Topological K-theory and Invertibility

by

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To

my brother,

Alex

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SUMMARY

We construct a relative version of the topological K-theory of dg categories, in the sense of (1), over an arbitrary quasi-compact, quasi-separated $\mathbb{C}$-scheme X. This has as input a Perf(X)-linear stable ∞-category and output a sheaf of spectra on $X(\mathbb{C})$, the space of complex points of X. We then characterize the values of this functor on inputs of the form Mod^{ω}_{A}, for A a derived Azumaya algebra over X. In such cases we show that this coincides with the α-twisted topological K-theory of $X(\mathbb{C})$ for some appropriately defined twist of K-theory. We use this to provide a topological analogue of a classical result of Quillen's on the algebraic K-theory of Severi-Brauer varieties.

CHAPTER 1

INTRODUCTION

Philosophically, this thesis attempts to contribute to a continually expanding dictionary between algebro-geometric and topological phenomena. The interplay of ideas goes both ways, as algebro-geometric objects are subsumed in a homotopy theoretic framework and objects in topology are often realized from algebraic origins.

Lying at the center of this cross-fertilization is the development of both algebraic and topological K-theory. Grothendieck was the first to define the algebraic K-group K_0, which he then used to formulate and prove the Grothendieck-Riemann-Roch theorem. Following Grothendieck's ideas, Michael Atiyah defined topo logical K-theory in order to study vector bundles on a topological space. Using the Bott-periodicity theorem, Atiyah was able to then give topological K-theory the structure of a generalized cohomology theory. In today's language, he constructed a spectrum whose zero space effectively represents vector bundles on X.

In the ensuing years, Quillen, Waldhausen and others extended Grothendieck's original ideas to define higher algebraic K-theory. Making essential use of the tools of stable homotopy theory, they defined the K-groups as homotopy groups of a certain spectrum, obtained in structured way from a category, broadly speaking, equipped with weak equivalences and a notion of exact sequences. Since then, many structural properties of algebraic K-theory have been singled out, which characterize it uniquely, up to contractible choice. The interested reader should consult

(2) for a modern characterization of algebraic K-theory as a spectrum valued functor of small *stable* ∞-categories, universal among those which send exact sequences of categories to cofiber sequences of spectra.

The impetus of K-theory can also be found in the development of the stable motivic category due to Morel and Voevodsky in (3), itself one of the culminations of the attempt to break down the barriers between topology and algebraic geometry. Indeed, the $\mathbb{A}^1$-invariant algebraic K-theory of schemes is representable in their category; there is an object KGL with the property that, for any suitable k-scheme X,

$$KH(X) \simeq Map_{\mathrm{SH}(X)}(\Sigma^\infty_{\mathbb{P}^1}(X), KGL) \simeq KH(X)$$

The stable motivic category SH_k thus provides a setting where algebraic K-theory acts as a generalized cohomology theory on the category of smooth schemes over a field k, occupying the role that the topological K-theory spectrum KU does in spectra. The correspondence doesn't end there; when k is the field of complex numbers, there is a realization functor

$$\mathbf{Betti}_{\mathbb{C}} : \mathrm{SH}_{\mathbb{C}} \to Sp$$

valued in the category of spectra; it is a theorem that $\mathbf{Betti}_{\mathbb{C}}(KGL) \simeq KU$.

1.1 Topological K-theory of complex dg-categories

In (1), the author constructs an invariant of dg categories over the complex numbers $K^{top}_{\mathbb{C}} : dgCat_{\mathbb{C}} \to Sp$ satisfying many of the same formal properties of algebraic K-theory. Indeed, via the framework of (2), this functor may be viewed as an algebra object in an appropriate ∞-category of *localizing invariants* with (nonconnective) algebraic K-theory as the unit. For any dg-category T, the topological K-theory spectrum $K^{top}(T)$ will be a module over the periodic complex K-theory spectrum. Perhaps rather surprisingly, the following equivalence holds for any scheme $X \in \mathrm{Sch}_{\mathbb{C}}$, separated of finite type:

$$K^{top}(\mathrm{Perf}(X)) \simeq KU(X(\mathbb{C})).$$

Here $\mathrm{Perf}(X)$ denotes dg-category of perfect complexes of $\mathcal{O}_X$-modules and $KU(X(\mathbb{C}))$ is topological K-theory of the associated space of complex points $X(\mathbb{C})$.

Our work here centers around this invariant together with its relative versions which we introduce.

1.2 Twisted sheaves in algebra and topology

The pattern of interdependence between disciplines is found in the notion of a twisted sheaf arising in both the algebraic and topological context. In topology, a twisted vector bundle is a version of an ordinary vector bundle, where the descent (or gluing) data is modified. Recall that on a space X, one can define a vector bundle $E \xrightarrow{\pi} X$ by specifiying vector bundles E_i over an open cover $\{U_i\}_{i \in I}$ equipped with isomorphisms

$$f_{ij} : E_i \to E_j$$

on $U_i \cap U_j$ satisfying the condition that $f_{ij} \circ f_{jk} = f_{ik}$. A twisted bundle is given by modifying this compatibily condition so that $f_{ij} f_{jk} f_{ik}^{-1} = \alpha_{ijk} \in \mathbb{C}^\times$. This gives rise to a $GL_1(\mathbb{C})$ cocycle, which in turn gives rise to a class in cohomology,

$$\alpha \in H^3(X, \mathbb{Z}).$$

On the algebraic side, we have an analogous notion of α-twisted sheaves (cf. (4; 5) If X is a scheme and $\alpha \in H^2_{\acute{e}t}(X, \mathbb{G}_m)$ is a class in étale cohomology represented by a Cech cocycle with respect to some cover $\{U_i\}$ we can define an α-twisted sheaf $\mathscr{F}$ analogously to the topological setting. Namely, over this Cech cover, one specifies a collection of sheaves $\{\mathscr{F}_i\}_{i \in I}$ together with isomorphisms $f_{ij} : \mathscr{F}_i \to \mathscr{F}_j$ when restricted to intersections. On triple intersections $U_i \cap U_j \cap U_k$ we have $f_{ij} f_{jk} f_{ik}^{-1} = \alpha_{ijk} \in \mathcal{O}_X^\times$ The category of α-twisted sheaves together with the obvious notion of sheaf homomorphism will then be an abelian category.

One key perspective in the study of twisted sheaves in algebraic geometry is that they may be understood, in sufficiently well-behaved cases, as modules over a certain sheaf of algebras $\mathcal{O}_A$, étale locally Morita equivalent to the base. Such a sheaf of algebras is called an *Azumaya algebra*. This correspondence is even more reliable upon passing to the setting of derived algebraic geometry. There one studies *derived Azumaya algebras* and their modules, which model the derived category of twisted sheaves.

1.3 Twisted K-theory; algebraic and topological

Let X be a topological space and let $\alpha \in H^3(X, \mathbb{Z})$ be as above. Furthermore, let $\mathrm{Vect}_\alpha(X)$ denote the category of α-twisted vector bundles over X. This category comes equipped with a suitable notion of an exact sequence; taking the Grothendieck group of this category gives one version of the *α-twisted topological K-theory* of X. This notion of K-theory was first introduced by Donovan and Karoubi in (6) wherein they defined a local system in $H^3(X, \mathbb{Z}) \times H^1(X, \mathbb{Z}/2) \times BBSU_\otimes(X)$ and then further developed by Rosenberg, Atiyah and others. See, for instance, (7). Often times, the twisted form of K-theory has a geometric interpretation. In particular if X is a space equipped with a bundle of projective spaces, one may define a form of twisted K-theory as the global sections of a certain bundle of KU module spectra twisted by the action of PU_n on the fibers. It is well known that π_0 of this spectrum coincides with the Grothendieck group construction applied to the exact category of α-twisted bundles on X for the resulting twist α.

There is an obvious notion of twisted algebraic K-theory as well. This is obtained by taking the K theory of $\mathrm{Perf}(X, \alpha)$, the category of perfect complexes of α-twisted sheaves of $\mathcal{O}_X$-modules, equivalently, $Perf(X, \mathcal{A})$, modules over the associated derived Azumaya algebra.

1.4 Statement of Results

In this work, we construct, for every scheme $X \in \mathrm{Sch}_{\mathbb{C}}$, a functor

$$K_X^{top} : \mathrm{Cat}^{\mathrm{perf}}(X) \to Shv_{Sp}(X(\mathbb{C}))$$

which associates, to a Perf(X)-linear dg-category, a sheaf of spectra on $X(\mathbb{C})$. By a Perf(X)-linear dg-category, we mean a module, in a suitably defined symmetric monoidal ∞-category of dg-categories, over Perf(X), perfect complexes over X. We refer to this as the *relative topological K-theory of X*. When $X = Spec(\mathbb{C})$, this reproduces the original version of topological K-theory due to Blanc.

Our first result identifies the topological K-theory of complexes of α-twisted sheaves on X with the twisted K-theory of $X(\mathbb{C})$.

Theorem 1.4.1. *Let X denote a quasi-compact, quasi-separated scheme over the complex numbers. Let $\alpha \in \pi_0\mathbf{Br}_0(X)$ be a Brauer class, with Perf$(X,\alpha) \in Cat^{perf}(X)$ the associated $Perf(X)$- linear category. Then there exists a functorial equivalence*

$$K_X^{top}(Perf(X,\alpha)) \simeq \underline{KU^{\widetilde{\alpha}}(X(\mathbb{C}))}.$$

Here, $\underline{KU^{\widetilde{\alpha}}(X(\mathbb{C}))}$ is the local system of invertible KU modules associated to a twist $\widetilde{\alpha} : X(\mathbb{C}) \to Pic_{KU}$ obtained functorially from α.

As a corollary we obtain the following result on the "absolute" topological K-theory.

Corollary 1.4.2. *Let $X \in Sch_{\mathbb{C}}$ be a quasi-compact, quasi separated scheme and let $\alpha \in H^2_{ét}(X, \mathbb{G}_m)$ be a torsion class in étale cohomology corresponding to an ordinary Azumaya algebra over X. Then*

$$K^{top}(Perf(X,\alpha)) \simeq KU^{\widetilde{\alpha}}(X(\mathbb{C}))$$

where $\widetilde{\alpha} \in H^3(X, \mathbb{Z})$ *is the class in singular cohomology obtained via the topological realization functor*

$$H^2_{\acute{e}t}(X, \mathbb{G}_m) = [X, B^2\mathbb{G}_m] \xrightarrow{||-||} [X(\mathbb{C}), ||B^2(\mathbb{G}_m)||] \simeq [X(\mathbb{C}), B^2(S^1)] = H^3(X(\mathbb{C}), \mathbb{Z}).$$

Last, we display an application of these results purely in the realm of topology. For this, we let X be a quasi compact scheme, and let $P \to X$ denote a Severi Brauer-scheme of relative dimension $n-1$ over X. This will mean that P is, étale locally on X, equivalent to projective space $\mathbb{P}^{n-1}_X$. As is well known, there is a sheaf of Azumaya algebras A, over X canonically associated to P. It is a classical theorem, due to Quillen in (8) that

$$K(P) \simeq K(X) \oplus K(A^{\otimes 1}) \oplus ... \oplus K(A^{\otimes n-1}).$$

In particular, this means that the K-groups of perfect complexes of $\mathcal{O}_P$-modules decompose as a direct sum of the K-groups of the categories of perfect complexes of $A^{\otimes n}$ modules where $A^{\otimes n}$ is the n-th tensor product of A.

We wonder whether the analogue of this result is true in the topological setting. More precisely, if X is a topological space, and if $\pi : P \to X$ is projective fiber bundle, does there exist a decomposition of the topological K-theory of P into summands involving the twisted topological K-theory of the base space? Using our methods, we prove the following affirmative result, as a generalization of the Leray-Hirsch theorem to this context:

Theorem 1.4.3. *Let X be a finite CW-complex. Let $\pi : P \to X$ be a bundle of rank $n-1$ projective spaces classified by a map $\alpha : X \to BPGL_n(\mathbb{C})$. Let $\tilde{\alpha} : X \to B^2\mathbb{C}^\times$ be the composition of this map along the map $BPGL_n(\mathbb{C}) \to B^2\mathbb{C}^\times \simeq K(\mathbb{Z}, 3)$ This gives rise to an element $\widetilde{\alpha} \in H^3(X, \mathbb{Z})$. Then the topological K-theory of the total space P decomposes as follows:*

$$KU^*(P) \simeq KU^*(X) \oplus KU^{\tilde{\alpha}}(X) ... \oplus KU^{\tilde{\alpha}^{n-1}}(X).$$

where $KU^{\widetilde{\alpha^k}}(X)$ denotes the twisted K-theory with respect to the class $\widetilde{\alpha}^k \in H^3(X, \mathbb{Z})$.

This topological analogue of Quillen's K-theoretic result has been hitherto unknown.

We would like to remark that Corollary 1.4.2 has been applied in (9) to show that for X a complex Enriques surface and Y a smooth complex variety such that there is an equivalence of derived categories $D^b(X, \alpha) \simeq D^b(Y, \beta)$ with $\alpha \neq 0$, then $X \cong Y$ and $\beta \neq 0$.

Finally, we would like to add that this thesis is based on the work in (10), which has been submitted for publication.

CHAPTER 2

HIGHER CATEGORICAL PRELIMINARIES

Throughout this thesis, we shall utilize the language of ∞-categories. We recall some basic definitions and review the notions that are central to the execution of this work. For a more thorough introduction to the theory one should consult (11; 12)

2.1 Presentable ∞-categories and ∞-topoi

We recall the definition of presentable ∞-categories. The categories we work with will often be either presentable, or will arise as subcategories of compact objects of presentable ∞-categories. Our arguments and constructions will frequently depend on this structure being present in the situation at hand.

Definition 2.1.1. Let $\mathcal{C}$ be an ∞-category and κ be an infinite regular cardinal. We may form the Ind-category $\mathrm{Ind}_\kappa(\mathcal{C})$ which is the formal completion of $\mathcal{C}$ under κ-filtered colimits. We say an ∞ category $\mathcal{D}$ is *accessible* if there exists a small ∞ category $\mathcal{C}$ and some regular cardinal κ such that $\mathcal{D} \simeq Ind_\kappa(\mathcal{C})$. If $\mathcal{C}$ is accessible and has all small colimits, then it is *presentable*

The collection of all presentable ∞-categories can be organized into an ∞-category Pr^{L} with morphisms consisting of those functors which are left adjoints. This category is closed monoidal in that the ∞-category of functors between any two presentable ∞-categories, $\mathrm{Fun}^{\mathrm{L}}(A, B)$ is itself presentable. We may also organize the collection of presentable ∞-categories where we keep track of the functors which are right adjoint; we denote the resulting ∞-category by $\mathcal{P}r^{\mathrm{R}}$.

According to (12), these two categories are anti-equivalent to each other. Furthermore, this identification gives the following useful description of the symmetric monoidal tensor product in $\mathcal{P}r^L$:

$$\mathcal{C} \otimes^{\mathrm{L}} \mathcal{D} \simeq Fun^R(\mathcal{C}^{op}, \mathcal{D})$$

We now give a characterization of ∞-topoi among presentable ∞-categories as those that satisfy a certain form of *descent*. A concise way to describe this is as follows, taken from (13):

Definition 2.1.2. Let $\mathcal{X}$ be a presentable ∞-category. Let $\mathcal{O}_{\mathcal{X}} := Fun(\Delta^1, \mathcal{X})$ denote the arrow category of $\mathcal{X}$ and let $\mathcal{O}_{\mathcal{X}} \to \mathcal{X}$ be functor sending a morphism to its target. This has the property of being a *Cartesian fibration*, in particular giving us a functor

$$\mathcal{X}^{op} \to \widehat{Cat_{\infty}}$$

We say that $\mathcal{X}$ is an ∞-topos if the following descent condition holds: if $T \simeq colim_{\alpha} T_{\alpha}$ is a colimit in $\mathcal{X}$ then there is an induced map

$$\mathcal{X}_{/T} \to \lim_{\alpha} \mathcal{X}_{/T\alpha}$$

which is an equivalence.

This definition in fact characterizes ∞-topoi uniquely amongst locally Cartesian closed presentable ∞-categories.

Definition 2.1.3. Given two ∞-topoi $\mathcal{X}$ and $\mathcal{Y}$, we define a *geometric morphism* $f : \mathcal{X} \to \mathcal{Y}$ to be an adjoint pair of functors

$$f^* : \mathcal{Y} \rightleftarrows \mathcal{X} : f_*$$

such that the left adjoint preserves finite limits.

For the purposes of this thesis, it is convenient to think of ∞-topoi as categories of sheaves on an ∞-category equipped with a Grothendieck topology. As discussed in (11, Section 6.2.2), a Grothendieck topology on an (essentially small) ∞-category $\mathcal{T}$ corresponds precisely to a Grothendieck topology (in the classical sense) on its homotopy category $\mathrm{h}\mathcal{T}$. We remark however that, unlike with ordinary topoi, not every ∞-topos is the category of sheaves on some ∞-category with a Grothendieck topology. We will not use too many intricacies of higher topos theory; we invite the reader from now to keep in mind the following specific examples that will arise: $Shv(X)$, the category of sheaves (of spaces) on a topological space X, or $Shv^{\acute{e}t}(X)$, the category of sheaves on the site of (smooth) schemes over a scheme X, endowed with the étale topology.

We collect here the definition of a locally constant object in an ∞-topos, as it appears in (12, Definition A.1.12).

