Problems Of Regularity In Models Arising From Fluid Dynamics.

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THESIS
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Chapter 1 is an introduction to incompressible fluid equations and questions of regularity. Chapter 2 is from a publication (A. Cheskidov and K. Zaya, Lower bounds of potential blow-up solutions of the three-dimensional Navier-Stokes equations in $\dot{H}^{3/2}$, J. Math. Phys., 57(2):023101, 7, 2016) that I co-authored with my advisor, Dr. Alexey Cheskidov. Chapter 3 is also from a publication (A. Cheskidov and K. Zaya, Regularizing effect of the forward energy cascade in the inviscid dyadic model, Proc. Amer. Math. Soc., 144(1):7385, 2016) that I co-authored with Dr. Alexey Cheskidov. Chapter 4 represents a paper I am the sole author of, which has been submitted for publication and has appeared on arXiv.org (K. Zaya, Regularity criterion for the three-dimensional Boussinesq equations, arXiv:1509.07434, 2015).

To my mother,

Lodia Zaya.

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KKZ

Summary

This work expands regularity results for equations related to fluid motion. First, we improve previously known lower bounds for Sobolev norms of potential blow-up solutions to the three-dimensional Navier-Stokes equations in the homogeneous Sobolev space $\dot{H}^{3/2}$. Next, we study the inviscid dyadic model of the Euler equations and prove some regularizing properties of the nonlinear term that occur due to forward energy cascade. We show every solution must have $\frac{3}{5}$ L^2-based (or $\frac{1}{10}$ L^3-based) regularity for all positive time. We conjecture this holds up to Onsager's scaling, where the L^2-based exponent is $\frac{5}{6}$ and the L^3-based exponent is $\frac{1}{3}$. Finally, we prove that a solution u to the three-dimensional Boussinesq equations does not blow-up at time T if $\|u_{\leq Q}\|_{B^1_{\infty,\infty}}$ is integrable on $(0,T)$, where $u_{\leq Q}$ represents the low modes of Littlewood-Paley projection of the velocity u.

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CHAPTER 1

Introduction

1. Incompressible Fluid Equations

The three-dimensional Euler equations describe the motion of an ideal inviscid fluid with velocity vector $u(x,t)$, density $\rho(x,t)$, and pressure $p(x,t)$, where x is the spatial variable in domain $\Omega = \mathbb{R}^3$ or $\Omega = \mathbb{T}^3$ and t is the time variable, $t \geq 0$. The Euler equations may be derived from Newton's second law of motion relating force F to mass m and acceleration a:

$$F = ma. \tag{1.1}$$

We apply (1.1) to V, an infinitesimal cube of fluid with volume $|V|$. The mass of the cube is given by

$$m = \rho(x,t)|V|. \tag{1.2}$$

Let $X = X(t)$ denote the position of the center of the fluid cube at time t. Then the velocity is given by

$$u(X(t),t) = \frac{\mathrm{d}}{\mathrm{d}t}X(t). \tag{1.3}$$

We use the velocity to find the acceleration a, which appears in the righthand-side of (1.1),

$$
\begin{aligned}
a(t) &= \frac{\mathrm{d}}{\mathrm{d}t} u(X(t), t) \\
&= \frac{\partial}{\partial t} u(X(t), t) + \nabla u(X(t), t) \cdot \frac{\mathrm{d}}{\mathrm{d}t} X(t) \\
&= \frac{\partial u}{\partial t} + \nabla u \cdot u \\
&= \frac{\partial u}{\partial t} + (u \cdot \nabla) u.
\end{aligned} \tag{1.4}
$$

Under the ideal fluid assumption, the force on the fluid cube comes from the normal force to the boundary of V and is due to the pressure:

$$F = -p(x, t)\mathbf{n}|A|, \tag{1.5}$$

where $\mathbf{n}$ is the outward normal vector to the boundary of the cube and $|A|$ is area of one side of the cube V. By the divergence theorem,

$$-\iint_{\partial V} p(x, t)\mathbf{n} \ \mathrm{d}S = -\iiint_V \nabla p \ \mathrm{d}V \tag{1.6}$$

Thus, pointwise, we have

$$F = -\nabla p. \tag{1.7}$$

One may derive the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \tag{1.8}$$

by conservation of mass. However, we will only consider fluids with constant density. Such fluids are referred to as incompressible. Without loss of generality by rescaling, assume

$$\rho \equiv 1, \tag{1.9}$$

which reduces (1.8) to

$$\nabla \cdot u = 0. \tag{1.10}$$

See [**29**] for a detailed derivation of the continuity equation for non-constant density $\rho(x,t)$. We insert (1.2), (1.4), and (1.7) into (1.1), divide by $|V|$, and couple the resulting equation with (1.10) to arrive at the three-dimensional incompressible Euler equations:

$$\begin{aligned} &\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p, \\ &\nabla \cdot u = 0. \end{aligned} \tag{1.11}$$

The first equation is referred to as the momentum equation and the second may be referred to as the incompressibility condition or the divergence-free condition.

For viscous flows, we must also consider the tangential component of the force, F_t, due to friction. We use Stokes theorem on a cube of fluid to model the total force due to friction and the divergence theorem to find

$$F_t = \iint_{\partial V} (\nabla u) \cdot \mathbf{n} \; \mathrm{d}S = \iiint_V \nabla \cdot (\nabla u) \; \mathrm{d}V = \iiint_V \Delta u \; \mathrm{d}V. \tag{1.12}$$

Thus, pointwise, the force due to friction is given by

$$F_t = \Delta u. \tag{1.13}$$

We add (1.13) to the righthand-side of the momentum equation in (1.11) to arrive at the three-dimensional incompressible Navier-Stokes equations:

$$\begin{aligned} &\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + \nu \Delta u, \\ &\nabla \cdot u = 0, \end{aligned} \tag{1.14}$$

where $\nu > 0$ denotes the kinematic viscosity of the fluid.

The three-dimensional incompressible Boussinesq equations account for the convective force of temperature on fluid flow with another term added to the righthand-side of the Navier-Stokes momentum equation and an additional evolution equation for the temperature $\theta(x,t)$:

$$\begin{aligned} &\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + \nu \Delta u + \theta e_3, \\ &\partial_t \theta + (u \cdot \nabla)\theta = k \Delta \theta, \\ &\nabla \cdot u = 0, \end{aligned} \tag{1.15}$$

where k is the thermal diffusivity coefficient and $e_3 = (0,0,1)^T$.

Other notable fluid equations are the magneto-hydrodynamics equations. The magneto-hydrodynamics equations incorporate the effects of magnetic fields on fluid flow to describe the motion of electrically charged fluids, such as liquid metals and plasmas. The magneto-hydrodynamics equations are given by:

$$\begin{aligned} &\frac{\partial u}{\partial t} + (u \cdot \nabla)u - (b \cdot \nabla)b = -\nabla p + \nu \Delta u, \\ &\frac{\partial b}{\partial t} + (u \cdot \nabla)b - (b \cdot \nabla)u = \mu \Delta b, \\ &\nabla \cdot u = 0, \quad \nabla \cdot b = 0, \end{aligned} \tag{1.16}$$

where $b = b(x,t)$ is the magnetic field and μ is the reciprocal of the magnetic Reynolds number.

2. Loss of Regularity and the Navier-Stokes Equations

The Navier-Stokes equations, known since the nineteenth century, are used for many applications in science and engineering, such as modeling weather and designing aircrafts. However, fundamental mathematical questions about solutions to the equations remain unanswered. One of the Clay Mathematics Institute Millennium Prize problems is on the existence and smoothness of the three-dimensional incompressible Navier-Stokes problem. The task is to determine given any smooth, divergence-free initial data $u(x,0) = u_0(t)$, if there exist smooth $p(x,t)$ and $u(x,t)$ that satisfy (1.14) for all time on the spatial domain Ω is $\mathbb{R}^3$ or $\mathbb{T}^3$ (or to negate this statement). This problem has proved to be exceedingly difficult to answer and remains completely open. One result of the challenge presented by this problem is the development of simplified toy models to explore the regularity and other properties of fluid equations. Shell models mimic energy cascade in turbulent flows. The dyadic model is an example of a shell model that has enjoyed progress and provided insight and intuition about the Navier-Stokes equations and the Euler equations. The inviscid dyadic model for the Euler equations is discussed in the next section and is the primary subject of the results in Chapter 3.

In Chapter 2, we focus on exploring properties of solutions to the Navier-Stokes equations that are presumed to lose regularity. A natural question to ask is: If we assume a smooth solution u loses regularity at time T^*, what is the rate of blow-up? In 1934, Leray [**40**] published his seminal work on the the fluid equations. He proved the existence of global weak solutions to (1.14) and proved that smooth solutions are unique in the class of Leray-Hopf solutions. He also showed that if $\|u(t)\|_{\dot{H}^1(\mathbb{R}^3)}$ is continuous on $[0, T^*)$ and blows up at time T^*, then

$$\|u(t)\|_{\dot{H}^1(\mathbb{R}^3)} \geq \frac{c}{(T^* - t)^{\frac{1}{4}}}. \tag{1.17}$$

After Leray proved (1.17), similar lower bounds were pursued and proven in $L^p(\mathbb{R}^3)$ for $3 < p < \infty$ [**32**], which extended the results to the Sobolev spaces $H^s(\mathbb{R}^3)$ for $\frac{1}{2} < s < \frac{3}{2}$ through Sobolev embedding. More recent work pushed progress on Sobolev norms for $s \geq \frac{3}{2}$ (a more detailed history and description is presented in Chapter 2). The main result of Chapter 2 is

THEOREM 1.1. *Let u be a smooth solution to* (1.14) *with finite energy initial data such that u loses regularity at time T^*. Then*

$$\|u\|_{\dot{H}^{3/2}(\Omega)} \geq \frac{c}{\sqrt{T^* - t}}, \tag{1.18}$$

for $0 \leq t < T^*$ *and* $\Omega = \mathbb{T}^3$ *or* $\Omega = \mathbb{R}^3$.

Such a bound was also presented in [**41**] and [**42**], which appeared shortly after Theorem 1.1 appeared in [**23**]. The method in Chapter 2 did not depend on rescaling arguments and thus works simultaneously for $\mathbb{R}^3$ or $\mathbb{T}^3$ via Littlewood-Paley decomposition. We stress the importance of the $H^{3/2}$ norm, which scales like the L^∞. No conclusion can be reached directly from Theorem 1.1 about the blow-up rate of the L^∞ norm, which is one of the most fundamental problems on regularity for the three-dimensional Navier-Stokes equations, but the connection between them is deep.

3. Regularity and the Inviscid Dyadic Model

The role of the nonlinear term is pivotal in the study of turbulent flows, which is another highly pursued, but still poorly understood topic in fluid dynamics. The basic principle proposed by Kolmogorov [**39**] behind turbulence is forward energy cascade. The theory asserts that energy moves from large to small scale structures called eddies, which can be roughly described as pockets of fluid with some uniting velocity structure. Energy moves without loss through the inertial range, which corresponds to low frequencies, until it reaches the dissipation range, which resides at high frequencies. Recent numerical and experimental data suggest deviation from Kolmogorov's original theory can be attributed to intermittency, which is when eddies do not occupy the whole space.

The study of turbulent flows remains quite difficult, but a great deal of insight can be found by studying shell models of the fluid equations. Shell models are designed to capture energy cascade in turbulent fluid flows. The sabra shell model (see [**25**] and [**26**]) and the dyadic model elicited a great deal of activity and insight recently. The dyadic model is a specific example where the nonlinearity is simplified to reflect just the local interactions between neighboring scales. It was initially introduced in 1974 by Desnianskii and Novikov [**28**] in the context of oceanography. The inviscid dyadic model shares two signature characteristics with the three-dimensional Euler equations: the formal conservation of energy and the scaling properties of the nonlinear term.

Kolmogorov predicted that energy cascade produces dissipation anomaly, which is characterized by the persistence of non-vanishing energy dissipation in the limit of vanishing viscosity. This phenomenon is possibly related to anomalous dissipation, which is failure of the energy to be conserved despite the absence of viscosity. Onsager conjectured that sufficiently rough solutions to Euler's equation can exhibit anomalous dissipation, but if the solution is smooth enough, then the energy should be conserved [**43**]. Anomalous dissipation and loss of regularity a priori seem unconnected, but a more discernible relationship exists in the context of the inviscid dyadic model and Onsager scaling. While results about the dyadic model can rarely be extended to answer questions about regularity of the actual fluid equations, turbulence, or Onsager's conjecture, they do give insight as to what mechanisms may be helping or impeding progress. The dyadic model functions as an illuminating testbed for methods and conjectures that may be more difficult to apply to the actual equations.

The total energy in the j^{th} shell is denoted by $a_j^2(t)$. Assuming only local interactions and extreme intermittency, one may model the flux through the j^{th} shell of radius λ_j as

$$\Pi_j = \lambda_j^{5/2} a_j^2 a_{j+1}, \tag{1.19}$$

leading to the inviscid system

$$a_j'(t) = \lambda_{j-1}^{5/2} a_{j-1}^2(t) - \lambda_j^{5/2} a_j(t) a_{j+1}(t), \qquad j = 1, 2, ... \tag{1.20}$$

with initial conditions $a_j(0) = a_j^0$ for $j = 0, 1, ...$ In Chapter 3, we prove

THEOREM 1.2. *For any positive solution to* (1.20) *with initial condition* $a(0)$ *in* l^2*,*

$$\sup_j \lambda_j^\theta a_j(t) < \infty, \tag{1.21}$$

for $t > 0$ *and* $\theta = 3/5$*.*

The approach used for Theorem 1.2 [**24**] was more dynamical, which made it possible to improve the previously known regularity results. The ultimate goal would be to show regularity for values of θ up to $5/6$, which corresponds to Onsager's scaling.

