\end{aligned}$$

and here the equality holds, for the same aforementioned reasons, when

$$f(p) = \frac{b-m}{b-a}\delta(p-a)+\left(1-\frac{b-m}{b-a}\right)\delta(p-b). \tag{B.2}$$

Appendix C

DISCRETE PROBABILITY DENSITY FUNCTION

This appendix follows a complete analogous procedure as in Appendix D of [12]. The minimization problem in Equation 3.36 can be solved by applying the Euler-Lagrange formula. This allows to get rid of the integral functions, so the Lagrangian is defined as follows

$$\mathcal{L}(f(p)) = f(p)\phi(p) + \lambda_1(f(p)\psi(p) - c) + \lambda_2(p(f(p) - m) + \lambda_3(f(p) - 1) - \mu(p)f(p) \tag{C.1}$$

where the first term is the functional to be minimized. The second, the third and the fourth terms comes from the constraints imposed by (a), (b) and (c), which are associated with the Lagrange multipliers λ_1, λ_2 and λ_3 respectively. The last term comes from the inequality of the constraint (d), and for the presence of this inequality the KKT conditions are used. $\mu(p) \geq 0$ is the KKT multiplier associated to the condition (d). Such condition can be rewritten as $-f(p) \leq 0$, that is the standard form for a minimization problem. The KKT condition consequently is

$$\frac{\partial \mathcal{L}}{\partial f} = \phi(p) + \lambda_1\psi(p) + \lambda_2 p + \lambda_3 - \mu(p) = 0 \tag{C.2}$$

$$\Rightarrow \mu(p) = \phi(p) + \lambda_1\psi(p) + \lambda_2 p + \lambda_3 \tag{C.3}$$

along with the conditions imposed by (a), (b), (c), (d), $\mu(p) \geq 0$ and the slackness condition $\mu(p)f(p) = 0$.

Appendix C (continued)

The function defined in Equation C.3 identifies a family of continuous functions, described by the parameters λ_1, λ_2 and λ_3. It is required by the conditions imposed previously that $\mu(p) \geq 0$, $\forall p \in [a, b]$, but also that $\mu(p)f(p) = 0$. Three possible behaviors for $\mu(p)$ can be considered:

- $\mu(p) > 0$, $\forall p \in [a, b]$, so, in order to comply to the slackness condition, $f(p)$ must be equal to 0 ($\forall p \in [a, b]$), which is a solution of no interest, because it means that the relay is always silent hence the rate R is 0.
- $\mu(p) = 0$, $\forall p \in [a, b]$. However this is impossible since $\mu(p)$ contains non-constant terms such as $\phi(p)$ and $\psi(p)$. This does not allow to find a non-zero measure subset of $[a, b]$ for which $\mu(p) = 0$.
- $\mu(p)$ is strictly positive in $[a, b]$ (and $f(p)$ is 0 consequently), except for a a discrete set of points $p_i \in [a, b]$, for which $\mu(p_i) = 0$ and $f(p) > 0$.

This is the only feasible option. It indicates that the maximizer $f(\cdot)$ is a discrete distribution: in particular it is composed of a set of probability masses located at p_i with magnitude π_i. In order to find the number of probability masses, the first derivative of $\mu(p)$ should be analyzed

$$\mu'(p) = \frac{k_1 p^2 + k_2 p + k_3}{(1 + \gamma_1 p)(1 + \gamma_2 p)} \tag{C.4}$$

where k_1, k_2 and k_3 depend on λ_1, λ_2, λ_3, γ_1 and γ_2. Since the numerator in Equation C.4 is a polynomial in p of degree two, it means that it can have up to two distinct solutions for p in $[a, b]$, which are local minima or maxima of the function $\mu(p)$. If the minimizer of Equation 3.36 is called $f^\star(p)$, then multiple cases are contemplated, by using Fermat and Weierstrass theorems:

Appendix C (continued)

1. The function has only one solution $p_A \in [a, b]$, which is not a local minimum or maximum. The p_A must be equal to a or b. Therefore the single probability mass has the form $f^\star(p) = \pi_1\delta(p - a)$ (or, $f^\star(p) = \pi_1\delta(p - b)$). That cannot be the solution because it has only one degree of freedom, π_1, which is not sufficient to satisfy the constraints (a), (b) and (c) at the same time.

2. $\mu(p)$ has again a single solution $p_A \in [a, b]$, which now is a local minimum, so the minimizer would be $f^\star(p) = \pi_1\delta(p - p_A)$. p_A and π_1 establish two degrees of freedom which again are not enough to satisfy the constraints imposed by (a), (b) and (c) all together.