Definition 2.1.4. Let $\mathcal{X}$ denote an arbitrary ∞-topos. Let $\mathcal{G}$ be an object of $\mathcal{X}$. Then $\mathcal{G}$ is *constant* if it is in the essential image of the unique geometric morphism $\pi^* : \mathcal{S} \to \mathcal{X}$ from the ∞-topos of spaces to $\mathcal{X}$. Furthermore, $\mathcal{G}$ is *locally constant* if there is a small collection of objects $\{V_\alpha \subseteq X\}$ satisfying the following:

1. The objects V_α cover $\mathcal{X}$. This means that we have an effective epimorphism $\sqcup V_\alpha \to \mathbf{1}$; here, $\mathbf{1}$ denotes the final object of $\mathcal{X}$.

2. For every $\alpha \in S$, the product $\mathcal{G} \times V_\alpha$ is a constant object of the ∞-topos $\mathcal{X}/_{V_\alpha}$.

2.2 Symmetric monoidal ∞-categories

We do not give a complete account of the theory of symmetric monoidal ∞-categories as we will only use a small part of the theory. One may find a more comprehensive treatment in (14, Chapter 2).

Let Γ denote the category of pointed finite sets, with morphisms pointed maps of finite sets. An ∞-operad is then an ∞-category $\mathcal{O}^{\otimes}$ and a functor

$$p : \mathcal{O}^{\otimes} \to N(\Gamma)$$

satisfying the conditions of (12, Definition 2.1.1.10). A symmetric monoidal ∞-category is then an ∞-operad $\mathcal{C}^{\otimes}$ together with a *cocartesian fibration of operads*(12, 2.1.2.18)

$$p : \mathcal{C}^{\otimes} \to N(\Gamma)$$

such that

- for each $n \geq 0$, the associated functors $\mathcal{C}^{\otimes}_{[n]} \to \mathcal{C}^{\otimes}_{[1]}$ determine an equivalence of ∞-categories $\mathcal{C}_{[n]} \simeq \mathcal{C}^{n}_{[1]}$, where $\mathcal{C}_{[n]} = p^{-1}([n])$.

Remark. We may reverse engineer this definition in order to obtain functors

$$\mathcal{C} \times \mathcal{C} \to \mathcal{C},$$

reminiscent of the 1-categorical notion of a symmetric monoidal category.

Remark. If $\mathcal{C}$ is a symmetric monoidal ∞-category, then its homotopy category will be symmetric monoidal in the 1-categorical sense.

We will occasionally study E_∞-algebras in a general symmetric monoidal ∞-category.

Definition 2.2.1. An E_∞-algebra in a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ is a section $\alpha \in Fun(N(\Gamma), \mathcal{C}^{\otimes})$ of the coCartesian fibration $p : \mathcal{C}^{\otimes} \to N(\Gamma)$.

At $[n] \in \Gamma$, such a section $\alpha([n])$ can be thought of as the n-fold product $\alpha([1])^n$ of the underlying object; the lifting condition for p-cocartesian edges gives us the multiplications.

We conclude this section by recalling the notion of a dualizable object in a symmetric monoidal ∞-category:

Definition 2.2.2. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category and let A be an object of $\mathcal{C}$. We say A is *dualizable* if there exists an object $A^\vee$ together with a evaluation map $\epsilon : A \otimes A^\vee \to 1$ and a coevaluation map $\eta : 1 \to A^\vee \otimes A$ such that the composition:

$$A \simeq A \otimes 1 \xrightarrow{1 \otimes \eta} A \otimes A^\vee \otimes A \xrightarrow{\epsilon \otimes 1} 1 \otimes A \simeq A$$

and

$$A^\vee \simeq 1 \otimes A^\vee \xrightarrow{\eta \otimes 1} A^\vee \otimes A \otimes A^\vee \xrightarrow{1 \otimes \epsilon} A^\vee \otimes 1 \simeq A^\vee$$

is equivalent to the identity

We say A is *invertible* if the evaluation and coevaluation morphisms are moreover equivalences.

2.3 Stable ∞-categories

Stable ∞-categories provide a natural framework for dealing with many categories arising from algebraic geometry and algebraic topology. Furthermore, they serve as natural inputs for the algebraic K-theory functor. The reader should consult (2) for a characterization of algebraic K-theory functor as an invariant of stable ∞-categories satisfying a certain universal property. We briefly recall their definition.

Definition 2.3.1. An ∞-category is *stable* if it has finite limits and colimits and if fiber sequences and cofiber sequences coincide. A functor $F : \mathcal{C} \to \mathcal{D}$ between two stable ∞-categories is *exact* if it preserves fiber sequences.

We may organize the collection of small, stable (idempotent complete) ∞-categories into an ∞-category, which we denote by $\mathrm{Cat}^{\mathrm{perf}}$. The morphisms in this category are the exact functors. Similarly, the collection of all stable presentable ∞-categories, with morphisms left adjoint functors, can be also organized into an ∞-category, denoted by $\mathcal{P}r^{L,st}$. We remark that, as in (12, Proposition 5.5.7.10) there is an ind-completion functor

$$Ind : Cat^{\mathrm{perf}} \to \mathcal{P}r^{L,st}$$

sending a small stable ∞-category $\mathcal{C}$ to $Ind(\mathcal{C})$, its formal cocompletion under ω-filtered colimits which will be a stable presentable category. This induces an equivalence between $\mathrm{Cat}^{\mathrm{perf}}$ and $\mathcal{P}r^{L,st,\omega}$, the subcategory of $\mathcal{P}r^{L,st}$ consisting of the compactly-generated stable presentable ∞-categories with morphisms left adjoint functors preserving compact object. We implictly use this identification throughout the course of this paper.

Remark. The ∞-category $\mathrm{Cat}^{\mathrm{perf}}$ is symmetric monoidal, with unit the ∞-category Sp^{ω} of finite spectra. This symmetric monoidal structure is characterized by the property that maps out of the tensor product $A \otimes B$ correspond to bifunctors out of the product $A \times B$ preserving colimits in each variable. Likewise, $\mathcal{P}r^{L,st}$ is symmetric monoidal with unit the ∞-category of spectra.

2.4 $\mathcal{E}$-linear ∞-categories

Definition 2.4.1. Let $\mathcal{E} \in \mathrm{Alg}_{\mathbb{E}_\infty}(Cat^{\mathrm{perf}})$ be a symmetric monoidal stable ∞-category. We set $\mathrm{Cat}^{\mathrm{perf}}(\mathcal{E}) := \mathrm{Mod}_{\mathcal{E}}(\mathrm{Cat}^{\mathrm{perf}})$. Similarly we may write $Cat_{\mathcal{E}} = \mathrm{Mod}_{Ind(\mathcal{E})}(\mathcal{P}r^{L,st})$. We refer to objects of this category as *$\mathcal{E}$-linear categories*; these are stable infinity categories $\mathcal{C}$ equipped with an exact functor $\mathcal{C} \otimes \mathcal{E} \to \mathcal{C}$.

Example 2.4.2. As an example, we take the ∞-category of *compact* R-modules $\mathcal{E} = Mod_R^{\omega}$ for R any $\mathbb{E}_\infty$ ring spectrum. We refer to these as R-linear ∞-categories.

Example 2.4.3. Fix a quasi-compact, quasi-separated scheme X. In this situation we set $\mathcal{E} = \mathrm{Perf}(X)$ the symmetric monoidal ∞-category of perfect complexes of $\mathcal{O}_X$-modules. We let $QCoh(X) \simeq Ind(Perf(X))$; this will be compactly generated by the perfect complexes.

Remark. For the following, we will actually need the assumption that $\mathcal{E}$ is *rigid*, so that all objects are dualizable. This is for instance satisfied in the above examples, which encompasses the situations we will deal with. Hence, we assume once and for all that $\mathcal{E}$ is rigid.

Let $\mathcal{C} \in Cat^{\mathrm{perf}}(\mathcal{E})$. Then, for every object $a \in \mathcal{C}$, we may define a functor $\mathcal{E} \to \mathcal{C}$ sending $e \to e \otimes a$. This functor preserves finite colimits and therefore admits an Ind-right adjoint, $\mathcal{C}^{\mathcal{E}}(a, -) : \mathcal{C} \to Ind(\mathcal{E})$. This gives $\mathcal{C}$ the structure of an $Ind(\mathcal{E})$ enriched category. For the example above, with $\mathcal{C}$ is an R-linear category, this means the mapping object, $\mathcal{C}(a, b)$ has the structure of an R-module for every object $a, b \in \mathcal{C}$.

We will be particularly concerned with $\mathcal{E}$-linear ∞-categories with the following properties which we now define:

Definition 2.4.4. Let $\mathcal{E} \in Alg_{\mathbb{E}_\infty}(Cat^{\mathrm{perf}})$ and let $\mathcal{C} \in Cat^{\mathrm{perf}}(\mathcal{E})$ We say $\mathcal{C}$ is

- *smooth* if $\mathcal{C}$ is compact as an object of $Ind(\mathcal{C}^{op} \otimes_{\mathcal{E}} \mathcal{C})$
- *proper* if for all objects a, b, the mapping object $\mathcal{C}(a, b) \in \mathcal{E}$ is compact as an object of $Ind(\mathcal{E})$.
- *saturated* if it is both smooth and proper.

Remark. We remark that if X is a smooth and proper scheme over $\mathbb{C}$, then its dg-category of perfect complexes, $\mathrm{Perf}(X)$, is smooth and proper as a $\mathbb{C}$-linear ∞-category. In fact, the two notions coincide. The reader should consult (15) for detailed proof of these facts.

We have the following characterization of saturated $\mathcal{E}$-linear ∞-categories, originally found in (2) whose proof may be found in (16).

Theorem 2.4.5. *Let $\mathcal{C} \in Cat^{perf}(\mathcal{E})$. Then $\mathcal{C}$ is smooth and proper if and only if it is dualizable as an object in $Cat^{perf}(\mathcal{E})$.*

2.5 DG-categories

We recall the definition of a dg-category.

Definition 2.5.1. A (k-linear) *dg-category* T is a category enriched over the category of chain complexes $Ch(K)$. In particular, for any two objects, x, y, there is a chain complex of morphisms $T(x, y)$. We require the composition morphisms to be k-linear.

A fairly comprehensive account of the theory of dg-categories may be found in (17). We describe how the theory of dg-categories can be subsumed into the framework of stable ∞-categories.

For a fixed commutative ring k there exists an ∞-category $Dg(k)$ encoding the homotopy theory of (small) dg-categories up to quasi-equivalences. See (17) for a more thorough discussion. There exists a "dg nerve" functor $N_{dg} : Dg(k) \to Cat_\infty$ and version for "big" dg categories. Moreover, a dg category $\mathcal{C}$ will be presentable (in the d.g. category sense) if and only if the associated ∞-category $N_{dg}(\mathcal{C})$ is presentable. Hence we can define the restriction $N_{dg} : Dg(k)^p \to \mathcal{P}r^{L,st}$ This functor is known to be conservative and reflects fully faithfullness. It also preserves the associated homotopy categories and therefore preserves the notion of compact generators. One can therefore restrict even further to obtain $N_{dg} : Dg(k)^{cc} \to \mathcal{P}r^{L,st,\omega}$; here, $Dg(k)^{cc}$ denotes compactly generated presentable dg categories.

Now, $Dg(k)^{cc} \simeq Dg(k)^{idem}$, the ∞-category of small dg-categories up to Morita equivalence In (18), it is shown that there is a factorization

$$Dg(k)^{cc} \to Mod_{Mod_k}(\mathcal{P}r^{L,st,\omega}) \to \mathcal{P}r^{L,st,\omega}$$

with the last map being the forgetful functor and the first map being an equivalence. Hence, we are able to identify $\mathrm{Cat}^{\mathrm{perf}}(k) \simeq Mod_{Mod_k}(\mathcal{P}r^{L,st,\omega})$ with the ∞-category of small idempotent-complete k-dg categories.

This identification allows us for us to work with dg-categories and categories of their invariants (for instance, their K-theory spectra) in the same setting.

2.6 $\mathcal{C}$-valued Sheaf categories

We will be dealing with sheaves valued in ∞-categories other than spaces. We collect here some basic properties of these ∞-categories of sheaves.

Definition 2.6.1. Let $\mathcal{X}$ be an ∞-topos and $\mathcal{C}$ be an arbitrary ∞ category containing all small limits. We define the category of $\mathcal{C}$ valued sheaves on $\mathcal{X}$ to be $Shv_{\mathcal{C}}(\mathcal{X}) := Fun^{lim}(\mathcal{X}^{op}, \mathcal{C})$, the category of limit-preserving functors from $\mathcal{X}^{op}$ to $\mathcal{C}$. Note that if $\mathcal{C}$ is presentable, this can be alternatively described as $Fun^R(\mathcal{X}^{op}, \mathcal{C})$, the internal hom in $\mathcal{P}r^R$ the ∞-category of presentable ∞-categories, together with right adjoints. Furthermore, by (12) this is equivalent to the tensor product $\mathcal{X} \otimes^L \mathcal{C}$ in $\mathcal{P}r^L$.

Example 2.6.2. As an example let $\mathcal{X} = Shv(X)$, the ∞-topos of sheaves on a topological space X and let $\mathcal{C} = Sp$, the ∞-category of spectra. Then $Shv_{Sp}(X)$ is the category of sheaves on the space X, valued in spectra.

Example 2.6.3. Let $\mathcal{X}$ be an arbitrary ∞-topos and let $\mathcal{C} = \mathrm{Cat}_{\infty}^{\mathrm{perf}}$ be the category of small stable, idempotent complete ∞ categories. This category itself is presentable (see (2)) and we can make sense of $Shv_{\mathrm{Cat}^{\mathrm{perf}}}(\mathcal{X})$ as $Fun^{R}(\mathcal{X}^{op}, \mathrm{Cat}^{\mathrm{perf}})$.

Let $\mathcal{T}$ be small ∞-category equipped with a Grothendieck topology, i.e. a Grothendieck topology on its underlying homotopy category. We abuse notation (justifiably) and denote $Shv_{\mathcal{C}}(Shv(\mathcal{T}))$ by $Shv_{\mathcal{C}}(\mathcal{T})$. Then the inclusion $Shv_{\mathcal{C}}(\mathcal{T}) \hookrightarrow \mathrm{Pre}_{\mathcal{C}}(\mathcal{T})$ admits a left adjoint; as we are in the context of presentable ∞-categories, the appropriate theory of Bousfield localization applies, displaying sheafification as a localization functor. Moreover, assuming $\mathcal{C}$ is symmetric monoidal, the morphisms which we invert by may be chosen so that this this is a symmetric monoidal functor; this gives us a natural symmetric monoidal structure on $\mathrm{Shv}_{\mathcal{C}}(\mathcal{T})$.

We now define a locally constant object, in analogy with the definition in the setting of ∞-topoi.

Definition 2.6.4. Let $\mathcal{X}$ be an arbitrary ∞-topos and let $\mathcal{C}$ be a presentable ∞-category. We say first $\mathcal{F} \in \mathrm{Shv}_{\mathcal{C}}(\mathcal{X})$ is *constant* if it lies in the image of the morphism $\mathcal{C} \to \mathrm{Shv}_{\mathcal{C}}(\mathcal{X})$ induced by the terminal geometric morphism.

We say $\mathcal{F}$ is *locally constant* if there is a collection of objects $\{U_\alpha\} \in \mathcal{X}$ which cover $\mathcal{X}$ in the sense of definition 2.1.4 and if for every $\alpha \in S$, $\phi^*(\mathcal{F})$ is constant in $\mathrm{Shv}_{\mathbb{C}}(\mathcal{X}_{/U_\alpha})$.

CHAPTER 3

TOPOLOGICAL K-THEORY OF $\mathbb{C}$-LINEAR ∞-CATEGORIES

The purpose of this chapter is to give an overview of topological K-theory of dg categories (originally defined in (1)) and describe some of its properties. First, we recall some essential facts about algebraic K-theory

3.1 Algebraic K-theory

As discussed in the introduction, K-theory was first defined as a functor of rings taking values in abelian groups; it was a breakthrough of Quillen's in (8) that these form the homotopy groups of a certain spectrum, functorially obtained from a category equipped with a suitable notion of exact sequence. In this thesis, we adopt the perspective that algebraic K-theory is a functor from *stable* ∞*-categories* to the category of spectra satisfying certain universal properties.

Definition 3.1.1. Let $\mathcal{C}$ be a small, stable ∞-category. Then there exists a spectrum, $K(\mathcal{C})$, unique up to contractible choice, which we refer to as the *algebraic* K*-theory of* $\mathcal{C}$.

The (connective) K-theory functor $K : \mathrm{Cat}_\infty^{\mathrm{perf}} \to Sp$ satisfies the following property: it sends split-exact sequences $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ of stable ∞-categories to fiber sequences $K(\mathcal{A}) \to K(\mathcal{B}) \to K(\mathcal{C})$ of spectra, making it into an *additive invariant.*

There exists a nonconnective version of the above constructions $\mathbb{K} : \mathrm{Cat}_\infty^{\mathrm{perf}} \to Sp$ which sends exact sequences of stable ∞-categories to fiber sequences of spectra, making it into a *localizing invariant.* There is a canonical map $K(\mathcal{C}) \to \mathbb{K}(\mathcal{C})$ which is an equivalence on non-negative homotopy groups, displaying $K(\mathcal{C})$ as the connective cover of $\mathbb{K}(\mathcal{C})$.

Remark. In (2), the authors characterize K-theory (both in the connective and nonconnective setting) as being universal with respect to the above properties. More precisely, they express it as a corepresentable functor in a suitable stable ∞-category of noncommutative motives $\mathbb{Mot}$, in turn built out of $\mathrm{Cat}_{\infty}^{\mathrm{perf}}$ the relations imposed by exactness. This is done using the theory of localization in the context of presentable ∞-categories. They then display it as the unit in $Fun^{loc}(\mathrm{Cat}_{\infty}^{\mathrm{perf}}, Sp)$, the symmetric monoidal ∞-category of localizing invariants, equivalently, $Fun^{L}(\mathbb{Mot}, Sp)$, the ∞-category of left adjoint functors out of $\mathbb{Mot}$.