4. Regularity Criteria and the Boussinesq Equations

The Boussinesq equations are important in the study of atmospheric sciences and they yield a wealth of interesting and difficult problems from a mathematical perspective. The regularity of (1.15) has been studied thoroughly on its own as well as in relation to the regularity of other systems of equations, such as the Navier-Stokes equations, Euler equations, and magneto-hydrodynamics equations. In three-dimensions, regularity criteria for (1.15) have been developed in many cases using many different methods. A more detailed survey of previous regularity results is contained in Chapter 4. The main result of Chapter 4 is the following regularity criterion for the three-dimensional Boussinesq equations:

THEOREM 1.3. *Let* $\big(u, \theta\big)$ *be a weak solution to* (1.15) *on* $[0, T]$*, assume* $\big(u, \theta\big)$ *is regular on* $(0, T)$*, and*

$$\|u_{\leq Q}\|_{B^1_{\infty,\infty}} \in L^1(0, T). \tag{1.22}$$

Then $\big(u(t), \theta(t)\big)$ *is regular on* $(0, T]$*.*

The key tool in the development of this regularity criterion is linked to the dissipation wave number $\Lambda(t)$, a natural tool to study dissipative equations, corresponding to the work of Kolmogorov [**39**]. Recent work utilizing Littlewood-Paley decomposition, the dissipation wave number, and determining modes have already provided key improvements to previously known regularity results for the surface quasi-geostrophic equations, the magneto-hydrodynamics equations, and the Navier-Stokes equations (see [**14**], [**15**], [**16**], and [**21**]) that used classical techniques.

5. Littlewood-Paley Decomposition

Fourier analysis methods, in particular Littlewood-Paley decomposition, applied to the Navier-Stokes and related equations have been a natural and productive fit. Joseph Fourier was reportedly very influential on Claude-Louis Navier, who in turn promoted the mathematical techniques developed his friend and teacher. Tremendous progress was achieved in the last decade by studying the above fluid equations with harmonic analysis tools. The methods used in the subsequent chapters center on Littlewood-Paley decomposition, which was introduced by Littlewood and Paley in the 1930s. We refer the reader to [**8**] for more on the history between Fourier and Navier and more information on the stages of the development of Fourier analysis and Littlewood-Paley applications to fluid equations.

Denote the wave numbers as $\lambda_q = 2^q$ (in some wave units). For $\psi \in C^\infty(\mathbb{R}^3)$, define

$$\psi(\xi) = \begin{cases} 1 & : \quad |\xi| \leq \frac{3}{4} \\ 0 & : \quad |\xi| \geq 1. \end{cases}$$

Next define $\phi(\xi) = \psi(\xi/\lambda_1) - \psi(\xi)$ and $\phi_q(\xi) = \phi(\xi/\lambda_q)$ for all q, and $\phi_{-1} = \psi(\xi)$. The ϕ_q's form a partition of unity. Then

$$u = \sum_{q=-\infty}^{\infty} u_q, \tag{1.23}$$

in the sense of distributions, where the u_q is the q^{th} Littlewood-Paley piece of u. On $\mathbb{R}^3$, the Littlewood-Paley pieces are defined as

$$u_q(x) = \int_{\mathbb{R}^3} u(x-y)\mathcal{F}^{-1}(\phi_q)(y)\,\mathrm{d}y, \tag{1.24}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform. On $\mathbb{T}^3$, the Littlewood-Paley pieces are given by

$$u_q(x) = \sum_{k\in\mathbb{Z}^3} \hat{u}(k)\phi_q(k)\,\mathrm{e}^{ik\cdot x}, \tag{1.25}$$

where (1.23) holds provided u has zero-mean. Moreover, $u_q = 0$ in the periodic case when $q < 0$.

We will use the notation

$$u_{\leq Q} = \sum_{q\leq Q} u_q, \tag{1.26}$$

$$u_{\geq Q} = \sum_{q\geq Q} u_q, \tag{1.27}$$

and

$$\tilde{u}_q = u_{q-1} + u_q + u_{q+1}. \tag{1.28}$$

Homogeneous Sobolev spaces are denoted by $\dot{H}^s$, for which the norm will be defined via Littlewood-Paley decomposition.

DEFINITION 1.4. *The homogeneous Sobolev norm of a function u is given by*

$$\|u\|_{\dot{H}^s} = \Big(\sum_{q=-\infty}^{\infty} \lambda_q^{2s}\,\|u_q\|_{L^2}^2\Big)^{\frac{1}{2}}. \tag{1.29}$$

Note that it corresponds to the nonhomogeneous Sobolev norm H^s in the periodic case. We also use Littlewood-Paley decomposition to define the homogeneous Besov norm:

DEFINITION 1.5. *The homogeneous Besov norm of a function u is given by*

$$\|u\|_{\dot{B}^s_{p,r}} = \left\| \left(\lambda_q^s \, \|u_q\|_{L^p} \right)_{q\in\mathbb{Z}} \right\|_{l^r}, \tag{1.30}$$

for $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$.

A function u belongs in one of those spaces if the associated norm in that space is finite.

6. Notation

The symbol $\lesssim$ (or $\gtrsim$) denotes that an inequality holds up to a constant:

$$A \lesssim B \Leftrightarrow A \leq cB, \tag{1.31}$$

where c is an absolute constant. Further, we will suppress notation for L^p norms as

$$\| \cdot \|_p := \| \cdot \|_{L^p(\Omega)}, \tag{1.32}$$

where Ω is the spatial domain specified per section.

CHAPTER 2

Lower Bounds of Potential Blow-Up Solutions of the Three-dimensional Navier-Stokes Equations in Sobolev Spaces

1. Background

In this chapter we focus on the three-dimensional incompressible Navier-Stokes equations and bound the blow-up rates of Sobolev norms of solutions that are assumed to lose regularity in finite time. The contents of this chapter were previously published as A. Cheskidov and K. Zaya, Lower bounds of potential blow-up solutions of the three-dimensional Navier-Stokes equations in $\dot{H}^{3/2}$, J. Math. Phys., 57(2), 7, 2016 (see [**23**]). First, recall the Navier-Stokes equations from Chapter 1:

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla p + \nu \Delta u, \\
&\nabla \cdot u = 0, \\
&u(x,0) = u_0(x),
\end{aligned}
\tag{2.1}
$$

where the velocity $u(x,t)$ and the pressure $p(x,t)$ are unknowns, $\nu > 0$ is the kinematic viscosity coefficient, the initial data $u_0(\cdot) \in L^2(\Omega)$, and the spatial domain $\Omega = \mathbb{T}^3$ or $\Omega = \mathbb{R}^3$.

In 1934, Leray [**40**] showed that if $\|u(t)\|_{H^1}$ is continuous on $[0, T^*)$ and blows up at time T^*, then

$$
\|u(t)\|_{\dot{H}^1(\mathbb{R}^3)} \geq \frac{c}{(T^* - t)^{\frac{1}{4}}}. \tag{2.2}
$$

Moreover, the bound for L^p norms for $3 < p < \infty$,

$$\|u(t)\|_{L^p(\mathbb{R}^3)} \geq \frac{c_p}{(T^* - t)^{\frac{p-3}{2p}}}, \tag{2.3}$$

have been known for a long time (see [**40**] and [**32**]). The Sobolev embedding

$$\dot{H}^s(\mathbb{R}^3) \subset L^{\frac{6}{3-2s}}(\mathbb{R}^3) \tag{2.4}$$

and (2.3) yield that

$$\|u(t)\|_{\dot{H}^s(\Omega)} \geq \frac{c}{(T^* - t)^{\frac{2s-1}{4}}}, \tag{2.5}$$

for $\frac{1}{2} < s < \frac{3}{2}$ and $\Omega = \mathbb{R}^3$. Robinson, Sadowski, and Silva extended (2.5) in [**48**] for $\frac{3}{2} < s < \frac{5}{2}$ for the whole space and in the presence of periodic boundary conditions. This bound is considered optimal for those values of s.

When $s > \frac{5}{2}$, Benameur [**6**] showed

$$\|u(t)\|_{\dot{H}^s(\mathbb{R}^3)} \geq \frac{c(s)\|u(t)\|_{L^2(\mathbb{R}^3)}^{\frac{3-2s}{3}}}{(T^* - t)^{\frac{s}{3}}}, \tag{2.6}$$

which was improved upon by Robinson, Sadowski, and Silva in [**48**] to

$$\|u(t)\|_{\dot{H}^s(\Omega)} \geq \frac{c(s)\|u_0\|_{L^2(\Omega)}^{\frac{5-2s}{5}}}{(T^* - t)^{\frac{2s}{5}}}, \tag{2.7}$$

when $\Omega = \mathbb{T}^3$ or $\Omega = \mathbb{R}^3$.

The border cases $s = \frac{3}{2}$ and $s = \frac{5}{2}$ required separate treatment. For $s = \frac{3}{2}$, Robinson, Sadowski, and Silva had an epsilon correction to the lower bound. In [**27**], Cortissoz, Montero, and Pinilla improved the bound for $s = \frac{3}{2}$ on $\mathbb{T}^3$, but they had a logarithmic correction:

$$\|u(t)\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^3)} \geq \frac{c}{\sqrt{(T^* - t)|\log(T^* - t)|}}. \tag{2.8}$$

For $s = \frac{5}{2}$, Cortissoz, Montero, and Pinilla [**27**] also found

$$\|u(t)\|_{\dot{H}^{\frac{5}{2}}(\Omega)} \geq \frac{c}{(T^* - t)|\log(T^* - t)|}, \tag{2.9}$$

when $\Omega = \mathbb{T}^3$ or $\Omega = \mathbb{R}^3$. In [**41**], the authors proved

$$\limsup_{t \to T^{*-}} (T^* - t)\|u(t)\|_{\dot{H}^{5/2}(\Omega)} \geq c. \tag{2.10}$$

The result of this chapter improves the bound for the $\dot{H}^{\frac{3}{2}}(\Omega)$-norm to the optimal bound (2.5) when $\Omega = \mathbb{R}^3$ or $\Omega = \mathbb{T}^3$. The method is not contingent on rescaling arguments and thus works simultaneously for $\mathbb{R}^3$ and $\mathbb{T}^3$ and differs from previous works because Littlewood-Paley decomposition is employed. The importance of the $H^{3/2}$ norm must be stressed because it scales to the L^∞ norm and corresponds to the uncovered limit of (2.5). The $H^{5/2}$ norm is also significant because $H^{5/2}$ is a critical space for the Euler equations and scales like $B^1_{\infty,\infty}$, the Beale-Kato-Majda space. Furthermore, the persistence of the logarithmic correction in estimate (2.9) is consistent with the recent result of Bourgain and Li [**7**] on the ill-posedness of the Euler equations in $H^{5/2}$.

REMARK 2.1. *The lower bound for the $\dot{H}^{\frac{3}{2}}$-norm of blow-up solutions was also presented in papers by Montero* [**42**] *and McCormick, Olson, Robinson, Rodrigo, Vidal-Lopez, and Zhou* [**41**], *which both appeared shortly after the results presented here.*

2. Bounding Blow-Up

We begin by testing the weak formulation of the Navier-Stokes equation with $\lambda_q^{2s}(u_q)_q$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda_q^{2s}\|u_q\|_2^2\right) = -\nu\,\lambda_q^{2s+2}\|u_q\|_2^2 + 2\,\lambda_q^{2s}\int \mathrm{Tr}[(u\otimes u)_q \cdot \nabla u_q]\ \mathrm{d}x. \tag{2.11}$$

In the typical fashion, we write

$$(u \otimes u)_q = u_q \otimes u + u \otimes u_q + r_q(u,u), \tag{2.12}$$

for $q > -1$, where the remainder function is given by

$$r_q(u,u)(x) = \int \mathcal{F}^{-1}(\phi_q)(y)(u(x-y) - u(x)) \otimes (u(x-y) - u(x))\ \mathrm{d}y. \tag{2.13}$$

Thus, we rewrite the nonlinear term as

$$\int \mathrm{Tr}[(u \otimes u)_q \cdot \nabla u_q]\,\mathrm{d}x = \int r_q(u,u) \cdot \nabla u_q\,\mathrm{d}x - \int u_q \cdot \nabla u_{\leq q+1} \cdot u_q\,\mathrm{d}x. \tag{2.14}$$

LEMMA 2.2. *The integral* (2.14) *corresponding to the nonlinear term in* (2.11) *is bounded above by*

$$\begin{aligned} \int \mathrm{Tr}[(u \otimes u)_q \cdot \nabla u_q]\ \mathrm{d}x \lesssim{}& \lambda_q^{-1} \|u_q\|_2 \sum_{p=-\infty}^{q} \lambda_p^2 \|u_p\|_4^2 \\ &+ \lambda_q \|u_q\|_2 \sum_{p=q+1}^{\infty} \|u_p\|_4^2 \\ &+ \|u_q\|_2^2 \sum_{p=-\infty}^{q+1} \lambda_p^{\frac{5}{2}} \|u_p\|_2. \end{aligned} \tag{2.15}$$

PROOF. We examine the two integrals on the right-hand side of (2.14) separately. By Hölder's inequality,

$$\int r_q(u,u) \cdot \nabla u_q\,\mathrm{d}x \lesssim \|r_q(u,u)\|_2\,\lambda_q\,\|u_q\|_2. \tag{2.16}$$

We use Littlewood-Paley decomposition and split the sum into low versus high modes to find

$$
\begin{aligned}
\|r_q(u,u)\|_2 &\lesssim \int_{\mathbb{R}^3} |\,\mathcal{F}^{-1}(\phi_q)(y)|\ \|u(x-y)-u(x)\|_4^2 \,\mathrm{d}y \\
&\lesssim \int_{\mathbb{R}^3} |\,\mathcal{F}^{-1}(\phi_q)(y)| \sum_{p=-\infty}^{q} \|(u(x-y)-u(x))_p\|_4^2 \,\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^3} |\,\mathcal{F}^{-1}(\phi_q)(y)| \sum_{p=q+1}^{\infty} \|(u(x-y)-u(x))_p\|_4^2 \,\mathrm{d}y\,.
\end{aligned}
\tag{2.17}
$$