3. $\mu(p)$ has two solutions, p_A and $p_B \in [a, b]$, and none of them is a local minimum. That means that p_A is located in a and p_B is located in b, so that the minimizer becomes $f^\star(p) = \pi_1\delta(p - a) + \pi_2\delta(p - b)$. The degrees of freedom are again two, π_1 and π_2, and for the same reason given in point 2, this solution is not acceptable.

4. The last case is when $\mu(p)$ has two solutions, p_A and $p_B \in [a, b]$, and one of them is a local minimum. The expression of the minimizer can be two: $f^\star(p) = \pi_1\delta(p - a) + \pi_2\delta(p - p_B)$ or $f^\star(p) = \pi_1\delta(p - p_A) + \pi_2\delta(p - b)$. Both the expressions are characterized by a triplet of degrees of freedom ($\{\pi_1, \pi_2, p_A\}$ and $\{\pi_1, \pi_2, p_B\}$ respectively) which are sufficient to meet the constraints imposed by (a), (b) and (c).

The minimizer expression is given in point 4 of the list, but actually they are two, so which must be chosen in Equation 3.37 is decided by the constant parameters γ_1 and γ_2. To show the

Appendix C (continued)

dependency on those two parameters, the minimizer formula can be generalized into a family of distributions of the type

$$f^{\star}(p, x, y) = \pi(x, y)\delta(p - x) + [1 - \pi(x, y)]\delta(p - y) \tag{C.5}$$

where $\pi(x, y) = \frac{y-m}{y-x} > 0$, with $m \leq y \leq b$ and $a \leq x \leq m$. The two expressions in Equation 3.37 are given by $f^{\star}(p, a, p_B)$ and $f^{\star}(p, p_A, b)$ respectively. Using this definition in Equation 3.36-(a), the constraint can be rewritten as

$$F(x, y) = \int_a^b f^{\star}(p, x, y)\psi(p)\mathrm{d}p = \pi(x, y)\psi(x) + [1 - \pi(x, y)]\psi(y) = c \tag{C.6}$$

and similarly the cost function becomes

$$G(x, y) = \int_a^b f^{\star}(p, x, y)\phi(p)\mathrm{d}p = \pi(x, y)\phi(x) + [1 - \pi(x, y)]\phi(y). \tag{C.7}$$

Since it must hold that $\psi(p) = \phi(p) + \eta(p)$, it must also hold $F(x, y) = G(x, y) + H(x, y)$, so it is necessary to define

$$H(x, y) = \pi(x, y)\eta(x) + [1 - \pi(x, y)]\eta(y) \tag{C.8}$$

Now some observations can be done:

- F and G are increasing functions of x and decreasing functions of y [13].

Appendix C (continued)

- $F(x, y) = c$ is the implicit equation of a curve, which will be called from now on $y_c(x)$, with $a \leq x \leq p_A$ and $p_B \leq y \leq b$. Having defined $F_x = \frac{\partial F}{\partial x}$ and $F_y = \frac{\partial F}{\partial y}$, the derivative of the function $y_c(x)$ is $y_c'(x) = \frac{dy_c(x)}{dx} = -\frac{F_x}{F_y}$. Using the just given definitions it can be stated that $y_c'(x) > 0$. By applying a similar reasoning to the function $G(x, y) = t$, which again is the implicit equation of $y_t(x)$, it can be demonstrated that y_t' is positive too.
- If the constant c is fixed, then it exist a value of t such that the two curves $y_c(x)$ and $y_t(x)$ meet in a common point (x^*, y^*).

The goal is to find the shared point P which gives the minimizer of the cost function. First of all, a general point P is considered, excluding the extremal points, so $P \neq (a, p_B)$ and $P \neq (p_A, b)$. Then two cases can be distinguished:

- If $y_c'(x) > y_t'(x)$ in P, it means that t is not the global minimum of the cost function in Equation 3.36. Then it must exists a value $\epsilon > 0$ for which the intersection of the two curves $y_c(x)$ and $y_{t-\epsilon}(x)$ happens at $P' = (x^* + \Delta x, y^* + \Delta y)$, and in such point the cost function $G(x^* + \Delta x, y^* + \Delta y) = t - \epsilon$ it is lower then in P. This is true for all the points $P = (x^*, y^*)$, so in the end the minimizer is found exactly in the extreme point which was excluded before, giving as a minimizer $f^\star(p, p_A, b)$ with a minimum $G(p_A, b)$.
- With a totally equivalent reasoning, in the case in which $y_c'(x) < y_t'(x)$ at P, the minimizer is found in the other extremal point and it is $f^\star(p, a, p_B)$ with the minimum $G(a, p_B)$.