There is an alternative approach to the theory of noncommutative motives in (19), which allows for a direct comparison with the "commutative " motives of Morel-Voevodsky.

3.2 Topological realization

Recall the functor

$$X \mapsto X(\mathbb{C})$$

associating to every scheme over the complex numbers its set of complex points. Every set $X(\mathbb{C}) := Map_{Sch/_{\mathbb{C}}}(Spec(\mathbb{C}), X)$ will come endowed with a natural topology, allowing us to view this as a functor landing in spaces. We then compose this with the singular functor $\mathrm{Sing}(-) : Top \to \mathcal{S}$.

This gives the following commutative triangle:

$$\begin{tikzcd} \mathrm{Aff}_{\mathbb{C}} \arrow[r] \arrow[d, hook] & \mathcal{S} \\ \mathrm{Pre}(\mathrm{Aff}_{\mathbb{C}}) \arrow[ur] & \end{tikzcd}$$

where the vertical arrow is the Yoneda embedding $\mathcal{Y} : \mathrm{Aff}_{\mathbb{C}} \to \mathrm{Pre}(\mathrm{Aff}_{\mathbb{C}})$ and the diagonal arrow is the left Kan extension of the "complex points functor". As this is a functor of presentable ∞-categories, we can take its stabilization to obtain the following functor:

$$|| - ||_{\mathbb{S}} : \mathrm{Pre}_{Sp}(\mathrm{Aff}_{\mathbb{C}}) \to Sp$$

which we refer to as spectral realization. Hence, given any presheaf of spectra $\mathcal{F} : \mathrm{Aff}_{\mathbb{C}}^{op} \to Sp$, there is a spectrum $||\mathcal{F}||_{\mathbb{S}}$ functorially associated to it.

An important feature of topological realization is that it factors through ètale sheaves. Namely, we have the following proposition:

Proposition 3.2.1. *Let $X \in Aff_{\mathbb{C}}$ be a scheme and let $U_\bullet \to V$ be an étale hypercover of X. Then $|| - ||$ sends*

$$colim_{\Delta^{op}} h_{U_\bullet} \to h_X$$

to an equivalence of spaces. Hence, $|| - ||$, and its stabilization $|| - ||_{\mathbb{S}}$, factor through the ∞-category of étale hypersheaves of spaces, resp. spectra.

Proof. This is (1, Theorem 3.4).

$\square$

3.2.1 Topological K-theory OF dg-categories

Definition 3.2.2. Let $T \in \mathrm{Cat}^{\mathrm{perf}}(\mathbb{C})$ be a $\mathbb{C}$-linear stable ∞-category, which by Chapter 2.5 , we may think of as a dg-category over $\mathbb{C}$. To T we associate a presheaf of spectra $\underline{K(T)} : \mathrm{Aff}_{\mathbb{C}} \to Sp$ sending

$$\mathrm{Spec}(A) \mapsto K(T \otimes_{\mathbb{C}} Mod_A^{\omega}),$$

the (nonconnective) K-theory of the category $T \otimes_{\mathbb{C}} Mod_A^{\omega}$ where "$\otimes_{\mathbb{C}}$" is the tensor product in $\mathrm{Cat}^{\mathrm{perf}}(\mathbb{C})$ described in Section 2.3. We define the *semi-topological k-theory*, $K^{st}(T) := ||\underline{K(T)}||_{\mathbb{S}}$ to be the spectral realization of this presheaf.

Given the relative abstractness of the above construction, it is perhaps rather surprising that this recovers connective complex K-theory

Proposition 3.2.3 (Blanc)**.** *There is an equivalence $K^{st}(\mathbb{1}) \simeq ku$, where ku the connective complex K-theory spectrum.*

Proof. We give a streamlined proof of this here. Let $\mathrm{Vect}(-)$ denote the (1-)stack of algebraic vector bundles. Then, for any commutative $\mathbb{C}$-algebra A, $K(\mathrm{Vect}(A)$ is the the K-theory of the symmetric monoidal category of algebraic vector bundles on $Spec(A)$. It is well known (see, e.g. (20, Theorem 1.11.7)) that in the setting of commutative rings or schemes, the "direct sum" K-theory $K(\mathrm{Vect}(A)$ computes the connective algebraic K-theory, $K(A)$. Hence, $K(\mathrm{Vect}(-)$ recovers the presheaf of spectra $\underline{K(-)}$.

We note that there is an adjunction $\mathcal{B} : Alg_{E_\infty}(Pre(\mathrm{Aff}_\mathbb{C}) \leftrightarrow Pre_{Sp}(\mathrm{Aff}_\mathbb{C}) : \Omega_\infty$ between E_∞-algebras in $\mathrm{Pre}(\mathrm{Aff}_\mathbb{C})$ and $\mathrm{Pre}_{Sp}(\mathrm{Aff}_\mathbb{C})$, induced by the standard adjunction between E_∞ spaces and connective spectra. The unit of the adjunction $E \mapsto \Omega_\infty \mathcal{B}(E)$ sends an $E \in Alg_{E_\infty}(\mathrm{Pre}(\mathrm{Aff}_\mathbb{C})$ to its group completion as an E_∞ object. By definition of direct sum K-theory, we have the equivalence $K(Vect(-) \simeq \mathcal{B}Vect(-)$.

We now recall that there is an étale local equivalence,

$$\bigsqcup_{n\geq 0} BGL_n \simeq Vect(-) \tag{3.1}$$

cf (21, Section 1); here BG denotes the classifying stack of the group object G. Note that $\sqcup_{n\geq 0} BGL_n$ is itself an E_∞ object in $\mathrm{Pre}(\mathrm{Aff}_\mathbb{C})$, induced by block sum of matrices. Putting this all together, we obtain the following chain of equivalences:

$$\begin{aligned}
||\underline{K(-)}||_\mathbb{S} &\simeq ||\mathcal{B}(\bigsqcup_{n\geq 0} BGL_n)|| && (3.2)\\
&\simeq \mathcal{B}||\bigsqcup_{n\geq 0} BGL_n)|| && (3.3)\\
&\simeq \mathcal{B}(\bigsqcup_{n\geq 0} B||GL_n||) && (3.4)\\
&\simeq \mathcal{B}(\bigsqcup_{n\geq 0} BGL_n(\mathbb{C})) && (3.5)\\
&\simeq ku && (3.6)
\end{aligned}$$

Here, the first equivalence follows from Equation 3.1 together with Proposition 3.2.1. The second equivalence follows from (1) where it is shown the the topological realization functor commutes with group completion functor. (Blanc works in the context of Γ-spaces to show this.) The third equivalence follows from the fact that topological realization is a left adjoint and therefore commutes with coproducts. Finally, it is a classical fact in topology that that the group completion in spaces of $\coprod_{n\geq 0} BGL_n(\mathbb{C}))$ is $BU \times \mathbb{Z}$.

□

Since $||-||_{\mathbb{S}}$ is symmetric monoidal, it follows that $K^{st}(T)$ is a ku-module for any dg-category T. Hence, there exists an action by the Bott element $\beta \in \pi_2(ku)$. The topological K-theory of T is defined as follows:

Definition 3.2.4. Let $T \in \mathrm{Cat}^{\mathrm{perf}}(\mathbb{C})$ and let $K^{st}(T)$ be semi-topological K-theory of T, as defined above. The *topological K-theory* of T is defined to be

$$K^{top}(T) := L_{KU}(K^{st}(T)) \simeq K^{st}(T) \wedge_{ku} KU,$$

the inversion of $K^{st}(T)$ with respect to the Bott element.

We collect here several structural features of topological K-theory.

Theorem 3.2.5. *Let $K^{top} : Cat^{perf}(\mathbb{C}) \to Sp$ be the functor of topological K-theory. Then,*

a. $K^{top}(-)$ commutes with filtered colimits, is Morita invariant and is a localizing invariant in that it sends exact sequences of dg-categories $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ to fiber sequences of spectra.

b. K^{top} *is lax symmetric monoidal.*

c. (Fundamental theorem of topological K-theory) If $X \xrightarrow{\phi} Spec(\mathbb{C})$ *is a separated scheme of finite type, then there exists a functorial equivalence*

$$K^{top}(Perf(X)) \simeq KU^*(X(C)).$$

Here, $Perf(X)$ *denotes the dg category (or* $\mathbb{C}$*-linear* ∞ *category) of perfect complexes of quasi-coherent* $\mathcal{O}_X$ *modules on* X*.*

Proof. Nonconnective K-theory is a localizing invariant; as homotopy colimits are computed pointwise in $\mathrm{Pre}_{Sp}(\mathrm{Aff}_{\mathbb{C}})$, this implies that

$$\underline{K(\mathcal{A})} \to \underline{K(\mathcal{B})} \to \underline{K(\mathcal{C})}$$

is a (co)fiber sequence of sheaves of spectra. We apply $||-||_{\mathbb{S}}$, which is a left adjoint functor and hence preserves cofiber sequences. As Bott inversion is itself a localization and hence a left adjoint, we conclude that

$$K^{top}(\mathcal{A}) \to K^{top}(\mathcal{B}) \to K^{top}(\mathcal{C})$$

is a (co)fiber sequence of spectra.

We remark that $||-||$ is symmetric monoidal; together with the lax monoidality of K, and the well known fact that Bott inversion is symmetric monoidal, this implies that K^{top} : $\mathrm{Cat}^{\mathrm{perf}}(\mathbb{C}) \to Sp$ is lax monoidal, which proves b.

For the proof of c, we refer the reader to (1, Theorem 1.1). The arguments there rely on the well-known characterization of smooth separated finite-type schemes as the dualizable objects in $SH_{\mathbb{C}}$, the stable motivic category over $\mathbb{C}$. This allows for a proof when X is smooth; the general case for a possibly singular scheme X follows from the fact, proven in (1), that K^{top} satisfies cdh-descent which allows (given the fact that we are in characteristic zero and may refine our cdh cover to a cover of smooth $\mathbb{C}$-schemes) us to infer the result from the smooth case. $\square$

CHAPTER 4

DERIVED AZUMAYA ALGEBRAS AND THE BRAUER STACK

To state our main result we first give a brief account of the theory of derived Azumaya algebras due to Toën in (22) and Antieau-Gepner in (23). We fix R to be an arbitrary base connective $\mathbb{E}_\infty$-ring. For our purposes R will be the Eilenberg-Maclane spectrum $H\mathbb{C}$ or commutative algebra over $H\mathbb{C}$.

4.1 Derived Azumaya algebras

Definition 4.1.1. An R algebra A is an Azumaya R-algebra if A is a compact generator of Mod_R and if the natural R-algebra map

$$A \otimes_R A^{op} \to End_R(A)$$

is an equivalence.

Since $End_R(A)$ is Morita equivalent to the base ring, this means that if A is an Azumaya algebra, then there exists some algebra A^{op} such that $A \otimes_R A^{op}$ is Morita equivalent to R. This gives rise to the following characterization of Azumaya algebras. See (23) for a proof.

Proposition 4.1.2. *Let $\mathcal{C}$ be a compactly generated R linear category. Then C is invertible in $Cat_{R,\omega} \simeq Cat^{perf}(R)$ if and only if $\mathcal{C}$ is equivalent to Mod_A for an Azumaya R-Algebra A.*

Azumaya algebras were classically defined with the property that they are equivalent to matrix algebras étale-locally. In the derived setting this can be stated as follows:

Theorem 4.1.3. *If A is a derived Azumaya algebra over a connective E_∞ ring R there exists a faithfully flat étale R-algebra S with the property that $A \otimes_R S$ is Morita equivalent to S.*

Proof. See for instance, (23, Theorem 5.11). □

4.2 The Brauer space

Proposition 4.1.2 allows us to define the *Brauer space* of R to be $\mathrm{Pic}(Cat_{R,\omega})$. However, we will give a "global" definition of the Brauer space, where the input is that of an étale ∞-sheaf, as opposed to a just an $\mathbb{E}_\infty$ ring. For this we work in the ∞-category of sheaves on the category $\mathrm{CAlg}_R^{cn}(Sp)$ of connective commutative algebras over R endowed with a version of the étale topology. We review the details of this category here.

Definition 4.2.1. A morphism $f : S \to T$ of commutative ring spectra is called flat if

$$\pi_0(f) : \pi_0 S \to \pi_0 T$$

is a flat morphism of ordinary discrete rings and if f induces isomorphisms

$$\pi_k S \otimes_{\pi_0 S} \pi_0 T \simeq \pi_k T$$

This allows us to transport the notion of an étale morphism to the context of $\mathbb{E}_\infty$-rings; namely we say a morphism $f : S \to T$ is étale if $\pi_0 f : \pi_0 S \to \pi_0 T$ is étale as a morphism of discrete rings. We may now endow $\mathrm{CAlg}_R^{cn}(Sp)^{op}$ with a Grothendieck topology with covering sieves corresponding to étale morphisms. We denote this category of sheaves with respect to this Grothendieck topology by $\mathrm{Shv}_R^{\acute{e}t}$. Our goal is to define a distinguished object in this ∞-topos, the *Brauer sheaf.*

In this vein, we set $\mathcal{C}at_R : \mathrm{Aff}_R^{op} \to Cat_\infty$ be the functor sending a commutative R-algebra S to Cat_S. Let $Cat_S^{desc} \subset Cat_S$ be the subcategory of S-linear categories satisfying étale hyperdescent. From this, we may define the subfunctor $\mathcal{C}at_{R,}^{desc} \subset \mathcal{C}at_R$ of R-linear categories with R hyper-descent, such that $S \mapsto Cat_S^{desc}$. By the arguments of (14, Theorem 7.5), this defines an étale hyperstack on $Shv_R^{\acute{e}t}$.

Let now $\mathcal{A}lg : \mathrm{Aff}_R^{op} \to Pr^L$ be the functor sending $Spec(S) \mapsto \mathrm{Alg}(\mathrm{Mod}_S)$; by (12) this defines a hyperstack on $Shv_R^{\acute{e}t}$. There exists an obvious morphism of stacks $\mathcal{A}lg \to \mathcal{C}at^{desc}$ induced by the functor $A \to Mod_A$ sending an algebra to its associated category of modules. Let $\mathcal{A}z$ be the subfunctor of $\mathcal{A}lg$ given by restricting to the subcategory of Azumaya algebras.

Finally, let $\mathbf{Az}, \mathbf{Alg}$ and $\mathbf{Pr}$ be the underlying ∞-sheaves (restrict to the maximal subgroupoid sectionwise). We define the Brauer stack as follows:

Definition 4.2.2. Let $\mathbf{Az} \to \mathbf{Pr}$ be the induced map of ∞-sheaves defined above. We define the *Brauer stack* to be the étale sheafification of the image of this map. Given an étale sheaf $X \in Shv_R^{\acute{e}t}$, we let $\mathbf{Br}(X) := Map_{Shv_R^{\acute{e}t}}(X, \mathbf{Br})$ denote the *Brauer space* of X. Finally, we write $Br(X) := \pi_0(\mathbf{Br}(X)$ as the *Brauer group* of X.

If $X = Spec(S)$ for S a connective E_∞-R-algebra then it is easy to see $\mathbf{Br}(X) \simeq Pic(Cat_{S,\omega})$ so we see that this definition agrees with the aforementioned one .

The Brauer stack may be viewed as a delooping of the Picard stack. Indeed, since Azumaya algebras are étale locally equivalent to the ground ring, it follows that $\mathbf{Br}$ is a connected sheaf; moreover it is equivalent to the classifying space of the trivial Brauer class, which is the map $\mathcal{M}od : Spec(R) \to \mathbf{Br} \to \mathbf{Pr}$ sending $Spec(S) \mapsto \mathrm{Mod}_S$. The sheaf of auto-equivalences of $\mathcal{M}$od is presicely the sheaf of line bundles. Hence,

$$\Omega \mathbf{Br} \simeq \mathbf{Pic}.$$

This identification allows us to better understand the homotopy of $\mathbf{Br}$. Recall the following split fiber sequence of étale sheaves:

$$\mathbf{BGL}_1 \to \mathbf{Pic} \to \mathbb{Z}.$$

This may be delooped to obtain the fiber sequence $\mathbf{B}^2\mathbf{GL}_1 \to \mathbf{Br} \to \mathbf{B}\mathbb{Z}$. This gives the following decomposition, for an étale sheaf X, of the derived Brauer group

$$\pi_0 \mathbf{Br}(X) \simeq H^2_{\acute{e}t}(X, \mathbb{G}_m) \times H^1_{\acute{e}t}(X, \mathbb{Z}).$$

Remark. Let X be a quasi-compact, quasi-separated scheme. Let $\alpha \in Br(X) = \pi_0(\mathbf{Br}(X))$ be a class in the Brauer group represented by a map $\alpha : X \to \mathbf{Br}$ in $Shv^{\acute{e}t}_{\mathbb{C}}$. By (23, Theorem 6.17), this map will lift to a map $\alpha \to \mathbf{Az}$ classifying a derived Azumaya algebra over X. We may therefore think of $H^2_{\acute{e}t}(X, \mathbb{G}_m) \times H^1_{\acute{e}t}(X, \mathbb{Z})$ as Morita equivalence classes of derived

Azumaya algebras. In particular we will often be concerned with Brauer classes living in the étale cohomology group $H^2_{ét}(X, \mathbb{G}_m)$.