We apply the Mean-Value Theorem on the low modes and the triangle inequality on the high modes to arrive at

$$
\begin{aligned}
\|r_q(u,u)\|_2 &\lesssim \int_{\mathbb{R}^3} |\,\mathcal{F}^{-1}(\phi_q)(y)|\ |y|^2 \sum_{p=-\infty}^{q} \|\nabla u_p\|_4^2 \,\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^3} |\,\mathcal{F}^{-1}(\phi_q)(y)| \sum_{p=q+1}^{\infty} \|u_p\|_4 \,\mathrm{d}y \\
&\lesssim \lambda_q^{-2} \sum_{p=-\infty}^{q} \lambda_p^2\, \|u_p\|_4^2 + \sum_{p=q+1}^{\infty} \|u_p\|_4
\end{aligned}
\tag{2.18}
$$

Thus,

$$
\begin{aligned}
\int r_q(u,u)\cdot \nabla u_q \,\mathrm{d}x &\lesssim \lambda_q^{-1}\, \|u_q\|_2 \sum_{p=-\infty}^{q} \lambda_p^2\, \|u_p\|_4^2 \\
&\quad + \lambda_q\, \|u_q\|_2 \sum_{p=q+1}^{\infty} \|u_p\|_4^2.
\end{aligned}
\tag{2.19}
$$

For the second term of (2.14), we use a similar process as above in addition to Bernstein's inequality in three dimensions, which says, for $1 \le p \le q$,

$$
\|u_j\|_q \lesssim \lambda_j^{3(\frac{1}{p}-\frac{1}{q})}\, \|u_j\|_p, \tag{2.20}
$$

to find

$$\int u_q \cdot \nabla u_{\le q+1} \cdot u_q \,\mathrm{d}x \lesssim \|u_q\|_2^2 \sum_{p=-\infty}^{q+1} \lambda_p \|u_p\|_\infty \tag{2.21}$$

$$\lesssim \|u_q\|_2^2 \sum_{p=-\infty}^{q+1} \lambda_p^{5/2} \|u_p\|_2.$$

Combining (2.19) and (2.21) yields the desired bound (2.15). □

Similar estimates were executed in [**13**] and [**22**]. We apply the bound obtained in Lemma 2.2 to write

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{q=-\infty}^{\infty} \left(\lambda_q^{2s} \|u_q\|_2^2 \right) \lesssim - \sum_{q=-\infty}^{\infty} \left(\nu\, \lambda_q^{2s+2} \|u_q\|_2^2 \right) + 2\big(A + B + C\big), \tag{2.22}$$

where

$$A = \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q} \lambda_q^{2s-1} \|u_q\|_2\, \lambda_p^2 \|u_p\|_4^2, \tag{2.23}$$

$$B = \sum_{q=-\infty}^{\infty} \sum_{p=q+1}^{\infty} \lambda_q^{2s+1} \|u_q\|_2 \|u_p\|_4^2, \tag{2.24}$$

$$C = \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q+1} \lambda_q^{2s} \|u_q\|_2^2\, \lambda_p^{5/2} \|u_p\|_2. \tag{2.25}$$

THEOREM 2.3. *Let u be a solution to* (2.1) *with finite energy initial data. Then for $s = \frac{3}{2}$, the solution u satisfies the Riccati-type differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{q=-\infty}^{\infty} \left(\lambda_q^3 \|u_q\|_2^2 \right) \lesssim \sum_{q=-\infty}^{\infty} \left(\lambda_q^3 \|u_q\|_2^2 \right)^2 \tag{2.26}$$

PROOF. We bound the nonlinear terms. First, we estimate (2.23) for $s = \frac{3}{2}$. We apply Bernstein's inequality in three-dimensions and we rewrite the sum

$$
\begin{aligned}
A &= \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q} \lambda_q^2 \, \|u_q\|_2 \, \lambda_p^2 \, \|u_p\|_4^2 \\
&\lesssim \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q} \lambda_q^2 \, \|u_q\|_2 \, \lambda_p^{7/2} \, \|u_p\|_2^2 \\
&= \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q} \lambda_{q-p}^{-1/2} \big(\lambda_q^{5/2} \, \|u_q\|_2 \big) \big(\lambda_p^3 \, \|u_p\|_2^2 \big).
\end{aligned} \tag{2.27}
$$

We apply the Cauchy-Schwartz inequality to yield

$$
A \lesssim \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q} \lambda_{q-p}^{-1/2} \left(\frac{\nu}{3} \lambda_q^5 \, \|u_q\|_2^2 \right) + \lambda_{q-p}^{-1/2} \left(\nu^{-1} \, \lambda_p^3 \, \|u_p\|_2^2 \right)^2. \tag{2.28}
$$

Next we sum in p for the first term and exchange the order of summation and sum in q for the second term of (2.28):

$$
A \lesssim \sum_{q=-\infty}^{\infty} \left(\nu^{-1} \, \lambda_q^3 \, \|u_q\|_2^2 \right)^2 + \frac{\nu}{3} \sum_{q=-\infty}^{\infty} \left(\lambda_q^5 \, \|u_q\|_2^2 \right). \tag{2.29}
$$

To estimate (2.24) when $s = \frac{3}{2}$, first we apply Bernstein's inequality for three-dimensions to find

$$
\begin{aligned}
B &= \sum_{q=-\infty}^{\infty} \sum_{p=q+1}^{\infty} \lambda_q^4 \, \|u_q\|_2 \|u_p\|_4^2 \\
&\lesssim \sum_{q=-\infty}^{\infty} \sum_{p=q+1}^{\infty} \lambda_q^4 \, \|u_q\|_2 \, \lambda_p^{3/2} \, \|u_p\|_2^2.
\end{aligned} \tag{2.30}
$$

We rewrite the sum to look like

$$
B \lesssim \sum_{q=-\infty}^{\infty} \sum_{p=q+1}^{\infty} \lambda_{p-q}^{-5/2} \big(\lambda_q^{3/2} \, \|u_q\|_2 \big) \big(\lambda_p^{3/2} \, \|u_p\|_2 \big) \big(\lambda_p^{5/2} \, \|u_p\|_2 \big). \tag{2.31}
$$

We apply Young's inequality with the exponents $\theta_1 = \theta_2 = 4$ and $\theta_3 = 2$ to yield

$$
\begin{aligned}
B \lesssim & \sum_{q=-\infty}^{\infty} \sum_{p=q+1}^{\infty} \lambda_{p-q}^{-5/2} \left(\nu^{-1}\, \lambda_q^3 \, \|u_q\|_2^2\right)^2 \\
& + \sum_{q=-\infty}^{\infty} \sum_{p=q+1}^{\infty} \lambda_{p-q}^{-5/2} \left(\nu^{-1}\, \lambda_p^3 \, \|u_p\|_2^2\right)^2 \\
& + \sum_{q=-\infty}^{\infty} \sum_{p=q+1}^{\infty} \lambda_{p-q}^{-5/2} \left(\frac{\nu}{3}\, \lambda_p^5 \, \|u_p\|_2^2\right).
\end{aligned} \tag{2.32}
$$

Next we sum in p for the first term and exchange the order of summation and sum in q for the second and third terms of (2.32). Note the summation in q converges:

$$
B \lesssim \sum_{q=-\infty}^{\infty} \left(\nu^{-1}\, \lambda_q^3 \, \|u_q\|_2^2\right)^2 + \sum_{p=-\infty}^{\infty} \left[\left(\nu^{-1}\, \lambda_p^3 \, \|u_p\|_2^2\right)^2 + \left(\frac{\nu}{3}\, \lambda_p^5 \, \|u_p\|_2^2\right)\right]. \tag{2.33}
$$

Thus we arrive at the bound

$$
B \lesssim \sum_{q=-\infty}^{\infty} \left(\nu^{-1}\, \lambda_q^3 \, \|u_q\|_2^2\right)^2 + \frac{\nu}{3} \sum_{q=-\infty}^{\infty} \left(\lambda_q^5 \, \|u_q\|_2^2\right). \tag{2.34}
$$

Finally, we estimate (2.25) for $s = \frac{3}{2}$. We rewrite the sum

(2.35)

$$
\begin{aligned}
C &= \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q+1} \lambda_q^3 \, \|u_q\|_2^2 \, \lambda_p^{5/2} \, \|u_p\|_2^2 \\
&= \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q+1} \lambda_{q-p}^{-\delta} \left(\lambda_q^{3/2} \, \|u_q\|_2\right)^{2-\delta} \left(\lambda_q^{5/2} \, \|u_q\|_2^2\right)^{\delta} \left(\lambda_p^{3/2} \, \|u_p\|_2\right)^{\delta} \left(\lambda_p^{5/2} \, \|u_p\|_2\right)^{1-\delta},
\end{aligned}
$$

where δ is a small positive number we can choose. We apply Young's inequality with

$$
\theta_1 = \frac{4}{2-\delta}, \quad \theta_2 = \frac{2}{\delta}, \quad \theta_3 = \frac{4}{\delta}, \quad \theta_4 = \frac{2}{1-\delta}, \tag{2.36}
$$

where we require $\delta < 1$ to ensure the exponents are all positive and indeed $\frac{1}{\theta_1} + \frac{1}{\theta_2} + \frac{1}{\theta_3} + \frac{1}{\theta_4} = 1$. Then we have

$$\begin{aligned} C \lesssim & \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q+1} \left[\lambda_{q-p}^{-\delta} \left(\nu^{-1} \lambda_q^3 \|u_q\|_2^2 \right)^2 + \lambda_{q-p}^{-\delta} \left(\frac{\nu}{6} \lambda_q^5 \|u_q\|_2^2 \right) \right] \\ & + \sum_{q=-\infty}^{\infty} \sum_{p=-\infty}^{q+1} \left[\lambda_{q-p}^{-\delta} \left(\nu^{-1} \lambda_p^3 \|u_p\|_2^2 \right)^2 + \lambda_{q-p}^{-\delta} \left(\frac{\nu}{6} \lambda_p^5 \|u_p\|_2^2 \right) \right], \end{aligned} \tag{2.37}$$

For the first two terms of (2.37), we sum in p. For the third and fourth terms, we exchange the order of summation and sum in q to arrive at

$$\begin{aligned} C \lesssim & \sum_{q=-\infty}^{\infty} \left[\left(\nu^{-1} \lambda_q^3 \|u_q\|_2^2 \right)^2 + \left(\frac{\nu}{6} \lambda_q^5 \|u_q\|_2^2 \right) \right] \\ & + \sum_{p=-\infty}^{\infty} \left[\left(\nu^{-1} \lambda_p^3 \|u_p\|_2^2 \right)^2 + \left(\frac{\nu}{6} \lambda_p^5 \|u_p\|_2^2 \right) \right]. \end{aligned} \tag{2.38}$$

Note δ positive ensures the summation in q converges. Rewriting the above inequality yields

$$C \lesssim \sum_{q=-\infty}^{\infty} \left(\nu^{-1} \lambda_q^3 \|u_q\|_2^2 \right)^2 + \frac{\nu}{3} \sum_{q=-\infty}^{\infty} \left(\lambda_q^5 \|u_q\|_2^2 \right). \tag{2.39}$$

We use the estimates (2.29), (2.34), and (2.39) in (2.22) with $s = \frac{3}{2}$ to get the Ricatti-type differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{q=-\infty}^{\infty} \left(\lambda_q^3 \|u_q\|_2^2 \right) \lesssim \sum_{q=-\infty}^{\infty} \left(\nu^{-1} \lambda_q^3 \|u_q\|_2^2 \right)^2. \tag{2.40}$$

$\square$

REMARK 2.4. *The method used to prove Theorem 2.3 works for* $\frac{1}{2} < s < \frac{5}{2}$. *Instead of* (2.26), *one must show*

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{q=-\infty}^{\infty} \left(\lambda_q^{2s} \|u_q\|_2^2 \right) \lesssim \sum_{q=-\infty}^{\infty} \left(\lambda_q^{2s} \|u_q\|_2^2 \right)^{\frac{2s+1}{2s-1}}. \tag{2.41}$$

In the proof for (2.41)*, one must treat the three cases* $\frac{1}{2} < s < \frac{3}{2}$*,* $s = \frac{3}{2}$*, and* $\frac{3}{2} < s < \frac{5}{2}$ *separately, but in analogous manners.*

THEOREM 2.5. *Let u be a smooth solution to* (2.1) *with finite energy initial data such that u loses regularity at time T^*. Then*

$$\|u(t)\|_{\dot{H}^{3/2}(\Omega)} \geq \frac{c}{\sqrt{T^* - t}}, \tag{2.42}$$

for $0 \leq t < T^*$ *and* $\Omega = \mathbb{T}^3$ *or* $\Omega = \mathbb{R}^3$*.*

PROOF. Let $y(t) = \|u(t)\|^2_{\dot{H}^{3/2}}$. By Theorem 2.3, y satisfies the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) \lesssim y(t)^2. \tag{2.43}$$

Rearranging the inequality and integrating from time t to blow-up time T^* yields

$$\int_{y(t)}^{\infty} \frac{\mathrm{d}w}{w^2} \lesssim \int_t^{T^*} \mathrm{d}\tau, \tag{2.44}$$

which becomes

$$\frac{1}{y(t)} \lesssim T^* - t. \tag{2.45}$$

Then, as desired

$$\|u(t)\|_{\dot{H}^{3/2}(\Omega)} \geq \frac{c}{\sqrt{T^* - t}}, \tag{2.46}$$

for $0 \leq t < T^*$ and $\Omega = \mathbb{T}^3$ or $\Omega = \mathbb{R}^3$. $\square$

REMARK 2.6. *The procedure in Theorem 2.5 can be applied to* (2.41) *for* $y(t) = \|u(t)\|^2_{\dot{H}^s(\Omega)}$ *to yield*

$$\|u(t)\|_{\dot{H}^s(\Omega)} \geq \frac{c}{(T^* - t)^{\frac{2s-1}{4}}}, \tag{2.47}$$

for $\frac{1}{2} < s < \frac{5}{2}$*,* $0 \leq t < T^*$*, and* $\Omega = \mathbb{R}^3$ *or* $\Omega = \mathbb{T}^3$*.*

CHAPTER 3

Regularizing Effect of the Forward Energy Cascade in the Inviscid Dyadic Model

1. Introduction to Shell Models

In this chapter, we study the regularizing effect of the forward energy cascade in the inviscid dyadic model of the Euler equations, which we recall from Chapter 1:

$$\begin{aligned} &\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p, \\ &\nabla \cdot u = 0, \end{aligned} \tag{3.1}$$

where the velocity vector field, $u(x,t)$, and the pressure scalar, $p(x,t)$, are unknowns. Regularity of the three-dimensional Euler equations is a compelling question as well. The inviscid dyadic model shares important characteristics with the three-dimensional Euler equations, namely formal conservation of energy and the scaling properties of the nonlinear term. The contents of this chapter were published as A. Cheskidov and K. Zaya, Regularizing effect of the forward energy cascade in the inviscid dyadic model, Proc. Amer. Math. Soc., 144(1):7385, 2016 (see [**24**]).