The problem now is to understand which minimizer must be chosen among the two. There are some parameters which discriminate the choice, in particular they are γ_1 and γ_2. In order to do

Appendix C (continued)

a comparison between the two derivatives $y_c'(x)$ and $y_t'(x)$, the definitions of F, G and H must be used. In fact $y_c'(x) = -\frac{F_x}{F_y} = -\frac{G_x+H_x}{G_y+H_y}$ and $y_t'(x) = -\frac{G_x}{G_y}$ where the partial derivatives of G and H are introduced, according to the notation used for F. The expressions of G_x, G_y, H_x and H_y can be easily derived also by observing that $\frac{\partial\pi}{\partial x} = \pi_x = \frac{\pi}{y-x}$ and $\frac{\partial\pi}{\partial y} = \pi_y = \frac{1-\pi}{y-x}$.

In the case $y_c'(x) \geq y_t'(x)$, the expression becomes

$$-\frac{G_x + H_x}{G_y + H_y} \geq -\frac{G_x}{G_y} \Longrightarrow -\frac{G_x}{G_y} \leq -\frac{H_x}{H_y} \tag{C.9}$$

Now it can be observed that $\phi(p)$ and $\eta(p)$ are the same function of the type $\log(1+\gamma p)$, the former with $\gamma = \gamma_1$ and the latter with $\gamma = \gamma_2$. Since G depends on $\phi(p)$ and H depends on $\eta(p)$, it means that $-\frac{G_x}{G_y} = \zeta(\gamma_1)$ and $-\frac{H_x}{H_y} = \zeta(\gamma_2)$, where ζ is a function which depends on the parameter inside the brackets. An important property of the function $\zeta(\gamma)$ is that it is an increasing function of γ. In fact, by imposing $\zeta'(\gamma) \geq 0$ and after doing some calculations and simplifications, it comes out that $\log\left(\frac{1+\gamma y}{1+\gamma x}\right)(2 + \gamma y + \gamma x) \geq 2\gamma(y - x)$. The right hand side of the previous inequality is a positive function, linear with γ ($\gamma \geq 0$). The left hand side instead is a convex positive function, tangent to the right hand side when $\gamma = 0$. That proofs the increasing behavior of ζ with γ. Therefore if $\gamma_1 \leq \gamma_2$, then $-\frac{G_x}{G_y} \leq -\frac{H_x}{H_y}$ and consequently $y_c(x) \leq y_t(x)$. Viceversa, if $\gamma_1 > \gamma_2$, then $y_c(x) < y_t(x)$.

CITED LITERATURE

1. Behboodi, A., Chaaban, A., Mathar, R., and Alouini, M. S.: On full duplex gaussian relay channels with self-interference. In 2016 IEEE International Symposium on Information Theory (ISIT), pages 1864–1868, July 2016.

2. Boyd, S. and Vandenberghe, L.: Convex Optimization. Cambridge University Press, 2004.

3. Cardone, M., Tuninetti, D., Knopp, R., and Salim, U.: On the gaussian half-duplex relay channel. IEEE Transactions on Information Theory, 60(5):2542–2562, May 2014.

4. Cover, T. and Gamal, A. E.: Capacity theorems for the relay channel. IEEE Transactions on Information Theory, 25(5):572–584, September 1979.

5. Gamal, A. E., Mohseni, M., and Zahedi, S.: Bounds on capacity and minimum energy-per-bit for awgn relay channels. IEEE Transactions on Information Theory, 52(4):1545–1561, April 2006.

6. He, X. and Yener, A.: End-to-end secure multi-hop communication with untrusted relays. IEEE Transactions on Wireless Communications, 12(1):1–11, January 2013.

7. Kang, Y. Y., Kwak, B. J., and Cho, J. H.: An optimal full-duplex af relay for joint analog and digital domain self-interference cancellation. IEEE Transactions on Communications, 62(8):2758–2772, Aug 2014.

8. Knopp, R. and Humblet, P. A.: Information capacity and power control in single-cell multiuser communications. In Communications, 1995. ICC '95 Seattle, 'Gateway to Globalization', 1995 IEEE International Conference on, volume 1, pages 331–335 vol.1, Jun 1995.

9. Korpi, D., Riihonen, T., Haneda, K., Yamamoto, K., and Valkama, M.: Achievable transmission rates and self-interference channel estimation in hybrid full-duplex/half-duplex mimo relaying. In 2015 IEEE 82nd Vehicular Technology Conference (VTC2015-Fall), pages 1–5, Sept 2015.