CHAPTER 5

STACKS OF PRESENTABLE ∞-CATEGORIES & ÉTALE K-THEORY

As announced in the introduction, we construct, for any quasi-compact, quasi separated scheme $\phi : X \to Spec(\mathbb{C})$, a relative version of topological K-theory $K_X^{top}(T)$ with input a Perf(X)-linear ∞-category and output a sheaf of KU-module spectra on $X(\mathbb{C})$. We will particularly be concerned with the values of this functor when T is the ∞-category of perfect complexes of $\mathcal{O}_X$-modules Perf(X, α), for $\alpha \in \pi_0\mathbf{Br}(X)$. In this section, we describe an equivalent definition of $\mathbf{Br}(X)$, which will appear more naturally within our constructions. We do this in the setting of sheaves of linear categories over the scheme X, a notion of independent interest.

Let $\mathrm{Cat}^{\mathrm{perf}} : CAlg_{\mathbb{C}} \to \widehat{Cat_\infty}$ denote the functor sending $R \mapsto \mathrm{Cat}_\infty^{\mathrm{perf}}(R)$. We may right-Kan extend this along the embedding $\mathrm{Aff}_{\mathbb{C}} \to Shv_{\mathbb{C}}^{\acute{e}t}$ to obtain the following commutative diagram of functors:

$$\begin{tikzcd} \mathrm{Aff}_{\mathbb{C}}^{op} \arrow[r, "\mathrm{Cat}^{\mathrm{perf}}(-)"] \arrow[d, hook] & \widehat{Cat_\infty} \\ Shv_{\mathbb{C}}^{\acute{e}t\,op}. \arrow[ur, "ShvCat^{\acute{e}t}(-)"'] & \end{tikzcd}$$

Hence, if X is any scheme, or more generally any étale sheaf we may informally describe an object $\mathcal{C} \in ShvCat^{ét}(X)$ as an assignment, to any $Spec(S) \in \mathrm{Aff}_{/X}$ an S-linear category $\mathcal{C}_S \in \mathrm{Cat}^{\mathrm{perf}}(S)$ together with equivalences

$$\mathcal{C}_{S_1} \otimes_{\mathrm{Cat}^{\mathrm{perf}}(S)} Mod^{\omega}_{S^2} \simeq \mathcal{C}_{S_2}$$

for any morphism of schemes $\phi : Spec(S_2) \to Spec(S_1)$ over X.

By (24, Theorem 1.5.2), the functor $ShvCat^{ét}(-)$ satisfies étale hyperdescent on the category; hence

$$ShvCat^{ét}(-) : Shv^{ét\,op}_{\mathbb{C}} \to \widehat{Cat_\infty}$$

is limit preserving.

Lemma 5.0.1. *$ShvCat^{ét}(X)$ is a symmetric monoidal ∞-category for any $X \in Shv^{ét}_{\mathbb{C}}$.*

Proof. This follows from the properties of right Kan extension, together with the fact that $\widehat{Cat_\infty}$ is itself symmetric monoidal. □

We now investigate the Picard space of this category.

Proposition 5.0.2. *There is a natural equivalence of spaces $Pic(ShvCat^{ét}(X)) \simeq \mathbf{Br}(X)$ for X a quasi-compact, quasi-separated scheme.*

Proof. By (13, Theorem 7.7), the Picard space functor Pic : $\widehat{Cat_\infty} \to \mathcal{S}$ itself preserves limits as it appears as a right adjoint in the adjoint pair

$$\mathrm{Pre} : CAlg^{grp}(\mathcal{S}) \rightleftarrows CAlg(\mathcal{P}r^{L,st,\omega}) : \mathrm{Pic},$$

where $Pre(X) := Fun_\infty(X, \mathcal{S})$. Hence, the composition, $\mathrm{Pic}(ShvCat^{\acute{e}t}(-))$ preserves limits, and therefore defines an étale hypersheaf of spaces. We display a map

$$\mathrm{Pic}ShvCat^{\acute{e}t} \to \mathbf{Br}$$

of objects in $Shv^{\acute{e}t}_{\mathbb{C}}$ and then argue that it is an equivalence. There is a natural map of functors

$$\mathrm{Pic}(\mathrm{Cat}^{\mathrm{perf}}(-)) \to \mathbf{Br} \to \mathbf{Pr}, \tag{5.1}$$

essentially by definition of the Brauer sheaf in Section 5, together with the fact that for every $R \in \mathrm{CAlg}(\mathbb{C})$, an invertible object $\mathcal{C} \in \mathrm{Pic}(\mathrm{Cat}^{\mathrm{perf}}(R)) \simeq \mathrm{Pic}(Cat_{R,\omega})$ corresponds to Mod_A where A is a derived Azumaya algebra over R. This gives the desired map. Note that the symmetric monoidal equivalence

$$Ind : \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{P}r^{L,st,\omega}$$

from section 2.3 induces the equivalence $\mathrm{Cat}_{\infty}^{\mathrm{perf}}(R) \simeq Cat_{R,\omega}$. Furthermore, recall from section 5, that

$$\mathrm{Pic}(Cat_{R,\omega}) \simeq \mathbf{Br}(R)$$

for every commutative $\mathbb{C}$-algebra R. Let $\{\Gamma_R\}$ be the collection of functors $\Gamma_R : Shv_{\mathbb{C}}^{\acute{e}t} \to \mathcal{S}$ with $\Gamma_R(\mathcal{C}) = (\mathcal{C}_R)$ for R ranging over all commutative, connective $\mathbb{C}$ algebras. This defines a conservative family of functors and therefore detects equivalences in $Shv_{\mathbb{C}}^{\acute{e}t}$. We conclude that Equation 5.1 is an equivalence. Hence we make the identification

$$\mathrm{Pic}(ShvCat^{\acute{e}t}(X)) \simeq \mathbf{Br}(X).$$

$\square$

For X a quasi-compact, quasi-separated scheme the ∞-category $ShvCat^{\acute{e}t}(X)$ admits a perhaps simpler description, as follows. Let $\mathrm{Perf}(X)$ denote the ∞-category of perfect complexes of $\mathcal{O}_X$ modules. This is well known to be a small, stable idempotent complete ∞-category. In the setting of section 2.4 we set

$$\mathrm{Cat}^{\mathrm{perf}}(X) := \mathrm{Mod}_{\mathrm{Perf}(X)}(\mathrm{Cat}_{\infty}^{\mathrm{perf}})$$

For X a general étale sheaf, it is not necessarily true that $\mathrm{Cat}^{\mathrm{perf}}(X) \simeq ShvCat^{\acute{e}t}(X)$. This is true whenever X is 1-affine in the sense of (24).

Theorem 5.0.3 (Gaitsgory)**.** *Let X be a scheme. Then there exists a symmetric monoidal adjunction*

$$\mathbf{Loc} : Cat^{perf}(X) \rightleftarrows ShvCat^{\acute{e}t}(X) : \mathbf{\Gamma_X}$$

If X is in particular quasi-compact, quasi-separated, then these are inverse equivalences and $\mathbf{\Gamma_X}$ is itself (strictly) symmetric monoidal.

Proof. This is (24, Theorem 2.1.1) □

Corollary 5.0.4. *Let X be a quasi-compact, quasi-separated scheme. There is an equivalence of spaces*

$$Pic(Cat^{perf}(X)) \simeq \mathbf{Br}(X).$$

Proof. By theorem 5.0.3, $Pic(\mathrm{Cat}^{\mathrm{perf}}(X)) \simeq Pic(ShvCat^{\acute{e}t}(X))$. By proposition 5.0.2,

$$\mathrm{Pic}(ShvCat^{\acute{e}t}(X)) \simeq \mathbf{Br}(X).$$

The equivalence follows. □

5.1 Étale sheafified K-theory

Fix X a quasi-compact, quasi-separated scheme over $Spec(\mathbb{C})$. Let Sm_X denote the (nerve of) the category of smooth schemes of finite type over X. We may equip this category with the étale topology (for a reminder on étale topology the interested reader should consult (25)). Our definition of relative topological K-theory will factor through the ∞-category of étale sheaves of spectra on this category which we now describe.

Definition 5.1.1. Let $\mathrm{Pre}(\mathrm{Sm}/X) := \mathrm{Fun}_\infty(\mathrm{Sm}^{op}_{/X}, \mathcal{S})$ denote the ∞-topos of presheaves of spaces on X. We define $Shv^{\acute{e}t}(X) \subset \mathrm{Pre}(\mathrm{Sm}_{/X})$ to be the full subcategory of hypersheaves of spectra with respect to the étale topology.

Definition 5.1.2. Using the notions from chapter 2.6 we may now define $Shv^{\acute{e}t}_{Sp}(X)$, the stable ∞-category of étale hypersheaves of spectra on $\mathrm{Sm}_{/X}$ to be

$$Shv^{\acute{e}t}_{Sp}(X) := \mathrm{Fun}^{lim}(Shv^{\acute{e}t}(X)^{op}, Sp) \simeq Shv^{\acute{e}t}(X) \otimes^L Sp$$

The forgetful functor $Shv^{\acute{e}t}_{Sp}(X) \subset \mathrm{Pre}_{Sp}(\mathrm{Sm}_{/X})$ admits a (symmetric monoidal) left adjoint

$$L_{\acute{e}t} : \mathrm{Pre}_{Sp}(\mathrm{Sm}_{/X}) \to Shv^{\acute{e}t}_{Sp}(X)$$

(cf. section 2.6) which we think of as *étale sheafification.* Let

$$\underline{K(-)} : \mathrm{Cat}^{\mathrm{perf}}(X) \to Pre_{Sp}(\mathrm{Sm}_{/X})$$

be the functor assigning to $\mathrm{Perf}(X)$-linear category $T \in \mathrm{Cat}^{\mathrm{perf}}(X)$, the presheaf of spectra

$$\underline{K(T)} : \mathrm{Sm}^{op}_{/X} \to Sp$$

sending a smooth scheme Y to the algebraic K-theory spectrum $K(\mathrm{Perf}(Y) \otimes_{\mathcal{O}_X} T)$.

It is not typically the case that $\underline{K(T)}$ is an étale sheaf, for any $T \in \mathrm{Cat}^{\mathrm{perf}}(X)$. Of course, this is because K-theory is known to not satisfy descent with respect the étale topology (see for example (26, Section 1.1)).

We can however, apply the aforementioned sheafification functor. We compose this with the algebraic K-theory functor $\underline{K(-)} : \mathrm{Cat}^{\mathrm{perf}}(X) \to \mathrm{Pre}_{Sp}(\mathrm{Sm}_{/X})$ to obtain a well defined functor which we denote by

$$\underline{K_X^{\acute{e}t}} : \mathrm{Cat}^{\mathrm{perf}}(X) \to Shv_{Sp}^{\acute{e}t}(X).$$

Work of (27) displays algebraic K-theory as a lax symmetric monoidal functor on stable ∞-categories; this in turn displays the functor $\underline{K(-)} : \mathrm{Cat}^{\mathrm{perf}}(X) \to \mathrm{Pre}_{Sp}(\mathrm{Sm}_{/X})$ as lax symmetric monoidal, with respect to the pointwise monoidal structure on $\mathrm{Pre}_{Sp}(X)$. Since sheafification is itself (strongly) symmetric monoidal, the composition $\underline{K^{\acute{e}t}}$ will itself be lax symmetric monoidal. In particular, this functor canonically factors through the forgetful functor

$$Mod_{\underline{K^{\acute{e}t}}(\mathbb{1})}\left(Shv_{Sp}^{\acute{e}t}(X)\right) \to Shv_{Sp}^{\acute{e}t}(X).$$

Here we make the identification $K_X^{\acute{e}t} := \underline{K_X^{\acute{e}t}(\mathbb{1})}$.

Since $K_X^{\acute{e}t}$ is only lax symmetric monoidal, it is not immediately clear that invertible objects in $\mathrm{Cat}^{\mathrm{perf}}(X)$ are sent to invertible objects in $Mod_{K_X^{\acute{e}t}}\left(Shv_{Sp}^{\acute{e}t}(X)\right)$. This is something we will need to know later. In order to prove this, we will use the étale local triviality result discussed in section.

Proposition 5.1.3. *The functor* $\underline{K_X^{\acute{e}t}} : Cat^{perf}(X) \to Mod_{K^{\acute{e}t}}\left(Shv_{Sp}^{\acute{e}t}(X)\right)$ *sends invertible objects in* $Cat^{perf}(X)$ *to invertible* $K_X^{\acute{e}t}$*-modules. In particular,* $\underline{K_X^{\acute{e}t}}$ *induces a map of Picard spaces:*

$$Pic(Cat^{perf}(X)) \to Pic(Mod_{K_X^{\acute{e}t}}\left(Shv_{Sp}^{\acute{e}t}(X)\right)).$$

Proof. Fix A, an invertible object of $\mathrm{Cat}^{\mathrm{perf}}(X)$ which by proposition 5.0.2 represents a corresponding element of the derived Brauer group $\pi_0\mathbf{Br}(X)$; we denote its inverse with respect to the monoidal structure by A^{-1}. Next, we apply $K_X^{\acute{e}t}$; we denote the corresponding sheaves of spectra $\underline{K_X^{\acute{e}t}(A)}$ and $\underline{K_X^{\acute{e}t}(A \otimes_{\mathcal{O}_X} A^{-1})} \simeq \underline{K^{\acute{e}t}(\mathbb{1})}$. Since the functor is lax symmetric monoidal, there is a map of sheaves of spectra (it is not necessarily an equivalence). Let us denote this map as

$$\pi : \underline{K^{\acute{e}t}(A)} \wedge_{K_X^{\acute{e}t}} \underline{K^{\acute{e}t}(A^{-1})} \to \underline{K^{\acute{e}t}(A \otimes_{\mathcal{O}_X} A^{-1})}.$$

By proposition 6 above, derived Azumaya algebras are étale locally trivial. Hence, the stalks on hensel local rings are Morita equivalent to the base, and their corresponding étale K-theory is equivalent to the étale K-theory of the local ring of the stalk. The functor

$$\phi_x^* : Shv_{Sp}^{\acute{e}t}(X) \to Sp$$

of taking stalks will be symmetric monoidal, as it is the left adjoint of (the stabilization of) a geometric morphism. Hence, the stalk of $\underline{K^{\acute{e}t}(A)} \wedge \underline{K^{\acute{e}t}(A^{-1})}$ is equivalent to the stalk of $K_X^{\acute{e}t}(\mathbb{1})$, namely the étale K-theory of the local ring at that point. Since the collection of stalk functors $\{\phi_x^*\}_{x\in X}$ forms a jointly conservative family of functors, this is enough to conclude that the map π above is an equivalence.

We have shown that $\underline{K_X^{\acute{e}t}(A\otimes A^{-1})} \simeq \underline{K_X^{\acute{e}t}(\mathbb{1})}$, the unit of the symmetric monoidal structure on $Mod_{K_X^{\acute{e}t}}(\mathrm{Shv}_{Sp}^{\acute{e}t}(X))$ thereby displaying $\underline{K_X^{\acute{e}t}(A)}$ as an invertible object in this category. $\square$

CHAPTER 6

RELATIVE TOPOLOGICAL K-THEORY

In this chapter we introduce our definition of relative topological K-theory. We will need to use a version of topological realization defined over an arbitrary quasi-compact, quasi-separated $\mathbb{C}$-scheme, which we now recall.

6.1 Topological Realization over a varying base scheme

Let Y be an complex analytic space. For out purposes, this means Y is a locally ringed topological space, locally equivalent to the vanishing locus of a collection of analytic functions $f_1, ..., f_k$. Let $\mathrm{AnSm}/_Y$ denote the category of *smooth analytic spaces* over Y. We give this category a Grothendieck topology; given $Z \in \mathrm{AnSm}/_Y$, we let a covering of Z be given by the standard covering of Z via open sets. These open sets will themselves be smooth analytic spaces over Y, as the composition of the inclusion with the structure map $f : Z \to Y$ will be smooth. This gives the structure of a Grothendieck site on AnSm/Y. Let $Shv(\mathrm{AnSm}/_Y)$ denote the ∞-category of sheaves of spectra on this site, with respect to this topology.

Let $Op(Y)$ denote the category of open subsets of Y. Let $\mathrm{Shv}_{Sp}(Y)$ denote the ∞-category of sheaves of spectra on Y, which is obtained as the stabilization of the ∞ topos of sheaves of spaces over Y. There is an evident inclusion of 1-categories $\iota : Op(Y) \subset \mathrm{AnSm}/Y$. Indeed, given an open subspace of Y, it naturally has the structure of an analytic space over Y as any open subset can be endowed canonically with the structure of an analytic space, smooth over the base space. The map of sites $Op(Y) \subset \mathrm{AnSm}_{/Y}$ induces the following adjunction between sheaf categories

$$\iota^* : Shv_{Sp}(Y) \rightleftarrows Shv(\mathrm{AnSm}/_Y) : \iota_*.$$

Let $\mathbb{D}^1 := \{z \in \mathbb{C} : |z| \leq 1\}$, $\mathbb{D}^n = (\mathbb{D}^1)^n$ and let $\mathbb{D}^n{}_Z := \mathbb{D}^n \times Z$ for any analytic space Z over Y. This is an complex analytic space over Y. We let $\mathrm{Shv}^{\mathbb{D}^1}_{Sp}(\mathrm{AnSm}/Y)$ be the full subcategory of sheaves F with the property that the pullback map $F(Z) \to F(\mathbb{D}^n_Z)$ is an equivalence for all smooth analytic spaces Z over Y. Ayoub proves for any sheaf of spectra F, that $\iota^*(F) \in \mathrm{Shv}^{\mathbb{D}^1}_{Sp}(\mathrm{AnSm}/Y)$. In fact, he proves the following:

Theorem 6.1.1 (Ayoub)**.** *The adjunction* $\iota^* : Shv_{Sp}(Y) \rightleftarrows Shv_{Sp}(AnSm/_Y) : \iota_*$ *induces an equivalence* $Shv_{Sp}(Y) \simeq Shv^{\mathbb{D}^1}_{Sp}(AnSm/Y)$*.*

Proof. This is found in (28, Theoreme 1.18). □

Heuristically speaking, this is because the category $Shv_{Sp}(\mathrm{AnSm}_{/Y})$ is generated by objects of the form $\{\mathcal{Y}(\mathbb{D}^n_U) \otimes A_{cst}\}_{n \in \mathbb{N}, U \in Op(Y)}$ where $\mathcal{Y}(\mathbb{D}^n_U)$ is the representable functor associated to U and A_{cst} is the constant sheaf associated to $A \in Sp$. Passing to $\mathbb{D}^1$-invariant sheaves reduces the generators to those of the form $\{U \otimes A_{cst}\}$.