Kolmogorov's [**39**] theory about turbulence centered around forward energy cascade, which asserts that energy moves from large to small scales. Small scales correspond to the the dissipation range, where the viscous forces dominate. For the Navier-Stokes equations the dissipation range is the only tool used to prove regularity of solutions, but the forward energy cascade might also be a mechanism to regularize solutions. For quasilinear scalar equations, the regularizing property of the nonlinear term has been studied by Tadmor and Tao in [**49**], but such results remain out of reach for the Euler or Navier-Stokes equations. Recently, Tao [**51**] proved blow-up for averaged Navier-Stokes equations by reducing the equations to a more complicated dyadic model where he introduced a delay in energy cascade. This delay seems to destroy the regularizing effect of the nonlinear term studied here and produces a strong blow-up.

Shell models are designed to capture energy cascade in turbulent fluid flows. The dyadic model is a specific example where the nonlinearity is simplified to reflect just the local interactions between neighboring scales. Although initially introduced in 1974 by Desnianskii and Novikov [**28**], other derivations have been since developed. We refer the reader to Chapter 1, Section 5 and [**17**] for explanation via Littlewood-Paley decomposition. Recent mathematical analysis has more recently led to several other results in the last decade, see for example [**2**], [**4**], [**12**], [**31**], [**37**], and [**38**].

The inviscid dyadic model is an infinite system of nonlinearly coupled ordinary differential equations constructed to mimic the behavior of the energy of solutions to the Euler equations in dyadic shells. In [**13**], Cheskidov, Constantin, Friedlander, and Shvydkoy examined the energy flux Π_j due to the nonlinearity in the Euler equations through the shell of radius $\lambda_j = 2^j$ and obtained the bound

$$|\Pi_j| \lesssim \sum_{i=-1}^{\infty} \lambda_{|j-i|}^{-\frac{2}{3}} \lambda_i \, \|u_i\|_3^3, \tag{3.2}$$

where u_i is a i^{th} Littlewood-Paley piece of u. Recall Bernstein's inequality in three dimensions, which says

$$\|u_j\|_q \lesssim \lambda_j^{3(\frac{1}{p}-\frac{1}{q})} \|u_j\|_p, \tag{3.3}$$

for $1 \le p \le q$. We assume $\|u_j\|_3 \sim \lambda_j^{\beta} \|u_j\|_2$ where $\beta \in [0, \frac{1}{2}]$ is the intermittency parameter. Kolmogorov's regime corresponds to $\beta = 0$, whereas $\beta = \frac{1}{2}$ gives extreme intermittency. Denote the total energy in the j^{th} shell by $a_j^2(t)$. As in [**17**], assuming only local interactions and extreme intermittency, we model the flux through the j^{th}

shell of radius λ_j as $\Pi_j = \lambda_j^{\frac{5}{2}} a_j^2 a_{j+1}$. This leads to the following inviscid system

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t} a_j(t) = \lambda_{j-1}^{\frac{5}{2}} a_{j-1}^2(t) - \lambda_j^{\frac{5}{2}} a_j(t) a_{j+1}(t), \qquad j = 1, 2, ...\\ &a_0(t) = 0,\end{aligned} \tag{3.4}$$

with initial conditions $a_j(0) = a_j^0$ for $j = 0, 1, ...$

As discussed in Chapter 1, Kolmogorov predicted that energy cascade produces dissipation anomaly, which is possibly related to (but not to be confused with) anomalous dissipation. Dissipation anomaly is the persistence of non-vanishing energy dissipation in the limit of vanishing viscosity, whereas anomalous dissipation is when energy is not conserved despite the absence of viscosity. Onsager [**43**] conjectured that if a solution to Euler's equations is smooth enough, then the energy should be conserved, however, rough solutions to Euler's equation may exhibit anomalous dissipation. A relationship between anomalous dissipation and loss of regularity is more evident in the context of the inviscid dyadic model. The regularity of solutions is related to the natural scaling of the equations and for the dyadic model we suspect that the natural space for regularity is the Onsager space. Despite the absence of viscosity, in the inviscid dyadic model, a solution with rough initial data immediately gains regularity. This is due to the forward energy cascade and the smoothing properties of the nonlinear term.

In [**18**] and [**19**], Cheskidov, Friedlander, and Pavlović showed that all the solutions of the forced inviscid dyadic model must have Onsager's regularity almost everywhere in time and confirmed anomalous dissipation and dissipation anomaly. They also showed that all solutions blow up in finite time in $H^{\frac{5}{6}}$. On the other hand, all solutions are in H^θ for almost all time for $\theta < \frac{5}{6}$. In [**3**], Barbato and Morandin studied the unforced inviscid model and showed Onsager regularity almost everywhere, as well. In addition, they demonstrated that solutions remain in $H^{\frac{1}{2}-}$ for all time. We improve their result by showing that regularity even closer to Onsager's is retained:

THEOREM 3.1. *For any positive solution to* (3.4) *with initial condition* $a(0)$ *in* l^2,

$$\sup_j \lambda_j^{\theta} a_j(t) < \infty \tag{3.5}$$

for $t > 0$ *and* $\theta = \frac{3}{5}$.

Barbato and Morandin proved the theorem for $\theta = \frac{1}{2}$ by finding an invariant region for solutions. The method presented below is different as we use a more dynamical approach which allows us to improve regularity for values of θ up to $\frac{3}{5}$. It is natural to conjecture that every solution must have exactly Onsager's regularity for all positive time and the ultimate goal would be to show regularity for values of θ up to $\frac{5}{6}$ (Onsager's scaling).

REMARK 3.2. *As a comparison to* L^3*-based regularity, our result* (3.5) *can be expressed as*

$$\sup_j \lambda_j^{q+\beta} a_j(t) < \infty \tag{3.6}$$

for $q = \frac{1}{10}$. *The ultimate Onsager scaling is* $q = \frac{1}{3}$.

2. Energy Conservation and Onsager's Conjecture

In this chapter, we will denote the energy norm simply by $|\cdot| := \|\cdot\|_{l^2}$. A solution $a(t)$ is called positive if $a_j(t) \geq 0$ for all $j \in \mathbb{N}$ and all time t. In [**19**] and [**3**], the authors proved solutions with positive initial data $a_j(0)$ remain positive for all time $t > 0$. Moreover in [**3**], Barbato and Morandin proved uniqueness for positive initial data. Thus we have

THEOREM 3.3. *Let* $a(t)$ *be a solution to* (3.4) *such that* $a_j(0) \geq 0$ *for all* $j \in \mathbb{N}$. *Then* $a_j(t) \geq 0$ *for all* $j \in \mathbb{N}$ *and all* $t > 0$.

Below, we illustrate why $\theta = \frac{5}{6}$ corresponds to Onsager's scaling by proving the following theorem (cf. [**13**], [**20**]):

THEOREM 3.4. *Let $a(t)$ be a positive solution to* (3.4) *such that*

$$\lim_{j\to\infty}\int_0^T \left(\lambda_j^{\frac{5}{6}}\, a_j(t)\right)^3 \, \mathrm{d}t = 0 \tag{3.7}$$

then $a(t)$ conserves energy on $[0,T]$.

Proof. We examine the total energy flux through the first J shells. We multiply equations (3.4) by $a_j(t)$, take the finite sum from $j = 0$ to $j = J$, and integrate over time for $0 \le t \le T$ to obtain

$$\int_0^t \sum_{j=0}^J a_j a_j' \, \mathrm{d}\tau = \int_0^t \sum_{j=0}^J \left(\lambda_{j-1}^{\frac{5}{2}}\, a_{j-1}^2 a_j - \lambda_j^{\frac{5}{2}}\, a_j^2 a_{j+1}\right) \, \mathrm{d}\tau\,. \tag{3.8}$$

The right-hand sum telescopes and we rewrite the left side

$$\int_0^t \sum_{j=0}^J \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}(a_j^2) \, \mathrm{d}\tau = -\int_0^t \lambda_J^{\frac{5}{2}}\, a_J^2 a_{J+1}\, \mathrm{d}\tau, \tag{3.9}$$

which yields

$$\frac{1}{2}\sum_{j=0}^J a_j^2(t) - \frac{1}{2}\sum_{j=0}^J a_j^2(0) = -\int_0^t \lambda_J^{\frac{5}{2}}\, a_J^2(\tau) a_{J+1}(\tau) \, \mathrm{d}\tau\,. \tag{3.10}$$

We apply Young's inequality to the integral on the righthand side of (3.10) to find

$$\begin{aligned} 0 \le \int_0^t \lambda_J^{\frac{5}{2}}\, a_J^2 a_{J+1} \, \mathrm{d}\tau &\le \int_0^t \lambda_J^{\frac{5}{2}} \left(\frac{(a_J^2)^{\frac{3}{2}}}{3/2} + \frac{(a_{J+1})^3}{3}\right) \mathrm{d}\tau \\ &\le \int_0^t \lambda_J^{\frac{5}{2}}\, a_J^3 \, \mathrm{d}\tau + \int_0^t \lambda_{J+1}^{\frac{5}{2}}\, a_{J+1}^3 \, \mathrm{d}\tau\,. \end{aligned} \tag{3.11}$$

Hence by our assumption,

$$\lim_{J\to\infty}\int_0^t \lambda_J^{\frac{5}{2}}\, a_J^2 a_{J+1} \, \mathrm{d}\tau = 0. \tag{3.12}$$

We take the limit of (3.10) as J goes to infinity to conclude that energy is conserved since $|a(t)|^2 = |a(0)|^2$. □

3. The Modified Galerkin Approximation with Flux

The strong and weak distances, denoted respectively by $\mathrm{d_S}$ and $\mathrm{d_W}$, are quantities given by

$$\mathrm{d_S}(a,b) := |a-b|, \tag{3.13}$$

and

$$\mathrm{d_W}(a,b) := \sum_{j=0}^{\infty} \frac{1}{\lambda^{(j)^2}} \frac{|a_j - b_j|}{1+|a_j-b_j|}. \tag{3.14}$$

DEFINITION 3.5. *The modified Galerkin approximation with flux, denoted by*

$$a^n(t) = (a_0^n(t), a_1^n(t), ..., a_n^n(t), 0, ...), \tag{3.15}$$

is a solution to the following finite system of ordinary differential equations:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} a_j^n - \lambda_{j-1}^{\frac{5}{2}}(a_{j-1}^n)^2 + \lambda_j^{\frac{5}{2}}\, a_j^n a_{j+1}^n = 0, \qquad j = 1,2,...,n-1,\\ &\frac{\mathrm{d}}{\mathrm{d}t} a_n^n - \lambda_{n-1}^{\frac{5}{2}}(a_{n-1}^n)^2 + \lambda^{\frac{5}{2}-2\theta}\, \lambda_n^{\frac{5}{2}-\theta}\, a_n^n = 0, \end{aligned} \tag{3.16}$$

with $a_j^n(0) = a_j^0$ for $j = 1,2,...,n,$ where θ is any positive number.

By a similar argument to Theorem 3.2 from [**19**], we obtain the following theorem:

THEOREM 3.6. *The sequence of the modified Galerkin approximation with flux converges to a solution of the dyadic model* (3.4)*.*

Proof. Denote $a(0) = a^0$, such that $a^0 \in l^2$ and let $T > 0$ be arbitrary. We will show that the modified Galerkin approximation with flux converges to a solution of (3.16) on $[0,T]$. We know there exists a unique solution $a^n(t)$ to (3.16) from the theory of ordinary differential equations. We will show the system of Galerkin approximations $\{a^n\}$ is weakly equicontinuous. There exists $M > 1$ such that $a_j^n(t) \leq$

M for any $t \in [0,T]$ and for all j and n. Then

$$\begin{aligned}\left|a_j^n(t) - a_j^n(s)\right| &\leq \left|\int_s^t \left(\lambda_{j-1}^{\frac{5}{2}}(a_{j-1}^n)^2(\tau) - \lambda_j^{\frac{5}{2}}\, a_j^n(\tau)a_{j+1}^n(\tau)\right) \mathrm{d}\tau\right| \\ &\leq \left(\lambda_{j-1}^{\frac{5}{2}}\, M^2 + \lambda_j^{\frac{5}{2}}\, M^2\right) |t-s|\,. \end{aligned} \tag{3.17}$$

Thus

$$\mathrm{d_W}\left(a^n(t), a^n(s)\right) = \sum_{j=0}^{\infty} \frac{1}{\lambda^{(j)^2}} \frac{\left|a_j^n(t) - a_j^n(s)\right|}{1 + \left|a_j^n(t) - a_j^n(s)\right|} \leq c\,|t-s|\,, \tag{3.18}$$

for some constant c independent of n. Then $\{a^n\}$ is an equicontinuous sequence in $C([0,T];l^2_W)$ with bounded initial data. The Arzelà-Ascoli theorem then implies that $\{a^n\}$ is relatively compact in $C([0,T];l^2_W)$. Passage to a subsequence yields a weakly continuous l^2-valued function $a(t)$ such that

$$a^{n_m} \to a \quad \text{as} \quad n_m \to \infty \quad \text{in} \quad C([0,T];l^2_W). \tag{3.19}$$

In particular, $a_j^{n_m} \to a_j(t)$ as $n_m \to \infty$ for all j and for all $t \in [0,T]$. Thus $a(0) = a^0$. Furthermore

$$a_j^{n_m}(t) = a_j^{n_m}(0) + \int_0^t \left(\lambda_{j-1}^{\frac{5}{2}}(a_{j-1}^{n_m})^2(\tau) - \lambda_j^{\frac{5}{2}}\, a_j^{n_m}(\tau)a_{j+1}^{n_m}(\tau)\right) \mathrm{d}\tau, \tag{3.20}$$

for $j \leq n_m - 1$. Now let $n_m \to \infty$. Then

$$a_j(t) = a_j(0) + \int_0^t \left(\lambda_{j-1}^{\frac{5}{2}}\, a_{j-1}^2(\tau) - \lambda_j^{\frac{5}{2}}\, a_j(\tau)a_{j+1}(\tau)\right) \mathrm{d}\tau\,. \tag{3.21}$$

Since $a_j(t)$ is continuous, then $a_j \in C^1([0,T])$ and it satisfies our inviscid dyadic system.