10. Lee, J. H.: Full-duplex relay for enhancing physical layer security in multi-hop relaying systems. IEEE Communications Letters, 19(4):525–528, April 2015.

CITED LITERATURE (continued)

11. Lee, S. H. and Chung, S. Y.: When is compress-and-forward optimal? In 2010 Information Theory and Applications Workshop (ITA), pages 1–3, Jan 2010.

12. Nordio, A., Chiasserini, C. F., and Viterbo, E.: Optimal power allocation strategies in two-hop x-duplex relay channel. IEEE Transactions on Communications, PP(99):1–1, 2018.

13. Nordio, A., Chiasserini, C., and Viterbo, E.: Optimal power allocation strategies in full-duplex relay networks. CoRR, abs/1708.01407, 2017.

14. Riihonen, T., Werner, S., and Wichman, R.: Hybrid full-duplex/half-duplex relaying with transmit power adaptation. IEEE Transactions on Wireless Communications, 10(9):3074–3085, September 2011.

15. Riihonen, T., Werner, S., and Wichman, R.: Mitigation of loopback self-interference in full-duplex mimo relays. IEEE Transactions on Signal Processing, 59(12):5983–5993, Dec 2011.

16. Sankaranarayanan, L., Kramer, G., and Mandayam, N. B.: Hierarchical sensor networks: capacity bounds and cooperative strategies using the multiple-access relay channel model. In 2004 First Annual IEEE Communications Society Conference on Sensor and Ad Hoc Communications and Networks, 2004. IEEE SECON 2004., pages 191–199, Oct 2004.

17. Shang, C. Y. A., Smith, P. J., Woodward, G. K., and Suraweera, H. A.: Linear transceivers for full duplex mimo relays. In 2014 Australian Communications Theory Workshop (AusCTW), pages 11–16, Feb 2014.

18. Shi, Q., Hong, M., Gao, X., Song, E., Cai, Y., and Xu, W.: Joint source-relay design for full-duplex mimo af relay systems. IEEE Transactions on Signal Processing, 64(23):6118–6131, Dec 2016.

19. Thomas, R. R., Cardone, M., Knopp, R., Tuninetti, D., and Maharaj, B. T.: A practical feasibility study of a novel strategy for the gaussian half-duplex relay channel. IEEE Transactions on Wireless Communications, 16(1):101–116, Jan 2017.

20. Wang, L. and Naghshvar, M.: On the capacity of the noncausal relay channel. IEEE Transactions on Information Theory, 63(6):3554–3564, June 2017.

CITED LITERATURE (continued)

21. Zafar, A., Shaqfeh, M., Alouini, M. S., and Alnuweiri, H.: Resource allocation for two source-destination pairs sharing a single relay with a buffer. IEEE Transactions on Communications, 62(5):1444–1457, May 2014.

22. Zlatanov, N., Sippel, E., Jamali, V., and Schober, R.: Capacity of the gaussian two-hop full-duplex relay channel with residual self-interference. IEEE Transactions on Communications, 65(3):1005–1021, March 2017.

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EDUCATION	
April 2018 - June 2018	Prisma Project: A1 German course offered by Bosch which selects only 16 master students, Politecnico di Torino, Italy
August 2017 - May 2018 (expected)	Master of science in Electrical and Computer Engineering, enrolled in the Double Degree program with Politecnico di Torino, University of Illinois at Chicago, USA
September 2016 - July/September 2018 (expected)	Master of science in Communications and computer networks engineering, Politecnico di Torino, Italy
September 2013 - July 2016	Bachelor's degree in Telecommunications engineering (103/110), Politecnico di Torino, Italy
2008-2013	Scientific high school diploma with bilingual (English and French) courses, liceo G. B. Bodoni, Italy
LANGUAGE SKILLS	
Italian	Native speaker
English	Full working proficiency
	2016 - IELTS examination (7.5/9)
	A.Y. 2017 One Semester of study abroad in Chicago, Illinois
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French	B2 level
	2013 - DELF certificate with a score of 75/100
TECHNICAL SKILLS	
Average level	Operative Systems: Linux, Windows.
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VITA (continued)

WORK EXPERIENCE AND PROJECTS

September 2016 - January 2017	Teaching assistant Part-time collaboration with professor Roberto Gaudino at Politecnico di Torino. Design of a digital transmission system (audio and images) on Matlab. Transcription of lecture material from paper to a digital format with Latex.
September 2013 - September 2016	Multidisciplinary class in support of high school students.

OTHER INFORMATION

Sport	Karate and running, at amateur level.
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