Now, fix a scheme $\phi : X \to \mathrm{Spec}(\mathbb{C})$. We define our realization functor on $Shv^{\acute{e}t}_{Sp}(X)$ landing in $Shv_{Sp}(X(\mathbb{C}))$. For this we must first pass to $\mathbb{A}^1$-invariant sheaves of spectra, which we now recall.

Definition 6.1.2. Let $\mathbb{A}^1_X = \mathbb{A}^1 \times_{Spec(\mathbb{C})} X$ denote the relative affine line. We define $Shv^{\acute{e}t,\mathbb{A}^1}_{Sp}(X)$ to be the full subcategory consisting of all $\mathcal{F} \in Shv^{\acute{e}t}_{Sp}(X)$ such that $\mathcal{F}(Y) \to \mathcal{F}(Y \times \mathbb{A}^1)$ is an equivalence for all $Y \in Sm_{/X}$. As described in say (29), there exists a left adjoint

$$L_{\mathbb{A}^1} : Shv^{\acute{e}t}_{Sp}(X) \to Shv^{\acute{e}t,\mathbb{A}^1}_{Sp}(X)$$

to the inclusion exhibiting this as a localization.

Recall from section 4, the functor $(-)(\mathbb{C}) : \mathrm{Sch}/_{Spec(\mathbb{C})} \to Top$ defined by

$$X \mapsto X(\mathbb{C})$$

where $X(\mathbb{C})$ inherits the structure of a complex analytic space. For every scheme X, we obtain a morphism of sites,

$$\mathrm{Sm}/X \to \mathrm{AnSm}_{/X(\mathbb{C})}$$

which induces the adjunction

$$An^*_X : Shv^{\acute{e}t}_{Sp}(X) \rightleftarrows Shv_{Sp}(\mathrm{AnSm}_{/X(\mathbb{C})}) : An_{X,*}.$$

If $X = Spec(\mathbb{C})$, this corresponds to the functor $|| - ||_{\mathbb{S}}$ described in section 4.

Remark. We remark that $An_X^*(\mathbb{A}^1) \simeq \mathbb{D}_X^1$. Hence An_X^* sends $\mathbb{A}^1$ invariant sheaves to $\mathbb{D}_X^1$ invariant sheaves of spectra on $Shv_{Sp}(\mathrm{AnSm}_{X(\mathbb{C})})$ and therefore descends to a functor

$$Shv_{Sp}^{\acute{e}t,\mathbb{A}^1}(X) \to Shv_{Sp}^{\mathbb{D}^1}(\mathrm{AnSm}_{/X(\mathbb{C})}) \simeq Shv_{Sp}(X(\mathbb{C})).$$

For the remainder of this paper we will take $\widetilde{An_X^*}$ to denote the composition

$$Shv_{Sp}^{\acute{e}t}(X) \xrightarrow{L_{\mathbb{A}^1}} Shv_{Sp}^{\acute{e}t,\mathbb{A}^1}(X) \to Shv_{Sp}^{\mathbb{D}^1}(\text{An-Sm}/(X(\mathbb{C}))) \simeq Shv_{Sp}((X(\mathbb{C})).$$

To state the following we recall the fact that $K_X^{\acute{e}t}$ is an E_∞ algebra in $\mathrm{Shv}_{Sp}^{\acute{e}t}(X)$.

Theorem 6.1.3. *$\widetilde{An^*}(K_X^{\acute{e}t})$ is equivalent to the connective topological K-theory sheaf $\underline{ku}^{X(\mathbb{C})}$ on $X(\mathbb{C})$, sending an open subspace $V \subset X$ to $F(\Sigma_+^\infty V, ku)$, the complex topological K-theory spectrum of V.*

Proof. By (1, Theorem 4.5),

$$\widetilde{An_{\mathbb{C}}^*}(K_{\mathbb{C}}^{\acute{e}t}) \simeq ||\underline{K}||_{\mathbb{S}} \simeq ku \in Sp$$

where $\underline{K}$ denotes the (non-connective) algebraic K-theory presheaf over $Spec(\mathbb{C})$ and ku is the connective topological k-theory spectrum. The first equivalence follows from the fact that $||-||_{\mathbb{S}}$ sends étale local equivalences in $\mathrm{Pre}_{Sp}(\mathrm{Sm}_{\mathbb{C}})$ to equivalences of spectra (cf. (1, Theorem 3.4)) and hence factors through étale sheaves. Now let $\phi : X \to Spec(\mathbb{C})$ be an arbitrary quasi-compact, quasi-separated scheme. We have the following commutative diagram of functors:

$$\begin{tikzcd} Shv^{ét}_{Sp}(\mathbb{C}) \arrow[r] \arrow[d, "\phi^*"] & Shv^{ét,\mathbb{A}^1}_{Sp}(\mathbb{C}) \arrow[r, "An^*"] \arrow[d] & Shv_{Sp}(*) \simeq Sp \arrow[d, "\phi^*_{an}"] \\ Shv^{ét}_{Sp}(X) \arrow[r] & Shv^{ét,\mathbb{A}^1}_{Sp}(X) \arrow[r, "An^*"] & Shv_{Sp}(X(\mathbb{C})) \end{tikzcd}$$

where the left horizontal arrows denote $\mathbb{A}^1$-localization. The left square commutes by the properties of localization. The right hand square commutes simply from the formal property that the following diagram of sites

$$\begin{tikzcd} \mathrm{Sm}_{/\mathbb{C}} \arrow[r] \arrow[d] & \mathrm{AmSm}_{/\mathbb{C}} \arrow[d] \\ \mathrm{Sm}_{/X} \arrow[r] & \mathrm{AnSm}_{/X(\mathbb{C})} \end{tikzcd}$$

commutes and induces a commutative diagram of left adjoints. The result will now follow from the following lemma and the fact that $\phi^*_{an}(ku) \simeq \underline{ku^{X(\mathbb{C})}}$. □

Lemma 6.1.4. *There exists an equivalence*

$$\phi^*(K^{ét}_{\mathbb{C}}) \simeq K^{ét}_X$$

in $Shv^{ét}_{Sp}(X)$.

Proof. Although this is immediately true when X is smooth, we need to show this in general. The corresponding statement is true at the level of connective K-theory; namely,

$$K^{cn}|_{\mathrm{Sm}_{/X}} \simeq \phi^*(K^{cn}|_{\mathrm{Sm}_{/\mathbb{C}}}) \tag{6.1}$$

as presheaves on $\mathrm{Sm}_{/X}$. To see this, recall that as a presheaf on $Sm_{/\mathbb{C}}, K^{cn}$ is the group completion of the E_∞ monoid

$$\coprod_{n\geq 0} BGL_n.$$

Each of the GL_n's are smooth and hence may be pulled back; because ϕ^* is colimit preserving, we may pullback $\coprod_{n\geq 0} BGL_n$. By (30, Lemma 5.5) the pullback functor will commute with group completion hence giving us the equivalence Equation 6.1.

Next, we claim that the nonconnective K-theory pulls back. For this, we recall the construction in (31, Section 2.5) of nonconnective (Bass-Thomason-Trobaugh) K-theory and show that it behaves well with respect to pull-back. There, for a scheme S, the nonconnective K-theory sheaf in $Shv^{Nis}_{Sp}(S)$ is constructed as the homotopy colimit of the following diagram

$$K^{cn}|_{\mathrm{Sm}_{/S}} \to F_S^{-1} \to F_S^{-2} \to ... \tag{6.2}$$

where the F^{-i}'s are constructed inductively from $K^{cn}|_{\mathrm{Sm}_{/S}}$.

In order to construct F^{-1} from $K^{cn}|_{\mathrm{Sm}/S}$ we study the following diagram in $Shv_{Sp}^{Nis}(S)$:

$$\begin{tikzcd} K^{cn} \arrow[r] \arrow[d] & \mathbf{R}\mathrm{Hom}(\Sigma^\infty(\mathbb{A}^1), K^{cn}) \arrow[d] \\ \mathbf{R}\mathrm{Hom}(\Sigma^\infty(\mathbb{A}^1), K^{cn}) \arrow[r] & \mathbf{R}\mathrm{Hom}(\Sigma^\infty(\mathbb{G}_m), K^{cn}). \end{tikzcd} \tag{6.3}$$

We let C_S denote the homotopy pushout of this diagram and define F_S^{-1} to be the homotopy fiber of the induced map $C \to \mathbf{R}\mathrm{Hom}(\Sigma^\infty(\mathbb{G}_m), K^{cn})$. Finally, we remark that there is a map $K^{cn} \to F_S^{-1}$, more generally $F_S^{-i} \to F_S^{-(i+1)}$ for every i, induced by the canonical map $S^1 \wedge \mathbb{G}_m \to K^{cn}$ given by the projective bundle formula; putting this together gives diagram Equation 6.2.

Since $\phi^* : Shv_{Sp}^{Nis}(\mathbb{C}) \to Shv_{Sp}^{Nis}(X)$ is a left adjoint functor, it preserves the homotopy colimit diagram Equation 6.2. Hence, it is enough show that the formation of F^{-i}'s is compatible with pullback. To this end we note that ϕ^* sends diagram Equation 6.3 in $Shv_{Sp}^{Nis}(\mathbb{C})$ to the corresponding diagram formed in $Shv_{Sp}^{Nis}(X)$. This follows from the fact that

$$\phi^*(K^{cn}|_{\mathrm{Sm}/\mathbb{C}}) \simeq K^{cn}|_{Sm/X}$$

and

$$\phi^*\mathbf{R}\mathrm{Hom}(A, B) \simeq \mathbf{R}\mathrm{Hom}(\phi^*A, \phi^*B)$$

for any $A, B \in Shv_{Sp}^{Nis}(\mathbb{C})$. Indeed, the functor $\phi^* : Shv_{Sp}^{Nis}(\mathbb{C}) \to Shv_{Sp}^{Nis}(X)$ fits into a Wirthmüller context (eg.it fulfills the conditions of (13, Theorem 6.4)) making ϕ^* into a closed monoidal functor. Hence, $\phi^*(C_{\mathbb{C}}) \simeq C_X$ and since ϕ^* is an exact functor of stable ∞-categories, the fiber F^{-1} is preserved. The same argument now applies to show that $\phi^*(F^i_{\mathbb{C}}) \simeq F^i_X$ for all i and therefore that $\phi^*(K|_{\mathrm{Sm}_{/\mathbb{C}}}) \simeq K|_{\mathrm{Sm}_{/X}}$.

Having concluded that the non-connective K-theory sheaf pulls back, we deduce the étale-sheafified version of this fact by appealing to fact that $K^{\acute{e}t}_{\mathbb{C}} \simeq L_{\acute{e}t}(\underline{K})$ in $Shv^{\acute{e}t}_{Sp}(\mathbb{C})$ and that étale sheafification commutes with pullback. □

Remark. By Theorem 6.1.3, the functor $\widetilde{An^*} : Shv^{\acute{e}t}_{Sp}(X) \to \mathrm{Shv}_{Sp}(X(\mathbb{C}))$, being symmetric monoidal, induces a functor $\widetilde{An^*} : Mod_{K^{\acute{e}t}}(Shv^{\acute{e}t}_{Sp}(X)) \to Mod_{ku_X}(Shv_{Sp})$.

6.2 Relative Topological K-theory

We are now in position to define the relative topological K-theory. Recall the fact that there exists a localization

$$Mod_{ku} \to Mod_{KU}$$

given precisely by $M \mapsto M \otimes_{ku} KU$. This induces a functor at the level of spectral sheaf categories; namely, we obtain an exact functor of stable ∞-categories

$$L_{KU} : Mod_{ku_X}[Shv_{Sp}(X(\mathbb{C}))] \to Mod_{KU_X}[Shv_{Sp}(X(\mathbb{C}))]$$

Definition 6.2.1. Let X be a quasi-compact, quasi-separated scheme over $\mathbb{C}$. Let $T \in Cat^{\mathrm{perf}}(X)$ be a Perf(X)-linear dg-category. We let

$$K^{top}_X(T) := L_{KU}(\widetilde{An^*_X}(\underline{K^{\acute{e}t}_X(T)}))$$

be the *relative topological* K*-theory* of T over X.

6.3 Relative Topological K-theory as a Motivic Realization

We give an alternate description of $K_X^{top}(T)$ as a *motivic realization* of the dg category T, in the setting of stable motivic homotopy theory.

Let $T \in \mathrm{Cat}^{\mathrm{perf}}(X)$. Then the nonconnective algebraic K-theory presheaf $\underline{K(T)}$ associated to it defined by:

$$Y \mapsto K(Perf(X) \otimes_{\mathcal{O}_X} T)$$

is a sheaf with respect to the Nisnevich topology on Sm_X. This follows from (14, Theorem 5.4)) where it is shown that the $\widehat{\mathcal{C}at_\infty}$-valued presheaf on $\mathrm{Sm}_{/X}$

$$Y \to Perf(Y) \otimes_{\mathcal{O}_X} T$$

satisfies descent in the étale topology and the well known fact that non-connective K-theory satisfies Nisnevich descent (eg. by (20)) Hence, we may view $\underline{K(T)}$ as an object in $Shv_{Sp}^{Nis}(X)$.

As in section 2, we let $L_{\acute{e}t} : Shv_{Sp}^{Nis}(X) \to Shv_{Sp}^{\acute{e}t}(X)$ denote the étale sheafication functor. By definition, the functor

$$\widetilde{An}^* \circ L_{\acute{e}t} : Shv_{Sp}^{Nis}(X) \to Shv_{Sp}(X(\mathbb{C}))$$

sends $\mathbb{A}^1$-equivalences of sheaves to equivalences in the target category. Hence it factors through

$$L_{\mathbb{A}^1} : Shv_{Sp}^{Nis}(X) \to Shv_{Sp}^{Nis,\mathbb{A}^1}(X)$$

by the universal property of $\mathbb{A}^1$ localization. Following the conventions in (31), we set $\underline{KH(T)} :=$ $L_{\mathbb{A}^1}(\underline{K(T)})$ to be the *homotopy K-theory* sheaf associated to T. We remark that $\underline{KH(T)}$ is a module over homotopy invariant K-theory $\underline{KH}$ and so inherits a multiplication by the Bott map β, where

$$\beta \in \pi_0 \mathrm{Map}_{Shv_{Sp}^{Nis,\mathbb{A}^1}(X)}(\mathbb{P}^1_X, \underline{KH})$$

is the map reflecting the projective bundle theorem. (see for example, the proof of proposition 4.3.2 in (1)).

We pass to SH(X), the stable motivic category over the scheme X. By (19, Corollary 2.22) we may define SH(X) as the stabilization with respect to $\mathbb{P}^1_X$:

$$\mathrm{SH}(X) = Stab_{\mathbb{P}^1_X}(Shv_{Sp}^{Nis,\mathbb{A}^1}(X)) := \lim(Shv_{Sp}^{Nis,\mathbb{A}^1}(X) \xleftarrow{\Omega_{\mathbb{P}^1_X}} Shv_{Sp}^{Nis,\mathbb{A}^1}(X) \ \leftarrow ...)$$

Via the arguments in section 2.16 in (31) we canonically associate an object $KGL(T) \in \mathrm{SH}(X)$ to $\underline{KH(T)}$; in effect, $KGL(T)$ will be the "constant spectrum" with structure maps given by

$$\beta : \underline{KH(T)} \to \Omega_{\mathbb{P}^1_X}(\underline{KH(T)} := Map_{Shv_{Sp}^{Nis,\mathbb{A}^1}(X)}(\mathbb{P}^1_X, \underline{KH(T)}) \simeq \underline{KH(T)}$$

By (see e.g. (28) or (19)) the universal property of $\mathrm{SH}(X)$ together with theorem 6.1.1, the functor $Y \mapsto \Sigma^{\infty}_{+}(Y(\mathbb{C}))$ uniquely induces a functor

$$\mathbf{Betti}_X : \mathrm{SH}(X) \to \mathrm{Shv}_{Sp}(X(\mathbb{C}))$$

This is symmetric monoidal, so that there is an map

$$\mathrm{Mod}_{KGL}[\mathrm{SH}(X)] \to \mathrm{Mod}_{KU_X}(\mathrm{Shv}_{Sp}(X(\mathbb{C}))).$$

We have the following proposition

Proposition 6.3.1. *There is an equivalence*

$$K_X^{top}(T) \simeq \mathbf{Betti}_X(KGL(T)).$$

Proof. By (28), the functor An_X^* sends $\mathbb{P}^1_X$ to the locally constant sheaf of spectra S^2_X associated to the sphere S^2. In particular, by definition of $\mathrm{SH}(X)$

$$\mathbf{Betti}_X : \mathrm{SH}(X) \to Shv_{Sp}(X(\mathbb{C}))$$

factors through the equivalence

$$\Omega^\infty_{S^2} : Shv_{Sp^{S^2}}(X(\mathbb{C})) \simeq Shv_{Sp}(X(\mathbb{C})),$$

where $Shv_{Sp^{S^2}}(X(\mathbb{C}))$ denotes the ∞-category of sheaves valued in S^2-spectra. We remark that this is canonically equivalent to $Shv_{Sp}(X(\mathbb{C}))$.