□

LEMMA 3.7. *If $a(t)$ solves* (3.4) *with initial condition $a(t_0) = a^0$, then $\tilde{a}(t) = \eta a(\eta t)$ is a solution to* (3.4) *with initial condition $\tilde{a}(t_0) = \eta a(\eta t_0) = \tilde{a}^0$.*

PROOF. For $j = 0$, the result is trivial. For $j = 1, 2, 3, ...$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} a_j(t) = \lambda_{j-1}^{\frac{5}{2}} a_{j-1}^2(t) - \lambda_j^{\frac{5}{2}} a_j(t) a_{j+1}(t). \tag{3.22}$$

So

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \tilde{a}_j(t) &= \eta^2 \frac{\mathrm{d}}{\mathrm{d}t} a_j(\eta t) \\ &= \eta^2 \left(\lambda_{j-1}^{\frac{5}{2}} a_{j-1}^2(\eta t) - \lambda_j^{\frac{5}{2}} a_j(\eta t) a_{j+1}(\eta t) \right) \\ &= \lambda_{j-1}^{\frac{5}{2}} (\eta a_{j-1}(\eta t))^2 - \lambda_j^{\frac{5}{2}} (\eta a_j(\eta t)) (\eta a_{j+1}(\eta t)) \\ &= \lambda_{j-1}^{\frac{5}{2}} \tilde{a}_{j-1}^2(t) - \lambda_j^{\frac{5}{2}} \tilde{a}_j(t) \tilde{a}_{j+1}(t). \end{aligned} \tag{3.23}$$

Thus $\tilde{a}(t)$ satisfies (3.4) with initial condition $\tilde{a}(t_0) = \eta a(\eta t_0) = \tilde{a}^0$. $\square$

4. Regularity of the Inviscid Dyadic Model

In this section, we study the regularity of positive solutions to the inviscid dyadic model. We apply the change of variables $c_j(t) = \lambda^{2\theta - \frac{5}{2}} \lambda_j^{\theta} a_j(t)$ to rewrite the equations as

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} c_j(t) = \lambda_j^{\frac{5}{2} - \theta} \left(c_{j-1}(t)^2 - \gamma\, c_j(t) c_{j+1}(t) \right), \qquad j = 1, 2, ..., \\ &c_0(t) = 0, \end{aligned} \tag{3.24}$$

where $\gamma = \lambda^{\frac{5}{2} - 3\theta}$. We choose

$$\theta = \frac{3}{5}. \tag{3.25}$$

THEOREM 3.8. *Let $a(t)$ be a positive solution to* (3.4). *There exists $\delta > 0$ such that if $c_j(0) \leq \delta < 1$ for any $j \in \mathbb{N}$, then $c_j(t) < 1$ for any $j \in \mathbb{N}$ and for all $t > 0$.*

PROOF. By the uniqueness proved in [**3**] and by Theorem 3.6, we have

$$a_j(t) = \lim_{m \to \infty} a_j^m(t), \tag{3.26}$$

where $a_j^m(t)$ is the m^{th} order Galerkin approximation of $a_j(t)$. So it suffices to prove the theorem for the Galerkin approximation $a^m(t)$. We will suppress the notation by omitting the index m.

Fix m and consider the m^{th} Galerkin approximation

$$c(t) = (c_0(t), c_1(t), ..., c_m(t), 0, ...). \tag{3.27}$$

Suppose for contradiction there exists $j_0 \in \mathbb{N}$ such that there is a time $T_0 > 0$ for which $c_{j_0}(T_0) = 1$ but $c_{j_0}(t) < 1$ for $0 < t < T_0$. Define the set of indices

$$I := \{j \in \mathbb{N} : j \leq m\}. \tag{3.28}$$

If $c_j(t) < 1$ for all $j \in I$ for any time $0 < t < T_0$, then let $n = j_0$. Otherwise, if there is a $j \in I$ such that $c_j(t) = 1$ for some time $0 < t \leq T_0$, then define $t_j > 0$ to be the time such that $c_j(t_j) = 1$ but $c_j(t) < 1$ for $0 < t < t_j \leq T_0$. If $c_j(t) < 1$ on $(0, T_0]$, then let $t_j = \infty$. Define

$$t^* := \min_{j \in I} t_j. \tag{3.29}$$

Define

$$n := \min\{j \in I : c_j(t^*) = 1\}, \tag{3.30}$$

and note that $n \neq 1$ since

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} c_1(t) = -\lambda^{\frac{5}{2}-\theta} \gamma c_1(t) c_2(t) < 0, \\ &c_1(0) \leq \delta < 1, \end{aligned} \tag{3.31}$$

as $c_1(t), c_2(t) \geq 0$ for all $t > 0$. Then $c_1(t)$ is a non-increasing function with initial value strictly below 1. Thus $c_1(t)$ cannot cross 1 and hence $t_1 = \infty$.

Thus, there is a fixed $n \in I$ such that $c_n(t) < 1$ for $0 < t < t^*$, $c_n(t^*) = 1$, and $c_j(t) < 1$ for all other $j \in I \backslash \{n\}$ and $0 < t < t^*$. We rescale time as $b_j(t) = c_j(\lambda_n^{\theta - \frac{5}{2}} t)$, which satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} b_j(t) = \lambda_{n-j}^{\theta-\frac{5}{2}}(b_{j-1}(t)^2 - \gamma\, b_j(t) b_{j+1}(t)), \tag{3.32}$$

where $b_j(0) < \delta$ and $b_j(t) < 1$ for all j and $0 < t < T^*$, where $T^* = \lambda_n^{\frac{5}{2}-\theta}\, t^*$.

For a very rough estimate for $b_n(t)$ for $t < t_0$, we first fix $k = 0.96$. Note $\delta < k < 1$. There exists time $t_0 > 0$ such that $k < b_n(t) < 1$ for $t_0 < t < T^*$ and $b_n(t_0) = k$. Recall our assumption on the initial data: $b_n(0) \le \delta$. So

$$\frac{\mathrm{d}}{\mathrm{d}t} b_n(t) = b_{n-1}(t)^2 - \gamma b_n(t) b_{n+1}(t) < 1, \tag{3.33}$$

since $b_{n-1}(t) < 1$ and $b_n(t), b_{n+1}(t) > 0$ for $0 < t < T^*$. Thus we have a lower bound on t_0: $t_0 \ge k - \delta$. Apply Gronwall's inequality backward in time for $b_n(t)$ to arrive at the following lower bound:

$$b_n(t) \ge k - t_0 + t \quad \text{for} \quad t \in [t_0 - k + \delta, t_0] \subseteq [\max\{0, t_0 - k\}, t_0]. \tag{3.34}$$

Next, we estimate $b_{n+1}(t_0)$. For $t \in [t_0 - k + \delta, t_0]$, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} b_{n+1}(t) &= \lambda^{\frac{5}{2}-\theta}\left(b_n(t)^2 - \gamma\, b_{n+1}(t) b_{n+2}(t)\right) \\ &\ge \lambda^{\frac{5}{2}-\theta}\left((k - t_0 + t)^2 - \gamma\, b_{n+1}(t)\right) \\ &= \lambda^{\frac{5}{2}-\theta}(k - t_0 + t)^2 - \lambda^{\frac{5}{2}-\theta}\,\gamma\, b_{n+1}(t). \end{aligned} \tag{3.35}$$

This yields the initial value problem:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} b_{n+1}(t) + \lambda^{\frac{5}{2}-\theta}\,\gamma\, b_{n+1}(t) \ge \lambda^{\frac{5}{2}-\theta}(k - t_0 + t)^2, \\ &b_{n+1}(t_0 - k + \delta) \ge 0. \end{aligned} \tag{3.36}$$

Apply Gronwall's inequality to find

$$\begin{aligned} b_{n+1}(t) &\geq b_{n+1}(t_0-k+\delta)\,\mathrm{e}^{-\int_{t_0-k+\delta}^{t}\lambda^{\frac{5}{2}-\theta}\,\gamma\,\mathrm{d}\tau} \\ &\qquad + \int_{t_0-k+\delta}^{t} \mathrm{e}^{-\int_s^t \lambda^{\frac{5}{2}-\theta}\,\gamma\,\mathrm{d}\tau}\,\lambda^{\frac{5}{2}-\theta}(k-t_0+s)^2\;\mathrm{d}s \\ &\geq \int_{t_0-k+\delta}^{t} \mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\,\gamma(t-s)}\,\lambda^{\frac{5}{2}-\theta}(k-t_0+s)^2\;\mathrm{d}s\,. \end{aligned} \tag{3.37}$$

An application of integration by parts yields

$$\begin{aligned} b_{n+1}(t) \geq \lambda^{\frac{5}{2}-\theta}\,\mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\,\gamma\,t}\Bigg[& \frac{(k-t_0+s)^2}{\lambda^{\frac{5}{2}-\theta}\,\gamma}\,\mathrm{e}^{\lambda^{\frac{5}{2}-\theta}\,\gamma\,s}\Bigg|_{t_0-k+\delta}^{t} \\ & - \int_{t_0-k+\delta}^{t} \frac{2(k-t_0+s)}{\lambda^{\frac{5}{2}-\theta}\,\gamma}\,\mathrm{e}^{\lambda^{\frac{5}{2}-\theta}\,\gamma\,s}\,\mathrm{d}s\Bigg]\,. \end{aligned} \tag{3.38}$$

We integrate by parts again to get

$$\begin{aligned} b_{n+1}(t) &\geq \frac{1}{\gamma}\,\mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\,\gamma\,t}\Bigg[\Bigg((k-t_0+s)^2\,\mathrm{e}^{\lambda^{\frac{5}{2}-\theta}\,\gamma\,s} - \frac{2(k-t_0+s)}{\lambda^{\frac{5}{2}-\theta}\,\gamma}\,\mathrm{e}^{\lambda^{\frac{5}{2}-\theta}\,\gamma\,s}\Bigg)\Bigg|_{t_0-k+\delta}^{t} \\ &\qquad + \int_{t_0-k+\delta}^{t} \frac{2}{\lambda^{\frac{5}{2}-\theta}\,\gamma}\,\mathrm{e}^{\lambda^{\frac{5}{2}-\theta}\,\gamma\,s}\,\mathrm{d}s\Bigg] \\ &= \frac{1}{\gamma}\,\mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\,\gamma\,t}\Bigg[(k-t_0+s)^2\,\mathrm{e}^{\lambda^{\frac{5}{2}-\theta}\,\gamma\,s} - \frac{2(k-t_0+s)}{\lambda^{\frac{5}{2}-\theta}\,\gamma}\,\mathrm{e}^{\lambda^{\frac{5}{2}-\theta}\,\gamma\,s} \\ &\qquad + \frac{2}{\lambda^{5-2\theta}\,\gamma^2}\,\mathrm{e}^{\lambda^{\frac{5}{2}-\theta}\,\gamma\,s}\Bigg]\Bigg|_{t_0-k+\delta}^{t} \\ &= \frac{(k-t_0+t)^2}{\gamma} - \frac{2(k-t_0+t)}{\lambda^{\frac{5}{2}-\theta}\,\gamma^2} + \frac{2}{\lambda^{5-2\theta}\,\gamma^3} \\ &\qquad - \mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\,\gamma(t-t_0+k-\delta)}\left[\frac{\delta^2}{\gamma} - \frac{2\delta}{\lambda^{\frac{5}{2}-\theta}\,\gamma^2} + \frac{2}{\lambda^{5-2\theta}\,\gamma^3}\right]. \end{aligned} \tag{3.39}$$

Thus we have

$$
\begin{aligned}
b_{n+1}(t_0) \geq & \left[\frac{k^2}{\gamma} - \frac{2k}{\lambda^{\frac{5}{2}-\theta}\,\gamma^2} + \frac{2}{\lambda^{5-2\theta}\,\gamma^3}\right] \\
& - \mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\,\gamma(k-\delta)} \left[\frac{\delta^2}{\gamma} - \frac{2\delta}{\lambda^{\frac{5}{2}-\theta}\,\gamma^2} + \frac{2}{\lambda^{5-2\theta}\,\gamma^3}\right] =: B(\delta).
\end{aligned} \tag{3.40}
$$

As δ tends to 0,

$$
B(\delta) \to \left[\frac{k^2}{\gamma} - \frac{2k}{\lambda^{\frac{5}{2}-\theta}\,\gamma^2} + \frac{2}{\lambda^{5-2\theta}\,\gamma^3}\right] - \frac{2\,\mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\,\gamma\,k}}{\lambda^{5-2\theta}\,\gamma^3} > 0.447. \tag{3.41}
$$

So there exists δ small enough that $B(\delta) \geq 0.447 := B$, which we will use as the bound on initial condition $b_{n+1}(t_0)$.