If we apply $\mathbf{Betti}_X$ to $KGL(T)$ we obtain the colimit of the following diagram

$$An^*_X(\underline{KH(T)}) \xrightarrow{\beta_T} \Omega^2(An^*_X(\underline{KH(T)})) \dot{\to} ..$$

This is precisely the formula for the L_{KU} localization of $An^*(\underline{KH(T)})$ at the level of $ku_{X(\mathbb{C})}$ modules in $Shv_{Sp}(X(\mathbb{C}))$, thereby giving us the equivalence

$$K^{top}_X(T) = L_{KU}(An^*_X(\underline{K(T)}) \simeq \mathbf{Betti}_X(KGL(T)).$$

□

6.4 Functoriality properties of relative topological K-theory

Let $\phi : Y \to X$ be a map of schemes. We have the following restriction/extension adjunction :

$$\phi^* : \mathrm{Cat}^{\mathrm{perf}}(X) \rightleftarrows \mathrm{Cat}^{\mathrm{perf}}(Y) : \phi_*$$

where $\phi^*(T) \simeq T \otimes_{\mathrm{Perf}(X)} \mathrm{Perf}(Y)$ Associated to the induced map of spaces $\phi : Y(\mathbb{C}) \to X(\mathbb{C})$, is the adjunction

$$\phi^* : \mathrm{Shv}_{Sp}(X(\mathbb{C}) \rightleftarrows \mathrm{Shv}_{Sp}(Y(\mathbb{C})) : \phi_*$$

Note that when $X = Spec(\mathbb{C})$, the functor $\phi_* : \mathrm{Shv}_{Sp}(X(\mathbb{C})) \to \mathrm{Shv}_{Sp}(Y(\mathbb{C})) \simeq Sp$ is none other than the functor sending a sheaf of spectra to its spectrum of global sections.

It is immediately clear, by the properties of restriction on smooth morphisms, that $K_X^{top} \circ \phi^* \simeq \phi^* \circ K_Y^{top}$ when $\phi : X \to Spec(\mathbb{C})$ is smooth. This equivalence, via the adjunction morphisms

$$\mathbb{1} \to \phi_*\phi^*, \phi^*\phi_* \to \mathbb{1}$$

gives rise to the following natural transformation

$$\eta : K_X^{top} \circ \phi_* \to \phi_* \circ K_Y^{top}$$

We do not yet know this to be an equivalence even for smooth Y. This issue is particularly transparent before we apply the Bott-localization functor L_{KU}. If we set $X = Spec(\mathbb{C})$, and let Y be an arbitrary $\mathbb{C}$-scheme, then by theorem 6.1.3,

$$\phi_* \widetilde{An_X^*}(\underline{K_X^{\acute{e}t}(\mathbb{1})}) \simeq ku(X(\mathbb{C})).$$

Meanwhile, the semi-topological K-theory, $K^{st}(X)$, is typically not equivalent to $ku(X(\mathbb{C}))$. Indeed $K^{st}_*(X)/n \simeq K_*(X)/n$ which is not true for connective complex K-theory with finite coefficients. We suspect however that applying Bott localization L_{KU} **does eliminate** this discrepancy and plan to investigate this in future work. The following proposition, which we will be critical for us here, serves as evidence for this hypothesis.

Proposition 6.4.1. *Let $\phi : Y \to X$ be a scheme, proper over X. Let Perf(Y) be the associated dg-category of perfect complexes, viewed as Perf(X)-linear category. Then,*

$$K^{top}_X(Perf(Y)) \simeq \phi_*(K^{top}_Y(\mathbb{1}_Y))$$

where $\mathbb{1}_Y$ is the unit in $Cat^{perf}(Y)$.

Proof. By proposition 6.3.1,

$$K^{top}(\mathrm{Perf}(Y)) \simeq \mathbf{Betti}_X(KGL(\mathrm{Perf}(Y)).$$

Hence, we argue using stable motivic homotopy theory. The map of schemes $\phi : Y \to X$ induces the push-forward $\phi_* : \mathrm{SH}(Y) \to \mathrm{SH}(X)$ fitting in the following diagram of functors, which we claim is commutative because of the properness assumption on ϕ:

$$\begin{array}{ccc} \mathrm{SH}(Y) & \xrightarrow{\mathbf{Betti}_Y} & Shv_{Sp}(Y(\mathbb{C})) \\ \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \phi_*} \\ \mathrm{SH}(X) & \xrightarrow{\mathbf{Betti}_X} & Shv_{Sp}(X(\mathbb{C})) \end{array} \tag{6.4}$$

To see this, we note that the functor $Shv^{an} : Sch_{/\mathbb{C}} \to CAlg(Pr^{L,st})$ given by

$$X \mapsto Shv_{Sp}(X(\mathbb{C}))$$

satisfies the six-functor formalism described for example in (28, Section 3). By (28, Theoreme 3.4), the Betti realization

$$\mathbf{Betti} : \mathrm{SH}^{\otimes} \to Shv^{an}$$

is a natural transformation of such functors with the property that $\mathbf{Betti}_X \circ \phi_! \simeq \phi_! \circ \mathbf{Betti}_Y$; here $\phi_!$ denotes pushforward with compact support. Because ϕ is proper, there is an equivalence of functors $\phi_! \simeq \phi_*$, hence proving commutativity of Equation 6.4.

Next we remark that

$$\phi_*(KGL(\mathbb{1}_Y)) \simeq \underline{KGL(\mathrm{Perf}(Y))},$$

the object in SH(X) associated to $\underline{KH(\mathrm{Perf}(Y))}$. This just follows from the formula for the pushforward in SH(X). To deduce the proposition, it is therefore enough to understand $\phi_*(\mathbf{Betti}_Y(KH(\mathbb{1}_Y))) \in Shv_{Sp}(X(\mathbb{C}))$.

By Theorem 6.1.3, $K_Y^{top}(\mathbb{1}_Y) = \mathbf{Betti}_Y(\underline{KH|_Y})$ is the sheaf of spectra on $Y(\mathbb{C})$ sending an open subset $U \subseteq Y(\mathbb{C})$ to $KU^*(U)$. Its push-forward $\phi_*(K_Y^{top}(\mathbb{1}_Y))$ will be the sheaf of spectra on $X(\mathbb{C})$ defined by the assignment

$$V \mapsto KU^*(\phi^{-1}(V))$$

for any open set $V \subseteq X(\mathbb{C})$, where $\phi : Y(\mathbb{C}) \to X(\mathbb{C})$ is the induced map on analytic spaces. By the commutativity of (Equation 6.4), we now conclude that

$$\begin{aligned} K_X^{top}(\mathrm{Perf}(Y)) &\simeq \mathbf{Betti}_X(\underline{KH(Y)}) \\ &\simeq \phi_*(\mathbf{Betti}_Y(\underline{KH_Y})) \\ &\simeq \phi_*(KU_Y) \\ &\simeq \phi_*(K_Y^{top}(\mathbb{1})). \end{aligned}$$

□

CHAPTER 7

LOCAL SYSTEMS AND TWISTED COHOMOLOGY THEORIES

We review the version of twisted topological K-theory we will be working with, in its modern homotopy theoretic formulation. Although we will be focusing on twists of KU, one may twist any cohomology theory E in an analogous manner. More of the general theory may be found in (32) or (33).

7.1 Local Systems

We first introduce the notion of local systems of objects of an ∞-category $\mathcal{C}$ on a space X. The category of local systems of KU-module spectra will play a central role in our version of twisted topological K-theory.

Definition 7.1.1. Let X be a space, (thought of as an ∞-groupoid) and let $\mathcal{C}$ denote an arbitrary ∞-category. We define the ∞-category of $\mathcal{C}$ valued local systems on X by $\mathrm{Loc}_X(\mathcal{C}) := \mathrm{Fun}(X, \mathcal{C})$

More generally, one may define a family of objects in a presentable ∞-category $\mathcal{C}$ parametrized by an object $X \in \mathcal{X}$ as

$$\mathrm{Shv}_{\mathcal{C}}(\mathcal{X}_{/X})$$

If $\mathcal{X} = \mathcal{S}$, we recover the category $\mathcal{S}_{/X}$ of local systems of a space.

We make a few remarks about $\mathrm{Loc}_X(\mathcal{C})$ for general categories $\mathcal{C}$. By the properties of taking functor categories (see, for example (12)) if $\mathcal{C}$ is stable, then so is $\mathrm{Loc}_X(\mathcal{C})$. Furthermore, $\mathrm{Loc}_X(\mathcal{C})$ will be symmetric monoidal if $\mathcal{C}$ is itself symmetric monoidal. The unit is precisely the constant functor $\mathbf{1} : X \to \mathcal{C}$ sending every zero simplex of X to the unit $\mathbf{1}_{\mathcal{C}} \in \mathcal{C}$ and every morphism to the identity morphism of $\mathbf{1}_{\mathcal{C}}$.

Fix an ∞-topos $\mathcal{X}$ and a presentable ∞-category $\mathcal{C}$. Recall from section 2.6 the definition of a locally constant object in category of sheaves $\mathrm{Shv}_{\mathcal{C}}(\mathcal{X})$. Local systems will be significant to us in part because they, in suitable situations, give another description of locally constant objects. Later on, certain arguments will become more transparent for us once we work in this setting.

We investigate this identification first when $\mathcal{C} = \mathcal{S}$. For now, let $\mathcal{X}$ be a general ∞-topos and let $\pi^* : \mathcal{S} \to \mathcal{X}$ be the left adjoint to the terminal geometric morphism. This functor preserves finite limits and therefore admits a pro-left adjoint $\pi_! : \mathcal{X} \to \mathrm{Pro}(\mathcal{S})$. We call $\pi_!(\mathbf{1})$ the *shape* of $\mathcal{X}$. We say that $\mathcal{X}$ is *locally of constant shape* if $\pi_!$ factors through the inclusion of constant pro-spaces $\mathcal{S} \to \mathrm{Pro}(\mathcal{S})$. In this setting, $\pi_! : \mathcal{X} \to \mathcal{S}$ will be a further left adjoint to π^*.

We now specialize to the case where X is a topological space which is locally contractible. The ∞-topos $Shv(X)$ will be locally of constant shape by (12) ; hence in this setting we have the morphism $\pi_! : Shv(X) \to \mathcal{S}$, left adjoint to π^*.

There is a canonical functor:

$$Shv(X) \simeq Shv(X)_{/\mathbf{1}} \xrightarrow{\pi_!} \mathcal{S}_{/\pi_!(\mathbf{1})}$$

which we denote as $\psi_!$. There is a right adjoint, which we denote by ψ^*. It can be described informally by :

$$\psi^*(Y) = \pi^*(Y) \times_{\pi^*\pi_!(\mathbf{1})} \mathbf{1}.$$

Theorem 7.1.2. *Let $\mathcal{X}$ be an ∞-topos which is locally of constant shape and let $\psi^* : \mathcal{S}_{/\pi_!(\mathbf{1})} \to \mathcal{X}$ be the above functor. Then ψ^* embeds $\mathcal{S}_{/\pi_!(\mathbf{1})}$ fully faithfully onto the the full subcategory of $\mathcal{X}$ spanned by the locally constant objects.*

Proof. This is (12, Theorem A.1.15). □

If X is again, locally contractible, then it satisfies the conditions of (12, Definition A.4.15) and therefore is of *singular shape*; this means that we may identify $\pi_!(\mathbf{1})$ with the simplicial set $\mathrm{Sing}(X)$.

We remark further that the identification of theorem 7.1.2 is symmetric monoidal. This follows from the fact that we can think of $\psi^* : \mathcal{S}_{/\pi_!(\mathbf{1})} \to Shv(X)$ as a composition of functors

$$\pi^* : \mathcal{S}_{/\pi_!(\mathbf{1})} \to Shv(X)_{/\pi^*\pi_!(\mathbf{1})}$$

followed by the change of base functor

$$Shv(X)_{/\pi^*\pi_!(\mathbf{1})} \to Shv(X)_{/\mathbf{1}} \simeq Shv(X)$$

induced by the unit of the adjunction $\mathbf{1} \to \pi^*\pi_!(\mathbf{1})$. Of course, each of these functors preserve cartesian products, and therefore preserve the relevant symmetric monoidal structure. To recap, we have displayed $\psi^* : Loc_{Sing(X)}(\mathcal{S}) \simeq \mathcal{S}_{/Sing(X)} \to Shv(X)$ as an algebra map in $\mathcal{P}r^L$.

To promote this to the level of spectra, we recall from (12) that the stabilization functor is functorial at the level of presentable ∞-categories; in our situation this means we can stabilize the adjunction

$$\psi_! : Shv(X) \leftrightarrow \mathcal{S}_{/\pi_!(\mathbf{1})} \simeq Loc_{\pi_!(\mathbf{1})}(\mathcal{S}) : \psi^*$$

to obtain the following adjunction of presentable stable ∞-categories

$$Stab(\psi_!) : Shv_{Sp}(X) \leftrightarrow Stab(\mathcal{S}_{/\pi_!(\mathbf{1})}) = Loc_{\pi_!(\mathbf{1})}(\mathcal{S}) : Stab(\psi^*).$$

It now follows from the functoriality of stabilization that this functor is fully faithful onto its image. Indeed as endofunctors on $Loc_X(Sp)$, the following chain of equivalences hold

$$Id \simeq Stab(\psi_! \circ \psi^*) \simeq Stab(\psi_!) \circ Stab(\psi^*)$$

Furthermore, $Stab(\psi^*)$ is symmetric monoidal. Indeed it may be displayed as the composition of symmetric monoidal functors

$$Loc_{\pi_!(\mathbf{1})}(Sp) \xrightarrow{Stab(\pi^*)} Shv_{Sp}(X)_{/\pi^*\pi_!(\mathbf{1})} \to Shv_{Sp}(X)$$

where the second map is the stabilization of the functor induced by pullback along $\mathbf{1} \to \pi^*\pi_!(\mathbf{1})$.

We may summarize the above discussion with the following proposition:

Proposition 7.1.3. *Let X be a (locally contractible) topological space. There exists a symmetric monoidal fully faithful right adjoint $\psi^* : Loc_X(Sp) \to Shv_{Sp}(X)$ with essential image the locally constant sheaves of spectra.*

Remark. In the above proposition, we may replace the category Sp with Mod_{KU}, or more generally Mod_E for any ring spectrum E.

7.2 Twisted topological K-theory

Let $X \in \mathcal{S}$ be a space and let $\alpha \in Loc_X(\mathrm{Pic}_{KU})$ be a map $\alpha : X \to \mathrm{Pic}_{KU}$. We take the composition $\tilde{\alpha} : X \to \mathrm{Pic}_{KU} \to \mathrm{Mod}_{KU}$. This *twist* classifies a bundle of invertible KU-modules over X.

Definition 7.2.1. We define the twisted KU-homology to be the Thom spectrum

$$M\alpha = \mathrm{colim}_X \tilde{\alpha}$$

and the twisted KU-cohomology to be

$$KU(X)_\alpha \simeq F_{KU}(M\alpha, KU),$$

the internal function spectrum. We call this the *twisted K-theory.* Alternatively, the twisted KU-cohomology may be defined as the spectrum of sections of this bundle i.e $\Gamma_X(\alpha) := Maps_{Loc_X(\mathrm{Mod}_{KU})}(1_X, -\tilde{\alpha})$.

Remark. The reader may have noticed a difference in signs between the two definitions. Indeed anti-equivalent in the sense that we must dualize the twist with respect to a canonical involution on the category $\mathrm{Loc}_X(\mathrm{Pic}_E)$ before taking Thom spectra in order to show that we get the same definition for cohomology.

The reader should consult (33, Section 5) for a more complete description of the theory.

CHAPTER 8

TOPOLOGICAL K-THEORY OF DERIVED AZUMAYA ALGEBRAS

8.1 Proof of the main theorem

Having set up the relevant machinery, we restate and prove our main theorem.

Theorem 8.1.1. *Let X denote a quasi separated, quasi compact scheme over the complex numbers. Let $\alpha \in \pi_0\mathbf{Br}(X)$ be a Brauer class, and $Perf(X, \alpha) \in Cat^{perf}(X)$ denote the associated Perf(X)- linear category. Then there exists a functorial equivalence*

$$K_X^{top}(Perf(X,\alpha)) \simeq \underline{KU^{\widetilde{\alpha}}(X(\mathbb{C}))}.$$

Here, $\underline{KU^{\alpha}(X(\mathbb{C}))}$ is the locally constant sheaf of KU-modules associated to a local system of invertible KU modules; this is in turn given by a twist $\widetilde{\alpha} : X(\mathbb{C}) \to Pic_{KU}$ obtained functorially from α.

We can rephrase the theorem as saying that there exist unique lifts making the following diagram of functors commute:

$$\begin{tikzcd}
 & & Loc_{\mathrm{Sing}(X(\mathbb{C}))}(Pic_{KU}) \arrow[d, hook] \\
\mathbf{Br}(X) \arrow[urr, dotted] \arrow[rr, dotted] \arrow[d, "K_X^{\acute{e}t}(-)"] & & \mathrm{Loc}_{\mathrm{Sing}(X(\mathbb{C}))}(\mathrm{Mod}_{KU}) \arrow[d, hook, "\psi^*"'] \\
\mathrm{Mod}_{K_X^{\acute{e}t}}(Shv_{Sp}^{\acute{e}t}(X)) \arrow[rr, "\widetilde{An_X^*}"] & & \mathrm{Shv}_{Mod_{KU}}(X(\mathbb{C}))
\end{tikzcd}$$

The existence of these lifts will follow once we show that $K_X^{top}(\mathrm{Perf}(X, \alpha))$ is both a locally constant sheaf, and is invertible as an object in $\mathrm{Shv}_{\mathrm{Mod}_{KU}}(X(\mathbb{C}))$. To this end we will use proposition 5.1.3 (and in particular, the étale-local triviality of derived Azumaya algebras) in an essential way.

Proposition 8.1.2. *Let $A \in Cat^{perf}(X)$ be Perf(X)-linear category over X corresponding to a derived Azumaya algebra. Then $K_X^{top}(A)$ is a locally constant sheaf of KU-module spectra on $X(\mathbb{C})$*

Proof. Let $x \in X(\mathbb{C})$ be a point. We will show that there exists some open neigborhood $x \in V$ for which the restriction $K_X^{top}(A)|_V$ is equivalent to KU_V in $Shv_{Sp}(V)$. The result will follow since KU_V is the sheafification of the constant sheaf on V sending all open sets to KU, and is hence locally constant.

By proposition 5.1.3, if A is a derived Azumaya algebra over X, representing a Brauer class $\alpha \in \pi_0\mathbf{Br}_0(X)$, its associated étale K-theory theory sheaf of spectra is étale locally equivalent to $K_X^{\acute{e}t}(\mathbb{1})$ (the étale K-theory sheaf of the base). This means that, for any $x \in X$, there exists an étale map $\phi : Spec(S) \to X$ with image containing x for for which

$$\phi^*(\underline{K^{\acute{e}t}(A)}) \simeq K^{\acute{e}t}|_{Spec(S)}.$$

Taking complex points, we obtain a map of spaces $\widetilde{\phi} : U := Spec(S)(\mathbb{C}) \to X(\mathbb{C})$. Let $V \subset X$ be any open subset of $\widetilde{\phi}(U) \subset X$ containing x. This will be our desired open neighborhood of x, for which the restriction of $K_X^{top}(A)$ is equivalent to the constant sheaf associated to KU. Since $\widetilde{\phi} : U \to X(\mathbb{C})$ is the realization of an étale morphism it is a local homeomorphism and therefore there exists a cover $\{U_i\}_{i \in I}$ of U such that each U_i is mapped homeomorphically onto its image. Choose some U_i with $x \in \widetilde{\phi}(U_i)$. We now have the following chain of equivalences in $Shv_{Sp}(U_i)$:

$$\begin{aligned} K_X^{top}(A)|_{U_i} &\simeq K_{Spec(S)}^{top}(A \otimes_{\mathcal{O}_X} Mod_S)|_{U_i} \\ &\simeq K_{Spec(S)}^{top}(\mathbb{1})|_{U_i} \\ &\simeq \underline{KU^U}|_{U_i} \\ &\simeq \underline{KU^{U_i}}, \end{aligned}$$

where the second equivalence follows from theorem 4.1.3 and the third and fourth follow from 6.1.3. We have displayed, for $x \in X$, an open set U_i over which the restriction $K_X^{top}(A)|_{U_i}$ is equivalent to the constant sheaf associated to KU. Hence, $K_X^{top}(A)$ is itself locally constant. □

Remark. Together with Proposition 7.1.3, this allows us to identify $K_X^{top}(A)$ with its associated local system in $Loc_{\mathrm{Sing}(X(\mathbb{C}))}(Mod_{KU})$.

Proposition 8.1.3. *Let Perf(X, α) be as above. Then $K_X^{top}(Perf(X, \alpha))$ is an invertible object of $Loc_{Sing(X(\mathbb{C}))}(Mod_{KU})$, that is*

$$K_X^{top}(Perf(X, \alpha)) \in Pic[Loc_{Sing(X(\mathbb{C}))}(Mod_{KU})].$$

Proof. As we showed in proposition 5.1.3, the associated étale K-theory sheaf $K^{\acute{e}t}(\mathrm{Perf}(X, \alpha)$ on $\mathrm{Shv}^{\acute{e}t}(X)$ is invertible as an object of $Mod_{K^{\acute{e}t}}(Shv^{\acute{e}t}_{Sp}(X))$. Since $\mathbb{A}^1$-localization and topological realization An^* is symmetric monodal by (28), it follows that $\widetilde{An^*}(K^{\acute{e}t}(\mathrm{Perf}(X, \alpha)))$ is invertible as well. Finally, the KU localization functor

$$L_{KU} : Shv_{\mathrm{Mod}_{ku}}(X(\mathbb{C})) \to Shv_{\mathrm{Mod}_{KU}}(X(\mathbb{C}))$$

is itself symmetric monoidal; combining all this, we conclude that

$$K_X^{top}(\mathrm{Perf}(X, \alpha)) = L_{KU}\widetilde{An^*_X}(K_X^{\acute{e}t}(\mathrm{Perf}(X, \alpha)$$

is an invertible object in $Shv_{\mathrm{Mod}_{KU}}(X(\mathbb{C}))$. □

Remark. The above proposition allows us to think of K_X^{top}(Perf(X,α) as an object in the Picard ∞-groupoid Pic($Loc_{X(\mathbb{C})}(Mod_{KU})$). We identify

$$\mathrm{Pic}(Loc_{X(\mathbb{C})}(Mod_{KU})) \simeq Loc_{X(\mathbb{C})}(\mathrm{Pic}_{KU})$$

where Pic_{KU} denotes the Picard space of Mod_{KU}. This follows from the pointwise symmetric monoidal structure on $Loc_{Sing(X(\mathbb{C}))}(\mathrm{Mod}_{KU}$: a local system $\alpha : \mathrm{Sing}(X(\mathbb{C})) \to Mod_{KU}$ will be invertible if and only if each simplex is sent to an invertible KU module.

Proof of Theorem 8.1.1. Let Perf(X, α) be the Perf(X) linear category of modules over the derived Azumaya algebra associated to $\alpha \in Br(X)$. By proposition 8.1.2, $K_X^{top}(Perf(X, \alpha))$ is locally constant; hence we may look at its image in $Loc_{\mathrm{Sing(X)}}(\mathrm{Mod}_{KU})$. By proposition 8.1.3, $K_X^{top}(\mathrm{Perf}(X, \alpha))$ is invertible as an object in the symmetric monoidal ∞-category $Loc_{\mathrm{Sing}(X(\mathbb{C}))}(\mathrm{Mod}_{KU})$. By the remarks above, we may therefore represent K_X^{top}(Perf(X,α)) by a local system $\widetilde{\alpha} : Sing(X(\mathbb{C})) \to \mathrm{Pic}_{KU} \to \mathrm{Mod}_{KU}$. This gives precisely the desired twist of K-theory. $\square$

8.2 Cohomological Brauer classes

We now restrict to the setting where our chosen Brauer class α lives in $H^2_{\acute{e}t}(X, \mathbb{G}_m)$. We show that in this case, the corresponding local system $\alpha : Sing(X(\mathbb{C})) \to \mathrm{Pic}_{KU}$ factors through the map $K(\mathbb{Z}, 3) \to Pic_{KU}$. Hence, the twist obtained in theorem 8.1.1 arises from a class

$\tilde{\alpha} \in H^3(X, \mathbb{Z})$. Our first order of business is then to study the homotopy of the space Pic_{KU}. It is straightforward to see that

$$\Omega(Pic_{KU}) \simeq GL_1(KU);$$

In other words, Pic_{KU} is a delooping of the space of units of KU. Indeed, for a general symmetric monoidal ∞-category $\mathcal{C}$ with unit $\mathbf{1} \in \mathcal{C}$ and any invertible object $X \in \mathcal{C}$, we have the following equivalences :

$$Map_{\mathcal{C}}(X, X) \simeq Map_{\mathcal{C}}(\mathbf{1} \otimes X, X) \simeq Map_{\mathcal{C}}(\mathbf{1}, X^{-1} \otimes X) \simeq Map_{\mathcal{C}}(\mathbf{1}, \mathbf{1})$$

where the second equivalence follows from the fact that we can view tensoring with X as an left adjoint functor (because it is invertible, hence dualizable). Its right adjoint is none other than tensoring with its dual, X^{-1}. The final equivalence holds because $X^{-1} \otimes X \simeq \mathbf{1}$. Now, recall that when forming the ∞-groupoid $Pic(\mathcal{C})$, we restrict to the subcategory of equivalences. This has the effect of restricting the endomorphism mapping spaces for each $X \in Pic(\mathcal{C})$ to the space $Aut(\mathbf{1})$ with path components corresponding to $\pi_0(End_{\mathcal{C}}(\mathbf{1}))^{\times} \subseteq \pi_0(End_{\mathcal{C}}(\mathbf{1}))$. In this particular case, where $\mathcal{C} = Mod_{KU}$, this means that

$$\pi_1(Pic_{KU}) = \pi_0(GL_1(KU)) = \pi_0(\Omega^{\infty}(R))^{\times};$$

$$\pi_n(Pic_{KU}) \simeq \pi_{n-1}(GL_1(KU)) \simeq \pi_{n-1}(\Omega^{\infty}(KU)).$$

Recall from (34) that there is a decomposition of infinite loop spaces:

$$GL_1(KU) \simeq K(\mathbb{Z}/2, 0) \times K(\mathbb{Z}, 2) \times BSU_{\otimes}$$

and therefore

$$BGL_1(KU) \simeq K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}, 3) \times BBSU_{\otimes}$$

Now, since $X(\mathbb{C})$ is connected (at least in our situation) a map $\alpha : Sing(X(\mathbb{C})) \to Pic_{KU}$ is equivalent to a map landing in the component of the identity $\alpha : Sing(X(\mathbb{C})) \to BGL_1(KU)$.

For the remainder of this section, we work in the ∞-topos, $Shv^{et}(\mathbb{C})$, of étale sheaves over $Spec(\mathbb{C})$. Earlier we showed that, to a given Azumaya algebra A, the associated étale K theory sheaf is invertible as a module over étale K-theory over the base. This implies that there exists a map of sheaves $\mathbf{Br} \to \widetilde{Pic}_{K^{et}}$ in $\mathrm{Shv}^{\acute{e}t}(X)$ where we think of $\widetilde{Pic}_{K^{et}}$ as the stack of invertible modules over étale K-theory. In particular, the space of sections over $Spec(R) \to Spec(\mathbb{C})$ is just the Picard space of the category of $K^{et}|_R$ modules in the étale ∞-topos of R.

As discussed above, the functor of taking complex points induces a morphism of topoi

$$|| - || : Shv_{et}(\mathbb{C}) \to \mathcal{S}.$$

We claim that

$$||\widetilde{\mathrm{Pic}_{K^{\acute{e}t}}}|| \simeq \mathrm{Pic}_{KU}.$$

Due to the decomposition of spaces (for a general symmetric ∞-category $\mathcal{C}$),

$$Pic(\mathcal{C}) \simeq \pi_0(Pic(\mathcal{C})) \times BAut(1),$$

it is enough to prove the following:

Proposition 8.2.1. *There is an equivalence of spaces* $||BGL_1(K^{et})|| \simeq BGL_1(KU)$.

Proof. It is enough to show that $||GL_1(K^{et})|| \simeq GL_1(ku)$. We have a following homotopy pullback square in $Shv_{\mathbb{C}}^{ét}$ defining the sheaf of units $GL_1(K^{ét})$:

$$\begin{array}{ccc} GL_1(K^{\acute{e}t}) & \longrightarrow & \Omega^\infty(K^{\acute{e}t}) \\ \downarrow & & \downarrow \\ \tau_0(\Omega^\infty(K^{\acute{e}t}))^\times & \longrightarrow & \tau_0(\Omega^\infty(K^{\acute{e}t})) \end{array}$$

where the bottom horizontal map is inclusion of the grouplike components into $\tau_0(\Omega^\infty(K^{\acute{e}t}))$. We will identify the image, under the realization functor, $|| - ||$ of this pullback square in $\mathcal{S}$ with the one defining $GL_1(ku)$. The proof will then follow since $|| - ||$ is a left adjoint to a geometric morphism of ∞-topoi and therefore preserves finite limits.

We first show that $||\Omega^\infty(K^{\acute{e}t})|| \simeq \Omega^\infty(ku)$. Recall, from the proof of Proposition 3.2.3 , the following chain of equivalences:

$$\begin{aligned}
||\Omega^\infty(K^{\acute{e}t})|| &\simeq ||(\bigsqcup_{n\geq 0} BGL_n)^{gp}|| \\
&\simeq (\bigsqcup_{n\geq 0} ||BGL_n||^{gp}) \\
&\simeq (\bigsqcup_{n\geq 0} BGL_n(\mathbb{C}))^{gp} \\
&\simeq BU \times \mathbb{Z}.
\end{aligned}$$

The second equivalence follows from (1) where it is shown the the topological realization functor commutes with group completion of an $\mathbb{E}_\infty$ space. (Blanc works in the context of Γ-spaces to show this.) The third equivalence follows from the fact that topological realization is a left adjoint and therefore commutes with coproducts. Next up, we show that $\Omega^\infty(K^{et}) \to \tau_0(\Omega^\infty(K^{et}))$ corresponds, upon applying $||-||$, with the zero truncation map $\Omega^\infty(ku) \to \tau_0(\Omega^\infty(ku))$. This is a consequence of (11, Proposition 5.5.6.28) where it is shown that for a colimit preserving, left exact functor $F : A \to B$ between presentable ∞-categories that is a left adjoint, there is a natural equivalence $F \circ \tau_0 \simeq \tau_0 \circ F$. Hence, upon applying topological realization to the right vertical arrow, we obtain the truncation map $\Omega^\infty(ku) \to \tau_0(\Omega^\infty(ku))$ as desired.

Finally, we show that

$$||\tau_0(\Omega^\infty(K^{\acute{e}t})^\times|| \to ||\tau_0(\Omega^\infty(K^{\acute{e}t})||$$

is the map

$$\tau_0(\Omega^\infty(ku))^\times \to \tau_0(\Omega^\infty(ku)).$$

For this we recall the fact that $\tau_0(K^{\acute{e}t})$ is the constant sheaf $\underline{\mathbb{Z}}$ and hence $\Omega^\infty(\tau_0(K^{\acute{e}t}))^\times$ is $\underline{\mathbb{Z}/2}$. Let $\pi^* : Shv^{\acute{e}t}(\mathbb{C}) \to \mathcal{S}$ be the left adjoint to the terminal geometric morphism, sending a space to its associated constant sheaf. If we take $||-||\circ\pi^*$, this is a left adjoint to geometric morphism $\mathcal{S} \to \mathcal{S}$; of course there is only one such morphism and it will be an equivalence. Hence, $||\pi^*(A)|| \simeq A$. Putting this all together, we conclude that $||\tau_0(\Omega^\infty(K^{\acute{e}t})||^\times \to ||\tau_0(\Omega^\infty(K^{\acute{e}t})||$ is equivalent to the inclusion $\mathbb{Z}/2 \to \mathbb{Z}$ of multiplicative monoids. $\square$

Corollary 8.2.2. *Let X be a quasi-compact, quasi separated scheme over $Spec(\mathbb{C})$ and let α be a class in $H^2_{\acute{e}t}(X, \mathbb{G}_m)$. Then the assignment*

$$\alpha \mapsto \widetilde{\alpha}$$

of theorem 8.1.1 is given precisely by the map on cohomology

$$H^2_{\acute{e}t}(X, \mathbb{G}_m) \to H^3(X(\mathbb{C}), \mathbb{Z}).$$

induced by the realization functor $||-|| : Shv^{\acute{e}t}_{\mathbb{C}} \to \mathcal{S}$.

Proof. By (23) there is a natural map $B^2\mathbb{G}_m \to \mathbf{Br}$. By the discussion preceding 8.2.1, taking étale K-theory induces the following map in $Shv^{\acute{e}t}_{\mathbb{C}}$.

$$\mathbf{Br} \to \widetilde{Pic_{K^{\acute{e}t}}}$$

We apply the realization functor $||-||$ to the composition

$$X \xrightarrow{\alpha} B^2\mathbb{G}_m \to \mathbf{Br} \to \widetilde{Pic_{K^{\acute{e}t}}}$$

and obtain, by proposition 8.2.1 the factorization of α

$$\mathrm{Sing}(X(\mathbb{C})) \to K(\mathbb{Z}, 3) \to Pic_{KU}$$

where the map $K(\mathbb{Z}, 3) \to Pic_{KU}$ is the canonical inclusion. $\square$

8.2.1 Absolute topological K-theory of Azumaya algebras

We now restate and prove our computation of the "absolute" K-theory of Azumaya algebras.

Theorem 8.2.3. *Let X be a separated scheme of finite type and let $\alpha \in H^2_{\acute{e}t}(X, \mathbb{G}_m)$ be a torsion class in étale cohomology corresponding to an ordinary (non-derived) Azumaya algebra over X. Then*

$$K^{top}(Perf(X, \alpha)) \simeq KU^{\widetilde{\alpha}}(X(\mathbb{C}))$$

where $\alpha \mapsto \widetilde{\alpha} \in H^3(X, \mathbb{Z})$ is the associated cohomological twist of corollary 8.2.2.

Remark. The reason 8.2.3 is non-trivial is, again, that while

$$\Gamma(\underline{KU^{\widetilde{\alpha}}(X(\mathbb{C}))} \simeq KU^{\widetilde{\alpha}}(X(\mathbb{C})),$$

we do not know yet that

$$\Gamma(K_X^{top}(\mathrm{Perf}(X,\alpha)) \simeq K^{top}(\mathrm{Perf}(X,\alpha))$$

By proposition 6.4.1 this is true for categories of the form $\mathrm{Perf}(Y) \in \mathrm{Cat}^{\mathrm{perf}}(X)$ for $Y \to X$, smooth and proper over X. We will deduce theorem 8.2.3 from this.