We also seek and estimate for $b_{n\pm1}(t)$ for $t_0 < t \leq T^*$. By our assumptions when $t > t_0$, in particular that $b_{n-2}(t) \leq 1$ and $b_n(t) \geq k$, we get the following inequality from equation (3.32):

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t}\, b_{n-1}(t) \leq \lambda^{\theta-\frac{5}{2}}\left(1 - k\,\gamma\, b_{n-1}(t)\right), \\
& b_{n-1}(t_0) \leq 1.
\end{aligned} \tag{3.42}
$$

Then by Gronwall's inequality,

$$
\begin{aligned}
b_{n-1}(t) &\leq b_{n-1}(t_0)\,\mathrm{e}^{-\int_{t_0}^{t} \lambda^{\theta-\frac{5}{2}}\,k\,\gamma\,\mathrm{d}\tau} + \int_{t_0}^{t} \lambda^{\theta-\frac{5}{2}}\,\mathrm{e}^{-\int_{s}^{t} \lambda^{\theta-\frac{5}{2}}\,k\,\gamma\,\mathrm{d}\tau}\ \mathrm{d}s \\
&\leq \mathrm{e}^{-\lambda^{\theta-\frac{5}{2}}\,k\,\gamma(t-t_0)} + \int_{t_0}^{t} \lambda^{\theta-\frac{5}{2}}\,\mathrm{e}^{-\lambda^{\theta-\frac{5}{2}}\,k\,\gamma(t-s)}\ \mathrm{d}s \\
&= \mathrm{e}^{-\lambda^{\theta-\frac{5}{2}}\,k\,\gamma(t-t_0)}\left(1 - \frac{1}{k\,\gamma}\right) + \frac{1}{k\,\gamma} =: \hat{b}_{n-1}(t).
\end{aligned} \tag{3.43}
$$

By our assumptions when $t_0 < t \leq T^*$, in particular that $b_{n+2}(t) \leq 1$ and $b_n(t) \geq k$, we get the following inequality from equation (3.32):

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t}\, b_{n+1}(t) \geq \lambda^{\frac{5}{2}-\theta}\left(k^2 - \gamma\, b_{n+1}(t)\right), \\
& b_{n+1}(t_0) \geq B.
\end{aligned} \tag{3.44}
$$

Again, by Gronwall's inequality,

$$
\begin{aligned}
b_{n+1}(t) &\geq b_{n+1}(t_0)\, \mathrm{e}^{-\int_{t_0}^{t} \lambda^{\frac{5}{2}-\theta}\,\gamma\,\mathrm{d}\tau} + \int_{t_0}^{t} \mathrm{e}^{-\int_{s}^{t} \lambda^{\frac{5}{2}-\theta}\,\gamma\,\mathrm{d}\tau}\, \lambda^{\frac{5}{2}-\theta}\, k^2 \;\, \mathrm{d}s \\
&\geq B\, \mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\,\gamma(t-t_0)} + \int_{t_0}^{t} \mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\,\gamma(t-s)}\, \lambda^{\frac{5}{2}-\theta}\, k^2 \;\, \mathrm{d}s \\
&= B\, \mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\,\gamma(t-t_0)} + \frac{k^2}{\gamma}\, \mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\,\gamma\, t}\, \mathrm{e}^{\lambda^{\frac{5}{2}-\theta}\,\gamma\, s}\Big|_{t_0}^{t} \\
&= \mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\,\gamma(t-t_0)} \left(B - \frac{k^2}{\gamma}\right) + \frac{k^2}{\gamma} =: \tilde{b}_{n+1}(t).
\end{aligned}
\tag{3.45}
$$

We finally estimate $b_n(t)$ for $t_0 < t \leq T^*$. We use the bounds on $b_{n\pm1}(t)$ from above to find an upperbound on $b_n(t)$:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\, b_n(t) = b_{n-1}^2(t) - \gamma\, b_n(t) b_{n+1}(t) \leq \hat{b}_{n-1}^2(t) - \gamma\, b_n(t)\tilde{b}_{n+1}(t), \\
&b_n(t_0) \leq k.
\end{aligned}
\tag{3.46}
$$

Another application of Gronwall's inequality yields

$$
\begin{aligned}
b_n(t) \leq& k\, \mathrm{e}^{-\int_{t_0}^{t} \gamma\, \tilde{b}_{n+1}\,\mathrm{d}\tau} + \int_{t_0}^{t} \mathrm{e}^{-\int_{s}^{t} \gamma\, \hat{b}_{n+1}\,\mathrm{d}\tau}\, \hat{b}_{n-1}^2\,\mathrm{d}s \\
=& k\, \mathrm{e}^{-\int_{t_0}^{t} \gamma\, \mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\,\gamma(\tau-t_0)}\left(B-\frac{k^2}{\gamma}\right)+k^2\,\mathrm{d}\tau} \\
&+ \int_{t_0}^{t} \mathrm{e}^{-\int_{s}^{t} \gamma\, \mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\,\gamma(\tau-t_0)}\left(B-\frac{k^2}{\gamma}\right)+k^2\,\mathrm{d}\tau} \left(\mathrm{e}^{-\lambda^{\theta-\frac{5}{2}}\,k\,\gamma(s-t_0)}\left(1-\frac{1}{k\,\gamma}\right)+\frac{1}{k\,\gamma}\right)^2 \mathrm{d}s \\
=& k\, \mathrm{e}^{\lambda^{\theta-\frac{5}{2}}\left(B-\frac{k^2}{\gamma}\right)\left(\mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\,\gamma(t-t_0)}-1\right)-k^2(t-t_0)} \\
&+ \int_{t_0}^{t} \mathrm{e}^{\lambda^{\theta-\frac{5}{2}}\left(B-\frac{k^2}{\gamma}\right)\left(\mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\,\gamma(t-t_0)}-\mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\,\gamma(s-t_0)}\right)-k^2(t-s)} \\
&\qquad\qquad \cdot \left(\mathrm{e}^{-\lambda^{\theta-\frac{5}{2}}\,k\,\gamma(s-t_0)}\left(1-\frac{1}{k\,\gamma}\right)+\frac{1}{k\,\gamma}\right)^2 \mathrm{d}s \\
=:& \beta(t).
\end{aligned}
\tag{3.47}
$$

Then

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\beta(t) =& k\,\mathrm{e}^{\lambda^{\theta-\frac{5}{2}}\left(B-\frac{k^2}{\gamma}\right)\left(\mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\gamma(t-t_0)}-1\right)-k^2(t-t_0)}\left(\left(k^2-B\,\gamma\right)\mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\gamma(t-t_0)}-k^2\right) \\
&+\left(\mathrm{e}^{-\lambda^{\theta-\frac{5}{2}}k\,\gamma(t-t_0)}\left(1-\frac{1}{k\,\gamma}\right)+\frac{1}{k\,\gamma}\right)^2 \\
&+\left(\left(k^2-B\,\gamma\right)\mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\gamma(t-t_0)}-k^2\right) \\
&\qquad\cdot\int_{t_0}^{t}\mathrm{e}^{\lambda^{\theta-\frac{5}{2}}\left(B-\frac{k^2}{\gamma}\right)\left(\mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\gamma(t-t_0)}-\mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\gamma(s-t_0)}\right)-k^2(t-s)} \\
&\qquad\cdot\left(\mathrm{e}^{-\lambda^{\theta-\frac{5}{2}}k\,\gamma(s-t_0)}\left(1-\frac{1}{k\,\gamma}\right)+\frac{1}{k\,\gamma}\right)^2\mathrm{d}s\,.
\end{aligned}
\tag{3.48}
$$

The exponent

$$
\lambda^{\theta-\frac{5}{2}}\left(B-\frac{k^2}{\gamma}\right)\left(\mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\gamma(t-t_0)}-\mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\gamma(s-t_0)}\right)
\tag{3.49}
$$

is nonnegative, thus

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\beta(t) \le& \left(\mathrm{e}^{-\lambda^{\theta-\frac{5}{2}}k\,\gamma(t-t_0)}\left(1-\frac{1}{k\,\gamma}\right)+\frac{1}{k\,\gamma}\right)^2 \\
&+\left(\left(k^2-B\,\gamma\right)\mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\gamma(t-t_0)}-k^2\right)\int_{t_0}^{t}\frac{1}{(k\,\gamma)^2}\mathrm{e}^{-k^2(t-s)}\,\mathrm{d}s \\
=&\left(\mathrm{e}^{-\lambda^{\theta-\frac{5}{2}}k\,\gamma(t-t_0)}\left(1-\frac{1}{k\,\gamma}\right)+\frac{1}{k\,\gamma}\right)^2 \\
&+\left(\left(1-\frac{B\,\gamma}{k^2}\right)\mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\gamma(t-t_0)}-1\right)\frac{1}{(k\,\gamma)^2}\left(1-\mathrm{e}^{-k^2(t-t_0)}\right) \\
=&\,\mathrm{e}^{-2\lambda^{\theta-\frac{5}{2}}k\,\gamma(t-t_0)}\left(1-\frac{1}{k\,\gamma}\right)^2+\frac{2}{k\,\gamma}\left(1-\frac{1}{k\,\gamma}\right)\mathrm{e}^{-\lambda^{\theta-\frac{5}{2}}k\,\gamma(t-t_0)}+\frac{1}{(k\,\gamma)^2}\mathrm{e}^{-k^2(t-t_0)} \\
&+\frac{1}{(k\,\gamma)^2}\left(1-\frac{B\,\gamma}{k^2}\right)\mathrm{e}^{-\lambda^{\frac{5}{2}-\theta}\gamma(t-t_0)}-\frac{1}{(k\,\gamma)^2}\left(1-\frac{B\,\gamma}{k^2}\right)\mathrm{e}^{-(\lambda^{\frac{5}{2}-\theta}\gamma+k^2)(t-t_0)}\,.
\end{aligned}
\tag{3.50}
$$

We have shown exponential decay for the derivative $\beta'(t)$ and thus it suffices to show $\beta(t) < 1$ on a finite interval, which can be accomplished easily numerically since $\beta(t)$ is given explicitly. Hence $b_n(t) < 1$ for all $t > 0$, which contradicts our assumption that $b_n(t)$ is the first b_j that crosses 1. Thus $b_j(t) < 1$ for any $j \in \mathbb{N}$ and for all $t > 0$. The conclusion extends to $c_j(t)$.

$\square$

This leads to our main result:

THEOREM 3.9. *Let $a(t)$ be a positive solution to (3.4) such that*

$$\sup_j \lambda_j^\theta a_j(0) = M \tag{3.51}$$

for some $M < \infty$, then

$$\sup_j \lambda_j^\theta a_j(t) < \frac{M}{\delta} \tag{3.52}$$

for all $t > 0$.

PROOF. By Lemma 3.7, we have that if $a_j(t)$ solves (3.4) with $a_j(0) = a_j^0$, then $\tilde{a}_j(t) = \eta a_j(\eta t)$ is a solution to (3.4) with initial condition $\tilde{a}_j(0) = \eta a_j(0) = \eta a_j^0$. In particular, this is true for $\eta = \frac{\delta}{M}$. Since $\sup_j \lambda_j^\theta a_j(0) = M$, we have

$$\sup_j \lambda_j^\theta \tilde{a}_j(0) = \eta M = \frac{\delta}{M} M = \delta. \tag{3.53}$$

Define

$$b_j(t) := \lambda_j^\theta \tilde{a}_j(t). \tag{3.54}$$

Then we have

$$\sup_j b_j(0) < \delta. \tag{3.55}$$

Given such an upper bound on the initial condition of $b_j(t)$, then recall that the theorem above yields

$$b_j(t) < 1 \tag{3.56}$$

for all $t > 0$. Then

$$\sup_j b_j(t) = \lambda_j^\theta \, \tilde{a}_j(t) = \lambda_j^\theta \, \eta a_j(t) < 1. \tag{3.57}$$

Therefore

$$\sup_j \lambda_j^\theta \, a_j(t) < \frac{1}{\eta} = \frac{M}{\delta} \tag{3.58}$$

for all $t > 0$. □

Similar to Theorem 10 in [**3**], we obtain the following

COROLLARY 3.10. *There exists a constant $k(\theta) > 0$ such that*

$$\sup_j \lambda_j^\theta \, a_j(t) < k(\theta)|a(0)|^{\frac{2}{3}} t^{-\frac{1}{3}}, \quad \text{for all} \quad t > 0 \tag{3.59}$$

for every positive solution $a(t)$ of (3.4) with $a(0)$ in l^2.