For the proof of this we recall the notion of a Severi-Brauer variety over a scheme X. One can associate, to a given cohomological Brauer class $\alpha \in H^2(X, \mathbb{G}_m)$ a scheme $f_\alpha : P \to X$ which is, étale locally, equivalent to projective space. By (35) there exists a *semi-orthogonal decomposition* of the category

$$\mathrm{Perf}(P) = \langle \mathrm{Perf}(X,\alpha), ...\mathrm{Perf}(X,\alpha^k)\rangle$$

where $\mathrm{Perf}(X,\alpha)$ is the derived ∞-category of α-twisted sheaves on X We will not recall exactly the precise definition of a semi-orthogonal decomposition. The significance of this for us lies in the fact that, there will be a decomposition of the algebraic K-theories:

$$K(\mathrm{Perf}(P)) \simeq K(\mathrm{Perf}(X,\alpha)) \oplus .... \oplus K(\mathrm{Perf}(X,\alpha^k))$$

This is because (connective) algebraic K-theory is an additive invariant and hence sends split exact sequences of stable ∞-categories to sums. This will be true when we pullback to other schemes as well: if $U \in Sch_{/X}$, then the corresponding decomposition exists for Perf$(U \times_X P)$ where the decomposition ranges over α_U-twisted sheaves on U.

Proof of Theorem 8.2.3. Since colimits are created objectwise in the category $Shv^{\acute{e}t}_{Sp}(X)$, and since sheafification $L : Pre_{Sp}(X) \to Shv^{\acute{e}t}_{Sp}(X)$ preserves colimits, we deduce from the above that

$$K^{\acute{e}t}_X(\underline{\mathrm{Perf}(P)}) \simeq K^{\acute{e}t}_X(\underline{\mathrm{Perf}(X, \alpha)}) \oplus .... \oplus K^{\acute{e}t}_X(\underline{\mathrm{Perf}(X, \alpha^k)})$$

is a sum in $\mathrm{Shv}^{\acute{e}t}_{Sp}(X)$. Recall that, from here, in order to define $K^{top}_X(-)$, we must first $\mathbb{A}^1$-localize and then apply the realization functors $An^*_X : Shv^{\mathbb{A}^1}_{Sp}(\mathcal{X}_{\acute{e}t}) \to Shv_{Sp}(X(\mathbb{C}))$. Finally we must apply the analogue of Bott inversion, $L_{KU} : \mathrm{Mod}_{ku}[\mathrm{Shv}_{Sp}(X(\mathbb{C}))] \to \mathrm{Mod}_{Ku}[\mathrm{Shv}_{Sp}(X(\mathbb{C}))]$. Each of these are left adjoint functors and hence preserve colimits. It follows that

$$K^{top}_X(\underline{\mathrm{Perf}(P)}) \simeq K^{top}_X(\underline{\mathrm{Perf}(X, \alpha)}) \oplus .... \oplus K^{top}_X(\underline{\mathrm{Perf}(X, \alpha^k)})$$

is a colimit in $\mathrm{Shv}_{Sp}(X(\mathbb{C})$. As taking global sections $\Gamma = \pi_* : Shv_{Sp}(X(\mathbb{C})) \to Sp$ is now an exact functor of stable ∞-categories we obtain a corresponding decomposition of the global sections.

By proposition 6.4.1,

$$\Gamma(K^{top}_X(\underline{\mathrm{Perf}(P)}_X) \simeq K^{top}(\underline{\mathrm{Perf}(P)}_{\mathbb{C}})$$

via the associated natural transformation. This isomorphism descends to the summands of both sides, hence we may conclude that

$$\Gamma(K_X^{top}(\mathrm{Perf}(X,\alpha)) = K^{top}(\mathrm{Perf}(X,\alpha))$$

Together theorem 8.1.1, this allows us to finally conclude that $K^{top}(\mathrm{Perf}(X,\alpha))) \simeq KU^{\widetilde{\alpha}}(X(\mathbb{C}))$, the $\widetilde{\alpha}$-twisted topological K-theory of the space $X(\mathbb{C})$. $\square$

CHAPTER 9

APPLICATIONS TO PROJECTIVE FIBER BUNDLES IN TOPOLOGY

We now prove our theorem on the topological K theory of projective space bundles. For the reader's convenience, we restate the theorem.

Theorem 9.0.1. *Let X be a finite CW-complex. Let $\pi : P \to X$ be a bundle of rank $n-1$ projective spaces classified by a map $\pi : X \to BPGL_n(\mathbb{C})$ and let $\tilde{\alpha} : X \to B^2\mathbb{G}_m$ be the composition of this map along the map $BPGL_n(\mathbb{C}) \to B^2\mathbb{G}_m \simeq K(\mathbb{Z}, 3)$ This gives rise to an element $\tilde{\alpha} \in H^3(X, \mathbb{Z})$ which we can use to define the twisted K-theory spectrum $KU^{\alpha}(X)$. Then there exits the following decomposition of spectra:*

$$KU^*(P) \simeq KU^*(X) \oplus KU^{\widetilde{\alpha}}(X) ... \oplus KU^{\widetilde{\alpha^{n-1}}}(X).$$

where $KU^{\widetilde{\alpha^k}}(X)$ denotes the twisted K-theory with respect to the class $\widetilde{\alpha^k} \in H^3(X, \mathbb{Z})$.

We use our results together with certain approximations of classifying spaces by algebraic varieties, due to Totaro. This allows us to reduce the theorem to an algebro-geometric setting, to which our previous results apply. We begin with the following lemma:

Lemma 9.0.2. *There is a weak homotopy equivalence*

$$BPGL_n(\mathbb{C}) \simeq colim_{i \to \infty} Y^i(\mathbb{C})$$

where $BPGL_n(\mathbb{C})$ denotes the classifying space of the projective linear group and each $Y^i(\mathbb{C})$ is the space of complex points of a smooth quasi projective variety over the complex numbers.

Proof. This is a reformulation of remark 1.4 of (36). Let W be a faithful representation of PGL_n; for instance we may choose the adjoint representation which is well known to be faithful. Then we let

$$V^N = Hom(\mathbb{C}^{N+n}, W) \simeq W^{N+n}$$

and let S be the closed subset in V^N of non-surjective linear maps. The group acts freely outside of S. Furthermore, the codimension of S goes to infinity as N goes to infinity. Now, $Y^N := (V^N \setminus S)/PGL_N$ exists as a smooth quasi-projective variety by (36, Remark 1.4) We may define maps

$$V^N \to V^{N+1}$$

by extending the linear maps in the obvious manner; it is clear that nonsurjective maps will be sent to nonsurjective maps. Moreover these maps will be equivariant with respect to the action. The colimit of this diagram is independent of the choice of faithful representation; moreover, it is shown in (3) to be $\mathbb{A}^1$- equivalent to the étale classifying space of PGL_n. Applying the induced functor $|| - || : H_{\mathbb{C}} \to \mathcal{S}$, we obtain the desired colimit diagram in spaces. □

Proof of Theorem 9.0.1. Let X be a finite CW complex, as in the statement of the theorem, equipped with a map $\alpha : X \to BPGL_n(\mathbb{C})$. By the above lemma, $BPGL_n(\mathbb{C}) \simeq \operatorname{hocolim}_{i\to\infty} Y^i(\mathbb{C})$. We may view X as a compact object in the homotopy category of spaces and therefore $\alpha : X \to BPGL_n(\mathbb{C})$ factors through some $Y^i(\mathbb{C}) \to BPGL_n(\mathbb{C})$. If we write this factorization as $\alpha = \beta \circ f$ for maps $f : X \to Y^i(\mathbb{C})$ and $\beta : Y^i(\mathbb{C}) \to BPGL_n(\mathbb{C})$ we see that the projective bundle P can be expressed as $f^*(P^i)$ the pullback of a projective space bundle over Y^i classified by the map $\beta : Y^i(\mathbb{C}) \to BPGL_n(\mathbb{C})$. It is important to note that, by the above lemma, the map β is in the image of the realization functor and therefore arises from some map of étale sheaves $\beta^{alg} : Y^i \to BPGL_n$, giving rise to a Severi-Brauer scheme over Y^i. The projective space bundle over $Y^i(\mathbb{C})$ can therefore be thought of as the space of complex points of this Severi-Brauer scheme.

The composition $Y^i(\mathbb{C}) \xrightarrow{\beta} BPGL_n(\mathbb{C}) \xrightarrow{\iota} K(\mathbb{Z}, 3)$ along the morphism gives rise to local system of invertible KU-module spectra over $Y^i(\mathbb{C})$. We obtain the following decomposition of local systems in $\mathrm{Loc}_{Y^i(\mathbb{C})}(\mathrm{Mod}_{KU})$:

$$\underline{KU(P^i)} \simeq \underline{KU(Y^i(\mathbb{C}))} \oplus \underline{KU^{\alpha}(Y^i(\mathbb{C}))} \oplus ... \oplus \underline{KU^{\alpha^{n-1}}(Y^i(\mathbb{C}))}.$$

by the results of the previous section. We seek to pull back this decomposition to $Loc_X(\mathrm{Mod}_{KU})$. This is straightforward; the map $f : X \to Y^i(\mathbb{C})$ induces an adjunction

$$f^* : Loc_{Y^i(\mathbb{C})}(Mod_K U) \leftrightarrow Loc_X(Mod_{KU}) : f_*$$

The functor f^*, being a left adjoint, preserves coproducts hence giving us a decomposition of local systems

$$f^*(\underline{KU(P^i)}) \simeq f^*(\underline{KU(Y^i(\mathbb{C}))}) \oplus f^*(\underline{KU^{\alpha}(Y^i(\mathbb{C}))}) \oplus ... \oplus f^*(\underline{KU^{\alpha^{n-1}}(Y^i(\mathbb{C}))}) \tag{9.1}$$

on X. It remains to identify these pullbacks with the corresponding local systems on X. To see that $f^*(\underline{KU(P^i)}) \simeq \underline{KU(P)}$ on X, note that the local system $\underline{KU(P^i)}$ is given by

$$Y \to \mathcal{S} \xrightarrow{KU^{(-)}} \mathrm{Mod}_{KU};$$

we obtain $f^*(\underline{KU(P^i)})$ by precomposing this with $X \xrightarrow{f} Y$. Of course the composition $X \xrightarrow{f} Y \to \mathcal{S}$ classifies the pull-back

$$P \simeq P^i \times_Y X$$

as a local system of spaces over X; composing with $\mathcal{S} \xrightarrow{KU^{(-)}} \mathrm{Mod}_{KU}$ gives us $\underline{KU^*(P)}$. The identification of the terms on the right hand side of Equation 9.1 is immediate as the pullback will be given by the following map obtained through precomposition by $X \xrightarrow{f} Y^i(\mathbb{C})$:

$$X \xrightarrow{f} Y^i(\mathbb{C}) \xrightarrow{\beta} BPGL_n(\mathbb{C}) \to K(\mathbb{Z}, 3) \to Pic_{KU} \hookrightarrow \mathrm{Mod}_{KU}$$

This is none other than the local system corresponding to $\alpha : X \to BPGL_n(\mathbb{C})$. The result is now immediate upon taking global sections. $\square$

CHAPTER 10

PREVIOUS WORK

As discussed in the main text, the formalism of topological K-theory of dg-categories was laid out by Anthony Blanc in (1). Foundational results on the theory of derived Azuamaya algebras can be found in (22) and (23) where the authors generalize the classical theories of Azumaya algebras and the Brauer group to the setting of simplicial commutative rings and E_∞ spectra, respectively. This dissertation applies ideas from the latter to theory the developed in the former.

This work is, in addition, heavily infuenced by the motivic perspective of algebraic K-theory set out in (2) and (19). The ideas in this thesis indirectly resulted from the author's attempts to apply this motivic perspective to topological K-theory by developing a framework of *topological noncommutative motives*. This remains an ongoing project.

CHAPTER 11

FUTURE WORK

11.1 Twisted Equivariant K-theory

The author has several ongoing working projects, continuing and extending the results of this thesis. In (37), the authors prove a generalization of Theorem 3.2.5, wherein they display a natural equivalence

$$K^{top})(\mathrm{Perf}(X/G)) \simeq K_M(X(\mathbb{C}))$$

for a global quotient X/G, of a scheme X by a reductive algebraic group G. The left hand side of this equivalence is the topological K-theory of the dg-category of perfect complexes on the stack X/G; the right hand side is the M-equivariant topological K-theory of $X(\mathbb{C})$ where $M \leq G(\mathbb{C})$ is a maximal compact subgroup of the complex points $G(\mathbb{C})$. The author is working on a twisted version of this equivalence.

Conjecture 11.1.1. *Let $\alpha \in H^2_{\acute{e}t}(X/G, \mathbb{G}_m)$ correspond to a sheaf of Azumaya algebras over the quotient stack X/G. Then there exists a natural equivalence*

$$K^{top}(Perf(X/G, \alpha) \simeq K_M(X(\mathbb{C}))_\alpha.$$

Here, $\mathrm{Perf}(X/G, \alpha)$ is the ∞-category of perfect complexes of modules over the corresponding Azumaya algebra and the right hand side is the twisted, equivariant K-theory of $X(\mathbb{C})$.

The author intends to formulate and prove a more general statement for Artin stacks (hence not necessarily global quotient stacks) involving the twisted complex K-theory of topological stacks.

11.2 Pushforward in twisted K-theory

It is shown in (33) that there exists a generalized pushforward map in twisted topological K-theory. In certain cases, for a proper map of spaces $f : X \to Y$, this gives rise to the following pushforward map in twisted K-theory:

$$f_* : KU_{f^*\alpha}(X) \to KU_\alpha(Y),$$

for $\alpha \in H^3(Y, \mathbb{Z})$. At the same time, if $f : X \to Y$ is a proper map of schemes, and Y an Azumaya algebra over Y, there exits a pushforward map $f_*^{cat} : \mathrm{Perf}(X, f^*A) \to \mathrm{Perf}(Y, A)$. We conjecture that this gives a categorification of the twisted pushforward map.

Conjecture 11.2.1. *Let X, Y be schemes over the complex numbers satisfying the conditions above. Then the map $f_* : KU_{f^*\alpha}(X) \to KU_\alpha(Y)$, is equivalent to $K^{top}(f_*^{cat}) : K^{top}(Perf(X, f^*A)) \to K^{top}(Perf(Y, A)$.*

This would result in an immediate proof of a twisted version of the following commutative diagram due to Atiyah-Hirzebruch in (38):

$$\begin{array}{ccc} K(X) & \longrightarrow & K(Y) \\ \downarrow & & \downarrow \\ KU(X(\mathbb{C})) & \longrightarrow & KU(Y(\mathbb{C})), \end{array}$$

displaying compatibility of topological and algebraic pushforwards on K-theory.

11.3 Structural results on the Topological K-theory of dg-categories

The author plans to investigate how topological K-theory of dg-categories interacts with the dg-category notions of *smoothness* and *properness* appearing in this thesis. Smooth and proper dg-categories occupy a central role in the setting of noncommutative (derived) algebraic geometry laid out by Kontsevich in (39) and generalize the classical notions of the smoothness and properness of a scheme.

Question 1. Let $T \in \mathrm{Cat}^{\mathrm{perf}}(\mathbb{C})$ be a smooth and proper dg category. Is its topological K-theory, $K^{top}(T)$, dualizable (hence compact) as a KU-module spectrum?

The analogue of this question in algebraic K-theory is false in general. It is known however to be true for Hochshild homology and periodic cyclic homology. This may be answerable for K^{top} since the identification $K^{top}(\mathrm{Perf}(X)) \simeq KU(X(\mathbb{C}))$ in (1, Theorem 1.1) gives rise to a Künneth formula for the topological K theory of dg categories of the form $\mathrm{Perf}(X)$ displaying the conjecture in such cases. I ask this question for relative topological K-theory.

Question 2. Let $T \in \mathrm{Cat}^{\mathrm{perf}}(X)$ be a smooth and proper $\mathrm{Perf}(X)$-linear dg-category. Is $K_X^{top}(T)$ dualizable as an object in $Shv_{Sp}(X(\mathbb{C}))$?

APPENDICES

Appendix A

INDEX OF NOTATION

$\mathrm{Cat}^{\mathrm{perf}}(X)$	The ∞-catgory of $\mathrm{Perf}(X)$-linear ∞-categories
$K^{cn}(-)$	Connective algebraic K-theory
$K(-)$	Nononnective algebraic K-theory
ku	Connective complex K-theory
$K^{\acute{e}t}(-)$	Étale sheafified K-theory
$K^{top}(-)$	Topological K-theory dg-categories
$K_X^{top}(-)$	Topological K-theory dg-categories, relative to a base scheme X.
$\mathrm{Perf}(X)$	The ∞-category of perfect complexes of $\mathcal{O}_X$-modules on X.
$\mathrm{Perf}(X, \alpha)$	The ∞-category of perfect complexes of α-twisted $\mathcal{O}_X$-modules on X.
$\mathrm{Pic}(\mathcal{C})$	The Picard ∞-groupoid of a symmetric monoidal ∞-category $\mathcal{C}$
$\mathcal{S}$	The ∞-category of spaces
Sp	The ∞-category of spectra

Appendix A (Continued)

$Shv(\mathcal{C})$	The ∞-category of sheaves of spaces on $\mathcal{C}$
$Shv_{Sp}(\mathcal{C})$	The ∞-category of sheaves of spectra on $\mathcal{C}$
SH_S	The ∞-category of motivic spectra over a scheme S.
$X(\mathbb{C})$	The topological space of complex points of a scheme X.

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