Proof. By Theorem 3.9,

$$\sup_j \lambda_j^\theta \, a_j(t) < \frac{1}{\delta} \sup_j \lambda_j^\theta \, a_j(s) \tag{3.60}$$

for all $s \in [0, t]$. By [**3**], there exists a constant $f(\theta) > 0$ such that

$$\mathscr{L}\left\{t > 0 : a_j(t) > l_j \text{ for some } j\right\} \leq f(\theta)|a(0)|^2 \sum_{j=1}^{\infty} \frac{1}{\lambda_j^{\frac{5}{2}} \, l_j^3}, \tag{3.61}$$

where $\mathscr{L}$ denotes the Lebesgue measure and $(l_j)_{j \geq 1}$ is any positive, non-increasing sequence. Let

$$l_j = \frac{f(\theta)^{\frac{1}{3}} M^{\frac{1}{3}} |a(0)|^{\frac{2}{3}}}{\lambda_j^\theta \, t^{\frac{1}{3}}}, \tag{3.62}$$

where M is such that

$$\sum_{j=1}^{\infty} \lambda_j^{3\theta - \frac{5}{2}} < M. \tag{3.63}$$

This series converges since $3\theta - \frac{5}{2} < 0$ by (3.25). Then we get

$$\begin{aligned}\mathscr{L}\left\{s > 0 : a_j(s) > l_j \text{ for some } j\right\} &\leq f(\theta)|a(0)|^2 \sum_{j=1}^{\infty} \frac{1}{\lambda_j^{\frac{5}{2}}} \frac{\lambda_j^{3\theta}\, t}{f(\theta)|a(0)|^2 M} \\ &= \frac{1}{M} \sum_{j=1}^{\infty} \lambda_j^{3\theta - \frac{5}{2}}\, t \qquad (3.64) \\ &< t.\end{aligned}$$

Thus for some $s \in [0, t]$, we have $a_j(s) \leq l_j$ for all j. Thus

$$\lambda_j^{\theta}\, a_j(s) \leq f(\theta)^{\frac{1}{3}} M^{\frac{1}{3}} |a(0)|^{\frac{2}{3}} t^{-\frac{1}{3}} \qquad (3.65)$$

for all j. Then

$$\begin{aligned}\sup_j \lambda_j^{\theta}\, a_j(s) &\leq f(\theta)^{\frac{1}{3}} M^{\frac{1}{3}} |a(0)|^{\frac{2}{3}} t^{-\frac{1}{3}} \\ &= k(\theta)|a(0)|^{\frac{2}{3}} t^{-\frac{1}{3}},\end{aligned} \qquad (3.66)$$

where k is a constant that depends on θ. This yields the result

$$\sup_j \lambda_j^{\theta}\, a_j(t) \leq \frac{1}{\delta} k(\theta)|a(0)|^{\frac{2}{3}} t^{-\frac{1}{3}} \qquad (3.67)$$

for all $t > 0$, as desired. $\square$

CHAPTER 4

Regularity Criterion for the Three-dimensional Boussinesq Equations

1. The Three-Dimensional Boussinesq Equations

We consider the three-dimensional incompressible Boussinesq equations, which we recall are given by

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + \nu \Delta u + \theta e_3, \tag{4.1}$$

$$\frac{\partial \theta}{\partial t} + (u \cdot \nabla)\theta = k \Delta \theta, \tag{4.2}$$

$$\nabla \cdot u = 0, \tag{4.3}$$

with initial data

$$u(x, 0) = u_0(x), \tag{4.4}$$

$$\theta(x, 0) = \theta_0(x), \tag{4.5}$$

where $x \in \mathbb{R}^3$, $t \geq 0$, $u = u(x,t)$ is the velocity vector, $p = p(x,t)$ is the pressure scalar, $\theta = \theta(x,t)$ is the temperature scalar, and the initial velocity is divergence free. The fluid kinematic viscosity is $\nu \geq 0$, the thermal diffusivity is $k \geq 0$, and $e_3 = (0, 0, 1)^T$. When θ vanishes, the system reduces to the incompressible Navier-Stokes equations, which can be further reduced to the incompressible Euler equations when $\nu = 0$. The work in this chapter has been submitted for publication and is published as an eprint as K. Zaya, Regularity criterion for the three-dimensional Boussinesq equations, arXiv:1509.07434, 2015 (see [**57**]).

2. Overview of Regularity Results

In three-dimensions, regularity criteria for (4.1) - (4.3) have been developed in many cases. In [**46**] and [**47**], Qiu, Du, and Yao developed Serrin-type regularity criteria for the Boussinesq equations, where in [**46**] they showed a smooth solution to (4.1) - (4.3) on time interval $[0,T)$ will remain smooth at time T if $u \in L^q\big(0,T;B^s_{p,\infty}(\mathbb{R}^3)\big)$ for $\frac{2}{q}+\frac{3}{p}=1+s$, $\frac{3}{s+1}<p\le\infty$, $-1<s\le 1$, and $(p,s)\neq(\infty,1)$. Ishimura and Morimoto [**36**] proved the Beale-Kato-Majda-like regularity criterion $\nabla u \in L^1\big(0,T;L^\infty(\mathbb{R}^3)\big)$. Later Fan and Zhou [**30**] studied the Boussinesq equations with partial viscosity and proved Beale-Kato-Majda-like regularity criteria in terms of the vorticity: $\nabla\times u \in L^1\big(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)\big)$. More regularity criteria in the three-dimensional case can be found in [**45**], [**52**], [**53**], [**55**], [**56**], and [**58**]. Regularity criteria in the two-dimensional case has also been studied at length [**1**], [**9**], [**10**], [**11**], [**33**], [**34**], [**35**], [**50**], and [**54**], to name just a few.

In [**5**], Beale, Kato, and Majda proved if

$$\int_0^T \|\nabla\times u\|_{L^\infty}\ \mathrm{d}t<\infty, \tag{4.6}$$

then a smooth solution to the Navier-Stokes equations on $(0,T)$ does not blow up at time T. This condition was weakened for the Euler equations by Planchon [**44**] and was improved for the three-dimensional Navier-Stokes equations by Cheskidov and Shvydkoy [**21**]. In [**15**], Cheskidov and Dai developed Beale-Kato-Majda-like, but weaker, regularity criterion for the three-dimensional magneto-hydrodynamics equations. In this chapter, we prove the following Beale-Kato-Majda-like regularity criterion for the three-dimensional Boussinesq equations:

THEOREM 4.1. *Let $\big(u,\theta\big)$ be a weak solution to* (4.1)-(4.3) *on* $[0,T]$, *assume* $\big(u,\theta\big)$ *is regular on* $(0,T)$, *and*

$$\|u_{\le Q}\|_{B^1_{\infty,\infty}}\in L^1(0,T). \tag{4.7}$$

Then $\big(u(t),\theta(t)\big)$ *is regular on* $(0,T]$.

REMARK 4.2. *We note that the above regularity criterion also recovers the previous known Prodi-Serrin-type regularity, in particular we improve upon the results in* [**46**], *by recovering the whole range, including the endpoint* $(p,s)=(\infty,1)$. *The result also covers* $u\in C\big((0,T];B^{-1}_{\infty,\infty}\big)$. *Further, the criterion in Theorem 4.1 improves previous Beale-Kato-Majda-like criterion since it only imposes a condition on the low modes of the projection of the velocity* u.

3. Definitions

We work in the class of weak solutions:

DEFINITION 4.3. *A weak solution of* (4.1)-(4.3) *on* $[0,T]$ *is a pair of functions* (u,θ), u *divergence free, in the class*

$$u,\theta\in C_w\big([0,T];L^2(\mathbb{R}^3)\big)\cap L^2\big(0,T;H^1(\mathbb{R}^3)\big) \tag{4.8}$$

such that

$$\begin{aligned}\big(u(t),\phi(t)\big)-\big(u_0,\phi(0)\big)&\\ =\int_0^t \big(u(s),\partial_s\phi(s)\big)+\nu\big(u(s),\Delta\phi(s)\big)&+\big(u(s)\cdot\nabla\phi(s),u(s)\big)+\big(\theta(s)e_3,\phi(s)\big)\ \mathrm{d}s\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\big(\theta(t),\phi(t)\big)-\big(\theta_0,\phi(0)\big)&\\ =\int_0^t \big(\theta(s),\partial_s\phi(s)\big)+k\big(\theta(s),\Delta\phi(s)\big)&+\big(u(s)\cdot\nabla\phi(s),\theta(s)\big)\ \mathrm{d}s,\end{aligned} \tag{4.10}$$

for all divergence free test functions $\phi\in C_0^\infty\big([0,T]\times\mathbb{R}^3\big)$.

DEFINITION 4.4. *A Leray-Hopf weak solution of* (4.1)-(4.5) *is regular on time interval* $\mathcal{I}$ *if the Sobolev norm* $\|u\|_{H^s}$ *is continuous for* $s>\frac{1}{2}$ *on* $\mathcal{I}$.

REMARK 4.5. *One can apply a standard bootstrap argument to show if a solution is regular, then u and θ are smooth.*

The development of our regularity criterion is linked to the dissipation wave number. Similarly to [**15**] and [**21**], we define

DEFINITION 4.6. *The dissipation wave number $\Lambda(t)$ is*

$$\Lambda(t) = \min\{\lambda_q : \lambda_p^{-1} \|u_p\|_\infty < c \min\{\nu, k\}, \forall p > q, q \geq 0\} \tag{4.11}$$

for absolute constant c.

Then $Q(t) \in \mathbb{N}$ is the index such that $\lambda_{Q(t)} = \Lambda(t)$. The time-dependent function $\Lambda(t)$ separates the low frequency inertial range, where the nonlinear term dominates the dynamics, from the high frequency dissipative range, where viscous forces take over. Work with the dissipation wave number and determining modes have provided key improvements to previous known regularity results for the surface quasi-geostrophic equations, the magneto-hydrodynamics equations, and the Navier-Stokes equations (see [**14**], [**15**], [**16**], and [**21**]).

REMARK 4.7. *Although the definition of $\Lambda(t)$ above is different than in* [**15**]*, one may nonetheless use similar estimates. For the Boussinesq equations, there is no restriction on the parameter r of the dissipation wave number $\Lambda_r(t)$ defined for the magneto-hydrodynamics equations in* [**15**]*. Instead, for the Boussinesq equations, one may let $r = \infty$, where as for magneto-hydrodynamics, one requires $2 < r < 6$.*

4. Proof of Theorem 4.1

Proof. We test (4.1) with $(u_q)_q$ and (4.2) with $(\theta_q)_q$. This yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_q\|_2^2 \leq -\nu\|\nabla u_q\|_2^2 + \int_{\mathbb{R}^3} \left(u \cdot \nabla u\right)_q \cdot u_q \,\mathrm{d}x - \int_{\mathbb{R}^3} \left(\theta e_3\right)_q \cdot u_q \,\mathrm{d}x, \tag{4.12}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_q\|_2^2 \leq -k\|\nabla \theta_q\|_2^2 + \int_{\mathbb{R}^3} \left(u \cdot \nabla \theta\right)_q \cdot \theta_q \,\mathrm{d}x. \tag{4.13}$$

We multiply (4.12) by $\lambda_q^{2s}$and (4.13) by $\lambda_q^{2\sigma}$, add them together, and sum over q to arrive at

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{q=-1}^{\infty}\left(\lambda_q^{2s}\,\|u_q\|_2^2+\lambda_q^{2\sigma}\,\|\theta_q\|_2^2\right)\leq-\sum_{q=-1}^{\infty}\left(\lambda_q^{2s}\,\nu\|\nabla u_q\|_2^2+\lambda_q^{2\sigma}\,k\|\nabla\theta_q\|_2^2\right)$$
$$+I_1+I_2+I_3, \tag{4.14}$$

where

$$I_1=\sum_{q=-1}^{\infty}\lambda_q^{2s}\int_{\mathbb{R}^3}\big(u\cdot\nabla u\big)_q\cdot u_q\,\mathrm{d}x, \tag{4.15}$$

$$I_2=-\sum_{q=-1}^{\infty}\lambda_q^{2s}\int_{\mathbb{R}^3}\big(\theta e_3\big)_q\cdot u_q\,\mathrm{d}x, \tag{4.16}$$

$$I_3=\sum_{q=-1}^{\infty}\lambda_q^{2\sigma}\int_{\mathbb{R}^3}\big(u\cdot\nabla\theta\big)_q\cdot\theta_q\,\mathrm{d}x. \tag{4.17}$$

For (4.15), we refer the reader to the computations carried out in [**21**] on the Navier-Stokes equations, where they show

$$|I_1|\lesssim c\nu\sum_{q\geq-1}\lambda_q^{2s+2}\,\|u_q\|_2^2+f(t)\sum_{q\geq-1}\lambda_q^{2s}\,\|u_q\|_2^2, \tag{4.18}$$

where

$$f(t)=\|u_{\leq Q(t)}(t)\|_{B^1_{\infty,\infty}}=\sup_{q\leq Q(t)}\lambda_q\,\|u_q(t)\|_\infty. \tag{4.19}$$

We use Young's inequality to estimate (4.16) as

$$|I_2|=\Big|\sum_{q\geq-1}\lambda_q^{2s}\int_{\mathbb{R}^3}\big(\theta e_3\big)_q\cdot u_q\,\mathrm{d}x\Big|\lesssim\sum_{q\geq-1}\lambda_q^{2s}\left(\|u_q\|_2^2+\|\theta_q\|_2^2\right). \tag{4.20}$$

For (4.17), we use a similar method as in [**15**]. First we decompose (4.17) into three parts:

$$
\begin{aligned}
I_3 =& \sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2\sigma}\int_{\mathbb{R}^3}(u_{\leq p-2}\cdot\nabla\theta_p)_q\theta_q \ \mathrm{d}x \\
&+\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2\sigma}\int_{\mathbb{R}^3}(u_p\cdot\nabla\theta_{\leq p-2})_q\theta_q \ \mathrm{d}x \\
&+\sum_{q\geq -1}\sum_{p\geq q-2}\lambda_q^{2\sigma}\int_{\mathbb{R}^3}(u_p\cdot\nabla\tilde{\theta}_p)_q\theta_q \ \mathrm{d}x \\
=& I_{3,1}+I_{3,2}+I_{3,3}.
\end{aligned}
\tag{4.21}
$$

One may denote the Littlewood-Paley operator as Δ_q, so the Littlewood-Paley pieces of a function u can also be denoted as $\Delta_q u = u_q$. By Bony's paraproduct and commutator notation, which says

$$[\Delta_q, u_{\leq p-2}\cdot\nabla]\theta_p = \Delta_q(u_{\leq p-2}\cdot\nabla\theta_p) - u_{\leq p-2}\cdot\nabla\Delta_q\theta_p,$$

one may further decompose $I_{3,1}$ as

$$
\begin{aligned}
I_{3,1} =& \sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2\sigma}\int_{\mathbb{R}^3}[\Delta_q, u_{\leq p-2}\cdot\nabla]\theta_p\theta_q \ \mathrm{d}x \\
&+\sum_{q\geq -1}\lambda_q^{2\sigma}\int_{\mathbb{R}^3}u_{\leq q-2}\cdot\nabla\theta_q\theta_q \ \mathrm{d}x \\
&+\sum_{q\geq -1}\sum_{|q-p|\leq 2}\lambda_q^{2\sigma}\int_{\mathbb{R}^3}(u_{\leq p-2}-u_{\leq q-2})\cdot\nabla\Delta_q\theta_p\theta_q \ \mathrm{d}x \\
=& I_{3,1,1}+I_{3,1,2}+I_{3,1,3}.
\end{aligned}
\tag{4.22}
$$

In [**15**], they note that their term equivalent to our $I_{3,1,1}$ can be estimated as

$$|I_{3,1,1}| \lesssim ck\sum_{q\geq Q+2}\lambda_q^{2\sigma+2}\left\|\theta_q\right\|_2^2 + f(t)\sum_{q\geq -1}\lambda_q^{2\sigma}\left\|\theta_q\right\|_2^2. \tag{4.23}$$

The second term, $I_{3,1,2} = 0$ because of the divergence-free condition on u. We also refer the reader to [**15**], where one can find

$$|I_{3,1,3}| + |I_{3,3}| \lesssim ck \sum_{q\geq -1} \lambda_q^{2\sigma+2} \|\theta_q\|_2^2 + f(t) \sum_{-1\leq q\leq Q+2} \lambda_q^{2\sigma} \|\theta_q\|_2^2. \tag{4.24}$$

For $I_{3,2}$, we use Hölder's inequality to find

$$|I_{3,2}| \lesssim \sum_{q>-1} \lambda_q^{2\sigma} \|u_q\|_\infty \sum_{|q-p|\leq 2} \|\theta_p\|_2 \sum_{p'\leq p-2} \lambda_{p'} \|\theta_{p'}\|_2. \tag{4.25}$$

Then we split the sum into high and low modes. For the high modes we use the definition of $\Lambda(t)$ and for the low modes we use $f(t)$ to find

$$\begin{aligned}
|I_{3,2}| \lesssim & ck \sum_{q>Q} \lambda_q^{2\sigma+1} \sum_{|q-p|\leq 2} \|\theta_p\|_2 \sum_{p'\leq p-2} \lambda_{p'} \|\theta_{p'}\|_2 \\
& + f(t) \sum_{-1\leq q\leq Q} \lambda_q^{2\sigma-1} \sum_{|q-p|\leq 2} \|\theta_p\|_2 \sum_{p'\leq p-2} \lambda_{p'} \|\theta_{p'}\|_2 \\
\lesssim & ck \sum_{q>Q-2} \lambda_q^{2\sigma+1} \|\theta_q\|_2 \sum_{p'\leq q} \lambda_{p'} \|\theta_{p'}\|_2 \\
& + f(t) \sum_{-1\leq q\leq Q+2} \lambda_q^{2\sigma-1} \|\theta_q\|_2 \sum_{p'\leq q} \lambda_{p'} \|\theta_{p'}\|_2.
\end{aligned} \tag{4.26}$$

We rearrange and apply Jensen's inequality to arrive at

$$\begin{aligned}
|I_{3,2}| \lesssim & ck \sum_{q>Q-2} \lambda_q^{\sigma+1} \|\theta_q\|_2 \sum_{p'\leq q} \lambda_{q-p'}^{\sigma} \lambda_{p'}^{\sigma+1} \|\theta_{p'}\|_2 \\
& + f(t) \sum_{-1\leq q\leq Q+2} \lambda_q^{\sigma} \|\theta_q\|_2 \sum_{p'\leq q} \lambda_{q-p'}^{\sigma-1} \lambda_{p'}^{\sigma} \|\theta_{p'}\|_2 \\
\lesssim & ck \sum_{q\geq -1} \lambda_q^{2\sigma+2} \|\theta_q\|_2^2 + f(t) \sum_{-1\leq q\leq Q+2} \lambda_q^{2\sigma} \|\theta_q\|_2^2,
\end{aligned} \tag{4.27}$$

for $\sigma < 0$.

The above estimates on the pieces of (4.17) yield

$$|I_3| \lesssim ck \sum_{q\geq -1} \lambda_q^{2\sigma+2} \|\theta_q\|_2^2 + f(t) \sum_{q\geq -1} \lambda_q^{2\sigma} \|\theta_q\|_2^2. \tag{4.28}$$

Inserting the estimate in (4.18), (4.20), and (4.28) into (4.14) yields

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{q\geq -1}\left(\lambda_q^{2s}\left\|u_q\right\|_2^2+\lambda_q^{2\sigma}\left\|\theta_q\right\|_2^2\right) \lesssim & -\nu\sum_{q\geq -1}\lambda_q^{2s+2}\left\|u_q\right\|_2^2-k\sum_{q\geq -1}\lambda_q^{2\sigma+2}\left\|\theta_q\right\|_2^2 \\
& + c\nu\sum_{q\geq -1}\lambda_q^{2s+2}\left\|u_q\right\|_2^2 \\
& + \big(f(t)+1\big)\sum_{q\geq -1}\lambda_q^{2s}\left\|u_q\right\|_2^2 \\
& + ck\sum_{q\geq -1}\lambda_q^{2\sigma+2}\left\|\theta\right\|_2^2+\sum_{q\geq -1}\lambda_q^{2s}\left\|\theta_q\right\|_2^2 \\
& + f(t)\sum_{q\geq -1}\lambda_q^{2\sigma}\left\|\theta_q\right\|_2^2.
\end{aligned}
\tag{4.29}
$$

For $2s\leq 2\sigma+2$,

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{q\geq -1}\left(\lambda_q^{2s}\left\|u_q\right\|_2^2+\lambda_q^{2\sigma}\left\|\theta_q\right\|_2^2\right) \leq & -\nu\sum_{q\geq -1}\lambda_q^{2s+2}\left\|u_q\right\|_2^2-k\sum_{q\geq -1}\lambda_q^{2\sigma+2}\left\|\theta_q\right\|_2^2 \\
& + C_1c\nu\sum_{q\geq -1}\lambda_q^{2s+2}\left\|u_q\right\|_2^2 \\
& + C_2(ck+1)\sum_{q\geq -1}\lambda_q^{2\sigma+2}\left\|\theta\right\|_2^2 \\
& + C_3\big(f(t)+1\big)\sum_{q\geq -1}\lambda_q^{2s}\left\|u_q\right\|_2^2+\lambda_q^{2\sigma}\left\|\theta_q\right\|_2^2,
\end{aligned}
\tag{4.30}
$$

where C_1, C_2, and C_3 are absolute constants. The choice $c=\min\{\frac{1}{C_1},\frac{1}{C_2}-\frac{1}{k}\}$ yields

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{\dot H^s}^2+\|\theta\|_{\dot H^\sigma}^2\right)\leq C(\nu,k,s,\sigma)\big(f(t)+1\big)\left(\|u\|_{\dot H^s}^2+\|\theta\|_{\dot H^\sigma}^2\right).
\tag{4.31}
$$

By Grönwall's inequality, we can show $\|u\|_{\dot H^s}^2+\|\theta\|_{\dot H^\sigma}^2$ remains bounded on $(0,T)$ for $\frac{1}{2}\leq s<1$ and $s-1<\sigma<0$ if

$$
\|u_{\leq Q}\|_{B^1_{\infty,\infty}}\in L^1(0,T),
\tag{4.32}
$$

as desired. $\square$

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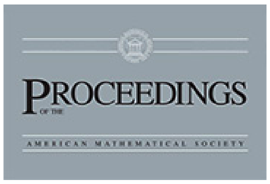

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Vita

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Education
B.S., Mathematics, University of Illinois at Chicago, Chicago, Illinois, 2009.
M.S., Mathematics, University of Illinois at Chicago, Chicago, Illinois, 2011.
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Publications

(1) A. Cheskidov and K. Zaya: Regularizing effect of the forward energy cascade in the inviscid dyadic model. *Proceedings of the American Mathematical Society*, 144(1):73-85, 2016.
(2) A. Cheskidov and K. Zaya: Lower bounds of potential blow-up solutions of the three-dimensional Navier-Stokes equations in $\dot{H}^{3/2}$. *Journal of Mathematical Physics*, 57(2), 2016.
(3) K. Zaya: Regularity criterion for the three-dimensional Boussinesq equations, *arXiv:1503.01784*, 2016.

Honors and Awards

(1) National Science Foundation support from grant DMS-1517583, PI Alexey Cheskidov, 2015.
(2) National Science Foundation support from grant DMS-1210896, PI Roman Shvydkoy, 2013-2015.
(3) OWLG Grant, Oberwolfach Leibniz Graduate Student, Mathematisches Forschungsinstitut Oberwolfach, 2015.
(4) Provosts Award for Graduate Research, University of Illinois at Chicago Graduate College, 2014.
(5) Victor Twersky Memorial Award, University of Illinois at Chicago Department of Mathematics, Statistics, and Computer Science, 2014.
(6) Teaching Award, University of Illinois at Chicago Department of Mathematics, Statistics, and Computer Science, 2011.
(7) Undergraduate Scholarship, University of Illinois at Chicago Department of Germanic Studies, 2008.
(8) Induction into the Delta Phi Alpha National German Honorary Society, University of Illinois at Chicago Department of Germanic Studies 2008.
(9) Association Award, University of Illinois at Chicago, 2007.

(10) Bernard Kurtin Fellowship, University of Illinois at Chicago Department of Mathematics, Statistics, and Computer Science, 2007.
(11) Educational Benefits Scholarship, Teamsters Union, 2005-2009.

Invited Presentations

(1) Speaker, Joint Mathematics Meetings: AMS Session on Partial Differential Equations, I, "Lower bounds of potential blow-up solutions of the three-dimensional Navier-Stokes equations in $\dot{H}^{3/2}$," Seattle, WA., January 2016.
(2) Speaker, Analysis and Applied Mathematics Seminar, "On regularity properties for fluid equations," University of Illinois at Chicago, Chicago, IL, November 2015.
(3) Speaker Harmonic Analysis and Partial Differential Equations Seminar, "On regularity properties for fluid equations," University of Virginia, Charlottesville, VA, November 2015.
(4) Poster Presentation, Analysis of Partial Differential Equations of Fluid Mechanics and Related Models Mini-school and Workshop, "Lower bounds for Sobolev norms of potential blow-up solutions to the three-dimensional Navier-Stokes equations," Rice University, Houston, TX, October 2015.
(5) Poster Presentation, American Mathematical Society Sectional Meeting #1112, "Lower bounds for Sobolev norms of potential blow-up solutions to the three-dimensional Navier-Stokes equations," Loyola University, Chicago, IL, October 2015.
(6) Poster Presentation, Mathematical Aspects of Hydrodynamics, "Lower bounds for Sobolev norms of potential blow-up solutions to the three-dimensional Navier-Stokes equations," Mathematisches Forschungsinstitut Oberwolfach, Oberwolfach, Germany, August 2015.
(7) Speaker, American Mathematical Society Sectional Meeting #1107, Special Session on Mathematical Fluid Dynamics and Turbulence, "Lower bounds for Sobolev norms of potential blow-up solutions to the three-dimensional Navier-Stokes equations," Georgetown University, Washington DC, March 2015.
(8) Speaker, Mathematics of Turbulence Long Program, "Regularizing effect of the forward energy cascade in the inviscid dyadic model," Institute for Pure and Applied Mathematics, Los Angeles, CA, September 2014.
(9) Speaker, Partial Differential Equations and Applied Mathematics Seminar, "Regularizing effect of the forward energy cascade in the inviscid dyadic model," Indiana University, Bloomington, IN, March 2014.
(10) Speaker, Joint Mathematics Meetings: AMS Special Session on Regularity Problems for Nonlinear PDEs Modeling Fluids and Complex Fluids, "Regularizing effect of the forward energy cascade in the inviscid dyadic model," Baltimore, MD, January 2014.
(11) Speaker, SIAM Conference on Analysis of Partial Differential Equations: Analysis of Nonlinear Differential Equations Arising in Fluid Dynamics Session, "Regularizing effect of the forward energy cascade in the inviscid dyadic model," Lake Buena Vista, FL, December 2013.

(12) Speaker, Analysis and Applied Mathematics Seminar, "Regularizing effect of the forward energy cascade in the inviscid dyadic model," University of Illinois at Chicago, Chicago, IL, November 2013.

Teaching Experience

Spring 2016	Math 165: Calculus for Business.
Spring 2014	Math 210: Calculus III.
Fall 2013	Math 165: Calculus for Business.
Spring 2013	Math 220: Introduction to Differential Equations.
Fall 2012	Math 220: Introduction to Differential Equations.
Summer 2012	Math 310: Applied Linear Algebra.
Spring 2012	Math 220: Introduction to Differential Equations.
Fall 2011	Math 181: Calculus II.
Summer 2011	Math 310: Applied Linear Algebra.
Spring 2011	Math 220: Introduction to Differential Equations.
Fall 2010	Math 181: Calculus II.
Summer 2010	Math 180: Calculus I.
Spring 2010	Math 180: Calculus I.
Fall 2009	Math 180: Calculus II.

Service

2015-2016	Mathematics Graduate Student Mentor.
August 2015	Mathematics Teaching Assistant Reviewer.
2013-2014	President of the UIC SIAM Student Chapter.
April 2013	Chicago Area SIAM Student Conference Co-organizer.
2012-2013	Graduate Analysis Seminar Co-organizer.
	Vice President of the UIC SIAM Student Chapter.
2011-2012	Treasurer of the UIC SIAM Student Chapter.
	President of the Mathematics Graduate Student Association.
2010-2011	Treasurer of the UIC SIAM Student Chapter.

Professional Membership

2010-2016	American Mathematical Society.
	Association for Women in Mathematics.
	Society for Industrial and Applied Mathematics.